1-1-2014

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Construction of Basis Vectors For Symmetric Irreducible Representations of $O(5) \supset O(3)$

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(Dated: August 24, 2018)

A recursive method for construction of symmetric irreducible representations of $O(2l+1)$ in the $O(2l+1) \supset O(3)$ basis for identical boson systems is proposed. The formalism is realized based on the group chain $U(2l+1) \supset U(2l-1) \otimes U(2)$, of which the symmetric irreducible representations are simply reducible. The basis vectors of the $O(2l+1) \supset O(2l-1) \otimes U(1)$ can easily be constructed from those of $U(2l+1) \supset U(2l-1) \otimes U(2) \supset O(2l-1) \otimes U(1)$ with no boson pairs, from which one can construct symmetric irreducible representations of $O(2l+1)$ in the $O(2l-1) \otimes U(1)$ basis when all symmetric irreducible representations of $O(2l-1)$ are known. As a starting point, basis vectors of symmetric irreducible representations of $O(5)$ are constructed in the $O_1(3) \otimes U(1)$ basis. Matrix representations of $O(5) \supset O_1(3) \otimes U(1)$, together with the elementary Wigner coefficients, are presented. After the angular momentum projection, a three-term relation in determining the expansion coefficients of the $O(5) \supset O(3)$ basis vectors in terms of those of the $O_1(3) \otimes U(1)$ is derived. The eigenvectors of the projection matrix with zero eigenvalues constructed according to the three-term relation completely determine the basis vectors of $O(5) \supset O(3)$. Formulae for evaluating the elementary Wigner coefficients of $O(5) \supset O(3)$ are derived explicitly. Analytical expressions of some elementary Wigner coefficients of $O(5) \supset O(3)$ for the coupling $(\tau\ 0) \otimes (1\ 0)$ with resultant angular momentum quantum number $L = 2\tau + 2 - k$ for $k = 0, 2, 3, \cdots, 6$ with a multiplicity 2 case for $k = 6$ are presented.

PACS numbers: 03.65.Fd, 02.20.Qs, 21.60.Fw, 02.30.Ik

I. INTRODUCTION

The orthogonal group $O(2l+1)$ and its Lie algebra occurs naturally in the classification of many-particle states of identical bosons with angular momentum l referred to as the l-bosons hereafter under the group chain $U(2l+1) \supset O(2l+1) \supset O(3)$, which is useful in atomic, molecular, and nuclear physics [1–3]. However, to construct $U(2l+1) \supset O(2l+1) \supset O(3)$ basis vectors is not easy mainly because the missing label problem in the reduction $O(2l+1) \downarrow O(3)$, which is not multiplicity-free in general. A non-trivial simplest case is to construct symmetric irreducible representations (irreps) of the $O(5)$ group in the $O(3)$ basis for identical d-bosons useful in the nuclear collective model [4, 5] and the interacting boson model for nuclei [6]. Because of its physical importance, there have been a lot of attempts to construct the $O(5) \supset O(3)$ basis vectors [7–22]. Most notably, Rowe, Hecht, and many others in a series of papers [23–26] established the vector-coherent-state (VCS) representations of $O(5) \supset O(3)$ and constructed the $O(5)$ spherical harmonics [27]. As shown in [27], the $O(5)$ spherical harmonics are useful for calculating the Wigner coefficients of $O(5) \supset O(3)$, which can also be computed in a number of other ways, for example, those shown in [28, 29].

There are many subgroup chains of $O(5)$, for example, those shown in [30]. Besides the VCS construction, the basis vectors of $O(5) \supset O(3)$ can be expanded in terms of any one of other group chains of $O(5)$. Similarly, for identical l-boson systems, basis vectors of $O(2l+1) \supset O(3)$ may be expanded in terms of those of $O(2l+1) \supset O(2l-1) \otimes U(1)$, which thus provides a systematic recursive procedure to construct the basis vectors of $O(2l+1) \supset O(3)$ starting with $l = 2$. In this paper, we focus on the $l = 2$ case to show how the procedure works.

II. THE $U(2l+1) \supset U(2l-1) \otimes U(2)$ BASIS FOR l-BOSONS

Let $b^\dagger_\mu$ (b_μ) ($\mu = -l, -l+1, \cdots, l$) be boson creation (annihilation) operators satisfying the following commutation relations:

$$[b_\mu,\ b_{\mu'}] = [b^\dagger_\mu,\ b^\dagger_{\mu'}] = 0, \quad [b_\mu,\ b^\dagger_{\mu'}] = \delta_{\mu\mu'}. \tag{1}$$

The $(2l+1)^2$ bilinear forms $\{b^\dagger_\mu b_{\mu'}\}$ or the equivalent $O(3)$ tensors $\left(b^\dagger \times \tilde{b}\right)^{(k)}_\mu$ with $k = 0,\ 1, \cdots, 2l$ and $\mu = k, k-1, \cdots, -k$ for fixed k, in which $\tilde{b}_\mu = (-)^{l-\mu} b_{-\mu}$, generate the $U(2l+1)$ algebra, where for convenience the

Lie group notation is also used to denote the corresponding Lie algebra. It is well known that $(2l+1)l$ operators $\left(b^{\dagger}\times\tilde{b}\right)^{(k)}_{\mu}$ with k = odd generate the subalgebra $O(2l+1)$. Moreover, $\left(b^{\dagger}\times\tilde{b}\right)^{(1)}_{\mu}$ are generators of the $O(3)$ subalgebra. In many physics applications, one needs to construct the $U(2l+1)$ basis adapted to the group chain $U(2l+1)\supset O(2l+1)\supset O(3)$. The reduction of $O(2l+1)\downarrow O(3)$ is not multiplicity-free except the trivial $l=1$ case.

Actually, there is a simple mathematical basis for $U(2l+1)$ when its Lie algebra is realized in terms of boson creation and annihilation operators. The $\left(b^{\dagger}\times\tilde{b}\right)^{(k)}_{\mu}$ with $k=1,\cdots,2l-2$ constructed from $b^{\dagger}_{\mu}$ (b_{μ}) with $\mu=-(l-1),-(l-1)+1,\cdots,l-1$ generate the $U(2l-1)$ subalgebra, while $J_{+}=b^{\dagger}_{l}b_{-l}$, $J_{-}=b^{\dagger}_{-l}b_{l}$, and $J_{0}=\frac{1}{2}(b^{\dagger}_{l}b_{l}-b^{\dagger}_{-l}b_{-l})$ generate the $U(2)$ subalgebra. Obviously, $U(2l-1)\otimes U(2)$ is a subgroup of $U(2l+1)$. For a given irrep $[n\dot{0}]$ of $U(2l+1)$, the reduction $U(2l+1)\downarrow U(2l-1)\otimes U(2)$ is simple with

$$\begin{array}{ccc} U(2l+1) & \downarrow & U(2l-1)\otimes U(2) \\ {[n\dot{0}]} & \downarrow & \oplus^{n}_{2J=0}\,[n-2J\;\dot{0}]\otimes J \end{array}, \tag{2}$$

where for simplicity we use the spinor quantum number J to label irreps of the $U(2)$, with the corresponding basis vectors denoted as

$$\left|\begin{array}{cc} \multicolumn{2}{c}{[n\dot{0}]} \\ {[n-2J\;\dot{0}]} & J \\ (\nu) & m_J \end{array}\right\rangle \equiv \left|\begin{array}{cc} {[n-2J\;\dot{0}]} & J \\ (\nu) & m_J \end{array}\right\rangle, \tag{3}$$

where (ν) stands for a set of quantum numbers needed to label the irrep $[n-2J\;\dot{0}]$ of $U(2l-1)$.

Then, the basis vectors of $U(2l+1)\supset O(2l+1)\supset O(3)$ can be expanded in terms of those of $U(2l+1)\supset U(2l-1)\otimes U(2)$ as

$$\left|\begin{array}{c} [n\dot{0}] \\ (\tau\dot{0}) \\ \alpha L M_L \end{array}\right\rangle = \sum_{(\nu)Jm_J} a^{(\nu)Jm_J}_{n\tau} \left|\begin{array}{cc} {[n-2J\;\dot{0}]} & J \\ (\nu) & m_J \end{array}\right\rangle, \tag{4}$$

where τ is the seniority quantum number for labeling the $O(2l+1)$ irrep, α is the multiplicity label needed to distinguish from basis vectors with the same angular momentum L, and $a^{(\nu)Jm_J}_{n\tau}$ is the corresponding expansion coefficient. We always assume that the basis vectors of $U(2l+1)\supset U(2l-1)\otimes U(2)$ are orthonormal.

In the construction of (4), the l-boson pairing operator defined as

$$P^{\dagger}_{l}=\sqrt{\frac{1}{2}}\sum_{\mu=-l}^{l}(-)^{l-\mu}b^{\dagger}_{\mu}b^{\dagger}_{-\mu} \tag{5}$$

is a useful construction that satisfies the following commutation relation

$$[P_l,\;P^{\dagger\xi}_{l}]=\xi P^{\dagger\xi-1}_{l}\left(2\sum_{\mu=-l}^{l}b^{\dagger}_{\mu}b_{\mu}+2\xi+2l-1\right). \tag{6}$$

The basis vectors of $U(2l+1)\supset O(2l+1)\supset O(3)$ with $n>\tau$ can be expressed by those with $n=\tau$ and the pairing operator (5) as [28, 31]

$$\left|\begin{array}{c} [n\dot{0}] \\ (\tau\dot{0}) \\ \alpha L M_L \end{array}\right\rangle = \left[\frac{(2\tau+2l-1)!!}{\xi!(2\tau+2\xi+2l-1)!!}\right]^{\frac{1}{2}} P^{\dagger\xi}_{l} \left|\begin{array}{c} [\tau\dot{0}] \\ (\tau\dot{0}) \\ \alpha L M_L \end{array}\right\rangle =$$

$$\left[\frac{(2\tau+2l-1)!!}{\xi!(2\tau+2\xi+2l-1)!!}\right]^{\frac{1}{2}} \sum_{(\nu)Jm_J} a^{(\nu)Jm_J}_{n\tau} P^{\dagger\xi}_{l} \left|\begin{array}{cc} {[\tau-2J\;\dot{0}]} & J \\ (\nu) & m_J \end{array}\right\rangle, \tag{7}$$

where $n=\tau+2\xi$, $\left|\begin{array}{c}[\tau\dot{0}]\\(\tau\dot{0})\\\alpha LM_L\end{array}\right\rangle$ is the l-boson pair vacuum state equivalent to the basis vectors of $O(2l+1)\supset O(3)$ satisfying

$$P_l\left|\begin{array}{c}[\tau\dot{0}]\\(\tau\dot{0})\\\alpha LM_L\end{array}\right\rangle\equiv P_l\left|\begin{array}{c}(\tau\dot{0})\\\alpha LM_L\end{array}\right\rangle=0. \tag{8}$$

It follows from this that once the orthonormal basis vectors of $U(2l-1)\supset O(2l-1)$ are constructed, those of $U(2l+1)\supset O(2l+1)\supset O(3)$ can be worked out according to Eq. (7), which provides a recursive procedure for constructing basis vectors of $U(2l+1)\supset O(2l+1)\supset O(3)$ from those of $U(2l-1)\supset O(2l-1)$ starting with $l=2$.

