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Finite Element Methods for Fourth Order Variational Inequalities

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FINITE ELEMENT METHODS FOR FOURTH ORDER VARIATIONAL INEQUALITIES

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Abstract

In this work we study finite element methods for fourth order variational inequalities. We begin with two model problems that lead to fourth order obstacle problems and a brief survey of finite element methods for these problems. Then we review the fundamental results including Sobolev spaces, existence and uniqueness results of variational inequalities, regularity results for biharmonic problems and fourth order obstacle problems, and finite element methods for the biharmonic problem. In Chapter 2 we also include three types of enriching operators which are useful in the convergence analysis. In Chapter 3 we study finite element methods for the displacement obstacle problem of clamped Kirchhoff plates. A unified convergence analysis is provided for C^1 finite element methods, classical nonconforming finite element methods and C^0 interior penalty methods. The key ingredient in the error analysis is the introduction of the auxiliary obstacle problem. An optimal $O(h)$ error estimate in the energy norm is obtained for convex domains. We also address the approximations of the coincidence set and the free boundary. In Chapter 4 we study a Morley finite element method and a quadratic C^0 interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates with general Dirichlet boundary conditions on general polygonal domains. We prove the magnitudes of the errors in the energy norm and the L^∞ norm are $O(h^\alpha)$, where $\alpha > 1/2$ is determined by the interior angles of the polygonal domain. Numerical results are also presented to illustrate the performance of the methods and verify the theoretical results obtained in Chapter 3 and Chapter 4. In Chapter 5 we consider an elliptic optimal control problem with state constraints. By formulating the problem as a fourth order obstacle problem with the boundary condition of simply supported plates, we study a quadratic C^0 interior penalty method and derive

the error estimates in the energy norm based on the framework we introduced in Chapter 3. The rate of convergence is derived for both quasi-uniform meshes and graded meshes. Numerical results presented in this chapter confirm our theoretical results.

Chapter 1
Introduction

1.1 Fourth Order Obstacle Problems

Variational inequalities arise from a wide range of application areas including mechanics and physics. Many practical problems, such as obstacle problems, optimal control problems, Stefan problems, unilateral problems, lead to variational inequalities. References on the theory and numerical analysis of variational inequalities include [56, 62, 66, 78, 80, 93]. We mainly focus on the numerical analysis for fourth order obstacle problems.

First, we consider the mathematical model for the bending of a clamped thin plate that satisfies the Kirchhoff-Love hypothesis, which is illustrated in Figure 1.1. The goal is to find the equilibrium position of the thin plate that lies above an obstacle under some external force.

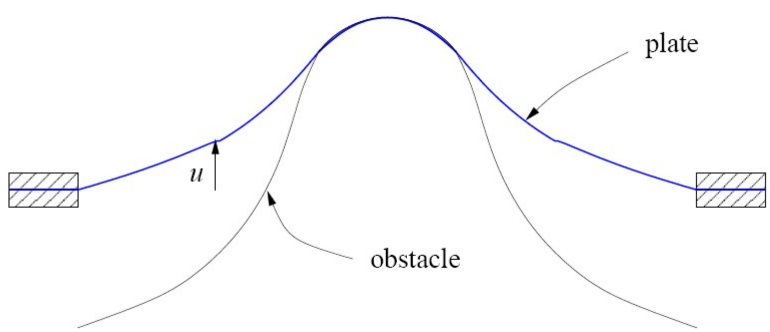


FIGURE 1.1. The displacement obstacle problem of a clamped Kirchhoff plate

Suppose the configuration domain Ω is a convex polygon in $\mathbb{R}^2$, u is the vertical displacement of the mid-surface of the thin plate, $f(x) \in L^2(\Omega)$ is the vertical load density divided by the flexural rigidity of the plate and $\psi(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ is the obstacle function such that $\psi(x) < 0$ on $\partial\Omega$.

By the principle of energy minimization, the obstacle problem for a thin plate can be stated as follows:

Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} G(v), \tag{1.1.1}$$

where

$$K = \{v \in H_0^2(\Omega) : v \geq \psi \text{ in } \Omega\}, \tag{1.1.2}$$

$$G(v) = \frac{1}{2}a(v,v) - (f,v), \tag{1.1.3}$$

$$a(v,w) = \int_\Omega D^2 v : D^2 w \, dx \text{ and } (f,v) = \int_\Omega f v \, dx. \tag{1.1.4}$$

Here $D^2 v : D^2 w = \sum_{i,j=1}^2 v_{x_i x_j} w_{x_i x_j}$ is the Frobenius inner product between the Hessian matrices of v and w.

Since $a(\cdot,\cdot)$ is symmetric and K is a nonempty closed convex subset of $H_0^2(\Omega)$, the problem (1.1.1) is equivalent to the following variational inequality:

Find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u) \qquad \forall v \in K. \tag{1.1.5}$$

In case of $K = H_0^2(\Omega)$, i.e, there is no obstacle for the plate problem, the variational inequality reduces to the variational equation:

$$a(u,v) = (f,v) \qquad \forall v \in H_0^2(\Omega). \tag{1.1.6}$$

Assume the solution u is smooth, say $u \in H^4(\Omega)$, then we can also formulate the problem (1.1.1) in the strong (or complementarity) form:

$$\Delta^2 u - f \geq 0, \quad u \geq \psi, \quad (\Delta^2 u - f)(u - \psi) = 0. \tag{1.1.7}$$

However, it was shown in [42,43,60,61] that the solution u in general only belongs to $H^3(\Omega) \cap C^2(\Omega)$ on a convex polygonal domain Ω and hence $\Delta^2 u$ is no longer an L^2 function. In fact u does not belong to $H^4_{loc}(\Omega)$ even when the data f, ψ and $\partial\Omega$ are smooth (cf. [42]). The strong (or complementarity) form of the variational inequality only holds in a weak sense:

$$\mu \geq 0, \quad u \geq \psi, \quad \int_\Omega (u-\psi)\, d\mu = 0, \tag{1.1.8}$$

where μ is a nonnegative Borel measure defined by

$$a(u,\phi) - (f,\phi) = \int_\Omega \phi\, d\mu \qquad \forall \phi \in H^2_0(\Omega).$$

Second, we consider a model elliptic distributed optimal control problem with pointwise state constraints:

$$\text{minimize} \quad J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2\, dx + \frac{\beta}{2}\int_\Omega u^2\, dx \tag{1.1.9}$$

$$\text{over} \quad (y,u) \in H^1_0(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad \begin{cases} -\Delta y = u & \text{in } \Omega, \\ \quad y \geq \psi & \text{a.e. in } \Omega. \end{cases} \tag{1.1.10}$$

Here we assume $\Omega \subseteq \mathbb{R}^2$ is a bounded convex polygonal domain, $y_d \in L^2(\Omega)$ is the desired state, $\beta > 0$ is a fixed parameter, and $\psi(x)$ is chosen as in the obstacle problem of clamped plates.

Since Ω is convex, the state y belongs to $H^2(\Omega)$ by elliptic regularity. Replacing u with $-\Delta y$ in (1.1.9), we can solve the problem by looking for the minimizer of the reduced functional

$$\hat{J}(y) = \frac{1}{2}\int_\Omega (y-y_d)^2\, dx + \frac{\beta}{2}\int_\Omega (\Delta y)^2\, dx, \tag{1.1.11}$$

in the set

$$K = \{y \in H^2(\Omega) \cap H^1_0(\Omega) : y \geq \psi \text{ in } \Omega\}. \tag{1.1.12}$$

Similar to the problem (1.1.1), we can also formulate the problem (1.1.11) as a fourth order variational inequality like the one in (1.1.5) with a different bilinear form and the closed convex set K defined in (1.1.12). In fact, the problem (1.1.11) is related to the displacement obstacle problem of simply supported plates. Hence we can also treat (1.1.11) as an obstacle problem. By solving the optimal state $\bar{y}$ from (1.1.11), we obtain the solution $(\bar{y}, \bar{u})$ of the optimal control problem, where $\bar{u} = -\Delta \bar{y}$.

1.2 Literature Review

If we replace the clamped plate with an elastic membrane and change the bilinear form $a(\cdot, \cdot)$ and the set K to be

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx,$$
$$K = \{v \in H_0^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega\},$$

the problem (1.1.1) becomes a second order obstacle problem and the corresponding variational inequality (1.1.5) is a second order variational inequality.

For this problem, finite element methods have been developed since the 1970s. The piecewise linear finite element method for homogeneous boundary condition was discussed in [59], where an $O(h)$ error estimate was obtained. The optimal $O(h)$ and $O(h^{3/2-\epsilon})$ $(\epsilon > 0)$ error estimates were obtained for linear and quadratic finite element methods for sufficiently smooth data with nonhomogeneous boundary condition in [39]. A mixed finite element method was also discussed in [40]. Recently, the results were extended to Discontinuous Galerkin methods in [106]. More references can be found in [48, 65, 66, 78, 100, 105]. However, the successful error analysis relies on the full regularity of the solution, i.e., $u \in H^2(\Omega)$ (cf. [37, 80, 93]). Then the following strong (or complementarity) form can be used in the error analysis:

$$-(\Delta u + f) \geq 0, \quad u \geq \psi, \quad -(\Delta u + f)(u - \psi) = 0.$$

The key here is that Δu belongs to $L^2(\Omega)$.

However, this is not the case for fourth order obstacle problem of a clamped plate since the strong (or complementarity) form of the variational inequality only exists in the weak sense (cf. (1.1.8)). Instead, we can only obtain a sub-optimal error estimate $O(h^{1/2})$ if we follow the standard finite element analysis developed for second order variational inequalities by using (1.1.8). Hence, the lack of H^4 regularity is a serious issue and it is the main difficulty for deriving the optimal $O(h)$ error estimate.

Mixed finite element methods for fourth order obstacle problems were discussed in [63, 66] without convergence rates. Nonconforming finite element methods were studied in [103, 104], where an $O(h)$ error estimate was obtained. However in [103, 104] $\Delta^2 u$ is erroneously treated as a function in $L^2(\Omega)$. Discontinuous Galerkin methods were also investigated in [3] under a mistaken H^4 regularity of u. Note that mixed finite element methods and the Morley finite element method have been proposed for fourth order variational inequality with curvature constraints (cf. [67, 101, 102]). However a thorough analysis of finite element methods for fourth order obstacle problem was still missing due to the lack of H^4 regularity.

State constrained elliptic optimal control problems arise in many practical applications. However the analysis of these problems are far from easy. Unlike the control constrained elliptic optimal control problems, the Lagrange multiplier associated with the problem (1.1.9) is only a measure in general (cf. [12, 13, 44]). The low regularity of the Lagrange multiplier makes the numerical computation and theoretical analysis difficult.

There are many numerical algorithms developed for this problem. In [11] an augmented Lagrangian method was considered. In [12] a primal-dual strategy was proposed. A Lavrentiev regularization technique was introduced in [99] so that the associated Lagrange multiplier is in $L^2(\Omega)$. Another approach is to formulate the problem as a free boundary problem and a level set method was used to solve the problem (cf. [74]).

However there are only a few papers that analyze finite element methods for state constrained elliptic optimal control problems. The convergence of finite element approximations were established in [45] for optimal control problems of semilinear elliptic equations with finite state constraints. An extension of the result to semilinear distributed and boundary control problems with a less regular setting for the states was investigated in [46]. In [53], a *priori* error estimates were obtained by piecewise linear finite element method and $O(h^{1-\epsilon})$ ($\epsilon > 0$ arbitrary) error estimates for the H^1 error of the state and the L^2 error of the control were obtained on sufficiently smooth domains. The analysis in [53] was based on a semi-discrete approach (cf. [75]). Similar results were also obtained by a fully discrete scheme (cf. [86]). Since the error analysis for the state and the error analysis for the control are coupled in [53, 86], the estimates for the H^1 error of the state and the L^2 error of the control have the same magnitude, which in the case of a rectangle with quasi-uniform meshes is also $O(h^{1-\epsilon})$.

The elliptic optimal control problem with state constraints is also solved as a fourth order variational inequality in [84] by a Morley finite element method and in [68] by a mixed finite element method. However the analysis in [68, 84] relies on additional assumptions on the active set first introduced in [13].

Other relevant references on numerical methods for state constrained elliptic optimal control problems can be found in [47, 54, 73, 76, 77, 87, 94].

1.3 Outline of the Dissertation

The main goal of this dissertation is to design and analyze finite element methods for fourth order obstacle problems. The outline and main contributions of this dissertation are as follows.

In Chapter 2 we review the existence and uniqueness of the solution of variational inequalities and also the regularity results for fourth order obstacle problems. We introduce three types of finite element methods for the biharmonic equations, which can be generalized to methods for variational inequalities. We also introduce different types of enriching operators, which are useful to estimate the distance between the finite element spaces and the Sobolev spaces.

In Chapter 3 we consider the displacement two-sided obstacle problem of clamped Kirchhoff plates for convex polygonal domains and homogeneous Dirichlet boundary conditions. We develop a unified convergence analysis for conforming finite element methods, classical nonconforming finite element methods and discontinuous Galerkin methods. First, we present a general framework for the analysis for these methods. Then an auxiliary obstacle problem is introduced in order to connect the continuous and discrete obstacle problems. We establish important properties for auxiliary obstacle problems which are useful for deriving the optimal $O(h)$ error estimate in the energy norm. An $O(h)$ error estimate in the L^∞ norm is also derived although it is not optimal. We also address the approximations of the coincidence set and the free boundary.

In Chapter 4 we extend the results obtained in Chapter 3 to the problem with general Dirichlet boundary conditions on general polygonal domains. Since different finite element methods have different treatments for the nonhomogeneous boundary conditions, we consider two different types of methods: a Morley finite element method and a quadratic C^0 interior penalty method. In each case, we show that the order of convergence in the energy norm and the L^∞ norm are both $O(h^\alpha)$ where $\alpha \in (1/2, 1]$ depends on the interior angle of Ω. We also present numerical results for both methods to verify the theoretical results.

In Chapter 5 we study a quadratic C^0 interior penalty method for an elliptic optimal control problem with state constraints on convex polygonal domains. First, we reformulate the problem as a fourth order variational inequality which can be treated as an obstacle problem of a simply supported plate. We show the error for the state in an H^2-like energy norm is $O(h^\alpha)$ on quasi-uniform meshes (where $\alpha \in (0, 1]$ is determined by the interior angles of the domain) and $O(h)$ on graded meshes. The error for the control in the L^2 norm has the same behavior. We present numerical results to illustrate the performance of the method.

All of the notations used in this dissertation are collected in a list after the references.

Chapter 2
Fundamentals

2.1 Sobolev Spaces

We briefly review the concept of Sobolev spaces. More details can be found in [1, 58, 69, 70].

Let Ω be a domain in Euclidean space $\mathbb{R}^d$, where d is a positive integer. For a real-valued Lebesgue measurable function f, we denote its Lebesgue integral by

$$\int_\Omega f(x)\,dx.$$

We introduce the notation

$$\begin{aligned} \|f\|_{L^p(\Omega)} &= \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p} \qquad 1 \le p < \infty, \\ \|f\|_{L^\infty(\Omega)} &= ess\ sup\{|f(x)| : x \in \Omega\} \end{aligned}$$

and define the Lebesgue spaces

$$L^p(\Omega) = \{f : \|f\|_{L^p(\Omega)} < \infty\} \qquad 1 \le p \le \infty.$$

We view the space $L^p(\Omega)$ as a set of equivalence classes of functions which can be different on a set of measure zero. For $1 \le p \le \infty$, $L^p(\Omega)$ is a Banach space.

Note that for $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(v, w)_{L^2(\Omega)} = \int_\Omega vw\,dx.$$

Let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω. For such functions, we can take arbitrary classical derivatives in Ω.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ with non-negative integers α_i $(i = 1, \cdots, d)$, the length of α is given by

$$|\alpha| = \sum_{i=1}^{d} \alpha_i.$$

For $\phi \in C_0^\infty(\Omega)$, denote by $D^\alpha \phi$ the classical partial derivative

$$\frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Definition 2.1. *The set of locally integrable functions is denoted by*

$$L^1_{loc}(\Omega) = \{f : f \in L^1(K) \quad \forall K \subset\subset \Omega\}.$$

Definition 2.2. *We say a function $f \in L^1_{loc}(\Omega)$ has an α^{th}-weak derivative $D^\alpha_w f$, provided there exists a function $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g(x)\phi(x)\, dx \qquad \forall \phi \in C_0^\infty(\Omega),$$

and denote $D^\alpha_w f = g$.

If $f \in C^{|\alpha|}(\Omega)$, the weak derivative $D^\alpha_w f$ exists and coincides with $D^\alpha f$. This can be easily seen from the integration by parts formula. From now on, we will omit the symbol w in the weak derivatives $D^\alpha_w f$ and use $D^\alpha f$ instead.

Definition 2.3. *Let k be a non-negative integer, and let $f \in L^1_{loc}(\Omega)$. Suppose that the weak derivatives $D^\alpha f$ exist for all $|\alpha| \leq k$. Define the Sobolev norm*

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \qquad 1 \leq p < \infty,$$

$$\|f\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

We define the corresponding Sobolev spaces as follows:

$$W^{k,p}(\Omega) = \{f \in L^1_{loc}(\Omega) : \|f\|_{W^{k,p}(\Omega)} < \infty\} \qquad 1 \leq p \leq \infty.$$

Definition 2.4. *Let k be a non-negative integer, and let $f \in W^{k,p}(\Omega)$. Define the Sobolev semi-norm*

$$|f|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \qquad 1 \le p < \infty,$$

$$|f|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|=k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

Theorem 2.5. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

For non-negative integer k and $p = 2$, $W^{k,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(v, w)_{W^{k,2}(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha v D^\alpha w \, dx.$$

We will denote this space by $H^k(\Omega)$ and write $\|\cdot\|_{H^k(\Omega)}$ and $|\cdot|_{H^k(\Omega)}$ as its norm and semi-norm.

Definition 2.6. *Let k be a non-negative integer. We define $H^k_0(\Omega)$ to be the closure of $C^\infty_0(\Omega)$ in $H^k(\Omega)$.*

Theorem 2.7. *Assume Ω is open. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$ for $1 \le p < \infty$.*

Theorem 2.8. *Let Ω to be a Lipschitz open subset of $\mathbb{R}^d$. Then $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $1 \le p < \infty$.*

It is important to establish the connection between the Sobolev spaces. For example, it is useful to know whether $W^{k,p}(\Omega)$ belongs to the Banach space $L^q(\Omega)$ or the continuous space $C(\bar{\Omega})$. We recall the following embedding theorems.

Definition 2.9. *Suppose X and Y are Banach spaces. We say X is continuously embedded in Y, provided*

(i) $X \subseteq Y$,

(ii) There exist a positive constant C such that

$$\|x\|_Y \leq C\|x\|_X \qquad x \in X.$$

Definition 2.10. *Suppose X and Y are Banach spaces. We say X is compactly embedded in Y, provided X is continuously embedded in Y and each bounded sequence in X is precompact in Y, i.e., every bounded sequence of X has a convergent subsequence in Y.*

Theorem 2.11. (Continuous Embedding Theorem) *Let k be a non-negative integer and $1 \leq p \leq \infty$. Assume Ω is a bounded Lipschitz domain, the following mappings are continuous embeddings:*

$$W^{k,p}(\Omega) \longrightarrow L^{p^\star}(\Omega), \quad \frac{1}{p^\star} = \frac{1}{p} - \frac{k}{d} \qquad \text{if } k < \frac{d}{p}, \tag{2.1.1}$$

$$W^{k,p}(\Omega) \longrightarrow L^q(\Omega), \quad \forall q \in [1, \infty) \qquad \text{if } k = \frac{d}{p}, \tag{2.1.2}$$

$$W^{k,p}(\Omega) \longrightarrow C(\bar{\Omega}) \qquad \text{if } k > \frac{d}{p}. \tag{2.1.3}$$

Theorem 2.12. (Compact Embedding Theorem) *Under the assumptions of Theorem 2.11, the following mappings are compact embeddings:*

$$W^{k,p}(\Omega) \longrightarrow L^q(\Omega), \quad \forall q \in [1, p^\star), \quad \frac{1}{p^\star} = \frac{1}{p} - \frac{k}{d} \qquad \text{if } k < \frac{d}{p}, \tag{2.1.4}$$

$$W^{k,p}(\Omega) \longrightarrow L^q(\Omega), \quad \forall q \in [1, \infty) \qquad \text{if } k = \frac{d}{p}, \tag{2.1.5}$$

$$W^{k,p}(\Omega) \longrightarrow C(\bar{\Omega}) \qquad \text{if } k > \frac{d}{p}. \tag{2.1.6}$$

Definition 2.13. *Let Ω be an open set in $\mathbb{R}^d$. For a real number $s = k + \gamma$ where k is a non-negative integer and $\gamma \in (0,1)$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ for $1 \leq p < \infty$ as follows:*

$$W^{s,p}(\Omega) = \left\{ f \in W^{k,p}(\Omega) : \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^{\frac{d}{p}+\gamma}} \in L^p(\Omega \times \Omega), \quad \text{for all } |\alpha| = k \right\}.$$

Note that the Sobolev space $W^{s,p}(\Omega)$ is also a Banach space. In case of $p = 2$, $W^{s,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(v, w)_{W^{s,2}(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha v D^\alpha w \, dx$$
$$+ \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha v(x) - D^\alpha v(y))(D^\alpha w(x) - D^\alpha w(y))}{|x-y|^{d+2\gamma}} dxdy,$$

and we denote this space by $H^s(\Omega)$.

From Theorem 2.11, $W^{k,p}(\Omega)$ is embedded in the continuous space $C(\bar{\Omega})$. Hence we can define the boundary values on $\partial\Omega$ for such functions. However, for a function in the Sobolev space $W^{k,p}(\Omega)$, in general it is not continuous and is only defined almost everywhere in Ω. Since the boundary $\partial\Omega$ has measure zero in the Lebesgue sense, the meaning of "boundary values" along $\partial\Omega$ needs to be clarified. Now we denote by Tr the operator defined by $Tr(v) = v|_{\partial\Omega}$ for smooth function v defined on $\bar{\Omega}$. The following is the Trace Theorem on smooth domains.

Theorem 2.14. (Trace Theorem) *Let Ω be a bounded open subset of $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. Assume that $s - \frac{1}{p}$ is not an integer, $s - \frac{1}{p} = l + \sigma$, $0 < \sigma < 1$, l an integer ≥ 0. Then the mapping*

$$v \longmapsto \left\{ Tr(v), Tr\left(\frac{\partial v}{\partial \nu}\right), \cdots, Tr\left(\frac{\partial^l v}{\partial \nu^l}\right) \right\}$$

defined for smooth function v on $\bar{\Omega}$, has a unique bounded liner extension from

$$W^{s,p}(\Omega) \quad onto \quad \prod_{j=0}^{l} W^{s-j-1/p,p}(\partial\Omega).$$

Here ν is the unit outward normal on $\partial\Omega$.

Remark 2.15. *In particular, there exists a constant C such that*

$$\|v\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)} \qquad \forall v \in H^1(\Omega),$$

where v on the left-hand side is understood in the sense of $Tr(v)$. Note that this estimate is also valid for a bounded polygonal open subset Ω of $\mathbb{R}^d$. Moreover, the Sobolev space $H_0^1(\Omega)$ can also be defined as the set of functions whose trace on $\partial\Omega$ is zero, i.e.,

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) : Tr(v) = 0 \text{ on } L^2(\partial\Omega) \right\}.$$

The general Trace Theorem on polygonal domains is more subtle and the details can be found in [69, 70].

Theorem 2.16. (Poincaré-Friedrichs Inequalities [90]) *Let Ω be a bounded domain with a Lipschitz boundary in $\mathbb{R}^d$. There exists a positive constant C such that*

$$\|v\|_{L^2(\Omega)} \leq C \left(\left| \int_{\partial\Omega} v \, ds \right| + |v|_{H^1(\Omega)} \right) \qquad \forall v \in H^1(\Omega), \tag{2.1.7}$$

$$\|v\|_{L^2(\Omega)} \leq C \left(\left| \int_{\Omega} v \, dx \right| + |v|_{H^1(\Omega)} \right) \qquad \forall v \in H^1(\Omega), \tag{2.1.8}$$

$$\|v\|_{H^2(\Omega)} \leq C \left(\left(\int_{\partial\Omega} |v|^2 \, ds \right)^{\frac{1}{2}} + |v|_{H^2(\Omega)} \right) \qquad \forall v \in H^2(\Omega). \tag{2.1.9}$$

From Theorem 2.16, it is clear that the semi-norm $|v|_{H^1(\Omega)}$ is equivalent to the norm $\|v\|_{H^1(\Omega)}$ for any $v \in H_0^1(\Omega)$ and the semi-norm $|v|_{H^2(\Omega)}$ is equivalent to the norm $\|v\|_{H^2(\Omega)}$ for any $v \in H^2(\Omega) \cap H_0^1(\Omega)$.

2.2 Existence and Uniqueness of the Solution of Variational Inequalities

In this section, we discuss the existence and uniqueness of the solution of elliptic variational inequalities (cf. [80, 83]). The theory of variational inequalities is typically formulated in terms of bilinear forms on Hilbert spaces. We start this section with the formulation of elliptic variational inequalities in a Hilbert space setting.

Let H be a real Hilbert space with the inner product $(\cdot, \cdot)$. The associated norm $\|\cdot\|$ is defined by

$$\|v\| = (v, v)^{\frac{1}{2}}$$

for any $v \in H$. Let H' denote the dual space of H and the pairing between H and H' is denoted by $\langle f, v \rangle$ for $f \in H'$ and $v \in H$.

Let $a(\cdot, \cdot)$ be a bilinear form on H, i.e., $a(\cdot, \cdot) : H \times H \longrightarrow \mathbb{R}$ is linear in each of the variables in the following sense:

$$a(u+v, w) = a(u, w) + a(v, w),$$

$$a(u, v+w) = a(u, v) + a(u, w),$$

$$a(\lambda u, v) = a(u, \lambda v) = \lambda a(u, v),$$

for any $u, v, w \in H$ and $\lambda \in \mathbb{R}$.

Definition 2.17. *The bilinear form $a(\cdot, \cdot)$ is bounded on H if there exists a constant $C_1 > 0$ such that*

$$|a(v, w)| \leq C_1 \|v\| \|w\| \qquad \forall v, w \in H. \tag{2.2.1}$$

Definition 2.18. *The bilinear form $a(\cdot, \cdot)$ is coercive on H if there exists a constant $C_2 > 0$ such that*

$$a(v, v) \geq C_2 \|v\|^2 \qquad \forall v \in H. \tag{2.2.2}$$

Let K be a nonempty closed convex subset of H and $f \in H'$. We consider the following variational inequality:
Find $u \in K$ such that

$$a(u, v-u) \geq \langle f, v-u \rangle \qquad \forall v \in K. \tag{2.2.3}$$

We want to prove the existence and uniqueness of the solution to the variational inequality (2.2.3) when the bilinear form $a(\cdot, \cdot)$ is bounded and coercive. First, we consider the case where $a(\cdot, \cdot)$ is a symmetric bilinear form, i.e.,

$$a(v, w) = a(w, v) \qquad \forall v, w \in H.$$

If the bilinear form $a(\cdot,\cdot)$ is symmetric, bounded and coercive, then it defines an equivalent norm $(a(\cdot,\cdot))^{1/2}$ on H. This motivates us to consider the simple case where the bilinear form reduces to the usual inner product on H, i.e., $a(\cdot,\cdot) = (\cdot,\cdot)$.

Lemma 2.19. *Let K be a nonempty closed convex subset of a Hilbert space H. Given any $x \in H$, there exists a unique $u \in K$ such that*

$$\|x - u\| = \inf_{v\in K} \|x - v\|. \tag{2.2.4}$$

Proof. Let $d = \inf\{\|x - v\| : v \in K\}$ and $v_k \in K$ be a minimizing sequence such that

$$\lim_{k\to\infty} \|x - v_k\| = d.$$

We claim there exists $u \in K$ such that $v_k \to u$ as $k \to \infty$. Then u satisfies (2.2.4) because of

$$\|x - u\| = \lim_{k\to\infty} \|x - v_k\| = d.$$

For this purpose, it suffices to show v_k is a Cauchy sequence. Take $v_m \in K$, From the parallelogram law and the fact that $\frac{v_k+v_m}{2} \in K$, we have

$$\begin{aligned}
0 &\le \|v_k - v_m\|^2 \\
&= \|v_k - x + x - v_m\|^2 \\
&= 2\|v_k - x\|^2 + 2\|x - v_m\|^2 - \|v_k + v_m - 2x\|^2 \\
&= 2\left[\|v_k - x\|^2 + \|x - v_m\|^2 - 2\left\|\frac{v_k + v_m}{2} - x\right\|^2\right] \\
&\le 2\left(\|v_k - x\|^2 + \|x - v_m\|^2 - 2d^2\right).
\end{aligned} \tag{2.2.5}$$

Taking $k, m \to \infty$, we conclude that

$$\lim_{k,m\to\infty} \|v_k - v_m\| = 0.$$

The existence of u follows from the completeness of H.

Now we show the uniqueness. Suppose there exist two elements $u, u' \in K$ satisfying (2.2.4). Then similar to (2.2.5), we have

$$0 \leq \|u - u'\|^2 \leq 2\left(\|u - x\|^2 + \|x - u'\|^2 - 2d^2\right) = 0.$$

Therefore we have $u = u'$. $\square$

Remark 2.20. *The element u that satisfies* (2.2.4) *is called the projection of x on H and we denote it by $u = \mathcal{P}_K x$.*

Lemma 2.21. *Let K be a nonempty closed convex subset of a Hilbert space H. Then the projection $u = \mathcal{P}_K x$ of x on K is characterized by*

$$(u, v - u) \geq (x, v - u) \qquad \forall v \in K. \tag{2.2.6}$$

Proof. Let $u = \mathcal{P}_K x$. For any $v \in K$, since K is convex, we know

$$(1 - t)u + tv = u + t(v - u) \in K \qquad \forall t \in [0, 1].$$

We consider the function $\Phi(t)$ defined by

$$\Phi(t) = \|x - u - t(v - u)\|^2 = \|x - u\|^2 - 2t(x - u, v - u) + t^2\|v - u\|^2.$$

It takes a minimum at $t = 0$ by (2.2.4). Hence we have $\Phi'(0) \geq 0$, i.e.,

$$(u, v - u) \geq (x, v - u) \qquad \forall v \in K.$$

Conversely, suppose u satisfies (2.2.6). For any $v \in K$, we have

$$0 \leq (u - x, v - x + x - u) \leq -\|x - u\|^2 + (u - x, v - x),$$

which means

$$\|x - u\|^2 \leq (u - x, v - x) \leq \|u - x\|\|v - x\|,$$

and hence

$$\|x - u\| \leq \|x - v\| \qquad \forall v \in K.$$

$\square$

Corollary 2.22. *Let K be a nonempty closed convex subset of a Hilbert space H. Then the projection operator $\mathcal{P}_K$ is nonexpansive, i.e.,*

$$\|\mathcal{P}_K x - \mathcal{P}_K x'\| \leq \|x - x'\| \qquad \forall x, x' \in K.$$

Proof. From Lemma 2.21, we have

$$(\mathcal{P}_K x, v - \mathcal{P}_K x) \geq (x, v - \mathcal{P}_K x) \qquad \forall v \in K, \tag{2.2.7}$$

$$(\mathcal{P}_K x', v - \mathcal{P}_K x') \geq (x', v - \mathcal{P}_K x') \qquad \forall v \in K. \tag{2.2.8}$$

We take $v = \mathcal{P}_K x'$ in (2.2.7) and $v = \mathcal{P}_K x$ in (2.2.8). Adding these inequalities, we obtain

$$\|\mathcal{P}_K x - \mathcal{P}_K x'\|^2 \leq (\mathcal{P}_K x - \mathcal{P}_K x', x - x') \leq \|\mathcal{P}_K x - \mathcal{P}_K x'\|\|x - x'\|,$$

from which we conclude that $\|\mathcal{P}_K x - \mathcal{P}_K x'\| \leq \|x - x'\|$. $\square$

Now we consider the variational inequality (2.2.3) for $a(\cdot, \cdot) = (\cdot, \cdot)$. By the Riesz representation theorem for Hilbert spaces, there exists $x \in H$ such that

$$\langle f, v\rangle = (x, v) \qquad \forall v \in H. \tag{2.2.9}$$

Therefore, (2.2.3) becomes

$$(u, v - u) \geq (x, v - u) \qquad \forall v \in K. \tag{2.2.10}$$

Then existence and uniqueness of the solution to this problem follow from Lemma 2.19 and Lemma 2.21.

