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Teaching Complex Numbers in High School

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ABSTRACT

One of the mathematics standards for high schools stated in the Common Core State Standards Initiative (CCSS, 2010 Appendix A, p. 60) is the understanding of the Complex Number system, performing arithmetic operations with complex numbers, representing complex numbers and their operations on the complex plane, and using complex numbers in polynomial identities and equations.

In this thesis, we trace back the history of complex numbers and formulate a set of problems based on the history. The thesis gives a description of the Common Core standards and goals. The exercises based on the history of complex numbers are shown to be in accordance with the Common Core standards. The different approaches to teaching complex numbers are described with examples. Their merits and shortfalls are discussed from a teacher's mathematical perspective. A list of Guidelines on teaching complex numbers for high school teachers is given in the last chapter of this thesis.

CHAPTER 1: INTRODUCTION

Complex numbers are introduced and taught in high school Algebra 2. Complex numbers are introduced primarily in order to extend the notion of roots of all quadratic equations. After this brief motivation, basic operations on complex numbers are taught in high schools. For many students, this is the last required course in mathematics where students are exposed to a combination of Algebra, Geometry, and Trigonometry.

Understanding the complex number system, performing arithmetic operations with complex numbers and representing complex numbers and their operations on the complex plane, is one of the Mathematics standards for high school highlighted in the Common Core State Standards Initiative (CCSSI, 2010 Appendix A, p. 60). For students to be able to fulfill all the standards required on this topic, teachers should have a deep content knowledge in teaching complex numbers. Using a combination of various approaches to teach the topic may help students to have a clear understanding of what complex numbers are, and the geometric transformations that correspond to the basic operations on complex numbers.

At present, the introduction of complex numbers is tricky and difficult for a variety of reasons. First, students do not perceive the need to solve any and every quadratic equation. Secondly, they have an aversion to numbers that are called “imaginary”. Thirdly, the basic operations (addition, subtraction, multiplication, and division) of complex numbers are taught first algebraically and then geometrically by plotting points on the complex plane, and using ideas from vector algebra, polar coordinates, and trigonometry. Being at the intersection of algebraic and geometric methods, students begin to view complex numbers as confusing and difficult.

The testimony of history is that complex numbers should be introduced to students as they were introduced to the mathematicians between the mid-16th century and the mid-19th century (Hersh p. 10). In this period, we find representations given for complex numbers, and when the geometric interpretation was offered, mathematical research on complex numbers flourished. The goal of this thesis is to provide teachers with teaching techniques and exercises that can be used to effectively teach complex numbers in high school Algebra 2.

The difficulty of the students in accepting and understanding complex numbers is one of the problems encountered in high school Algebra 2. *Hence, this thesis is undertaken in order to help teachers explore ways to help students to have a deeper understanding of complex numbers.*

In the next chapter, a rigorous explanation of complex numbers is discussed along with a short history of complex numbers in order to familiarize ourselves with the origin of complex numbers and the contributions of the mathematicians who developed complex numbers. Chapter 3 discusses the Common Core State Standards for complex numbers that students are expected to know. In chapter 4, the different approaches to teach complex numbers are presented. A discussion on the different ways to represent complex numbers is given with the hope that students may understand complex numbers via these representations. In chapter 5, the findings of this thesis and suggestions to teachers in teaching complex numbers are given.

CHAPTER 2: COMPLEX NUMBERS

We will start with the mathematical definition and construction of complex numbers. This would be the way in which a mathematician would understand complex numbers.

Definition. A field **F** is a set that contains at least two elements and is equipped with two operations, addition and multiplication defined on it such that the following properties are satisfied:

1. Closure: If a and b are elements of **F**, then $a + b \in$ **F** and $a \cdot b \in$ **F** where the symbol "+" denotes addition, and "·" , multiplication. This property means that adding a and b, the sum is an element of **F**. Likewise, the expression we obtain by multiplying a and b is an element of F. In short, **F** is closed under addition and multiplication.
2. Commutativity: If a and b are elements of **F**, $a + b = b + a$ and $ab = ba$.

   This means that the order in which elements are added or multiplied is immaterial.

3. Associativity: If a, b, c are elements of **F**, $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$.

   Associativity deals with the groupings – the different groupings of the elements do not affect the answer.

4. Distributivity: If a, b, c are elements of **F**, $a(b + c) = ab + ac$

   When we multiply an element a by the sum of two elements, the result is equal to the element obtained when the element a is multiplied to each summand separately and then the resulting elements are added up.

5. Identity: There exist elements 0 and 1 in **F** such that $0 + a = a$ and $1a = a$ for all $a \in$ **F**, and $0 \neq 1$.

The element 0 is called the additive identity and 1, the multiplicative identity.

6. Inverse: If a ϵ **F**, there exists an element, denoted –a ϵ **F** such that –a + a = 0. Also, for all a ≠ 0, there exists an element a^{-1} ϵ **F** such that a · a^{-1} = 1. The element a^{-1} is also written as 1/a.

The element '-a' is called the additive inverse, and 1/a, the multiplicative inverse of "a".

In each example below, the usual operations of addition and multiplication of numbers is used.

1. The set {0,1} is a field with these operations.

+	0	1
0	0	1
1	1	0

$\mathbf{F_2}$

*	0	1
0	0	0
1	0	1

$\mathbf{F_2}$

2. The set of all rational numbers, denoted **Q,** is a field.
3. The set of all real numbers is a field.

Remark.

After defining the concept of a field, teachers in Algebra 2 can assign the above examples as exercises. After this, teachers can assign the following problem that requires careful thinking, and writing skills.

Exercise. Prove or disprove:

The set **Z** of all integers is a field.

The set **R** of all real numbers has more structure. For instance, it **is** an ordered field, that is, given any a, b ϵ **R**, then, a < b or a = b or a > b (trichotomy property). The ordering given by the relation < is transitive. That is, if a < b and b < c then a < c.

Note that the definition of **R** requires more hypotheses than the ones listed above. We will not list these hypotheses here. Instead, we will rely on a working knowledge of the set **R.**

So far, we haven't raised the following more fundamental question:

What is a number? We all profess to have an understanding of the concept of a number. For instance, 5 is a number. The color "red" is not a number. To answer the question, let us consider 2 and ¼ . Define 2 as the symbol for the transformation of doubling. The symbol ¼ is defined as the transformation of quartering. In this way, we define numbers as scaling transformations. (see B. Mazur, 2003). The symbol or number 5, for instance scales up while $^1/_5$ scales down. Note that in defining these transformations, the operation of multiplication is involved. We will take this up this line of reasoning in defining complex numbers after the geometric approach has been introduced.

Let i be a symbol denoting an element in a field with the property that $i^2 = -1$. Such a symbol is not a real number. We can also show this by the method of contradiction.

Suppose that $i \in$ **R.** Then we know that i is greater than zero, equal to zero, or less than zero. If we take i to be greater than zero, then $i^2 = i \cdot i > 0$ since the product of two positive numbers is positive. That is, $-1 > 0$ which is false. Therefore, i cannot greater than 0.

Similar contradictions can be arrived at by supposing that if $i = 0$, and $i < 0$. Thus, we can conclude that i is not positive, zero, or negative. By the trichotomy property, it follows that i is not a real number.

Suppose there were a field that contained **R** and also contained some element, call it i, with the property that $i^2 = -1$.

Question: What other elements would this field have to contain?

Answer: bi (for any b $\in$ **R),** and a + bi (for any a, b $\in$ **R**)

The set should be {a + bi / a, b $\in$ **R**} closed under addition and multiplication and it should have additive and multiplicative inverses (of non-zero elements) and identities.

Before embarking further on questions about this field, we will focus on the basic operations of addition and multiplication on it.

If b $\in$ **R**, let us write bi and call it as multiplication of b and i.

Let $(b + c)i = bi + ci$ for any b, c $\in$ **R**.

Let a + bi be the expression that we get upon addition of a and bi.

Define the set $\mathbb{C}$ = {a + bi such that a ϵ **R** and b ϵ **R**}.

We call any element of $\mathbb{C}$ as a complex number. Define addition and multiplication on $\mathbb{C}$ by the following rules:

(a + bi) + (c + di) = (a + c) + i(b + d) (1)

and

(a + bi) (c + di) = ac + bc$i$ + ad$i$ + bdi^2

= (ac – bd) + i(ad + bc) (2)

where we have used i^2 = -1 in arriving at equation (2)

The multiplicative inverse of a non-zero complex number a + bi is given by

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)}$$

since $\frac{a-bi}{a-bi} = 1$. Upon algebraically simplification of the denominator, we get

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

In general, for any two complex numbers a + bi and c + di,

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$= \frac{ac+bci-adi-bdi^2}{c^2+d^2}$$

$$= \frac{(ac+bd)+ i(bc-ad)}{c^2+d^2}.$$

Consider all expressions of the form a + bi where a and b are real numbers. This set of expressions equipped with addition and multiplication defined by equations (1) and (2) constitutes a field, as can be verified. The verification can be an exercise for students. This field is denoted by $\mathbb{C}$, and is known as the field of complex numbers.

History of Complex Numbers

In an article written by Steven Strogatz, one finds that for more than 2,500 years, mathematicians have been obsessed with obtaining the roots of polynomial equations. The story of their struggle to find the "roots" of increasingly complicated equations is one of the great stories in the history of human thought. [Strogatz, 2010] Until the 1700's, mathematicians believed that square roots of negative numbers did not exist. According to them, such a number does not exist since any number multiplied to itself has to be non-negative. Taking a step back, it is worthwhile to note that even as late as in the 17th century, well known mathematicians such as John Wallis viewed negative numbers with suspicion. For understanding the reason behind the suspicion, we refer the reader to the fascinating book by Nahin. (Nahin, p. 14). The idea of square roots of negative numbers was naturally viewed as wild and unthinkable.

During the early days of mathematical history, if mathematicians, in the process of solving an equation, reached a point that involved taking the square root of a negative number, he or she would just stop since such an expression was meaningless. One such instance happened in 50 AD, when Heron of Alexandria was examining the volume of a truncated pyramid (frustum of a pyramid). However, at this time, even negative numbers were not "discovered" or used.

