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Which mean do you mean?: an exposition on means

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WHICH MEAN DO YOU MEAN?
AN EXPOSITION ON MEANS

A Thesis
Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Master of Science

in
The Department of Mathematics

by
Mabrouck K. Faradj
B.S., L.S.U., 1986
August, 2004
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This thesis is dedicated to my wife Marianna for sacrificing so much of her self so that I may realize my dreams. It would not have been done without her support.
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Abstract

The objective of this thesis is to give a brief exposition on the theory of means. In Greek mathematics, means are intermediate values between two extremes, while in modern mathematics, a mean is a measure of the central tendency for a set of numbers. We begin by exploring the origin of the antique means and list the classical means. Next, we present an overview of the theories of binary means and $n$-ary means. We include a general discussion on axiomatic systems for means and present theorems on properties that characterize the most common types of means.
Chapter 1. Introduction

In this chapter we give a brief introduction to the origins of the arithmetic, geometric, and harmonic means.

1.1 The Origins of the Term Mean

According to "Webster’s New Universal Dictionary", the term mean is used to refer to a quantity that is between the values of two or more quantities. The term mean is derived from the French root word mien whose origin is the Latin word medius, a term used to refer to a place, time, quantity, value, kind, or quality which occupies a middle position. The most common usage of the term mean is to express the average of a set of values. The term average, from the French word averie, is itself rich in history and has extended usage. The term average was used in medieval Europe to refer to a taxing system levied by a liege lord on a vassal or a peasant. The word average is derived from the Arabic awariyah, which translates as goods damaged in shipping. In the late middle ages, average was used in France and Italy to refer to financial loss resulting from damaged goods, where it came to specify the portion of the loss borne by each of the many people who invested in the ship or its cargo. In this usage, it is the amount individually paid by each of the investors when a loss is divided equally among them. The notion of an average is very useful in commerce, science, and legal pursuits; thus, it is not surprising that several possible kinds of averages have been invented so that a wide array of choices of an intermediate value for a given set of values is available to the user to select from.

1.2 Antique Means

The earliest documented usage of a mean was in connection with arithmetic, geometry, and music. In the 5th century B.C., the Greek mathematician Archytas gave a definition of the three commonly used means of his time in his treatise on music:

we have the arithmetic mean when, of three terms, the first exceeds the second by the same amount as the second exceeds the third; the geometric mean when the first is to the
second as the second is to the third; the harmonic mean when the three terms are such that by what ever part of itself the first exceeds the second, the second exceeds the third by the same part of the third. (Thomas, 1939, p. 236)

This can be translated to modern terms as follows. Let $a$ and $b$ be two whole numbers such that $a > b$ and $A$, $G$, and $H$ are the arithmetic, geometric, and harmonic means of $a$ and $b$ respectively. Then

1. $a - A = A - b \implies A = \frac{a+b}{2},$
2. $\frac{a}{G} = \frac{G}{b} \implies G = \sqrt{ab},$
3. $\frac{a-H}{a} = \frac{H-b}{b} \implies H = \frac{2ab}{a+b}.$

The origins of the names given to the antique means are obscured by time. The first of these means, and probably the oldest, is the arithmetic mean. To the ancient Greeks, the term $\alphaριθµητικς$ refers to the art of counting, and so, fittingly, they referred to what we commonly call the average as the arithmetic mean since it pertains to finding a number that is intermediate to a given pair of natural numbers. As for the name given to the geometric mean, it appears that the Pythagorean school coined the term mean proportional, i.e., the geometric mean, to refer to the measure of an altitude drawn from the right angle to the hypotenuse of a right triangle. The source of the name given to the harmonic mean can only be found in legends. The Roman Boethius (circa 5 A.D.) tells us of a legend about Pythagoras who on passing a blacksmith shop was struck by the fact that the sounds caused by the beating of different hammers on the anvil formed a fairly musical whole. This observation motivated Pythagoras to investigate the relation between the length of a vibrating string and the musical tone it produced. He observed that different harmonic musical tones are produced by particular ratios of the length of the vibrating string to its whole. He concluded, according to the legend, that the musical harmony produced was to be found in particular ratios of the length of the vibrating string. Thus to the Pythagoreans, who believed that all knowledge can be reduced to relations between numbers, musical harmony
occurred because certain ratio of numbers that lie between two extremes are harmonic, and thus the term harmonic mean was given to that value.

**Proposition 1.** Suppose $0 < a \leq b$. Let $A := \frac{a+b}{2}$, $G := \sqrt{ab}$, and $H := \frac{2ab}{a+b}$. Then $a \leq A \leq b$, $a \leq G \leq b$, and $a \leq H \leq b$.

**Proof.** Since $0 < a \leq b$, then $a + b \leq 2b$; therefore, $\frac{a+b}{2} \leq b$. Similarly, $2a \leq a + b$; therefore, $a \leq \frac{a+b}{2}$. Therefore, $a \leq A \leq b$. Hence, $\frac{1}{b} \leq \frac{2}{a+b} \leq \frac{1}{a}$. Therefore, $a \leq \frac{2ab}{a+b} \leq b$, and $a \leq H \leq b$. If $a \leq b$, $a^2 \leq ab \leq b^2$; therefore, $a \leq \sqrt{ab} \leq b$. Hence $a \leq G \leq b$. \qed

### 1.3 Geometric Interpretation of the Antique Means

Since geometry is the ancient Greeks’ preferred venue of scientific investigation, Greek mathematicians produced numerous geometric treatises that related the three antique means to each other by using straight edge and compass construction. An excellent example can be found in Schild (1974) and reproduced here:

**Example 1.1.** Suppose $a$ and $b$ are two whole numbers. Let $A$, $G$, and $H$ be the arithmetic, geometric, and harmonic means respectively of $a$ and $b$. Then by using a straight edge and compass we can illustrate that $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, and $H = \frac{2ab}{a+b}$. Draw the line segment $LMN$ with $LM = a$ and $MN = b$ (see figure 1.1). With $LN$ as diameter, draw a semi circle with center $O$ and fix $P$ on its circumference. Draw $MQ$ perpendicular to $OP$ and $MP$ perpendicular to $LN$. Then $OP = A$, $MP = G$, and $QP = H$. To show this is true, we give the following argument. Since $OP$ is the radius of the circle whose diameter is $LN$, then $OP = \frac{1}{2}(a+b) = A$, and since $(MP)^2 = (LM)(MN) = ab$, then $MP = \sqrt{ab} = G$. Let $\alpha = \angle POM$. Observe that $\angle QMP = \alpha$ and $\triangle POM$ is similar to $\triangle PMQ$; thus, $\frac{PQ}{PM} = \frac{PM}{PQ}$. Therefore, $PQ = \frac{(PM)^2}{PQ} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b} = H$.

In figure 1.1, observe what happens if $(a+b)$ remains fixed, i.e., segment $LN$ is fixed, and $M$ is allowed to move. As $M$ moves toward $N$, both $G$ and $H$ decrease. As $M$ moves towards $O$, both $G$ and $H$ increase. If $M$ coincides with $O$, i.e., $a = b$, then $A = G = H$. This may have been the motivation for investigating the inequality between the three means.
1.4 Antique Means Inequality

In this section we will present several proofs of the inequality:

$$H \leq G \leq A \quad (1.1)$$

Of the numerous useful inequalities in mathematics, the arithmetic-geometric mean inequality occupies a special position, not only from a historical standpoint, but also on account of its frequent usage in different mathematical proofs. We will give a more in-depth discussion about this inequality in Chapters 3 and 4. At this point, it suffices to say that there have been numerous proofs given for the above inequality over the centuries.

We begin our discussion by presenting an informal argument of the inequality. Referring back to Figure 1.1, we note that $|\sin \alpha| \leq 1$. From $\triangle OPM$, we have $\sin \alpha = \frac{MP}{OP}$, and from $\triangle PQM$, we have $\sin \alpha = \frac{QP}{MP}$. Therefore $MP = OP \sin \alpha \Rightarrow G = A \sin \alpha$. Hence

$$G \leq A, \quad (1.2)$$

and $QP = MP \sin \alpha \Rightarrow H = G \sin \alpha$. Hence

$$H \leq G. \quad (1.3)$$

From 1.2 and 1.3, we get 1.1.

However, since the above argument uses trigonometry, it does not reflect the spirit of the ancient proofs for this inequality. In Figure 1.2, we present an illustration that captures the fundamental
character of this inequality in mathematics, which may have motivated the ancient mathematicians to establish proofs of the arithmetic-geometric mean inequality (Gallant 1977). The inequality as illustrated by Figure 1.2 requires only rudimentary knowledge of geometry to prove. Now we give a more modern algebraic proof for the geometric-arithmetic mean inequality.

![Figure 1.2. Proof Without Words: A Truly Algebraic Inequality.](image)

### Theorem 1
For any nonnegative numbers \(a\) and \(b\), \(\sqrt{ab} \leq \frac{a+b}{2}\), with equality holding if and only if \(a = b\).

**Proof.** Let \(a = c^2\) and \(b = d^2\). Then \(\frac{a+b}{2} \geq \sqrt{ab}\) becomes \(\frac{c^2+d^2}{2} \geq cd\), or equivalently, \(\frac{c^2+d^2}{2} - cd \geq 0\). This is equivalent to \(c^2 - 2cd + d^2 \geq 0\) which is in turn equivalent to \((c-d)^2 \geq 0\). Since the square of any real number is nonnegative, we see that the inequality stated in the theorem is indeed true. Equality holds if and only if \(c = d\), that is \(c = d\), or equivalently, if and only if, \(a = b\).

We use the result from theorem 1 to establish an inequality between the harmonic and geometric means of any two nonnegative numbers.

### Corollary 1.2
For any nonnegative numbers \(a\) and \(b\), \(\frac{2ab}{a+b} \leq \sqrt{ab}\), with equality holding if and only if \(a = b\).

**Proof.** Since \(\sqrt{ab} \leq \frac{a+b}{2}\), then \(2\sqrt{ab} \leq (a+b)\). Therefore, \(2ab \leq (a+b)\sqrt{ab}\), and \(\frac{2ab}{a+b} \leq \sqrt{ab}\). 

\[ \square \]
From theorem 1 and corollary 1.2, we have

\[ H \leq G \leq A. \]  \hspace{1cm} (1.4)
Chapter 2. Classical Means

In this chapter we will explore the origins of the theory of binary means. The chapter includes two lists of the classical binary means as given by Greek mathematicians. The following list gives the names of Greek mathematician and the approximate dates of their work on means. It is helpful to the understanding of the historical development of the theory means in the ancient Greek world (Smith 1951).

Thales, 600 B.C.  Pythagoras, 540 B.C.  Archytas, 400 B.C.  Plato, 380 B.C.
Eudoxus, 370 B.C.  Eudemus, 335 B.C.  Euclid, 300 B.C.  Archimedes, 230 B.C.
Heron, 50 A.D.  Nicomachus, 100 A.D.  Theon, 125 A.D.  Porphyrius, 275 A.D.
Pappus, 300 A.D.  Iamblichus, 325 A.D.  Proclus, 460 A.D.  Boethius, 510 A.D.

2.1 The History of Classical Means

In this section we will give a brief discussion on what motivated Greek mathematicians to study and develop a doctrine for means by presenting the rationale given by prominent Greek mathematicians who touched on the history of the theory of means in their work and the opinions of Greek mathematics scholars on this matter.

According to Gow (1923), by Plato’s time numbers were grouped into two general categories. First, as single numbers categorized by their attributes such as odd, even, triangular, perfect, excessive, defective, amicable etc. Second, numbers were viewed as groups comprised of numbers that are either in series or proportions. The ancient Greeks viewed means as a special case of proportions (Allemann 1877, Thomas 1939, Gow 1923). Smith (1951) writes, "Early [Greek] writers spoke of an arithmetic proportion, meaning $b - a = d - c$ as in 2, 3, 4, 5, and of geometric proportion, meaning $a : b = c : d$ as in 2, 4, 5, 10, and a harmonic proportion, meaning $\frac{1}{b} - \frac{1}{a} = \frac{1}{d} - \frac{1}{c}$ as in $\frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \frac{1}{5}$." In his comments on paradigms of ancient Greek mathematics, Allemann (1877) says, "when two quantities were compared [in Greek mathematics], the basis for the comparison seems to be either how much the one is greater than the other, i.e., an arithmetic
ratio, or how many times is the one contained in the other, i.e., their geometrical ratio." Allemann (1877) claims that this type of comparison of ratios would naturally lead to the theory of means because for any three positive magnitudes, be it lines or numbers, \(a, b,\) and \(c,\) if \(a - b = b - c,\) the three magnitudes are in arithmetical proportion, but if \(a : b :: b : c,\) they are in geometrical proportion. Allemann’s claim seems to be supported by the work of Nicomachus in "Introduction to Arithmetic". In this work, Nicomachus began his discourse on means by giving the definition that distinguished a ratio from a proportion. He referred to the latter as the composition of two ratios. He then stated that when one term appears on both sides of a proportion, as in \(\frac{a}{b} = \frac{b}{c},\) the proportion is known as a continued proportion. The proportion is called disjunct when the middle terms are different. The highest term in a continued proportion is called the consequent, the least is called the antecedent, and the middle term is the mean, \(\mu\varepsilon\sigma\tau\iota\varepsilon\varsigma,\) which is medius when translated into Latin and from which the word mean is derived (Gow 1923).

As we have noted above, Greek mathematics viewed means as a special proportion involving three magnitudes; therefore, it is appropriate that we begin our review of the history of development of means by mentioning that Proclus attributed to Thales the beginning of the doctrine of proportions (Allemann 1877). Thales established the theorem that equiangular triangles have proportional sides (Allemann 1877). In "Introduction to Arithmetic", Nicomachus writes, "the knowledge of proportions is particularly important for the study of ancient mathematicians." This can be taken to mean that the doctrine of proportions played an important role in the development of Greek mathematics. Maziarz (1968) comments on the natural development of the theory of proportionals in Greek mathematics by saying, "If a point is a unit in a position, then a line is made of points. Consequently, the ratio of two given segments is merely the ratio of the number of points in each. Moreover, because any magnitude involves a ratio between the number of units it contains and the unit itself, and, thus, the comparison of two magnitudes implies either 2 or 4 ratios." By points, Maziarz seems to imply the tick marks that would be made if the segments were divided into many small equal units.
From the historical perspective, the ancient sources of Greek mathematics history that we have referenced do not mention when the arithmetic mean was first developed. However, they offer various explanations as to when the geometric and harmonic means were first introduced. Allemann (1877) states that ancient sources (Iamblichus, Nicomachus, Proclus) point to Eudoxus as the one who established the harmonic mean and to Pythagoras as the one who established the notion of a mean proportional between two given lines.