III. MATRIX REPRESENTATIONS OF $O(5)$ IN THE $O_1(3)\times U(1)$ BASIS

In the following, we use (7) to construct the basis vectors of $O(5)\supset O_2(3)$ from those of the $O(5)\supset O_1(3)\otimes U(1)$ as the starting point, where the quantum numbers of $O_2(3)\equiv O(3)$ are exactly those of the angular momentum of the d-boson system, of which the creation operators are expressed as $\{b^\dagger_{-2},b^\dagger_{-1},\cdots,b^\dagger_2\}$. The procedure involves two steps: (i) Firstly, we construct the basis vectors of $O(5)\supset O_1(3)\otimes U(1)$ from those of $U(5)\supset U(3)\otimes U(2)\supset O_1(3)\otimes U(1)$. (ii) Then, we expand the basis vectors of $O(5)\supset O(3)$ in terms of those of $O(5)\supset O_1(3)\otimes U(1)$.

In this case, generators of $O_1(3)$ are written in the canonical form as

$$l_+=\sqrt{2}(b^\dagger_1 b_0+b^\dagger_0 b_{-1}),\ l_-=(l_+)^\dagger,\ l_0=b^\dagger_1 b_1-b^\dagger_{-1}b_{-1}, \tag{9}$$

which satisfy the commutation relations

$$[l_+,\ l_-]=2l_0,\ [l_0,\ l_\pm]=\pm l_\pm. \tag{10}$$

Similarly, generators of $O(3)$ are written as

$$L_+=\sqrt{\frac{3}{2}}l_++\sqrt{2}(b^\dagger_2 b_1+b^\dagger_{-1}b_{-2}),\ L_-=(L_+)^\dagger,\ L_0=l_0+4J_0. \tag{11}$$

The orthonormal basis vectors of $U(3)\supset O_1(3)\supset O_1(2)$ and those of the $U(2)\supset U(1)$ are well known [32, 33]:

$$\left|\begin{array}{c}[r+2\xi\ \dot{0}]\\r\\m_r\end{array}\right\rangle=\left[\frac{(2r+1)!!}{\xi!(2r+2\xi+1)!!}\right]^{\frac{1}{2}}P_1^{\dagger\xi}\left|\begin{array}{c}r\\m_r\end{array}\right\rangle=$$

$$\left[\frac{2^{r+m_r}(2r+1)!!(r+m_r)!(r-m_r)!r!}{\xi!(2r+2\xi+1)!!(2r)!}\right]^{\frac{1}{2}}P_1^{\dagger\xi}\sum_x\frac{b_1^{\dagger x}b_0^{\dagger r+m_r-2x}b_{-1}^{\dagger x-m_r}}{2^x(x-m_r)!x!(r+m_r-2x)!}|0\rangle \tag{12}$$

for the $U(3)\supset O_1(3)\supset O_1(2)$, where $|0\rangle$ is the boson vacuum state, and

$$\left|\begin{array}{c}J\\m_J\end{array}\right\rangle=\frac{b_2^{\dagger J+m_J}b_{-2}^{\dagger J-m_J}}{\sqrt{(J+m_J)!(J-m_J)!}}|0\rangle \tag{13}$$

for the $U(2)\supset U(1)$.

According to (7), (12), and (13), the $O(5)\supset O_1(3)\otimes U(1)$ basis vectors may be expanded in terms of those $U(5)\supset U(3)\otimes U(2)\supset O_1(3)\otimes U(1)$ as

$$\left|\begin{array}{c}(r+2m_J+t\ 0)\\r\ m_r,\ m_J\end{array}\right\rangle=\sum_{\xi=0}^{t/2}a_\xi^{t,r,m_J}\left[\frac{(2r+1)!!}{\xi!(2r+2\xi+1)!!}\right]^{\frac{1}{2}}P_1^{\dagger\xi}\left|\begin{array}{cc}[r\ \dot{0}] & m_J+t/2-\xi\\r & m_J\\m_r & \end{array}\right\rangle, \tag{14}$$

where t is an even integer, which should satisfy

$$P_2 \left| \begin{array}{c} (r+2m_J+t\ 0) \\ r\ m_r,\ m_J \end{array} \right\rangle = \left(\sqrt{2}b_2 b_{-2} - P_1\right) \left| \begin{array}{c} (r+t+2m_J\ 0) \\ r\ m_r,\ m_J \end{array} \right\rangle = 0. \tag{15}$$

Eq. (15) leads to the following relation:

$$a_{\xi+1}^{t,r,m_J} = \left[\frac{(4m_J+t-2\xi)(t-2\xi)}{2(\xi+1)(2r+2\xi+3)}\right]^{\frac{1}{2}} a_{\xi}^{t,r,m_J}. \tag{16}$$

Using Eq. (16), we have

$$a_{\xi}^{t,r,m_J} = \left[\frac{(4m_J+t)!!(2r+1)!!t!!}{(4m_J+t-2\xi)!!(2\xi)!!(2r+2\xi+1)!!(t-2\xi)!!}\right]^{\frac{1}{2}} a_0^{t,r,m_J}. \tag{17}$$

Substituting (17) into (14), one has

$$\left| \begin{array}{c} (r+2m_J+t\ 0) \\ r\ m_r,\ m_J \end{array} \right\rangle = \sum_{\xi=0}^{t/2} \left[\frac{(4m_J+t)!!(2r+1)!!^2 t!!}{(4m_J+t-2\xi)!!(2\xi)!!(2r+2\xi+1)!!^2(t-2\xi)!!\xi!}\right]^{\frac{1}{2}} \times$$

$$a_0^{t,r,m_J} P_1^{\dagger\xi} \left| \begin{array}{cc} [r\ \dot{0}] & \\ r & m_J+t/2-\xi \\ m_r & m_J \end{array} \right\rangle. \tag{18}$$

The normalization condition of (18) leads to the following expression

$$\left| \begin{array}{c} (\tau\ 0) \\ r\ m_r,\ m_J \end{array} \right\rangle = \sum_{\xi=0}^{t/2} \left[\frac{(2\tau+1-t)!!(4m_J+t)!!(2r+t+1)!!t!!}{(2\tau+1)!!(4m_J+t-2\xi)!!(2\xi)!!(2r+2\xi+1)!!(t-2\xi)!!}\right]^{\frac{1}{2}} \times$$

$$\left| \begin{array}{cc} [r+2\xi\ 0] & \\ r & m_J+t/2-\xi \\ m_r & m_J \end{array} \right\rangle = \sum_{\xi=0}^{t/2} b_{\xi}^{\tau,r,m_J,t} \left| \begin{array}{cc} [r+2\xi\ 0] & \\ r & m_J+t/2-\xi \\ m_r & m_J \end{array} \right\rangle \tag{19}$$

with $\tau = r+2m_J+t$. In derivation of (19), the identity

$$\sum_{\xi=0}^{t/2} \frac{(4m_J+t)!!t!!}{(4m_J+t-2\xi)!!(2\xi)!!(2r+2\xi+1)!!(t-2\xi)!!} = \frac{(2\tau+1)!!}{(2\tau-t+1)!!(2r+t+1)!!} \tag{20}$$

is used, and the overall phase of (19) is thus fixed. It is clear from the construction of (19) that the branching rule of $O(5) \downarrow O_1(3) \otimes U(1)$ for the symmetric irrep $(\tau\ 0)$ of $O(5)$ is given by

$$r+2m_J = \ \tau,\ \tau-2,\ \tau-4,\ \cdots, \left\{ \begin{array}{l} 0 \text{ when } \tau \text{ is even}, \\ 1 \text{ when } \tau \text{ is odd}. \end{array} \right. \tag{21}$$

Under the $O(5) \supset O_1(3) \times U(1)$ basis, the boson operators $\{b_1^\dagger, b_0^\dagger, b_{-1}^\dagger, b_2^\dagger, b_{-2}^\dagger\}$ are rank-1 irreducible tensor operators of $O(5)$ with $T_{1\mu;0}^{(10)} = b_\mu^\dagger$ for $\mu = 1, 0, -1$, and $T_{00;\pm\frac{1}{2}}^{(10)} = b_{\pm 2}^\dagger$. Since these irreducible tensor operators appear in (11), we need matrix elements of them under the $O(5) \supset O(3)_1 \times O(2)$ basis in order to make the angular momentum projection.

By using the explicit expression (19) and Wigner-Eckart theorem, one finds

$$b_2^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=\sum_{\xi=0}^{t/2}b_\xi^{\tau,r,m_J,t}\left\langle\begin{array}{cc}m_J+t/2-\xi & 1/2\\ m_J & 1/2\end{array}\right|\left.\begin{array}{c}m_J+t/2-\xi+1/2\\ m_J+1/2\end{array}\right\rangle\times$$

$$\sqrt{2m_J+t-2\xi+1}\left|\begin{array}{cc}[r+2\xi\ 0] & m_J+t/2-\xi+1/2\\ r & m_J+1/2\\ m_r & \end{array}\right\rangle=$$

$$\sum_{\xi=0}^{t/2}b_\xi^{\tau,r,m_J,t}\sqrt{\frac{1}{2}(4m_J+t-2\xi+2)}\left|\begin{array}{cc}[r+2\xi\ 0] & m_J+t/2-\xi+1/2\\ r & m_J+1/2\\ m_r & \end{array}\right\rangle, \tag{22}$$

where $\left\langle\begin{array}{cc}m_J+t/2-\xi & 1/2\\ m_J & 1/2\end{array}\right|\left.\begin{array}{c}m_J+t/2-\xi+1/2\\ m_J+1/2\end{array}\right\rangle$ is the CG coefficient of $U(2)$, from which we obtain

$$\left\langle\begin{array}{c}(\tau+1\ 0)\\ r\ m_r,m_J+1/2\end{array}\right|b_2^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=\sqrt{\frac{(\tau+r+2m_J+3)(\tau-r+2m_J+2)}{2(2\tau+3)}}. \tag{23}$$