A truncated square pyramid with a and b as edge lengths of the top and bottom squares and height h, has volume

$$V = \frac{1}{3} h (a^2 + ab + b^2) \qquad (1)$$

The Egyptian geometers knew this formula. Note that the height h is not observable readily. Hence, the question was whether h can be written in terms of slant height c, and edge lengths a, b. A simple use of right-angled triangles yields

$$h = \sqrt{c^2 - 2\left(\frac{b-a}{2}\right)^2} \qquad (2)$$

Heron of Alexandria tried to use formulas (1) and (2) to find the volume of the frustum of a pyramid with a = 28, b = 4 and c = 15. Heron wrote

$h = \sqrt{225 - 2(12)^2}$ which was written as $h = \sqrt{81 - 144}$. In fact, Heron would write this as $h^2 + 144 = 81$ so that negative numbers do not appear. Instead of writing the next step as $\sqrt{-63}$, Heron recorded $\sqrt{63}$. The value recorded in stereometria appears as $\sqrt{63}$. The correct answer is $\sqrt{-63}$, a complex number. The problem is on calculating height and volume. Clearly, height cannot be $\sqrt{63}\, i$. This is an instance where complex numbers are not useful.

Exercise.

By drawing a truncated pyramid, show the details of the derivation of equation (2).

Two centuries later, Diophantus, a well-known algebraist considered the following problem: Given a right triangle with area 7 square units and perimeter 12 units, find its side lengths.

Thus,

$$\tfrac{1}{2} bh = 7 \qquad (3)$$

$$b + h + \sqrt{b^2 + h^2} = 12 \qquad (4)$$

Diophantus called b as $\frac{1}{x}$ so that h = 14x by equation (3).

Therefore, equation (4) becomes $\frac{1}{x} + 14x + \sqrt{\frac{1}{x^2} + 196x^2} = 12$

Multiply throughout by x to obtain $1 + 14x^2 + \sqrt{1 + 196x^4} = 12x$. Rearranging the terms, we get $\sqrt{1 + 196x^4} = 12x - 1 - 14x^2$.

Squaring both sides and then simplifying, we obtain

$$336x^3 - 172x^2 + 24x = 0.$$

Therefore, $x(336x^2 - 172x + 24) = 0$. Since x is a strictly positive number, we have

$$336x^2 - 172x + 24 = 0.$$

The quadratic formula gives

$$x = \frac{172 \pm \sqrt{(172)^2 - 4(336)(24)}}{2 \, x \, 336}$$

$$= \frac{43 \pm \sqrt{-167}}{168}.$$

Hence, Diophantus wrote that solving this quadratic equation is impossible. Thus, the problem has no solution. In fact, there is no right triangle with area 7 square units and perimeter 12 units. Here, teachers can pose the following questions to students.

Exercise.

(i) Carry out the details of “squaring and simplifying” that are mentioned above.

(ii) In Diophantus’ problem, the perimeter of the right triangle is larger than the area of a triangle. Is this the reason for the non-existence of a solution? Give your reasons.

The next instance in recorded history when one encountered complex numbers arose in solving cubic equations. The Italian algebraist Scipione del Ferro (1465 – 1526) knew how to solve a depressed cubic equation, that is, one of the form

$$x^3 + px = q \qquad (5)$$

where p and q are positive real numbers. The word “depressed” means that the x^2 term is absent in the cubic polynomial. His method is quite original and will be described in detail after the presentation of the history of cubic equations. del Ferro was interested only in finding the real root of (5). A real, positive root is guaranteed since equation (5) can be written as f(x) = 0 where $f(x) = x^3 + px – q$. Since $f(0) = - q < 0$ and f(x) becomes large as x becomes large. Hence, the graph of y = f(x) crosses the x axis at some $x > 0$.

To continue with the history, del Ferro kept his solution a secret, known only to a handful of his students. This allowed an Italian mathematician Niccolo Fontana (1500 – 1577) popularly known as Tartaglia to rediscover del Ferro’s solution and solve equations of the form $x^3 + px^2 = q$ as well. However, the next major step in solving cubic equations was made by a well-known Italian mathematician, Girolamo Cardano (1501 – 1576) who devised a method to reduce any cubic equation to a depressed cubic equation. Cardano obtained explicit expressions for the real roots of cubic equations. Much to his dismay, his mathematical expressions involved square root of negative numbers. Cardano formally used such expressions in his calculations. The word “formally” refers to “symbolically”, and is not a flattering word to a mathematician.

One of his problems is on dividing 10 into two parts such that their product is 40, he writes

$x (10 – x) = 40$

Therefore, $x^2 – 10x + 40 = 0$ whose roots are $5 \pm\sqrt{-15}$.

Nonetheless, Cardano calculated the product of the roots to obtain 40.

Exercise. Show that the product of the roots is equal to 40.

He called such calculations as mental tortures, and wrote of square roots of negative numbers as follows: “So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.” Cardano was bewildered by square roots of negative numbers appearing in his formula for cubic equations that clearly had only real solutions.

We will go back to the cubic equations studied by Cardano. The perplexing question that bothered Cardano with cubic equations was resolved by an Italian engineer – Architect Rafael Bombelli (1526 – 1572). Bombelli noticed that Cardano’s expression involved the addition of two cube roots of complex conjugates, that is, $\sqrt[3]{a+bi}$ + $\sqrt[3]{a-bi}$. Bombelli’s breakthrough was in realizing that this expression is equal to a real number since a real solution exists. Therefore, if the cube roots of complex conjugates are themselves complex conjugates, say c + d*i* and c – d*i*, then their sum would be 2c, a real number.

Bombelli worked out an example to show that cube root of complex conjugates are complex conjugates. This was truly a historical moment of epic proportions in the history of mathematics.This was an instance where complex numbers are useful in getting a real solution. We now proceed to the mathematical details of the work of del Ferro, Cardano, and Bombelli.

Consider $az^3 + bz^2 + cz + d = 0$

where $a \neq 0$. By dividing throughout by a, we can, without loss of generality, set a = 1 and consider the equation

$$z^3 + bz^2 + cz + d = 0. \qquad (1)$$

Let us also look at the equation $\left(x - \frac{b}{3}\right)^3 + b\left(x - \frac{b}{3}\right)^2 + c\left(x - \frac{b}{3}\right) + d = 0.$ (2)

Equation (2) is equivalent to equation (1). Indeed, if z is a solution of equation (1), then z $+\frac{b}{3}$ is a solution of equation (2). Conversely, if ***x*** solves equation (2), then x - $\frac{b}{3}$ is a solution of equation (1). Thus, solutions of equation (1) lead to solutions of equation (2), and the converse statement holds as well.

Consider the left side of equation (2).

$$\left(x-\frac{b}{3}\right)^3+b\left(x-\frac{b}{3}\right)^2+c\left(x-\frac{b}{3}\right)+d$$

$$=\left(x^3+\frac{3b^2x}{9}-\frac{3x^2b}{3}-\frac{b^3}{27}\right)+\ b\ \left(x^2+\frac{b^2}{9}-\frac{2bx}{3}\right)+c\left(x-\frac{b}{3}\right)+d$$

The x^2 terms cancel out; continuing,

$$=\quad x^3+\frac{b^2x}{3}-\frac{b^3}{27}+\frac{b^3}{9}-\frac{2b^2x}{3}+cx-\frac{bc}{3}+d$$

$$=\quad x^3+\ \left(c-\ \frac{b^2}{3}\right)x+(\frac{2b^3}{27}-\frac{bc}{3}+d)$$

$$=\quad 0.$$

Thus, we obtain

$$x^3+\left(c-\ \frac{b^2}{3}\right)x+\left(\frac{2b^3}{27}-\frac{bc}{3}\ +d\right)=\ 0$$ which can be written as

$$x^3+px+q=0 \qquad (3)$$

which is a depressed cubic equation. This idea is a brilliant idea due to Cardano.

Let us write $x = u + v$, therefore $v = x - u$. Let us impose the condition that

$$uv = -\frac{p}{3}.$$

Given x, we need

$$u(x - u) = -\frac{p}{3}$$

i.e. $$ux - u^2 = -\frac{p}{3}.$$

i.e. $$u^2 - ux = \frac{p}{3}.$$

By completing the square on the left side of the above equation, we obtain

$$u^2 - ux + \frac{x^2}{4} = \frac{p}{3} + \frac{x^2}{4}.$$

Therefore, $\left(u - \frac{x}{2}\right)^2 = \frac{p}{3} + \frac{x^2}{4}$, which yields

$$u - \frac{x}{2} = \pm\sqrt{\frac{p}{3} + \frac{x^2}{4}}.$$ Thus

$$u = \frac{x}{2} \pm \sqrt{\frac{p}{3} + \frac{x^2}{4}}.$$

If p > 0, u will be a real number. In general, u may involve the square root of a negative number.

Then, with $x = u + v$ and $uv = -\frac{p}{3}$ equation (3) becomes

$$(u+v)^3 + p(u+v) + q = 0\ .$$

i.e. $u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0.$

$u^3 + v^3 + (3uv + p)(u+v) + q = 0.$

i.e. $u^3 + v^3 + q = 0$ (4) since

$3uv + p = 0$, by the condition on u,v.

Since $= -\frac{p}{3u}$, we can write (4) in the form $u^3 - \frac{p^3}{27u^3} + q = 0.$

Multiply throughout by u^3 to obtain $u^6 + qu^3 - \frac{p^3}{27} = 0.$

This is a quadratic equation in *u*^{3}.

Therefore $u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$. This is the idea of del Ferro.

Having found u^3 , we take the cube roots to find the solutions for ***u*.**

Now $v = -\frac{p}{3u}$ so that v can be ascertained, once u is known.