It is interesting to note that some facets of the theory of means appear in various ancient Greek texts. Some of these were intended as mathematics treatises, such as the collection of books that constitute Euclid’s work known as the "Elements", but others did not have an apparent mathematical purpose. One such example, noted by Maziarz (1968), can be found in passages of "Timaeus" known as "The Construction the world-soul." In this section of the book, Plato attempts to construct the arithmetical continuum using two geometric progressions 1, 2, 4, 8 and 1, 3, 9, 27; then filling in the intervals between these numbers with the arithmetic and harmonic means. By successive duplication of the two progressions and filling in with the appropriate combination of arithmetic and harmonic means, all numbers can be generated, but not in their natural order. Another example can be found in Aristotle’s "Metaphysics". In this work, Aristotle describes Plato’s notion of distributive justice as, "The just in this sense is a mean between two extremes that are disproportionate, since the proportionate is a mean, and the just is proportionate. This kind of proportion is termed by mathematicians geometrical proportion."

From the above examples, one gets the sense that to the ancient Greeks, the theory of means and proportions may not have been just a mere mathematical concept since some aspects of the theory of means was also reflected in their literature, philosophy, and religion.

2.2 The Development of Classical Means Theory

It appears that the classical means were developed over a long period of time by the gradual addition of seven more means to the first three (Heath 1963). In all his work, Euclid only uses the three antique means (Allemann 1887, Gow 1923). However, by first century A.D., we know that
Greek mathematicians referred to ten means. All the sources reviewed (Allemann 1887, Beman 1910, Heath 1921, Gow 1923, Thomas 1939, Smith 1951) suggest that Greek mathematicians generated these means by considering three quantities \(a, b,\) and \(c,\) such that \(a > b > c.\) They assumed \(b\) to be the mean and formed three positive differences with the \(a, b,\) and \(c:\)

\[ (a - b), \quad (b - c), \quad \text{and} \quad (a - c). \]

Then they formed a proportion by equating a ratio of two of these differences to a ratio of two of the original magnitudes, \(a, b,\) and \(c.\) For example, \(b\) is the harmonic mean of \(a\) and \(c\) when \(\frac{a-b}{b-c} = \frac{\varepsilon}{\epsilon}.\) Nicomachus in "Introduction to Arithmetic" (Gow 1923) goes on to say: "Pythagoras, Plato, and Aristotle knew only six kinds of [continued] proportions: the arithmetic, geometric, and harmonic means, and their subcontraries, which have no names. Later writers added four more." Greek mathematicians referred to certain classical means as contrary and subcontrary means because these means were seen to be in a contrary (opposite) order from the arithmetic mean when compared to the geometric or harmonic means (Oxford English Dictionary 2004).

In his work "In Nicomachus" (Heath 1921), Iamblichus says, "the first three [antique means] only were known to Pythagoras, the second three were invented by Eudoxus." The remaining four, Iamblichus attributed to the later Pythagoreans. He adds that all ten were treated in the Euclidean manner by Pappus. Gow (1923) states that the number of continued proportions was raised to ten and kept at that number because the number ten was held by the ancient Greek mathematicians to be the most perfect number. He adds, "how else can we explain the fact that the golden mean, which Nicomachus calls the most perfect and embracing of all proportions, was left out from the list of means."

All these testimonies point to the conclusion that the theory of means in Greek mathematics was well established by the First Century. Our main complete source for ancient Greek mathematics’ theory of means is Boethius’ commentary on the works of Pappus and Nicomachus. In this work,
Boethius credits Nicomachus and Pappus as the main Greek mathematicians who dealt with means from a theoretical perspective (Smith 1951).

2.3 Nicomachus’ List of Means

The earliest known treatment of classical means as an independent body of knowledge was given by Nicomachus in "Introduction to Arithmetic" (Allemann 1887, Heath 1921, Gow 1923, Thomas 1939, Smith 1951). Allemann, Gow, Heath, and Thomas concluded (seemingly independent of each other) that Nicomachus proceeded to develop his list as follows:

He began his list by commenting on the continued arithmetical proportion \( a - b = b - c \). This suggests that \( a - b : b - c :: a : a \), which allows us to make a connection to other means. Gow (1923) remarks, "In a continued geometric proportion, \( a : b :: b : c \), he notices that \( a - b : b - c :: a : b \). Finally, the three magnitudes, \( a, b, c \), are in harmonic proportion if \( a - b : b - c :: a : c \)." A similar approach was used by Archytas (as cited by Porphyrius in his commentary on Ptolemy’s "Harmonics") when discussing the three antique means in terms of three magnitudes in continued arithmetic, geometric, and harmonic proportions (Thomas 1939).

Gow (1923) also points out that Nicomachus failed to mention that the arithmetic, geometric, and harmonic means of two numbers are in geometric proportion: \( \frac{a+b}{2} : \sqrt{ab} : \frac{2ab}{a+b} \). In Thomas’ translation of Nicomachus’ "Introduction to Arithmetic" (Thomas 1939), Nicomachus introduces the seven other means using the same treatment as the one mentioned above. (The reader may wish to refer to Table 2.3 for a compact summary of the following.)

The fourth mean, which is also called the subcontrary by reason of its being reciprocal and antithetical to the harmonic, comes about when of the three terms the greatest bears the same ratio to the least as the difference of the lesser terms bears to the difference of the greater, as in the case of 3, 5; 6 (Thomas, 1939, p. 119).

Nicomachus introduces the fifth mean as the subcontrary mean to the geometric mean,

The fifth [mean] exists when of the three terms, the middle bears to the least the same ratio as their difference bears to the difference between the greatest and the middle
terms, as in the case of 2, 4; 5, for 4 is double 2, the middle term is double the least, and 2 is double 1, that is the difference of the least terms is double the difference of the greatest. What makes it subcontrary to the geometric mean is this property, that in the case of the geometric mean the middle term bears to the lesser the same ratio as the excess of the greater term over the middle bears to that of the middle term over the lesser, while in the case of this mean a contrary relation holds (Thomas, 1939, p. 121).

Nicomachus introduces the sixth mean as,

The sixth mean comes about when of the three terms the greatest bears the same ratio to the middle as the excess of the middle term over the least bears to the excess of the greatest term over the middle as in the case of 1, 4; 6, for in each case the ratio is sesquialter [3 : 2]. No doubt, it is called subcontrary to the geometric mean because the ratios are reversed, as in the case of the fifth mean (Thomas, 1939, p. 121).

Nicomachus introduces the last 4 means by saying,

By playing about with the terms and their differences certain men discovered four other means which do not find a place in the writings of the ancients, but which nevertheless can be treated briefly in some fashion, although they are superfluous refinements, in order not to appear ignorant. The first of these, or the seventh in the complete list, exists when the greatest term bears the same relation to the least as their difference bears to the difference of the lesser terms, as in the case of 6, 8; 9, for the ratio of each is seen by compounding the terms to be the sesquialter. The eighth mean, or the second of these, comes about when the greatest term bears to the least the same ratio as the difference of the extreme bears to the difference of the greater terms, as in the case of 6, 7; 9, for here the two ratios are the sesquialter. The ninth mean in the complete series, and the third in the number of those more recently discovered, comes about when there are three terms and the middle bears to the least the same ratio as the difference between the extremes bears to the difference between the least terms, as 4, 6; 7. Finally, the tenth in the
complete series, and the fourth in the list set out by the moderns, is seen when in three terms the middle term bears to the least the same ratio as the difference between the extremes bears to the difference of the greater terms, as in the case of 3, 5; 8, for the ratio in each couple is the super-bi-partient \([5 : 3]\) (Thomas, 1939, p. 121).

<table>
<thead>
<tr>
<th>Mean</th>
<th>TABLE 2.1. Nicomachus’ Means Proportion</th>
<th>Numbers Exhibiting the Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>(a - b : b - c :: a : a)</td>
<td>2, 4, 6</td>
</tr>
<tr>
<td>Geometric</td>
<td>(a - b : b - c :: a : b)</td>
<td>4, 2, 1</td>
</tr>
<tr>
<td>Harmonic</td>
<td>(a - b : b - c :: a : c)</td>
<td>6, 3, 2</td>
</tr>
<tr>
<td>Cont. Harmonic</td>
<td>(b - c : a - b :: a : c)</td>
<td>3, 5, 6</td>
</tr>
<tr>
<td>Cont. Geometric</td>
<td>(b - c : a - b :: b : c)</td>
<td>2, 4, 5</td>
</tr>
<tr>
<td>Subcon. Geometric</td>
<td>(b - c : a - b :: a : b)</td>
<td>1, 4, 6</td>
</tr>
<tr>
<td>Seventh</td>
<td>(a - c : b - c :: a : c)</td>
<td>6, 8, 9</td>
</tr>
<tr>
<td>Eighth</td>
<td>(a - c : a - b :: a : c)</td>
<td>6, 7, 9</td>
</tr>
<tr>
<td>Ninth</td>
<td>(a - c : b - c :: b : c)</td>
<td>4, 6, 7</td>
</tr>
<tr>
<td>Tenth</td>
<td>(a - c : a - b :: b : c)</td>
<td>3, 5, 8</td>
</tr>
</tbody>
</table>

(Thomas 1939)

### 2.4 Pappus’ List of Means

Pappus used a different approach than Nicomachus when presenting his list of means (Heath 1921, Thomas 1939). Both Heath and Thomas state that the means on Pappus’ list are similar to those presented by Nicomachus, but in a different order after the sixth mean. Means number 8, 9, and 10 in Nicomachus’ list are respectively numbers 9, 10, and 7 on Pappus’ list. Moreover, Pappus omits mean number 7 on Nicomachus’ list and gives as number 8 an additional mean equivalent to the proportion \(c : b :: c - a : c - b\). Therefore, the two lists combined give five additional means to the first six.

In Thomas’ translation (1939) of Pappus’ work known as "Collections III", Pappus introduces his discussion on means as a response to a question posed by an uninformed geometer. He
demonstrates his answer by the construction of the three means in a semicircle (see figure 1.1). Pappus shows, in a series of propositions, that given three terms \( \alpha, \beta, \) and \( \gamma \) in geometrical progression (Heath 1921 uses "in geometric proportion"), it is possible to form from them three other terms \( a, b, \) and \( c \) which are integral linear combination of \( \alpha, \beta, \) and \( \gamma \) such that \( b \) is one of the classical means. The solutions to Pappus’s equations are shown in Table 2.2. The linear

<table>
<thead>
<tr>
<th>Mean</th>
<th>( a, b, c )</th>
<th>Numbers exhibiting the mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>( a = 2\alpha + 3\beta + \gamma )</td>
<td>6, 4, 2</td>
</tr>
<tr>
<td></td>
<td>( b = \alpha + 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Geometric</td>
<td>( a = \alpha + 2\beta + \gamma )</td>
<td>4, 2, 1</td>
</tr>
<tr>
<td></td>
<td>( b = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \gamma )</td>
<td></td>
</tr>
<tr>
<td>Harmonic</td>
<td>( a = 2\alpha + 3\beta + \gamma )</td>
<td>6, 3, 2</td>
</tr>
<tr>
<td></td>
<td>( b = 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Subcontrary</td>
<td>( a = 2\alpha + 3\beta + \gamma )</td>
<td>6, 5, 2</td>
</tr>
<tr>
<td></td>
<td>( b = 2\alpha + 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Fifth</td>
<td>( a = \alpha + 3\beta + \gamma )</td>
<td>5, 4, 2</td>
</tr>
<tr>
<td></td>
<td>( b = \alpha + 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Sixth</td>
<td>( a = \alpha + 3\beta + 2\gamma )</td>
<td>6, 4, 1</td>
</tr>
<tr>
<td></td>
<td>( b = \alpha + 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \alpha + \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Seventh</td>
<td>( a = \alpha + \beta + \gamma )</td>
<td>3, 2, 1</td>
</tr>
<tr>
<td></td>
<td>( b = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \gamma )</td>
<td></td>
</tr>
<tr>
<td>Eighth</td>
<td>( a = 2\alpha + 3\beta + \gamma )</td>
<td>6, 4, 3</td>
</tr>
<tr>
<td></td>
<td>( b = \alpha + 2\beta + \gamma )</td>
<td></td>
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<tr>
<td></td>
<td>( c = 2\beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Ninth</td>
<td>( a = \alpha + 2\beta + \gamma )</td>
<td>4, 3, 2</td>
</tr>
<tr>
<td></td>
<td>( b = \alpha + \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td>Tenth</td>
<td>( a = \alpha + \beta + \gamma )</td>
<td>3, 2, 1</td>
</tr>
<tr>
<td></td>
<td>( b = \beta + \gamma )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( c = \gamma )</td>
<td></td>
</tr>
</tbody>
</table>

(Heath 1921, Thomas 1939)

The equations shown in Table 2.2 are modern equivalents of the literal translation of the Greek version of Pappus. For example (Thomas 1939), in the case of the geometric mean mentioned in Table 2.2, the literal translation of Pappus’ words would be, "To form \( a \) take \( \alpha \) once, \( \beta \) twice, and \( \gamma \) once; and to form \( b \) we have to take \( \beta \) once and \( \gamma \) once; and to form \( c \) we take \( \gamma \) once." Notice also that the examples given by Pappus for the proportions formed by his equations sometimes differ.
from those given by Nicomachus. For example for the fourth mean, Nicomachus gave 3, 5, and 6 as an example for a solution, while Pappus gave 2, 5, and 6 as a solution.

Pappus’ exposition on means by using equations may be better understood from the perspective that proportions were used in those days to solve equations. Using Proclus’ commentary on Euclid as a reference, Klein (1966) states, "Greek mathematics’ usage of proportions can be compared to the modern sense of construction of an equation, and an equation may be viewed as a solution of a proportion. This may be due to the understanding of ratios, proportions, and harmony on the basis of a common mathematical property." Beman (1910) claims that the mathematicians of Alexandria understood equations of second degree mostly in the form of proportions. If we express Pappus’ method in modern terms, Pappus is parameterizing means by quadratics and, equivalently, giving quadratic polynomials to illustrate the relation among terms in the various means. For example, to calculate the harmonic mean, using three quantities in geometric progression is equivalent to using \( \alpha = 1, \beta = x, \) and \( \gamma = x^2 \); thus, given \( a = 2 + 3x + x^2, \) \( b = 2x + x^2, \) and \( c = x + x^2, \) we have

\[
\frac{2ac}{a+c} = \frac{2(2+3x+x^2)(x+x^2)}{2+3x+x^4+x+x^2} = \frac{2x(x+1)(x^2+3x+2)}{2(x+1)^2} = \frac{x(x^2+3x+2)}{x+1} = x(x+2) = x^2 + 2x = b.
\]

### 2.5 A Modern Reconstruction of the Classical Means

In this section, we will use a similar approach to the one used by Nicomachus to generate the classical means by considering three positive quantities \( a, b, \) and \( c \) such that \( a > b > c \), and we wish to make \( b \) the mean of \( a \) and \( c \). We will form three positive differences with these quantities: \( (a - b), (b - c), \) and \( (a - c) \). Then we will form a proportion by equating a ratio of two of these differences to a ratio of two of the original quantities (not necessarily distinct). For example, if we set the ratio \( \frac{a-b}{b-c} \) equal to the ratio \( \frac{a}{b} \), the result is \( b^2 = ac \), which represents the geometric mean. If you look at all the possible ways of doing this, several of them are automatically ruled out by the assumed inequality of \( a, b, \) and \( c \). The ones that are not (necessarily) ruled out are the eleven means summarized below (Madden 2000, Heath 1963):

1. \( \frac{(a-b)}{(b-c)} = \frac{a}{a} = \frac{b}{b} = \frac{c}{c} \), we have the arithmetic mean \( b = \frac{a+c}{2} \).
2. \( \frac{(a-b)}{(b-c)} = \frac{b}{c} = \frac{a}{b} \), we have the geometric mean \( b = \sqrt{ac} \).