While

$$b_{-2}^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=\sum_{\xi=0}^{t/2}b_\xi^{\tau,r,m_J,t}\sqrt{\frac{1}{2}(t-2\xi+2)}\left|\begin{array}{cc}[r+2\xi\ 0] & m_J+t/2-\xi+1/2\\ r & m_J-1/2\\ m_r & \end{array}\right\rangle, \tag{24}$$

from which we have

$$\left\langle\begin{array}{c}(\tau+1\ 0)\\ r\ m_r,m_J-1/2\end{array}\right|b_{-2}^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=\sqrt{\frac{(\tau+r-2m_J+3)(\tau-r-2m_J+2)}{2(2\tau+3)}}. \tag{25}$$

Similarly, we have

$$b_1^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=$$

$$\sum_{\xi=0}^{t/2}b_\xi^{\tau,r,m_J,t}\sqrt{\frac{(r+m_r+1)(r+m_r+2)(2r+2\xi+3)}{2(2r+1)(2r+3)}}\left|\begin{array}{cc}[r+2\xi+1\ 0] & m_J+t/2-\xi\\ r+1 & m_J\\ m_r+1 & \end{array}\right\rangle+$$

$$\sum_{\xi=0}^{t/2}b_\xi^{\tau,r,m_J,t}\sqrt{\frac{(r-m_r)(r-m_r-1)(2\xi+2)}{2(2r+1)(2r-1)}}\left|\begin{array}{cc}[r+2\xi+1\ 0] & m_J+t/2-\xi\\ r-1 & m_J\\ m_r+1 & \end{array}\right\rangle, \tag{26}$$

from which we get

$$\left\langle\begin{array}{c}(\tau+1\ 0)\\ r+1\ m_r+1,m_J\end{array}\right|b_1^\dagger\left|\begin{array}{c}(\tau\ 0)\\ r\ m_r,\ m_J\end{array}\right\rangle=$$

$$\sqrt{\frac{(\tau+r+2m_J+3)(\tau+r-2m_J+3)(r+m_r+1)(r+m_r+2)}{2(2\tau+3)(2r+3)(2r+1)}} \tag{27}$$

and

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r-1\ m_r+1, m_J \end{matrix} \middle| b_1^\dagger \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r,\ m_J \end{matrix} \right\rangle =$$

$$\sqrt{\frac{(\tau-r+2m_J+2)(\tau-r-2m_J+2))(r-m_r)(r-m_r-1)}{2(2\tau+3)(2r+1)(2r-1)}} \tag{28}$$

By using the similar procedure, we also get

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r+1\ m_r-1, m_J \end{matrix} \middle| b_{-1}^\dagger \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r,\ m_J \end{matrix} \right\rangle =$$

$$\sqrt{\frac{(\tau+r+2m_J+3)(\tau+r-2m_J+3)(r-m_r+1)(r-m_r+2)}{2(2\tau+3)(2r+3)(2r+1)}}, \tag{29}$$

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r-1\ m_r-1, m_J \end{matrix} \middle| b_{-1}^\dagger \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r,\ m_J \end{matrix} \right\rangle =$$

$$\sqrt{\frac{(\tau-r+2m_J+2)(\tau-r-2m_J+2)(r+m_r)(r+m_r-1)}{2(2\tau+3)(2r+1)(2r-1)}}, \tag{30}$$

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r+1\ m_r, m_J \end{matrix} \middle| b_0^\dagger \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r,\ m_J \end{matrix} \right\rangle =$$

$$\sqrt{\frac{(\tau+r+2m_J+3)(\tau+r-2m_J+3)(r-m_r+1)(r+m_r+1)}{(2\tau+3)(2r+3)(2r+1)}}, \tag{31}$$

and

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r-1\ m_r, m_J \end{matrix} \middle| b_0^\dagger \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r,\ m_J \end{matrix} \right\rangle =$$

$$-\sqrt{\frac{(\tau-r+2m_J+2)(\tau-r-2m_J+2)(r+m_r)(r-+m_r)}{(2\tau+3)(2r+1)(2r-1)}}. \tag{32}$$

As is well-known, the matrix elements of single-boson operators (23), (25), and (27)-(32) are key in deriving matrix elements of the $O(5)$ generators. Thus, the matrix representations of $O(5) \supset O_1(3) \otimes U(1)$ are completely determined.

Using the Racah factorization lemma [3], which is also called the generalized Wigner-Eckart theorem, we have

$$\left\langle \begin{matrix} (\tau+1\ 0) \\ r'\ m_r', m_J \end{matrix} \middle| T^{(10)}_{1\mu;0} \middle| \begin{matrix} (\tau\ 0) \\ r\ m_r, m_J \end{matrix} \right\rangle = \left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ r,\ m_J & 1,\ 0 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ r',\ m_J \end{matrix} \right\rangle \langle r m_r, 1\mu | r' m_r' \rangle \, \langle (\tau+1\ 0) \| T^{(10)} \| (\tau\ 0) \rangle, \tag{33}$$

where $\left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ r,\ m_J & 1,\ 0 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ r',\ m_J \end{matrix} \right\rangle$ is the elementary Wigner coefficient or called Isoscalar Factor (ISF) of $O(5) \supset O_1(3) \otimes U(1)$, $\langle r m_r, 1\mu | r' m_r' \rangle$ is the CG coefficient of $O_1(3)$, and $\langle (\tau+1\ 0) \| T^{(10)} \| (\tau\ 0) \rangle$ is the $O(5)$-reduced matrix element satisfying

$$\langle (\tau' 0) \| T^{(10)} \| (\tau 0) \rangle = \delta_{\tau',\tau+1}\sqrt{\tau+1} = \sqrt{\frac{\dim(\tau\ 0)}{\dim(\tau+1\ 0)}} \langle (\tau 0) \| U^{(10)} \| (\tau' 0) \rangle, \tag{34}$$

in which $\dim(\tau\ 0) = (\tau+1)(\tau+2)(2\tau+3)/6$ is the dimension of the $O(5)$ irrep $(\tau\ 0)$, while $U^{(10)}_{11,0} = b_{-1}$, $U^{(10)}_{10,0} = -b_0$, $U^{(10)}_{1-1,0} = b_1$, $U^{(10)}_{0,1/2} = -b_{-2}$, and $U^{(10)}_{0,-1/2} = -b_2$. Combining Eqs. (33), (34), and the symmetry properties of $O_1(3)$ CG coefficients, we have

$$\left\langle \begin{matrix} (\tau+1\ 0) & (1\ 0) \\ r',\ m_J & 1,\ 0 \end{matrix} \middle| \begin{matrix} (\tau\ 0) \\ r,\ m_J \end{matrix} \right\rangle = (-1)^{r+1-r'} \sqrt{\frac{(2r'+1)\dim(\tau 0)}{(2r+1)\dim(\tau+1\ 0)}} \left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ r,\ m_J & 1,\ 0 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ r',\ m_J \end{matrix} \right\rangle . \tag{35}$$

Similarly, we also have

$$\left\langle \begin{matrix} (\tau+1\ 0) & (1\ 0) \\ r,\ m'_J & 0, \pm 1/2 \end{matrix} \middle| \begin{matrix} (\tau\ 0) \\ r,\ m_J \end{matrix} \right\rangle = -\sqrt{\frac{\dim(\tau 0)}{\dim(\tau+1\ 0)}} \left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ r,\ m_J & 0, \mp 1/2 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ r,\ m'_J \end{matrix} \right\rangle . \tag{36}$$

All nonzero elementary Wigner coefficients of the $O(5) \supset O_1(3) \otimes U(1)$ are listed in Table I. These are useful for calculating matrix elements of the $O(5)$ irreducible tensor operators in the $O(5) \supset O_1(3) \otimes U(1)$ basis.

TABLE I: Wigner coefficients $\left\langle \begin{matrix} (\tau'\ 0) & (1\ 0) \\ r',\ m'_J & \mu,\ m \end{matrix} \middle| \begin{matrix} (\tau\ 0) \\ r,\ m_J \end{matrix} \right\rangle$ of $O(5) \supset O_1(3) \otimes U(1)$.

τ' \ $r',\ m'_J$; $\mu,\ m$	$r,\ m_J - 1/2$; $0,\ 1/2$	$r,\ m_J + 1/2$; $0,\ -1/2$
$\tau+1$	$-\left[\frac{(\tau+r-2m_J+3)(\tau-r-2m_J+2)}{2(\tau+3)(2\tau+5)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\tau+r+2m_J+3)(\tau-r+2m_J+2)}{2(\tau+3)(2\tau+5)}\right]^{\frac{1}{2}}$
$\tau-1$	$\left[\frac{(\tau+r+2m_J+1)(\tau-r+2m_J)}{2\tau(2\tau+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\tau+r-2m_J+1)(\tau-r-2m_J)}{2\tau(2\tau+1)}\right]^{\frac{1}{2}}$
τ' \ $r',\ m'_J$; $\mu,\ m$	$r-1,\ m_J$; $1,\ 0$	$r+1,\ m_J$; $1,\ 0$
$\tau+1$	$\left[\frac{(\tau-r+2m_J+2)(\tau-r-2m_J+2)r}{(\tau+3)(2\tau+5)(2r+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\tau+r+2m_J+3)(\tau+r-2m_J+3)(r+1)}{(\tau+3)(2\tau+5)(2r+1)}\right]^{\frac{1}{2}}$
$\tau-1$	$\left[\frac{(\tau+r+2m_J+1)(\tau+r-2m_J+1)r}{\tau(2\tau+1)(2r+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\tau-r+2m_J)(\tau-r-2m_J)(r+1)}{\tau(2\tau+1)(2r+1)}\right]^{\frac{1}{2}}$

IV. THE BASIS VECTORS OF $O(5) \supset O(3)$

The basis vectors of $O(5) \supset O(3) \supset O(2)$ can now be expanded in terms of those of the $O(5) \supset O_1(3) \otimes U(1)$ with the restriction $m_r + 4m_J = M_L$. For a given angular momentum quantum number $L = 2\tau - k$ with $M_L = L$, the quantum numbers of the $O_1(3) \supset O_1(2)$ and that of $U(1)$ may be parameterized as

$$\left| \begin{matrix} (\tau\ 0) \\ \zeta, L = M_L = 2\tau - k \end{matrix} \right\rangle = \sum_{q,t} c^{(\zeta)}_{q,t}(\tau,k) \left| \begin{matrix} (\tau\ 0) \\ k-q, k-2q+2t;\ (\tau-k+q-t)/2 \end{matrix} \right\rangle , \tag{37}$$

where ζ is the multiplicity label needed in the reduction $O(5) \downarrow O(3)$, which will be omitted if the reduction is simple, $c^{(\zeta)}_{q,t}(\tau,k)$ is the corresponding expansion coefficient, and $k = 0, 1, 2, \cdots, 2\tau$. (37) should satisfy

$$L_+ \left| \begin{array}{c} (\tau\ 0) \\ \zeta, L = M_L = 2\tau - k \end{array} \right\rangle = \left(\sqrt{\frac{3}{2}} l_+ + \sqrt{2}(b^\dagger_2 b_1 + b^\dagger_{-1} b_{-2}) \right) \left| \begin{array}{c} (\tau\ 0) \\ \zeta, L = M_L = 2\tau - k \end{array} \right\rangle = 0. \tag{38}$$