From equation (4), we obtain

$$v^3 = -q - u^3$$

$$= -q - \frac{\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)}{2}$$

$$= \frac{-q \mp \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$

Thus, if

$$u^3 = \frac{-q+\sqrt{q^2+\frac{4p^3}{27}}}{2}, \text{ then}$$

$$v^3 = \frac{-q-\sqrt{q^2+\frac{4p^3}{27}}}{2}.$$

These are not necessarily real numbers since $q^2 + \frac{4p^3}{27}$ may happen to be negative. If u^3 and v^3 are complex conjugates, then Bombelli conjectured that *u* and *v* are themselves complex conjugates. This is one of the greatest insights in the history of Mathematics. He used an example given below.

Example: Consider $x^3 - 15x - 4 = 0$ which is in the form of equation (3). Bombelli knew that $x = 4$ is a solution. By Cardano's method, we identify

$$u^3 = \frac{4 \pm \sqrt{16 - \frac{4}{27}(15)^3}}{2}$$

$$u^3 = 2 \pm \frac{\sqrt{-484}}{2}$$

$$= 2 \pm 11i$$

Therefore, u^3 is a complex number and v^3 is its conjugate.

Bombelli conjectured that their cube roots are complex conjugates. This is true in general. We will prove this fact in the section on geometric approach to complex numbers.

Thus,

$x = u + v$ happens to be a real solution though u,v are not. In other words, let u and v be $\alpha + i\beta$ and $\alpha - i\beta$ respectively. Since 4 is a root in this example,

$$4 = u + v$$

$$= (\alpha + i\beta) + (\alpha - i\beta)$$

$$= 2\alpha$$

Therefore, $\alpha = 2$.

Also, $(\alpha + i\beta)^3 = 2 + 11i$ from the previous page. Expanding the left hand side in the above equation and recalling that $\alpha = 2$, we get

$$8 + 12i - 6\beta^2 - \beta^3 i = 2 + 11i$$

Equating the real parts in the above equation,

$$8 - 6\beta^2 = 2 \qquad (6)$$

Thus, $\beta = 1$ is a solution of (6) and hence

$$\alpha + i\beta = 2 + i \text{ and}$$

$$\alpha - i\beta = 2 - i.$$

Adding them, we get the real root 4 of the cubic equation.

Important Ideas In Solving Cubic Equations

1. Cardano knew that the x^2 term in a cubic equation $x^3 + bx^2 + cx + d = 0$ can be made to vanish by replacing x by $x - \frac{b}{3}$. This leads to a depressed cubic equation $x^3 + px + q = 0$.

2. It is del Ferro's idea to write x as $u + v$ with $uv = -\frac{p}{3}$. This gives us a quadratic equation in the variable u^3.

3. Bombelli's idea consists in showing that cube roots of two complex conjugates are themselves conjugates.

   Teachers can spur the interest of students by assigning the following exercises that are connected to the above historical developments.

   The following exercises are given to stimulate students' thinking:

   (i) Use Cardano's method to reduce the cubic equation $x^3 - 3x^2 + x - 3 = 0$ to a depressed cubic equation.

   (ii) Given a cubic polynomial $x^3 + 3x^2 - 21x - 21 = 0$, find the depressed cubic equation.

   (iii) For the depressed equation obtained in (ii), use del Ferro's idea to find u^3 and v^3.

The phrase "imaginary number" for square root of a negative number was not used until the 17th century. In 1637, Rene Descartes first used the word "imaginary" as an adjective. In the next century, Leonhard Euler, a world renowned Swiss mathematician, introduced the letter *i* to denote the imaginary unit. He described the solutions of the equation $x^n + 1 = 0$ as vertices of a regular polygon in the plane.

Mathematicians had to develop useful ways to think about square roots of negative quantities. In 1673, John Wallis invented a simple way to represent imaginary numbers as points in a plane. He started from the familiar representation of real numbers as a line, with the positive numbers on the right of 0 and the negative numbers to the left of 0. Then he introduced another line, intersecting the first at 0 and at right angles to the first. Along this new line he placed the purely imaginary numbers, that is, multiples of *i*. This is the y-axis in rectangular coordinate system on the plane. Wallis' approach solved the problem of giving a geometric meaning to imaginary numbers.

The first one to agree with Wallis was Jean Robert Argand in 1806. He wrote the procedure that John Wallis invented for graphing complex numbers on a number plane, known as the complex plane. However, the person who made this idea popular was the famous German mathematician Carl Friedrich Gauss. He also made popular the use of the term complex number for numbers of the form a + b*i*. Finally, in 1833, an Irish mathematician named William Rowan Hamilton showed that pairs of real numbers with an appropriately defined multiplication form a number system. He said that Euler's previously mysterious "*i*" can be interpreted as one of these pairs of numbers. When points on the complex plane were written in polar coordinates (r, θ) instead of the rectangular coordinates (x, y), the operations of multiplication and division of complex numbers were understood better. Euler defined the complex exponential and published his proof of the important identity $e^{ix} = \cos x + i \sin x$, where x is measured in radians, in the year 1748 (Nahin, 1998). This is the period when the modern formulation of complex numbers began and flourished.

Throughout the 1800s, many mathematicians such as Karl Weierstrass, Richard Dedekind, and Henri Poincaré, have published deep and interesting results on complex numbers and functions of a complex variable. Though complex numbers were initially introduced to solve polynomial equations, their usefulness has transcended beyond our expectations and has opened new areas of mathematics such as Complex and Fourier Analysis.

We end this chapter by collecting the exercises mentioned so far, which are based on, or motivated by the history of complex numbers.

Summary of problems from this chapter

Show that the set {0,1} is a field if addition and multiplication are defined in a binary system.

1. Prove or disprove:

   The set **Z** of all integers is a field.

2. By drawing a truncated pyramid, show the details of the derivation of equation (2).

   $$h = \sqrt{c^2 - 2\left(\frac{b-a}{2}\right)^2} \qquad (2)$$

3. (i) Carry out the details of "squaring and simplifying" that are mentioned in page 10.

   (ii) In Diophantus' problem, the perimeter of a right triangle is larger than the area of a triangle. Is this the reason for the non-existence of a solution? Give your reasons.

   In Cardano's problem on page 12, show that the product of the roots is equal to 40.

4. Use Cardano's method to reduce the cubic equation $x^3 - 3x^2 + x - 3 = 0$

   to a depressed cubic equation.

5. Given the cubic polynomial $x^3 + 3x^2 - 21x - 21 = 0$, find the depressed cubic equation.

6. For the depressed equation obtained in (7), use del Ferro's idea to find u^3 and v^3

CHAPTER 3: COMMON CORE STATE STANDARDS AND GOALS

This chapter discusses the common core state standards for teaching complex numbers as well as operations on complex numbers that students are expected to know at the level of high school Algebra 2.

Common Core State Standards for Complex Numbers

The Common Core State Standards are a set of shared goals and standards developed through a bipartisan, state-led initiative spearheaded by state superintendents and state governors. The standards reflect the collective expertise of hundreds of teachers, education researchers, mathematicians, and state content experts from across the country [K 8 Publishers' Criteria, CCSSM, Spring 2013]. These standards are aligned with the expectations of colleges, workforce training programs, and employers that promote equity by ensuring that all students are well prepared to collaborate and compete with their peers in the United States and abroad. These educational standards are the learning goals that students should know and should be able to do at each grade level. The standards define what students should understand and be able to do in their study of mathematics.

According to the Common Core State Standards Initiative, "the Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The eight standards for mathematical practice are as follows:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

These standards describe ways in which developing student practitioners of the discipline of mathematics can increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. In this respect, those content standards which set an expectation of understanding are potential “points of intersection between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.”

The following are the standards that students are expected to learn or be able to do after their teachers have taught them complex numbers. Students are required to find the exact answer to each given problem and are expected to explain how and why their answer was correct. With these standards, it is expected that students will have the ability to think independently. Students are also challenged to question, elaborate and communicate to deepen their understanding of the concept and operations on complex numbers. Although some of these standards may be hard for students to attain, it is expected that teachers would be able to find ways for the students to achieve these goals.

According to the Common Core State Standards on complex numbers, the students are expected to know the following:

Perform arithmetic operations with complex numbers.

Know that there is a complex number *i* such that $i^2 = -1$, and every complex number has the form a + b*i* with a and b real.

Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

1. Find the conjugate of a complex number; use conjugates to find the moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

2. Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
3. Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.
4. Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Use complex numbers in polynomial identities and equations.

5. Solve quadratic equations with real coefficients that have complex solutions.
6. Extend polynomial identities to complex numbers.

7. Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

The standards also establish what students need to learn, but it does not say how teachers should teach those standards. It is the teacher's decision on what method to use or what works best in the classroom for students to understand those standards. Teachers will devise their own lesson plans and curriculum, and tailor their instruction to the individual needs of the students in their classrooms. Segmenting and pacing a lesson depends on teachers' knowledge of their students and their different learning needs. Because of the spiraling of the Common Core and the depth of each standard, teachers need to 'unwrap' or deconstruct the standards to make clear the concepts and skills embedded within, and plan smaller instructional learning progressions.

Through analyzing the formative assessments tied to the deconstructed standards, teachers can better understand the learning needs of their students and structure lessons to best support student learning. Moreover, these measures are also expected to help ensure that the students will have the necessary skills and knowledge for them to be successful.

The basic types of problems that students are expected to solve under common core are given below.

1. Add $(2 + 3i)$ and $(1 – 6i)$.

2. Subtract $- 4 – i$ from $(5 – 2i)$.

3. Multiply 5 by i.

4. Multiply $2i$ by i.

5. Using the distributive law for multiplication, multiply $5 – 7i$ by i.

6. Using the distributive law for multiplication, multiply $(2 – i)$ by $(3 + 4i)$.

7. Divide $(1 + 2i)$ by $(3 - 4i)$.

8. Find the conjugate of $1 + i$

9. Simplify the following and write the result in the form $a + bi$, with a and b real numbers.

a. $(9 + 3i)(3 - 6i)$ b. $(8 + i)(1 - 8i)$ c. $(2 + 3i)(4 - i) - 5(2 - i)$

Some exercises that lead to thought-provoking result and critical thinking skills are as follows:

10. Find out what is wrong with the following calculation.

$$1 = \sqrt{1}$$

$$= \sqrt{(-1)(-1)}$$

$$= \sqrt{-1}\ \sqrt{-1}$$

$$= i \cdot i$$

$$= i^2$$

$$= -1$$

11. Express the following in standard form, that is, in the form $a + bi$: $(1 + i)^2$, $(1 + i)^3$, $(1 + i)^4$, etc. Will you ever get an answer that is a real number? Explain your reasoning.