3. \( \frac{(a-b)}{(b-c)} = \frac{a}{c} \); we have the harmonic mean \( b = \frac{2}{\frac{1}{a} + \frac{1}{c}} \).

4. \( \frac{(a-b)}{(b-c)} = \frac{c}{a} \); we have the contra-harmonic mean \( b = \frac{a^2 + c^2}{a + c} \).

5. \( \frac{(a-b)}{(b-c)} = \frac{a}{b} \); we have the first contra-geometric mean \( b = \frac{a - c + \sqrt{a^2 - 2ac + 5c^2}}{2} \).

6. \( \frac{(a-b)}{(b-c)} = \frac{b}{a} \); we have the second contra-geometric mean \( b = \frac{c - a + \sqrt{5a^2 - 2ac + c^2}}{2} \).

7. \( \frac{(b-c)}{(a-c)} = \frac{c}{a} ; b = \frac{2ac - c^2}{a} \). This mean is on Nicomachus’ list but not Pappus’ list.

8. \( \frac{(b-c)}{(a-c)} = \frac{c}{b} ; b = \frac{c + \sqrt{4ac - 3c^2}}{2} \).

9. \( \frac{(a-b)}{(a-c)} = \frac{c}{a} ; b = \frac{a^2 - ac + c^2}{a} \).

10. \( \frac{(a-b)}{(a-c)} = \frac{b}{a} ; b = \frac{a^2}{2a - c} \). This mean is on Pappus’ list but not Nicomachus’ list.

11. \( \frac{(a-b)}{(a-c)} = \frac{c}{b} ; b = a - c \).

Note that some of these means are not very robust definitions of means. For example, if one uses the 11th mean on our list to find the mean of 5 and 4, then \( M(5, 4) = 1 \), which is not between 5 and 4. Note also that using the 5th mean on our list to find the mean of 1 and 2, we obtain the celebrated golden number \( \Phi = \frac{1}{618} \ldots \) However, as we will show in the next section, the above list does not exhaust all the means known to the ancient Greek world.

### 2.6 Other Means of the Ancient Greeks

In this section, we point out that Greek mathematicians continued to develop new means which were never included among the classical means. Nicomachus referred to a special mean obtained by the division of a segment into what he called "the most perfect proportions". This mean, which we will call \( b \), can be expressed by the division of a segment of magnitude \( a \) into two parts: A greater part, \( b \), and a lesser part, \( a - b \), in such a fashion that the ratio of \( a \) to \( b \) is equal to the ratio
of $b$ to $a - b$. Hence the proportion:

$$a : b = b : (a - b)$$

This, in turn, leads to the quadratic equation $b^2 + ab - a^2 = 0$, The positive root of which is $b = \frac{1}{2}a\sqrt{5} - 1$. A special solution of this equation is when $a = 1, b$ is the celebrated number $\Phi$.

The mathematicians of Alexandria referred to other quantities as means. For example Heron’s mean.

**Definition 2.3.** Suppose $a$ and $c$ are positive numbers. Then Heron’s mean is

$$b = \frac{a + \sqrt{ac} + c}{3}.$$

To check that Heron’s mean of any two positive values is always between these two values, let $0 < a < c$. Then by the arithmetic-geometric mean inequality, $a < \sqrt{ac} < \frac{a + c}{2} < c$. Thus

$$a = \frac{3a}{3} < \frac{2a + \sqrt{ac}}{3} < \frac{a + \sqrt{ac} + c}{3} < \frac{2c + \sqrt{ac}}{3} < \frac{3c}{3} = c.$$

Heron’s mean is used in calculating the volume of a pyramidal frustum (a prismatoid figure formed by chopping off the top of a pyramid), where $a$ and $c$ are the bottom and top areas respectively of the pyramidal frustum.

The centroidal mean is another example of a mean produced by ancient Greek mathematics which was not included in the list of classical means. This mean was developed by Archimedes for his work on centroids.

**Definition 2.4.** Let $a$ and $c$ be two natural numbers. Then the centroidal mean of $a$ and $c$ is

$$b = \frac{2(a^2 + ac + c^2)}{3(a + c)}.$$
We shall demonstrate that the centroidal mean of two positive values is always between these two values. Let $0 < a < c$. Using the inequality from proposition 1, we have $a \leq \frac{2ac}{a+c} \leq c$; therefore,

$$\frac{a}{3} + \frac{2(a^2 + c^2)}{3(a+c)} \leq \frac{2ac}{3(a+c)} + \frac{2(a^2 + c^2)}{3(a+c)} \leq \frac{c}{3} + \frac{2(a^2 + c^2)}{3(a+c)}.$$

Now, $a < c \Rightarrow a^2 + ac < a^2 + c^2 \Rightarrow a \leq \frac{a^2 + c^2}{a+c} \Rightarrow a < \frac{a}{3} + \frac{2(a^2 + c^2)}{3(a+c)}$. A similar argument can be used to show $c \geq \frac{c}{3} + \frac{2(a^2 + c^2)}{3(a+c)}$. Hence, $a \leq \frac{2(a^2 + c^2)}{3(a+c)} \leq c$. 

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Chapter 3. Binary Means

In this chapter we will give a contemporary definition for binary means and present an overview of the development of the theory of binary means. The chapter includes an exposition of the most common types of binary means. The chapter concludes with a summary on inequalities among binary means.

3.1 The Theory of Binary Means

Based upon the sources reviewed, ancient Greek mathematics’ treatment of means tended to be limited to finding the mean of two magnitudes, be it line segments, areas, or volumes. Berlinghoff (2002) claims that this limited view on means in Greek mathematics may have stemmed from their interest in geometry, where means of magnitudes of segments, areas, and volumes are intermediate value between the two extremes. Therefore, finding the mean of more than two such magnitudes was a problem that was not encountered because such a mean would not represent intermediate value between two extremes.

In modern times, this outlook has changed. The arithmetic mean, geometric mean, and the harmonic mean came to be viewed as specific cases of a general function not of just two variables but also of $n$-variables. In this chapter we will limit our discussion to binary means and postpone our dealing with $n$-ary means to the next chapter.

Huntington (1927) cites work published by R. Schimmack in 1909 which treats means as a continuous function that satisfies given restrictions. Dodd (1933) credits B. de Finetti’s 1931 work with formulating specific criteria that a mean function must satisfy. Although in both cases, the function referred to is an $n$ variable function, a similar view may be extended to two-variable mean functions. Borwein (1987) lists postulates for a mean function of two variables, $f(a,b)$, similar to the restrictions cited by Huntington and Dodd for a mean function of $n$-variables. We will use Borwein’s (1987) definition and criteria for binary means to develop a definition for a generalized binary mean. Next we will subject the classical means to the criteria we have
developed for binary means to conclude whether or not these means can be considered means in our refined sense. We will also introduce the modern notion of power means, and show the various inequalities that relate binary means to each other. We will conclude the chapter with examples of other functions that generate binary means.

We begin by introducing the term isotone (Borwein 1987), which we will subsequently use to identify a specific property for functions of two variables.

**Definition 3.5.** Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a function. $f$ is isotone if for each $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$, $f(a,x)$ and $f(x,b)$ are monotone increasing functions of $x$.

To demonstrate that $f(a,b)$ is isotone, we fix one variable, say $a$, and show that $f(a,x)$ is monotone increasing as a function of $x$. Then we appeal to the same argument for $b$.

**Definition 3.6.** A binary mean function, $f$, is a positive real valued function, $f(a,b)$, of two strictly positive real variables $a$ and $b$ that satisfies the following postulates:

- **CR** $f$ is a continuous and real valued function.
- **IS** $f$ is a isotone.
- **IN** $f$ is internal, i.e., $\min(a,b) \leq f(a,b) \leq \max(a,b)$.
- **DI** $f$ is diagonal, i.e., $f(a,b) = a$ or $f(a,b) = b$ if and only if $a = b$.
- **HO** $f$ is homogeneous, i.e., $f(\lambda a, \lambda b) = \lambda f(a,b)$, where $\lambda \geq 0$.
- **SY** $f$ is symmetric, i.e., $f(a,b) = f(b,a)$.

Remark: Note that $HO$ permits us to write $M(a,b) = aM(1, \frac{b}{a})$, a useful result utilized in proofs of many theorems on means.

### 3.2 Classical Means as Binary Mean Functions

We will revisit our eleven classical means to explore which of these satisfy the binary mean postulates listed in definition 3.6. For the following arguments, we will assume that $a$ and $b$ are positive real numbers such that $a < b$ and $M$ is the mean.

**Proposition 2.** If $M$ is equal to $A$, $G$, or $H$, then $M$ is a binary mean function.
Proof. Suppose $0 < a \leq b$ and $M(a, b)$ is $A, G,$ or $H$. Clearly, $M$ satisfies $CR$. To show $M$ satisfies $IS$, fix $a$ and let $0 < x_1 < x_2$. Now we show $M(a, x_1) < M(a, x_2)$. If $M = A$ or $M = H$, then $M(a, x_1) < M(a, x_2)$ because $\frac{a + x_1}{2} < \frac{a + x_2}{2}$ and $\frac{2ax_1}{a + x_1} < \frac{2ax_2}{a + x_2}$ by properties of addition and multiplication of positive numbers. If $M = G$, we have $x_1 < x_2$ and $ax_1 < ax_2$. Thus, $\sqrt{ax_1} < \sqrt{ax_2}$, since the square root function is monotone increasing. Thus $M(a, x_1) < M(a, x_2)$.

A similar argument can be used to show $M(x, b)$ is monotone increasing. $M$ satisfies $IN$ by proposition 1. $M$ satisfies $DI$. This can be checked by substituting $b = a$ in the definition of $A, G,$ and $H$. $M$ satisfies $HO$ by the distributive property of multiplication over addition. $M$ satisfies $SY$ by the commutative properties of addition and multiplication. $\square$

The remaining eight classical means are not binary mean functions according to definition 3.6. To substantiate this claim, we take each in turns.

1. The contra-harmonic mean, $M = \frac{a^2 + b^2}{a + b}$, fails to satisfy $IS$. For example, $M(6, 2) = 5 = M(6, 3)$.

2. The contra-geometric mean, $M = (a - b + \sqrt{a^2 - 2ab + 5b^2})/2$ fails to satisfy $SY$. For example, $M(1, 2) = \frac{1 - 2 + \sqrt{1 - 4 + 20}}{2} = \frac{1 + \sqrt{17}}{2}$. On the other hand, $M(2, 1) = \frac{2 - 1 + \sqrt{16 - 4 + 5}}{2} = \frac{1 + \sqrt{17}}{2}$.

3. The subcontra-geometric mean, $M = (b - a + \sqrt{5a^2 - 2ab + b^2})/2$, fails to satisfy $SY$. For an example, $M(1, 4) = \frac{4 - 1 + \sqrt{5 - 8 + 16}}{2} = \frac{3 + \sqrt{13}}{2}$. On the other hand, $M(4, 1) = \frac{1 - 4 + \sqrt{80 - 8 + 1}}{2} = \frac{-3 + \sqrt{73}}{2}$.

4. $M = \frac{2ab - b^2}{a}$. $M$ fails to satisfy $SY$. For example, $M(1, 2) = \frac{4 - 4}{2} = 0$. On the other hand, $M(2, 1) = \frac{4 - 1}{2} = \frac{3}{2}$.

5. $M = \frac{b + \sqrt{4ab - 3b^2}}{2}$ also fails $SY$.

6. $M = \frac{a^2 - ab + b^2}{a}$. Clearly, this mean fails to satisfy $SY$.

7. $M = \frac{a^2}{2a - b}$. $M$ fails to satisfy $SY$. 21
8. \( M = b - a \). Clearly, \( M \) fails to satisfy \( SY \).

Therefore, of the eleven classical means, only the antique means are considered binary mean functions according to definition 3.6 for a mean.

### 3.3 Binary Power Means

Another representation for binary mean functions is known as power means. The root-mean-square (also known as the Euclidean mean), \( R(a, b) = \sqrt{\frac{a^2 + b^2}{2}} \), may have been the first example of this new class of means (Lin 1974).

**Definition 3.7.** Suppose \( r > 0 \), \( a > 0 \), and \( b > 0 \). Then the \( r^{th} \) power mean of \( a \) and \( b \), denoted \( M_r(a, b) \), is \( \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \).

**Theorem 2.** Let \( a > 0 \), \( b > 0 \), and \( r \neq 0 \). The function \( M_r(a, b) = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \) is a binary mean function.

**Proof.** \( M_r(a, b) \) satisfies \( CR \). Clearly \( M_r(a, b) \) is continuous for \( r > 0 \), \( a > 0 \) and \( b > 0 \) since it is a composition of continuous functions. \( M_r(a, b) \) satisfies \( IS \). Fix \( a \). Let \( x_1 \) and \( x_2 \) be any positive numbers such that \( 0 < x_1 < x_2 \). If \( r > 0 \), then
\[
\left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} < \frac{x_1^r + x_2^r}{2} < \frac{a^r + b^r}{2}.
\]
Therefore,
\[
\left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} < \frac{a^r + b^r}{2}.
\]
Similarly, if \( r < 0 \), we have
\[
a^r < \frac{a^r + b^r}{2} < b^r.
\]
Fix \( a < b \). If \( r > 0 \), then we have \( a < \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} < b \) since
\[
a^r < \frac{a^r + b^r}{2} < b^r.
\]
Similarly, if \( r < 0 \), we have \( a^r > \frac{a^r + b^r}{2} > b^r \) and \( a < \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} < b \). \( M_r(a, b) \) satisfies \( DI \). Suppose \( \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} = a \). Then we have \( \frac{a^r + b^r}{2} = a^r \) which implies \( b = a \). Similarly, if \( b = a \), then \( \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} = a \). Therefore, \( M_r(a, b) = a \) if and only if \( b = a \). \( M_r(a, b) \) satisfies \( HO \). Fix \( \lambda > 0 \). Then
\[
M_r(\lambda a, \lambda b) = \left( \frac{(\lambda a)^r + (\lambda b)^r}{2} \right)^{\frac{1}{r}} = \lambda \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}}.
\]
\( M_r(a, b) \) satisfies \( SY \), since
\[
M_r(a, b) = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} = \left( \frac{b^r + a^r}{2} \right)^{\frac{1}{r}} = M_r(b, a).
\]

Note that the arithmetic mean, the harmonic mean, and the root-mean-square are power mean functions by direct substitution in \( M_r = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \) with the appropriate value for \( r \):

1. \( r = 1 \), then \( M_1(a, b) \) yields the arithmetic mean \( A = \frac{a + b}{2} \).
2. \( r = -1 \), then \( M_{-1}(a, b) \) yields the harmonic mean \( H = \frac{2ab}{a+b} \).