According to the Racah factorization lemma [3], by using the $O(5)$-reduced matrix element (34) and the Wigner coefficients shown in Table I, it can easily be proven that the condition (38) leads to the following three-term recurrence relation for the expansion coefficients $c^{(\zeta)}_{q,t}(\tau,k)$ needed in (37):

$$\left[\frac{(2k-3q+2t+2)(2k-3q+2t+3)(2k-2q+t+3)(2\tau-2k+2q-t)}{(2k-2q+1)(2k-2q+3)}\right]^{\frac{1}{2}} c^{(\zeta)}_{q-1,t}(\tau,k) +$$

$$[3(q-2t)(2k-3q+2t+1)]^{\frac{1}{2}}\, c^{(\zeta)}_{q,t}(\tau,k) + \left[\frac{(q-2t-1)(q-2t-2)(t+2)(2\tau-t+1)}{(2k-2q-1)(2k-2q+1)}\right]^{\frac{1}{2}} c^{(\zeta)}_{q+1,t+2}(\tau,k) = 0. \tag{39}$$

The boundary conditions for integers q and even integer t can be obtained from the allowed quantum number $m_r = k - 2q + 2t$ of $O_1(2)$ under the reduction of $O_1(3) \supset O_1(2)$ and allowed $m_J = (n - k + q - t)/2$ of $U(1)$ under the reduction of $U(2) \supset U(1)$ according to (37), which can be specified as

$$k - q \geq |k - 2q + 2t| \tag{40}$$

with $0 \leq q \leq k$ and $0 \leq t \leq \mathrm{Int}[k/2]$, where $\mathrm{Int}[x]$ is the integer part of x. A set of allowed $(q,\ t)$ combinations for given k are listed in Table II for $0 \leq k \leq 10$, which is generated by a simple Mathematica code according to (40).

TABLE II: Allowed $(q,\ t)$ combinations in the basis vectors (37) of $O(5) \supset O(3)$ for $L = 2\tau - k$ expanded in terms of those of $O_1(3) \otimes U(1)$ and the corresponding multiplicity $\mathrm{Multi}(\tau,k)$ for $\tau \geq 10$ and $k \leq 10$, where d is the total number of terms needed in the expansion.

k	(q,t)	d	$\mathrm{Multi}(\tau,k)$
0	$(0,0)$	1	1
1	-	0	0
2	$(0,0),(1,0)$	2	1
3	$(0,0),(1,0),(2,0)$	3	1
4	$(0,0),(1,0),(2,0),(4,2)$	4	1
5	$(0,0),(1,0),(2,0),(3,0),(4,2)$	5	1
6	$(0,0),(1,0),(2,0),(3,0),(4,0),(4,2),(5,2)$	7	2
7	$(0,0),(1,0),(2,0),(3,0),(4,0),(4,2),(5,2),(6,2)$	8	1
8	$(0,0),(1,0),(2,0),(3,0),(4,0),(4,2),(5,0),(5,2),(6,2),(8,4)$	10	2
9	$(0,0),(1,0),(2,0),(3,0),(4,0),(4,2),(5,0),(5,2),(6,0),(6,2),(7,2),(8,4)$	12	2
10	$(0,0),(1,0),(2,0),(3,0),(4,0),(4,2),(5,0),(5,2),(6,0),(6,2),(7,2),(8,2),(8,4),(9,4)$	14	2

Practically, one can construct a matrix equation of (39) with

$$\mathbf{P}(\tau,k)\mathbf{c}^{(\zeta)}(\tau,k) = \lambda \mathbf{c}^{(\zeta)}(\tau,k), \tag{41}$$

where the transpose of $\mathbf{c}^{(\zeta)}(\tau,k)$ is arranged as $\left(\mathbf{c}^{(\zeta)}(\tau,k)\right)^{\mathrm{T}} = (c^{(\zeta)}_{0,0}(\tau,k), c^{(\zeta)}_{1,0}(\tau,k), \cdots, c^{(\zeta)}_{4,0}(\tau,k), c^{(\zeta)}_{4,2}(\tau,k), \cdots)$, of which some examples are shown in Table II. Entries of the angular momentum projection matrix $\mathbf{P}(\tau,k)$ can easily be read out from Eq. (39). The components of eigenvector $\mathbf{c}^{(\zeta)}(\tau,k)$ corresponding to $\lambda = 0$ provide the expansion coefficients $\{c^{(\zeta)}_{q,t}(\tau,k)\}$ of (37). Once the matrix $\mathbf{P}(\tau,k)$ is constructed, it can be verified that the number of $\lambda = 0$ solutions of Eq. (41) for sufficiently large τ equals exactly to the number of rows of $\mathbf{P}(\tau,k)$ with all entries zero. However, some entries of $\mathbf{P}(\tau,k)$ will be zero or become complex for some specific values of τ. In such cases, nonzero solution of $\{c^{(\zeta)}_{q,t}(\tau,k)\}$ does not exist, which will be examined for $\tau \leq 8$ cases separately in the following. Furthermore, $\left(\mathbf{c}^{(\zeta')}(\tau,k)\right)^{\mathrm{T}} \cdot \mathbf{c}^{(\zeta)}(\tau,k) \neq 0$ when the multiplicity is greater than 1 mainly because the projection matrix $\mathbf{P}(\tau,k)$ is nonsymmetric. Therefore, the basis vectors (37) constructed from the expansion coefficients obtained according to (39) are non-orthogonal with respect to the multiplicity label ζ. The Gram-Schmidt process may be adopted in order to construct orthonormalized basis vectors of $O(5) \supset O(3)$.

On the other hand, for given $L = 2\tau - k$ of $O(3)$, the number of $\lambda = 0$ solutions, Multi(τ,k), of Eq. (41) with $\zeta = 1, 2, \cdots,$ Multi(τ,k) equals exactly to the multiplicity in the reduction $O(5) \downarrow O(3)$ for the symmetric irrep $(\tau\ 0)$ of $O(5)$, which may be calculated in the following way: Let $Q_\tau(k)$ be the number of different τ-partitions of the positive integer k with $k = \sum_{i=1}^{\tau} \xi_i$, where $4 \geq \xi_1 \geq \xi_2 \geq \cdots \geq \xi_\tau \geq 0$. Then, Multi$(\tau,k) = Q_\tau(k) + Q_{\tau-2}(k-5) - Q_\tau(k-1) - Q_{\tau-2}(k-4)$, where $Q_\tau(0) = 1$ and $Q_\tau(v) = 0$ when $v < 0$ may be defined for convenience in the computation. The corresponding Multi(τ,k) for given k and $\tau \geq 10$ are also shown in the last column of Table II.

In the following, we list some $\mathbf{P}(\tau,k)$ matrices and the corresponding expansion coefficients $\{c^{(\zeta)}_{q,t}(\tau,k)\}$. There is always a freedom in choosing the global phase. In our calculation, we always set $c_{0,0}(\tau,k) > 0$, while the relative phase is completely determined by the eigen-equation (41).

When $k = 0$, $\mathbf{P}(\tau,0) = 0$ with $c_{0,0}(\tau,0) = 1$, which is trivial corresponding to one unique highest weight state of the symmetric irrep $(\tau\ 0)$ of $O(5) \supset O(3)$ with $L = 2\tau$. When $k = 1$, $\mathbf{P}(\tau,1) = 2\sqrt{3}$ which requires $c_{0,0}(\tau,1) = 0$. Namely, there is no basis vector for the symmetric irrep $(\tau\ 0)$ of $O(5) \supset O(3)$ with $L = 2\tau - 1$. When $k = 2$,

$$\mathbf{P}(\tau,2) = \begin{pmatrix} 0 & 0 \\ 2\sqrt{2\tau-2} & \sqrt{6} \end{pmatrix} \tag{42}$$

with $(\mathbf{c}(\tau,2))^{\mathrm{T}} = (c_{0,0}(\tau,2), c_{1,0}(\tau,2))$. Since there is one row with all entries zero, the multiplicity of $L = 2\tau - 2$ is Multi$(\tau,2) = 1$ for $\tau > 1$. The normalized expansion coefficients are $c_{0,0}(\tau,2) = \sqrt{\frac{3}{4\tau-1}}$, $c_{1,0}(\tau,2) = -\sqrt{\frac{4(\tau-1)}{4\tau-1}}$ for $\tau > 1$. Though arbitrary $c_{0,0}(\tau,2)$ is a possible solution of (42) when $\tau = 1$, only $\mathbf{c}(\tau,2) = 0$ is valid according to the branching rule of $O(5) \downarrow O(3)$.

For $k = 3$,

$$\mathbf{P}(\tau,3) = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{6(2\tau-4)} & 2\sqrt{3} & 0 \\ 0 & \sqrt{2(2\tau-2)} & \sqrt{6} \end{pmatrix}. \tag{43}$$

Since there is one row with all entries zero in (43) when $\tau > 2$, the multiplicity of $L = 2\tau - 3$ is Multi$(\tau,3) = 1$ for $\tau > 2$. The normalized nonzero expansion coefficients corresponding to $\lambda = 0$ are $c_{0,0}(\tau,3) = \sqrt{\frac{3}{(2\tau-1)(\tau-1)}}$, $c_{1,0}(\tau,3) = -\sqrt{\frac{3(\tau-2)}{(2\tau-1)(\tau-1)}}$, $c_{2,0}(\tau,3) = \sqrt{\frac{2(\tau-2)}{2\tau-1}}$ for $\tau > 2$. Though arbitrary $c_{0,0}(\tau,3)$ is a possible solution of (43) when $\tau = 2$, only $\mathbf{c}(\tau,3) = 0$ is valid according to the branching rule of $O(5) \downarrow O(3)$.