CHAPTER 4: DIFFERENT APPROACHES

Chapter 4 discusses the various teaching methods that can be used by high school teachers to teach complex numbers and the basic representations and operations for complex numbers. In describing the various approaches, the required mathematical details are given.

1. Traditional Approach

Traditional approach is where the teacher has the power and responsibility in the classroom. This is usually in the form of lectures where students simply carry out the rules and procedures that the teacher gives in the classroom. The teacher introduces the concept of complex numbers and states the facts. Students are just given the required information. For instance, the students are asked to add the complex numbers $4 + 3i$ and $2 + 7i$. They are told to add the real part and the imaginary part separately to obtain the result $6 + 10i$. As for multiplication, if the complex numbers $1 + 2i$ and $-5 + 3i$ are to be multiplied, the students will do the required expansion and then simplify their answer. This is the algebraic approach to complex numbers. Though it is strictly regimental, this approach is expedient in introducing and teaching the basic operations on complex numbers.

The traditional method is not a recommendation by Common Core State Standards. It stunts students' thinking skills. This approach introduces Argand diagram for displaying complex numbers graphically as a point in rectangular coordinates on the plane. Since students have already a prior knowledge of plotting ordered pairs in a coordinate plane, the use of the Argand diagram is easy for them. It is similar to what they learned in Algebra 1 though in the present context, we have the real and the imaginary axis for the plane instead of the x-axis and the y-axis. The representation consists in identifying complex number $a + bi$ by the point (a,b) on the plane. When a complex number $a + bi$ is given, we call a as the real part and b as the imaginary part. It is represented by the point (a, b) on the plane. Conversely, any point (a, b) on the plane can be identified with a complex number a + bi. For this reason, we call the plane as the complex plane in this context. If the real part is positive, the movement is to the right along the real axis and if it is negative, it is moved to the left of the real axis. If the imaginary part is positive, the movement is up along the imaginary axis, and down if it is negative.

The traditional method is one of the common ways used by teachers though it is not recommended by experts on Math education. This teaching approach is totally inadequate by itself since it neither builds the mind's eye nor does it motivate students to learn this topic. This method doesn't help students to understand multiplication and division of complex numbers.

2. Vector Approach

In addition to representation by points in the Argand diagram where a + b*i* is identified with the point (a,b) on the plane, complex numbers can also be represented by vectors in the Argand diagram. Since Argand diagram uses rectangular coordinates, it ties in well with the representation of complex numbers by vectors.

The rules for adding and subtracting complex numbers are mirrored by rules for adding and subtracting vectors. Teachers who adopt this method explain to the students that complex numbers behave exactly like two dimensional vectors. Hahn (1994) recommends representing complex numbers by points or vectors in the Argand diagram depending on which identification is more useful for a given context (Hahn, 1994).

The complex number z = a + b*i* is represented by the directed line segment which starts from (0, 0) and terminates at (a, b) in the complex plane. Such a vector is known as a position vector. Its magnitude is $\sqrt{a^2 + b^2}$, namely the Euclidean length of the line segment. A vector is determined by its magnitude and direction. In general, a vector with a given magnitude and direction can be drawn with any point on the plane as its starting point. However, a position vector has (0, 0) as its starting point.

The sum of the two vectors z_1 and z_2 is represented by the diagonal of the parallelogram formed if the two original vectors are taken as the adjacent sides of the parallelogram. Figure 1 illustrates the sum of two complex numbers represented by vectors z_1 and z_2.

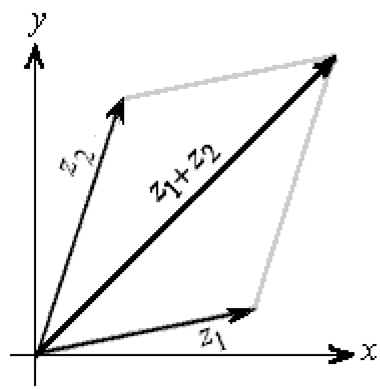


Figure 1 Sum of two complex numbers represented by vectors

If a + bi is a given complex number, then the vector representation of it is given by the line segment that starts at the origin and terminates at the point (a, b). Call it as v_1. Let v_2 be the vector that corresponds to the complex number c + d*i*. Then $v_1 + v_2$ is the vector that starts at (0, 0) and terminates at (a + c, b + d).

The difference between two vectors is obtained by finding $z_1 + (- z_2)$. If the position vector for z_2 has terminal point (a, b), then the position vector for $- z_2$ has terminal point (-a, -b). Figure 2 shows the negative of a vector while figure 3 represents the vector $z_1 – z_2$.

We can represent the vector $z_1 – z_2$ starting from any point on the plane as long as its direction and magnitude are not changed. A given two- dimensional method can be drawn using any point on the plane as the starting point. This non-uniqueness tends to confuse students.

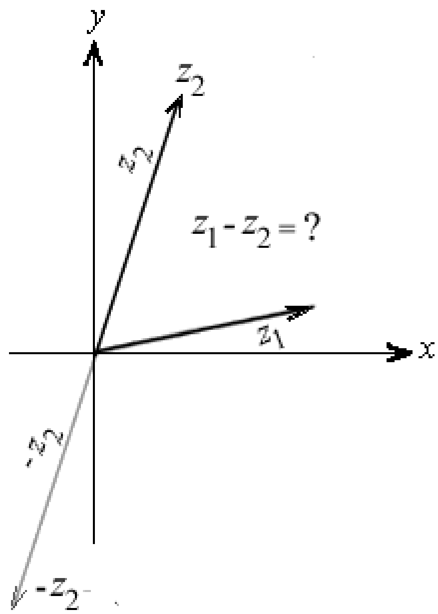


Figure 2   Negative vector (- z_2)

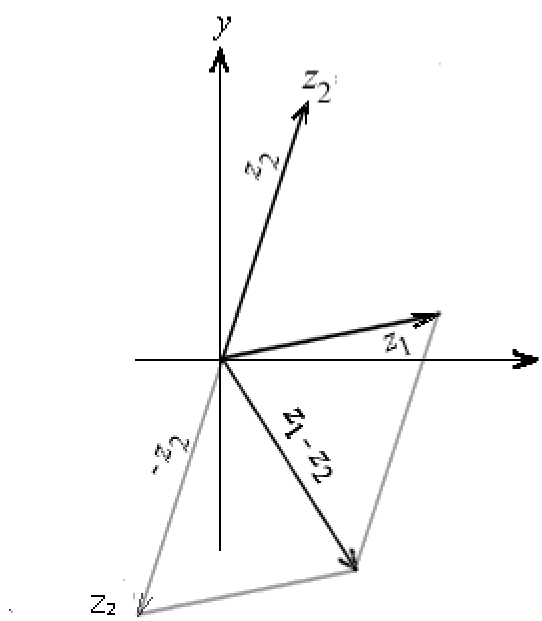


Figure 3 z_1 - z_2 = z_1 + $(-z_2)$

Figure 4 below shows another representation for z_1 - z_2

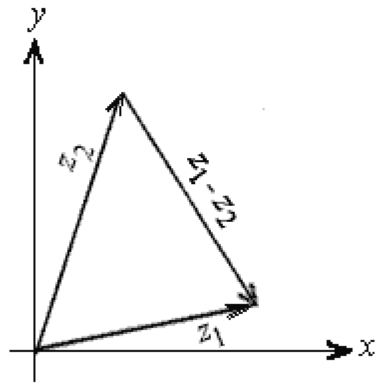


Figure 4 The difference of two vectors z_1 and z_2

The vector approach can be mentioned in a classroom, if time permits such a discussion.

Teaching complex numbers with the use of vectors may be helpful to students because of the visual image it provides. However, I was not able to use this method. The reason was that the students did not have prior knowledge of vectors that are usually discussed in a Physics course. Besides, most of my students were still having trouble with plotting and reading the graphs.

3. Geometric Approach

Another way to represent a complex number z = a + bi is by means of polar coordinates (r, θ). Rather than using the real part and the imaginary parts to represent complex numbers, we can also use its distance from the origin, and the angle measured counterclockwise from positive real axis to the position vector with terminal point (a, b). This distance of (a, b) to the origin is usually denoted by *r*, and the angle formed from the positive x-axis to the position vector is denoted by θ. r is the absolute value (or norm or modulus) of the complex number z, and its argument, θ, is the angle measured in the counterclockwise direction from the positive side of x-axis. The coordinates of the rectangular and polar form are related as follows:

$a = r \cos \theta$ and $b = r \sin \theta$

with $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$, if $a \neq 0$

so that $z = a + bi$

$= r\cos\theta + i\, r\sin\theta$ or

$z = r(\cos\theta + i\sin\theta)$

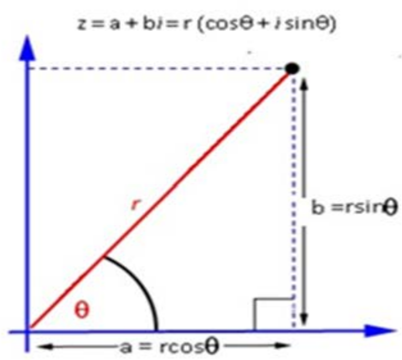


Figure 5 Complex Plane

In Figure 5 above, r is called the modulus of z and θ is called the argument. The form

$z = r(\cos\theta + i\sin\theta)$ is the polar form of a complex number.

For instance, let's take an example of expressing the complex number $3 + 5i$ in polar form.

Since the polar form of $z = a + bi$ is $z = r(\cos\theta + i\sin\theta)$ we can find the absolute value of r.

$r = |z| = \sqrt{a^2 + b^2}$

$= \sqrt{3^2 + 5^2}$

$= \sqrt{34}$

$= 5.83$

Now, we can find the argument θ.