3. \( r = 2 \), then \( M_2(a, b) \) yields the root-mean-square \( R = \sqrt{\frac{a^2+b^2}{2}} \).

We now show that the geometric mean is a limit of power mean functions.

**Theorem 3.** \( \lim_{r \to 0} M_r(a, b) = \sqrt{ab} \).

**Proof.** Observe that \( \lim_{r \to 0} \left( \frac{a^r + b^r}{2} \right)^\frac{1}{r} = \lim_{r \to 0} \exp \left\{ \left( \frac{1}{r} \right) \ln \left( \frac{a^r + b^r}{2} \right) \right\} = \exp \left\{ \lim_{r \to 0} \left( \frac{1}{r} \right) \ln \left( \frac{a^r + b^r}{2} \right) \right\} \).

Applying L’Hôpital rule, \( \lim_{r \to 0} \left( \frac{\ln(a^r + b^r)/2}{r} \right) = \lim_{r \to 0} \left( \frac{\frac{d}{dr}((\ln(a^r + b^r)/2))}{1} \right) = \lim_{r \to 0} \left( \frac{2}{a^r + b^r} \right) \left( \frac{a^r \ln a + b^r \ln b}{2} \right) = \frac{\ln a + \ln b}{2} \). Therefore, \( \exp \left\{ \lim_{r \to 0} \left( \frac{1}{r} \right) \ln \left( \frac{a^r + b^r}{2} \right) \right\} = \exp \left\{ \frac{\ln a + \ln b}{2} \right\} = \sqrt{ab} \).

**Definition 3.8.** Let \( a \) and \( b \) be any positive numbers. Then \( M_0(a, b) := \sqrt{ab} \).

With the development of this representation for means, ways had to be found to compare these means to each other and to the already established ones. This led to the establishment of some of the most well-known inequalities in mathematics.

**Theorem 4.** If \( a, b, \) and \( r \) are positive numbers, then \( M_0(a, b) \leq M_r(a, b) \). With equality holding if and only if \( a = b \).

**Proof.** Note that \( a = b \iff \sqrt{ab} = \left( \frac{a^r + b^r}{2} \right)^\frac{1}{r} \). Suppose \( a < b \), then \( M_0(a, b) = \sqrt{ab} \) and

\[
M_r(a, b) = \left( \frac{a^r + b^r}{2} \right)^\frac{1}{r}.
\]

Observe that \( \sqrt{ab} = (ab)^\frac{1}{2} \). Then \( (ab)^\frac{1}{2} = (a^r b^r)^\frac{1}{2} \) and

\[
\left[ \left( \frac{a^r + b^r}{2} \right)^\frac{1}{r} \right]^r = \frac{a^r + b^r}{2}.
\]

By the arithmetic-geometric mean inequality, \( (a^r b^r)^\frac{1}{2} < \frac{a^r + b^r}{2} \). Therefore, \( M_0(a, b) < M_r(a, b) \).}

**Theorem 5.** If \( a, b, r, \) and \( s \) are positive numbers such that \( r < s \), then \( M_r(a, b) < M_s(a, b) \).

The proof we present is a modified version of the proof given in Schaumberger (1988) for \( n \)-ary power means.
Proof. Let \( x > 0 \) and \( f(x) = rx^s + (s-r) - sx^r \). We note that \( f(x) \) has an absolute minimum only at \( x = 1 \) (since \( f'(x) = rsx^{s-1} - rsx^{r-1} = 0 \) only at \( x = 1 \) and \( f''(1) = rs(s-r) > 0 \)). Observe that \( f(1) = 0 \); therefore, \( f(x) = rx^s + (s-r) - sx^r \geq 0 \). Hence,

\[
rx^s + (s-r) \geq sx^r, \quad (3.5)
\]

with equality holding if and only if \( x = 1 \). Let \( T = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \). Put \( x_1 = \frac{a}{T} \) and \( x_2 = \frac{b}{T} \). By substituting for \( x_1 \) and \( x_2 \) in equation 3.5 successively for \( x \) and adding, we obtain

\[
r \left[ \left( \frac{a}{T} \right)^s + \left( \frac{b}{T} \right)^s \right] + 2s - 2r \geq s \left[ \left( \frac{a}{T} \right)^r + \left( \frac{b}{T} \right)^r \right].
\]

Hence, \( r \left[ \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \right] + 2s - 2r \geq s \left[ \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \right] \). But

\[
T' = \frac{a^r + b^r}{2} \text{. Therefore, } \left[ \frac{a^r + b^r}{2} \right] \geq 2. \text{ Hence } \frac{a^r + b^r}{2} \geq T^s, \text{ and this implies } \frac{a^r + b^r}{2} \geq \left( \frac{a^r + b^r}{2} \right) \frac{1}{r}. \text{ This leads to } \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \geq \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}} \geq T^s. \text{ Therefore, } M_r(a,b) \leq M_s(a,b). \]

\[\square\]

### 3.4 The Logarithmic Binary Mean

The logarithmic mean is encountered in various applications such as in investigation of heat transfer, fluid mechanics (Lin 1974), and the distribution of electrical charge on a conductor (Stolarsky 1975).

**Definition 3.9.** Let \( a > 0 \) and \( b > 0 \). Then

\[
L(a, b) = \begin{cases} 
\frac{a - b}{\ln a - \ln b} & \text{if } a \neq b \\
a & \text{if } a = b
\end{cases}
\]

**Theorem 6.** \( L \) is a binary mean.

Proof. First we prove that \( L(a, b) \) satisfies CR. Clearly, \( L(a, b) \) is continuous on \((0, \infty) \times (0, \infty)\) except maybe on the line \( a = b \). To show that \( L(a, b) \) is continuous when \( a = b \), we note first that

\[
\lim_{u \to 1} \frac{u - 1}{\ln u} = 1 \text{ by L'Hôpital's rule. Thus } \lim_{(y,u) \to (a,1)} y \frac{u - 1}{\ln u} = a; \text{ therefore, by substituting } \frac{y}{u} \text{ for } u,
\]

we have

\[
\lim_{(x,y) \to (a,a)} \left( \frac{y(x - 1)}{\ln \frac{x}{y}} \right) = a. \text{ So, } \lim_{(x,y) \to (a,a)} L(x,y) = a. \text{ Now we show } L(a,b) \text{ satisfies IS. Fix } a > 0 \text{ and let } x \in (0, \infty). \text{ Let } g(x) := L(a,x) = \frac{x - a}{\ln x - \ln a} = \frac{x}{\ln x} - \frac{a}{\ln a}. \text{ We must show that } g(x) \text{ is monotone}.
\]

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increasing. It suffices to show $g'(x) > 0$, except possibly at finitely many points. When $x \neq a$,
\[
g'(x) = \frac{\ln \frac{x}{a} - \left(1 - \frac{x}{a}\right)}{(\ln \frac{x}{a})^2};
\]
so, it suffices to show $\frac{\ln \frac{x}{a} - \left(1 - \frac{x}{a}\right)}{(\ln \frac{x}{a})^2} > 0$, except at finitely many points. Let
\[
h(u) = -\ln u - 1 + u.
\]
We need to show $h(u) > 0$, except at finitely many points. By examining $h'$, we see that $h$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Since $h(1) = 0$, we have proved what is needed. We show that $L(a,b)$ satisfies $IN$, i.e., $\min(a,b) \leq L(a,b) \leq \max(a,b)$. Let $f(x) = \ln(x)$. Then for any $0 < a < b$, by the Mean Value Theorem, there exists a $t$ in $[a,b]$ such that $f'(t) = \frac{\ln b - \ln a}{b-a}$. Therefore, $\frac{1}{t} = \frac{\ln b - \ln a}{a-b}$. Hence $t = \frac{a-b}{\ln a - \ln b}$ and $a \leq t \leq b$. That $L(a,b)$ satisfies $DI$ is evident from the definition of $L(a,b)$. We show that $L(a,b)$ satisfies $HO$. Let $\lambda > 0$. Then $L(\lambda a, \lambda b) = \frac{\lambda a - \lambda b}{\ln \lambda a - \ln \lambda b} = \frac{\lambda(a-b)}{\ln a + \ln \lambda - \ln a - \ln \lambda} = \lambda L(a,b)$. We show that $L(a,b)$ satisfies $SY$. $L(a,b) = \frac{a-b}{\ln a - \ln b} = \frac{-(b-a)}{-(\ln b - \ln a)} = \frac{b-a}{\ln b - \ln a} = L(b,a)$. □

The following theorem establishes an inequality between $L$, $A$ and $G$.

**Theorem 7.** If $a > 0$ and $b > 0$ such that $a \neq b$, then $G(a,b) < L(a,b) < A(a,b)$.

The following proof was given by Carlson (1972)

**Proof.** If $t > 0$, the inequality of the arithmetic and geometric mean implies that
\[
t^2 + t(a+b) + \left(\frac{a+b}{2}\right)^2 > t^2 + t(a+b) + ab > t^2 + 2t(ab)^{\frac{1}{2}} + ab.
\]
Thus
\[
\int_0^\infty \frac{dt}{(t+2)^{\frac{3}{2}}} < \int_0^\infty \frac{dt}{(t+a)(t+b)} < \int_0^\infty \frac{dt}{(t+\sqrt{ab})^2}.
\]
Evaluating the middle integral by the method of partial fractions, we find
\[
\frac{2}{a+b} < \frac{1}{a-b} \lim_{R \to \infty} |\ln(t+b) - \ln(t+a)|_R^R < \frac{1}{\sqrt{ab}}.
\]
This implies
\[
\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}.
\]
□

Based upon the results obtained above, we have the following inequality that relates the harmonic mean, geometric mean, logarithmic mean, arithmetic mean, and the root-mean-square.

**Corollary 3.10.** Let $a > 0$ and $b > 0$ such that $a > b$, then $H \leq G \leq L \leq A \leq R$.

It is interesting to note that the logarithmic mean does not quite lend itself to a natural generalization to $n$ variables (Pittenger 1985). This mean fails a particular axiom (namely the associativity axiom) for $n$-ary means. However, due to the use of this $n$-ary mean in various applications such as in defining average temperatures and analysis of index numbers in
economics, a theoretical framework for the generalization of the logarithmic mean of \( n \) variables has been recently developed. We refer the interested reader to the work of Pittenger (1985) for more information on \( n \)-ary logarithmic means.

### 3.5 Representation of Links between Binary Means

Eves (2003) gives an excellent geometric link between various binary means using a trapezoid. Let \( a > b > 0 \). Suppose a trapezoid has parallel sides \( a \) and \( b \) as shown in figure 3. The various means can be ranked in size relative to each other as the lengths of vertical segments. The segment whose length is:

- The harmonic mean, \( H \), passes through the intersection of the diagonals.
- The geometric mean, \( G \), divides the trapezoid into two similar trapezoids.
- The Heronian mean, \( N \), is one third of the way from the arithmetic mean to the geometric mean.
- The arithmetic mean, \( A \), bisects the sides of the trapezoid.
- The centroidal mean, \( T \), passes through the centroid of the trapezoid.
- The root-mean-square, \( R \), bisects the area of the trapezoid.

![Figure 3.3. Binary Means as Parts of a Trapezoid](image-url)
• The contra-harmonic mean, $C$, is as far to the right of the arithmetic mean as the harmonic mean is to the left of it.

### 3.6 Other Binary Means

Interest in generating different binary means functions continued to grow into the late 20th century as other functions of two variables were found that satisfy given criteria for a desired mean function. Borwein (1987) defined a class of binary mean functions, $M_p(a, b)$, that is derived from a mean function, $M(a, b)$, that satisfies the postulates given in Section 3.1. This class of binary means is determined by the formula

$$M_p(a, b) := \frac{M(a^p, b^p)}{M(a^{p-1}, b^{p-1})},$$

where $p \in R$. We refer the reader to Borwein (1987) for the proof that $M_p(a, b)$ satisfies the postulates given in Section 3.1.

Example of such binary means include (Borwein 1987):

• Lehmer means. Let $a, b > 0$ and $p \in R$. Then Lehmer means, $L_p$, is defined as

$$L_p(a, b) = \frac{a^p + b^p}{a^{p-1} + b^{p-1}}.$$

Observe that $L_1 = A$ and $L_\frac{1}{2} = G$.

• Gini means. Let $a, b > 0$ and $r \neq s$. Then Gini mean, $G_{(s, r)}(a, b)$, is defined as

$$G_{(s, r)}(a, b) = \left(\frac{a^s + b^s}{a^r + b^r}\right)^{\left(\frac{1}{s-r}\right)}$$

• Stolarsky’s Means. Let $a, b > 0$ and $p \neq 0, 1$. Then the Stolarsky’s Mean, $S_p(a, b)$, is defined as

$$S_p(a, b) = \left(\frac{a^p + b^p}{p(a - b)}\right)^{\left(\frac{1}{p-1}\right)}$$
Observe that $S_0(a,b) = \lim_{p \to 0} S_p(a,b) = \frac{b-a}{\ln b - \ln a}$, which is the logarithmic mean. And

\[ S_1(a,b) = \lim_{p \to 1} S_p(a,b) = e^{-1}(a^p b^{-p})^{\frac{1}{p-1}}, \] which is also known as the identric mean.

We refer the interested reader to Borwein (1987) for more information on the means listed above.

Mays (1983) investigated conditions under which binary means can be associated with a single variable function. In this work, Mays developed an idea presented by Moskovitz (1933). Mays also pointed out some errors contained in Moskovitz (1933). Given a function $f$ from $(0, \infty)$ into $\mathbb{R}$, Mays (1983) defines $M_f(a,b)$ to be the $X$-intercept of the line connecting $(a, f(a))$ and $(b, -f(b))$. See Figure 3.4 (reproduced from Mays (1983)). Clearly $M_f(a,b)$ satisfies $IN$ and $SY$.