Theorem 2.23. *If $a(\cdot, \cdot)$ is a symmetric bounded coercive bilinear form on H and K is a nonempty closed convex subset of H, then given any $f \in H'$, there exists a unique solution of* (2.2.3).

Proof. Since $(a(v,v))^{1/2}$ defines a norm equivalent to $\|v\|$, we can view H as a Hilbert space associated with the inner product $a(\cdot,\cdot)$. Then the result follows from the same argument as discussed above. $\square$

Next we consider the general case where the bilinear form $a(\cdot,\cdot)$ is not necessarily symmetric. Moreover, we only assume $a(\cdot,\cdot)$ is coercive on the set $K-K$, i.e., there exists a constant $C_2 > 0$ such that

$$a(v-w, v-w) \geq C_2\|v-w\|^2 \qquad \forall v, w \in K. \tag{2.2.11}$$

Theorem 2.24. *Let K be a nonempty closed convex subset of a Hilbert space H and $a(\cdot,\cdot)$ be a bounded bilinear form on H which is coercive on the set $K-K$. Given $f \in H'$, there exists a unique solution u of* (2.2.3)*.*

Proof. First we establish the uniqueness of the solution. In fact, let u_1, u_2 be two solutions of (2.2.3). Then we have

$$a(u_1, v-u_1) \geq \langle f, v-u_1\rangle \qquad \forall v \in K, \tag{2.2.12}$$

$$a(u_2, v-u_2) \geq \langle f, v-u_2\rangle \qquad \forall v \in K. \tag{2.2.13}$$

Putting $v = u_2$ in (2.2.12) and $v = u_1$ in (2.2.13) and adding together, we have

$$C_2\|u_1-u_2\|^2 \leq a(u_1-u_2, u_1-u_2) \leq 0.$$

Hence $u_1 = u_2$.

Next, we prove existence. By the Riesz representation theorem for Hilbert spaces there exists a linear operator $A : H \longrightarrow H$ such that

$$a(v,w) = (Av, w) \qquad \forall v, w \in H. \tag{2.2.14}$$

The boundedness of the bilinear form implies the boundedness of the operator A, i.e.,

$$\|Av\| \leq C_1\|v\| \qquad \forall v \in H.$$

Given any $x \in K$, let $\Phi(x) \in H$ be defined by

$$(\Phi(x), v) = (x, v) - \rho\,[a(x, v) - \langle f, v\rangle] \qquad \forall v \in H,$$

where $\rho > 0$ will be chosen later. For any $x_1, x_2 \in K$, we have

$$(\Phi(x_1) - \Phi(x_2), v) = (x_1 - x_2 - \rho(A(x_1 - x_2), v)) \qquad \forall v \in H.$$

Therefore, by the coercivity on the set $K - K$ and boundedness of A, we obtain

$$\begin{aligned}
\|\Phi(x_1) - \Phi(x_2)\|^2 &= \|x_1 - x_2 - \rho A(x_1 - x_2)\|^2 \\
&= \|x_1 - x_2\|^2 + \rho^2\|A(x_1 - x_2)\|^2 - 2\rho(A(x_1 - x_2), x_1 - x_2) \\
&\leq \|x_1 - x_2\|^2 + \rho^2 C_1^2\|x_1 - x_2\|^2 - 2\rho C_2\|x_1 - x_2\|^2 \qquad (2.2.15) \\
&= [1 - (2\rho C_2 - \rho^2 C_1^2)]\|x_1 - x_2\|^2.
\end{aligned}$$

We choose $\rho \in (0, \frac{2C_2}{C_1^2})$ such that $1 - (2\rho C_2 - \rho^2 C_1^2) \in (0, 1)$. Then it follows from (2.2.15) that Φ defines a contraction mapping.

Since $\mathcal{P}_K$ is nonexpansive by Corollary 2.22, we know $\mathcal{P}_K\Phi : K \longrightarrow K$ is also a contraction mapping. Applying the Banach fixed point theorem, there exists a unique $u \in K$ such that $\mathcal{P}_K\Phi(u) = u$.

By Lemma 2.21 and the definition of Φ, we have the following variational inequality

$$\begin{aligned}
(\mathcal{P}_K\Phi(u), v - \mathcal{P}_K\Phi(u)) &\geq (\Phi(u), v - \mathcal{P}_K\Phi(u)) \\
&= (u, v - \mathcal{P}_K\Phi(u)) \\
&\qquad - \rho[a(u, v - \mathcal{P}_K\Phi(u)) - \langle f, v - \mathcal{P}_K\Phi(u)\rangle].
\end{aligned}$$

By replacing $\mathcal{P}_K\Phi(u)$ with u in the above equation, we finally show that $u = \mathcal{P}_K\Phi u \in K$ satisfies

$$a(u, v - u) \geq \langle f, v - u\rangle \qquad \forall v \in K.$$

$\square$

Remark 2.25. *The proof of Theorem* 2.24 *combines the result of the symmetric case and a fixed point theorem. In particular, when $K = H$ and $a(\cdot, \cdot)$ is coercive on H, then Theorem* 2.24 *reduces to the Lax-Milgram Theorem, i.e., there exists a unique solution $u \in H$ to the variational equation*

$$a(u, v) = \langle f, v \rangle \qquad \forall v \in H. \tag{2.2.16}$$

Remark 2.26. *The proof of Theorem* 2.24 *also gives a natural algorithm for computing the solution of* (2.2.3)*:*

- *Let $u^0 \in K$, $k = 0$,*
- *Set $u^{k+1} = \mathcal{P}_K(u^k - \rho(Au^k - x))$ for $0 < \rho < \frac{2C_2}{C_1^2}$,*
- *Set $k = k + 1$, return step 2.*

Here A and x are defined in (2.2.14) *and* (2.2.9)*. If $\mathcal{P}_K$ is easy to compute, then the sequence u^k converges to u. This method is also known as the gradient projection method.*

The following corollary is similar to Corollary 2.22 and we omit the proof.

Corollary 2.27. *Let K be a nonempty closed convex subset of a Hilbert space H and $a(\cdot, \cdot)$ be a bounded bilinear form on H which is coercive on the set $K - K$. Let $u_f \in K$ be the unique solution of* (2.2.3) *for $f \in H'$. Then the nonlinear map*

$$f \longmapsto u_f$$

is Lipschitz continuous, i.e.,

$$\|u_f - u_g\| \leq \frac{1}{C_2}\|f - g\|_{H'} \qquad \forall f, g \in H,$$

where C_2 is a coercivity constant given in (2.2.11)*.*

2.3 Regularity Results

In this section, we review the regularity results for both biharmonic problems and fourth order obstacle problems.

2.3.1 Regularity of Biharmonic Problems

We consider the biharmonic equation with two types of boundary conditions:

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{2.3.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{2.3.1b}$$

and

$$\Delta^2 u = f \qquad \text{in } \Omega, \tag{2.3.2a}$$

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega, \tag{2.3.2b}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain, n is the unit outer normal of Ω, and $f \in L^2(\Omega)$. The boundary value problem (2.3.1) is related to the bending of clamped Kirchhoff plates and (2.3.2) is related to the bending of simply supported Kirchhoff plates.

The weak formulation of the biharmonic problems is to find $u \in V$ such that

$$a(u,v) = \int_\Omega f v \, dx \qquad \forall v \in V, \tag{2.3.3}$$

where

$$a(v,w) = \int_\Omega D^2 v : D^2 w \, dx, \tag{2.3.4}$$

here $D^2 v : D^2 w = \sum_{i,j=1}^2 v_{x_i x_j} w_{x_i x_j}$ is the inner product of the Hessian matrices of v and w, and

$$V = H_0^2(\Omega) \text{ for (2.3.1)}, \tag{2.3.5}$$

$$V = H^2(\Omega) \cap H_0^1(\Omega) \text{ for (2.3.2)}. \tag{2.3.6}$$

The bilinear form $a(\cdot,\cdot)$ is bounded and coercive on V by the Poincaré-Friedrichs inequalities (Theorem 2.16). By the Lax-Milgram Theorem (Remark 2.25), there exists a unique solution u to (2.3.3) for the choices of V in (2.3.5) or (2.3.6).

First we consider the regularity result on smooth domains.

Theorem 2.28. *Let Ω be a smooth domain, $f \in L^2(\Omega)$ and u be the solution of* (2.3.3) *for V in* (2.3.5) *or* (2.3.6)*. Then $u \in H^4(\Omega)$ and*

$$\|u\|_{H^4(\Omega)} \le C_\Omega \|f\|_{L^2(\Omega)}.$$

Next, we consider a polygonal domain Ω. In this case, the solution u in general does not belong to $H^4(\Omega)$.

Theorem 2.29. *Assume $f \in L^2(\Omega)$ and Ω is a polygonal domain. Let u be the solution of* (2.3.3) *for V in* (2.3.5)*. Then $u \in H^{2+\alpha}(\Omega)$ for some $\alpha \in (1/2, 2]$. The value of α depends on the interior angles at the corners of Ω. In particular, $\alpha > 1$ for convex Ω. Moreover, we have*

$$\|u\|_{H^{2+\alpha}(\Omega)} \le C_\Omega \|f\|_{L^2(\Omega)}. \tag{2.3.7}$$

If u is the solution of (2.3.3) *for V in* (2.3.6)*. Then $u \in H^{2+\alpha}(\Omega)$ for some $\alpha \in (0, 2]$, here α can be close to 0 even when the domain is convex. In particular, $\alpha = 1$ if the largest interior angle of Ω is less than or equal to $\pi/2$. Moreover,* (2.3.7) *is also valid in this case.*

More details of the elliptic regularity result for the biharmonic equations can be found in [14, 51, 69, 70, 81, 85, 89].

2.3.2 Regularity of Fourth Order Obstacle Problems

In this section, we review the regularity theory of fourth order obstacle problems. We assume $\Omega \subseteq \mathbb{R}^2$ is a polygonal domain, $f \in L^2(\Omega)$, $\psi_1, \psi_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$\psi_1 < \psi_2 \text{ on } \bar{\Omega} \quad \text{and} \quad \psi_1 < 0 < \psi_2 \text{ on } \partial\Omega. \tag{2.3.8}$$

We define the set

$$K = \{v \in V : \psi_1 \le v \le \psi_2 \text{ in } \Omega\}, \tag{2.3.9}$$

and consider the following problem:

Find $u \in K$ such that

$$a(u, v-u) \ge (f, v-u) \qquad \forall v \in K, \tag{2.3.10}$$

where $a(\cdot,\cdot)$ is defined in (2.3.4).

If we choose $V = H_0^2(\Omega)$ in (2.3.9), then the variational inequality (2.3.10) is the displacement obstacle problem of clamped Kirchhoff plates. If we choose $V = H^2(\Omega) \cap H_0^1(\Omega)$, then (2.3.10) is the displacement obstacle problem of simply supported Kirchhoff plates.

Since $a(\cdot,\cdot)$ is symmetric, bounded and coercive on $V = H_0^2(\Omega)$ (or $H^2(\Omega) \cap H_0^1(\Omega)$) and K is a nonempty closed convex subset of $V = H_0^2(\Omega)$ (or $H^2(\Omega) \cap H_0^1(\Omega)$), the problem (2.3.10) has a unique solution by Theorem 2.23.

The following interior regularity theory can be found in [34, 42, 43, 60, 61].

Theorem 2.30. *Let u be the solution to the variational inequality* (2.3.10) *under our assumptions on f and the obstacle functions ψ_1, ψ_2. We have*

$$u \in H^3_{loc}(\Omega) \cap C^2(\Omega).$$

Remark 2.31. *The C^2 regularity result is obtained for the one-obstacle problem and $f = 0$ in* [42, 61]. *But it can be extended to the two-obstacle problem with nonhomogeneous right-hand side f under our assumptions on ψ_1, ψ_2* (cf. [34]). *In general, $u \notin C^2(\Omega)$ if we only assume $\psi_1 \le \psi_2$ in Ω* (cf. [43]) *and $u \notin H^4_{loc}(\Omega)$ even when the data is smooth* (cf. [42]).

Note that the above interior regularity result is independent of the boundary $\partial\Omega$. Now we consider the boundary regularity result. In view of (2.3.8), we have $\psi_1 < u < \psi_2$ in a neighborhood $\mathcal{N}$ of $\partial\Omega$, i.e., u is unconstrained in $\mathcal{N}$. Hence $\Delta^2 u = f$ in $\mathcal{N}$ and then $u \in H^{2+\alpha}(\mathcal{N})$ for some α determined by the interior angles of the domain and the choice of the Sobolev space V. Therefore, we can combine the interior regularity result in Theorem 2.30 and the elliptic regularity theory for boundary value problems in Theorem 2.29 to obtain the following corollary.

Corollary 2.32. *Let u be the solution to the variational inequality* (2.3.10) *where $V = H_0^2(\Omega)$ and under our assumptions on the data. Then $u \in H^{2+\alpha}(\Omega) \cap C^2(\Omega)$ for some $\alpha \in (1/2, 1]$ determined by the interior angles of Ω. In particular, $u \in H^3(\Omega) \cap C^2(\Omega)$ if Ω is convex.*

Let u be the solution to the variational inequality (2.3.10) *where $V = H^2(\Omega) \cap H_0^1(\Omega)$, then $u \in H^{2+\alpha}(\Omega) \cap C^2(\Omega)$ for some $\alpha \in (0, 1]$ determined by the interior angles of Ω. In particular, $u \in H^3(\Omega) \cap C^2(\Omega)$ if the largest interior angle of Ω is less than or equal to $\pi/2$.*

2.4 Finite Element Methods for the Biharmonic Problem

In this section, we consider finite element methods for the biharmonic equation with the boundary condition of clamped Kirchhoff plates, i.e., (2.3.3) for $V = H_0^2(\Omega)$. These methods form the basis for solving the fourth order obstacle problem. For simplicity, we assume $\Omega \subseteq \mathbb{R}^2$ is a bounded convex polygon. Then the solution $u \in H^3(\Omega)$ by Theorem 2.29.

2.4.1 C^1 Conforming Finite Element Methods

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h, i.e., there exists a constant κ such that

$$\frac{h_T}{\rho_T} \leq \kappa \qquad \forall T \in \mathcal{T}_h, \tag{2.4.1}$$

where ρ_T is the diameter of the largest ball contained in $\bar{T}$.

Let V_h be the Hsieh-Clough-Tocher macro finite element space [50]. The degrees of freedom of $v \in V_h$ consist of (i) the values of the derivatives of v up to order 1 at the interior vertices, (ii) the values of the normal derivative of v at the midpoints of the edges in the interior edges of $\mathcal{T}_h$.

Remark 2.33. *Note that $V_h \subseteq C^1(\bar{\Omega})$. The nodal variables for this element are depicted in Figure* 2.1*. In the figure, the solid dot represents the value of a shape function, the circle represents the values of first order derivatives of a shape function, and the arrow represents the value of the normal derivative of a shape function. For any $v \in V_h$, its restriction to each $T \in \mathcal{T}_h$ is a piecewise cubic polynomial on the three triangles formed by the center and the vertices of T.*

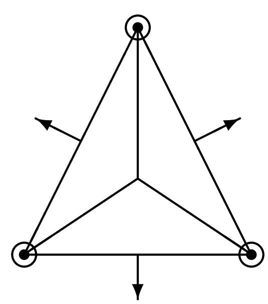


FIGURE 2.1. Degrees of freedom for the Hsieh-Clough-Tocher macro element

Now we consider the following discrete problem:

Find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v) \qquad \forall v \in V_h, \tag{2.4.2}$$

where $a(\cdot,\cdot)$ is defined in (2.3.4) and $f \in L^2(\Omega)$. The existence and uniqueness of the solution to (2.4.2) follow from the Lax-Milgram Theorem (Remark 2.25) and the fact that $V_h \subseteq H_0^2(\Omega)$.

Combining (2.3.3) and (2.4.2), we obtain the Galerkin orthogonality relation:

$$a(u-u_h, v) = 0 \qquad \forall v \in V_h. \tag{2.4.3}$$

Next, we want to estimate the error $\|u-u_h\|_{H^2(\Omega)}$. The following is an abstract error estimate which is also called the Céa Lemma (cf. Theorem 2.8.1 in [28]).

Theorem 2.34. **(Céa)** *Let u be the solution of* (2.3.3) *and u_h be the solution of* (2.4.2)*. Then we have*

$$\|u-u_h\|_{H^2(\Omega)} \leq \frac{C_1}{C_2} \min_{v\in V_h} \|u-v\|_{H^2(\Omega)}, \tag{2.4.4}$$

where C_1 is the continuity constant and C_2 is the coercivity constant of $a(\cdot,\cdot)$.

Proof. For any $v \in V_h$, from the boundedness and coercivity of $a(\cdot,\cdot)$ and the Galerkin orthogonality relation (2.4.3), we have

$$\begin{aligned}
C_2\|u-u_h\|^2_{H^2(\Omega)} &\leq a(u-u_h, u-u_h) \\
&= a(u-u_h, u-v) + a(u-u_h, v-u_h) \\
&= a(u-u_h, u-v) \\
&\leq C_1\|u-u_h\|_{H^2(\Omega)}\|u-v\|_{H^2(\Omega)}.
\end{aligned}$$

Hence,

$$\|u-u_h\|_{H^2(\Omega)} \leq \frac{C_1}{C_2}\|u-v\|_{H^2(\Omega)}.$$

Since $v \in V_h$ is arbitrary, the estimate (2.4.4) is obtained. $\square$

Let $\Pi_h : H_0^2(\Omega) \longrightarrow V_h$ be a quasi-local interpolation operator (cf. [49, 64, 95]). We have the following interpolation error estimate.

Lemma 2.35. *Let u be the solution of* (2.3.3)*. There exists a positive constant C independent of h such that*

$$\|u - \Pi_h u\|_{H^2(\Omega)} \leq Ch|u|_{H^3(\Omega)}. \tag{2.4.5}$$

Combining the Céa Lemma and Lemma 2.35, we have a concrete error estimate in the following context.

Theorem 2.36. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex polygonal domain. Let u be the solution of* (2.3.3) *and u_h be the solution of* (2.4.2)*. We have*

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}. \tag{2.4.6}$$

Remark 2.37. *The results in this subsection also apply to other conforming finite element methods, such as the quintic Argyris finite element* [5]*, the Bogner-Fox-Schmit element* [15, 48]*, etc.*

2.4.2 Nonconforming Finite Element Methods

Let V_h be the Morley finite element space associated with $\mathcal{T}_h$ (cf. [88]), i.e.,

$$\begin{aligned} V_h = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}_2(T), v &\text{ is continuous at the vertices and vanishes} \\ &\text{at the vertices along } \partial\Omega,\ \partial v/\partial n \text{ is continuous at the midpoints of edges} \\ &\text{and vanishes at the midpoints along } \partial\Omega\}. \end{aligned}$$

The degrees of freedom for this element are depicted in Figure 2.2.

Note that $V_h \nsubseteq C(\bar{\Omega})$, hence $V_h \nsubseteq H_0^2(\Omega)$ is a nonconforming finite element space. The bilinear form $a(\cdot,\cdot)$ is not well-defined on $V_h \times V_h$, so we need to define a discrete bilinear form $a_h(\cdot,\cdot)$ for the discrete problem:

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 w \, dx \qquad \forall v, w \in V_h. \tag{2.4.7}$$

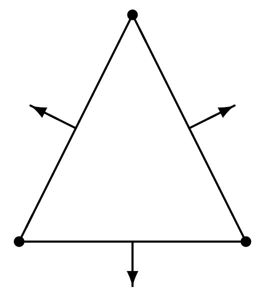


FIGURE 2.2. Degrees of freedom for the Morley element

Since for any $v \in V_h$, $v|_T$ is a quadratic polynomial, the above definition is well-defined. Moreover, we have

$$a_h(v,w) = a(v,w) \qquad \forall v,w \in H_0^2(\Omega). \tag{2.4.8}$$

In the error analysis, we use the following energy norm:

$$\|v\|_h = \sqrt{a_h(v,v)} \qquad \forall v \in V_h. \tag{2.4.9}$$

It is easy to show that the discrete bilinear form $a_h(\cdot,\cdot)$ satisfies the boundedness and coercivity conditions with respect to the energy norm $\|\cdot\|_h$, i.e., there exist positive constants C_1, C_2 such that

$$a_h(v,w) \le C_1\|v\|_h\|w\|_h \qquad \forall v,w \in V_h + H_0^2(\Omega), \tag{2.4.10}$$

$$a_h(v,v) \ge C_2\|v\|_h^2 \qquad \forall v \in V_h. \tag{2.4.11}$$

In fact, the constants $C_1 = C_2 = 1$ in this case.

The Morley finite element method for (2.3.3) is as follows:

Find $u_h \in V_h$ such that

$$a_h(u_h,v) = (f,v) \qquad \forall v \in V_h. \tag{2.4.12}$$

The boundedness and coercivity of $a_h(\cdot,\cdot)$ guarantee there exists a unique solution u_h to (2.4.12).

Now we give an abstract error estimate for nonconforming finite element methods which can be viewed as a generalization of the Céa Lemma.

Lemma 2.38 (Berger-Scott-Strang lemma [9])**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded polygonal domain. Let u be the solution of* (2.3.3) *and u_h be the solution of* (2.4.12)*. We have*

$$\|u - u_h\|_h \leq C \left(\inf_{v \in V_h} \|u - v\|_h + \sup_{w \in V_h \backslash \{0\}} \frac{|a_h(u, w) - (f, w)|}{\|w\|_h} \right). \tag{2.4.13}$$

Proof. For any $v \in V_h$, we have by the triangle inequality

$$\|u - u_h\|_h \leq \|u - v\|_h + \|v - u_h\|_h. \tag{2.4.14}$$

Then by (2.4.10), (2.4.11) and (2.4.12), we can write

$$\begin{aligned}
C_2 \|v - u_h\|_h^2 &\leq a_h(v - u_h, v - u_h) \\
&= a_h(v - u, v - u_h) + a_h(u - u_h, v - u_h) \\
&= a_h(v - u, v - u_h) + (a_h(u, v - u_h) - (f, v - u_h)) \\
&\leq C_1 \|u - v\|_h \|v - u_h\|_h + (a_h(u, v - u_h) - (f, v - u_h)).
\end{aligned}$$

Hence

$$\begin{aligned}
\|v - u_h\|_h &\leq \frac{1}{C_2} \left(C_1 \|u - v\|_h + \frac{|a_h(u, v - u_h) - (f, v - u_h)|}{\|v - u_h\|_h} \right) \\
&\leq \frac{1}{C_2} \left(C_1 \|u - v\|_h + \sup_{w \in V_h \backslash \{0\}} \frac{|a_h(u, w) - (f, w)|}{\|w\|_h} \right). \qquad (2.4.15)
\end{aligned}$$

Since $v \in V_h$ is arbitrary, we obtain the estimate from (2.4.14) and (2.4.15). $\square$

Remark 2.39. *Note that the second term of* (2.4.13) *would be* 0 *if $V_h \subseteq H_0^2(\Omega)$. Hence this term reflects the nonconforming error.*

Let $\Pi_h : H_0^2(\Omega) \longrightarrow V_h$ be the interpolation operator defined by the following conditions:

$$(\Pi_h \zeta)(p) = \zeta(p) \qquad \forall \zeta \in H_0^2(\Omega), \tag{2.4.16a}$$

$$\int_e \frac{\partial(\Pi_h \zeta)}{\partial n} ds = \int_e \frac{\partial \zeta}{\partial n} ds \qquad \forall \zeta \in H_0^2(\Omega), \tag{2.4.16b}$$

for any internal vertex p of $\mathcal{T}_h$ and any internal edge e of $\mathcal{T}_h$. The following interpolation estimate is standard (cf. [48, 82]):

$$\|\zeta - \Pi_h \zeta\|_h \le Ch|\zeta|_{H^3(\Omega)} \qquad \forall \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{2.4.17}$$

In view of (2.4.17), it remains to estimate the second term of the right-hand side of (2.4.13). In fact we have the following concrete error estimate.

Theorem 2.40. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex polygonal domain. Let u be the solution of* (2.3.3) *and u_h be the solution of* (2.4.12). *We have*

$$\|u - u_h\|_h \le Ch\|f\|_{L^2(\Omega)}. \tag{2.4.18}$$

Remark 2.41. *The proof of Theorem* 2.40 *can be found in* [6, 34, 97]. *See also Remark* 3.18.

Remark 2.42. *There are some other nonconforming finite elements such as the Zienkiewicz element* [8], *the de Veubeke element* [52], *the Adini element* [2], *and the incomplete biquadratic element* [96], *etc.*

2.4.3 C^0 Interior Penalty Methods

In this subsection, we consider C^0 interior penalty methods (also known as continuous/discontinuous Galerkin methods) which were introduced in [57] for smooth domains and further investigated in [29] for polygonal domains. More discussions can be found in the survey article [23]. Other C^0 discontinuous Galerkin methods for fourth order problems are also discussed in [79, 107]. C^0 interior penalty methods have certain advantages over classical finite element methods for fourth order problems, such as their simplicity and symmetric positive-definiteness, the existence of isoparametric C^0 interior penalty methods for curved domains [27], and the existence of natural preconditioners. For simplicity, we only focus on a quadratic C^0 interior penalty method. The presentation below follows [23, 29].

Let $\mathcal{T}_h$ be a regular triangulation of Ω. First, we introduce some notations for later reference.

- h_T is the diameter of the triangle T.
- h is the mesh parameter proportional to $\max_{T\in\mathcal{T}_h} h_T$.
- v_T is the restriction of the function v to the triangle T.
- $\mathcal{E}_h$ is the set of the edges of the triangles in $\mathcal{T}_h$.
- $\mathcal{E}_h^i$ is the subset of $\mathcal{E}_h$ consisting of edges interior to Ω.
- $\mathcal{E}_h^b$ is the subset of $\mathcal{E}_h$ consisting of edges along $\partial\Omega$.
- $|e|$ is the length of an edge e.
- $\mathcal{T}_e$ is the set of the triangles in $\mathcal{T}_h$ that share the common edge e.

Let V_h be the $\mathbb{P}_2$ Lagrange finite element (cf. Figure 2.3) space associated with $\mathcal{T}_h$ whose members vanish on $\partial\Omega$. It is clear that $V_h \subseteq C(\bar{\Omega})$ but $V_h \nsubseteq C^1(\bar{\Omega})$, hence $V_h \subseteq H_0^1(\Omega)$ but $V_h \nsubseteq H_0^2(\Omega)$. In this sense, the finite element space V_h is nonconforming.

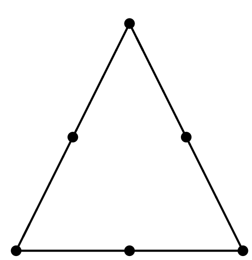


FIGURE 2.3. Degrees of freedom for the $\mathbb{P}_2$ Lagrange element

The construction and analysis of the quadratic C^0 interior penalty method require the concepts of jumps and means of normal derivatives across the edges in

$\mathcal{T}_h$, which are defined for functions in the piecewise Sobolev space

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v_T = v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\},$$

where s is a positive real number to be specified later.

Let $e \in \mathcal{E}_h^i$, $\mathcal{T}_e = \{T_-, T_+\}$, and n_e be the unit normal of e pointing from T_- to T_+. We define on e

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_+}{\partial n_e^2}\bigg|_e + \frac{\partial^2 v_-}{\partial n_e^2}\bigg|_e\right) \qquad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{2.4.19}$$

$$\left[\!\!\left[\frac{\partial v}{\partial n}\right]\!\!\right] = \frac{\partial v_+}{\partial n_e}\bigg|_e - \frac{\partial v_-}{\partial n_e}\bigg|_e \qquad \forall v \in H^2(\Omega, \mathcal{T}_h), \tag{2.4.20}$$

where $v_\pm = v|_{T_\pm}$. For the analysis of the C^0 interior penalty methods, we also need

$$\left\{\!\!\left\{\frac{\partial v}{\partial n_e}\right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial v_+}{\partial n_e}\bigg|_e + \frac{\partial v_-}{\partial n_e}\bigg|_e\right) \qquad \forall v \in H^2(\Omega, \mathcal{T}_h), \tag{2.4.21}$$

$$\left[\!\!\left[\frac{\partial^2 v}{\partial n_e^2}\right]\!\!\right] = \frac{\partial^2 v_+}{\partial n_e^2}\bigg|_e - \frac{\partial^2 v_-}{\partial n_e^2}\bigg|_e \qquad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{2.4.22}$$

$$\left[\!\!\left[\frac{\partial^2 v}{\partial n_e \partial t_e}\right]\!\!\right] = \frac{\partial^2 v_+}{\partial n_e \partial t_e}\bigg|_e - \frac{\partial^2 v_-}{\partial n_e \partial t_e}\bigg|_e \qquad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{2.4.23}$$

where the unit tangent vector t_e is obtained by rotating n_e by a counterclockwise right-angle.

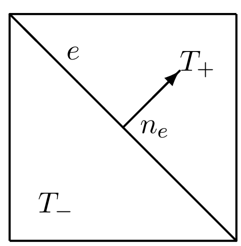


FIGURE 2.4. Choice of n_e in the definitions of average and jump operators

Remark 2.43. *Note that the definitions of* $\{\!\{\partial^2 v/\partial n^2\}\!\}$ *and* $[\![\partial v/\partial n]\!]$, *which appear in* C^0 *interior penalty methods, are independent of the choice of* $T_\pm$ *(or* n_e*). On the other hand, the definitions of* $\{\!\{\partial v/\partial n_e\}\!\}$, $[\![\partial^2 v/\partial n_e^2]\!]$ *and* $[\![\partial^2 v/\partial n_e \partial t_e]\!]$, *which appear only in the analysis, do depend on the choice of* $T_\pm$ *(or* n_e*).*

On $e \in \mathcal{E}_h^b$ with $\mathcal{T}_e = \{T\}$, we choose n_e to be the unit normal of e that points towards the outside of Ω and define

$$\left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\} = \left.\frac{\partial^2 v_T}{\partial n_e^2}\right|_e \qquad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{2.4.24}$$

$$\left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] = -\left.\frac{\partial v_T}{\partial n_e}\right|_e \qquad \forall v \in H^2(\Omega, \mathcal{T}_h), \tag{2.4.25}$$

where $v_T = v|_T$.

We formulate the quadratic C^0 interior penalty method for (2.3.3):
Find $u_h \in V_h$ such that

$$a_h(u_h, v) = (f, v) \qquad \forall v \in V_h, \tag{2.4.26}$$

where

$$\begin{aligned} a_h(v, w) = &\sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 w \, dx + \sum_{e \in \mathcal{E}_h} \int_e \{\!\!\{\partial^2 v/\partial n^2\}\!\!\} [\![\partial w/\partial n]\!] ds \\ &+ \sum_{e \in \mathcal{E}_h} \int_e \{\!\!\{\partial^2 w/\partial n^2\}\!\!\} [\![\partial v/\partial n]\!] ds + \sigma \sum_{e \in \mathcal{E}_h} |e|^{-1} \int_e [\![\partial v/\partial n]\!] [\![\partial w/\partial n]\!] ds, \end{aligned} \tag{2.4.27}$$

and $\sigma > 0$ is a penalty parameter.