Since $a > 0$, use the formula $\theta = \tan^{-1} \frac{b}{a}$.

$$\theta = \tan^{-1} \frac{5}{3}$$

$$\approx 1.03$$

Hence, the polar form of $3 + 5i$ is about $5.83(\cos 1.03 + i \sin 1.03)$

Let us give another example: Suppose, the given complex number is z = -9 and the objective is to find the polar form of this complex number. Here, it is understood that the imaginary part is 0 and so we have the real part which is -9. Also, since it is a negative real number, the argument is π. Now, to compute for r, we can use the same formula that we used in the previous example. Therefore:

$$r = |z| = \sqrt{a^2 + b^2}$$

$$= \sqrt{(-9)^2 + 0^2}$$

$$= \sqrt{81}$$

$$r = |z| = 9.$$

Thus,

$$z = 9(\cos(\pi) + i \sin(\pi))$$

Note that for $z = a + bi$, if $a = 0$, the bi is purely imaginary. If $b>0$, then $\theta = \frac{\pi}{2}$. If $b < 0$, conclude that $\theta = \frac{3\pi}{2}$.

The polar representation is very useful when we multiply or divide complex numbers. Given the complex numbers $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$, we can find the product

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2)$$

$= \quad r_1 r_2 (\cos \theta_1 \cos \theta_2 + \cos \theta_1 i \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2)$

$= \quad r_1 r_2 \ [\cos \theta_1 \cos \theta_2 + (-1) \sin \theta_1 \sin \theta_2 + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$

$= \quad r_1 r_2 \ [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$

Here, we see the formula for the cosine of the sum of two angles and the sine of the sum of two angles. Thus,

$z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$

It is amazing to see how useful and intriguing the above formula is. Based on the formula above, students can see that in multiplying two complex numbers, we are not just multiplying everything. Rather, we are adding the arguments or the angles, and multiplying the moduli or the values of r.

For instance, if students are asked to find the product of these two given complex numbers $z_1 = 5(\cos \pi/3 + i \sin \pi/3)$, and $z_2 = 4 (\cos 2\pi/3 + i \sin 2\pi/3)$

Then using the same formula above,

$z_1 \cdot z_2 = 5 \cdot 4 (\cos (\pi/3 + 2\pi/3) + i \sin (\pi/3 + 2\pi/3))$

$= 20 (\cos \pi + i \sin \pi)$

The above formula will make it easier for students to multiply of complex numbers instead of going through many steps.

Similarly, given the same two complex numbers $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, we find the quotient as:

$$\frac{z_1}{z_2} = \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)}$$

$$= \frac{r_1 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{r_2 (\cos \theta_2 + i \sin \theta_2)(\cos \theta_2 - i \sin \theta_2)}$$

$$= \frac{r_1\ [\cos\theta_1 \cos\theta_2 - i \cos\theta_1 \sin\theta_2 + i \sin\theta_1 \cos\theta_2 - i^2 \sin\theta_1 \sin\theta_2]}{r_2\ [(\cos\theta_2)^2 - i \cos\theta_1 \sin\theta_2 + i \cos\theta_2 \sin\theta_2 - i^2 (\sin\theta_2)^2]}$$

$$= \frac{r_1\ [(\cos\theta_1 \cos\theta_2 - (-1) \sin\theta_1 \sin\theta_2 + i (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)]}{r_2 [\cos^2\theta_2 - (-1) \sin^2\theta_2]}$$

$$= \frac{r_1\ [(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2)]}{r_2 [\cos^2\theta_2 + \sin^2\theta_2]}$$

Here, we have the formula for the cosine of the difference of two angles and the sine of the difference of two angles. Using the Pythagorean identity, we arrived at

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Based from the above formula, we are dividing the moduli or the r values, and subtracting the arguments or the angles.

For example, let us use the same given $z_1 = 5(\cos \pi/3 + i \sin\pi/3)$, and $z_2 = 4 (\cos 2\pi/3 + i \sin 2\pi/3)$ in finding the quotient of two complex numbers z_1 and z_2

$$\frac{z_1}{z_2} = \frac{4}{5} [\cos(\pi/3 - 2\pi/3) + i \sin(\pi/3 - 2\pi/3)]$$

$$= \frac{4}{5} [\cos(-\pi/3) + i \sin(-\pi/3)]$$

$$= \frac{4}{5} (\cos \pi/3 - i \sin \pi/3)$$

The above formula will help students in dealing with division of complex number in an easier way. It would also increase their interest in complex numbers since even the basic operations are intriguing. In terms of transformations, when two complex numbers are multiplied, the result is obtained by an appropriate rotation (counterclockwise) and an appropriate expansion or contraction. If $r>1$, it is an expansion, otherwise, contraction. In Chapter 2, we have mentioned that a number can be viewed as a transformation (B. Mazur, p. 190). For instance, the number 3 can be thought of as the transformation that triples any given number. The number *i*, for instance is the transformation of rotation of the complex plane counterclockwise by 90^0. In fact, when the number *i* is viewed as such a transformation, there is nothing "imaginary" about it. When numbers are viewed as transformations, complex numbers are very simple to understand.

4. Using Information and Communication Technology (ICT)

Mathematics educators are increasingly leveraging on ICT today to develop students' interest in the subject and enhance their learning experience (Wong, 2009). Here, the graphing calculator is employed in the tool mode, and students use technology to learn and apply mathematics. Wong (2009) advocates this mode because it fosters conceptual understanding. It also enables students to learn through hands-on experience, instead of being presented with information in a didactic manner.

Video has become an integral part of students' online experience and YouTube is one of the best examples. YouTube is a powerful help for students and they are very familiar with this internet resource. Inside the classroom, this is also a great help to increase students' understanding of a certain topic.

Another resource available on the internet on introducing complex numbers is "John and Betty's Journey into Complex Numbers" (Bower, n.d.). This electronic book is a story about two children, John and Betty, who solved a series of problems designed to introduce complex numbers and the Argand diagram. This online book motivates learning and introduces complex numbers, in a way, that is intuitive and enjoyable for students.

Information technology is a must nowadays. Students learn quickly with the help of technology in the classroom. It is good for them to explore, discover, and learn the concept on their own. However, in using this method, the students do not have enough hands on experience in solving problems. In my Algebra 2 class, the students are able to explore and learn things on their own with the use of the graphing calculators. In some ways, when we discussed complex numbers, and they are allowed to use graphing calculators, they were able to appreciate for a while the importance of learning and understanding complex numbers. They also find it easier to comprehend the complicated concept of complex numbers when we used the online resources available for them in the classroom.

5. Learning via Constructive Projects

Fractals are a never-ending pattern that are "self-similar" — they look the same under different levels of magnification (scaling); the parts that make it up look the same as the larger shape. It is created by repeating a simple algorithm over and over in an ongoing feedback loop. This is essentially an iteration process where the procedure starts with a number and is then fed into a formula. We obtain a result, and feed this result back into the formula in order to obtain the next output. This process is repeated. Fractals use complex numbers to generate their images. If suitable software is available to use in the computer area of the classroom, students can explore various formulas to generate different kinds of images.

We start with an example to illustrate what an iteration means.

Example of a simple iteration with a complex number:

$f(x) = 4x + (5 - 3i)$; starting value of $x = 0$.

$f(0) = 5 - 3i$

$f(5 – 3i) = \quad 4(5 – 3i) + (5 – 3i)$

$= 20 - 15i + 5 - 3i$

$= 25 - 18i$

$f(25 - 18i) = 4(25 - 18i) + (5 - 3i)$

$= 100 - 72i + 5 - 3i$

$= 105 - 75i$

$f(105 - 75i) = 4(105 - 75i) + (5 - 3i)$

$= 420 - 300i + 5 - 3i$

$= 425 - 303i$ and so on…

The Julia Set

Julia set is the collection of all complex numbers for which chaotic behavior of a complex function occurs.

We will consider only quadratic functions of the form

$$f_c(z) = z^2 + c.$$

where $c, z \in \mathbb{C}$. The orbit of $z \in \mathbb{C}$ refers to the sequence $\{ f_c^k(z)\}$ where c is fixed in $\mathbb{C}$, and f_c^k is the k-fold composition of f_c with itself. By chaotic behavior, we mean sensitivity of orbits of nearby points.

Fix $c \in \mathbb{C}$. Consider the set of all z's whose orbits remain bounded. If we color this set, then we obtain the filled Julia set. The Julia set is the boundary of the filled Julia set.

An exercise for students is:

Exercise: Consider $f_o(z) = z^2$.

(i) Consider the orbit for $z = i/2$. Does it stay bounded?

(ii) Explore the orbit for z = i. Is it bounded?

(iii) Explore the orbit for z = 1 + i. Does it stay bounded?

(iv) Find the Julia set for f_o.

In this exercise, students learn the behavior of the complex squaring function and its dynamics. Understanding the Julia set for various $c \in \mathbb{C}$ results in various visually pleasing pictures and is an important first step in nonlinear discrete complex dynamics.

One can use the following criterion in identifying the Julia set.

Result**.** Suppose that $|z| \geq |c| > 2$. Then we have $|fc^n(z)| \to \infty$ as $n \to \infty$.

Proof.

$|z|^2 = |z^2| = |z^2 + c - c|$

$\leq |z^2 + c| + |c|$

$\leq |z^2 + c| + |z|$ by hypothesis

Therefore,

$|z|^2 - |z| \leq |z^2 + c|$.

That is, $|z|(|z| - 1) \leq |z^2 + c|$ --- (1)

Again, by our hypothesis,

$|z| - 1 > 1$. Hence, $|z| - 1 > 1 + a$ for some $a > 0$.

Using this in inequality (1), we obtain $|z^2 + c| > (1 + a)|z|$.

Applying this argument repeatedly, one has $|fc^n(z)| > (1 + a)^n |z|$. Thus, the orbit of z tends to ∞. The proof is over.

Remark.

If $|c| > 2$, then $|f(0)| = |c| > 2$. Hence, the orbit of 0 escapes to ∞ if $|c| > 2$.

(i) Suppose $|z| > \max\{|c|, 2\}$. Then $|fc^n(z)| \to \infty$ as $n \to \infty$

The simplest approach to approximately compute the Julia set is to form a fine grid in some region in the plane. For a point, (e.g., the center) z of each grid, compute the orbit and decide whether it escapes to ∞. If it does not, then the point z is inside or on the Julia set.