We find a formula for $M_f(a,b)$ by calculating the slope of the line through $(a,f(a))$ and $(b,-f(b))$ in two ways:

\[ \frac{f(a)}{a-M_f(a,b)} = \frac{f(b)}{b-M_f(a,b)}. \]

Solving for $M_f(a,b)$, we get:

\[ M_f(a,b) = \frac{af(b) + bf(a)}{f(a) + f(b)}. \] (3.6)

**Theorem 8.** $M_f = M_g \iff g = kf$ for some $k > 0$

**Proof.** If $g = kf$, $k$ cancels in the right hand side of $M_{kf}(a,b) = \frac{af(b) + bf(a)}{kf(a) + kf(b)} = M_g = M_f$.

If $g \neq kf$, pick $a$, $b$, and $k$ so that $g(a) = kf(a)$ but $g(b) \neq kf(b)$. Then if $M_f(a,b) = M_g(a,b)$, we have $\left[ \frac{af(b) + bf(a)}{f(a) + f(b)} \right] = \left[ \frac{ag(b) + bkf(a)}{kf(a) + g(b)} \right]$. Therefore, $af(a)(kf(b) - g(b)) = bf(a)(kf(b) - g(b))$.

Since $kf(b) - g(b) \neq 0$ and $f(a) \neq 0$, then $a = b$, a contradiction. \qed

**Corollary 3.11.** If $M = M_f$, then there exists $\overline{f}$ such that $M = M_{\overline{f}}$ and $\overline{f} = 1$.

**Proof.** Let $\overline{f} = \frac{f(x)}{f(1)}$. \qed

Corollary 3.11 allows us to assume, without loss of generality, when associating $M_f$ with a given function $f$ that $f(1) = 1$.

**Definition 3.12.** Let $f$ be a function in one variable. $f$ is multiplicative if the domain of $f$ is closed under multiplication and $f(xy) = f(x)f(y)$ for every $x, y$ in the domain of $f$. 28
Lemma 3.13. Suppose \( f : (0, \infty) \to \mathbb{R} \) and \( f(1) = 1 \). Then \( f \) is multiplicative if and only if

\[
f(a)f(\lambda b) = f(\lambda a)f(b)
\]

for all \( a, b, \) and \( \lambda \in (0, \infty) \).

Proof. Note that the condition 3.7 implies \( f(\lambda b) = f(\lambda)f(b) \) for all \( \lambda, b \in (0, \infty) \). Conversely, if \( f \) is multiplicative, then \( f(a)f(\lambda b) = f(a)f(\lambda)f(b) = f(\lambda a)f(b) \).

Theorem 9. Suppose \( f(1) = 1 \). Then \( M_f \) is homogeneous if and only if \( f \) is multiplicative.

Proof. By equation 3.6, \( \forall \ a, \ b, \ \lambda > 0 \)

\[
\lambda M_f(a, b) = M_f(\lambda a, \lambda b) \iff \lambda \left( \frac{af(b) + bf(a)}{f(a) + f(b)} \right) = \frac{\lambda af(\lambda b) + \lambda bf(\lambda a)}{f(\lambda a) + f(\lambda b)} \iff
\]

\[
af(b)f(\lambda a) + af(b)f(\lambda b) + bf(a)f(\lambda a) + bf(a)f(\lambda b) =
\]

\[
af(a)f(\lambda b) + bf(a)f(\lambda a) + af(b)f(\lambda b) + bf(b)f(\lambda a) \iff
\]

\[
af(b)f(\lambda a) + bf(a)f(\lambda b) = af(a)f(\lambda b) + bf(b)f(\lambda a) \iff
\]

\[
(a - b)f(b)f(\lambda a) = (a - b)f(a)f(\lambda b) \iff f(b)f(\lambda a) = f(\lambda b) \iff f \) is multiplicative. \qed
We now explore some ideas that are motivated by Mays (1983). Let $M$ be a function of two variables (not necessarily a mean). We define

$$F_M(x,y) := \frac{M(x,y) - y}{x - M(x,y)}.$$  

Similarly, if $F$ is a function of two variables, we define

$$M_F(x,y) := \frac{xF(x,y) + y}{1 + F(x,y)}.$$  

Note that if $M$ is a mean, $F$ is the kind of ratio that was considered by the Greeks in developing the classical means.

**Theorem 10.** $F_M = F$ and $M_F = M$ as functions on $\{(x,y) \mid x \neq y\}$. If we choose $M$ and $F$ such that $M_F = M$ and $F_M = F$, then:

1. $M$ is homogeneous if and only if $F$ is “projective”, i.e., $F(x,y) = F(\lambda x, \lambda y)$.

2. $M$ is intermediate if and only if $F$ is positive.

3. $M$ is symmetric if and only if $F(x,y)F(y,x) = 1$.

**Proof.** We show $F_M = F$. $F_M = M_F = \frac{M(x,y) - y}{x - M(x,y)} = \frac{xF(x,y) - y}{x + xF(x,y) - y} = \frac{(x-y)F}{(x-y)} = F$. We show $M_F = M$. $M_F = M_F = \frac{M_F(x,y) + y}{1 + F(x,y)} = \frac{x(F(x,y) + y)}{x + M_F(x,y) - y} = \frac{(x-y)M}{(x-y)} = M$. Now we show that $M$ is homogeneous if and only if $F(x,y) = F(\lambda x, \lambda y)$. Suppose $M$ is homogeneous. Then

$$F(\lambda x, \lambda y) = M(\lambda x, \lambda y) = \frac{\lambda M(x,y) - \lambda y}{\lambda x - M(x,y)} = \frac{\lambda M(x,y)}{\lambda x - M(x,y)} = \frac{M(x,y) - y}{x - M(x,y)} = F_M(x,y).$$

Now suppose $F(x,y) = F(\lambda x, \lambda y)$. $M(\lambda x, \lambda y) = \frac{\lambda x F(\lambda x, \lambda y) + \lambda y}{1 + F(\lambda x, \lambda y)} = \lambda \left( \frac{x F(x,y) + y}{1 + F(x,y)} \right) = \lambda M(x,y)$. We now show $M$ is intermediate if and only if $F > 0$. $x(1 + F) > xF + y > y$ and $\frac{M - y}{x - M} > 0 \iff m$ is between $x$ and $y$.

To show $M$ is symmetric if and only if $F(x,y)F(y,x) = 1$, let $F = F(x,y)$ and $\overline{F} = F(y,x)$.

Observe that $M_F$ is symmetric $\iff \frac{\overline{F} + y}{1 + \overline{F}} = \frac{\overline{F} + x}{1 + \overline{F}} \iff xF + yF + y\overline{F} + x\overline{F} + \overline{F} = yF + x + y\overline{F} + Fx \iff x + yF\overline{F} = y + xF\overline{F} \iff (x-y) = (x-y)F\overline{F} \iff F\overline{F} = 1$. \qed
Mays addressed the problem of when a given binary mean $m$ can be expressed as $M_f$ for some $f(x)$. We now use $F$ and $M$ to give a more general solution to the problem than Mays’.

**Proposition 3.** Suppose $M$ is any function of two variables $x$ and $y$ such that $x \neq y$. Then $M = M_f$ if and only if

$$F_M(x,y) = \frac{f(y)}{f(x)}.$$ 

**Proof.** Suppose $M = M_f$. Then $m = \frac{xF(x,y) + y}{1 + F(x,y)} = \frac{xf(x) + yf(y)}{f(x) + f(y)}$. Therefore,

$$xf(x)F(x,y) + yf(x) + xf(y)F(x,y) + yf(y) = yf(x)f(x,y) + yf(x) + xf(y)F(x,y) + xf(y),$$

and $xf(x)F(x,y) + yf(y) = yf(x)F(x,y) + xf(y)$. Hence, $(x - y)f(x)F(x,y) = (x - y)f(y)$, which implies $F(x,y) = \frac{f(y)}{f(x)}$. Conversely, we can show if $F(x,y) = \frac{f(y)}{f(x)}$, then

$$M_f = \frac{xF(x,y) + y}{1 + F(x,y)} = \frac{xf(x) + yf(y)}{f(x) + f(y)}$$

by using a similar argument to the one given above. Therefore, if we let $M = M_f$, we have proved what is needed. □
Chapter 4. \( n \)-ary Means

The chapter begins with a brief discussion of the historical development of \( n \)-ary means. Next we present an overview of postulates of \( n \)-ary means, starting with postulates for the arithmetic mean and follow their evolution into postulates for generalized means. We discuss the translation invariance property for \( n \)-ary means. We present examples of various types of \( n \)-ary means. We conclude the chapter with a brief discussion of the theory of inequalities among \( n \)-ary means.

4.1 Historical Overview

One of the earliest known references concerning the arithmetic mean of several numbers is given by Iamblichus in a treatise on what we call now number theory. In this work, "The Theology of Arithmetic," Iamblichus outlines an example involving finding the arithmetic mean of the numbers 1 to 9:

In the first place, we must set out in a row the sequence of numbers from the monad up to nine: 1, 2, 3, 4, 5, 6, 7, 8, 9. Then we must add up the amount of all of them together, and since the row contains nine terms, we must look for the ninth part of the total to see if it is already naturally present among the numbers in the row; and we will find that the property of being [one] ninth [of the sum] only belongs to the [arithmetic] mean itself (Heath, 1921, p.82).

As we have mentioned in Chapter 1, in the middle ages the term average referred to the equal apportionment of a loss or expense incurred by a ship (or its cargo), in which case the individual compensation made by the owners (or insurers) of a ship or its cargo is in proportion to the value of their respective interests. This notion of average represents the most documented usage of the arithmetic mean during that period.

In the 17th century, astronomers were making several observations of specific cosmic events for confirmation purposes. They were faced with the problem of combining observations to come up with a single value that best represented the true value of the quantity being measured. Hald
(1998) states that for a long period of time the usual practice [by astronomers] for estimating the true value was to select the best among several observations of the same object, the best being defined by such criteria as the occurrence of good observational conditions, the exertion of special care, and so on. Gradually, however, it became common practice to use the arithmetical mean as an estimate of the true value. No theoretical foundation for this practice seems to have existed before the works of Simpson and Lagrange in the mid 18th century.

The problem of finding the best estimate of an unknown parameter from a set of \( n \) direct observations of that parameter may be very difficult. It depends on the distribution of the parameter, and (as Gauss showed) only when the distribution is normal is the arithmetic mean in every case the best estimate. (The precise sense of "best" is itself a complex problem that we shall avoid addressing). Therefore, the arithmetic mean is not always the best choice for averaging a set of observational data. Stevens (1955) states that in choosing a method of averaging physical magnitudes, one fundamental issue to be considered is the natural method of combining them.

Where magnitudes are naturally combined by taking sums, the arithmetic mean is meaningful and may be useful. However, where positive magnitudes are naturally combined by taking products, the geometric average may be the most appropriate to use.

To establish a general framework for the presentation of the various means and the postulates on \( n \)-ary means, we introduce the following:

**Convention 4.14.** Let \( X \subseteq R \), where \( R \) is the set of real numbers and \( i \in N \). We consider a sequence of functions \( f_i : X^i \to R \). For convenience, we sometimes write \( f(a, b, \ldots, c) \) letting \( f \) represent the appropriate \( f_i \). We call \( X \) the domain of \( f \).

It will be useful for us at this point to define the analogues for the most common means encountered:

**Definition 4.15.** The arithmetic mean, \( A = \{A_i\}_{i \in N} \) is:

\[
A(x_1, x_2, \ldots, x_n) = A = \frac{x_1 + x_2 + \ldots + x_n}{n}.
\]
• **The geometric mean,** $G$ is:

\[ G(x_1, x_2, \ldots, x_n) = G = (x_1 x_2 \cdots x_n)^{\frac{1}{n}}. \]

• **The harmonic mean,** $H$ is:

\[ H(x_1, x_2, \ldots, x_n) = H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} = \frac{1}{A(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})}. \]

• **The root-mean-square,** $R$, of the sequence is:

\[ R(x_1, x_2, \ldots, x_n) = R = \left( \frac{1}{n} (x_1^2 + x_2^2 + \cdots + x_n^2) \right)^{\frac{1}{2}}. \]

**Remark 4.16.** Note that $A$ and $R$ are defined for $X = R$. While $G$ is defined only for $X = R_{\geq 0} := \{ x \in R \mid x \geq 0 \}$, and $H$ is defined for $X = R_{> 0} := \{ x \in R \mid x > 0 \}$.

The first three means shown above are clearly similar to the classical means, while the root-mean-square is an obvious generalization of the binary root-mean-square. The arithmetic mean and the root-mean-square are widely used in mechanics (as in the definitions of the center of gravity and radius of gyration), and in the modern theory of statistics. The root-mean-square of the differences of some variable from its arithmetic mean is the standard deviation\[ \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}. \]

The geometric mean is used in the construction of index numbers in economics. The harmonic mean is little used, except in special investigations (Huntington 1927).

A natural generalization of these means is referred to as power means, sometimes known as Cauchy means (Bullen 1928):
Definition 4.17. Let \( r \in R \setminus \{0\} \). The \( r \)th power mean is:

\[
M_r = \left[ \frac{1}{n} \sum_{i=1}^{n} x_i^r \right]^{\frac{1}{r}}.
\]

Remark 4.18. Note that the domain of \( M_r \) always contains \( \{ x \in R \mid x > 0 \} \), but in some cases it is larger. Unless specified otherwise, we will take \( M_r \) to refer to a sequence of functions with domain \( \{ x \in R \mid x > 0 \} \). Thus \( M_r \) satisfies axiom PO (see below).

In the cases \( r = -1, 1, \) and \( 2 \), \( M_r \) is the harmonic mean, the arithmetic mean and the root-mean-square respectively. Although \( r \) is defined to be nonzero, the following theorem establishes that the geometric mean is a limiting case of \( M_r \) as \( r \) tends to 0.

Theorem 11. Let \( x_1, \ldots, x_n \) be positive real numbers. Then

\[
\lim_{r \to 0} M_{(x_1, x_2, \ldots, x_n)} = (x_1 x_2 \ldots x_n)^{\frac{1}{r}}.
\]

The following proof is from Burrows (1986).