For $k = 4$,

$$\mathbf{P}(\tau,4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{8(2\tau-6)} & 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{2(2\tau-4)} & 3\sqrt{2} & 0 \\ 0 & 0 & \sqrt{\frac{4(\tau-1)}{3}} & \sqrt{\frac{4(2\tau+1)}{3}} \end{pmatrix}. \tag{44}$$

Since there is one row with all entries zero in (44) when $\tau > 3$, the multiplicity of $L = 2\tau - 4$ is Multi$(\tau,4) = 1$ for $\tau > 3$. The normalized nonzero expansion coefficients corresponding to $\lambda = 0$ are $c_{0,0}(\tau,4) = \sqrt{\frac{27(2\tau+1)}{(2\tau-3)(4\tau-3)(4\tau-5)}}$, $c_{1,0}(\tau,4) = -\sqrt{\frac{24(2\tau+1)(\tau-3)}{(2\tau-3)(4\tau-3)(4\tau-5)}}$, $c_{2,0}(\tau,4) = \sqrt{\frac{32(2\tau+1)(\tau-2)(\tau-3)}{3(2\tau-3)(4\tau-3)(4\tau-5)}}$, $c_{4,2}(\tau,4) = -\sqrt{\frac{32(\tau-1)(\tau-2)(\tau-3)}{3(2\tau-3)(4\tau-3)(4\tau-5)}}$ for $\tau > 3$. Similar to the $k = 3$ case, though arbitrary $c_{0,0}(\tau)$ is a possible solution of (44) when $\tau = 3$, only $\mathbf{c}(\tau,4) = 0$ is valid according to the branching rule of $O(5) \downarrow O(3)$. Namely, $L = 2$ does not occur in the reduction $(3\ 0) \downarrow L$. Moreover, one entry in (44) becomes complex when $\tau = 2$ which must be ruled out since complex solutions obviously violate the branching rule of $O(5) \downarrow O(3)$.

For $k=5$,

$$\mathbf{P}(\tau,5)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{10(2\tau-8)} & 2\sqrt{6} & 0 & 0 & \\ 0 & \sqrt{6(2\tau-6)} & \sqrt{30} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{12(2\tau-4)}{5}} & 3\sqrt{2} & \sqrt{\frac{4(2\tau+1)}{15}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{20(\tau-1)}{3}} \end{pmatrix}. \tag{45}$$

Since there is also one row with the entries all zero in (45) when $\tau>4$, the multiplicity of $L=2\tau-5$ is $\mathrm{Multi}(\tau,5)=1$ for $\tau>4$. The normalized nonzero expansion coefficients corresponding to $\lambda=0$ are $c_{0,0}(\tau,5)=\sqrt{\frac{90}{(2\tau-3)(4\tau-7)(\tau-2)}}$, $c_{1,0}(\tau,5)=-\sqrt{\frac{75(\tau-4)}{(2\tau-3)(4\tau-7)(\tau-2)}}$, $c_{2,0}(\tau,5)=\sqrt{\frac{30(\tau-3)(\tau-4)}{(2\tau-3)(4\tau-7)(\tau-2)}}$, $c_{3,0}(\tau,5)=-\sqrt{\frac{8(\tau-3)(\tau-4)}{(2\tau-3)(4\tau-7)}}$, $c_{4,2}(\tau,5)=0$ for $\tau>4$. Similar to the discussions shown in the previous examples, only $\tau>4$ is allowed for the $k=5$ case.

For $k=6$,

$$\mathbf{P}(\tau,6)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{12(2\tau-10)} & \sqrt{30} & 0 & 0 & 0 & 0 & \\ 0 & \sqrt{8(2\tau-8)} & \sqrt{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{30(2\tau-6)}{7}} & 6 & 0 & \sqrt{\frac{4(2\tau+1)}{35}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{72(2\tau-4)}{5}} & 2\sqrt{3} & 0 & \sqrt{\frac{4(2\tau+1)}{5}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{28(2\tau-4)}{5}} & \sqrt{6} \end{pmatrix}. \tag{46}$$

Since there are two rows with all entries zero in (44) when $\tau>5$, the multiplicity of $L=2\tau-6$ is $\mathrm{Multi}(\tau,6)=2$ for $\tau>5$. The normalized nonzero expansion coefficients corresponding to $\lambda=0$ in this case are

$$c_{0,0}^{(\zeta=1)}(\tau,6)=\sqrt{\tfrac{2205}{f(\tau)}},\ c_{1,0}^{(\zeta=1)}(\tau,6)=-\sqrt{\tfrac{1764(\tau-5)}{f(\tau)}},\ c_{2,0}^{(\zeta=1)}(\tau,6)=\sqrt{\tfrac{672(\tau-4)(\tau-5)}{f(\tau)}},$$

$$c_{3,0}^{(\zeta=1)}(\tau,6)=-\sqrt{\tfrac{160(\tau-3)(\tau-4)(\tau-5)}{f(\tau)}},\ c_{4,0}^{(\zeta=1)}(\tau,6)=\sqrt{\tfrac{32(\tau-2)(\tau-3)(\tau-4)(\tau-5)}{f(\tau)}},\ c_{4,2}^{(\zeta=1)}(\tau,6)=0,\ \ c_{5,2}^{(\zeta=1)}(\tau)=0,$$

where $f(\tau)=32\tau^4-288\tau^3+1024\tau^2-1692\tau+1065$, and

$$c_{0,0}^{(\zeta=2)}(\tau,6)=\sqrt{\tfrac{405(2\tau+1)}{128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465}},\ \ c_{1,0}^{(\zeta=2)}(\tau,6)=-\sqrt{\tfrac{324(2\tau+1)(\tau-5)}{128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465}},$$

$$c_{2,0}^{(\zeta=2)}(\tau,6)=\sqrt{\tfrac{864(2\tau+1)(\tau-4)(\tau-5)}{7(128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465)}},\ c_{3,0}^{(\zeta=2)}(\tau,6)=-\sqrt{\tfrac{128(2\tau+1)(\tau-3)(\tau-4)(\tau-5)}{5(128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465)}},$$

$$c_{4,0}^{(\zeta=2)}(\tau,6)=0,\ \ c_{4,2}^{(\zeta=2)}(\tau,6)=-\sqrt{\tfrac{288(\tau-3)(\tau-4)(\tau-5)}{7(128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465)}},$$

$$c_{5,2}^{(\zeta=2)}(\tau,6)=\sqrt{\tfrac{384(\tau-2)(\tau-3)(\tau-4)(\tau-5)}{5(128\tau^4-1376\tau^3+5608\tau^2-10042\tau+6465)}}.$$

One can verify the $\left(\mathbf{c}^{(\zeta=1)}(\tau,6)\right)^T\cdot\mathbf{c}^{(\zeta=2)}(\tau,6)\neq 0$. After the Gram-Schmidt orthonormalization, we have

$$\bar{\mathbf{c}}^{\chi=1}(\tau,6)=\mathbf{c}^{(\zeta=1)}(\tau,6);$$

$$\bar{c}_{0,0}^{\chi=2}(\tau,6)=-\frac{12(3\tau-5)\sqrt{10(2\tau+1)(\tau-3)(\tau-4)(\tau-5)}}{\sqrt{(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)(\tau-2)f(\tau)}},$$

$$\bar{c}_{1,0}^{\chi=2}(\tau,6)=\frac{24(\tau-5)(3\tau-5)\sqrt{2(2\tau+1)(\tau-3)(\tau-4)}}{\sqrt{(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)(\tau-2)f(\tau)}},$$

$$\bar{c}_{2,0}^{\chi=2}(\tau,6)=-\frac{32(3\tau-5)(\tau-4)(\tau-5)\sqrt{3(2\tau+1)(\tau-3)}}{\sqrt{7(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)(\tau-2)f(\tau)}},$$

$$\bar{c}_{3,0}^{\chi=2}(\tau,6)=\frac{(64\tau^4-896\tau^3+4448\tau^2-9244\tau+6705)\sqrt{(2\tau+1)}}{\sqrt{5(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)(\tau-2)f(\tau)}},$$

$$\bar{c}_{4,0}^{\chi=2}(\tau,6) = \frac{\sqrt{(2\tau+1)}(64\tau^3-480\tau^2+1172\tau-915)}{\sqrt{(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)f(\tau)}},$$
$$\bar{c}_{4,2}^{\chi=2}(\tau,6) = \frac{3\sqrt{f(\tau)}}{\sqrt{7(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)(\tau-2)}},$$
$$\bar{c}_{5,2}^{\chi=2}(\tau,6) = -\frac{2\sqrt{3f(\tau)}}{\sqrt{5(4\tau-5)(4\tau-7)(4\tau-9)(2\tau-5)}}.$$

Instead of $\mathbf{c}^{(\zeta)}(\tau,6)$, the basis vectors (37) with the expansion coefficients $\bar{\mathbf{c}}^{(\chi)}(\tau,6)$ for $\tau>5$ are orthonormal with respect to the new multiplicity label χ.

Similar to discussions in previous examples, the expansion coefficients $c_{0,0}(\tau,6)$, $c_{1,0}(\tau,6)$, and $c_{2,0}(\tau,6)$ become zero when $3\leq\tau\leq5$ for $L=2\tau-6$. The effective projection matrix $\mathbf{P}(\tau,6)$ in this case becomes

$$\mathbf{P}(\tau,6) = \begin{pmatrix} 6 & 0 & \sqrt{\frac{4(2\tau+1)}{35}} & 0 \\ \sqrt{\frac{72(2\tau-4)}{5}} & 2\sqrt{3} & 0 & \sqrt{\frac{4(2\tau+1)}{5}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{28(2\tau-4)}{5}} & \sqrt{6} \end{pmatrix} \tag{47}$$

with the remaining nonzero components of $\mathbf{c}^T(\tau,6)$ arranged as $\{c_{3,0}(\tau,6),c_{4,0}(\tau,6),c_{4,2}(\tau,6),c_{5,2}(\tau,6)\}$. Obviously, the multiplicity of $L=2\tau-6$ when $3\leq\tau\leq5$ becomes 1 with the normalized nonzero expansion coefficients

$$c_{3,0}(\tau,6) = \sqrt{\frac{2\tau+1}{5(18\tau^2+91\tau-190)}},\ \ c_{4,0}(\tau,6) = -\sqrt{\frac{9(2\tau+1)(\tau-2)}{18\tau^2+91\tau-190}},$$
$$c_{4,2}(\tau,6) = -\frac{3\sqrt{7}}{\sqrt{18\tau^2+91\tau-190}},\ \ c_{5,2}(\tau,6) = \frac{14\sqrt{3(\tau-2)}}{\sqrt{5(18\tau^2+91\tau-190)}}.$$

For $k=7$

$$\mathbf{P}(\tau,7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{14(2\tau-12)} & 6 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \sqrt{10(2\tau-10)} & 3\sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\sqrt{14(2\tau-8)} & 3\sqrt{6} & 0 & \frac{2}{3}\sqrt{\frac{2\tau+1}{7}} & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{\frac{5(2\tau-6)}{7}} & 6 & 0 & \sqrt{\frac{12(2\tau+1)}{35}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{4(2\tau-4)}{5}} & 0 & 0 & \sqrt{\frac{8(2\tau+1)}{5}} \\ 0 & 0 & 0 & 0 & 0 & 3\sqrt{\frac{6(2\tau-6)}{7}} & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{14(2\tau-4)}{5}} & \sqrt{6} \end{pmatrix}. \tag{48}$$