We will measure the discretization error by the energy norm defined by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e \in \mathcal{E}_h} |e| \left\| \{\!\!\{\partial^2 v/\partial n^2\}\!\!\} \right\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[\![\partial v/\partial n]\!]\|_{L^2(e)}^2, \tag{2.4.28}$$

for any $v \in V_h$.

Remark 2.44. *Note that both the discrete bilinear form $a_h(\cdot, \cdot)$ in (2.4.27) and the energy norm $\|\cdot\|_h$ in (2.4.28) are well-defined for functions in the space $H^s(\Omega, \mathcal{T}_h) + V_h$ for $s > \frac{5}{2}$. In particular, the solution u belongs to $H^3(\Omega)$ for a convex polygonal domain. Hence we are able to evaluate the error $\|u - u_h\|_h$ in the energy norm.*

Next, we discuss the well-posedness of the quadratic C^0 interior penalty method (2.4.26). By the Cauchy-Schwarz inequality we have

$$\sum_{e\in\mathcal{E}_h}\Big|\int_e \{\!\!\{\partial^2 v/\partial n^2\}\!\!\}[\![\partial w/\partial n]\!]\,ds\Big| \tag{2.4.29}$$
$$\leq \Big(\sum_{e\in\mathcal{E}_h}|e|\,\big\|\{\!\!\{\partial^2 v/\partial n^2\}\!\!\}\big\|^2_{L^2(e)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial w/\partial n]\!]\|^2_{L^2(e)}\Big)^{\frac{1}{2}}$$

for any $v, w \in H^s(\Omega,\mathcal{T}_h)+V_h$ for $s>\frac{5}{2}$. Hence we obtain the following boundedness of the $a_h(\cdot,\cdot)$:

$$|a_h(v,w)| \leq C_1\|v\|_h\|w\|_h \qquad \forall v,w\in H^3(\Omega,\mathcal{T}_h)+V_h, \tag{2.4.30}$$

where C_1 is a positive constant independent of h.

Now we introduce a weaker mesh-dependent norm $|\cdot|_{H^2(\Omega,\mathcal{T}_h)}$:

$$|v|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}, \tag{2.4.31}$$

Lemma 2.45. *The mesh-dependent norms defined in* (2.4.28) *and* (2.4.31) *are equivalent on* V_h.

Proof. By the trace theorem with scaling and standard inverse estimates, we have

$$\sum_{e\in\mathcal{E}_h}|e|\,\big\|\{\!\!\{\partial^2 v/\partial n^2\}\!\!\}\big\|^2_{L^2(e)} \leq C\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} \qquad \forall v\in V_h. \tag{2.4.32}$$

Hence we have established the equivalence of the two different norms on V_h. $\square$

Combining (2.4.29) and (2.4.32), we apply the arithmetic-geometric mean inequality to obtain

$$a_h(v,v) \geq \sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)} - C\Big(\sum_{T\in\mathcal{T}_h}|v|^2_{H^2(T)}\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}\Big)^{\frac{1}{2}}$$
$$+\,\sigma\sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)} \tag{2.4.33}$$

$$\geq \frac{1}{2} \sum_{T \in \mathcal{T}_h} |v|^2_{H^2(T)} + \left(\sigma - \frac{C^2}{2}\right) \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}$$
$$\geq \frac{1}{2} |v|^2_{H^2(\Omega,\mathcal{T}_h)} \qquad \forall v \in V_h,$$

if we choose the penalty parameter σ to be large enough.

From now on we assume σ is sufficiently large so that the above inequality holds, then by Lemma 2.45, we obtain the coercivity condition for $a_h(\cdot,\cdot)$ with respect to the energy norm $\|\cdot\|_h$:

$$a_h(v,v) \geq C_2 \|v\|_h^2 \qquad \forall v \in V_h. \tag{2.4.34}$$

In view of (2.4.30) and (2.4.34), the quadratic C^0 interior penalty method (2.4.26) is well-defined. Moreover, we have the consistency relation:

$$a_h(u,v) = (f,v) \qquad \forall v \in V_h. \tag{2.4.35}$$

Remark 2.46. *The consistency relation* (2.4.35) *is easy to verify for smooth solution, i.e.,* $u \in H^4(\Omega)$. *However, we only have* H^3 *regularity for a convex polygonal domain* Ω. *But it turns out that* (2.4.35) *can be established by using the singular function representation of* u. *Details of the proof can be found in* [29].

By (2.4.30) and (2.4.34), we can still apply Lemma 2.38. But the second term of the right-hand side of (2.4.13) vanishes because of the Galerkin orthogonality condition (2.4.35). Therefore, we have the following abstract error estimate

Lemma 2.47. *Let* u *be the solution of* (2.3.3) *and* u_h *be the solution of* (2.4.26). *Then we have*

$$\|u - u_h\|_h \leq C \inf_{v \in V_h} \|u - v\|_h. \tag{2.4.36}$$

Let Π_h be the nodal interpolation operator from $H_0^2(\Omega)$ to V_h. The following interpolation error estimate is well-known (cf. [28, 48])

$$\sum_{m=0}^{2} h_T^{2m} |\zeta - \Pi_h \zeta|^2_{H^m(T)} \leq C h^6 |\zeta|^2_{H^3(\Omega)} \tag{2.4.37}$$

for all $T \in \mathcal{T}_h$ and $\zeta \in H^3(T)$. Using (2.4.37) and the trace theorem with scaling, we obtain

$$\|\zeta - \Pi_h \zeta\|_h \le Ch|\zeta|_{H^3(\Omega)} \qquad \forall \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{2.4.38}$$

Now we provide a concrete error estimate based on Lemma 2.47 and (2.4.38):

Theorem 2.48. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded convex polygonal domain. Let u be the solution of* (2.3.3) *and u_h be the solution of* (2.4.26)*. We have*

$$\|u - u_h\|_h \le Ch\|f\|_{L^2(\Omega)}. \tag{2.4.39}$$

Next we derive a useful integration by parts formula that will be frequently used in later chapters. Let $T \in \mathcal{T}_h$, $v \in \mathbb{P}_2(T)$ and $w \in H^2(T)$. We have

$$\begin{aligned}\int_T D^2 v : D^2 w \, dx &= -\int_T \nabla(\Delta v) \cdot \nabla w \, dx + \int_{\partial T} [(D^2 v)n] \cdot \nabla w \, ds \\ &= \int_{\partial T} \left[\left(\frac{\partial^2 v}{\partial n^2} \right) \left(\frac{\partial w}{\partial n} \right) + \left(\frac{\partial^2 v}{\partial n \partial t} \right) \left(\frac{\partial w}{\partial t} \right) \right] ds,\end{aligned}$$

where $\partial/\partial n$ (resp. $\partial/\partial t$) denotes the outward normal derivative (resp. counterclockwise tangential derivative) along ∂T. By summing up the integration by parts formula over all the triangles in $\mathcal{T}_h$, we find

$$\begin{aligned}\sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 w \, dx = &- \sum_{e \in \mathcal{E}_h^i} \int_e \left(\left[\!\left[\frac{\partial^2 v}{\partial n_e^2} \right]\!\right] \left\{\!\!\left\{ \frac{\partial w}{\partial n_e} \right\}\!\!\right\} + \left[\!\left[\frac{\partial^2 v}{\partial n_e \partial t_e} \right]\!\right] \frac{\partial w}{\partial t_e} \right) ds \\ &- \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial v}{\partial n^2} \right\}\!\!\right\} \left[\!\left[\frac{\partial w}{\partial n} \right]\!\right] ds\end{aligned} \tag{2.4.40}$$

for all $v \in \tilde{V}_h$ and $w \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)$. Here $\tilde{V}_h$ is the $\mathbb{P}_2$ Lagrange finite element space without boundary conditions.

2.5 Enriching Operators

In this section we discuss the enriching operators which can be used to measure the distance between the finite element space V_h and the Sobolev space $H^2(\Omega)$. Such operators were first introduced in the context of domain decomposition algorithms for nonconforming finite element methods (cf. [18–20]). The presentation in this section follows [21, 29, 31, 33]. Other references on this topic include [22, 23, 30].

In addition to the notations introduced in Section 2.4.3, we will also need the following notations throughout the dissertation.

- $\mathcal{T}_T$ is the set of triangles sharing a vertex with T.
- $\mathcal{T}_p$ is the set of triangles sharing the common vertex p.
- $\mathcal{V}_T$ is the set of the tree vertices of T.
- $\mathcal{S}_T$ is the interior of the closure of $\bigcup_{T' \in \mathcal{T}_T} T'$.
- $\mathcal{E}_{\mathcal{V}(T)}$ is the set of the edges in $\mathcal{T}_T$ sharing a vertex with T.
- $\mathcal{E}^i_{\mathcal{V}(T)}$ is the set of the edges in $\mathcal{E}^i_h$ emanating from the vertices of T.
- Let $e \in \mathcal{E}^b_h$. Then T_e is the triangle in $\mathcal{T}_h$ with e as its edge and $v_{T_e} = v|_{T_e}$.

2.5.1 Enriching Operator for the Morley Element

Let V_h be the Morley finite element space associated with $\mathcal{T}_h$ (cf. Section 2.4.2) and W_h be the Hsieh-Clough-Tocher macro finite element space associated with $\mathcal{T}_h$ (cf. Section 2.4.1). The enriching operator

$$E_h : V_h \longrightarrow W_h \subseteq H^2_0(\Omega)$$

is defined by

$$(E_h v)(p) = v(p), \tag{2.5.1}$$
$$\frac{\partial(E_h v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m), \tag{2.5.2}$$
$$[\partial^\beta(E_h v)](p) = \frac{1}{|\mathcal{T}_p|}\sum_{T\in\mathcal{T}_p}(\partial^\beta v_T)(p) \qquad |\beta| = 1, \tag{2.5.3}$$

where p and m are internal vertices and midpoints of $\mathcal{T}_h$.

The following properties of the enriching operator E_h are established in [21]:

$$\sum_{m=0}^{2} h_T^{2m}|v - E_h v|^2_{H^m(T)} \le C h_T^4 \sum_{T'\in\mathcal{T}_T} |v|^2_{H^2(T')} \quad \forall v \in V_h, \tag{2.5.4}$$

$$\sum_{m=0}^{2} h_T^{m}|\zeta - E_h \Pi_h \zeta|_{H^m(T)} \le C h_T^{2+s} |\zeta|_{H^{2+s}(\mathcal{S}_T)} \qquad \forall \zeta \in H^{2+s}(\Omega)\cap H_0^2(\Omega), \tag{2.5.5}$$

where $s \in (1/2, 1]$.

It follows from (2.4.9), (2.5.4), (2.5.5), the trace theorem with scaling and standard inverse estimates that

$$\|v - E_h v\|_{L_2(\Omega)} + h\Big(\sum_{T\in\mathcal{T}_h} |v - E_h v|^2_{H^1(T)}\Big)^{\frac{1}{2}} + h^2|E_h v|_{H^2(\Omega)} \le C h^2 \|v\|_h, \tag{2.5.6}$$

$$\sum_{m=0}^{2} h^{m}|\zeta - E_h \Pi_h \zeta|_{H^m(\Omega)} \le C h^{2+s} |\zeta|_{H^{2+s}(\Omega)}, \tag{2.5.7}$$

for any $v \in V_h$ and any $\zeta \in H^{2+s}(\Omega)\cap H_0^2(\Omega)$, $s \in (1/2, 1]$.

2.5.2 Enriching Operators for the Quadratic Lagrange Element

Case 1. $E_h : V_h \longrightarrow H_0^2(\Omega)$

Let V_h be the $\mathbb{P}_2$ Lagrange finite element space associated with $\mathcal{T}_h$ whose members vanish on $\partial\Omega$ and $\tilde{W}_h$ be the $\mathbb{P}_6$ Argyris finite element space (cf. [5]) associated with $\mathcal{T}_h$. The degrees of freedom of $w \in \tilde{W}_h$ (cf. Figure 2.5) consist of (i) the values of the derivatives of w up to second order at the vertices of $\mathcal{T}_h$, (ii) the values of w at the midpoints of the edges of $\mathcal{T}_h$ and at the center of the triangles of $\mathcal{T}_h$, and (iii) the values of the normal derivatives of w at two nodes on each edge in $\mathcal{E}_h$. Note that in Figure 2.5, the larger circle represents the values of the second order derivatives of a shape function.

The enriching operator

$$E_h : V_h \longrightarrow \tilde{W}_h \cap H_0^2(\Omega)$$

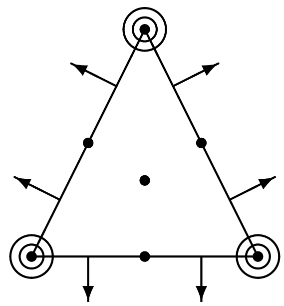


FIGURE 2.5. Degrees of freedom for the $\mathbb{P}_6$ Argyris element

can be constructed by averaging as follows. For $v \in V_h$, the degrees of freedom of $E_h v$ at any node (with the exception of the degrees of freedom along $\partial\Omega$ that involve differentiation) are defined to be the average of the corresponding degrees of freedom of v from the triangles of $\mathcal{T}_h$ that share the node. Since v is continuous at the vertices, midpoints, and centers, $E_h v = v$ at these nodes. In particular we have

$$(E_h v)(p) = v(p), \tag{2.5.8}$$

where p is any vertex of $\mathcal{T}_h$.

To ensure that $E_h v \in H_0^2(\Omega)$, we take the normal derivative of $E_h v$ at the nodes on the boundary edges to be 0. Similarly we assign the value 0 to all first order derivatives of $E_h v$ at the vertices on $\partial\Omega$, and at a corner of Ω we also assign the value 0 to all the second order derivatives of $E_h v$. Finally we define $\partial^2(E_h v)/\partial t^2$ and $\partial^2(E_h v)/\partial t \partial n$ to be 0 at the vertices on $\partial\Omega$ that are not one of the corners of Ω, and we define the remaining second order derivative $\partial^2(E_h v)/\partial n^2$ at such a vertex by averaging. (Here $\partial/\partial t$ and $\partial/\partial n$ are the differentiations in the tangential and normal directions along $\partial\Omega$, respectively.)

The proofs of the following properties of E_h can be found in [29]. Letting $T \in \mathcal{T}_h$ be arbitrary, we have

$$\sum_{m=0}^{2} h_T^{2m}|v - E_h v|^2_{H^m(T)}$$
$$\leq C\left(\sum_{T'\in\mathcal{T}_T} h_{T'}^4|v|^2_{H^2(T')} + h_T^4 \sum_{e\in\mathcal{E}_{\mathcal{V}(T)}} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}\right), \tag{2.5.9}$$

$$\sum_{m=0}^{2} h_T^m|\zeta - E_h\Pi_h\zeta|_{H^m(T)} \leq Ch_T^{2+s}|\zeta|_{H^{2+s}(S_T)}, \tag{2.5.10}$$

for any $v \in V_h$ and any $\zeta \in H^{2+s}(\Omega) \cap H_0^2(\Omega)$, $s \in (0, 1]$.

It then follows from (2.4.28), (2.5.9), (2.5.10), the trace theorem with scaling and standard inverse estimates that

$$\|v - E_h v\|_{L^2(\Omega)} + h|v - E_h v|_{H^1(\Omega)} + h^2|E_h v|_{H^2(\Omega)} \leq Ch^2\|v\|_h, \tag{2.5.11}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\, \|\{\!\{\partial(v - E_h v)/\partial n_e\}\!\}\|^2_{L^2(e)} \leq C\|v\|_h^2, \tag{2.5.12}$$

$$\sum_{m=0}^{2} h^m|\zeta - E_h\Pi_h\zeta|_{H^m(\Omega)} \leq Ch^{2+s}|\zeta|_{H^{2+s}(\Omega)}, \tag{2.5.13}$$

$$\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|\partial(\zeta - E_h\Pi_h\zeta)/\partial n_e\|^2_{L^2(e)} \leq Ch^{2s}|\zeta|^2_{H^{2+s}(\Omega)}, \tag{2.5.14}$$

for any $v \in V_h$ and any $\zeta \in H^{2+s}(\Omega) \cap H_0^2(\Omega)$, $s \in (0, 1]$.

Case 2. $E_h : V_h \longrightarrow H^2(\Omega) \cap H_0^1(\Omega)$

Now we construct an enriching operator E_h that maps V_h into $H^2(\Omega) \cap H_0^1(\Omega)$, which will be useful for the convergence analysis in Chapter 5.

The construction of E_h is given in the following procedure. For convenience, we first construct the enriching operator $E_h : \tilde{V}_h \longrightarrow \tilde{W}_h$, where $\tilde{V}_h$ is the $\mathbb{P}_2$ Lagrange finite element space without boundary conditions.

(i) Let N be a degree of freedom associated with an interior node p. We define

$$N(E_h v) = \frac{1}{|\mathcal{T}_p|}\sum_{T\in\mathcal{T}_p} N(v_T).$$

(ii) Let N be a degree of freedom (involving the normal derivative) associated with a boundary node interior to an edge $e \in \mathcal{E}_h^b$. We define

$$N(E_h v) = N(v_{T_e}).$$

(iii) Let p be a boundary node which is not a corner of Ω such that p is the common endpoint of two edges $e_1, e_2 \in \mathcal{E}_h^b$. For any degree of freedom N associated with p, we define

$$N(E_h v) = \frac{1}{2}\left[N(v_{T_{e_1}}) + N(v_{T_{e_2}})\right].$$

(iv) Let p be a corner of Ω. Then p is the common endpoint of $e_1, e_2 \in \mathcal{E}_h^b$. Let t_j (resp. n_j) be a unit tangent (resp. normal) of e_j. We define

$$\begin{aligned}
(E_h v)(p) &= v(p), \\
(\partial(E_h v)/\partial t_j)(p) &= (\partial v_{T_{e_j}}/\partial t_j)(p) \qquad \text{for } j = 1, 2, \\
(\partial^2(E_h v)/\partial t_j^2)(p) &= (\partial^2 v_{T_{e_j}}/\partial t_j^2)(p) \qquad \text{for } j = 1, 2, \\
(\partial^2(E_h v)/\partial n_1 \partial t_1)(p) &= (\partial^2 v_{T_{e_1}}/\partial t_1 \partial n_1)(p).
\end{aligned}$$

Remark 2.49. *We can also replace the last equation in (iv) by*

$$(\partial^2(E_h v)/\partial n_2 \partial t_2)(p) = (\partial^2 v_{T_{e_2}}/\partial t_2 \partial n_2)(p).$$

It is also easy to check that

$$E_h v \in W_h = \tilde{W}_h \cap H_0^1(\Omega) \subseteq H^2(\Omega) \cap H_0^1(\Omega) \quad \text{if } v \in V_h \ (= \tilde{V}_h \cap H_0^1(\Omega)). \tag{2.5.15}$$

Since v is continuous at the vertices, E_h preserves the nodal values at the vertices of $\mathcal{T}_h$, i.e.,

$$(E_h v)(p) = v(p) \qquad \forall v \in V_h, \tag{2.5.16}$$

for any vertex p of $\mathcal{T}_h$.

Moreover we have the following local approximation properties of E_h and $E_h \Pi_h$:

Lemma 2.50. *For any $T \in \mathcal{T}_h$, we have*

$$\sum_{m=0}^{2} h_T^{2m} |v - E_h v|_{H^m(T)}^2$$
$$\leq C h_T^4 \Big(\sum_{T' \in \mathcal{T}_T} |v|_{H^2(T')}^2 + \sum_{e \in \mathcal{E}_{\mathcal{V}(T)}^i} |e|^{-1} \| [\![\partial v / \partial n]\!] \|_{L^2(e)}^2 \Big), \tag{2.5.17}$$

$$\sum_{m=0}^{2} h_T^{m} |\zeta - E_h \Pi_h \zeta|_{H^m(T)} \leq C h_T^{2+s} |\zeta|_{H^{2+s}(\mathcal{S}_T)}, \tag{2.5.18}$$

for any $v \in V_h$ and any $\zeta \in H^{2+s}(\mathcal{S}_T)$, $s \in [0,1]$.

Proof. Let $T \in \mathcal{T}_h$ be arbitrary. Since $v = E_h v$ at the vertices and the center of T, we have, by scaling,

$$\begin{aligned} \|v - E_h v\|_{L^2(T)}^2 \leq & C h_T^4 \Big(\sum_{p \in \mathcal{V}_T} |\nabla (v - E_h v)(p)|^2 + \sum_{p \in \mathcal{N}_T} \Big| \frac{\partial (v - E_h v)}{\partial n}(p) \Big|^2 \\ & + \sum_{p \in \mathcal{V}_T} h_T^2 |D^2 (v - E_h v)(p)|^2 \Big) \qquad \forall v \in \tilde{V}_h, \end{aligned} \tag{2.5.19}$$

where $\mathcal{N}_T$ is the set of the six nodes on ∂T associated with the degrees of freedom of the $\mathbb{P}_6$ Argyris finite element that involve the normal derivative.

Let $p \in \mathcal{V}_T$ be interior to Ω. Since the tangential derivative of $v - E_h v$ is continuous across element boundaries, it follows from the definition of E_h and a standard inverse estimate that

$$\begin{aligned} |\nabla (v - E_h v)(p)|^2 &= \Big| \frac{1}{|\mathcal{T}_p|} \sum_{T' \in \mathcal{T}_p} (\nabla v_T(p) - \nabla v_{T'}(p)) \Big|^2 \\ &\leq C \sum_{T', T'' \in \mathcal{T}_p} |\nabla v_{T'}(p) - \nabla v_{T''}(p)|^2 \\ &\leq C \sum_{e \in \mathcal{E}_p^i} |e|^{-1} \| [\![\partial v / \partial n]\!] \|_{L^2(e)}^2, \end{aligned} \tag{2.5.20}$$

where $\mathcal{E}_p^i$ is the set of edges in $\mathcal{E}_h^i$ sharing p as a common endpoint and the pair of $T', T'' \in \mathcal{T}_p$ in the second summation of (2.5.20) shares an edge $e \in \mathcal{E}_p^i$. Similarly, the estimate (2.5.20) is also true for a boundary vertex $p \in \mathcal{V}_T$. In this case we can connect the triangle T to a triangle with a boundary edge through a sequence of triangles in $\mathcal{T}_h$.

Let $p \in \mathcal{N}_T$. If p is a boundary node, then $|\partial(v - E_h v)/\partial n(p)| = 0$ by the definition of E_h. Otherwise we have, by a standard inverse estimate,

$$|\partial(v - E_h v)/\partial n(p)|^2 \leq C|e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)} \tag{2.5.21}$$

for some $e \in \mathcal{E}_h^i$.

Finally, let $p \in \mathcal{V}_T$. If p is an interior vertex of $\mathcal{T}_h$, we conclude from the inverse estimate that

$$\begin{aligned} |D^2(v - E_h v)(p)|^2 &= \Big|\frac{1}{|\mathcal{T}_p|}\sum_{T' \in \mathcal{T}_p} D^2(v_T - v_{T'})(p)\Big|^2 \\ &\leq C \sum_{T' \in \mathcal{T}_p} h_{T'}^{-2}|v|^2_{H^2(T')}. \end{aligned} \tag{2.5.22}$$

Similarly, the estimate (2.5.22) holds if p is a boundary node.

Combining (2.5.19)–(2.5.22), we obtain the estimate (2.5.17) for $m = 0$:

$$|v - E_h v|^2_{L^2(T)} \leq Ch_T^4\Big(\sum_{T' \in \mathcal{T}_T} |v|^2_{H^2(T')} + \sum_{e \in \mathcal{E}^i_{\mathcal{V}(T)}} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}\Big). \tag{2.5.23}$$

The other estimates in (2.5.17) now follow from (2.5.23) and standard inverse estimates.

Next, the interpolation property of Π_h (2.4.37) and (2.5.17) imply that $E_h\Pi_h$ is a bounded linear operator from $H^{2+s}(\mathcal{S}_T)$ to $H^2(T)$ for $s \in [0, 1]$. Furthermore, by the definition of E_h, we have

$$E_h\Pi_h\zeta = \zeta \quad \text{on } T \qquad \forall \zeta \in \mathbb{P}_2(\mathcal{S}_T). \tag{2.5.24}$$

Hence the estimate (2.5.18) follows from the Bramble-Hilbert lemma (cf. [16, 55]).

$\square$

Using (2.5.17) and the trace theorem with scaling, the following global estimates are satisfied

$$\|v - E_h v\|_{L^2(\Omega)} + h|v - E_h v|_{H^1(\Omega)} + h^2|E_h v|_{H^2(\Omega)} \leq Ch^2\|v\|_h, \tag{2.5.25}$$

$$\sum_{e\in\mathcal{E}_h} |e|^{-1}\| \{\!\{\partial(v - E_h v)/\partial n_e\}\!\} \|^2_{L^2(e)} \leq C\|v\|_h^2, \tag{2.5.26}$$

for any $v \in V_h$.

Chapter 3

Finite Element Methods for the Obstacle Problem of Clamped Plates

In Chapter 1 we saw the equivalence between the displacement obstacle problem of a clamped plate (1.1.1) and a fourth order variational inequality (1.1.5). For simplicity, we only considered the one obstacle case in (1.1.1) and (1.1.5). Now we consider a general case which includes two-sided obstacle functions:

Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} G(v), \tag{3.0.1}$$

where

$$K = \{v \in H_0^2(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ in } \Omega\}, \tag{3.0.2}$$

$G(v) = \frac{1}{2}a(v,v) - (f,v)$ and $a(\cdot,\cdot)$ is defined in (1.1.4). Here $\Omega \subseteq \mathbb{R}^2$ is a bounded convex polygon domain, $f \in L^2(\Omega)$, $\psi_1, \psi_2 \in C^2(\Omega) \cap C(\bar{\Omega})$, $\psi_1 < \psi_2$ on $\bar{\Omega}$, and $\psi_1 < 0 < \psi_2$ on $\partial\Omega$.

Remark 3.1. *We can also use an equivalent formulation of* (3.0.1) *where the bilinear form $a(\cdot,\cdot)$ is given by $a(v,w) = \int_\Omega (\Delta v)(\Delta w)\,dx$. But the choice of $a(\cdot,\cdot)$ in* (1.1.4) *is more appropriate for nonconforming finite element methods since the corresponding norms provide more local information.*

The obstacle problem (3.0.1) has a unique solution that is also uniquely determined by the variational inequality

$$a(u, v-u) \geq (f, v-u) \qquad \forall v \in K. \tag{3.0.3}$$

By the regularity theory result in Corollary 2.32, we know $u \in H^3(\Omega) \cap C^2(\Omega)$.

In this chapter we provide a general framework for finite element methods for the obstacle problem (3.0.1). A unified convergence analysis is carried out in Section

3.3 after we introduce an auxiliary obstacle problem in Section 3.2. Approximation results of the coincidence set and the free boundary are also addressed in Section 3.4. The presentation below follows [34].

3.1 A General Framework

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h. The piecewise Sobolev space $H^3(\Omega, \mathcal{T}_h)$ is defined by

$$H^3(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v_T = v|_T \in H^3(T) \quad \forall T \in \mathcal{T}_h\}.$$

Remark 3.2. *The energy space $H^3(\Omega, \mathcal{T}_h)$ is needed for C^0 interior penalty methods* (*cf. Section* 2.4.3). *For the classical nonconforming finite element methods* (*cf. Section* 2.4.2), *we can use the larger space $H^2(\Omega, \mathcal{T}_h)$.*

Let V_h be a finite element space associated with $\mathcal{T}_h$ such that the functions in V_h are continuous at $p \in \mathcal{V}_h$, where $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$. Let $a_h(\cdot, \cdot)$ be a symmetric bilinear form on $H^3(\Omega, \mathcal{T}_h)$ such that

$$a_h(v, w) = a(v, w) \qquad \forall v, w \in H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h). \tag{3.1.1}$$

We assume there exists a norm $\|\cdot\|_h$ on $V_h + (H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h))$ such that

$$|a_h(v, w)| \leq C_1 \|v\|_h \|w\|_h \qquad \forall v, w \in V_h + H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h), \tag{3.1.2}$$

$$a_h(v, v) \geq C_2 \|v\|_h^2 \qquad \forall v \in V_h. \tag{3.1.3}$$

From now on we use C (with or without subscripts) to denote a generic positive constant independent of h that can take different values at different appearances.

We assume there exists an operator Π_h: $H^2(\Omega) \longrightarrow V_h$ such that

$$(\Pi_h \zeta)(p) = \zeta(p) \qquad \forall \zeta \in H_0^2(\Omega),\ p \in \mathcal{V}_h, \tag{3.1.4}$$

$$\|\zeta - \Pi_h \zeta\|_h \leq Ch|\zeta|_{H^3(\Omega)} \qquad \forall \zeta \in H^3(\Omega) \cap H_0^2(\Omega). \tag{3.1.5}$$

Furthermore we assume there exists an operator

$$E_h : V_h \longrightarrow H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h), \tag{3.1.6}$$

such that for any $v \in V_h$ and $\zeta \in H^3(\Omega) \cap H_0^2(\Omega)$,

$$(E_h v)(p) = v(p) \qquad \forall p \in \mathcal{V}_h, \tag{3.1.7}$$

$$\|v - E_h v\|_{L_2(\Omega)} + h\Big(\sum_{T\in\mathcal{T}_h} |v - E_h v|_{H^1(T)}^2\Big)^{\frac{1}{2}} + h^2 |E_h v|_{H^2(\Omega)} \leq Ch^2 \|v\|_h, \tag{3.1.8}$$

$$|a_h(\zeta, v - E_h v)| \leq Ch|\zeta|_{H^3(\Omega)} \|v\|_h, \tag{3.1.9}$$

$$\sum_{m=0}^{2} h^m |\zeta - E_h \Pi_h \zeta|_{H^m(\Omega)} \leq Ch^3 |\zeta|_{H^3(\Omega)}. \tag{3.1.10}$$

Next, we present three examples of finite element methods that satisfy the assumptions we made above. Hence the framework developed in this chapter can be applied to C^1 finite element methods, classical nonconforming finite element methods and discontinuous Galerkin methods.

Example 3.3. (C^1 Finite Element Methods) *We take V_h to be the Hsieh-Clough-Tocher macro element space or the quintic Argyris finite element space (cf. Section* 2.4.1*), $a_h(\cdot,\cdot) = a(\cdot,\cdot)$, $\|\cdot\|_h = \sqrt{a(\cdot,\cdot)} = |\cdot|_{H^2(\Omega)}$, Π_h to be a quasi-local interpolation operator* (*cf.* [49,64,95])*, and E_h to be the natural injection. The properties of* (3.1.7)–(3.1.10) *are trivial in this case.*

Example 3.4. (Classical Nonconforming Finite Element Methods) *Let $V_h \subseteq L^2(\Omega)$ be the Morley finite element space* (*cf. Section* 2.4.2). *We take*

$$a_h(v, w) = \sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 w \, dx,$$

$\|\cdot\|_h = \sqrt{a_h(\cdot,\cdot)}$, Π_h *to be the interpolation operator defined in* (2.4.16*a*)–(2.4.16*b*), *and E_h to be the enriching operator defined in Section* 2.5.1, *where we also discussed the properties* (3.1.7), (3.1.8) *and* (3.1.10) (*cf.* (2.5.1), (2.5.6) *and* (2.5.7)). *The* *following lemma showed that* (3.1.9) *is a consequence of integration by parts and* (3.1.8).