Proof. Let \( y(r) = \frac{1}{n} \sum_{i=1}^{n} x_i^r \), where \( r \neq 0 \). Then \( y'(r) = \frac{1}{n} \sum_{i=1}^{n} x_i^r \ln x_i \). By the Mean Value Theorem \( y(r) = y(0) + ry'(\theta) \), where \( 0 < \theta < r \). Hence \( y(r) = 1 + ry'(\theta) \). Now,

\[
\frac{1}{r} \ln y(r) = \frac{1}{r} \ln(1 + ry'(\theta)) = y'(\theta) + o(r) = \frac{1}{r} (ry'(\theta) + o(r^2))
\]

\[
= \lim_{r \to 0} \frac{1}{r} \ln y(r)
\]

\[
= y'(0) = \frac{1}{n} \sum_{i=1}^{n} \ln x_i.
\]

Since \( y'(r) \) is continuous. Therefore, taking the antilogarithm of this last result we get

\[
\lim_{r \to 0} y(r) = (x_1 x_2 \ldots x_n)^{\frac{1}{r}}.
\]

\( \square \)
4.2 The Axiomatic Theory of $n$-ary Means

Beginning around 1900, several authors took up the problem of finding and analyzing axiomatic characteristics for various $n$-ary means. The following postulates appear in the various postulate systems we have reviewed, and, thus, we give them special labels for convenience. We demand the equations to be true when the terms are defined, i.e. when all arguments belong to $X$.

PO The domain of $f$ is $X = \left\{ x \in R \mid x > 0 \right\}$ and $f_i(x_1, x_2, \ldots, x_i) > 0$.

SY $f$ is symmetric, i.e., it is independent of the order in which the $n$ quantities $x_1, x_2, \ldots, x_n \in X$, are taken, i.e.,

$$f(x_1, x_2, \ldots, x_i, x_j, \ldots, x_n) = f(x_1, x_2, \ldots, x_j, x_i, \ldots, x_n).$$

DI $f$ is diagonal, i.e., $f(a, a, a, \ldots, a) = a$.

IN $f$ is internal, i.e., $a \leq f(x_1, \ldots, x_n) \leq b$ if $a \leq x_i \leq b$ for all $i$.

HO $f$ is homogeneous, i.e., for all $k$,

$$f(kx_1, kx_2, kx_3, kx_4, \ldots, kx_n) = kM(x_1, x_2, \ldots, x_n), \text{ where } x_i \subseteq X.$$  

OD $f$ is odd, i.e., $f(-x_1, \ldots, -x_n) = -f(x_1 \ldots x_n)$ (Note that this a special case of HO).

TR $f$ is translation invariant, i.e, $f(k + x_1, \ldots, k + x_n) = k + f(x_1 \ldots x_n)$ for any $k$.

AS $f$ is "associative" in the sense that $f(x_1, x_2, \ldots, x_n) = f(f_i, \ldots, f_i, x_{i+1}, \ldots x_n)$, where $f_i = f(x_1, \ldots x_i)$.

AS$_2$ $f(x_1, x_2, x_3, x_4, \ldots, x_n) = M(m, m, x_3, x_4, \ldots, x_n)$, where $m = f(x_1, x_2)$.

The earliest approach to the theory of means by using the postulation method is Schimmack (1909 p. 128). He gave a set of axioms that completely characterize the arithmetic mean of $n$ positive numbers. Specifically, he proved the following theorem (Schimmack 1909):

**Theorem 12.** Let $f$ be a sequence of functions such that $f$ satisfies TR, OD, SY, and AS. Then $f$ is the arithmetic mean.

We refer the reader Schimmack (1909) for an elegant proof of the above theorem. Beetle (1915) established the complete independence of Schimmack’s postulates. Beetle (1915) states, "The
The notion of complete independence is much more restrictive than the requirement of independence. The requirement for the latter is that no one property is a logical consequence of any of the others. However, these properties are not necessarily devoid of interrelations. For example, it may well be that non-possession of one property implies possession of another. Complete independence implies neither any one of them, nor its negative, is a logical consequence of any combination formed by the others and their negatives." To show the complete independence of Schimmack’s four postulates, Beetle (1915) proved the existence of $2^4$ types of systems, $f$, each defined and real valued for all real values of its arguments, in which at least one system possess any given combination of the properties but does not possess the remaining properties.

Grattan-Guiness (2000) refers to Huntington as one of the major American postulationists whose main mathematical interest was developing axiomatic systems for various mathematical concepts and establishing their consistency, independence, completeness, and equivalence. Huntington (1927) extended Schimmack’s work by considering functions $f$ that satisfy the general postulates given below. He established the independence of these postulates in a manner similar Beetle’s (1915) method in establishing the independence of Schimmack’s postulates.

**Definition 4.19.** Huntington’s general postulates are: PO, HO, DI, SY, and AS$_2$. We call $f$ a Huntington mean if it satisfies Huntington’s general postulates.

Huntington concerned himself with the arithmetic mean, geometric mean, the harmonic mean and the root-mean-square. We will generalize some of his results to power means, which we defined earlier.

**Theorem 13.** Suppose $M_r$ is the $r^{th}$ power mean, where $r$ is a nonzero real number. Then $M_r$ is a Huntington mean.

**Proof.** $M_r$ satisfies PO, since each $x_i$ is positive.

$M_r$ satisfies SY by the commutative law of addition.
Mr satisfies HO. Let $k > 0$. Then $M_r(kx_1, kx_2, \ldots, kx_n) = \left[ \frac{k' x'_1 + k' x'_2 + \cdots + k' x'_n}{n} \right]^\frac{1}{k} = k \left[ \frac{x'_1 + x'_2 + \cdots + x'_n}{n} \right]^\frac{1}{k} = k M_r.$

$M_r$ satisfies DI, since $M_r(a, a, \ldots, a) = a.$

$M_r$ satisfies AS$_2$. Suppose $M_r(x_1, x_2) = m.$ Then $m = \frac{x_1^p + x_2^p}{2}.$ Therefore,

$$M_r(x_1, x_2, x_3, x_4, \ldots, x_n) = \left[ \frac{x'_1 + x'_2 + \cdots + x'_n}{n} \right]^\frac{1}{t} = \left[ \frac{\frac{x'_1 + x'_2 + \cdots + x'_n}{2} + x_3 + \cdots + x_n}{n} \right]^\frac{1}{t} = \left[ \frac{m + m + x'_3 + \cdots + x'_n}{n} \right]^\frac{1}{t} = M_r.$$  

\[ \square \]

**Theorem 14.** $G$ is a Huntington mean.

**Proof.** $G = (x_1 x_2 \ldots x_n)^\frac{1}{n}$. Then $G$ satisfies PO, since the product and powers of positive numbers are positive. $G$ satisfies SY, since multiplication is commutative. To show $G$ satisfies HO, let $k > 0$. Then $((kx_1)(kx_2)\ldots(kx_n))^\frac{1}{n} = (k^n(x_1 x_2 x_3 \ldots x_n))^\frac{1}{n} = k(x_1 x_2 x_3 \ldots x_n)^\frac{1}{n} = k G$. $G$ satisfies DI, since $G(a, a, \ldots, a) = a$. $G$ satisfies AS$_2$. Suppose $G_2 = (x_1 x_2)^\frac{1}{2} = m \Rightarrow x_1 x_2 = m^2$. Then $G = (x_1 x_2 x_3 \ldots x_n)^\frac{1}{n} = G = (x_1 x_2 x_3 \ldots x_n)^\frac{1}{2} = (m^2 x_3 \ldots x_n)^\frac{1}{n}$

As we have mentioned earlier, Huntington (1927) established several other properties that completely characterize each of the four means, $A$, $G$, $H$, and $R$. The following theorem summarizes some of the results that Huntington presented.

**Theorem 15.** Let $f$ be a Huntington mean. Then:

a) $f = A$ if and only if

$$f(1 - x_1, 1 - x_2, \ldots, 1 - x_n) = 1 - f(x_1, x_2, \ldots, x_n).$$

b) $f = H$ if and only if

$$f(\frac{x_1}{x_1 - 1}, \frac{x_2}{x_2 - 1}, \ldots, \frac{x_n}{x_n - 1}) = \frac{f(x_1, x_2, \ldots, x_n)}{f(x_1, x_2, \ldots, x_n) - 1}.$$  

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c) \( f = G \) if and only if
\[
f\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) = \frac{1}{f(x_1, x_2, \ldots, x_n)}.
\]

d) \( f = R \) if and only if
\[
f((1-x_1^2)^{\frac{1}{r}}, (1-x_2^2)^{\frac{1}{r}}, \ldots, (1-x_n^2)^{\frac{1}{r}}) = (1 - (f(x_1, x_2, \ldots, x_n))^r)^{\frac{1}{r}}.
\]

The following theorem generalizes parts (a), (b), and (d) of theorem 15.

**Theorem 16.** Let \( f \) be a Huntington mean. Suppose \( r \neq 0 \). Then \( f = M_r \) if and only if
\[
f((1-x_1^r)^{\frac{1}{r}}, (1-x_2^r)^{\frac{1}{r}}, \ldots, (1-x_n^r)^{\frac{1}{r}}) = (1 - (f(x_1, x_2, \ldots, x_n))^r)^{\frac{1}{r}}.
\]

for all \( 0 < x_i < 1 \).

**Proof.** First we prove that if \( f = M_r \), then \( M_r \) satisfies equation 4.8. We have
\[
M_r((1-x_1^r)^{\frac{1}{r}}, (1-x_2^r)^{\frac{1}{r}}, \ldots, (1-x_n^r)^{\frac{1}{r}}) = \left(\frac{((1-x_1^r)^{\frac{1}{r}} + (1-x_2^r)^{\frac{1}{r}} + \cdots + (1-x_n^r)^{\frac{1}{r}})^r}{n}\right)^{\frac{1}{r}}
\]
\[
= \left(1 - \frac{x_1^r + x_2^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}} = (1 - (M_r(x_1, x_2, \ldots, x_n))^r)^{\frac{1}{r}}.
\]

To prove the converse, assume equation 4.8. First, we show \( f(a, b) = M_r(a, b) \).
\[
f(a, b) = (a^r + b^r)^{\frac{1}{r}} f\left(\frac{a}{(a^r + b^r)^{\frac{1}{r}}}, \frac{b}{(a^r + b^r)^{\frac{1}{r}}}\right)
\]
(by HO)
\[
= (a^r + b^r)^{\frac{1}{r}} f\left(1 - \frac{b^r}{(a^r + b^r)^{\frac{1}{r}}}, 1 - \frac{a^r}{(a^r + b^r)^{\frac{1}{r}}}\right)
\]
(by algebraic identities)
\[
= (a^r + b^r)^{\frac{1}{r}} \left(1 - \left\{f\left(\frac{b}{(a^r + b^r)^{\frac{1}{r}}}, \frac{a}{(a^r + b^r)^{\frac{1}{r}}}\right)\right\}^r\right)^{\frac{1}{r}}
\]
(by 4.8.3.6.)
\[
= ((a^r + b^r) - f(b, a)^r)^{\frac{1}{r}}
\]
(by HO & simple manipulation).
Thus \( f(a, b)^r = (a^r + b^r) - f(b, a)^r \). So \( 2f(a, b)^r = (a^r + b^r) \), and \( f(a, b)^r = \frac{a^r + b^r}{2} \). Therefore,

\[
f(a, b) = \left( \frac{a^r + b^r}{2} \right)^{\frac{1}{r}}.
\]

Now we prove that \( f = M_r \) for any number of arguments. Suppose that \( x_1, x_2, \ldots, x_n \) are given. We claim that for \( k = 1, \ldots, n \) there is a \( q \) such that

\[
x_1^r + x_2^r + \ldots + x_k^r - (k - 1)q^r > 0.
\]

To prove the claim, note that \( x_1^r + x_2^r + \ldots + x_k^r - (k - 1)q^r > 0 \) for \( k = 2, \ldots, n \) if and only if \( x_1^r + x_2^r + \ldots + x_k^r > (k - 1)q^r \) if and only if \( \frac{x_1^r + x_2^r + \ldots + x_k^r}{k-1} > q^r \), for \( k = 2, \ldots, n \)

\[
\text{if and only if } \begin{cases} \left( \frac{x_1^r + x_2^r + \ldots + x_k^r}{k-1} \right)^{\frac{1}{r}} > q \quad \text{when } r > 0 \\ \left( \frac{x_1^r + x_2^r + \ldots + x_k^r}{k-1} \right)^{\frac{1}{r}} < q \quad \text{when } r < 0 \end{cases}
\]

for \( k = 2, \ldots, n \).

So it is only necessary to pick \( q \) satisfying finitely many inequalities. The claim is thus proved.

Now,

\[
M_r(q, (x_1^r + x_2^r - q^r)^{\frac{1}{r}}) = M_r(x_1, x_2).
\]

Let \( Z_k = (x_1^r + x_2^r + \ldots + x_k^r - (k - 1)q^r)^{\frac{1}{r}} \). Our choice of \( q \), ensures that \( Z_k \) is the \( r^{th} \) root of a positive number. Also, from \( f(a, b) = M_r(a, b) \), we have \( f(x_1, x_2) = f(q, Z_2) \), and in general

\[
f(Z_k, x_{k+1}) = f(q, Z_{k+1}) \text{, for } k = 1, \ldots, n - 1.
\]

So, from AS2, we have
\[ f(x_1, x_2, \ldots, x_n) = f(q, Z_2, x_3, \ldots, x_n) \]
\[ = f(q, q, Z_3, x_4, \ldots, x_n) \]
\[ = \ldots \]
\[ = f(q, q, q, \ldots, q, Z_n) \]  \hspace{1cm} (4.9)

Now, put \( a = M_r(x_1, \ldots, x_n) \). Then

\[ a = f(a, a, \ldots, a) \]  \hspace{1cm} by DI
\[ = f(q, q, \ldots, q, (na^r - (n-1)q^r)^{\frac{1}{r}}) \]  \hspace{1cm} by 4.9 and \( x_i = a \)
\[ = f(x_1, x_2, \ldots, x_n) \]  \hspace{1cm} by 4.9

\[ \square \]

Now we give a proof part (c) of theorem 15.