Since there is also one row with all entries zero in (48) when $\tau>6$, the multiplicity of $L=2\tau-7$ is $\mathrm{Multi}(\tau,7)=1$ for $\tau>6$. The normalized nonzero expansion coefficients corresponding to $\lambda=0$ are

$$c_{0,0}(\tau,7) = 9\sqrt{\frac{35(2\tau+1)}{(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)(\tau-3)}},\ \ c_{1,0}(\tau,7) = -21\sqrt{\frac{5(2\tau+1)(\tau-6)}{(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)(\tau-3)}},$$
$$c_{2,0}(\tau,7) = 35\sqrt{\frac{2(2\tau+1)(\tau-5)(\tau-6)}{3(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)(\tau-3)}},\ c_{3,0}(\tau,7) = -36\sqrt{\frac{(2\tau+1)(\tau-4)(\tau-5)(\tau-6)}{7(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)(\tau-3)}},$$
$$c_{4,0}(\tau,7) = 8\sqrt{\frac{2(2\tau+1)(\tau-4)(\tau-5)(\tau-6)}{5(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)}},\ c_{4,2}(\tau,7) = -4\sqrt{\frac{2(\tau-4)(\tau-5)(\tau-6)}{3(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)(\tau-3)}},$$
$$c_{5,2}(\tau,7) = 4\sqrt{\frac{6(\tau-4)(\tau-5)(\tau-6)}{7(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)}},\ c_{6,2}(\tau,7) = -8\sqrt{\frac{(\tau-2)(\tau-4)(\tau-5)(\tau-6)}{5(4\tau-11)(4\tau-9)(2\tau-3)(2\tau-5)}}.$$

Using the similar procedure exemplified in the previous $k=6$ case, one can verify that no nonzero solution exists for $L=2\tau-7$ with $4\leq\tau\leq6$.

For $k = 8$,

$$\mathbf{P}(\tau, 8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\sqrt{2\tau - 14} & \sqrt{42} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{12(2\tau - 12)} & \sqrt{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{90(2\tau-10)}{11}} & 6\sqrt{2} & 0 & \sqrt{\frac{4(2\tau+1)}{99}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{14(2\tau-8)}{3}} & \sqrt{60} & 0 & 0 & \sqrt{\frac{4(2\tau+1)}{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{12(2\tau-6)}{7}} & 0 & \sqrt{30} & 0 & \sqrt{\frac{24(2\tau+1)}{35}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{88(2\tau-8)}{9}} & 0 & 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{36(2\tau-6)}{7}} & 3\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{14(2\tau-4)}{15}} & \sqrt{\frac{8(2\tau-1)}{3}} \end{pmatrix}. \quad (49)$$

Since there is two rows with all entries zero in (49) when $\tau > 7$, the multiplicity of $L = 2\tau - 8$ is $\mathrm{Multi}(\tau, 8) = 2$ for $\tau > 7$. The normalized nonzero expansion coefficients corresponding to $\lambda = 0$ are

$$c_{0,0}^{(\zeta=1)}(\tau,8) = -\sqrt{\frac{114345}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}}, \quad c_{1,0}^{(\zeta=1)}(\tau,8) = \sqrt{\frac{87120(\tau-7)}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}},$$

$$c_{2,0}^{(\zeta=1)}(\tau,8) = -\sqrt{\frac{31680(\tau-6)(\tau-7)}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}}, \quad c_{3,0}^{(\zeta=1)}(\tau,8) = \sqrt{\frac{7200(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}},$$

$$c_{4,0}^{(\zeta=1)}(\tau,8) = -\sqrt{\frac{1120(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}}, \quad c_{4,2}^{(\zeta=1)}(\tau,8) = 0,$$

$$c_{5,0}^{(\zeta=1)}(\tau,8) = \sqrt{\frac{64(\tau-3)(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)}}, \quad c_{5,2}^{(\zeta=1)}(\tau,8) = 0, \quad c_{6,2}^{(\zeta=1)}(\tau,8) = 0, \quad c_{8,4}^{(\zeta=1)}(\tau,8) = 0;$$

and

$$c_{0,0}^{(\zeta=2)}(\tau,8) = \sqrt{\frac{8505(2\tau+1)(2\tau-1)}{(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{1,0}^{(\zeta=2)}(\tau,8) = -\sqrt{\frac{6480(2\tau+1)(2\tau-1)(\tau-7)}{(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{2,0}^{(\zeta=2)}(\tau,8) = \sqrt{\frac{25920(2\tau+1)(2\tau-1)(\tau-6)(\tau-7)}{11(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{3,0}^{(\zeta=2)}(\tau,8) = -\sqrt{\frac{512(2\tau+1)(2\tau-1)(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{4,0}^{(\zeta=2)}(\tau,8) = \sqrt{\frac{2048(2\tau+1)(2\tau-1)(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{35(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{4,2}^{(\zeta=2)}(\tau,8) = -\sqrt{\frac{5184(2\tau-1)(\tau-5)(\tau-6)(\tau-7)}{11(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{5,0}^{(\zeta=2)}(\tau,8) = 0, \quad c_{5,2}^{(\zeta=2)}(\tau,8) = \sqrt{\frac{512(2\tau-1)(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{6,2}^{(\zeta=2)}(\tau,8) = -\sqrt{\frac{2048(2\tau-1)(\tau-3)(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{7(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}},$$

$$c_{8,4}^{(\zeta=2)}(\tau,8) = \sqrt{\frac{1024(\tau-2)(\tau-3)(\tau-4)(\tau-5)(\tau-6)(\tau-7)}{5(4\tau-13)(256\tau^5-4800\tau^4+35248\tau^3-122724\tau^2+195220\tau-110355)}}.$$

After the Gram-Schmidt orthonormalization, we have

$$\bar{\mathbf{c}}^{(\chi=1)}(\tau,8) = \mathbf{c}^{(\zeta=1)}(\tau,8);$$

$$\bar{c}_{0,0}^{(\chi=2)}(\tau,8) = \sqrt{\frac{30240(3\tau-7)^2(2\tau+1)(2\tau-1)(\tau-5)(\tau-6)(\tau-7)}{(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}_{1,0}^{(\chi=2)}(\tau,8) = -\sqrt{\frac{23040(3\tau-7)^2(\tau-7)^2(2\tau+1)(2\tau-1)(\tau-5)(\tau-6)}{(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}_{2,0}^{(\chi=2)}(\tau,8) = \sqrt{\frac{92160(3\tau-7)^2(\tau-6)^2(\tau-7)^2(2\tau+1)(2\tau-1)(\tau-5)}{11(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}_{3,0}^{(\chi=2)}(\tau,8) = -\sqrt{\frac{(128\tau^4-2624\tau^3+19312\tau^2-59716\tau+63735)^2(2\tau+1)(2\tau-1)}{(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}_{4,0}^{(\chi=2)}(\tau,8) = \sqrt{\frac{(256\tau^4-5568\tau^3+42224\tau^2-132612\tau+142695)^2(2\tau+1)(2\tau-1)(\tau-4)}{35(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}^{(\chi=2)}_{4,2}(\tau,8)=-\sqrt{\frac{162[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515](2\tau-1)}{11(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)}},$$

$$\bar{c}^{(\chi=2)}_{5,0}(\tau,8)=\sqrt{\frac{4(64\tau^3-720\tau^2+2636\tau-3045)^2(2\tau+1)(2\tau-1)(\tau-4)}{(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}^{(\chi=2)}_{5,2}(\tau,8)=\sqrt{\frac{16[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515](2\tau-1)(\tau-4)}{(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)(\tau-3)}},$$

$$\bar{c}^{(\chi=2)}_{6,2}(\tau,8)=-\sqrt{\frac{64(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)^2(2\tau-1)(\tau-4)}{7(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}},$$

$$\bar{c}^{(\chi=2)}_{8,4}(\tau,8)=\sqrt{\frac{32(32\tau^4-416\tau^3+2128\tau^2-5044\tau+4515)^2(\tau-2)(\tau-4)}{5(4\tau-13)(4\tau-11)(4\tau-9)(4\tau-7)(2\tau-5)(2\tau-7)[4\tau(2\tau-13)(4\tau^2-26\tau+97)+4515]}}.$$

When $4\leq\tau\leq 7$, similar to discussions in previous examples, the expansion coefficients $c_{0,0}(\tau,8)$, $c_{1,0}(\tau,8)$, and $c_{2,0}(\tau,8)$ become zero for $L=2\tau-8$. The effective projection matrix $\mathbf{P}(\tau,8)$ in this case is reduced as

$$\mathbf{P}(\tau,8)=\begin{pmatrix} 6\sqrt{2} & 0 & \sqrt{\frac{4(2\tau+1)}{99}} & 0 & 0 & 0 & 0\\ \sqrt{\frac{14(2\tau-8)}{3}} & \sqrt{60} & 0 & 0 & \sqrt{\frac{4(2\tau+1)}{21}} & 0 & 0\\ 0&0&0&0&0&0&0\\ 0 & \sqrt{\frac{12(2\tau-6)}{7}} & 0 & \sqrt{30} & 0 & \sqrt{\frac{24(2\tau+1)}{35}} & 0\\ 0 & 0 & \sqrt{\frac{88(2\tau-8)}{9}} & 0 & 3\sqrt{2} & 0 & 0\\ 0&0&0&0&\sqrt{\frac{36(2\tau-6)}{7}} & 3\sqrt{2} & 0\\ 0&0&0&0&0&\sqrt{\frac{14(2\tau-4)}{15}} & \sqrt{\frac{8(2\tau-1)}{3}} \end{pmatrix} \tag{50}$$

with the remaining nonzero components of $\mathbf{c}^T(\tau,8)$ arranged as $(c_{3,0}(\tau,8), c_{4,0}(\tau,8), c_{4,2}(\tau,8), c_{5,0}(\tau,8), c_{5,2}(\tau,8),$ $c_{6,2}(\tau,8), c_{8,4}(\tau,8))$. It can be shown that no nonzero solution of $\mathbf{c}(\tau,8)$ exists from (50) when $\tau=4$. For $5\leq\tau\leq 7$, $L=2\tau-8$ occurs only once with the orthonormalized expansion coefficients

$$c_{3,0}(\tau,8)=\sqrt{\frac{(2\tau+1)(2\tau-1)}{(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},\quad c_{4,0}(\tau,8)=-\sqrt{\frac{289(2\tau+1)(2\tau-1)(\tau-4)}{35(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},$$

$$c_{4,2}(\tau,8)=-\sqrt{\frac{1782(2\tau-1)}{(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},\quad c_{5,0}(\tau,8)=-\sqrt{\frac{36(2\tau+1)(2\tau-1)(\tau-3)(\tau-4)}{(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},$$

$$c_{5,2}(\tau,8)=\sqrt{\frac{1936(2\tau-1)(\tau-4)}{(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},\quad c_{6,2}(\tau,8)=-\sqrt{\frac{7744(2\tau-1)(\tau-3)(\tau-4)}{7(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}},$$

$$c_{8,4}(\tau,8)=\sqrt{\frac{3872(\tau-2)(\tau-3)(\tau-4)}{5(4\tau-13)(36\tau^3+620\tau^2-2517\tau+2023)}}.$$

As shown from the above examples, it seems that the orthonormalized expansion coefficients $\bar{\mathbf{c}}^{(\chi)}(\tau,k)$ can always be expressed by polynomials of τ. But the expression becomes much more complicated with increasing of k, especially for non-multiplicity-free cases. Anyway, $\lambda=0$ solutions of (41) determined by (39) for given τ and k completely determine the expansion coefficients $\bar{\mathbf{c}}^{(\chi)}(\tau,k)$, of which a numerical algorithm can easily be implemented for the purpose. The results are consistent with the multiplicities calculated from the well-known $O(5)\downarrow O(3)$ branching rule for symmetric irrep $(\tau\ 0)$ of $O(5)$ shown in [7, 8, 34] with $L=2p,\ 2p-2, 2p-3,\cdots,p$ and $p=\tau,\tau-3,\tau-6,\cdots,p_{\min}$, where $p_{\min}=0,1,2$.