Lemma 3.5. *The property* (3.1.9) *is valid for the Morley finite element.*

Proof. For any $v \in V_h$ and $\zeta \in H^3(\Omega)$. Since the piecewise linear vector field ∇v is continuous at the midpoints of the interior edges in $\mathcal{E}_h$ and vanishes at the midpoints of the boundary edges in $\mathcal{E}_h$ and $E_h v \in C^1(\bar{\Omega}) \cap H_0^2(\Omega)$, we have

$$
\begin{aligned}
a_h(\zeta, v - E_h v) &= \sum_{T \in \mathcal{T}_h} \int_T D^2\zeta : D^2(v - E_h v)\, dx \\
&= -\sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta\zeta) \cdot \nabla(v - E_h v)\, dx \\
&\quad + \sum_{T \in \mathcal{T}_h} \int_{\partial T} D^2\zeta : [\nabla(v - E_h v) \otimes n_T]\, ds \\
&= -\sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta\zeta) \cdot \nabla(v - E_h v)\, dx \\
&\quad + \sum_{e \in \mathcal{E}_h} \int_e [D^2\zeta - \overline{(D^2\zeta)_e}] : [[\nabla(v - E_h v) \otimes n]]_e\, ds,
\end{aligned} \tag{3.1.11}
$$

where $\overline{(D^2\zeta)_e}$ is the average of $D^2\zeta$ along e and $[[\nabla(v - E_h v) \otimes n]]_e$ is the sum of $\nabla(v - E_h v) \otimes n_T$ over the triangles that share e as a common edge.

The two terms of the right-hand side of (3.1.11) can be estimated as follows:

$$
\begin{aligned}
\Big| \sum_{T \in \mathcal{T}_h} \int_T \nabla(\Delta\zeta) \cdot \nabla(v - E_h v)\, dx \Big| &\le |\Delta\zeta|_{H^1(\Omega)} \Big(\sum_{T \in \mathcal{T}_h} |v - E_h v|^2_{H^1(T)} \Big)^{\frac{1}{2}} \\
&\le Ch|\zeta|_{H^3(\Omega)} \|v\|_h
\end{aligned} \tag{3.1.12}
$$

by the Cauchy-Schwarz inequality and (3.1.8);

$$
\begin{aligned}
&\Big| \sum_{e \in \mathcal{E}_h} \int_e [D^2\zeta - \overline{(D^2\zeta)_e}] : [[\nabla(v - E_h v) \otimes n]]_e\, ds \Big| \\
&\quad \le \Big(\sum_{e \in \mathcal{E}_h} |e|^{-1} \|D^2\zeta - \overline{(D^2\zeta)_e}\|^2_{L^2(e)} \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}_h} |e| \|[[\nabla(v - E_h v) \otimes n]]_e\|^2_{L^2(e)} \Big)^{\frac{1}{2}} \\
&\quad \le Ch|\zeta|_{H^3(\Omega)} \|v\|_h
\end{aligned} \tag{3.1.13}
$$

by the Cauchy-Schwarz inequality, the trace theorem with scaling, a standard interpolation error estimate and (3.1.8).

Combining (3.1.11)–(3.1.13), we have

$$|a_h(\zeta, v - E_h v)| \leq Ch|\zeta|_{H^3(\Omega)}\|v\|_h.$$

$\square$

Example 3.6. (C^0 Interior Penalty Methods) *For simplicity, we consider the quadratic case. Let $V_h \subseteq H_0^1(\Omega)$ be the $\mathbb{P}_2$ Lagrange finite element space (cf. Section* 2.4.3*). We take $a_h(\cdot,\cdot)$ to be the bilinear form defined in* (2.4.27) *and the energy norm $\|\cdot\|_h$ as defined in* (2.4.28)*. It is clear that property* (3.1.1) *is valid since $[\![\partial v/\partial n]\!] = 0$ on $e \in \mathcal{E}_h$ for any $v \in H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h)$ and then the integrals involving the jumps and averages on the edges vanish. We take Π_h to be the nodal interpolation operator and E_h to be an enriching operator defined in Case* 1 *of Section* 2.5.2*, where the properties* (3.1.7)*,* (3.1.8) *and* (3.1.10) *are discussed* (*cf.* (2.5.8)*,* (2.5.11) *and* (2.5.13))*. We provide a proof for property* (3.1.9) *in the following lemma.*

Lemma 3.7. *The property* (3.1.9) *is valid for the quadratic C^0 interior penalty method.*

Proof. Since $\zeta \in H^3(\Omega) \cap H_0^2(\Omega)$, we have $[\![\partial\zeta/\partial n]\!] = 0$ on $e \in \mathcal{E}_h$. Hence for any $v \in V_h$,

$$\begin{aligned}
a_h(\zeta, v - E_h v) &= \sum_{T\in\mathcal{T}_h} \int_T D^2\zeta : D^2(v - E_h v)\,dx \\
&\quad + \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\partial^2\zeta/\partial n^2\}\!\}[\![\partial(v - E_h v)/\partial n]\!]\,ds \\
&= -\sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta\zeta)\cdot\nabla(v - E_h v)\,dx \qquad (3.1.14)
\end{aligned}$$

$$+\sum_{T\in\mathcal{T}_h}\int_{\partial T}[(D^2\zeta)n]\cdot\nabla(v-E_hv)\,ds$$
$$+\sum_{e\in\mathcal{E}_h}\int_e\{\!\!\{\partial^2\zeta/\partial n^2\}\!\!\}[\![\partial(v-E_hv)/\partial n]\!]ds.$$

It follows from the H^3 regularity of ζ and the fact $v-E_hv\in H_0^1(\Omega)$ that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\int_{\partial T}(D^2\zeta)n\cdot\nabla(v-E_hv)\,ds\\ &=\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(\frac{\partial^2\zeta}{\partial n^2}\frac{\partial(v-E_hv)}{\partial n}+\frac{\partial^2\zeta}{\partial n\partial t}\frac{\partial(v-E_hv)}{\partial t}\right)ds \qquad (3.1.15)\\ &=-\sum_{e\in\mathcal{E}_h}\int_e\{\!\!\{\partial^2\zeta/\partial n^2\}\!\!\}[\![\partial(v-E_hv)/\partial n]\!]ds.\end{aligned}$$

Combining (3.1.14) and (3.1.15), we obtain from (3.1.8)

$$\begin{aligned}|a_h(\zeta,v-E_hv)|&=\left|-\sum_{T\in\mathcal{T}_h}\int_T\nabla(\Delta\zeta)\cdot\nabla(v-E_hv)\,dx\right|\\ &\le Ch|\zeta|_{H^3(\Omega)}\|v\|_h. \qquad (3.1.16)\end{aligned}$$

□

The finite element method for (3.0.1) is:

Find $u_h\in K_h$ such that

$$u_h=\operatorname*{argmin}_{v\in K_h}G_h(v), \qquad (3.1.17)$$

where $G_h(v)=\frac{1}{2}a_h(v,v)-(f,v)$,

$$K_h=\{v\in V_h:\psi_1(p)\le v(p)\le\psi_2(p)\quad\forall p\in\mathcal{V}_h\}, \qquad (3.1.18)$$

and $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$.

Remark 3.8. *We only require the pointwise constraints for $v\in K_h$ to be satisfied at the vertices of the triangulation $\mathcal{T}_h$. However, it is also possible to impose additional constraints on other degrees of freedom of the finite elements. For example, we can add constraints on the midpoints of the triangulation for the quadratic C^0 interior penalty method.*

Since $a_h(\cdot,\cdot)$ is symmetric positive definite on V_h, the unique solution u_h to (3.1.17) is also characterized by the following variational inequality

$$a_h(u_h, v - u_h) \geq (f, v - u_h) \qquad \forall v \in K_h. \tag{3.1.19}$$

The finite element function u_h provides an approximation to u. We give a preliminary error estimate in the following lemma whose proof only involves the discrete variational inequality (3.1.19).

Lemma 3.9. *There exist positive constants $C_\dagger$ and $C_\ddagger$ independent of h such that*

$$\|u - u_h\|_h^2 \leq C_\dagger \|u - \Pi_h u\|_h^2 + C_\ddagger [a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)]. \tag{3.1.20}$$

Proof. From (3.1.2), (3.1.3), (3.1.19) and the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
\|\Pi_h u - u_h\|_h^2 &\leq C_2 a_h(\Pi_h u - u_h, \Pi_h u - u_h) \\
&= C_2[a_h(\Pi_h u - u, \Pi_h u - u_h) + a_h(u, \Pi_h u - u_h) - a_h(u_h, \Pi_h u - u_h)] \\
&\leq C_1 C_2 \|\Pi_h u - u\|_h \|\Pi_h u - u_h\|_h + C_2[a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)] \\
&\leq \frac{1}{2}\|\Pi_h u - u_h\|_h^2 + C_3 \|\Pi_h u - u\|_h^2 \\
&\qquad + C_2[a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)],
\end{aligned}$$

and hence

$$\|\Pi_h u - u_h\|_h^2 \leq 2C_3 \|\Pi_h u - u\|_h^2 + 2C_2[a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)].$$

Finally the estimate (3.1.20) follows from the triangle inequality. $\square$

3.2 An Auxiliary Obstacle Problem

In view of (3.1.5), it only remains to find an estimate for the second term on the right-hand side of (3.1.20):

$$a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h).$$

This term involves functions in K_h and K. So far we have only applied the discrete variational inequality (3.1.19). In order to apply the continuous variational inequality (3.0.3), it is important to build a connection between the two constrained sets K_h and K. By (3.0.2) and (3.1.18), we know $K_h \subseteq H_0^1(\Omega)$ and $K \subseteq H_0^2(\Omega)$, which means they are in different spaces. However, by (3.1.6) and (3.1.7), we have

$$E_h v \in \tilde{K}_h \qquad \forall v \in K_h,$$

where $\tilde{K}_h \subseteq H_0^2(\Omega)$ is defined by

$$\tilde{K}_h = \{v \in H_0^2(\Omega) : \psi_1(p) \leq v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h\}. \tag{3.2.1}$$

Note that $\tilde{K}_h$ is well-defined since $H_0^2(\Omega)$ is embedded in $C(\bar{\Omega})$.

Now we consider the following auxiliary obstacle problem:
Find $\tilde{u}_h \in \tilde{K}_h$ such that

$$\tilde{u}_h = \operatorname*{argmin}_{v \in \tilde{K}_h} G(v), \tag{3.2.2}$$

where $G(v) = \frac{1}{2}a(v, v) - (f, v)$ and $a(\cdot, \cdot)$ is given by (1.1.4).

Note that $\tilde{K}_h$ is a closed convex subset of $H_0^2(\Omega)$ and $K \subseteq \tilde{K}_h$. The unique solution of (3.2.2) is characterized by the variational inequality

$$a(\tilde{u}_h, v - \tilde{u}_h) \geq (f, v - \tilde{u}_h) \qquad \forall v \in \tilde{K}_h. \tag{3.2.3}$$

In the following we will evaluate the difference between u and $\tilde{u}_h$. First, we estimate the distance between $\tilde{u}_h$ and K through several lemmas.

Lemma 3.10. *There exists a positive constant C independent of h such that*

$$\|\tilde{u}_h\|_{H^2(\Omega)} \leq C. \tag{3.2.4}$$

Proof. Since $K \subseteq \tilde{K}_h$, we have

$$G(\tilde{u}_h) \leq G(u). \tag{3.2.5}$$

By the Cauchy-Schwarz inequality, a Poincaré-Friedrichs inequality (cf. Theorem 2.16) and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}\frac{1}{2}|\tilde{u}_h|^2_{H^2(\Omega)} &\leq G(u) + \int_\Omega f\tilde{u}_h\,dx \\ &\leq G(u) + \|f\|_{L^2(\Omega)}\|\tilde{u}_h\|_{L^2(\Omega)} \\ &\leq G(u) + C\|f\|^2_{L^2(\Omega)} + \frac{1}{4}|\tilde{u}_h|^2_{H^2(\Omega)},\end{aligned}$$

from which we obtain the estimate (3.2.4). □

Lemma 3.11. *The solution $\tilde{u}_h$ of* (3.2.2) (*or* (3.2.3)) *converges uniformly on Ω to the solution of* (3.0.1) (*or* (3.0.3)) *as $h \downarrow 0$.*

Proof. It suffices to show that given any sequence $h_n \downarrow 0$ there exists a subsequence h_{n_k} such that $\tilde{u}_{h_{n_k}}$ converges uniformly on Ω to u as $k \to \infty$.

From (3.2.4), u_h is uniformly bounded, then it follows from the weak sequential compactness of bounded subsets of a Hilbert space that there exists a subsequence h_{n_k} such that

$$\tilde{u}_{h_{n_k}} \text{ converges weakly to some } u_* \in H^2_0(\Omega) \text{ as } k \to \infty. \tag{3.2.6}$$

By Theorem 2.12, $H^2(\Omega)$ is compactly embedded in $C(\bar{\Omega})$. Hence the weak convergence of $\tilde{u}_{h_{n_k}}$ implies that

$$\tilde{u}_{h_{n_k}} \text{ converges uniformly on } \Omega \text{ to } u_* \text{ as } k \to \infty. \tag{3.2.7}$$

In view of (3.2.1) and the fact that the set $\cup_{j\geq k}\mathcal{V}_{h_{n_j}}$ is dense in Ω for any k, we obtain by (3.2.7) the relation $\psi_1 \leq u_* \leq \psi_2$ on Ω, i.e., $u_* \in K$.

The convexity and continuity of the functional G imply it is also weakly lower semi-continuous (cf. [36]). Therefore, by (3.2.4) and (3.2.6),

$$G(u_*) \leq \liminf_{k\to\infty} G(\tilde{u}_{h_{n_k}}) \leq G(u).$$

Then we conclude that $u_* = u$ by the uniqueness of the solution to (3.0.1). Finally we establish the uniform convergence by (3.2.7). $\square$

Let I_i $(i = 1, 2)$ be the coincidence set of the obstacle problem (3.0.1) defined by

$$I_i = \{x \in \Omega : u(x) = \psi_i(x)\}. \tag{3.2.8}$$

The boundary condition of u and the assumptions on ψ_i imply that the compact sets $\partial\Omega$, I_1 and I_2 are mutually disjoint.

For any positive τ, let the compact set $I_{i,\tau}$ $(i = 1, 2)$ be defined by

$$I_{i,\tau} = \{x \in \bar{\Omega} : \text{dist } (x, I_i) \leq \tau\}. \tag{3.2.9}$$

We can choose $\tau_i > 0$ $(i = 1, 2)$ small enough so that the compact sets $I_{i,2\tau_1}$, $I_{i,2\tau_2}$ and $\partial\Omega$ remain mutually disjoint.

Lemma 3.12. *There exist positive numbers h_0, γ_1 and γ_2 such that*

$$\tilde{u}_h(x) - \psi_1(x) \geq \gamma_1 \qquad \text{if } x \in \bar{\Omega} \text{ and dist } (x, I_1) \geq \tau_1, \tag{3.2.10}$$

$$\psi_2(x) - \tilde{u}_h(x) \geq \gamma_2 \qquad \text{if } x \in \bar{\Omega} \text{ and dist } (x, I_2) \geq \tau_2, \tag{3.2.11}$$

provided $h \leq h_0$.

Proof. Since $u - \psi_1$ is strictly positive on the compact set

$$\{x \in \bar{\Omega} : \text{dist } (x, I_i) \geq \tau\},$$

the estimate (3.2.10) is an immediate consequence of Lemma 3.11. Similarly we obtain the estimate (3.2.11). $\square$

Let $\mathcal{I}_h$ be the nodal interpolation operator for the conforming $\mathbb{P}_1$ finite element space associated with $\mathcal{T}_h$. The constraints on $\tilde{u}_h$ can be rewritten as

$$\mathcal{I}_h\psi_1 \leq \mathcal{I}_h\tilde{u}_h \leq \mathcal{I}_h\psi_2 \quad \text{ on } \bar{\Omega}. \tag{3.2.12}$$

Let the number $\delta_{h,i}$ $(i = 1, 2)$ be defined by

$$\delta_{h,i} = \|(\tilde{u}_h - \mathcal{I}_h\tilde{u}_h) + (\mathcal{I}_h\psi_i - \psi_i)\|_{L^\infty(I_{i,\tau_i})}. \tag{3.2.13}$$

Since $\psi_i \in C^2(\Omega)$ and the compact set I_{i,τ_i} is disjoint from $\partial\Omega$, Taylor's Theorem implies that

$$\|\psi_i - \mathcal{I}_h\psi_i\|_{L^\infty(I_{i,\tau_i})} \leq Ch^2 \qquad \text{for } i = 1, 2. \tag{3.2.14}$$

By the standard interpolation error estimate (cf. [28,48]) and (3.2.4), we also have

$$\|\tilde{u}_h - \mathcal{I}_h\tilde{u}_h\|_{L^\infty(\Omega)} \leq Ch|\tilde{u}_h|_{H^2(\Omega)} \leq Ch. \tag{3.2.15}$$

Combining (3.2.14) and (3.2.15), we obtain

$$\delta_{h,i} \leq Ch \qquad \text{for } i = 1, 2. \tag{3.2.16}$$

Lemma 3.13. *Let h_0, γ_1 and γ_2 be as in Lemma 3.12, then there exists a positive constant C independent of h such that*

$$|u - \tilde{u}_h|^2_{H^2(\Omega)} \leq C(\delta_{h,1} + \delta_{h,2}) \tag{3.2.17}$$

provided $h \leq h_0$.

Proof. We have, by (3.2.12) and (3.2.13),

$$\begin{aligned}
\tilde{u}_h(x) - \psi_1(x) &= (\tilde{u}_h(x) - \mathcal{I}_h\tilde{u}_h(x)) + (\mathcal{I}_h\tilde{u}_h(x) - \mathcal{I}_h\psi_1(x)) \\
&\quad + (\mathcal{I}_h\psi_1(x) - \psi_1(x)) \qquad (3.2.18) \\
&\geq -\delta_{h,1} \qquad \forall x \in I_{i,\tau_1}, \\
\psi_2(x) - \tilde{u}_h(x) &= (\psi_2(x) - \mathcal{I}_h\psi_2(x)) + (\mathcal{I}_h\psi_2(x) - \mathcal{I}_h\tilde{u}_h(x)) \\
&\quad + (\mathcal{I}_h\tilde{u}_h(x) - \tilde{u}_h) \qquad (3.2.19) \\
&\geq -\delta_{h,2} \qquad \forall x \in I_{2,\tau_2}.
\end{aligned}$$

In view of (3.2.16), we may assume

$$\max_{i=1,2} \delta_{h,i} < \min_{i=1,2} \gamma_i \tag{3.2.20}$$

for $h \leq h_0$.

Let ϕ_i $(i = 1, 2)$ be a function in $C^\infty(\bar{\Omega})$ such that

$$0 \leq \phi_i \leq 1 \qquad \text{on } \bar{\Omega}, \tag{3.2.21}$$

$$\phi_i = 1 \qquad \text{on } I_{i,\tau_i}, \tag{3.2.22}$$

$$\phi_i = 0 \qquad \text{on } \bar{\Omega} \setminus I_{i,2\tau_i}. \tag{3.2.23}$$

We claim that

$$\hat{u}_h = \tilde{u}_h + \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2 \in K. \tag{3.2.24}$$

Since the compact sets $I_{1,2\tau_1}$, $I_{2,2\tau_2}$ and $\partial\Omega$ are mutually disjoint, we have $\hat{u}_h \in H_0^2(\Omega)$ from (3.2.23). Now we need to show $\hat{u}_h$ satisfies the constraints for every point in Ω, i.e.,

$$\psi_1(x) \leq \hat{u}_h(x) \leq \psi_2(x) \qquad \forall x \in \Omega. \tag{3.2.25}$$

We will consider three possibilities: (i) $x \in \Omega \setminus (I_{1,2\tau_1} \cup I_{2,2\tau_2})$, (ii) $x \in I_{1,2\tau_1}$ and (iii) $x \in I_{2,2\tau_2}$.

For any $x \in \Omega \setminus (I_{1,2\tau_1} \cup I_{2,2\tau_2})$, it follows from the definition of $\hat{u}_h$ and (3.2.23) that $\hat{u}_h(x) = \tilde{u}_h(x)$. Hence (3.2.25) holds for $x \in \Omega \setminus (I_{1,2\tau_1} \cup I_{2,2\tau_2})$ by (3.2.10) and (3.2.11).

For any $x \in I_{1,2\tau_1}$, we have by (3.2.11), (3.2.20), (3.2.21) and (3.2.23)

$$\hat{u}_h(x) = \tilde{u}_h(x) + \delta_{h,1}\phi_1(x) \leq \psi_2(x) - \gamma_2 + \delta_{h,1} < \psi_2(x).$$

Furthermore, if $x \in I_{1,2\tau_1} \setminus I_{1,\tau_1}$, then

$$\hat{u}_h(x) = \tilde{u}_h(x) + \delta_{h,1}\phi_1(x) \geq \psi_1(x) + \gamma_1 > \psi_1(x)$$

by (3.2.10) and (3.2.21). If $x \in I_{1,\tau_1}$, then

$$\hat{u}_h(x) = \tilde{u}_h(x) + \delta_{h,1} \geq \psi_1(x)$$

by (3.2.18) and (3.2.22). Therefore (3.2.25) holds for $x \in I_{1,2\tau_1}$.

Similarly we can show that (3.2.25) also holds for $x \in I_{2,2\tau_2}$.

Now we complete the proof by using (3.0.3) and (3.2.3)

$$\begin{aligned}
|u - \tilde{u}_h|^2_{H^2(\Omega)} &= a(u - \tilde{u}_h, u - \tilde{u}_h) \\
&= a(u, u - \tilde{u}_h) - a(\tilde{u}_h, u - \tilde{u}_h) \\
&\leq a(u, u - \tilde{u}_h) - (f, u - \tilde{u}_h) \\
&= [a(u, u - \hat{u}_h) - (f, u - \hat{u}_h)] + [a(u, \hat{u}_h - \tilde{u}_h) - (f, \hat{u}_h - \tilde{u}_h)] \\
&\leq a(u, \hat{u}_h - \tilde{u}_h) - (f, \hat{u}_h - \tilde{u}_h) \\
&= a(u, \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2) - (f, \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2) \\
&\leq C(\delta_{h,1} + \delta_{h,2}).
\end{aligned}$$

□

Remark 3.14. *In fact K is a subset of $\tilde{K}_h$ and the problems* (3.0.1) *and* (3.2.2) *involve the same functional G. Hence we can consider u as an internal approximation of $\tilde{u}_h$. It was shown in* [7] *that the distance between $\tilde{u}_h$ and u is bounded by the square root of the distance between $\tilde{u}_h$ and K. According to* (3.2.24) *the distance between $\tilde{u}_h$ and K is bounded by*

$$|\delta_{h,1}\phi_1 - \delta_{h,2}\phi_2|_{H^2(\Omega)} \leq C(\delta_{h,1} + \delta_{h,2}),$$

and hence the estimate (3.2.17) *agrees with the result in* [7].

From (3.2.16) and (3.2.17), we only have $O(h^{\frac{1}{2}})$ estimate for $|u - \tilde{u}_h|^2_{H^2(\Omega)}$. In fact, by combining the C^2 regularity of u and (3.2.17), we are able to provide better estimates than those obtained in (3.2.16) and (3.2.17). This will be accomplished by the following lemma.

Lemma 3.15. *Let h_0, γ_1 and γ_2 be as in Lemma* 3.12*, then there exists a positive constant C independent of h such that*

$$\delta_{h,i} \leq Ch^2, \tag{3.2.26}$$

$$|u - \tilde{u}_h|_{H^2(\Omega)} \leq Ch, \tag{3.2.27}$$

provided $h \leq h_0$.

Proof. Since $u \in C^2(\Omega)$ and the compact set I_{i,τ_i} is disjoint from $\partial\Omega$, we have, by Taylor's Theorem,

$$\|u - \mathcal{I}_h u\|_{L^\infty(I_{i,\tau_i})} \leq Ch^2 \qquad \text{for } i = 1, 2. \tag{3.2.28}$$

By a standard interpolation error estimate (cf. [28, 48]) and (3.2.28), we find

$$\begin{aligned} |\ \tilde{u}_h - \mathcal{I}_h \tilde{u}_h\|_{L^\infty(I_{i,\tau_i})} &\leq \|(u - \tilde{u}_h) - \mathcal{I}_h(u - \tilde{u}_h)\|_{L^\infty(\Omega)} + \|u - \mathcal{I}_h u\|_{L^\infty(I_{i,\tau_i})} \\ &\leq C(h|u - \tilde{u}_h|_{H^2(\Omega)} + h^2), \end{aligned}$$

which together with (3.2.13) and (3.2.14) implies

$$\delta_{h,i} \leq C(h|u - \tilde{u}_h|_{H^2(\Omega)} + h^2) \qquad \text{for } i = 1, 2. \tag{3.2.29}$$

It now follows from (3.2.17), (3.2.29) and the arithmetic-geometric mean inequality that

$$|u - \tilde{u}_h|^2_{H^2(\Omega)} \leq Ch^2 + \frac{1}{2}|u - \tilde{u}_h|^2_{H^2(\Omega)},$$

and hence we have

$$|u - \tilde{u}_h|_{H^2(\Omega)} \leq Ch.$$

Finally we obtain from (3.2.27) and (3.2.29) that

$$\delta_{h,i} \le Ch^2 \qquad \text{for } i = 1, 2.$$

$\square$

Remark 3.16. *Even though we only consider obstacle problems for convex domains and homogeneous Dirichlet boundary conditions in this subsection, the results remain valid for general polygonal domains and general boundary conditions because the arguments here only require (i) $\tilde{K}_h$ is a closed convex subset of $H^2(\Omega)$, (ii) $K \subseteq \tilde{K}_h$, (iii) the separation of the obstacle functions and the boundary displacement (cf. (4.1.1)), (iv) the C^2 smoothness assumptions on the obstacle functions, (v) $u \in C^2(\Omega)$.*

3.3 A Unified Convergence Analysis

With Lemma 3.15 and the two variational inequalities (3.0.3) and (3.2.3), we can now complete the convergence analysis of the finite element methods.

In our analysis, we will also need the estimate

$$|a(\zeta, v)| \le C|\zeta|_{H^3(\Omega)}|v|_{H^1(\Omega)} \qquad \forall \zeta \in H^3(\Omega),\ v \in H_0^2(\Omega). \tag{3.3.1}$$

In fact, this is an immediate consequence of integration by parts formula:

$$|a(\zeta, v)| = \left| -\int_\Omega \nabla(\Delta\zeta)\cdot\nabla v\, dx \right| \le C|\zeta|_{H^3(\Omega)}|v|_{H^1(\Omega)}.$$

From (3.1.20), it suffices to estimate $a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)$. The following lemma reduces the estimation of this term to an estimate at the continuous level.

Lemma 3.17. *There exists a positive constant C independent of h such that*

$$\begin{aligned} &a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h) \\ \le& Ch\|\Pi_h u - u_h\|_h + [a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h))]. \end{aligned} \tag{3.3.2}$$

Proof. Since $E_h(\Pi_h u - u_h) \in H_0^2(\Omega) \cap H^3(\Omega, \mathcal{T}_h)$, we have

$$a_h(u, E_h(\Pi_h u - u_h)) = a(u, E_h(\Pi_h u - u_h))$$

by (3.1.1). Because $\Pi_h u - u_h \in V_h$, it follows from (3.1.8) and (3.1.9) that

$$\begin{aligned}
&a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h) \\
&= [a_h(u, (\Pi_h u - u_h) - E_h(\Pi_h u - u_h)) \\
&\qquad - (f, (\Pi_h u - u_h) - E_h(\Pi_h u - u_h))] \\
&\qquad + [a_h(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h))] \\
&\le Ch\|\Pi_h u - u_h\|_h + [a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h))].
\end{aligned} \tag{3.3.3}$$

□

Remark 3.18. *When the obstacles are absent we have* $K = H_0^2(\Omega)$, $K_h = V_h$, *and the finite element methods proposed here become finite element methods for the biharmonic problem. We are able to derive a concise unified analysis of finite element methods by the introduction of the enriching operator* E_h. *In fact, we have*

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) = 0. \tag{3.3.4}$$

Then from (3.1.20), (3.3.2) *and the arithmetic-geometric mean inequality, we obtain*

$$\|u - u_h\|_h^2 \le C(\|u - \Pi_h u\|_h^2 + h^2) + \frac{1}{2}\|u - u_h\|_h^2,$$

and hence

$$\|u - u_h\|_h \le Ch,$$

by (3.1.5).

In the presence of obstacles, (3.3.4) *does not hold. However we are still able to derive an estimate of this term by using the properties of the auxiliary obstacle problem.*

Lemma 3.19. *There exists a positive constant C independent of h such that*

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \leq C(h^2 + h\|\Pi_h u - u_h\|_h). \tag{3.3.5}$$

Proof. Since E_h is a linear operator, we have

$$\begin{aligned} a(u, E_h(\Pi_h u - u_h)) &- (f, E_h(\Pi_h u - u_h)) \\ &= [a(u, E_h\Pi_h u - u) - (f, E_h\Pi_h u - u)] \\ &\quad + [a(u, u - E_h u_h) - (f, u - E_h u_h)]. \end{aligned} \tag{3.3.6}$$

The first term on the right-hand side of (3.3.6) can be estimated by (3.3.1) and (3.1.10):

$$\begin{aligned} a(u, &E_h\Pi_h u - u) - (f, E_h\Pi_h u - u) \\ &\leq Ch|u|_{H^3(\Omega)}|u - E_h\Pi_h u|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\|u - E_h\Pi_h u\|_{L^2(\Omega)} \\ &\leq Ch^2 + Ch^3 \leq Ch^2. \end{aligned} \tag{3.3.7}$$

For the second term on the right-hand side of (3.3.6), since $\hat{u}_h \in K$, we apply (3.0.3) and (3.2.26) to obtain

$$\begin{aligned} &a(u, u - E_h u_h) - (f, u - E_h u_h) \\ &= [a(u, u - \hat{u}_h) - (f, u - \hat{u}_h)] + [a(u, \hat{u}_h - E_h u_h) - (f, \hat{u}_h - E_h u_h)] \\ &\leq a(u, \hat{u}_h - E_h u_h) - (f, \hat{u}_h - E_h u_h) \\ &= [a(u, \hat{u}_h - \tilde{u}_h) - (f, \hat{u}_h - \tilde{u}_h)] + [a(u, \tilde{u}_h - E_h u_h) - (f, \tilde{u}_h - E_h u_h)] \\ &\leq \delta_{h,1}[a(u, \phi_1) - (f, \phi_1)] - \delta_{h,2}[a(u, \phi_2) - (f, \phi_2)] \\ &\qquad + [a(u, \tilde{u}_h - E_h u_h) - (f, \tilde{u}_h - E_h u_h)] \\ &\leq Ch^2 + [a(u, \tilde{u}_h - E_h u_h) - (f, \tilde{u}_h - E_h u_h)]. \end{aligned} \tag{3.3.8}$$

Note that

$$|u - E_h u_h|_{H^2(\Omega)} \leq |u - E_h\Pi_h u|_{H^2(\Omega)} + |E_h\Pi_h u - E_h u_h|_{H^2(\Omega)} \tag{3.3.9}$$

$$\leq C(h + \|\Pi_h u - u_h\|_h)$$

by (3.1.8) and (3.1.10). Since $E_h u_h \in \tilde{K}_h$, we can now combine (3.2.3), (3.2.27) and (3.3.9) to obtain

$$\begin{aligned}
&a(u, \tilde{u}_h - E_h u_h) - (f, \tilde{u}_h - E_h u_h) \\
&= [a(u - \tilde{u}_h, \tilde{u}_h - u) + a(u - \tilde{u}_h, u - E_h u_h)] \\
&\quad + [a(\tilde{u}_h, \tilde{u}_h - E_h u_h) - (f, \tilde{u}_h - E_h u_h)] \\
&\leq Ch|u - E_h u_h|_{H^2(\Omega)} \qquad (3.3.10) \\
&\leq C(h^2 + h\|\Pi_h u - u_h\|_h).
\end{aligned}$$

Note that here we have used the obvious fact that $a(u - \tilde{u}_h, \tilde{u}_h - u) \leq 0$.