A simple observation is that the filled Julia set of f_c when $|c|$ > 2 lies fully inside the disk {z: $|z|$ < $|c|$}.

A result that is well-known is that the Julia set and the filled Julia set coincide and form a totally disjoint set of points when $|c|$ > 2.

There are also bifurcations that arise when c = 0.25 and c = - 0.75. Hence, there are abrupt changes in the approximate pictures of Julia set near these values of c.

Exercise. For each of the following c values use the computer to draw approximately the filled Julia set. Is the picture one piece or many isolated pieces?

(i) c = 0.24

(ii) c = 0.26

The Mandelbrot Set

The Mandelbrot set was named after the French-American mathematician Benoit Mandelbrot. The Mandelbrot set M is defined as the collection of all values of c that are complex numbers for which the filled Julia set for z = 0 is connected. In other words,

M = {c ϵ ℂ: $|f_c^n(0)|$ does not diverse to ∞}.

The plotting of all points in the set creates a striking shape or form in which each part is a scaled copy of the whole. As each complex number can be represented as a point on the complex plane, it is possible to create a colorful picture of the Mandelbrot set. This is done by dividing the complex plane into a fine grid of squares, then testing one number in each square (usually the center) to see if it is in the Mandelbrot set. If it is, the square will be colored; otherwise, it will be white. By plotting millions of c values that are part of the Mandelbrot set, the computer creates an intricate picture. Since the Mandelbrot set is a huge collection of points plotted on the complex plane, they could easily be just in one color. However, one can see the Mandelbrot set is given in many colors.

Fix $c \in \mathbb{C}$. Then $f_c(0) = c$.

Therefore, $f^2c\,(0) = c^2 + c$

$f^3c\,(0) = (c^2 + c)^2 + c$, etc.

We thus obtain a sequence of complex numbers starting with $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$

If this sequence is bounded (use a computer program to check this), then *c* is an element of the Mandelbrot set. (A sequence of points in the Argand plane is bounded if there is some fixed circle that contains all the points.)

A complex number, *c*, is in the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of z_n never exceeds a certain number (that number depends on *c*) however large *n* gets. When $c = i$ for instance, we obtain the sequence $0, i, (-1 + i), -i, (-1 + i), -i, \ldots$, which is bounded since each term in absolute value is ≤ 2. Then, *i* belongs to the Mandelbrot set. On the contrary, for example, letting $c = 1$ gives the sequence 0, 1, 2, 5, 26,…, which tends to infinity so that 1 is not an element of the Mandelbrot set as this sequence is unbounded.

To compute the Mandelbrot set, one starts with a rectangular grid in the complex plane. The grid can be inside the square with center (0,0) and side length 4 since M is contained in the circle of radius 2 (by the escape criterion given earlier). For each c in this grid, we compute $f_c(0)$, $f_c^2(0)$, etc. If this orbit does not escape to ∞, then $c \in M$. As an algorithm, we can set the number of iterations as N. Thus, for each c in the grid, compute the first N points of the orbit of o under f_c. If $|f_c{}^n(0)| > 2$ for some $n \leq N$. Then stop the iterations and declare that c is not in M. If $|f_c{}^n(0)| \leq 2$ for all $n \leq N$, then $c \in M$. This is an approximate way to find M. Mandelbrot set gives us one of the most intricate shapes in discrete nonlinear complex dynamics. The Mandelbrot set has a large cardioids with periodic bulbs attached to it. For more details on the terminology and generation of these decorations, we refer the reader to R. L. Devancy.

Teachers may use the Mandelbrot set as a visual tool to motivate students in appreciating the concept of complex numbers. Since students have already an idea of plotting complex numbers, it will not be hard for them to plot the Mandelbrot set using a computer program. Then, let the students color their result. Most students like coloring as they are attracted to beautiful intricate patterns.

Using Graphing Calculator to graph the Mandelbrot Set

Mandelbrot set can be sketched with the use of a computer, but students can also use graphing calculator to do this. The MANDELBR program draws an approximation of the Mandelbrot Set. With the use of TI-83/TI-84, this can be done by the students. Here is how it goes:

Turn the calculator and press PRGM. Then select the option Mandelbr and press enter twice in order to run the program. A prompt appears asking for the stage of the iteration and this stage must be an integer greater than 4. The greater the number, the closer the approximation is to the real Mandelbrot Set. The program then sets up some variables. Then it will start drawing a line around the Mandelbrot Set although it will take several minutes to complete the picture. After the picture is completed, the stage of the iteration is shown in the top left corner of the display. (National Science Foundation Project)

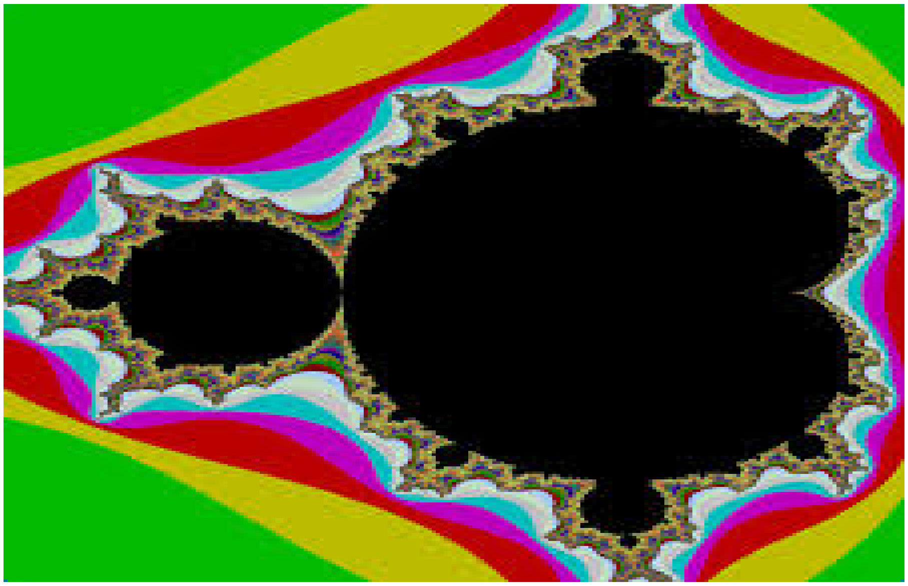


Figure 6

www.cs.princeton.edu

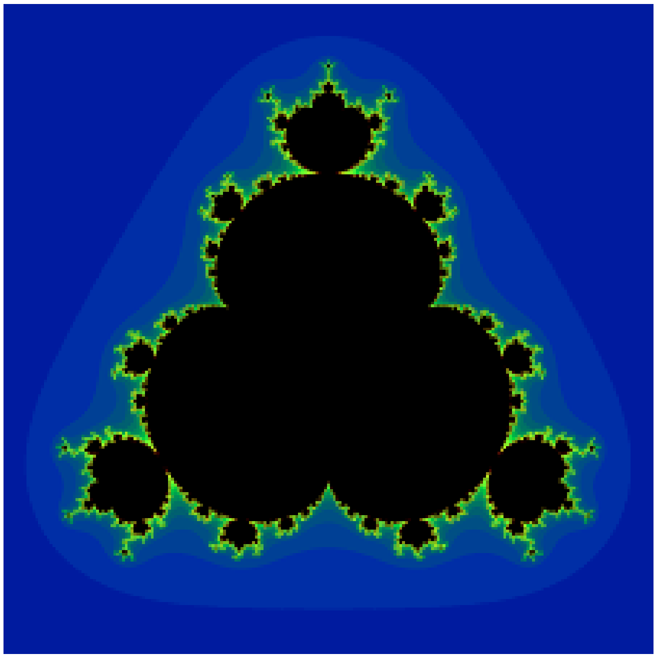


Figure 7

www.relativitybook.com

Pictures of Mandelbrot Set

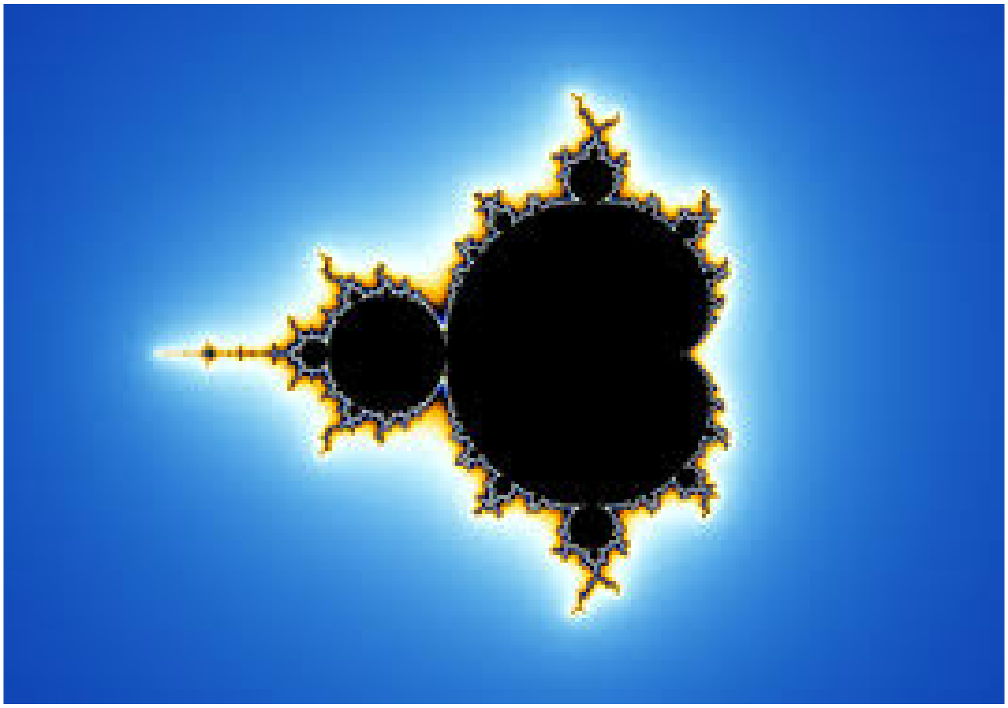


Figure 8

hqwallbase.com

6. Discovery Approach

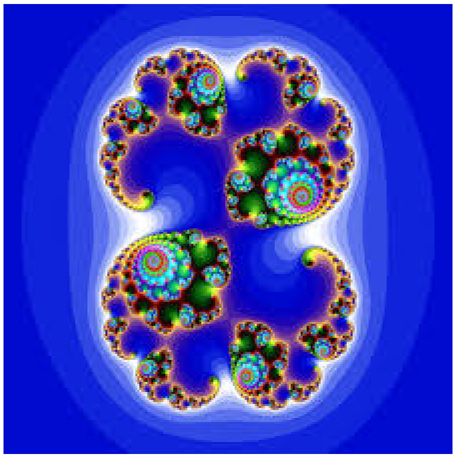
In this approach, students are guided to discover knowledge on their own by working on various activities (Bruner, 1974). This was carried out in teaching the geometric effect of adding two complex numbers through a series of questions. (The Open University, 1981, p. 12).