Moreover, as shown in [35], there is an arbitrary $SO(\text{Multi}(\tau,\text{k}))$ rotational transformation with respect to the multiplicity labels $\chi=1,2,\cdots,\text{Multi}(\tau,\text{k})$. When $\text{Multi}(\tau,\text{k})=2$ for example, let $|\chi=1\rangle=\left|\begin{matrix}(\tau\ 0)\\ \chi=1, L=M_L=2\tau-k\end{matrix}\right\rangle$ and $|\chi=2\rangle=\left|\begin{matrix}(\tau\ 0)\\ \chi=2, L=M_L=2\tau-k\end{matrix}\right\rangle$ be orthonormalized basis vectors of $O(5)\supset O(3)$. New vectors $\{|\bar{\chi}\rangle\}$ after an $SO(2)$ rotation with respect to the multiplicity labels with

$$|\bar{\chi}=1\rangle=\cos\theta|\chi=1\rangle-\sin\theta|\chi=2\rangle,$$

$$|\bar{\chi}=2\rangle=\sin\theta|\chi=1\rangle+\cos\theta|\chi=2\rangle \tag{51}$$

are also orthonormalized basis vectors of $O(5) \supset O(3)$, where $0 \leq \theta \leq 2\pi$. As a result, non-multiplicity-free Wigner coefficients of $O(5) \supset O(3)$ may be numerically different when they are derived by using different methods.

V. SOME ELEMENTARY WIGNER COEFFICIENTS OF $O(5) \supset O(3)$

Once the expansion coefficients $\bar{\mathbf{c}}^{(\chi)}(\tau, k)$ are obtained, one can easily calculate matrix elements of d-boson creation operators $\{b^\dagger_{-2}, b^\dagger_{-1}, \cdots, b^\dagger_2\}$ in the $O(5) \supset O(3)$ basis. Since $\{b^\dagger_{-2}, b^\dagger_{-1}, \cdots, b^\dagger_2\}$ is rank-1 and rank-2 irreducible tensor operators of $O(5)$ and $O(3)$, respectively, using the Racah factorization lemma, we have

$$\left\langle \begin{array}{c} (\tau+1\ 0) \\ \chi', L = M_L = 2\tau+2-k' \end{array} \middle| b^\dagger_\mu \middle| \begin{array}{c} (\tau\ 0) \\ \chi, L = M_L = 2\tau - k \end{array} \right\rangle = \sqrt{\tau+1} \left\langle \begin{array}{cc} (\tau\ 0) & (1\ 0) \\ \chi,\ 2\tau-k & 2 \end{array} \middle| \begin{array}{c} (\tau+1\ 0) \\ \chi',\ 2\tau+2-k' \end{array} \right\rangle \times$$

$$\langle 2\tau-k,\ 2\tau-k;\ 2\ \mu | 2\tau+2-k',\ 2\tau+2-k' \rangle, \tag{52}$$

where the condition $k' = k + 2 - \mu$ should be satisfied to keep the $O(3)$ CG coefficient $\langle 2\tau-k,\ 2\tau-k;\ 2\ \mu | 2\tau+2-k',\ 2\tau+2-k' \rangle$ nonzero in order to derive the corresponding elementary $O(5) \supset O(3)$ Wigner coefficient $\left\langle \begin{array}{cc} (\tau\ 0) & (1\ 0) \\ \chi,\ 2\tau-k & 2 \end{array} \middle| \begin{array}{c} (\tau+1\ 0) \\ \chi',\ 2\tau+2-k' \end{array} \right\rangle$. After the left hand side of Eq. (52) is expanded in terms of $O(5) \supset O_1(3) \otimes U(1)$ basis vectors according to (37) with orthonormalized expansion coefficients $\bar{\mathbf{c}}^{(\chi)}(\tau)$, we obtain

$$\left\langle \begin{array}{cc} (\tau\ 0) & (1\ 0) \\ \chi,\ 2\tau-k & 2 \end{array} \middle| \begin{array}{c} (\tau+1\ 0) \\ \chi',\ 2\tau-k+\mu \end{array} \right\rangle = \sqrt{\frac{(4\tau+\mu-2k+3)!(4\tau+\mu-2k-2)!}{(4\tau+2\mu-2k)!(4\tau-2k)!(4\tau+2\mu-2k+1)(\tau+1)}} \times$$

$$\sum_{q't'qt} \bar{c}^{(\chi')}_{q't'}(\tau+1, k+2-\mu) \bar{c}^{(\chi)}_{qt}(\tau, k) \times$$

$$\left\langle \begin{array}{c} (\tau+1\ 0) \\ k+2-\mu-q'\ \ k+2-\mu-2q'+2t', \frac{1}{2}(\tau-1-k+\mu+q'-t') \end{array} \middle| b^\dagger_\mu \middle| \begin{array}{c} (\tau\ 0) \\ k-q,\ k-2q+2t,\ \frac{1}{2}(\tau-k+q-t) \end{array} \right\rangle \tag{53}$$

for $\mu = 2, 1, 0, -1, -2$, where the matrix elements of d-boson operators under the $O(5) \supset O_1(3) \otimes U(1)$ basis in the sum are all given in Sec. III. For the specific values of μ, (53) can be simplified with

$$\left\langle \begin{array}{cc} (\tau\ 0) & (1\ 0) \\ \chi,\ 2\tau-k & 2 \end{array} \middle| \begin{array}{c} (\tau+1\ 0) \\ \chi',\ 2\tau+2-k \end{array} \right\rangle = \sum_{qt} \bar{c}^{(\chi')}_{qt}(\tau+1, k) \bar{c}^{(\chi)}_{qt}(\tau, k) \sqrt{\frac{(2\tau+3-t)(2\tau-2k+2q-t+2)}{2(\tau+1)(2\tau+3)}}, \tag{54}$$

$$\left\langle \begin{array}{cc} (\tau\ 0) & (1\ 0) \\ \chi,\ 2\tau-k & 2 \end{array} \middle| \begin{array}{c} (\tau+1\ 0) \\ \chi',\ 2\tau+1-k \end{array} \right\rangle = \sum_{qt} \bar{c}^{(\chi')}_{qt}(\tau+1, k+1) \bar{c}^{(\chi)}_{qt}(\tau, k) \times$$

$$\sqrt{\frac{(2\tau-k+2)(2\tau+3-t)(2k-2q+t+3)(2k-3q+2t+2)(2k-3q+2t+1)}{2(\tau+1)(2\tau+3)(2\tau-k)(2k-2q+1)(2k-2q+3)}} +$$

$$\sum_{qt} \bar{c}^{(\chi')}_{q+2\ t+2}(\tau+1, k+1) \bar{c}^{(\chi)}_{qt}(\tau, k) \sqrt{\frac{(2\tau-k+2)(2\tau-2k+2q-t)(t+2)(q-2t)(q-2t-2)}{2(\tau+1)(2\tau+3)(2\tau-k)(2k-2q+1)(2k-2q-1)}}, \tag{55}$$

$$\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi,\ 2\tau-k & 2 & \chi',\ 2\tau-k \end{array} \right\rangle = \sum_{qt} \bar{c}^{(\chi')}_{q+1\ t}(\tau+1,k+2)\bar{c}^{(\chi)}_{qt}(\tau,k)\times$$

$$\sqrt{\frac{(4\tau-2k+3)(2\tau+3-t)(2\tau-k+1)(2k-2q+t+3)(2k-3q+2t+1)(q-2t+1)}{(\tau+1)(2\tau+3)(2\tau-k)(4\tau-2k-1)(2k-2q+1)(2k-2q+3)}} + \sum_{qt} \bar{c}^{(\chi')}_{q+3\ t+2}(\tau+1,k+2)\times$$

$$\bar{c}^{(\chi)}_{qt}(\tau,k)\sqrt{\frac{(4\tau-2k+3)(2\tau-k+1)(2\tau-2k+2q-t+2)(t+2)(2k-3q+2t)(q-2t)}{(\tau+1)(2\tau+3)(4\tau-2k-1)(2\tau-k)(2k-2q+1)(2k-2q-1)}}, \tag{56}$$

$$\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi,\ 2\tau-k & 2 & \chi',\ 2\tau-k-1 \end{array} \right\rangle = \sum_{qt} \bar{c}^{(\chi')}_{q+2\ t}(\tau+1,k+3)\bar{c}^{(\chi)}_{qt}(\tau,k)\times$$

$$\sqrt{\frac{(4\tau-2k+1)(2\tau+3-t)(2\tau-k+1)(2k-2q+t+3)(2k-3q+2t+2)(q-2t+1)}{2(\tau+1)(2\tau+3)(2\tau-k-1)(4\tau-2k-1)(2k-2q+1)(2k-2q+3)}} + \sum_{qt} \bar{c}^{(\chi')}_{q+4\ t+2}(\tau+1,k+3)\times$$

$$\bar{c}^{(\chi)}_{qt}(\tau,k)\sqrt{\frac{(4\tau-2k+1)(2\tau-k+1)(2\tau-2k+2q-t+2)(t+2)(2k-3q+2t)(2k-3q+2t-1)}{2(\tau+1)(2\tau+3)(4\tau-2k-1)(2\tau-k-1)(2k-2q+1)(2k-2q-1)}}, \tag{57}$$

$$\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi,\ 2\tau-k & 2 & \chi',\ 2\tau-k-2 \end{array} \right\rangle = \sum_{qt} \bar{c}^{(\chi')}_{q+4\ t+2}(\tau+1,k+4)\bar{c}^{(\chi)}_{qt}(\tau,k)\sqrt{\frac{(4\tau-2k+1)(4\tau-2k+1)(t+2)}{2(\tau+1)(2\tau+3)(4\tau-2k-3)}}. \tag{58}$$