**Proof.** First, we show that

\[ G\left( \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \frac{1}{G(x_1, x_2, \ldots, x_n)}. \]  \hspace{1cm} (4.10)

We have \( G(x_1, x_2, x_3, \ldots, x_n) = (x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}} \). Therefore,

\[ G\left( \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \left( \frac{1}{x_1 x_2 \ldots x_n} \right)^{\frac{1}{n}} \]
\[ = \left( \frac{1}{x_1 x_2 \ldots x_n} \right)^{\frac{1}{n}} \]
\[ = \frac{1}{(x_1 x_2 \ldots x_n)^{\frac{1}{n}}} \]
\[ = \frac{1}{G(x_1, x_2, x_3, \ldots, x_n)}. \]
Now we prove the converse. Suppose
\[ f \left( \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \frac{1}{f(x_1, x_2, \ldots, x_n)}. \] (4.11)

Let \( a \) and \( b \) be positive numbers. Using equation 4.11 for \( n = 2 \), we have
\[ f \left( \frac{1}{a}, \frac{1}{b} \right) = \frac{1}{f(a, b)} \]

Multiplying by \( ab \) and using \( HO \), we get
\[ f(b, a) = f \left( \frac{ab}{a}, \frac{ab}{b} \right) = \frac{ab}{f(a, b)} \]

But by \( SY \) and \( HO \), the left hand side is \( f(a, b) = f(b, a) \). Thus \( f(a, b)^2 = ab \) and
\[ f(a, b) = (ab)^{\frac{1}{2}}. \] (4.12)

\[
\begin{align*}
f(x_1, x_2, x_3, \ldots, x_n) &= f(1, x_1, x_2, x_3, \ldots, x_n) \\
&= f(1, 1, x_1, x_2, x_3, \ldots, x_n) \\
&= \ldots \\
&= f(1, 1, \ldots, 1, x_1, x_2, \ldots, x_n). \\
\end{align*}
\]

Set \( a = (x_1 x_2 \ldots x_n)^{\frac{1}{n}} \) and let each of the \( x_i = a \), we see that \( f(a, a, \ldots, a) = f(1, 1, \ldots, 1, a^n) = a \), by \( DI \). Hence
\[
f(x_1, x_2, x_3, \ldots, x_n) = (x_1 x_2 x_3 \ldots x_n)^{\frac{1}{n}}.
\]
Another milestone work on the properties of generalized \( n \)-ary means that is contemporary with Huntington (1927) is Nagumo (1929). Nagumo proved the following theorem:

**Theorem 17.** Suppose \( M \) is a sequence of functions of real numbers satisfying the postulates: SY, DI, AS, and IN in addition, the property:

For any \( x_1 < x_2 \), we demand \( x_1 < M(x_1, x_2) < x_2 \) (a strengthening of IN).

Then \( M \) is of the form:

\[
M(x_1, x_2, \ldots, x_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \frac{\varphi(x_i)}{n}\right),
\]

where \( \varphi(x) \) is a continuous monotone increasing function with inverse \( \varphi^{-1} \).

Dodd (1934) proved the complete independence of the postulates for means of the type given by Nagumo (1929).

### 4.3 Translation Invariance Property of \( n \)-ary Means

Hoehn & Niven (1985) proved that certain means other than \( A \), while failing to satisfy \( TR \), satisfy "translational inequalities". Their theorem is:

**Theorem 18.** Let \( A \) be the arithmetic mean, \( G \) be the geometric mean, \( H \) be the harmonic mean, and \( R \) be the root-mean-square. Let \( a_1, a_2, \ldots, a_n, x \) be positive numbers, where the \( a_i \)'s are not all equal. Then:

(i) \( A(x+a_1, x+a_2, \ldots, x+a_n) = x + A(a_1, a_2, \ldots, a_n) \).

(ii) \( G(x+a_1, x+a_2, \ldots, x+a_n) > x + G(a_1, a_2, \ldots, a_n) \).

(iii) \( H(x+a_1, x+a_2, \ldots, x+a_n) > x + H(a_1, a_2, \ldots, a_n) \).

(iv) \( R(x+a_1, x+a_2, \ldots, x+a_n) < x + R(a_1, a_2, \ldots, a_n) \).

First Hoehn & Niven proved lemma 4.20, in which they also proved part (i) of theorem 18 and used the results from the lemma to prove the parts (ii) of theorem 18.
We will present the proof from Hoehn & Niven (1985) for lemma 4.20 and part (ii) of theorem 18. After this, we shall state and prove a theorem about $M_r$ that generalizes parts (iii) and (iv) of theorem 18.

**Proof.** In part (ii) of theorem 18, we must show:

$$G(x + a_1, x + a_2, \ldots, x + a_n) > x + G(a_1, a_2, \ldots, a_n).$$

First apply the mean value theorem to the differentiable function $G(x)$ on the interval $[0, c]$. Thus,

$$\frac{G(a_1 + c, a_2 + c, \ldots, a_n + c) - G(a_1, a_2, \ldots, a_n)}{c} = G'(a_1 + \theta, a_2 + \theta, \ldots, a_n + \theta)$$

for $0 < \theta < c$. But $G' > 1$ by part (b) of lemma 4.20. Therefore,

$$G(a_1 + c, a_2 + c, \ldots, a_n + c) - G(a_1, a_2, \ldots, a_n) > c + G_n(a_1, a_2, \ldots, a_n).$$

**Lemma 4.20.** Let $a_1, a_2, \ldots, a_n, x$ be positive real numbers, where the $a_i$’s are not all equal. Then:

(a) $\frac{dA}{dx} = 1$, and

(b) $\frac{dG}{dx} > 1$.

**Proof.** Part (a):

First, we will establish that $A(x) = A(x + a_1, x + a_2, \ldots, x + a_n) = x + A(a_1, a_2, \ldots, a_n)$.

$$A(x + a_1, x + a_2, \ldots, x + a_n) = \frac{x + a_1 + x + a_2 + \ldots + x + a_n}{n}$$

$$= \frac{nx + a_1 + a_2 + \ldots + a_n}{n}$$

$$= \frac{nx}{n} + \frac{a_1 + a_2 + \ldots + a_n}{n}$$

$$= x + A(a_1, a_2, \ldots, a_n).$$

Therefore, differentiating $A(x)$ with respect to the variable $x$ we have

$$\frac{dA}{dx} = \frac{d}{dx}A(x + a_1, x + a_2, \ldots, x + a_n) = x + A(a_1, a_2, \ldots, a_n) = 1.$$
Part (b):

For the geometric mean, note that the derivative of the product \( \prod_{i=1}^{n} (a_i + x) \) is the \((n - 1)\) elementary symmetric polynomial of the \((a_i + x)\), denoted here by \( S_{n-1} \). For example, in the case \( n = 3 \) the polynomial \( S_2 \) would be \( S_2 = (a_1 + x)(a_2 + x) + (a_1 + x)(a_3 + x) + (a_2 + x)(a_3 + x) \). In general, \( S_{n-1} = \sum_{i=1}^{n} (\prod_{j \neq i} (a_j + x)) \).

Let \( G(x) = G(x + a_1, x + a_2, \ldots, x + a_n) \); therefore \( G^n = \prod_{i=1}^{n} (a_i + x) \).

Thus, \( \frac{d}{dx} G^n = nG^{n-1} \frac{dG}{dx} = S_{n-1} \).

To prove that \( \frac{d}{dx} G^n > 1 \), it suffices to show

\[
\frac{S_{n-1}}{n} > G^{n-1}.
\]

This is nothing more than the arithmetic-geometric mean inequality applied to the \( n \) terms of \( S_{n-1} = \sum_{i=1}^{n} (\prod_{j \neq i} (a_j + x)) \). The geometric mean of these \( n \) terms is the \( n \)th root of their product, which is \( [(a_1 + x)(a_2 + x) \ldots (a_n + x)]^{n-1} \), or \( G^{n(n-1)} \), because each \((a_j + x)\) appears in exactly \( n - 1 \) terms of \( S_{n-1} \). \( \square \)

**Theorem 19.** Suppose \( M_r(x_1, x_2, \ldots, x_n) = (\frac{x_1^r + x_2^r + \ldots + x_n^r}{n})^{\frac{1}{r}} \), where \( r \neq 0 \). Also, suppose \( a_1, a_2, \ldots, a_n \) are positive real numbers (not all the same).

Then \( M_r(x + a_1, x + a_2, \ldots, x + a_n) > M_r(a_1, a_2, \ldots, a_n) \) if \( r < 1 \), and

\( M_r(x + a_1, x + a_2, \ldots, x + a_n) < M_r(a_1, a_2, \ldots, a_n) \) if \( r > 1 \).

To prove the above theorem 19, we will use lemma 4.21 (the proof of lemma 4.21 will follow below).

**Proof.** When \( r < 1 \). By lemma 4.21, \( \frac{d}{dx} m_r(x) > 1 \). Then \( M_r \) is a monotone increasing function, and together with \( a_1, a_2, \ldots, a_n, x \) being positive real numbers, we have:

\( M_r(x + a_1, x + a_2, \ldots, x + a_n) > M_r(a_1, a_2, \ldots, a_n) \).

When \( r > 1 \). Similarly, by lemma 4.21, \( \frac{d}{dx} m_r(x) < 1 \). Then \( M_r \) is monotone a decreasing function, and together with \( a_1, a_2, \ldots, a_n, x \) being positive real numbers, we have:
Lemma 4.21. Let $a_1, a_2, \ldots, a_n, x$ be positive real numbers, where the $a_i$'s are not all equal, and $r \neq 0$.

Suppose $m_r(x) = M_r(x + a_1, x + a_2, \ldots, x + a_n)$:

Then $\frac{d}{dx} m_r(x) > 1$ when $r < 1$, and

$\frac{d}{dx} m_r(x) < 1$ when $r > 1$.

Proof. Suppose $0 < r < s$. Then $M_r(y_1, y_2, \ldots, y_n) = \left(\frac{y_1^r + y_2^r + \cdots + y_n^r}{n}\right)^{\frac{1}{r}}$,

and using the inequality, we have

$\left(\frac{y_1^r + y_2^r + \cdots + y_n^r}{n}\right)^{\frac{1}{r}} \leq \frac{y_1^s + y_2^s + \cdots + y_n^s}{n}.$

Hence, $m_r(x) > m_{r-1}(x)$. This implies that

$$(m_r(x))^{r-1} \begin{cases} > \frac{1}{n}((x + a_1)^{r-1} + (x + a_2)^{r-1} + \cdots + (x + a_n)^{r-1}) & \text{as } r > 1 \\ < & r < 1 \end{cases}$$

Now $(m_r(x))^{r-1} \frac{r}{n}((x + a_1)^{r} + (x + a_2)^{r} + \cdots + (x + a_n)^{r})$.

So $r(m_r(x))^{r-1} \frac{d}{dx} m_r(x) = \frac{r}{n}((x + a_1)^{r-1} + (x + a_2)^{r-1} + \cdots + (x + a_n)^{r-1})$;

Thus,

$$\frac{d}{dx} m_r(x) = \frac{1}{n} \frac{(x + a_1)^{r-1} + (x + a_2)^{r-1} + \cdots + (x + a_n)^{r-1}}{(m_r(x))^{r-1}} \begin{cases} > \frac{1}{n}((x + a_1)^{r-1} + (x + a_2)^{r-1} + \cdots + (x + a_n)^{r-1}) & \text{as } r > 1 \\ < & r < 1 \end{cases}$$

Based upon the theorem 18 and lemma 4.20, Hoehn & Niven (1985) went on to note that for every positive value $x$, $(G(x) - A(x)) < 0$ although $(G(x) - A(x))$ is an increasing function, and $(R(x) - A(x)) > 0$ although $(R(x) - A(x))$ is a decreasing function. This observation motivated
Hoehn & Niven (1985) to look at the limits of these functions as \( x \) tends to infinity. Hoehn & Niven (1985) noted that the equation:

\[
\lim_{x \to \infty} [F(x + a_1, x + a_2, \ldots, x + a_n) - A(x + a_1, x + a_2, \ldots, x + a_n)] = 0
\]

holds with any one of \( A, G, H, \) or \( R \) in place of \( F \). Hoehn & Niven (1985) established the result by the following theorem:

**Theorem 20.** \( \lim_{x \to \infty} [F(x + a_1, x + a_2, \ldots, x + a_n) - x] = A(a_1, a_2, \ldots, a_n) \) holds with any one of \( A, G, H, \) or \( R \) in place of \( F \).

**Proof.** We begin with the case of \( F(x) = A(x) \). We have

\[
\lim_{x \to \infty} [F(x + a_1, x + a_2, \ldots, x + a_n) - x] = \lim_{x \to \infty} [A(x + a_1, x + a_2, \ldots, x + a_n) - x].
\]

But \( x = A(x + a_1, x + a_2, \ldots, x + a_n) - A(a_1, a_2, \ldots, a_n) \) by lemma 4.21. Therefore,

\[
\lim_{x \to \infty} [A(x + a_1, x + a_2, \ldots, x + a_n) - x] = A(a_1, a_2, \ldots, a_n).
\]

In the other three cases, it suffices to show \( F = H(x) \) and \( F = R(x) \) and using the inequality \( H < G < R \) for the case \( F = G \). To prove \( F = H \), we use the result from calculus:

\[
\lim_{x \to \infty} \frac{c_m x^m + c_{m-1} x^{m-1} + \ldots + c_0}{k_m x^m + c_{m-1} x^{m-1} + \ldots + k_0} = \frac{c_m}{k_m}. \tag{4.13}
\]

assuming \( k_0 \neq 0 \), we write \( H = H(x + a_1, x + a_2, \ldots, x + a_n) \) in the form \( H = \frac{nG^n}{S_{n-1}} \) where \( G^n \) is the product given above and \( S_{n-1} \) is as above. Thus we write

\[
\lim_{x \to \infty} (H - x) = \lim_{x \to \infty} \frac{nG^n - xS_{n-1}}{S_{n-1}}. \tag{4.14}
\]

Now \( S_{n-1} \) is a polynomial in \( x \) of degree \( n - 1 \). The coefficient of \( x^{n-1} \) is \( n \), and the coefficient of \( x^{n-2} \) is \((n - 1)(a_1 + a_2 + \ldots + a_n)\). The coefficient of \( x^n \) in \( G^n \) is 1, and \( x^{n-1} \) is \((a_1 + a_2 + \ldots + a_n)\). Hence, in \( nG^n - xS_{n-1} \), the terms of degree \( n \) cancel, and by applying 4.13 to 4.14, we have \( \lim_{x \to \infty} (H - x) = \lim_{x \to \infty} \frac{nG^n - xS_{n-1}}{S_{n-1}} \), and we get \( \lim_{x \to \infty} (H - x) = \frac{(a_1 + a_2 + \ldots + a_n)}{n} = A \).
To prove the case \( F = R \), we write \( \lim_{x \to \infty} [R(x + a_1, x + a_2, \ldots, x + a_n) - x] = \lim(R - x) = \lim \frac{R^2 - x^2}{R + x} \).

Expanding \( R^2 - x^2 \) as a quadratic polynomial in \( x \), we notice that the \( x^2 \) terms cancel, so that

\[
R^2 - x^2 = \frac{2(a_1 + a_2 + \ldots + a_n)}{n} + \frac{(a_1^2 + a_2^2 + \ldots + a_n^2)}{n}.\]

It follows that

\[
\frac{R^2 - x^2}{R + x} = \frac{2(a_1 + a_2 + \ldots + a_n)}{nR} + \frac{a_1^2 + a_2^2 + \ldots + a_n^2}{nx}.
\]

From the calculation, we have divided the numerator and denominator by \( x \). As \( x \to \infty \) we note that \( \lim \frac{R}{x} = 1 \) from the calculation

\[
\lim_{x \to \infty} [R(x + a_1, x + a_2, \ldots, x + a_n) - x] = \lim(R - x) = \lim \frac{R^2 - x^2}{R + x} = \frac{2(a_1 + a_2 + \ldots + a_n) + 0}{1 + 0} = A.
\]

### 4.4 Inequality Among \( n \)-ary Means

One of the most prominent property of \( n \)-ary means, at least from the theoretical standpoint, is of course the inequalities between the various means. This topic has attracted the attention of many mathematician as evident by the richness of the research in that area. We begin the section with the arithmetic-geometric mean inequality for \( n \)-ary means. Cauchy appears to have been the first to state the theorem in its most general form. The theorem is listed in Cauchy’s Cours d’Analyse (p.458-459), which appeared in 1821. We will present two different proofs for this inequality.