The estimate (3.3.5) follows from (3.3.6)–(3.3.8), and (3.3.10). □

Finally we have the following error estimate in the energy norm.

Theorem 3.20. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_h \leq Ch. \qquad (3.3.11)$$

Proof. It follows from (3.1.5), (3.1.20), (3.3.2), (3.3.5), the triangle inequality and the arithmetic-geometric mean inequality that

$$\begin{aligned}
\|u - u_h\|_h^2 &\leq C(h^2 + h\|\Pi_h u - u_h\|_h) \\
&\leq C(h^2 + h\|u - u_h\|_h) \\
&\leq Ch^2 + \frac{1}{2}\|u - u_h\|_h^2,
\end{aligned}$$

which implies the estimate (3.3.11). □

3.4 Approximations of the Coincidence Set and the Free Boundary

In this section we consider the approximations of the coincidence set (resp. free boundary) by the discrete coincidence sets (resp. discrete free boundaries). We begin with an error estimate in the L^∞ norm.

Theorem 3.21. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch. \tag{3.4.1}$$

Proof. We start with

$$\begin{aligned} \|u - u_h\|_{L^\infty(\Omega)} \leq \|u - \Pi_h u\|_{L^\infty(\Omega)} + \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)} \\ + \|E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)}. \end{aligned} \tag{3.4.2}$$

The right-hand side of (3.4.2) can be estimated as follows:

$$\|u - \Pi_h u\|_{L^\infty(\Omega)} \leq Ch^2 |u|_{H^3(\Omega)} \tag{3.4.3}$$

by a standard interpolation error estimate;

$$\begin{aligned} \|(\Pi_h u - u_h) &- E_h(\Pi_h u - u_h)\|_{L^\infty(\Omega)} \\ &= \max_{T\in\mathcal{T}_h} \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^\infty(T)} \\ &\leq \max_{T\in\mathcal{T}_h} h_T^{-1} \|(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)\|_{L^2(T)} \\ &\leq Ch\|\Pi_h u - u_h\|_h \end{aligned} \tag{3.4.4}$$

by an inverse inequality and (2.5.4) (or (2.5.9));

$$\begin{aligned} \|E_h(\Pi_h u - u_h)\|_{L_\infty(\Omega)} &\leq C\|E_h(\Pi_h u - u_h)\|_{H^2(\Omega)} \\ &\leq C\|\Pi_h u - u_h\|_h. \end{aligned} \tag{3.4.5}$$

by the Sobolev inequality (cf. Theorem 2.11), a Poincaré-Friedrichs inequality (cf. Theorem 2.16), and (3.1.8).

Combining (3.4.2)–(3.4.5) and Theorem 3.20, we obtain (3.4.1). □

Remark 3.22. *Numerical results in Chapter* 4 *indicate that the estimate* (3.4.1) *is not sharp.*

Let I_i $(i = 1, 2)$ be the coincidence sets of the obstacle problem (3.0.1) (or (3.0.3)) defined in (3.2.8) and let $F_i = \partial I_i$ $(i = 1, 2)$ be the free boundaries.

The discrete coincidence sets $I_{h,i}$ $(i = 1, 2)$ are defined by

$$I_{h,1} = \{x \in \Omega : u_h(x) - \psi_1(x) \le \tau_h\}, \tag{3.4.6}$$

$$I_{h,2} = \{x \in \Omega : \psi_2(x) - u_h(x) \le \tau_h\}, \tag{3.4.7}$$

where

$$\tau_h = \rho \|u - u_h\|_{L_\infty(\Omega)}, \tag{3.4.8}$$

and ρ can be any number > 1. Observe that (3.2.8) and (3.4.6)–(3.4.7) imply, for $i = 1, 2$,

$$I_i \subseteq I_{h,i} \quad \text{ and } \quad I_{h,i} \setminus I_i \subseteq \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \le 2\tau_h\}. \tag{3.4.9}$$

It follows from the assumptions of ψ_i $(i = 1, 2)$ and (3.4.1) that the discrete coincidence sets are disjoint compact subsets of Ω if h is sufficiently small, which is assumed to be the case. We define $F_{h,i} = \partial I_{h,i}$ $(i = 1, 2)$ to be the discrete free boundaries.

We can obtain an approximation result for the coincidence set under the following non-degeneracy assumption (cf. [38, 41, 91]):
There exist positive numbers μ_1 and μ_2 such that

$$|\{x \in \Omega : 0 < |u(x) - \psi_i(x)| \le \epsilon\}| \le C\epsilon^{\mu_i} \quad (i = 1, 2) \tag{3.4.10}$$

for $\epsilon > 0$ sufficiently small, where $|A|$ denotes the Lebesgue measure of the set A and the positive constant C is independent of ϵ. Let

$$I_i \Delta I_{h,i} = (I_i \setminus I_{h,i}) \cup (I_{h,i} \setminus I_i)$$

be the symmetric difference of I_i and $I_{h,i}$. From (3.4.9), we know $I_i \Delta I_{h,i} = I_{h,i} \setminus I_i$. Combine this with (3.4.9) and (3.4.10), we have

$$|I_i \Delta I_{h,i}| \leq C\tau_h^{\mu_i} \quad \text{for } i = 1, 2. \tag{3.4.11}$$

The approximation result for the free boundary requires the following stronger non-degeneracy assumption (cf. [38, 41, 91]):

There exist positive numbers μ_1 and μ_2 such that

$$\{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq \epsilon\} \subseteq \{x \in \Omega : \text{dist}(x, F_i) \leq C\epsilon^{\mu_i}\} \tag{3.4.12}$$

for $i = 1, 2$ and $\epsilon > 0$ sufficiently small, where the positive constant C is independent of ϵ.

Observe that (3.2.8) and (3.4.6)–(3.4.7) imply, for $i = 1, 2$,

$$F_{h,i} \subseteq \{x \in \Omega : 0 < |u(x) - \psi_i(x)| \leq 2\tau_h\} \tag{3.4.13}$$

since $|u_h(x) - \psi_i(x)| = \tau_h$ for any $x \in F_{h,i}$ and $\tau_h > \|u - u_h\|_{L^\infty(\Omega)}$ (except in the trivial case where $u = u_h$). It then follows from (3.4.12) and (3.4.13) that

$$F_{h,i} \subseteq \{x \in \Omega : \text{dist}(x, F_i) \leq C\tau_h^{\mu_i}\}, \tag{3.4.14}$$

i.e., the discrete free boundary $F_{h,i}$ is within a tubular neighborhood of the continuous free boundary F_i whose width is $O(\tau^{\mu_i})$.

Remark 3.23. *For second order elliptic obstacle problems, it is possible to establish* (3.4.10) *and* (3.4.12) *under appropriate assumptions on the obstacle functions and* *the load function* (*cf.* [41, 91])*. But such results are not available for plates, where limited theoretical results can be found in* [42, 92]*. Hence* (3.4.10) *and* (3.4.12) *are assumptions that need to be verified for individual plate obstacle problems.*

Remark 3.24. *The results obtained in this subsection also work for general polygonal domains and general boundary conditions.*

Chapter 4

Extension to General Polygonal Domains with General Dirichlet Boundary Conditions

In Chapter 3 we presented a general framework for finite element methods for the displacement obstacle problem of clamped plates and obtained $O(h)$ error estimates in the energy norm and the L^∞ norm. Note that the results we obtained were on convex polygonal domains with homogeneous Dirichlet boundary conditions. In this chapter, we will extend the results in Chapter 3 to general polygonal domains with general Dirichlet boundary conditions. The general obstacle problem will be introduced in Section 4.1. Since different finite element methods have different treatments for nonhomogeneous boundary conditions, we will only focus on a Morley finite element method [32] and a quadratic C^0 interior penalty method [31] in Section 4.2 and 4.3, respectively. A generalized finite element method was also discussed in [25]. Numerical results are provided in Section 4.4 to illustrate the performance of the methods. The presentation in this chapter follows [31, 32].

4.1 A General Fourth Order Obstacle Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $f(x) \in L^2(\Omega)$, $g(x) \in H^4(\Omega)$, and $\psi_1(x)$, $\psi_2(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ be the obstacle functions such that

$$\psi_1 < \psi_2 \text{ in } \Omega \text{ and } \psi_1 < g < \psi_2 \text{ on } \partial\Omega. \tag{4.1.1}$$

We consider the following two-sided displacement obstacle problem with general Dirichlet boundary conditions:

Find $u \in K$ such that

$$u = \operatorname*{argmin}_{v \in K} G(v), \tag{4.1.2}$$

where

$$K = \{v \in H^2(\Omega) : v - g \in H_0^2(\Omega),\ \psi_1 \le v \le \psi_2 \text{ in } \Omega\}, \tag{4.1.3}$$

$G(v) = \frac{1}{2}a(v,v) - (f,v)$, and $a(\cdot,\cdot)$ is given by (1.1.4).

Note that K is a nonempty closed convex subset of $H^2(\Omega)$, and the symmetric bilinear form $a(\cdot,\cdot)$ is bounded on $H^2(\Omega)$ and coercive on the set $K - K = \{v - w : v, w \in K\} \subseteq H_0^2(\Omega)$. Therefore it follows from Theorem 2.24 that the problem (4.1.2) has a unique solution $u \in K$ which is also uniquely determined by the variational inequality

$$a(u, v-u) \ge (f, v-u) \qquad \forall v \in K. \tag{4.1.4}$$

From Corollary 2.32, the solution of (4.1.2) belongs to $H^{2+\alpha}(\Omega) \cap C^2(\Omega)$ for some $\alpha \in (1/2, 1]$ determined by the interior angles of Ω. We will refer to α as the index of elliptic regularity. For nonconvex polygonal domains, α is less than 1. This makes the convergence analysis for finite element methods more delicate.

As in Chapter 3, the key of error analysis is the introduction of an auxiliary obstacle problem. In fact we can still consider the following intermediate obstacle problem:

Find $\tilde{u}_h \in \tilde{K}_h$ such that

$$\tilde{u}_h = \operatorname*{argmin}_{v \in \tilde{K}_h} G(v), \tag{4.1.5}$$

where

$$\tilde{K}_h = \{v \in H^2(\Omega) : v - g \in H_0^2(\Omega),\ \psi_1(p) \le v(p) \le \psi_2(p) \quad \forall p \in \mathcal{V}_h\}, \tag{4.1.6}$$

$G(v) = \frac{1}{2}a(v,v) - (f,v)$, and $a(\cdot,\cdot)$ is given by (1.1.4).

It is clear that $K \subset \tilde{K}_h$ and $\tilde{K}_h$ is a closed convex subset of $H^2(\Omega)$. Then there exists a unique solution $\tilde{u}_h$ to the problem (4.1.5) which is also characterized by the variational inequality

$$a(\tilde{u}_h, v - \tilde{u}_h) \geq (f, v - \tilde{u}_h) \qquad \forall v \in \tilde{K}_h. \tag{4.1.7}$$

The connection between (4.1.2) and (4.1.5) is given by the following properties of $\tilde{u}_h$ (cf. Remark 3.16):

$$|u - \tilde{u}_h|_{H^2(\Omega)} \leq Ch, \tag{4.1.8}$$

and there exist $h_0 > 0$ such that

$$\hat{u}_h = \tilde{u}_h + \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2 \in K \qquad \forall h \leq h_0, \tag{4.1.9}$$

where $\phi_1, \phi_2 \in C_0^\infty(\Omega)$ and the positive numbers $\delta_{h,1}$ and $\delta_{h,2}$ satisfy

$$\delta_{h,i} \leq Ch^2. \tag{4.1.10}$$

4.2 A Morley Finite Element Method

In this section we solve (4.1.2) by using a Morley finite element method.

4.2.1 Discrete Problem

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h. Let $\tilde{V}_h$ be the Morley finite element space associated with $\mathcal{T}_h$ and V_h be the subspace of $\tilde{V}_h$ with vanishing degrees of freedom on $\partial\Omega$. Furthermore, let $\Pi_h : H^2(\Omega) \longrightarrow \tilde{V}_h$ be the interpolation operator defined in (2.4.16a)–(2.4.16b) that also includes the boundary vertices and boundary edges. We have the following interpolation error estimate (cf. [48, 82])

$$\|\zeta - \Pi_h\zeta\|_h \leq Ch^\alpha|\zeta|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega). \tag{4.2.1}$$

We consider the discrete obstacle problem:

Find $u_h \in K_h$ such that

$$u_h = \underset{v \in K_h}{\operatorname{argmin}}\, G_h(v), \tag{4.2.2}$$

where

$$K_h = \{v \in \tilde{V}_h : v - \Pi_h g \in V_h,\ \psi_1(p) \le v(p) \le \psi_2(p) \quad \forall p \in \mathcal{V}_h\}, \qquad (4.2.3)$$

$$G_h(v) = \frac{1}{2}a_h(v,v) - (f,v), \qquad (4.2.4)$$

and $a_h(\cdot,\cdot)$ is given by (2.4.7).

Similar to the continuous case, $a_h(\cdot,\cdot)$ is symmetric positive definite on the set $K_h - K_h \subseteq V_h$. Therefore the discrete problem (4.2.2) is well-posed and its unique solution is also characterized by the variational inequality

$$a_h(u_h, v - u_h) \ge (f, v - u_h) \qquad \forall v \in K_h. \qquad (4.2.5)$$

4.2.2 Error Estimates

As discussed in Section 3.2 and 3.3, it is important to connect the continuous obstacle problem (4.1.2) and the auxiliary obstacle problem (4.1.5). However the enriching operator E_h defined there fails to map K_h to $\tilde{K}_h$ unless $g = 0$. Since $v - \Pi_h g \in V_h$ for any $v \in K_h$, we can define an operator $T_h : K_h \longrightarrow H^2(\Omega)$ by

$$T_h v = g + E_h(v - \Pi_h g) \qquad \forall v \in K_h. \qquad (4.2.6)$$

The following properties of T_h are useful for the convergence analysis.

Lemma 4.1. *We have*

$$T_h : K_h \longrightarrow \tilde{K}_h, \qquad (4.2.7)$$

and, for any $v \in K_h$ and $\zeta \in H^{2+\alpha}(\Omega) \cap K$,

$$|T_h\Pi_h\zeta - T_h v|_{H^2(\Omega)} \le C\|\Pi_h\zeta - v\|_h, \qquad (4.2.8)$$

$$\sum_{m=0}^{2} h^m|\zeta - T_h\Pi_h\zeta|_{H^m(\Omega)} \le Ch^{2+\alpha}|\zeta - g|_{H^{2+\alpha}(\Omega)}. \qquad (4.2.9)$$

Proof. Let $v \in K_h$. Note that $T_h v - g = E_h(v - \Pi_h g)$ belongs to $H_0^2(\Omega)$ by (4.2.3) and (3.1.6), and

$$(T_h v)(p) = g(p) + [E_h(v - \Pi_h g)](p) = g(p) + (v - \Pi_h g)(p) = v(p) \qquad \forall p \in \mathcal{V}_h$$

by (3.1.4) and (3.1.7). It then follows from (4.2.3) and (4.1.6) that $T_h v \in \tilde{K}_h$.

From (4.2.6), we have

$$T_h\Pi_h\zeta - T_h v = E_h(\Pi_h\zeta - v) \qquad \forall v \in K_h, \tag{4.2.10}$$

$$\zeta - T_h\Pi_h\zeta = (\zeta - g) - E_h\Pi_h(\zeta - g) \qquad \forall \zeta \in H^{2+\alpha}(\Omega) \cap K. \tag{4.2.11}$$

Since $\Pi_h\zeta - v \in K_h - K_h \subseteq V_h$ and $\zeta - g \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$, the properties (4.2.8) and (4.2.9) follow from (2.5.6) and (2.5.7). □

The remaining part of this section is devoted to deriving the error estimate for $\|u - u_h\|_h$. Hence we begin with the abstract estimate (cf. (3.1.20)):

$$\|u - u_h\|_h^2 \le C_\dagger \|u - \Pi_h u\|_h^2 + C_\ddagger[a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)]. \tag{4.2.12}$$

In view of (4.2.1), it only remains to estimate the second term on the right-hand side of (4.2.12). We will follow the framework of the error analysis in Section 3.3. However, since we are dealing with general polygonal domains with general Dirichlet boundary conditions, some technical lemmas will be needed.

Recall that the solution u of (4.1.2) belongs to $H^{2+\alpha}(\Omega)$. Note also that $H^{2+\alpha}(\Omega)$ can be obtained from $H^2(\Omega)$ and $H^3(\Omega)$ by the real method of interpolation. More precisely, we have (cf. [1, 98])

$$H^{2+\alpha}(\Omega) = \left[H^2(\Omega), H^3(\Omega)\right]_{\alpha,2}. \tag{4.2.13}$$

Lemma 4.2. *We have*

$$|a_h(u, v - E_h v)| \le C h^\alpha \|u\|_{H^{2+\alpha}(\Omega)} \|v\|_h \qquad \forall v \in V_h. \tag{4.2.14}$$

Proof. Given any $v \in V_h$, we have, by (3.1.8),

$$|a_h(\zeta, v - E_h v)| \le C\|\zeta\|_{H^2(\Omega)} \|v\|_h \qquad \forall \zeta \in H^2(\Omega). \tag{4.2.15}$$

Now suppose $\zeta \in H^3(\Omega)$. By (3.1.9), we find

$$|a_h(\zeta, v - E_h v)| \le Ch|\zeta|_{H^3(\Omega)}\|v\|_h \qquad \forall \zeta \in H^3(\Omega). \tag{4.2.16}$$

Combining (4.2.15), (4.2.16), (4.2.13) and the interpolation between Sobolev spaces, we have

$$|a_h(\zeta, v - E_h v)| \le Ch^\alpha \|\zeta\|_{H^{2+\alpha}(\Omega)}\|v\|_h \qquad \forall \zeta \in H^{2+\alpha}(\Omega).$$

Taking $\zeta = u$, we obtain the estimate (4.2.14). □

Remark 4.3. *The estimate* (4.2.14) *can be viewed as a generalization of* (3.1.9).

Lemma 4.4. *We have*

$$|a(u, T_h\Pi_h u - u)| \le Ch^{2\alpha}\|u\|_{H^{2+\alpha}(\Omega)}|u - g|_{H^{2+\alpha}(\Omega)}. \tag{4.2.17}$$

Proof. The Cauchy-Schwarz inequality and (4.2.9) imply

$$|a(\zeta, T_h\Pi_h u - u)| \le Ch^\alpha \|\zeta\|_{H^2(\Omega)}|u - g|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^2(\Omega). \tag{4.2.18}$$

Since $T_h\Pi_h u - u \in H_0^2(\Omega)$ (cf. (4.2.11)), we also have, by integration by parts and (4.2.9),

$$\begin{aligned}|a(\zeta, T_h\Pi_h u - u)| &= \left|\int_\Omega \nabla(\Delta\zeta)\cdot\nabla(T_h\Pi_h u - u)\,dx\right| \\ &\le Ch^{1+\alpha}\|\zeta\|_{H^3(\Omega)}|u - g|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^3(\Omega).\end{aligned} \tag{4.2.19}$$

It follows from (4.2.18), (4.2.19), (4.2.13) and the interpolation between Sobolev spaces that

$$|a(\zeta, T_h\Pi_h u - u)| \le Ch^{2\alpha}\|\zeta\|_{H^{2+\alpha}(\Omega)}|u - g|_{H^{2+\alpha}(\Omega)}. \tag{4.2.20}$$

Taking $\zeta = u$, we obtain (4.2.17). □

Now we are able to estimate the second term on the right-hand side of (4.2.12).

Lemma 4.5. *There exists a positive constant C independent of h such that*

$$\begin{aligned} & a(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h) \\ \leq & Ch^\alpha \|\Pi_h u - u_h\|_h + [a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h))] \end{aligned} \tag{4.2.21}$$

Proof. The proof is similar to Lemma 3.17. In fact, since $\Pi_h u - u_h \in V_h$, we can still insert $E_h(\Pi_h u - u_h)$ into left-hand side of (4.2.21). Using Lemma 4.2 and (3.1.8), we obtain (4.2.21). □

The next lemma is analogous to Lemma 3.19. However, since a new operator T_h is used for connecting the discrete and the auxiliary obstacle problems, we also include the proof here.

Lemma 4.6. *There exists a positive constant C independent of h such that*

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \leq C(h^{2\alpha} + h\|\Pi_h u - u_h\|_h). \tag{4.2.22}$$

Proof. We use (4.2.10) to write

$$\begin{aligned} a(u, E_h(\Pi_h u - u_h)) &= a(u, T_h \Pi_h u - T_h u_h) \\ &= a(\tilde{u}_h, T_h \Pi_h u - T_h u_h) + a(u - \tilde{u}_h, T_h \Pi_h u - T_h u_h) \end{aligned} \tag{4.2.23}$$

and note that

$$\begin{aligned} |a(u - \tilde{u}_h, T_h \Pi_h u - T_h u_h)| &\leq C|u - \tilde{u}_h|_{H^2(\Omega)} |T_h \Pi_h u - T_h u_h|_{H^2(\Omega)} \\ &\leq Ch\|\Pi_h u - u_h\|_h \end{aligned}$$

by (4.1.8) and (4.2.8). From (4.1.7) and (4.2.7) we have

$$\begin{aligned} a(\tilde{u}_h, T_h \Pi_h u - T_h u_h) &= a(\tilde{u}_h, T_h \Pi_h u - \tilde{u}_h) + a(\tilde{u}_h, \tilde{u}_h - T_h u_h) \\ &\leq a(\tilde{u}_h, T_h \Pi_h u - \tilde{u}_h) + (f, \tilde{u}_h - T_h u_h). \end{aligned}$$

Moreover we have

$$a(\tilde{u}_h, T_h\Pi_h u - \tilde{u}_h) = a(\tilde{u}_h - u, T_h\Pi_h u - \tilde{u}_h) + a(u, T_h\Pi_h u - u) \\ + a(u, u - \tilde{u}_h) \tag{4.2.24}$$

and observe that

$$\begin{aligned} |a(\tilde{u}_h - u, T_h\Pi_h u - \tilde{u}_h)| &\le |\tilde{u}_h - u|_{H^2(\Omega)}(|T_h\Pi_h u - u|_{H^2(\Omega)} + |u - \tilde{u}_h|_{H^2(\Omega)}) \\ &\le Ch^{1+\alpha} \end{aligned}$$

by (4.1.8) and (4.2.9). By Lemma 4.4, we also have

$$|a(u, T_h\Pi_h u - u)| \le Ch^{2\alpha}.$$

Hence we obtain from (4.2.10) and (4.2.23) the relation

$$\begin{aligned} &a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \\ &\quad = a(u, E_h(\Pi_h u - u_h)) - (f, T_h(\Pi_h u - u_h)) \\ &\quad \le C(h^{2\alpha} + h\|\Pi_h u - u_h\|_h) + [a(u, u - \tilde{u}_h) - (f, T_h\Pi_h u - \tilde{u}_h)]. \end{aligned} \tag{4.2.25}$$

Now we use (4.1.9) and (4.1.4) to derive

$$\begin{aligned} a(u, u - \tilde{u}_h) &= a(u, u - \hat{u}_h) + \delta_{h,1}a(u, \phi_1) - \delta_{h,2}a(u, \phi_2) \\ &\le (f, u - \hat{u}_h) + \delta_{h,1}a(u, \phi_1) - \delta_{h,2}a(u, \phi_2) \\ &= (f, u - T_h\Pi_h u) + (f, T_h\Pi_h u - \tilde{u}_h) \\ &\quad - \delta_{h,1}[(f, \phi_1) - a(u, \phi_1)] + \delta_{h,2}[(f, \phi_2) - a(u, \phi_2)], \end{aligned} \tag{4.2.26}$$

and we note that, by (4.2.9) and (4.1.10),

$$|(f, u - T_h\Pi_h u)| \le \|f\|_{L^2(\Omega)}\|u - T_h\Pi_h u\|_{L^2(\Omega)} \le Ch^{2+\alpha},$$

$$|\delta_{h,1}[(f, \phi_1) - a(u, \phi_1)]| + |\delta_{h,2}[(f, \phi_2) - a(u, \phi_2)]| \le Ch^2.$$

Therefore we deduce from (4.2.25) the relation

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \leq C(h^{2\alpha} + h\|\Pi_h u - u_h\|_h),$$

which is the estimate (4.2.22). □

Theorem 4.7. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_h \leq Ch^\alpha. \tag{4.2.27}$$

Proof. It follows from (4.2.1), (4.2.12), (4.2.21), (4.2.22), the triangle inequality and the arithmetic-geometric mean inequality that

$$\begin{aligned} \|u - u_h\|_h^2 &\leq C(h^{2\alpha} + h^\alpha\|\Pi_h u - u_h\|_h) \\ &\leq C(h^{2\alpha} + h^\alpha\|u - u_h\|_h) \\ &\leq Ch^{2\alpha} + \frac{1}{2}\|u - u_h\|_h^2. \end{aligned}$$

Hence we obtain the estimate (4.2.27). □

The following error estimate in the L^∞ norm is based on (3.1.8), (4.2.1), Theorem 4.7, standard inverse estimates and the Continuous Embedding Theorem (cf. Theorem 2.11). The proof is similar to Theorem 3.21 in Section 3.3 and is thus omitted.

Theorem 4.8. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^\alpha. \tag{4.2.28}$$

4.3 A Quadratic C^0 Interior Penalty Method

In this section we solve (4.1.2) by using a quadratic C^0 interior penalty method.

4.3.1 Discrete Problem

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h and let $\tilde{V}_h \subseteq H^1(\Omega)$ be the $\mathbb{P}_2$ Lagrange finite element space associated with $\mathcal{T}_h$. We consider the following C^0 interior penalty method for (4.1.2):

Find $u_h \in K_h$ such that

$$u_h = \operatorname*{argmin}_{v \in K_h} G_h(v), \tag{4.3.1}$$

where

$$\begin{aligned} G_h(v) = \frac{1}{2}\Big(\sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 v\, dx + 2 \sum_{e \in \mathcal{E}_h} \int_e \{\!\{\partial^2 v/\partial n^2\}\!\} [\![\partial(v-g)/\partial n]\!] ds \\ + \sigma \sum_{e \in \mathcal{E}_h} |e|^{-1} \int_e [\![\partial(v-g)/\partial n]\!] [\![\partial(v-g)/\partial n]\!] ds \Big) - (f, v), \end{aligned} \tag{4.3.2}$$

$$K_h = \{v \in \tilde{V}_h : v - \Pi_h g \in H_0^1(\Omega),\ \psi_1(p) \le v(p) \le \psi_2(p) \quad \forall p \in \mathcal{V}_h\}, \tag{4.3.3}$$

and σ is a positive penalty parameter. Here $\Pi_h : C(\bar{\Omega}) \longrightarrow \tilde{V}_h$ is the nodal interpolation operator for the $\mathbb{P}_2$ Lagrange finite element space such that (3.1.4) holds.

Remark 4.9. *While the approximation of the boundary condition $u = g$ is included in the definition of K_h, the approximation of the boundary condition $\partial u/\partial n = \partial g/\partial n$ is enforced by the penalty term in $G_h(\cdot)$. This is different from the Morley finite element method discussed in Section* 4.2.

Remark 4.10. *The functional $G_h(\cdot)$ is motivated by the bilinear form for C^0 interior penalty methods for the biharmonic equation with general Dirichlet boundary conditions.*

Since $[\![\partial g/\partial n]\!] = 0$ on interior edges, we can reformulate (4.3.1) as follows:

Find $u_h \in K_h$ such that

$$u_h = \operatorname*{argmin}_{v \in K_h} \left[\frac{1}{2} a_h(v, v) - F(v) \right], \tag{4.3.4}$$

where

$$
\begin{aligned}
F(v) &= (f, v) + \sum_{e\in\mathcal{E}_h^b} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} + \frac{\sigma}{|e|} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] \right) \left[\!\!\left[\frac{\partial g}{\partial n} \right]\!\!\right] ds \\
&= (f, v) + \sum_{e\in\mathcal{E}_h^b} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} + \frac{\sigma}{|e|} \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right] \right) \left[\!\!\left[\frac{\partial u}{\partial n} \right]\!\!\right] ds, \qquad (4.3.5)
\end{aligned}
$$

and $a_h(\cdot,\cdot)$ is given by (2.4.27).

We will measure the discretization error by the energy seminorm $\|\cdot\|_h$ defined in (2.4.28). The seminorm $\|\cdot\|_h$ is well-defined on the space $\tilde{V}_h + H^{2+\alpha}(\Omega)$ where $\alpha \in (1/2, 1]$ is the index of elliptic regularity. Moreover, it is a norm on the space $V_h \supseteq (K_h - K_h)$ where V_h is a subspace of $\tilde{V}_h$ with vanishing degrees of freedom on $\partial\Omega$.

Note that Lemma 2.45 is still true for any $v \in \tilde{V}_h$, hence we have

$$
a_h(v, v) \geq C\|v\|_h^2 \qquad \forall v \in \tilde{V}_h, \qquad (4.3.6)
$$

provided that σ is sufficiently large. From (4.3.6) we see that $a_h(\cdot,\cdot)$ is positive definite on the set $K_h - K_h \subseteq V_h$. Therefore the discrete obstacle problem (4.3.4) has a unique solution characterized by the discrete variational inequality

$$
a_h(u_h, v - u_h) \geq F(v - u_h) \qquad \forall v \in K_h. \qquad (4.3.7)
$$

Next we recall some useful properties of the nodal interpolation operator Π_h. Similar to (2.4.37) and (2.4.38), we have (cf. [28, 48])

$$
\sum_{m=0}^{2} h_T^m |\zeta - \Pi_h\zeta|_{H^m(T)} \leq C h_T^{2+\alpha} |\zeta|_{H^{2+\alpha}(T)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega),\ T \in \mathcal{T}_h, \qquad (4.3.8)
$$

$$
\|\zeta - \Pi_h\zeta\|_h \leq C h^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega). \qquad (4.3.9)
$$

In particular, we have

$$
\sum_{e\in\mathcal{E}_h^i} |e|^{-1} \|[\![\partial(\Pi_h\zeta)/\partial n]\!]\|_{L^2(e)}^2 = \sum_{e\in\mathcal{E}_h^i} |e|^{-1} \|[\![\partial(\zeta - \Pi_h\zeta)/\partial n]\!]\|_{L^2(e)}^2
$$

$$\leq \|\zeta - \Pi_h\zeta\|_h^2 \leq Ch^{2\alpha}|\zeta|^2_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega). \tag{4.3.10}$$

Lemma 4.11. *We have*

$$\sum_{e\in\mathcal{E}_h^i} |e| \, \big\| [\![\partial^2(\Pi_h\zeta)/\partial^2 n_e]\!] \big\|^2_{L^2(e)} \leq Ch^{2\alpha}|\zeta|^2_{H^{2+\alpha}(\Omega)} \qquad \forall \zeta \in H^{2+\alpha}(\Omega). \tag{4.3.11}$$

Proof. Let $e \in \mathcal{E}_h^i$ be arbitrary and Q_e be the quadrilateral formed by the two triangles in $\mathcal{T}_e$. By the Bramble-Hilbert lemma (cf. [16,55]), there exists $z_e \in \mathbb{P}_2(Q_e)$ such that $|\zeta - z_e|_{H^2(Q_e)} \leq C|e|^\alpha|\zeta|_{H^{2+\alpha}(Q_e)}$. Combining this estimate with (4.3.8) and a standard inverse estimate, we find

$$\begin{aligned}
\sum_{e\in\mathcal{E}_h^i} |e| \, \big\| [\![\partial^2(\Pi_h\zeta)/\partial^2 n_e]\!] \big\|^2_{L^2(e)} &= \sum_{e\in\mathcal{E}_h^i} |e| \, \big\| [\![\partial^2(z_e - \Pi_h\zeta)/\partial^2 n_e]\!] \big\|^2_{L^2(e)} \\
&\leq C \sum_{e\in\mathcal{E}_h^i} \sum_{T\in\mathcal{T}_e} |z_e - \Pi_h\zeta|^2_{H^2(T)} \\
&\leq C \sum_{e\in\mathcal{E}_h^i} \sum_{T\in\mathcal{T}_e} \Big(|z_e - \zeta|^2_{H^2(T)} + |\zeta - \Pi_h\zeta|^2_{H^2(T)} \Big) \\
&\leq Ch^{2\alpha} \sum_{e\in\mathcal{E}_h^i} \sum_{T\in\mathcal{T}_e} |\zeta|^2_{H^{2+\alpha}(Q_e)} \leq Ch^{2\alpha}|\zeta|^2_{H^{2+\alpha}(\Omega)}.
\end{aligned}$$

□

4.3.2 Error Estimates

In this section we provide the error analysis of the quadratic C^0 interior penalty method (4.3.4).