(i) (a) Mark 1 , i, and 1 + i on an Argand diagram.

(b) Add 2 + i to each of the above complex numbers, and mark the new numbers on the same diagram.

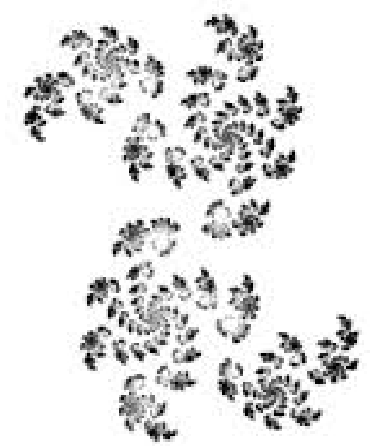
(ii) What effect, geometrically, does adding 2 + i have on the position of a complex number in the Argand diagram?

(iii) What is the geometric interpretation of adding a + bi to a complex number?



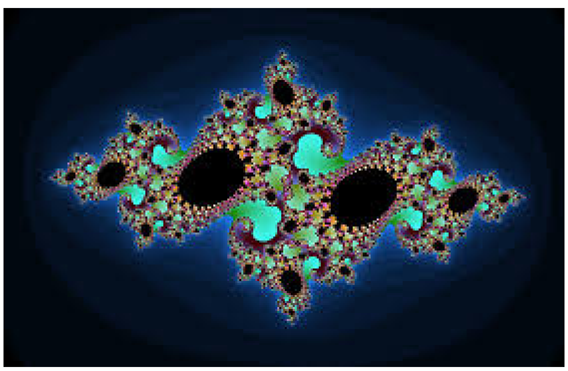
c = 0.285

From Wikimedia.org



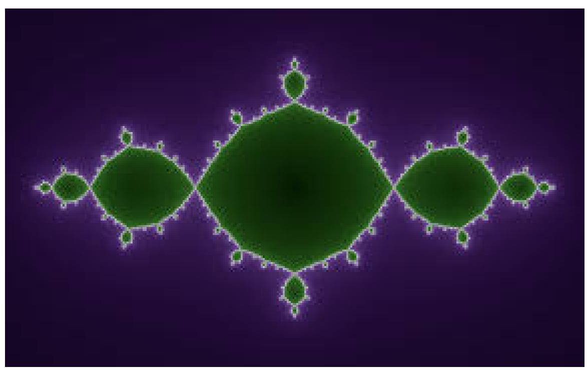
c = 0.360284 + 0.100376*i*

from Wikipedia.org



C = -0.75 + 01*i*

www.ifp.illinois.edu



c = -1

commons.wikimedia.org

Pictures of the Julia Set

According to an article written by Hui and Lam, an advantage of the discovery approach is that students have a greater ownership in their learning. Moreover, this avoids procedural understanding. However, one of the disadvantages of this approach is students may learn the wrong mathematics or become frustrated not knowing what to create [Hui and Lam, 2013].

There may be different approaches in the teaching of complex numbers; each one has its own advantage and disadvantage. It is important for students to see the visual representation of what complex numbers are and how they are solved. This way, they can easily learn the concept of complex numbers. Combining the traditional approach with other approaches will definitely lead to a better understanding of complex numbers by the students.

In the case of my students, they had difficulty meeting all the standards in the common core. The primary reason is that of the gaps brought about by the lack of their algebraic foundation and their problem with the notion of complex numbers. This is especially true when we discussed multiplication and division of complex numbers. Students find it very hard to get the correct answer when they multiply because they always forget the rule that $i^2 = -1$. When it comes to division, they stop whenever they encounter longer steps than usual especially if they are asked to simplify their final answer.

Just like when dealing with operations with real numbers, operations can be performed in complex numbers following the rules. For instance, when adding and subtracting complex numbers, only similar terms can be combined. As for multiplication of complex numbers, distributive property may be used. Of these operations, it was multiplication and division that students find very hard to remember. Multiplication which involves distributive property was hard for them especially when it comes to i^2. Students always forget that $i^2 = -1$ and that is their common mistake. As for division, that was the hardest part especially since it involves conjugation. Some of them just won't continue or will just stop doing division because they don't know the next step. This is the part where most of my students had difficulty.

CHAPTER 5: CONCLUSIONS

The topic of complex numbers is one which in my opinion, stands at a much higher level of sophistication than other topics in Algebra 2. The students encounter difficulties with (i) accepting the existence of the number *i* that satisfies $i^2 = -1$, (ii) overcoming the anxieties created by the unfortunate choice of words such as "imaginary" and "complex" in reference to certain numbers, (iii) mixing the ideas from algebra, geometry and trigonometry and (iv) being unaware of the origin, history, and usefulness of complex numbers. As teachers, it is our job to remove the skepticism prevalent among students concerning complex numbers.

- Based on my research, the traditional method is not a recommended method though it is the quickest way provided the students are willing to do mathematics as they are told. This method does not lead to further the students' thinking at a deeper level and stunts their motivation to learn the topic. The geometric method creates more interest in students towards complex numbers. This method does not mean simply plotting points in the Argand plane. It is very useful in teaching the basic operations on complex numbers. For instance, multiplication of two complex numbers $r_1(\cos \theta_1 + i \sin \theta_1)$, and $r_2(\cos \theta_2 + i \sin \theta_2)$ can be taught as finding the point which is r_1r_2 distance away from the origin and which subtends an angle of $\theta_1 + \theta_2$ from the positive x-axis, measured counterclockwise. Thus, multiplication leads to a transformation that involves rotation, and expansion or contraction. Both algebraic and geometric methods may be taught so that students learn the topic well.
- The vector method may be used by teachers to illustrate addition and subtraction of complex numbers since students get a visual representation of complex numbers in the Argand plane. However, it is not useful for teaching multiplication and division.

- It is recommended that teachers use the origin and history of complex numbers and exercises based on the historical problems so that students learn with a heightened sense of motivation. When students realize that mathematicians had difficulties in understanding complex numbers, they feel they are not alone in viewing square roots of negative numbers with suspicion and skepticism. This feeling of inclusion goes a long way in developing their interest in Mathematics.
- Intriguing problems can be given to students that make them think and appreciate complex numbers. For instance, a problem such as “Solve $x^2 = i$” or “Solve $x^4 = -81$” allows them to think and discuss mathematics. Students should be taught to realize that the introduction of complex numbers is sufficient to solve any polynomial equation.
- The “Mandelbrot set” and the “Julia set” may be used by teachers to motivate students and arouse their interest in understanding complex numbers. Creating the Mandelbrot set is an appealing way for students to apply their understanding of complex numbers in generating visual patterns that are intricate.

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APPENDIX A: SOLUTIONS TO PROBLEMS ON COMPLEX NUMBERS

The first 12 problems were formulated by Dr. James Madden.

1. Add $(2 + 3i)$ and $(1 – 6i)$.

In adding two complex numbers, we simply add the real parts and the imaginary parts separately. So,

$2 + 1 + 3i – 6i$

$= 3 – 3i$.

2. Subtract $- 4 – i$ from $(5 – 2i)$.

   When subtracting two complex numbers, we follow the same rule when we subtract integers, that is we change the sign of the subtrahend and proceed to addition. Hence,

   $5 – 2i – (-4 – i)$

   $= 5 – 2i + 4 + i$

   $= 9 - i$

3. Multiply 5 by i.

   Similar to the rules in multiplying polynomials, just multiply 5 by i so the result is $5i$.

4. Multiply $2i$ by i.

   Multiplying $2i$ by i gives $2i^2$. But $i^2 = -1$. Therefore, the result is -2.

5. Using the distributive law for multiplication, multiply $5 – 7i$ by i.

   $i(5 – 7i)$

   $= 5i – 7i^2$

   $= 5i – 7(-1)$

   $= 5i + 7$

= 7 + 5*i*.

6. Using the distributive law for multiplication, multiply (2 – *i*) by (3 + 4*i*).

   Here, use the distributive property to get the product of (2 – *i*) and (3 + 4*i*).

   (2 – *i*) (3 + 4*i*)

   = 2(3) + 2(4*i*) – 3(*i*) – *i*(4*i*).

   = $6 + 8i - 3i - 4i^2$

   = 6 + 5*i* – 4(-1)

   = 6 + 5*i* + 4

   = 10 + 5*i*

7. Divide (1 + 2*i*) by (3 - 4*i*).

   In dividing complex numbers, multiply both the numerator and the denominator by the conjugate. Then perform multiplication and finally, simplify.

   $\frac{1+2i}{3-4i} \cdot \frac{3+4i}{3+4i}$

   $= \frac{3+4i+6i+8i^2}{9-16i^2}$

   $= \frac{3+10i+8(-1)}{9-16(-1)}$

   $= \frac{3+10i-8}{9+16}$

   $= \frac{-5+10i}{25}$

   $= \frac{-5}{25} + \frac{10i}{25}$

   $= \frac{-1}{5} + \frac{2i}{5}$

8. Find the conjugate of $1 + i$.

In finding the conjugate of $1 + i$, the real part stays the same and simply take the opposite of the imaginary part. That is, $1 - i$.