Then we give some commentary on this inequality. Next we extend this inequality to include the harmonic mean, and we close the section with a theorem on inequalities between power means.

**Theorem 21.** Let \( x_1, x_2, x_3, \ldots, x_n \) be non-negative real numbers. Then

\[
\frac{1}{n} \sum (x_1 + x_2 + \ldots + x_n) \geq (x_1x_2 \ldots x_n)^\frac{1}{n},
\]

and equality occurs if and only if all the \( x_i \)'s are equal.

The first proof we present is by Chrystal (1916). We chose this proof because it was used in a number of college freshman algebra textbooks around the turn of the 20th century and represents an elementary approach to proving the inequality.

**Proof.** Let \( a, b, \ldots, k \) be a sequence of \( n \) non-negative real numbers. Consider their geometric mean, \((ab\ldots k)^\frac{1}{n}\). If \( a, b, \ldots, k \) are not all equal, replace their greatest and least of them, say \( a \) and \( k \), by \( \frac{a + k}{2} \).
Then
\[
\left(\frac{a+k}{2}\right)^2 > ak,
\]
and the result has been to increase the geometric mean of the sequence of numbers, while the
arithmetic mean of the \(n\) quantities, \(\frac{a+k}{2}, b, c, \ldots \frac{a+k}{2}\), is clearly the same as the arithmetic mean of
\(a, b, c, \ldots k\). If the new set of \(n\) quantities are not equal, replace the greatest and the least as before,
and so on. By repeating the process sufficiently often, we can make all the quantities as nearly
equal as we please, and then the geometric mean of the sequence of numbers becomes equal to
the arithmetic mean. But, since the arithmetic mean remained unaltered throughout, and the
geometric mean has been increased at each step, it follows that the first geometric mean, namely
\((abc\ldots k)^\frac{1}{n}\), is less than the arithmetic mean, \(\frac{a+b+\ldots +k}{n}\). \(\Box\)

The next proof we have selected is the one given by Thacker (1851). We chose this proof because
it is concurrent with Chrystal’s proof, but it shows an approach that is more mathematically
balanced yet still straight forward.

**Proof.** If \(x > 0\) and \(n\) is an integer. Then using the binomial theorem, we note the following:

\[
(1 + \frac{x}{n})^n = 1 + x + \frac{1 - \frac{1}{n}}{2}x^2 + \frac{1 - \left(\frac{1}{n}\right)^2}{2 \cdot 3}x^3 + \ldots, \text{and} \quad (4.15)
\]

\[
(1 + \frac{x}{n-1})^{n-1} = 1 + x + \frac{1 - \frac{1}{n-1}}{2}x^2 + \frac{1 - \left(\frac{1}{n-1}\right)^2}{2 \cdot 3}x^3 + \ldots \quad (4.16)
\]

Note that for any positive integer \(n\), we have the following inequalities

\[
1 - \frac{1}{n} > 1 - \frac{1}{n-1},
\]

\[
1 - \frac{2}{n} > 1 - \frac{2}{n-1},
\]

\[
\ldots > \ldots
\]
Hence every term involving \( n \) in the series expansion in 4.15 is greater than the corresponding term in the series expansion in 4.16, and since the terms of series in 4.15 and 4.16 are positive, we have

\[
(1 + \frac{x}{n})^n > (1 + \frac{x}{n-1})^{n-1}.
\]

(4.17)

Now, let there be \( n \) positive quantities \( a_1, a_2, \ldots, a_n \) arranged in order of magnitude with \( a_1 \) being the least. Then

\[
\left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^n = a_1^n \left(1 + \frac{a_1 + a_2 + \ldots + a_n - na_1}{na_1}\right)^n
\]

\[
> a_1^n \left(\frac{1 + a_1 + a_2 + \ldots + a_n - na_1}{a_1(n-1)}\right)^{n-1}
\]

by 4.17

Note that

\[
a_1^n \left(\frac{1 + a_1 + a_2 + \ldots + a_n - na_1}{a_1(n-1)}\right)^{n-1} = a_1 \left(\frac{a_1 + a_2 + \ldots + a_n}{a_1(n-1)}\right)^{n-1}.
\]

Therefore, we have

\[
\left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)^n > 1
\]

\[
a_1 \left(\frac{a_2 + a_3 + \ldots + a_n}{n-1}\right)^{n-1} > 1
\]

\[
\left(\frac{a_2 + a_3 + \ldots + a_n}{n-1}\right)^{n-1} > 1
\]

\[
a_2 \left(\frac{a_3 + a_4 + \ldots + a_n}{n-2}\right)^{n-2} > 1
\]

\[
a_3 \left(\frac{a_4 + a_5 + \ldots + a_n}{n-3}\right)^{n-3} > 1
\]

\[
\vdots
\]

\[
\frac{a_{n-1} + a_n}{2} > a_{n-1}a_n
\]

Hence, by multiplication we get

\[
\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) > (a_1a_2\ldots a_n), \text{ or}
\]

\[
\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) > (a_1a_2\ldots a_n)^{\frac{1}{n}}.
\]
We have given two proofs for this inequality for several reasons. In the first place, the inequality is interesting in that it can be established in a large number of ways. There are literally dozens of different proofs for the arithmetic-geometric inequality that are based on ideas representing a great variety of sources. In the second place, it has a fundamental role in the theory of inequalities and is the keystone on which many other very important results rest. In the third place, we can use some of its consequences to solve a number of maximization and minimization problems (Beckenbach 1961). Therefore, from the historical perspective and its ubiquitousness in mathematics, this inequality truly deserves to be listed first.

Just about every aspect of this inequality has been investigated. Tung (1975) proved that the upper and lower bounds of the difference between the arithmetic and the geometric means of \( n \) quantities can be given as follows:

**Theorem 22.** Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. Suppose \( t \) and \( T \) are the smallest and largest values respectively of the given \( n \) quantities, and
\[
c = \frac{\ln \left( \frac{T}{T - t} \right) \ln T}{\ln T}.
\]
Then
\[
(n^{-1}(\sqrt{T} - \sqrt{t})^2) \leq A - G \leq (ct + (1 - c)T - t^c T^{1-c})
\]

We refer the reader to Tung (1975) for the proof of this theorem. Others mathematicians sought more exotic inequalities between the arithmetic and geometric means. For example, Kedlaya (1994) proved a conjecture made by F. Holland in 1992 which states the following:

**Theorem 23.** Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. The arithmetic mean of the numbers \( x_1, \sqrt{x_1 x_2}, \sqrt[3]{x_1 x_2 x_3}, \ldots, \sqrt[n]{x_1 x_2 \ldots x_n} \)
does not exceed the geometric mean of the numbers

\[ \frac{x_1 \cdot x_2}{2}, \frac{x_1 \cdot x_2 \cdot x_3}{3}, \ldots, \frac{x_1 \cdot x_2 \cdot \ldots \cdot x_n}{n} \]

Now we establish the inequality between $G$ and $H$.

**Theorem 24.** Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then $H \leq G$, and $H = G$ if and only if all the $x_i$’s are equal.

**Proof.** Pick $a_1, a_2, \ldots, a_n$ to be positive real numbers.

For the numbers $a_1, a_2, \ldots, a_n$, by the arithmetic-geometric means inequality, we have

\[
\left( \frac{1}{a_1} \frac{1}{a_2} \ldots \frac{1}{a_n} \right)^\frac{1}{n} \leq \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \]

Then

\[
\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \leq \left( a_1 a_2 \ldots a_n \right)^{\frac{1}{n}}.
\]

Equality holds here if and only if $a_1^{-1} = a_2^{-1} = \ldots = a_n^{-1}$. \hfill \Box

Note by theorems 21 and 24, we have:

\[ H \leq G \leq A, \]

and $H = G = A$ if and only if all the $x_i$’s are equal.

The following theorem establishes a generalized inequality between power means.

**Theorem 25.** Suppose $p$ and $q$ are nonzero real numbers such that $p > q$. Then for any positive numbers $a_1, a_2, \ldots, a_n$,

\[
\left( \frac{a_1^p + a_2^p + \ldots + a_n^p}{n} \right)^\frac{1}{p} \geq \left( \frac{a_1^q + a_2^q + \ldots + a_n^q}{n} \right)^\frac{1}{q},
\]

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with equality holding if and only if $a_1 = a_2 \ldots = a_n$.

We present the proof given by Schaumberger (1988).

**Proof.** Suppose $p > q > 0$ or $0 > p > q$, and let $f(x) = qx^p + (p - q) - px^q$. Then for $x > 0$:

$$qx^p + p - q \geq px^q$$

with equality holding if and only if $x = 1$. This follows from $f(x) = qx^p + (p - q) - px^q$ has an absolute minimum at $x = 1$ (because $f''(x) = qpx^{p-1} - qpx^{q-1}$ vanishes if and only if $x = 1$, and $f''(1) = (qp(p - q) > 0)$.

Now consider any $n$ positive numbers $a_1, a_2, \ldots, a_n$ and let

$$A = (\frac{a_1^q + a_2^q + \ldots + a_n^q}{n})^{\frac{1}{q}}.$$  \hspace{1cm} (4.19)

Substituting $x_i = \frac{a_i}{A}$ for $1 \leq i \leq n$ in equation 4.19 and adding we get

$$q(\frac{a_1^p + a_2^p + \ldots + a_n^p}{Ap}) + pn - qn \geq p(\frac{a_1^q + a_2^q + \ldots + a_n^q}{A^q}).$$

Since $a_1^q + a_2^q + \ldots + a_n^q = nA^q$, it follows that

$$q(\frac{a_1^p + a_2^p + \ldots + a_n^p}{Ap}) \geq qn$$ \hspace{1cm} (4.20)

If $p > q > 0$, then equation 4.19 gives

$$\frac{a_1^p + a_2^p + \ldots + a_n^p}{n} \geq A^p,$$

Which can be written as

$$\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)^{\frac{1}{q}}.$$  

If $0 > p > q$, then division by $q$ reverses the inequality and equation 4.19 reduces to
\[
\frac{a_1^p + a_2^p + \ldots + a_n^p}{n} \leq A^p.
\]

Raising both sides to the \(\frac{1}{p}\) power reverses the inequality again, and we have

\[
\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)^{\frac{1}{q}}.
\]

Note, furthermore, that equality holds if and only if each of the substituted values \(x_i = \frac{a_i}{A}\) equals 1, which is equivalent to \(a_1 = a_2 = \ldots = a_n\).

Let \(p > 0 > q\). Then for \(x > 0\), \(f(x) = qx^p + (p - q)x^q\) has an absolute maximum at \(x = 1\). Thus

\[
qx^p + (p - q) \leq px^q \quad (4.21)
\]

with equality holding if and only if \(x = 1\). Again let

\[
A = \left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)^{\frac{1}{q}}.
\]

Substituting \(x_i = \frac{a_i}{A}\) successively in in equation 4.21, and adding the inequalities, we obtain

\[
q\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{A^p}\right) + pn - qn \leq p\left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{A^q}\right) = pn.
\]

Thus,

\[
q\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{A^p}\right) \leq qn.
\]

Since \(0 > q\), division by \(q\) reverses the inequality and leads to

\[
\left(\frac{a_1^p + a_2^p + \ldots + a_n^p}{n}\right)^{\frac{1}{p}} \geq \left(\frac{a_1^q + a_2^q + \ldots + a_n^q}{n}\right)^{\frac{1}{q}}.
\]
Equality holds if and only if each of the substituted values \( x_i = \frac{a_i}{A} \) equals 1, which is equivalent to

\[ a_1 = a_2 \ldots = a_n \]
Chapter 5. Conclusion

The history of means is long and laden with details as means are used by ordinary people, experts, academicians, and scientists to express a representative number that typifies a set of values. The antique means appear to have been well known by the dawn of Greek mathematics. After the 4th century A.D., the theory of means appears to have come to a standstill. With Pappus’s documentation of the ten known means of his time in the 4th century A.D., it seems that the interest in the theory of means had waned. Boethius is the last known ancient writer to mention the eleven classical means as part of his work "Arithmetic", which is a commentary on works of Nicomachus and Pappus (Gow 1923). The next reference to the classical means after Boethius appeared as a quotation by Ocreatus in his work "Prolgus In Helceph," which was written in the 12th or 13th century A.D. (Heath 1921). Thus, it seems reasonable to speculate that the only means that may have drawn any interest to be mentioned in mathematics literature between the 5th century and the 16th century are the three antique means which were treated as the only types of means. That the antique means continued to be of interest from the 5th century to the 16th century comes as no surprise, since the arithmetic mean is often useful in commercial transactions, the geometric mean was preserved by the use of the mean proportional in geometry, and the harmonic mean is closely tied to music theory.

In Middle Ages, the golden section became a favorite topic of theological speculation. Many learned people, inspired by the arguments of the Pythagorians and Platonists, sought and found in this proportion a key to the mystery of creation, declaring that extreme and mean ratios were the very principle which the Supreme Architect had adopted in the cosmic and global design; hence, the title divine proportion bestowed upon this ratio (Dantzig 1955).

The late renaissance’ interest in science and ancient Greek mathematics brought about a new interest in the theory of means as particular aspects of some physical phenomena can best be expressed by using their mean values. Advances in the theory of statistics has shown that the
arithmetic mean can be used as a representative of a set of observation data (Buhler 1944). This outlook carried over quite naturally to finding new ways to express a mean of a set of values that best fits the purpose at hand, and thus theory of means was reborn. This rich history motivates us to dig deeper into the concept of means to understand the underlying foundation in which it is couched.

We close by noting that there is a proliferation of means in mathematics of which we only touched on a few. This proliferation may be attributed to the fact that new ways to express a mean of a set of numbers arise in applications continuously. Unfortunately, our discussion did not include many other types of means that play a pivotal role in mathematics research such as the rich area of iterated means, e.g. Gauss’ arithmetic-geometric mean, non-symmetric means, and weighted means. Means are connected with diverse areas of mathematics research from Fourier Series, error measurements, to aggregation and social choice. With each new mean developed, ways have to be found that will relate this mean to the ones already known.
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