By (4.3.3) and (4.1.6), the operator defined in (4.2.6) still connects the set K_h and $\tilde{K}_h$ through

$$T_h K_h \subseteq \tilde{K}_h.$$

Of course we need to use the enriching operator E_h defined in Case 1 of Section 2.5.2. Moreover the properties (4.2.8) and (4.2.9) in Lemma 4.1 remain valid.

For any $v \in \tilde{V}_h$ and $w \in V_h$, since $w - E_h w \in H^2(\Omega, \mathcal{T}_h) \cap H_0^1(\Omega)$, $w - E_h w$ vanishes at the vertices and $v \in \mathbb{P}_2(T)$, we have for any edge e of $T \in \mathcal{T}_h$,

$$\int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n_e \partial t_e} \right]\!\!\right] \frac{\partial(w - E_h w)}{\partial t} \, ds = \left[\!\!\left[\frac{\partial^2 v}{\partial n_e \partial t_e} \right]\!\!\right] \int_e \frac{\partial(w - E_h w)}{\partial t} \, ds = 0.$$

Hence we obtain the following integration by parts formula from (2.4.40):

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2(w - E_h w) \, dx = &- \sum_{e \in \mathcal{E}_h^i} \int_e \left[\!\!\left[\frac{\partial^2 v}{\partial n_e^2} \right]\!\!\right] \left\{\!\!\left\{ \frac{\partial(w - E_h w)}{\partial n_e} \right\}\!\!\right\} ds \\ &- \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(w - E_h w)}{\partial n} \right]\!\!\right] ds, \quad (4.3.12) \end{aligned}$$

for any $v \in \tilde{V}_h$ and $w \in V_h$.

Next we derive a basic estimate for $u - u_h$, where u (resp. u_h) is the solution of (4.1.2) (resp. (4.3.4)). Note that the estimate is similar to the ones obtained in Lemma 3.9 and (4.2.12).

Lemma 4.12. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_h^2 \leq 2\|u - \Pi_h u\|_h^2 + C[a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)]. \qquad (4.3.13)$$

Proof. Since $\Pi_h u \in K_h$, we deduce from (4.3.6) and (4.3.7) that

$$\begin{aligned} \|u - u_h\|_h^2 &\leq 2\|u - \Pi_h u\|_h^2 + 2\|\Pi_h u - u_h\|_h^2 \\ &\leq 2\|u - \Pi_h u\|_h^2 + C a_h(\Pi_h u - u_h, \Pi_h u - u_h) \\ &\leq 2\|u - \Pi_h u\|_h^2 + C[a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)]. \end{aligned}$$

□

In view of (4.3.9), it remains only to estimate the second term on the right-hand side of (4.3.13). The following two lemmas will be needed.

Lemma 4.13. *We have*

$$|a(u, T_h \Pi_h u - u)| \leq C h^{2\alpha} \|u\|_{H^{2+\alpha}(\Omega)} |u - g|_{H^{2+\alpha}(\Omega)}. \qquad (4.3.14)$$

Remark 4.14. *The proof of Lemma* 4.13 *is the same as Lemma* 4.4*. Another proof without using the interpolation between Sobolev spaces can be found in* [31]*.*

Lemma 4.15. *We have*

$$\left|\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2(v-E_h v)\,dx+\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\}[\![\partial v/\partial n]\!]ds\right|$$
$$\le Ch^\alpha\|v\|_h \qquad \forall v\in V_h. \tag{4.3.15}$$

Proof. Let $v\in V_h$ be arbitrary. It follows from (4.3.12) and the fact $[\![\partial(E_h v)/\partial n]\!]=0$ on $e\in\mathcal{E}_h$ that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h u):D^2(v-E_h v)\,dx+\sum_{e\in\mathcal{E}_h}\int_e \{\!\{\partial^2(\Pi_h u)/\partial n^2\}\!\}[\![\partial v/\partial n]\!]ds\\ &\quad=-\sum_{e\in\mathcal{E}_h^i}\int_e\left[\!\!\left[\frac{\partial^2(\Pi_h u)}{\partial n_e^2}\right]\!\!\right]\left\{\!\!\left\{\frac{\partial(v-E_h v)}{\partial n_e}\right\}\!\!\right\}ds.\end{aligned}$$

Moreover, we have, by Lemma 4.11 and (2.5.12),

$$\begin{aligned}&\left|\sum_{e\in\mathcal{E}_h^i}\int_e[\![\partial^2(\Pi_h u)/\partial n_e^2]\!]\{\!\{\partial(v-E_h v)/\partial n_e\}\!\}\,ds\right|\\ &\le\left(\sum_{e\in\mathcal{E}_h^i}|e|\|[\![\partial^2(\Pi_h u)/\partial n_e^2]\!]\|_{L^2(e)}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{E}_h^i}|e|\|\{\!\{\partial(v-E_h v)/\partial n_e\}\!\}\|_{L^2(e)}^2\right)^{\frac12}\\ &\le Ch^\alpha\|v\|_h.\end{aligned}$$

The lemma follows from the above inequality. □

We are now ready to bound the second term on the right-hand side of (4.3.13).

Lemma 4.16. *There exists a positive constant C independent of h such that*

$$a_h(\Pi_h u,\Pi_h u-u_h)-F(\Pi_h u-u_h) \tag{4.3.16}$$
$$\le Ch^\alpha\|\Pi_h u-u_h\|_h+[a(u,E_h(\Pi_h u-u_h))-(f,E_h(\Pi_h u-u_h))]$$

Proof. By (2.4.27) and (4.3.5), we have

$$
\begin{aligned}
& a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h) \\
= & \sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2(\Pi_h u - u_h)\,dx \\
& \quad + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2(\Pi_h u)}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] ds \\
& \quad\quad + \sum_{e\in\mathcal{E}_h} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2(\Pi_h u - u_h)}{\partial n^2} \right\}\!\!\right\} + \frac{\sigma}{|e|} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] \right) \left[\!\!\left[\frac{\partial(\Pi_h u - u)}{\partial n} \right]\!\!\right] ds \\
& \quad\quad - (f, \Pi_h u - u_h),
\end{aligned}
\tag{4.3.17}
$$

where we have also used the fact that $[\![\partial u/\partial n]\!] = 0$ on interior edges.

By (2.4.28) and (4.3.10) we have

$$
\begin{aligned}
& \left| \sum_{e\in\mathcal{E}_h} \int_e \left(\left\{\!\!\left\{ \frac{\partial^2(\Pi_h u - u_h)}{\partial n^2} \right\}\!\!\right\} + \frac{\sigma}{|e|} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] \right) \left[\!\!\left[\frac{\partial(\Pi_h u - u)}{\partial n} \right]\!\!\right] ds \right| \\
\le & C \left[\sum_{e\in\mathcal{E}_h} \left(|e| \left\| \left\{\!\!\left\{ \frac{\partial^2(\Pi_h u - u_h)}{\partial n^2} \right\}\!\!\right\} \right\|_{L^2(e)}^2 + \frac{1}{|e|} \left\| \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] \right\|_{L^2(e)}^2 \right) \right]^{\frac{1}{2}} \\
& \quad \times \left(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \| [\![\partial(u - \Pi_h u)/\partial n]\!] \|_{L^2(e)}^2 \right)^{\frac{1}{2}} \\
\le & C h^\alpha \|\Pi_h u - u_h\|_h.
\end{aligned}
\tag{4.3.18}
$$

The sum of the first two terms on the right-hand side of (4.3.17) can be rewritten as

$$
\begin{aligned}
& \sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2(\Pi_h u - u_h)\,dx + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2(\Pi_h u)}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] ds \\
& \quad = \sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2 E_h(\Pi_h u - u_h)\,dx \\
& \quad\quad + \sum_{T\in\mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2[(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)]\,dx \\
& \quad\quad\quad + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2(\Pi_h u)}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] ds,
\end{aligned}
\tag{4.3.19}
$$

and it follows from Lemma 4.15 that

$$\left| \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2[(\Pi_h u - u_h) - E_h(\Pi_h u - u_h)]\, dx \right.$$
$$\left. + \sum_{e \in \mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2(\Pi_h u)}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\Pi_h u - u_h)}{\partial n} \right]\!\!\right] ds \right| \leq Ch^\alpha \|\Pi_h u - u_h\|_h. \quad (4.3.20)$$

Combining (4.3.17)-(4.3.20), we have

$$\begin{aligned} &a_h(\Pi_h u, \Pi_h u - u_h) - F(\Pi_h u - u_h)\, dx \\ \leq& Ch^\alpha \|\Pi_h u - u_h\|_h + \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2 E_h(\Pi_h u - u_h)\, dx \qquad (4.3.21) \\ &\quad - (f, \Pi_h u - u_h). \end{aligned}$$

The second term on the right-hand side of (4.3.21) can be rewritten as

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u) : D^2 E_h(\Pi_h u - u_h)\, dx \\ =& \sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 E_h(\Pi_h u - u_h)\, dx + \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u - u) : D^2 E_h(\Pi_h u - u_h)\, dx \\ =& \, a(u, E_h(\Pi_h u - u_h)) + \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u - u) : D^2 E_h(\Pi_h u - u_h)\, dx \qquad (4.3.22) \end{aligned}$$

and observe that

$$\left| \sum_{T \in \mathcal{T}_h} \int_T D^2(\Pi_h u - u) : D^2 E_h(\Pi_h u - u_h)\, dx \right| \leq Ch^\alpha \|\Pi_h u - u_h\|_h \qquad (4.3.23)$$

by (4.3.9) and (2.5.11).

Moreover, we have, by (2.5.11),

$$(f, (\Pi_h u - u_h) - E_h(\Pi_h u - u_h)) \leq Ch^2 \|\Pi_h u - u_h\|_h. \qquad (4.3.24)$$

Combining (4.3.21)-(4.3.24), we obtain the estimate (4.3.16). □

The following lemma is the same as Lemma 4.6 whose proof involves the properties of T_h and Lemma 4.13. We omit the proof.

Lemma 4.17. *There exists a positive constant C independent of h such that*

$$a(u, E_h(\Pi_h u - u_h)) - (f, E_h(\Pi_h u - u_h)) \leq C(h^{2\alpha} + h\|\Pi_h u - u_h\|_h). \quad (4.3.25)$$

Similar to Theorem 4.7 and Theorem 4.8, we obtain the error estimates in the energy norm and the L^∞ norm. The convergence rates for the quadratic C^0 interior penalty method are the same as those obtained for the Morley finite element method.

Theorem 4.18. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_h \leq Ch^\alpha. \quad (4.3.26)$$

Theorem 4.19. *There exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^\alpha. \quad (4.3.27)$$

4.4 Numerical Results

In this section we evaluate the performance of the Morley finite element method (4.2.2) and the quadratic C^0 interior penalty method (4.3.4). For simplicity we only consider one-obstacle problems. The algorithm we use to solve the discrete obstacle problems is an active set algorithm developed in [71]. We denote by $\psi(x)$ the lower obstacle function and solve the discrete obstacle problems on uniform triangulations, where the length h_j of the horizontal/vertical edge in $\mathcal{T}_j$ is 2^{-j} for Example 1–3 and $2^{-(j+1)}$ for Example 4. In addition, we take the penalty parameter σ to be 5 for the quadratic C^0 interior penalty method (cf. [26]).

Example 1

We construct an example with a known exact solution to validate the numerical results. We begin with the plate obstacle problem on the disc $\{x : |x| < 2\}$ with $f = 0$, $\psi(x) = 1 - |x|^2$, and homogeneous Dirichlet boundary conditions. This problem is rotationally invariant and can be solved exactly. The exact solution is given by

$$u(x) = \begin{cases} C_1|x|^2 \ln|x| + C_2|x|^2 + C_3 \ln|x| + C_4 & r_0 < |x| < 2, \\ 1 - |x|^2 & |x| \leqslant r_0, \end{cases} \tag{4.4.1}$$

where $r_0 \approx 0.18134452$, $C_1 \approx 0.52504063$, $C_2 \approx -0.62860904$, $C_3 \approx 0.01726640$, $C_4 \approx 1.04674630$.

We then consider the obstacle problem on $\Omega = (-0.5, 0.5)^2$ whose exact solution is the restriction of u to Ω. For this problem $f = 0$, $\psi(x) = 1 - |x|^2$, and the (nonhomogeneous) Dirichlet boundary data are determined by u.

Let u_j be the numerical solution of the jth level discrete obstacle problem and let $e_j = \Pi_j u - u_j$, where Π_j is the interpolation operator for the jth level Morley (or $\mathbb{P}_2$ Lagrange) finite element space. The energy norm on the jth level is denoted by $\|\cdot\|_{h_j}$. We evaluate the error $\|e_j\|_{h_j}$ in the energy norm and the error $\|e_j\|_\infty = \max_{p\in\mathcal{V}_j} |e_j(p)|$ for the Morley finite element method and $\|e_j\|_\infty = \max_{p\in\mathcal{N}_j} |e_j(p)|$ for the quadratic C^0 interior penalty method, where $\mathcal{N}_j$ is the set of the nodal points (vertices and midpoints) in the jth level triangulation. We also compute the rates of convergence in these norms by

$$\beta_h = \ln(\|e_{j-1}\|_{h_{j-1}}/\|e_j\|_{h_j})/\ln(2) \quad \text{and} \quad \beta_\infty = \ln(\|e_{j-1}\|_\infty/\|e_j\|_\infty)/\ln(2).$$

The numerical results are presented in Table 4.1 for the Morley finite element method and Table 4.2 for the quadratic C^0 interior penalty method. The error estimate in the energy norm is of order $O(h)$, which agrees with the theoretical estimate in Theorem 4.7 for the Morley finite element method. For the quadratic C^0 interior penalty method, the magnitude of the energy norm error is $O(h^{1.5})$, which is better than the error estimate in Theorem 4.18. We believe this is likely due to the effects of superconvergence because the solution u is a piecewise C^∞

function and we use uniform grids in the computation. The l_∞ error estimate is $O(h^2)$ for both methods, which is better than the theoretical results in Theorem 4.8 and Theorem 4.19.

TABLE 4.1. Energy norm errors and l_∞ errors for Example 1 of the obstacle problem of clamped plates (Morley)

j	$\|e_j\|_{h_j}/\|u_8\|_{h_8}$	β_h	$\|e_j\|_\infty$	β_∞
1	1.57E−01		0.00E+00	
2	2.13E−01	−0.44	1.65E−03	−−
3	1.31E−01	0.70	6.46E−04	1.35
4	7.14E−02	0.87	1.97E−04	1.71
5	3.66E−02	0.96	5.38E−05	1.87
6	1.85E−02	0.98	1.54E−05	1.81
7	9.30E−03	0.99	3.37E−06	2.19
8	4.66E−03	1.00	8.69E−07	1.95

TABLE 4.2. Energy norm errors and l_∞ errors for Example 1 of the obstacle problem of clamped plates (C^0IP)

j	$\|e_j\|_{h_j}/\|u_8\|_{h_8}$	β_h	$\|e_j\|_\infty$	β_∞
1	3.44E−02		1.08E−02	
2	1.81E−02	0.92	3.52E−03	1.61
3	6.17E−03	1.55	6.27E−04	2.49
4	2.19E−03	1.50	1.48E−04	2.09
5	9.25E−04	1.24	7.52E−05	0.97
6	3.64E−04	1.34	2.63E−05	1.52
7	1.25E−04	1.54	6.76E−06	1.96
8	4.64E−05	1.43	1.70E−06	1.99

We also consider the approximation of the coincidence set and the free boundary. From (4.4.1), we see that the continuous coincidence set is

$$I = \{x \in \Omega : |x| \le r_0\},$$

and the continuous free boundary is

$$F = \{x \in \Omega : |x| = r_0\}.$$

Moreover the continuous solution u is C^3 outside the continuous coincidence set according to (4.4.1). A simple calculation using Taylor's theorem shows that the assumptions (3.4.10) and (3.4.12) are valid for $\mu = 1/3$.

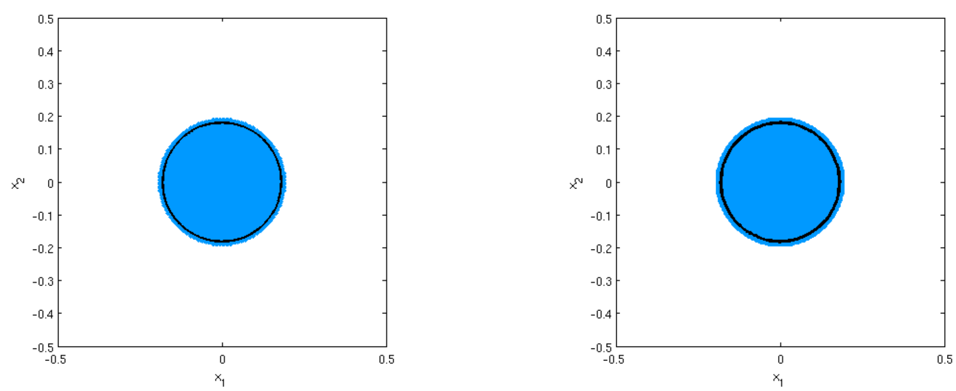
For simplicity, we take jth level discrete coincidence set to be

$$I_j = \{p \in \mathcal{V}_j : u_j(p) - \psi(p) \leq \|e_j\|_\infty\} \tag{4.4.2}$$

for the Morley finite element method, and

$$I_j = \{p \in \mathcal{N}_j : u_j(p) - \psi(p) \leq \|e_j\|_\infty\} \tag{4.4.3}$$

for the quadratic C^0 interior penalty method. The discrete coincidence sets in level 8 for two methods are displayed in Figure 4.1, where the circle represents the exact free boundary F.



(a) I_8 (Morley) (b) I_8 (C^0IP)

FIGURE 4.1. Discrete coincidence sets and exact free boundary for Example 1 of the obstacle problem of clamped plates

We compute the convergence rates for the coincidence set and the free boundary by

$$\beta_c = \ln(m_{j-1}/m_j)/\ln(2) \text{ and } \beta_b = \ln(d_{j-1}/d_j)/\ln(2),$$

where m_j is the Lebesgue measure of $I \Delta I_j$ and $d_j = \text{dist}(F_j, F)$, and tabulate the results in Table 4.3 and Table 4.4. According to (3.4.11) and (3.4.14), the magnitude of the errors should be $O(\|e_j\|_\infty^\mu) = O(h^{2/3})$, which is in agreement with the numerical results.

TABLE 4.3. Approximations of the coincidence set and the free boundary for Example 1 of the obstacle problem of clamped plates (Morley)

level	m_j	β_c	d_j	β_b
4	1.79E−02		4.40E−02	
5	1.11E−02	0.69	2.83E−02	0.64
6	8.37E−03	0.40	2.18E−02	0.38
7	4.77E−03	0.81	1.27E−02	0.78
8	3.02E−03	0.66	8.14E−03	0.64

TABLE 4.4. Approximations of the coincidence set and the free boundary for Example 1 of the obstacle problem of clamped plates (C^0IP)

level	m_j	β_c	d_j	β_b
4	1.79E−02		4.40E−02	
5	1.42E−02	0.33	3.57E−02	0.30
6	9.35E−03	0.61	2.42E−02	0.56
7	5.81E−03	0.69	1.54E−02	0.65
8	3.57E−03	0.70	9.58E−03	0.68

In the following examples, the exact solutions are not known, we take $\tilde{e}_j = u_{j-1} - u_j$ and compute the rates of convergence $\tilde{\beta}_h$ and $\tilde{\beta}_\infty$ by

$$\tilde{\beta}_h = \ln(\|\tilde{e}_{j-1}\|_{h_{j-1}}/\|\tilde{e}_j\|_{h_j})/\ln(2) \text{ and } \tilde{\beta}_\infty = \ln(\|\tilde{e}_{j-1}\|_\infty/\|\tilde{e}_j\|_\infty)/\ln(2),$$

where $\|\tilde{e}_j\|_\infty = \max_{p\in\mathcal{V}_{j-1}} |\tilde{e}_j(p)|$ for the Morley finite element method and $\|\tilde{e}_j\|_\infty = \max_{p\in\mathcal{N}_j} |\tilde{e}_j(p)|$ for the quadratic C^0 interior penalty method.

Example 2

In this example we take $\Omega = (-0.5, 0.5)^2$, $f = g = 0$ and $\psi(x) = 1 - 5|x|^2 + |x|^4$. The numerical results are presented in Table 4.5 and Table 4.6. It is observed that the magnitude of energy norm error is $O(h)$, as predicted by Theorem 4.7 and Theorem 4.18. The results for the l_∞ norm errors suggest that the correct estimate for $\|u - u_h\|_{L^\infty(\Omega)}$ is $O(h^2)$ for this example.

TABLE 4.5. Energy norm errors and l_∞ errors for Example 2 of the obstacle problem of clamped plates (Morley)

j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.98E−01		−−	
2	4.95E−01	−0.31	0.00E+00	−−
3	5.24E−01	−0.08	1.41E−02	−−
4	4.40E−01	0.25	9.08E−03	0.64
5	2.97E−01	0.57	5.72E−03	0.67
6	1.70E−01	0.81	1.95E−03	1.56
7	8.89E−02	0.93	5.21E−04	1.90
8	4.51E−02	0.98	1.46E−04	1.83

TABLE 4.6. Energy norm errors and l_∞ errors for Example 2 of the obstacle problem of clamped plates (C^0IP)

j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.24E−01		1.00E+00	
2	4.54E−01	−0.49	3.44E−01	1.54
3	4.99E−01	−0.14	5.97E−02	2.53
4	3.83E−01	0.38	2.61E−02	1.19
5	1.96E−01	0.97	3.66E−03	2.84
6	9.27E−02	1.08	1.29E−03	1.50
7	4.47E−02	1.05	4.17E−04	1.63
8	2.19E−02	1.03	1.02E−04	2.03

The discrete coincidence sets for levels 7-8 are displayed in Figure 4.2, where we replace e_j with $\tilde{e}_j$ in (4.4.2) and (4.4.3). It is observed that I_j converges to a domain with a smooth boundary and the correct symmetry for both methods. Since $\Delta^2\psi > 0$ in this example, the noncoincidence set $\Omega \setminus I$ is connected (cf. [42]). This is confirmed by Figure 4.2.

Example 3

In this example we take $\Omega = (-0.5, 0.5)^2$, $f = g = 0$ and $\psi(x) = 1 - 5|x|^2 - |x|^4$. The numerical results for $\|\tilde{e}_j\|_{h_j}$, $\|\tilde{e}_j\|_\infty$, $\tilde{\beta}_h$, and $\tilde{\beta}_\infty$ are presented in Table 4.7 and Table 4.8. It is observed that the magnitude of the energy norm error is $O(h)$. The

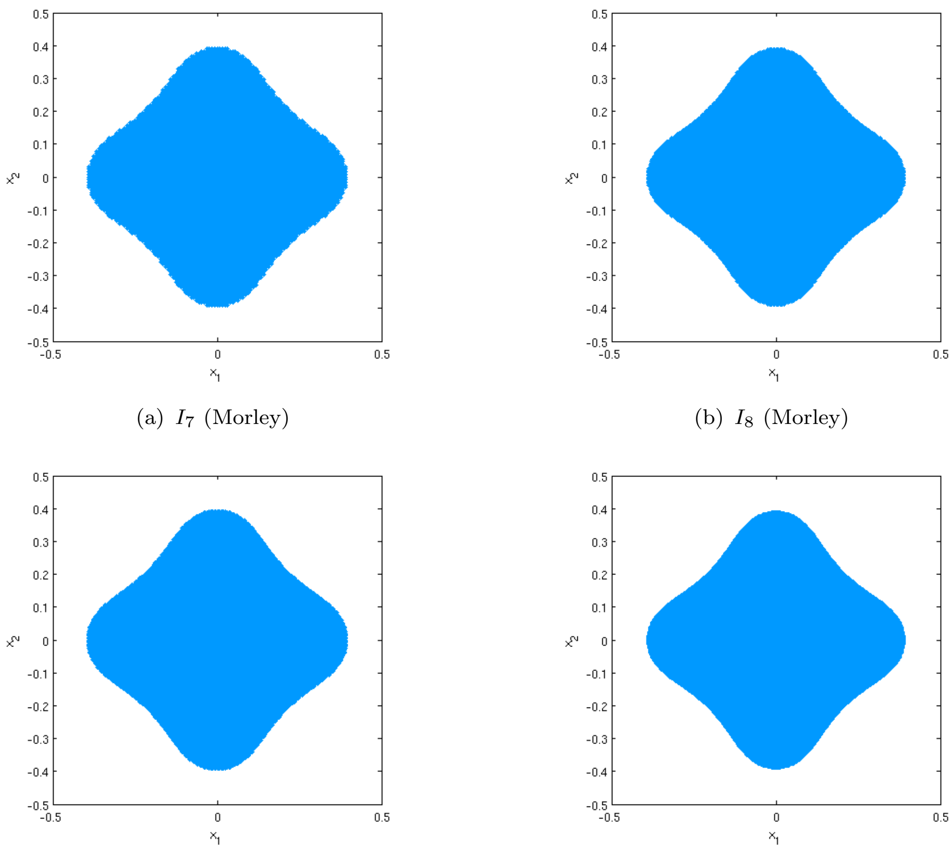


FIGURE 4.2. Discrete coincidence sets for Example 2 of the obstacle problem of clamped plates

results for the l_∞ norm error suggest that the correct estimate for $\|u - u_h\|_{L^\infty(\Omega)}$ is $O(h^2)$ for this example.

TABLE 4.7. Energy norm errors and l_∞ errors for Example 3 of the obstacle problem of clamped plates (Morley)

j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	4.19E−01		−−	
2	5.11E−01	−0.29	0.00E+00	−−
3	5.29E−01	−0.05	3.16E−02	−−
4	4.26E−01	0.31	1.02E−02	1.63
5	2.68E−01	0.67	5.19E−03	0.98
6	1.47E−01	0.86	1.91E−03	1.44
7	7.60E−02	0.96	5.20E−04	1.88
8	3.83E−02	0.99	1.25E−04	2.06

TABLE 4.8. Energy norm errors and l_∞ errors for Example 3 of the obstacle problem of clamped plates (C^0IP)

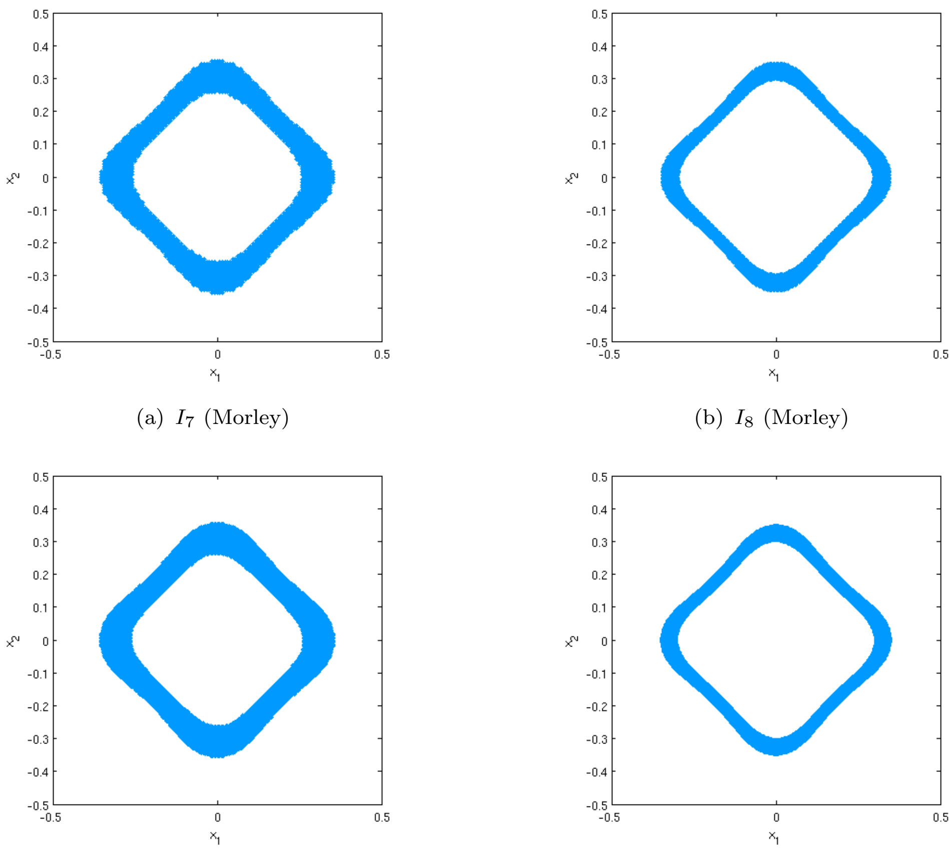
j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.41E−01		1.00E+00	
2	4.76E−01	−0.48	3.33E−01	1.59
3	5.11E−01	−0.10	7.26E−02	2.20
4	3.39E−01	0.59	2.53E−02	1.52
5	1.69E−01	1.00	7.65E−03	1.73
6	7.91E−02	1.10	1.62E−03	2.24
7	3.86E−02	1.03	5.82E−04	1.48
8	1.89E−02	1.03	1.10E−04	2.40

Note that this example is similar to Example 2 except in the sign of the term $|x|^4$ that appears in the obstacle function. Here we have $\Delta^2\psi < 0$ in Ω, and hence the interior of the coincidence set must be empty since $\Delta^2 u$ (in the sense of distributions) is a nonnegative measure (cf. [42]). This is confirmed by the pictures of the discrete coincidence sets I_7 and I_8 displayed in Figure 4.3. Moreover, the free boundary appears to be smooth.

Example 4

In this example we take Ω to be the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$, $f = g = 0$, and $\psi(x) = 1 - [((x_1 + 0.25)/0.2)^2 + (x_2/0.35)^2]$.

The energy norm errors and l_∞ norm errors are presented in Table 4.9 and Table 4.10. The magnitude of the observed energy norm error is consistent with Theorem 4.7 and Theorem 4.18, since the index of elliptic regularity α is less than 1 for the L-shaped domain. In fact we have $\alpha \approx 0.5445$, and the energy norm error at level 7 has not reached the asymptotic region. The results for the l_∞ norm errors suggest the convergence rate for the L^∞ norm is $O(h^{2\alpha})$ for the Morley finite element method and $O(h^{1+\alpha})$ for the quadratic C^0 interior penalty method.