9. Simplify the following and write the result in the form a + b*i*, with a and b real numbers.

a. $(9 + 3i)(3 - 6i)$

$= 9(3) - 9(6i) + 3(3i) - 3i(6i)$

$= 27 - 54i + 9i - 18i^2$

$= 27 - 45i - 18(-1)$

$= 27 - 45i + 18$

$= 45 - 45i$.

b. $(8 + i)(1 - 8i)$

$= 8(1) - 8(8i) + i(1) - i(8i)$

$= 8 - 64i + i - 8i^2$

$= 8 - 63i - 8(-1)$

$= 8 - 63i + 8$

$= 16 - 63i$.

c. $(2 + 3i)(4 - i) - 5(2 - i)$

$= 2(4) - 2(i) + 4(3i) - i(3i) - 5(2) - 5(-i)$

$= 8 - 2i + 12i - 3i^2 - 10 + 5i$

$= 8 - 10 + 10i - 3(-1) + 5i$

$= -2 + 15i + 3$

$= 1 + 15i$

Express the following in standard form, that is, in the form a + b*i*:

$(1 + i)^2$, $(1 + i)^3$, $(1 + i)^4$, etc.

10. $(1 + i)^2$

$= 1 + 2i + i^2$

$= 1 + 2i - 1$

$= 2i.$

11. $(1 + i)^3$

$= 1 + 3i + 3i^2 + i^3$

$= 1 + 3i + 3(-1) + i^2 \cdot i$

$= 1 + 3i - 3 + (-1)i$

$= -2 + 2i$

12. $(1 + i)^4$

$= 1 + 4i + 6i^2 + 4i^3 + i^4$

$= 1 + 4i + 6(-1) + 4(-1)i + (-1)^2$

$= 1 + 4i - 6 - 4i + 1$

$= -4$

13. Find out what is wrong with the following calculation.

$$1 = \sqrt{1}$$

$$= \sqrt{(-1)(-1)}$$

$$= \sqrt{-1}\ \sqrt{-1}$$

$$= i \cdot i$$

$$= i^2$$

$= -1$

The square root function has two different values for each input. For instance,

$i \cdot i = -1$ and $-i \cdot -i = -1$.

Therefore, $\sqrt{-1}$ as a function of -1 assumes two values *i* and –*i*.

For such functions, we cannot build equalities by fixing the value of the function as we please in each step.

APPENDIX B: SOLUTIONS TO HISTORICAL PROBLEMS

The next problems are collection of exercises which are based on, or motivated by the history of complex numbers. These problems are aligned with the Common Core sate standards.

1. Show that the set {0,1} is a field if addition and multiplication are defined in a binary system.

By the table for addition and multiplication on page 4, the set {0,1} is closed under addition and multiplication. Commutativity and associativity properties are easy to check. Note that 0 is the additive identity and 1 is the multiplicative identity. The additive inverse of 0 is 0 and the multiplicative inverse of 1 is 1 itself. To check distributivity property, let us take any a, b, and c from the set {0, 1}.

If a = 0, then

$0\,(b + c) = 0$ by the multiplication table;

$= 0b + 0c.$

If a = 1, then $1(b + c) = b + c$ by the multiplication table;

$= 1 \cdot b + 1 \cdot c.$

Thus, {0,1} is a field.

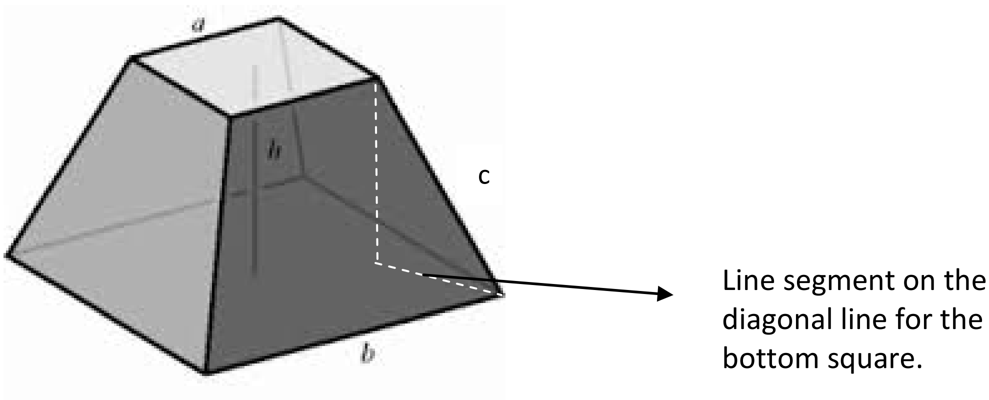
2. Prove or disprove:

The set **Z** of all integers is a field.

The set **Z** is not a field under the usual operations of addition and multiplication since multiplicative inverse of each element of **Z** fails to be in **Z.** For example, 2 ϵ **Z**. Its multiplicative inverse, ½ is not in **Z**.

3. By drawing a truncated pyramid, show the details of the derivation of equation (2).

$$h = \sqrt{c^2 - 2\left(\frac{b-a}{2}\right)^2} \qquad (2)$$



The length of this segment is

$$= \frac{\sqrt{2}\,b}{2} - \frac{\sqrt{2}\,a}{2}$$

Length of half the diagonal for the bottom square ($\frac{\sqrt{2}\,b}{2}$); ½ the length of the diagonal for the top square ($\frac{\sqrt{2}\,a}{2}$)

$$= \sqrt{2}\left(\frac{b-a}{2}\right)$$

Therefore,

$$h = \sqrt{c^2 - 2\left(\frac{b-a}{2}\right)^2}$$

4. (i) Carry out the details of “squaring and simplifying” that are mentioned in page 10.

$$\sqrt{1 + 196x^4} = 12x - 1 - 14x^2$$

Squaring both sides, $1 + 96x^4 = 144x^2 + 1 + 196x^4 - 24x + 28x^2 - 336x^3$

Rearranging terms,

Therefore, $336x^3 - 172x^2 + 24x = 0$.

(ii) In Diophantus' problem, the perimeter of a right triangle is larger than the area of a triangle. Is this the reason for the non-existence of a solution? Give your reasons.

No. This is not the reason. Side lengths can be ascertained even if the perimeter of a right triangle is larger than its area. For instance, the right triangle with side lengths 1, 1, $\sqrt{2}$ has a perimeter 2 + $\sqrt{2}$ which is larger than its area, ½ .

5. In Cardano's problem on page 12, show that the product of the roots is equal to 40.

The roots are $5 \pm \sqrt{15}\, i$

Therefore, $(5 + \sqrt{15}\, i)\,(5 - \sqrt{15}\, i)$

$= 25 + 5\,(\sqrt{15}\,)\, i - 5\sqrt{15}\, i + (\sqrt{15}\, i)\,(-\sqrt{15}\, i)$

$= 25 - 15i^2$

$= 40.$

6. Use Cardano's method to reduce the cubic equation $x^3 - 3x^2 + x - 3 = 0.$ to a depressed cubic equation.

Consider

$$x^3 - 3x^2 + x - 3 = 0.$$

$$b = -3.$$

Replace x by x - $\frac{b}{3}$, That is, x - $\frac{(-3)}{3}$ = x + 1 in the cubic polynomial. Then

$$(x + 1)^3 - 3(x + 1)^2 + (x + 1) - 3 = 0.$$

Therefore, we get

$$x^3 + 3x^2 + 3x + 1 - 3(x^2 + 2x + 1) + (x + 1) - 3 = 0.$$

Therefore,

$$x^3 + 3x^2 + 3x + 1 - 3x^2 - 6x - 3 + x + 1 - 3 = 0$$

which simplifies to $x^3 – 2x – 4 = 0$. This is the depressed cubic equation.

7. Given the cubic polynomial $x^3 + 3x^2 - 21x – 21 = 0$, find the depressed cubic equation.

For the cubic equation $x^3 + 3x^2 -21x - 21 = 0$, b = 3

Therefore, $x - \frac{b}{3} = x – 1$.

Therefore, $(x – 1)^3 + 3(x -1)^2 – 21(x -1) – 21 = 0$ gives

$x^3 – 3x^2 + 3x – 1 + 3(x^2 – 2x + 1) – 21x + 21 – 21 = 0$.

Therefore, $x^3 – 24x + 2 = 0$ upon simplification.

8. For the depressed equation obtained in (7), use del Ferro's idea to find u^3 and v^3.

If $x^3 – 24x + 2 = 0$ is identified with $x^3 + px + q = 0$, then

p = -24

q = 2

Therefore, del Ferro's formulas yield

$$u^3 = \frac{-2 \pm \sqrt{4 + \frac{4}{27}(-24)^3}}{2} \qquad v^3 = \frac{-2 \mp \sqrt{4 + \frac{4}{27}(-24)^3}}{2}$$

$$u^3 = \frac{-2 \pm \sqrt{4 - 2048}}{2} \qquad v^3 = \frac{-2 \mp \sqrt{4 - 2048}}{2}$$

$$u^3 = \frac{-2 \pm \sqrt{-2044}}{2} \qquad v^3 = \frac{-2 \mp \sqrt{-2044}}{2}$$

$u^3 = \frac{-2 \pm 2\sqrt{-511}}{2}$ $v^3 = \frac{-2 \mp 2\sqrt{-511}}{2}$

$u^3 = -1 \pm 22.61\, i$ $u^3 = -1 \mp 22.61\, i$

VITA

Esperanza Chavez is a native of the Philippines. She finished her Bachelor of Secondary Education major in Mathematics at St. Joseph's College, Quezon City, Philippines in 1994 and earned her Master of Arts in Education major in Mathematics at The National Teachers College, Manila, Philippines in 1999. She was a full scholar for four years while in college and graduated as top eight among the graduating class of 1994. She spent her first two years as a high school math teacher in Indang, Cavite, Philippines and the next 12 years in teaching college students before she came to America where she is now a math teacher at Broadmoor High School. She is married and has two wonderful daughters – Charlene and Michaella.