(a) I_7 (Morley) (b) I_8 (Morley)

(c) I_7 (C^0IP) (d) I_8 (C^0IP)

FIGURE 4.3. Discrete coincidence sets for Example 3 of the obstacle problem of clamped plates

TABLE 4.9. Energy norm errors and l_∞ errors for Example 4 of the obstacle problem of clamped plates (Morley)

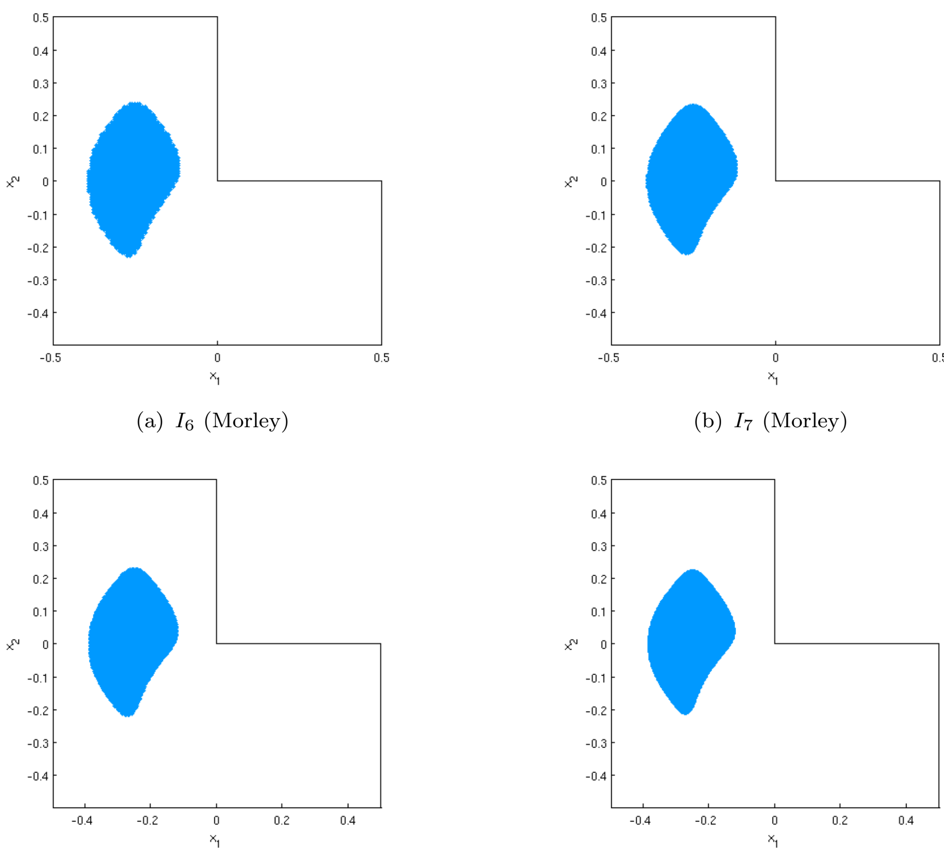
j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	4.92E−01		−−	
2	5.34E−01	−0.12	3.64E−02	−−
3	5.26E−01	0.02	1.75E−02	1.05
4	3.61E−01	0.54	1.57E−02	0.16
5	2.12E−01	0.77	9.51E−03	0.72
6	1.17E−01	0.86	4.70E−03	1.02
7	6.48E−02	0.85	2.25E−03	1.07

The discrete coincidence sets for levels 6-7 are depicted in Figure 4.4. It is observed that I_j converges to a domain with a smooth boundary. The noncoincidence

TABLE 4.10. Energy norm errors and l_∞ errors for Example 4 of the obstacle problem of clamped plates (C^0IP)

j	$\|\tilde{e}_j\|_{h_j}/\|u_8\|_{h_8}$	$\tilde{\beta}_h$	$\|\tilde{e}_j\|_\infty$	$\tilde{\beta}_\infty$
1	3.88E−01		1.00E+00	
2	5.71E−01	−0.56	2.11E−01	2.24
3	4.47E−01	0.35	4.52E−02	2.22
4	2.32E−01	0.94	1.40E−02	1.69
5	1.17E−01	0.99	5.43E−03	1.37
6	6.23E−02	0.91	1.72E−03	1.66
7	3.52E−02	0.82	5.89E−04	1.54

set is connected, which agrees with the result in [42] since $\Delta^2\psi = 0$ in Ω in this example.



(a) I_6 (Morley) (b) I_7 (Morley)

(c) I_6 (C^0IP) (d) I_7 (C^0IP)

FIGURE 4.4. Discrete coincidence sets for Example 4 of the obstacle problem of clamped plates

Chapter 5

A Quadratic C^0 Interior Penalty Method for an Elliptic Optimal Control Problem with State Constraints

In this chapter we consider an elliptic optimal control problem with pointwise state constraints which can be formulated as a fourth order variational inequality for the state. We solve it by a quadratic C^0 interior penalty method and derive the error estimate for the state in an H^2-like energy norm on quasi-uniform meshes and graded meshes. Numerical results are given in Section 5.5 which verify the theoretical estimates. The presentation in this chapter follows [33].

5.1 The Continuous Problem and Regularity Results

5.1.1 An Equivalent Fourth Order Variational Inequality Formulation

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$, $y_d \in L^2(\Omega)$, $\gamma \geq 0, \beta > 0$ be constants. The following problem is a model elliptic distributed optimal control problem with pointwise state constraints:

$$\text{minimize} \quad J(y,u) = \frac{\gamma}{2}\int_\Omega (y - y_d)^2\,dx + \frac{\beta}{2}\int_\Omega u^2\,dx \tag{5.1.1}$$

$$\text{over} \quad (y,u) \in H_0^1(\Omega) \times L^2(\Omega)$$

$$\text{subject to} \quad \begin{cases} -\Delta y = u & \text{in } \Omega \\ \psi_1 \leq y \leq \psi_2 & \text{a.e. in } \Omega. \end{cases} \tag{5.1.2}$$

Here the functions $\psi_1(x), \psi_2(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy

$$\psi_1 < \psi_2 \text{ in } \Omega \qquad \text{and} \qquad \psi_1 < 0 < \psi_2 \text{ on } \partial\Omega. \tag{5.1.3}$$

Note that (cf. Theorem 2.2.1 in [70])

$$\int_\Omega (\Delta v)(\Delta w)\,dx = \int_\Omega D^2 v : D^2 w\,dx \qquad \forall v, w \in H^2(\Omega) \cap H_0^1(\Omega). \tag{5.1.4}$$

Combining this with (1.1.11), we can solve the optimal control problem (5.1.1) by looking for the minimizer of the reduced functional

$$\hat{J}(y) = \frac{\gamma}{2}\int_\Omega (y - y_d)^2\,dx + \frac{\beta}{2}\int_\Omega (D^2y : D^2y)\,dx, \tag{5.1.5}$$

in the set

$$K = \{y \in H^2(\Omega) \cap H_0^1(\Omega) : \psi_1 \le y \le \psi_2 \text{ in } \Omega\}. \tag{5.1.6}$$

A simple calculation shows that this is equivalent to the following problem: Find $\bar{y} \in K$ such that

$$\bar{y} = \operatorname*{argmin}_{y \in K} G(y), \tag{5.1.7}$$

where

$$G(y) = \frac{1}{2}\mathcal{A}(y, y) - (f, y), \tag{5.1.8}$$

$$\mathcal{A}(v, w) = \int_\Omega \left[\beta(D^2v : D^2w) + \gamma vw\right]\,dx, \tag{5.1.9}$$

$f = \gamma y_d$, and $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

Since (5.1.3) implies that K is a nonempty closed convex subset of $H^2(\Omega)\cap H_0^1(\Omega)$ and the bilinear form $\mathcal{A}(\cdot,\cdot)$ is symmetric, bounded and coercive on $H^2(\Omega)\cap H_0^1(\Omega)$, we can apply the standard theory in Chapter 2 to conclude that the problem (5.1.7) has a unique solution $\bar{y} \in K$ characterized by the variational inequality

$$\mathcal{A}(\bar{y}, y - \bar{y}) \ge (f, y - \bar{y}) \qquad \forall y \in K. \tag{5.1.10}$$

The solution of the optimal control problem is then given by $(\bar{y}, \bar{u})$, where $\bar{u} = -\Delta\bar{y}$. Note that (5.1.7) becomes the displacement obstacle problem for simply supported Kirchhoff plates if we take γ to be 0. For this reason we will also refer to (5.1.7) as an obstacle problem. Therefore, the results in this chapter cover both obstacle problems for simply supported plates and optimal control problems with pointwise state constraints.

5.1.2 Regularity Results

Note that the bilinear form $\mathcal{A}(\cdot,\cdot)$ includes a low order term, but the results in Theorem 2.30 are still valid. Hence the solution $\bar{y}$ of (5.1.7) belongs to $H^3_{loc}(\Omega)\cap C^2(\Omega)$ under our assumptions on the functions y_d, ψ_1 and ψ_2. From Corollary 2.32, we have $\bar{y}\in H^{2+\alpha}(\Omega)$ for some $\alpha\in(0,1]$ determined by the interior angles of Ω. Moreover, the boundary conditions of simply supported plates allow us to use the elliptic regularity theory for the Laplace operator [51,69,89]. Let $p_1,\dots,p_L$ be the corners of Ω and ω_l be the interior angle at p_l for $1\le l\le L$. We have the following regularity result for $\bar{y}$ (cf. [14,33,108]):

(i) The function $\Delta\bar{y}$ belongs to $H^1_0(\Omega)$. Therefore $\bar{u}=-\Delta\bar{y}$ belongs to $H^1_0(\Omega)$ for the problem (5.1.1).

(ii) Let $\alpha_l>0$ be determined by

$$\begin{cases}\alpha_l=1 & \text{if } \omega_l\le\frac{\pi}{2},\\[2ex] \alpha_l<\left(\frac{\pi}{\omega_l}\right)-1 & \text{if } \frac{\pi}{2}<\omega_l<\pi.\end{cases}\tag{5.1.11}$$

Then $\bar{y}\in H^{2+\alpha_l}(\mathcal{N}_l)$, where $\mathcal{N}_l(\subseteq\Omega)$ is a neighborhood of p_l. In fact, we have $\alpha=\min_{1\le l\le L}\alpha_l$ so that $\bar{y}\in H^{2+\alpha}(\Omega)$.

(iii) The solution $\bar{y}$ can be represented as the sum of a regular part and a singular part $\bar{y}=\bar{y}_R+\bar{y}_S$, where $\bar{y}_R\in H^3(\Omega)\cap H^1_0(\Omega)$, $\Delta\bar{y}_R\in H^1_0(\Omega)$ and $\bar{y}_S$ have the following properties.

- The singular part $\bar{y}_S$ is an H^3 function away from the corners of Ω where the angles are greater than $\pi/2$.
- We define

$$\varphi_l=r_l^{\pi/\omega_l}\sin((\pi/\omega_l)\theta_l).\tag{5.1.12}$$

Here (r_l, θ_l) are the polar coordinates at p_l such that the two edges of Ω emanating from p_l are given by $\theta_l = 0$ and $\theta_l = \omega_l$. Then $\bar{y}_S$ is a multiple of φ_l in a neighborhood $\mathcal{N}_l$ of a corner p_l where $\omega_l > \pi/2$.

– The function $\Delta \bar{y}_S$ belongs to $H_0^1(\Omega)$.

5.2 A Quadratic C^0 Interior Penalty Method

5.2.1 Triangulation

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω with mesh size h that is regular. We consider both quasi-uniform and graded triangulations. For a quasi-uniform triangulation $\mathcal{T}_h$, we have

$$h_T \approx h \qquad \forall T \in \mathcal{T}_h. \tag{5.2.1}$$

For a graded triangulation $\mathcal{T}_h$, we have

$$h_T \approx h\Phi(c_T) \qquad \forall T \in \mathcal{T}_h, \tag{5.2.2}$$

where c_T is the center of T and

$$\Phi(x) = \prod_{l=1}^{L} |p_l - x|^{1-\alpha_l}, \tag{5.2.3}$$

where the grading parameters $\alpha > 0$ are determined in (5.1.11).

Note that (5.2.2) and (5.2.3) imply

$$h_T^{\alpha_l} \approx h \tag{5.2.4}$$

if $T \in \mathcal{T}_h$ touches the corner p_l.

Remark 5.1. *The construction of regular triangulations that satisfy* (5.2.2) *can be found in* [4, 17, 24].

For φ_l defined in (5.1.12), we have $r_l^{1-\alpha_l}(\partial^\mu \varphi_l) \in L^2(\mathcal{N}_l)$ for $|\mu| = 3$. Then we obtain

$$\Phi(\partial^\mu \bar{y}_S) \in L^2(\Omega) \text{ for } |\mu| = 3 \text{ and hence } \Phi(\partial^\mu \bar{y}) \in L^2(\Omega) \text{ for} |\mu| = 3. \tag{5.2.5}$$

5.2.2 The Discrete Problem

Let $V_h \subseteq H_0^1(\Omega)$ be the $\mathbb{P}_2$ Lagrange finite element space associated with $\mathcal{T}_h$ whose members vanish on $\partial\Omega$. To define the quadratic C^0 interior penalty method for (5.1.7), we need the jumps and averages of normal derivatives for functions defined in the piecewise Sobolev spaces. For $e \in \mathcal{E}_h^i$, we use the same definitions as in (2.4.19)–(2.4.22). We also need the following definitions in the convergence analysis. For a boundary edge $e \in \mathcal{E}_h^b$ with $\mathcal{T}_e = \{T\}$, we define on e

$$\left\{\!\!\left\{ \frac{\partial v}{\partial n_e} \right\}\!\!\right\} = \frac{\partial v_T}{\partial n_e}\bigg|_e \qquad \forall v \in H^2(\Omega, \mathcal{T}_h), \tag{5.2.6}$$

$$\left[\!\!\left[\frac{\partial^2 v}{\partial n_e^2} \right]\!\!\right] = -\frac{\partial^2 v_T}{\partial n_e^2}\bigg|_e \qquad \forall v \in H^s(\Omega, \mathcal{T}_h), s > \frac{5}{2}, \tag{5.2.7}$$

where n_e is the unit normal of e pointing towards the outside of Ω and $v_T = v|_T$.

We define the bilinear form $a_h(\cdot,\cdot)$ on $V_h \times V_h$ by

$$\begin{aligned} a_h(v,w) = &\sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2 w\, dx + \sum_{e\in\mathcal{E}_h^i} \int_e \{\!\!\{\partial^2 v/\partial n^2\}\!\!\} [\![\partial w/\partial n]\!]\, ds \\ &+ \sum_{e\in\mathcal{E}_h^i} \int_e \{\!\!\{\partial^2 w/\partial n^2\}\!\!\} [\![\partial v/\partial n]\!]\, ds + \sigma \sum_{e\in\mathcal{E}_h^i} |e|^{-1} \int_e [\![\partial v/\partial n]\!][\![\partial w/\partial n]\!]\, ds, \end{aligned} \tag{5.2.8}$$

where $\sigma > 0$ is a penalty parameter. The bilinear form only involves the integration of averages and jumps on interior edges (cf. (2.4.27)).

Similar to (2.4.32), we have, by the trace theorem with scaling and standard inverse estimates,

$$\sum_{e\in\mathcal{E}_h^i} |e|\, \big\|\{\!\!\{\partial^2 v/\partial n^2\}\!\!\}\big\|_{L^2(e)}^2 \le C \sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 \qquad \forall v \in V_h. \tag{5.2.9}$$

Therefore, for sufficiently large σ, we have (cf. (2.4.33))

$$a_h(v,v) \ge C\Big(\sum_{T\in\mathcal{T}_h} |v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h^i} |e|^{-1} \|[\![\partial v/\partial n]\!]\|_{L^2(e)}^2 \Big) \qquad \forall v \in V_h. \tag{5.2.10}$$

The discrete bilinear form that approximates $\mathcal{A}(\cdot,\cdot)$ is then given by

$$\mathcal{A}_h(v,w) = \beta a_h(v,w) + \gamma(v,w), \tag{5.2.11}$$

and

$$\|v\|_h = \Big[\beta\Big(\sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}\Big) + \gamma\|v\|^2_{L^2(\Omega)}\Big]^{\frac{1}{2}} \qquad (5.2.12)$$

is the mesh-dependent energy norm. It follows from (5.2.9)–(5.2.12) that

$$|\mathcal{A}_h(v,w)| \le C\|v\|_h\|w\|_h \qquad \forall v,w\in V_h, \qquad (5.2.13)$$

$$\mathcal{A}_h(v,v) \ge C\|v\|_h^2 \qquad \forall v\in V_h, \qquad (5.2.14)$$

provided σ is large enough, which we assume to be the case from now on.

Note also that

$$\|v\|^2_{H^1(\Omega)} \le C\Big(\sum_{T\in\mathcal{T}_h} |v|^2_{H^2(T)} + \sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial v/\partial n]\!]\|^2_{L^2(e)}\Big) \qquad (5.2.15)$$

for any $v \in H^2(\Omega,\mathcal{T}_h)\cap H^1_0(\Omega)$ by a Poincaré-Friedrichs inequality [35], and hence

$$\|v\|_{H^1(\Omega)} \le C\|v\|_h \qquad \forall v\in H^2(\Omega,\mathcal{T}_h)\cap H^1_0(\Omega)\ (\supseteq V_h + H^2(\Omega)\cap H^1_0(\Omega)). \quad (5.2.16)$$

We now define the discrete problem for (5.1.7):

Find $\bar{y}_h \in K_h$ such that

$$\bar{y}_h = \operatorname*{argmin}_{y_h\in K_h} G_h(y_h), \qquad (5.2.17)$$

where

$$G_h(y_h) = \frac{1}{2}\mathcal{A}_h(y_h,y_h) - (f,y_h), \qquad (5.2.18)$$

$$K_h = \{v\in V_h : \psi_1(p)\le v(p)\le \psi_2(p) \quad \forall p\in\mathcal{V}_h\}. \qquad (5.2.19)$$

Let Π_h be the nodal interpolation operator for the $\mathbb{P}_2$ Lagrange finite element space. Then Π_h maps $H^2(\Omega)\cap H^1_0(\Omega)$ into V_h and K into K_h. Hence K_h is a nonempty closed convex subset of V_h. By (5.2.14), the bilinear form $\mathcal{A}(\cdot,\cdot)$ is symmetric positive definite. Therefore, the discrete problem (5.2.17) has a unique solution $\bar{y}_h\in K_h$ characterized by the discrete variational inequality

$$\mathcal{A}_h(\bar{y}_h, y_h-\bar{y}_h) \ge (f, y_h-\bar{y}_h) \qquad \forall y_h\in K_h. \qquad (5.2.20)$$

We have the following local interpolation error estimate (cf. [28, 48])

$$\sum_{m=0}^{2} h_T^m |\zeta - \Pi_h \zeta|_{H^m(T)} \le C h_T^{2+s} |\zeta|_{H^{2+s}(T)} \qquad \forall \zeta \in H^{2+s}(\Omega),\ T \in \mathcal{T}_h \tag{5.2.21}$$

for any $s \in [0,1]$.

Next we give global interpolation error estimates for the solution $\bar{y}$ of (5.1.7).

Lemma 5.2. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \Pi_h \bar{y}\|_h \le C h^\tau, \tag{5.2.22}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Proof. The estimate for a quasi-uniform $\mathcal{T}_h$ is a standard result (cf. [29]). We will only focus on a graded $\mathcal{T}_h$. Let $\mathcal{T}_h^I$ be the set of triangles in $\mathcal{T}_h$ that do not touch any corner of Ω and $\mathcal{T}_h^C = \mathcal{T}_h \setminus \mathcal{T}_h^I = \bigcup_{1 \le l \le L} \mathcal{T}_{h,l}^C$, where $\mathcal{T}_{h,l}^C$ is the set of triangles that touch the corner p_l.

Since $\bar{y}_T \in H^3(T)$ for $T \in \mathcal{T}_h^I$ (cf. (*iii*) in Section 5.1.2), we obtain from (5.2.21), (5.2.2) and (5.2.5) that

$$\begin{aligned}\sum_{T \in \mathcal{T}_h^I} \sum_{m=0}^{2} h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 &\le C \sum_{T \in \mathcal{T}_h^I} (\Phi^{-2}(c_T) h_T^2) \Phi^2(c_T) |\bar{y}|_{H^3(T)}^2 \\ &\le C h^2. \end{aligned} \tag{5.2.23}$$

Let $T \in \mathcal{T}_{h,l}^C$ be a triangle that touches a corner p_l. Then $\bar{y} \in H^{2+\alpha_l}(T)$ (cf. (*ii*) in Section 5.1.2). We have, by (5.2.21),

$$\sum_{m=0}^{2} h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 \le C h_T^{2\alpha_l} |\bar{y}|_{H^{2+\alpha_l}(T)}^2 \qquad \forall T \in \mathcal{T}_{h,l}^C, \tag{5.2.24}$$

which together with (5.2.4) imply that

$$\sum_{T \in \mathcal{T}_h^C} \sum_{m=0}^{2} h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2 = \sum_{l=1}^{L} \sum_{T \in \mathcal{T}_{h,l}^C} \sum_{m=0}^{2} h_T^{2(m-2)} |\bar{y} - \Pi_h \bar{y}|_{H^m(T)}^2$$

$$\leq Ch^2. \tag{5.2.25}$$

Combine (5.2.23) and (5.2.25), we obtain

$$\sum_{T\in\mathcal{T}_h}\sum_{m=0}^{2} h_T^{2(m-2)}|\bar{y}-\Pi_h\bar{y}|^2_{H^m(T)} \leq Ch^2. \tag{5.2.26}$$

By the trace theorem with scaling and (5.2.26), we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|[\![\partial(\bar{y}-\Pi_h\bar{y})/\partial n]\!]\|^2_{L^2(e)} &\leq \sum_{T\in\mathcal{T}_h}\Big(h_T^{-2}|\bar{y}-\Pi_h\bar{y}|^2_{H^1(T)}+|\bar{y}-\Pi_h\bar{y}|^2_{H^2(T)}\Big)\\ &\leq Ch^2. \end{aligned}\tag{5.2.27}$$

The estimate for graded $\mathcal{T}_h$ now follows from (5.2.12), (5.2.15), (5.2.26) and (5.2.27). □

5.3 An Auxiliary Problem

In this section we introduce the following auxiliary problem:

Find $\bar{y}_h^* \in K_h^*$ such that

$$\bar{y}_h^* = \underset{y_h^*\in K_h^*}{\operatorname{argmin}}\, G(y_h^*), \tag{5.3.1}$$

where

$$K_h^* = \{v \in H^2(\Omega)\cap H_0^1(\Omega) : \psi_1(p) \leq v(p) \leq \psi_2(p) \quad \forall p \in \mathcal{V}_h\}, \tag{5.3.2}$$

and $G(\cdot)$ is defined in (5.1.8).

The problem (5.3.1) has a unique solution $\bar{y}_h^*$ characterized by the variational inequality

$$\mathcal{A}(\bar{y}_h^*, y_h^* - \bar{y}_h^*) \geq (f, y_h^* - \bar{y}_h^*) \qquad \forall y_h^* \in K_h^*. \tag{5.3.3}$$

Similar to the discussion in Section 4.1, we can establish the connection between (5.1.7) and (5.3.1). In fact, the results in Section 3.2 can be generalized to general boundary conditions (cf. Remark 3.16). Hence there exist two nonnegative functions $\phi_1, \phi_2 \in C_0^\infty(\Omega)$ and a positive number h_0 such that for any $h \le h_0$ we can find two positive numbers $\delta_{h,1}$ and $\delta_{h,2}$ with the following properties:

$$\hat{y}_h = \bar{y}_h^* + \delta_{h,1}\phi_1 - \delta_{h,2}\phi_2 \in K \quad \text{and} \quad \delta_{h,i} \le Ch^2. \tag{5.3.4}$$

Moreover, we have

$$|\bar{y}_h^* - \bar{y}|_{H^2(\Omega)} \le Ch. \tag{5.3.5}$$

Now we can connect K_h and K_h^* by the enriching operator E_h that maps V_h into $H^2(\Omega) \cap H_0^1(\Omega)$ (cf. Case 2 in Section 2.5.2). The properties of (2.5.15)–(2.5.16) together with (5.2.19) and (5.3.2) imply that

$$E_h K_h \subseteq K_h^*. \tag{5.3.6}$$

Finally the quasi-local estimate (2.5.18) implies the following result for the solution $\bar{y}$ of (5.1.7). We omit the proof due to its similarity with the proof of Lemma 5.2.

Lemma 5.3. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - E_h\Pi_h\bar{y}\|_{L^2(\Omega)} + h|\bar{y} - E_h\Pi_h\bar{y}|_{H^1(\Omega)} + h^2|\bar{y} - E_h\Pi_h\bar{y}|_{H^2(\Omega)} \le Ch^{2+\tau}, \tag{5.3.7}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

5.4 Error Estimates

The convergence analysis in this section follows the framework of Section 4.3.2. We begin with the following integration by parts formula that holds for $v, w \in V_h$:

$$\sum_{T\in\mathcal{T}_h} \int_T D^2 v : D^2(w - E_h w)\,dx = -\sum_{e\in\mathcal{E}_h} \int_e \left[\!\left[\frac{\partial^2 v}{\partial n_e^2} \right]\!\right] \left\{\!\!\left\{ \frac{\partial(w - E_h w)}{\partial n_e} \right\}\!\!\right\} ds$$

$$-\sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2 v}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial(w-E_h w)}{\partial n}\right]\!\!\right] ds. \quad (5.4.1)$$

Similar to (4.3.12), the formula (5.4.1) can be derived from (2.4.40), (5.2.6) and (5.2.7).

From Theorem 4.12, we have the following basic estimate for $\bar{y}-\bar{y}_h$:

$$\|\bar{y}-\bar{y}_h\|_h^2 \le 2\|\bar{y}-\Pi_h\bar{y}\|_h^2 + C[\mathcal{A}_h(\Pi_h\bar{y},\Pi_h\bar{y}-\bar{y}_h)-(f,\Pi_h\bar{y}-\bar{y}_h)]. \quad (5.4.2)$$

In view of Lemma 5.2 and (5.4.2), we can complete the error analysis by bounding the second term on the right-hand side of (5.4.2). For this purpose, we need several technical lemmas.

Lemma 5.4. *There exists a positive constant C independent of h such that*

$$\sum_{e\in\mathcal{E}_h}|e|\,\big\|[\![\partial^2(\Pi_h\bar{y})/\partial n_e^2]\!]\big\|_{L^2(e)}^2 \le Ch^{2\tau}, \quad (5.4.3)$$

where $\tau=\alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau=1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Proof. We will use the representation $\bar{y}=\bar{y}_R+\bar{y}_S$ (cf. Section 5.1.2). Since $\bar{y}_R\in H^3(\Omega)$ and $\partial\bar{y}_R/\partial n^2=\Delta\bar{y}_R=0$ on $\partial\Omega$, we have, by (5.2.21) and the trace theorem with scaling that

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}|e|\,\big\|[\![\partial^2(\Pi_h\bar{y}_R)/\partial n_e^2]\!]\big\|_{L^2(e)}^2 &= \sum_{e\in\mathcal{E}_h}|e|\,\big\|[\![\partial^2(\Pi_h\bar{y}_R-\bar{y}_R)/\partial n_e^2]\!]\big\|_{L^2(e)}^2\\ &\le Ch^2|\bar{y}_R|_{H^3(\Omega)}^2. \quad (5.4.4)\end{aligned}$$

Next we focus on the estimate for the singular part $\bar{y}_S$. Let

$$\begin{aligned}\mathcal{E}_h^R=\{e\in\mathcal{E}_h : e &\text{ is not an edge of any triangle that touches a corner of }\Omega\\ &\text{whose angle is greater than }\pi/2\}\end{aligned}$$

and $\mathcal{E}_h^S = \mathcal{E}_h \setminus \mathcal{E}_h^R$. Note that the number of edges in $\mathcal{E}_h^S$ is bounded by a constant determined by the minimum angle of $\mathcal{T}_h$.

From Section 5.1.2, we know $\bar{y}_S$ belongs to $H^3(T)$ whenever T is away from the corners of Ω where the angles are greater than $\pi/2$. Together with the fact that $\partial \bar{y}_S/\partial n^2 = \Delta \bar{y}_S = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_h^R} |e| \left\| [\![\partial^2(\Pi_h \bar{y}_S)/\partial n_e^2]\!] \right\|_{L^2(e)}^2 &= \sum_{e\in\mathcal{E}_h^R} |e| \left\| [\![\partial^2(\Pi_h \bar{y}_S - \bar{y}_S)/\partial n_e^2]\!] \right\|_{L^2(e)}^2 \\ &\le C \sum_{e\in\mathcal{E}_h^R} \sum_{T\in\mathcal{T}_e} (h_T^2 \Phi^{-2}(c_T))\Phi^2(c_T)|\bar{y}_S|_{H^3(T)}^2,\end{aligned}$$

where the function Φ is defined in (5.2.3). Hence by (5.2.1)–(5.2.3), and (5.2.5) we have

$$\sum_{e\in\mathcal{E}_h^R} |e| \left\| [\![\partial^2(\Pi_h \bar{y}_S)/\partial n_e^2]\!] \right\|_{L^2(e)}^2 \le Ch^{2\tau}, \tag{5.4.5}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ satisfies (5.2.2)–(5.2.3).

Let $e \in \mathcal{E}_h^S$ be an edge of a triangle that touches a corner p_l of Ω where the angle $w_l \in (\pi/2, \pi)$. It follows from the trace theorem with scaling that

$$\begin{aligned}|e| \left\| [\![\partial^2(\Pi_h \bar{y}_S)/\partial n_e^2]\!] \right\|_{L^2(e)}^2 &\le C \sum_{T\in\mathcal{T}_e} |\Pi_h \bar{y}_S|_{H^2(T)}^2 \\ &\le C \sum_{T\in\mathcal{T}_e} (|\Pi_h \bar{y}_S - \bar{y}_S|_{H^2(T)}^2 + |\bar{y}_S|_{H^2(T)}^2).\end{aligned}$$

Since $\bar{y}_S \in H^{2+\alpha}(T)$, we have

$$|\Pi_h \bar{y}_S - \bar{y}_S|_{H^2(T)} \le Ch_T^{\alpha_l},$$

and by a direct calculation using (5.1.12), we also have

$$|\bar{y}_S|_{H^2(T)} \le Ch_T^{(\pi/w_l)-1} \le Ch_T^{\alpha_l}.$$

Therefore, we have by (5.2.1) and (5.2.4),

$$\sum_{e\in\mathcal{E}_h^S} |e| \left\| [\![\partial^2(\Pi_h \bar{y}_S)/\partial n_e^2]\!] \right\|_{L^2(e)}^2 \le Ch^{2\tau}, \tag{5.4.6}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ satisfies (5.2.2)–(5.2.3).

Finally the estimate (5.4.3) follows from (5.4.4)–(5.4.6). $\square$

Lemma 5.5. *There exists a positive constant C independent of h such that*

$$\left| a_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - \int_\Omega D^2\bar{y} : D^2 E_h(\Pi_h\bar{y} - \bar{y}_h)\,dx \right| \le Ch^\tau \|\Pi_h\bar{y} - \bar{y}_h\|_h, \quad (5.4.7)$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Proof. Since both $[\![\partial E_h(\Pi_h\bar{y} - \bar{y}_h)/\partial n]\!]$ and $[\![\partial\bar{y}/\partial n]\!]$ are equal 0 on $e \in \mathcal{E}_h^i$, we have, by (5.2.8),

$$\begin{aligned}
a_h(\Pi_h\bar{y},\Pi_h\bar{y} - \bar{y}_h) &= \sum_{T\in\mathcal{T}_h}\int_T D^2\bar{y} : D^2E_h(\Pi_h\bar{y} - \bar{y}_h)\,dx \\
&+ \sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h\bar{y} - \bar{y}) : D^2E_h(\Pi_h\bar{y} - \bar{y}_h)\,dx \\
&+ \sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h\bar{y}) : D^2E_h[(\Pi_h\bar{y} - \bar{y}_h) - (\Pi_h\bar{y} - \bar{y}_h)]\,dx \qquad (5.4.8)\\
&+ \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2(\Pi_h\bar{y})}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial[(\Pi_h\bar{y} - \bar{y}_h) - E_h(\Pi_h\bar{y} - \bar{y}_h)]}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2(\Pi_h\bar{y} - \bar{y}_h)}{\partial n^2}\right\}\!\!\right\} \left[\!\!\left[\frac{\partial(\Pi_h\bar{y} - \bar{y})}{\partial n}\right]\!\!\right] ds \\
&+ \sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial(\Pi_h\bar{y} - \bar{y}_h)}{\partial n}\right]\!\!\right] \left[\!\!\left[\frac{\partial(\Pi_h\bar{y} - \bar{y})}{\partial n}\right]\!\!\right] ds.
\end{aligned}$$

For the second term on the right-hand side of (5.4.8), by Lemma 5.2 and (2.5.25) we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h\bar{y} - \bar{y}) : D^2E_h(\Pi_h\bar{y} - \bar{y}_h)\,dx \\
&\qquad \le \Big(\sum_{T\in\mathcal{T}_h} |\Pi_h\bar{y} - \bar{y}|^2_{H^2(T)}\Big)^{\frac{1}{2}} |E_h(\Pi_h\bar{y} - \bar{y})|_{H^2(\Omega)} \qquad (5.4.9)\\
&\qquad \le Ch^\tau\|\Pi_h\bar{y} - \bar{y}\|_h.
\end{aligned}$$

The fifth and sixth terms on the right-hand side of (5.4.8) can be estimated by Lemma 5.2 and (5.2.12):

$$
\begin{aligned}
&\Big|\sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2(\Pi_h\bar{y}-\bar{y}_h)}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial(\Pi_h\bar{y}-\bar{y})}{\partial n}\right]\!\!\right] ds\Big| \\
&\qquad \le \Big(\sum_{e\in\mathcal{E}_h^i}|e|\,\|\{\!\{\partial^2(\Pi_h\bar{y}-\bar{y}_h)/\partial n^2\}\!\}\|_{L^2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\|[\![\partial(\Pi_h\bar{y}-\bar{y})/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad \le C\Big(\sum_{T\in\mathcal{T}_h}|\Pi_h\bar{y}-\bar{y}_h|_{H^2(T)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\|[\![\partial(\Pi_h\bar{y}-\bar{y})/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}} \qquad (5.4.10) \\
&\qquad \le Ch^\tau\|\Pi_h\bar{y}-\bar{y}_h\|_h,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\sum_{e\in\mathcal{E}_h^i}\frac{\sigma}{|e|}\int_e \left[\!\!\left[\frac{\partial(\Pi_h\bar{y}-\bar{y}_h)}{\partial n}\right]\!\!\right]\left[\!\!\left[\frac{\partial(\Pi_h\bar{y}-\bar{y})}{\partial n}\right]\!\!\right] ds\Big| \\
&\qquad \le C\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\|[\![\partial(\Pi_h\bar{y}-\bar{y}_h)/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\|[\![\partial(\Pi_h\bar{y}-\bar{y})/\partial n]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad \le Ch^\tau\|\Pi_h\bar{y}-\bar{y}_h\|_h. \qquad (5.4.11)
\end{aligned}
$$

Now we use (2.5.26), (5.4.1) and Lemma 5.4 to estimate the sum of the third and fourth terms on the right-hand side of (5.4.8) by

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T D^2(\Pi_h\bar{y}) : D^2E_h[(\Pi_h\bar{y}-\bar{y}_h)-(\Pi_h\bar{y}-\bar{y}_h)]\,dx \\
&\qquad\qquad + \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2(\Pi_h\bar{y})}{\partial n^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial[(\Pi_h\bar{y}-\bar{y}_h)-E_h(\Pi_h\bar{y}-\bar{y}_h)]}{\partial n}\right]\!\!\right] ds \\
&\qquad = -\sum_{e\in\mathcal{E}_h}\int_e \left[\!\!\left[\frac{\partial^2(\Pi_h\bar{y})}{\partial n_e^2}\right]\!\!\right]\left\{\!\!\left\{\frac{\partial[(\Pi_h\bar{y}-\bar{y}_h)-E_h(\Pi_h\bar{y}-\bar{y}_h)]}{\partial n_e}\right\}\!\!\right\} ds \qquad (5.4.12) \\
&\qquad \le \Big(\sum_{e\in\mathcal{E}_h}|e|\,\|[\![\partial^2(\Pi_h\bar{y})/\partial n_e^2]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad\qquad \times\Big(\sum_{e\in\mathcal{E}_h}\frac{1}{|e|}\,\|\{\!\{\partial[(\Pi_h\bar{y}-\bar{y}_h)-E_h(\Pi_h\bar{y}-\bar{y}_h)]/\partial n_e\}\!\}\|_{L^2(e)}^2\Big)^{\frac{1}{2}} \\
&\qquad \le Ch^\tau\|\Pi_h y-y_h\|_h,
\end{aligned}
$$

The lemma follows from (5.4.8)–(5.4.12). $\square$

Lemma 5.6. *There exists a positive constant C independent of h such that*

$$\mathcal{A}(\bar{y}, E_h\Pi_h\bar{y} - \bar{y}) \le Ch^{1+\tau}, \tag{5.4.13}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Proof. Since $\Delta\bar{y} \in H_0^1(\Omega)$ (cf. Section 5.1.2), we obtain from (5.1.4) and Lemma 5.3 that

$$\begin{aligned}\int_\Omega D^2\bar{y} : D^2(E_h\Pi_h\bar{y} - \bar{y})\,dx &= \int_\Omega (\Delta\bar{y})(\Delta(E_h\Pi_h\bar{y} - \bar{y}))\,dx \\ &= \int_\Omega \nabla(\Delta\bar{y}) \cdot \nabla(E_h\Pi_h\bar{y} - \bar{y})\,dx \\ &\le C|E_h\Pi_h\bar{y} - \bar{y}|_{H^1(\Omega)} \le Ch^{1+\tau}.\end{aligned} \tag{5.4.14}$$

Moreover Lemma 5.3 also implies

$$(\bar{y}, E_h\Pi_h\bar{y} - \bar{y}) \le Ch^{2+\tau}. \tag{5.4.15}$$

Therefore the lemma follows from (5.2.11), (5.4.14) and (5.4.15). $\square$

Lemma 5.7. *There exists a positive constant C independent of h such that*

$$\begin{aligned}&\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (f, \Pi_h\bar{y} - \bar{y}_h) \\ \le &Ch^\tau\|\Pi_h\bar{y} - \bar{y}_h\|_h + [\mathcal{A}(\bar{y}, E_h(\Pi_h\bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h\bar{y} - \bar{y}_h))],\end{aligned} \tag{5.4.16}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Proof. From (5.1.9) and (5.2.11) we have

$$\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (f, \Pi_h\bar{y} - \bar{y}_h)$$

$$
\begin{aligned}
&= \beta \left[a_h(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - \int_\Omega (D^2 \bar{y} : D^2 E_h(\Pi_h \bar{y} - \bar{y}_h))\, dx \right] \\
&\quad + \gamma \left[(\Pi_h \bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) \right] \\
&\quad\quad - (f, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
&\quad\quad\quad + \mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h \bar{y} - \bar{y}_h)).
\end{aligned} \tag{5.4.17}
$$

Note that

$$
\begin{aligned}
(\Pi_h \bar{y}, &\Pi_h \bar{y} - \bar{y}_h) - (\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
&= (\Pi_h \bar{y} - \bar{y}, \Pi_h \bar{y} - \bar{y}_h) + (\bar{y}, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h)) \\
&\le h^2 \|\Pi_h \bar{y} - \bar{y}_h\|_h
\end{aligned} \tag{5.4.18}
$$

by (5.2.21) and (2.5.25). We also have, by (2.5.25),

$$
|(f, (\Pi_h \bar{y} - \bar{y}_h) - E_h(\Pi_h \bar{y} - \bar{y}_h))| \le C h^2 \|\Pi_h \bar{y} - \bar{y}_h\|_h. \tag{5.4.19}
$$

By Lemma 5.5 and (5.4.17)–(5.4.19), we obtain Lemma 5.7. □

From Lemma 5.7, it remains to estimate the second term on the right-hand side of (5.4.16). Note that the functions involved there belong to $H^2(\Omega) \cap H_0^1(\Omega)$, hence we can apply Lemma 5.6, the properties of the auxiliary solution (5.3.4)–(5.3.5) and Lemma 5.3 to obtain the following analog of Lemma 4.17.

Lemma 5.8. *There exists a positive constant C independent of h such that*

$$
\mathcal{A}(\bar{y}, E_h(\Pi_h \bar{y} - \bar{y}_h)) - (f, E_h(\Pi_h \bar{y} - \bar{y}_h)) \le C(h^{2\tau} + h\|\Pi_h \bar{y} - \bar{y}_h\|_h). \tag{5.4.20}
$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Now we are able to derive the error estimates for the state in the energy norm and the L^∞ norm. The proof of the following theorems are similar to Theorem 3.20 and Theorem 3.21.

Theorem 5.9. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_h \leq Ch^\tau, \tag{5.4.21}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Theorem 5.10. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq Ch^\tau, \tag{5.4.22}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

For the problem (5.1.1), we can take the piecewise constant function

$$\bar{u}_h = -\Delta_h \bar{y}_h$$

to be an approximation of the optimal control $\bar{u}$, where Δ_h is the piecewise Laplacian with respect to $\mathcal{T}_h$. By (5.2.12), (5.2.16) and Theorem 5.9, we immediately have the following corollary.

Corollary 5.11. *There exists a positive constant C independent of h such that*

$$|\bar{y} - \bar{y}_h|_{H^1(\Omega)} + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch^\tau, \tag{5.4.23}$$

where $\tau = \alpha$ if $\mathcal{T}_h$ is quasi-uniform and $\tau = 1$ if $\mathcal{T}_h$ is graded according to (5.2.2)–(5.2.3).

Remark 5.12. *The norms $\|\cdot\|_{L^\infty(\Omega)}$ and $|\cdot|_{H^1(\Omega)}$ are weaker than the energy norm $\|\cdot\|_h$. We expect to have a better convergence rate for these norms. Hence the estimates for $\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)}$ and $|\bar{y} - \bar{y}_h|_{H^1(\Omega)}$ are not sharp (cf. the numerical results in Section* 5.5).

5.5 Numerical Results

In this section we present several numerical examples for the problem (5.1.7) with $\psi_1(x) = -\infty$ to demonstrate the performance of the quadratic C^0 interior penalty method. The computational domain for the first two examples is the square $(-0.5, 0.5) \times (-0.5, 0.5)$. The discrete problems are defined on uniform triangulations $\mathcal{T}_j$ with mesh parameter $h_j = 2^{-j}$ (the length of the horizontal and vertical edges) for $1 \leq j \leq 8$ and the penalty parameter σ is chosen to be 5. The solutions of the discrete problems are denoted by $\bar{y}_j$ ($1 \leq j \leq 8$), which are obtained by a primal-dual active set algorithm [10, 72].

Example 1

In this example we validate our numerical scheme by solving (5.1.7) with a known solution. Similar to Example 1 in Section 4.4, we begin with the obstacle problem on the disc $\{x : |x| < 2\}$ with $\gamma = 0$, $\beta = 1$, $f = 0$ and $\psi_2(x) = 1 - |x|^2$. The exact solution is given by

$$y_\dagger(x) = \begin{cases} C_1|x|^2 \ln|x| + C_2|x|^2 + C_3 \ln|x| + C_4 & r_0 < |x| < 2, \\ \frac{1}{2}|x|^2 - 1 & |x| \leqslant r_0, \end{cases} \tag{5.5.1}$$

where $r_0 \approx 0.31078820$, $C_1 \approx -0.26855864$, $C_2 \approx 0.45470930$, $C_3 \approx -0.02593989$, $C_4 \approx -1.05625438$.

Let $\bar{y}$ be the restriction of $y_\dagger$ to $\Omega = (-0.5, 0.5)^2$. Then we have

$$\bar{y} = \underset{y \in \tilde{K}}{\operatorname{argmin}} \Big[\frac{1}{2}\int_\Omega (D^2 y : D^2 y)\, dx - \int_{\partial\Omega} \Big(\frac{\partial^2 y_\dagger}{\partial n^2}\Big)\Big(\frac{\partial y}{\partial n}\Big)\, ds\Big], \tag{5.5.2}$$

where

$$\tilde{K} = \{v \in H^2(\Omega) : v - y_\dagger \in H_0^1(\Omega) \text{ and } v \leq \psi_2 \text{ in } \Omega\}.$$

Therefore $\bar{y}$ is the solution of an obstacle problem for a simply supported plate with nonhomogeneous boundary conditions.

As in the case of clamped plates (cf. Section 4.3), our results for simply supported plates with homogeneous boundary conditions (Theorem 5.9 and Theorem 5.10) can be extended to the nonhomogeneous case. Let $\tilde{V}_h$ be the $\mathbb{P}_2$ Lagrange finite element space associated with the triangulation $\mathcal{T}_h$. The discrete problem for (5.5.2) is to find

$$\bar{y}_h = \underset{y_h \in \tilde{K}_h}{\operatorname{argmin}} \Big[\frac{1}{2} a_h(y_h, y_h) - \sum_{e \in \mathcal{E}_h^b} \int_e \Big(\frac{\partial^2 y_\dagger}{\partial n^2}\Big)\Big(\frac{\partial y_h}{\partial n}\Big)\, ds\Big], \tag{5.5.3}$$

where

$$\tilde{K}_h = \{v \in \tilde{V}_h : v - \Pi_h y_\dagger \in H_0^1(\Omega) \text{ and } v(p) \le \psi_2(p) \quad \forall p \in \mathcal{V}_h\}.$$

Let Π_j be the Lagrange nodal interpolation operator for the jth level finite element space. We evaluate the error $e_j = \Pi_j \bar{y} - \bar{y}_j$ in the energy norm $\|\cdot\|_{h_j}$ and in the l_∞ norm $\|\cdot\|_\infty$ defined by $\|e_j\|_\infty = \max_{p \in \mathcal{N}_j} |e_j(p)|$. We also compute the order of convergence in these norms by the formulas

$$\ln(\|e_{j-1}\|_{h_{j-1}}/\|e_j\|_{h_j})/\ln(2) \quad \text{and} \quad \ln(\|e_{j-1}\|_\infty/\|e_j\|_\infty)/\ln(2).$$

The numerical results are presented in Table 5.1. The order of convergence in the energy norm is 1.5, which is similar to the case of clamped plates. The order of convergence in the l_∞ norm is close to 2, which is better than 1 as predicted by Theorem 5.10.

We plot the discrete coincidence sets I_7 and I_8 in Figure 5.1, where

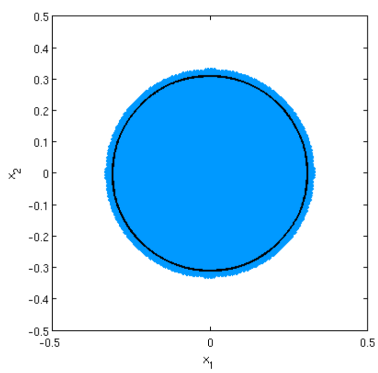
$$I_j = \{p \in \mathcal{N}_j : \bar{y}_j(p) \ge \psi_2(p) - \|e_j\|_\infty\}.$$

It is evident that the discrete coincidence sets (resp. free boundaries) are converging to the exact coincidence set (resp. free boundary).

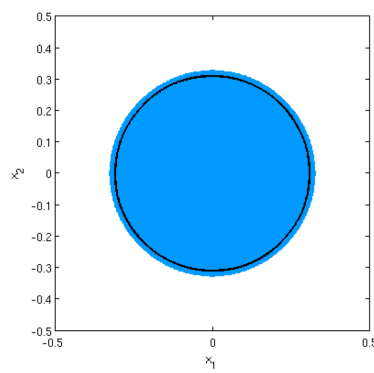
The other examples concern an optimal control problem with state constraints that come from [12, 84]. The value of γ is taken to be 1. Since the exact solutions

TABLE 5.1. Energy norm errors and l_∞ errors for Example 1 of the obstacle problem of simply supported plates

j	$\frac{\|e_j\|_{h_j}}{\|\bar{y}_8\|_{h_8}}$	order	$\|e_j\|_\infty$	order
1	2.18E-01		7.09E-03	
2	6.83E-02	1.68	5.97E-04	3.57
3	3.14E-02	1.12	5.72E-04	0.07
4	9.86E-03	1.67	1.16E-04	2.31
5	3.85E-03	1.36	3.55E-05	1.71
6	1.45E-03	1.40	1.07E-05	1.73
7	5.42E-04	1.42	3.31E-06	1.69
8	1.99E-04	1.45	8.97E-07	1.89



(a) I_7



(b) I_8

FIGURE 5.1. Discrete coincidence sets for Example 1 of the obstacle problem of simply support plates

are not known, we take $\tilde{e}_{\bar{y},j} = \bar{y}_{j-1} - \bar{y}_j$ and evaluate the error of the state $\|\tilde{e}_{\bar{y},j}\|_{h_j}$ in the energy norm, $|\tilde{e}_{\bar{y},j}|_{H^1}$ in the $H^1(\Omega)$ seminorm and $\|\tilde{e}_{\bar{y},j}\|_\infty$ in the l_∞ norm defined by $\|\tilde{e}_{\bar{y},j}\|_\infty = \max_{p\in\mathcal{N}_j} |\tilde{e}_{\bar{y},j}(p)|$. The approximations of the optimal control are given by the piecewise constant function $\bar{u}_j = -\Delta_j \bar{y}_j$, where Δ_j is the piecewise Laplace operator with respect to $\mathcal{T}_j$. We take $\tilde{e}_{\bar{u},j} = \bar{u}_{j-1} - \bar{u}_j$ and evaluate the error of the control $\|\tilde{e}_{\bar{u},j}\|_{L^2}$ in the $L^2(\Omega)$ norm. The order of convergence is generated by the formulas

$$\ln(\|\tilde{e}_{\bar{y},j-1}\|/\|\tilde{e}_{\bar{y},j}\|)/\ln(2) \quad \text{and} \quad \ln(\|\tilde{e}_{\bar{u},j-1}\|/\|\tilde{e}_{\bar{u},j}\|)/\ln(2).$$

Example 2

In this example we take $\psi_2(x) = 0.1$, $y_d(x) = \sin(2\pi(x_1+0.5)(x_2+0.5))$, $\beta = 10^{-3}$. The errors for the approximations of the state and the control are given in Table 5.2. Figure 5.2 contains the plots for the discrete state $\bar{y}_8$ and the discrete control $\bar{u}_8$.

TABLE 5.2. State and control errors for Example 2 of an optimal control problem with state constraints

j	$\frac{\|\tilde{e}_{y,j}\|_{h_j}}{\|\bar{y}_8\|_{h_8}}$	order	$\|\tilde{e}_{\bar{y},j}\|_\infty$	order	$\frac{\vert\tilde{e}_{y,j}\vert_{H^1}}{\vert\bar{y}_8\vert_{H^1}}$	order	$\frac{\|\tilde{e}_{u,j}\|_{L^2}}{\|\bar{u}_8\|_{L^2}}$	order
1	2.70E+00		6.32E-01		3.29E+00		1.74E+00	
2	1.24E+00	1.12	1.22E-01	2.37	1.15E+00	1.51	1.06E+00	0.72
3	6.76E-01	0.88	3.71E-02	1.72	3.39E-01	1.77	6.80E-01	0.64
4	3.46E-01	0.97	7.24E-03	2.36	9.21E-02	1.88	3.57E-01	0.93
5	1.75E-01	0.98	2.57E-03	1.50	2.66E-02	1.79	1.76E-01	1.01
6	8.64E-02	1.02	7.40E-04	1.79	7.28E-03	1.87	8.58E-02	1.05
7	4.27E-02	1.02	1.97E-04	1.91	1.86E-03	1.97	4.19E-02	1.03
8	2.12E-02	1.01	4.95E-05	1.99	4.67E-04	1.99	2.08E-02	1.01

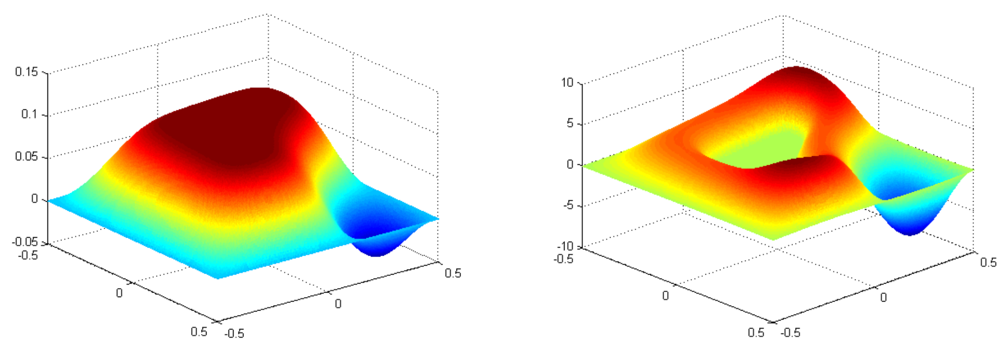


FIGURE 5.2. The discrete state $\bar{y}_8$ (left) and control $\bar{u}_8$ (right) for Example 2 of an optimal control problem with state constraints

The numerical results in Table 5.2 confirm the error estimate for $\|\bar{y} - \bar{y}_h\|_h$ in Theorem 5.9 and the error estimate for $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ in Corollary 5.11, since the index of elliptic regularity $\alpha = 1$ for a rectangular domain. On the other hand the order of convergence for $\bar{y}_h$ is 2 for both the $L^\infty(\Omega)$ norm and the $H^1(\Omega)$

seminorm, which is better than the order of convergence predicted by Theorem 5.10 and Corollary 5.11. The plots of the state and control in Figure 5.2 agree with those obtained in [12, 84].

Example 3

In this example we take Ω to be the pentagonal domain obtained from the square $(-0.5, 0.5)^2$ by deleting the triangle with vertices $(0.5, 0)$, $(0.5, 0.5)$ and $(0, 0.5)$. We use the same data as Example 2. The mesh parameter for the jth level uniform triangulation $\mathcal{T}_j$ is $h_j = 2^{-(j+1)}$. The errors for the approximate state $\bar{y}_j$ and approximate control $\bar{u}_j$ are presented in Table 5.3. Since the index of elliptic regularity α for the pentagonal domain can be taken to be any number less than 1/3, the results in Table 5.3 agree with Theorem 5.9. However, for this example the magnitude of the l_∞ error of the state seems to be $O(h^{2\alpha})$ and the magnitude of the $H^1(\Omega)$ error of the state seems to be $O(h)$.

We also plot the discrete state $\bar{y}_8$ and control $\bar{u}_6$ in Figure 5.3.

TABLE 5.3. State and control errors for Example 3 of an optimal control problem with state constraints

j	$\frac{\|\tilde{e}_{y,j}\|_{h_j}}{\|\bar{y}_8\|_{h_8}}$	order	$\|\tilde{e}_{\bar{y},j}\|_\infty$	order	$\frac{\|\tilde{e}_{y,j}\|_{H^1}}{\|\bar{y}_8\|_{H^1}}$	order	$\frac{\|\tilde{e}_{u,j}\|_{L^2}}{\|\bar{u}_8\|_{L^2}}$	order
1	1.27E+00		1.25E-01		1.16E+00		1.12E+00	
2	7.31E-01	0.80	3.71E-02	1.77	3.48E-01	1.73	7.78E-01	0.53
3	3.61E-01	1.02	4.69E-03	2.99	8.28E-02	2.07	3.81E-01	1.03
4	1.96E-01	0.88	1.37E-03	1.78	2.22E-02	1.90	1.97E-01	0.95
5	1.18E-01	0.73	3.44E-04	2.00	6.53E-03	1.76	1.13E-01	0.81
6	7.90E-02	0.57	1.50E-04	1.20	2.13E-03	1.61	7.36E-02	0.62
7	5.77E-02	0.45	7.71E-05	0.96	8.76E-04	1.28	5.30E-02	0.47
8	4.42E-02	0.39	4.63E-05	0.74	4.57E-04	0.94	4.03E-02	0.40

Example 4

In this example we solve the same problem in Example 5 on graded meshes obtained from a uniform triangulation $\mathcal{T}_0$ of the pentagonal domain by the refinement process

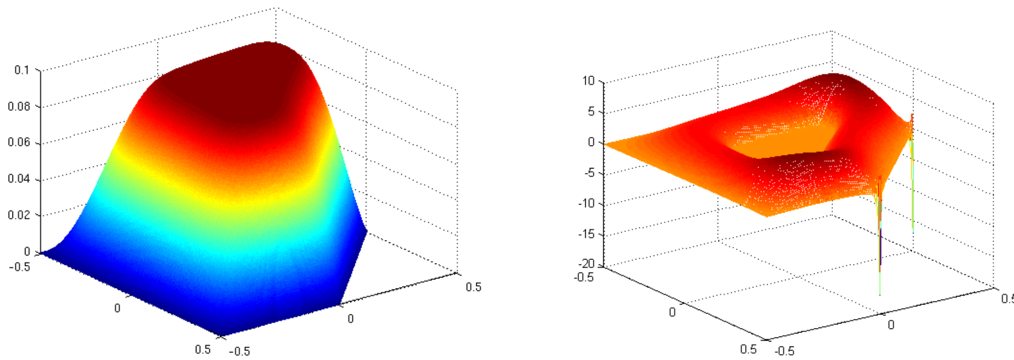


FIGURE 5.3. The discrete state $\bar{y}_8$ (left) and control $\bar{u}_6$ (right) for Example 3 of an optimal control problem with state constraints

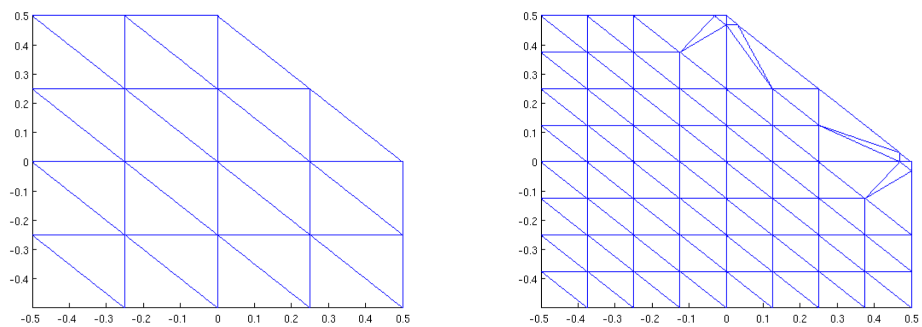


FIGURE 5.4. Triangulation $\mathcal{T}_0$ (left) and $\mathcal{T}_1$ (right) for the pentagonal domain

in [17] (cf. Figure 5.4), and we take the penalty parameter σ to be 20. The errors for the approximate state $\bar{y}_j$ and approximate control $\bar{u}_j$ are reported in Table 5.4. We can see that the order of convergence for the state in the energy norm and for the control in the $L^2(\Omega)$ norm is about 1, which agrees with Theorem 5.9 and Corollary 5.11. The order of convergence for the state in the l_∞ norm and the $H^1(\Omega)$ seminorm is about 1.5, which is better than the order of convergence predicted by Theorem 5.10 and Corollary 5.11.

The discrete state $\bar{y}_7$ and control $\bar{u}_3$ are depicted in Figure 5.5.

TABLE 5.4. State and control errors for Example 4 of an optimal control problem with state constraints

j	$\frac{\|\tilde{e}_{y,j}\|_{h_j}}{\|\bar{y}_7\|_{h_7}}$	order	$\|\tilde{e}_{\bar{y},j}\|_\infty$	order	$\frac{\vert\tilde{e}_{y,j}\vert_{H^1}}{\vert\bar{y}_7\vert_{H^1}}$	order	$\frac{\|\tilde{e}_{u,j}\|_{L^2}}{\|\bar{u}_7\|_{L^2}}$	order
1	5.15E-01		2.16E-02		2.25E-01		5.57E-01	
2	3.43E-01	0.59	8.52E-03	1.34	1.10E-01	1.03	3.78E-01	0.56
3	1.98E-01	0.79	3.93E-03	1.12	4.50E-02	1.29	2.15E-01	0.81
4	1.10E-01	0.86	1.53E-03	1.36	1.50E-02	1.58	1.15E-01	0.90
5	6.01E-02	0.87	6.23E-04	1.30	5.73E-03	1.39	6.22E-02	0.89
6	3.28E-02	0.87	2.22E-04	1.49	1.95E-03	1.56	3.35E-02	0.89
7	1.78E-02	0.88	7.57E-05	1.55	6.61E-04	1.56	1.80E-02	0.90

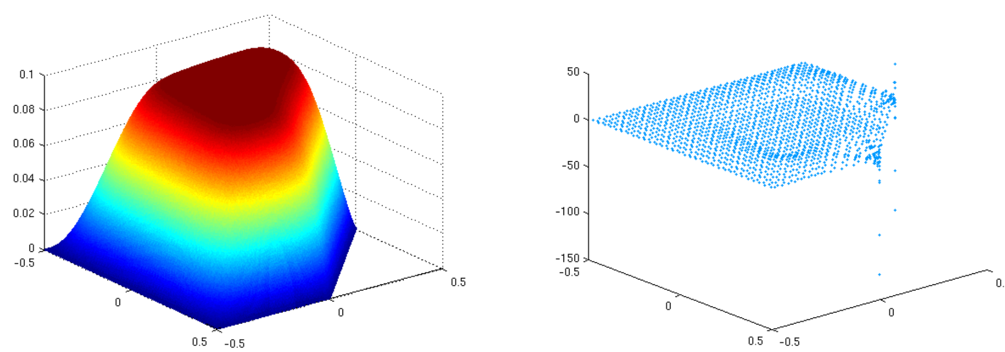


FIGURE 5.5. The discrete state $\bar{y}_7$ (left) and control $\bar{u}_3$ (right) for Example 4 of an optimal control problem with state constraints

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Notations

- $|e|$ is the length of an edge e.
- $\mathcal{E}_h$ is the set of the edges of the triangles in $\mathcal{T}_h$.
- $\mathcal{E}_h^b$ is the subset of $\mathcal{E}_h$ consisting of edges along $\partial\Omega$.
- $\mathcal{E}_h^i$ is the subset of $\mathcal{E}_h$ consisting of edges interior to Ω.
- $\mathcal{E}_{\mathcal{V}(T)}$ is the set of the edges in $\mathcal{T}_T$ sharing a vertex with T.
- $\mathcal{E}_{\mathcal{V}(T)}^i$ is the set of the edges in $\mathcal{E}_h^i$ emanating from the vertices of T.
- h_T is the diameter of the triangle T.
- h is the mesh parameter proportional to $\max_{T\in\mathcal{T}_h} h_T$.
- $\mathcal{N}_h$ is the set of the vertices and midpoints of $\mathcal{T}_h$.
- $\mathcal{N}_T$ is the set of the vertices and midpoints of T.
- $\mathcal{S}_T$ is the interior of the closure of $\bigcup_{T'\in\mathcal{T}_T} T'$.
- T_e is the triangle in $\mathcal{T}_h$ that contains $e \in \mathcal{E}_h^b$ as an edge.
- $\mathcal{T}_e$ is the set of the triangles in $\mathcal{T}_h$ that share the common edge e.
- $\mathcal{T}_h$ is a simplicial triangulation of Ω.
- $\mathcal{T}_p$ is the set of triangles sharing the common vertex p.
- $\mathcal{T}_T$ is the set of triangles sharing a vertex with T.
- v_T is the restriction of the function v to the triangle T.

- $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$.

- $\mathcal{V}_T$ is the set of the three vertices of T.

Vita

Yi Zhang was born in 1983, in Yongding, Fujian Province, China. He finished his undergraduate studies at Wuhan University June 2006. He earned a master of science degree in mathematics from Wuhan University in June 2008. In August 2008 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2013.