For multiplicity-free cases, our results are consistent with those derived in [27] up to a phase. Let $\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ L_1 & 2 & L \end{array} \right\rangle_{\mathrm{R}}$ be multiplicity-free Wigner coefficients of $O(5) \supset O(3)$ obtained numerically in [27]. The $O(5) \supset O(3)$ Wigner coefficients derived from (54)-(58) can be expressed as

$$\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ L_1 & 2 & L \end{array} \right\rangle = (-)^{L_1+2-L} \left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ L_1 & 2 & L \end{array} \right\rangle_{\mathrm{R}}. \tag{59}$$

While non-multiplicity-free Wigner coefficients of $O(5) \supset O(3)$ derived from (54)-(58) are numerically different as compared to the corresponding numerical results shown in [27]. But they all satisfy the orthonormality condition:

$$\sum_{\chi_1 L_1} \left| \left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi_1\ L_1 & 2 & \chi\ L \end{array} \right\rangle \right|^2 = 1. \tag{60}$$

As discussed at the end of previous section, though non-multiplicity-free Wigner coefficients derived from different methods may be different in values, they are equivalent up to an $SO(\mathrm{Multi}(\tau,k))$ rotational transformation. Furthermore, similar to the symmetry property of $O(5) \supset O_1(3) \otimes U(1)$ discussed in Sec. III, the $O(5) \supset O(3)$ Wigner coefficients satisfy the following symmetry relations as discussed in many papers, for example in [27, 28]:

$$\left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi_1\ L_1 & 2 & \chi\ L \end{array} \right\rangle = (-)^{L_1+2-L} \left\langle \begin{array}{cc|c} (1\ 0) & (\tau\ 0) & (\tau+1\ 0) \\ 2 & \chi_1\ L_1 & \chi\ L \end{array} \right\rangle \tag{61}$$

and

$$\left\langle \begin{array}{cc|c} (\tau+1\ 0) & (1\ 0) & (\tau\ 0) \\ \chi\ L & 2 & \chi_1\ L_1 \end{array} \right\rangle = (-)^{L_1+2-L}\sqrt{\frac{\dim(\tau\ 0)(2L+1)}{\dim(\tau+1\ 0)(2L_1+1)}} \left\langle \begin{array}{cc|c} (\tau\ 0) & (1\ 0) & (\tau+1\ 0) \\ \chi_1\ L_1 & 2 & \chi\ L \end{array} \right\rangle. \tag{62}$$

Some analytical expressions of elementary $O(5) \supset O(3)$ Wigner coefficients for the coupling $(\tau\ 0) \otimes (1\ 0)$ with resultant $O(3)$ quantum number $L = 2\tau+2-k$ and $k \leq 6$ are shown in Tables III and IV, in which only $\tau > k_1$ and $\tau > k-1$ cases related with $L_1 = 2\tau - k_1$ and $L = 2\tau+2-k$, respectively, are shown.

TABLE III: Elementary $O(5) \supset O(3)$ Wigner coefficients $\left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ L_1 & 2 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ L \end{matrix} \right\rangle$.

L_1	$L = 2\tau - 1$	$L = 2\tau$	$L = 2\tau + 2$
2τ	$\sqrt{\frac{2(4\tau+1)(\tau-1)}{(4\tau-1)(2\tau-1)(\tau+1)}}$	$-\sqrt{\frac{2(2\tau+1)}{(4\tau-1)(\tau+1)}}$	1
$2\tau - 2$	$\sqrt{\frac{3(2\tau+1)}{(4\tau-1)(\tau+1)(\tau-1)}}$	$\sqrt{\frac{(\tau-1)(4\tau+3)}{(4\tau-1)(\tau+1)}}$	0
$2\tau - 3$	$\sqrt{\frac{\tau(2\tau+1)(\tau-2)}{(2\tau-1)(\tau+1)(\tau-1)}}$	0	0
L_1	$L = 2\tau - 2$	$L = 2\tau - 3$	
2τ	$-\sqrt{\frac{32\tau(\tau-1)(\tau-2)}{(4\tau-1)(4\tau-3)(2\tau+3)(2\tau-1)(\tau+1)}}$	0	
$2\tau - 2$	$-\sqrt{\frac{4(2\tau+1)^2(4\tau+1)(\tau-2)}{(4\tau-1)(4\tau-5)(2\tau+3)(\tau+1)(\tau-1)}}$	$\sqrt{\frac{2(4\tau-1)(\tau-2)(\tau-3)}{(4\tau-5)(2\tau-3)(\tau+1)(\tau-1)}}$	
$2\tau - 3$	$\sqrt{\frac{4(4\tau+1)(4\tau-1)}{(2\tau+3)(2\tau-3)(2\tau-1)(\tau+1)(\tau-1)}}$	$-\sqrt{\frac{2(2\tau+1)^2(\tau-3)}{(4\tau-7)(2\tau-3)(\tau+1)(\tau-1)}}$	
$2\tau - 4$	$\sqrt{\frac{(\tau-3)(2\tau-1)(2\tau+1)(4\tau+1)(4\tau-1)}{(4\tau-3)(4\tau-5)(2\tau+3)(2\tau-3)(\tau+1)}}$	$\sqrt{\frac{6(2\tau+1)(2\tau-1)}{(4\tau-5)(2\tau-3)(\tau+1)(\tau-2)}}$	
$2\tau - 5$	0	$\sqrt{\frac{(\tau-1)(\tau-4)(4\tau-3)(2\tau-1)}{(4\tau-7)(2\tau-3)(\tau+1)(\tau-2)}}$	

TABLE IV: Elementary $O(5) \supset O(3)$ Wigner coefficients $\left\langle \begin{matrix} (\tau\ 0) & (1\ 0) \\ \chi_1\ L_1 & 2 \end{matrix} \middle| \begin{matrix} (\tau+1\ 0) \\ \chi\ L = 2\tau - 4 \end{matrix} \right\rangle$.

χ_1, Ł$_1$	$\chi = 1$	$\chi = 2$
$2\tau - 2$	0	$\sqrt{\frac{3f(\tau+1)}{(4\tau-7)(4\tau-5)(2\tau+3)(2\tau-3)(\tau+1)(\tau-1)}}$
$2\tau - 3$	$\sqrt{\frac{32(4\tau-5)(2\tau-1)(\tau-1)^2(\tau-3)(\tau-4)}{(4\tau-7)(\tau+1)(\tau-2)f(\tau+1)}}$	$\sqrt{\frac{(4\tau-9)^2(4\tau-3)(4\tau-1)(2\tau+1)^2(2\tau-1)}{(4\tau-7)(2\tau+3)(2\tau-3)(\tau+1)(\tau-1)f(\tau+1)}}$
$2\tau - 4$	$-\sqrt{\frac{6(4\tau-5)^2(4\tau-3)(2\tau+1)(\tau-4)}{(4\tau-9)(\tau+1)(\tau-2)f(\tau+1)}}$	$\sqrt{\frac{192(4\tau-1)(2\tau+1)(2\tau-1)^2(\tau-1)(\tau-3)^3}{(4\tau-9)(4\tau-5)(2\tau+3)(2\tau-3)(\tau+1)f(\tau+1)}}$
$2\tau - 5$	$\sqrt{\frac{12(2\tau+1)^2(2\tau-1)^2(2\tau-3)^2}{(4\tau-7)(2\tau-5)(\tau+1)(\tau-2)f(\tau+1)}}$	$-\sqrt{\frac{96(4\tau-5)(4\tau-3)(4\tau-1)(\tau-1)(\tau-3)(\tau-4)}{(4\tau-7)(2\tau+3)(2\tau-3)(2\tau-5)(\tau+1)f(\tau+1)}}$
$\chi_1 = 1,\ 2\tau - 6$	$\sqrt{\frac{(\tau-5)f(\tau+1)}{(\tau+1)f(\tau)}}$	0
$\chi_1 = 2,\ 2\tau - 6$	$\sqrt{\frac{32(4\tau-7)(4\tau-5)(2\tau+1)(2\tau-1)^2(2\tau-3)^2(\tau-3)(\tau-4)}{(4\tau-9)(2\tau-5)(\tau+1)(\tau-2)f(\tau+1)f(\tau)}}$	$\sqrt{\frac{(4\tau-3)(4\tau-1)(2\tau+1)(2\tau-3)(\tau-1)f(\tau)}{(4\tau-9)(4\tau-7)(2\tau+3)(2\tau-5)(\tau+1)f(\tau+1)}}$

$f(\tau) = 32\tau^4 - 288\tau^3 + 1024\tau^2 - 1692\tau + 1065.$

VI. CONCLUSION

In this paper, a recursive method for construction of symmetric irreps of $O(2l+1)$ in an $O(2l+1) \supset O(3)$ basis for identical boson systems is proposed. The formalism is realized based on the group chain $U(2l+1) \supset U(2l-1) \otimes U(2)$, for which the symmetric irreps are simply reducible. Within this framework, the basis vectors of the $O(2l+1) \supset O(2l-1) \otimes U(1)$ are constructed from those of $U(2l+1) \supset U(2l-1) \otimes U(2) \supset O(2l-1) \otimes U(1)$ with no boson pairs, and from these one can deduce symmetric irreps of $O(2l+1)$ in the $O(2l-1) \otimes U(1)$ basis when all symmetric irreps of $O(2l-1)$ are known.

As a starting point, basis vectors of symmetric irreps of $O(5)$ are constructed in the $O_1(3)\otimes U(1)$ basis. Matrix representations of $O(5)\supset O_1(3)\otimes U(1)$, together with the elementary Wigner coefficients, are then generated, and after angular momentum projection, a three-term relation for determining the expansion coefficients of the $O(5)\supset O(3)$ basis vectors expanded in terms of those of the $O_1(3)\otimes U(1)$ is derived. The eigenvectors with zero eigenvalues of the projection matrix constructed according to the three-term relation completely determine the basis vectors of $O(5)\supset O(3)$, which enables one to derive analytical expressions of elementary Wigner coefficients of $O(5)\supset O(3)$ with the formulae shown in (54)-(58). Some simple elementary Wigner coefficients of $O(5)\supset O(3)$ are presented as examples. An algorithm that satisfies the three-term relation (39) can be readily determined. As far as the elementary Wigner coefficients of $O(5)\supset O(3)$ are concerned, the procedure shown in this paper seems simpler than the method shown in [27] using the overlap integrals of $O(5)$ spherical harmonic functions and the recursive method proposed in [28].

Using the matrix representations of $O(5)\supset O_1(3)\otimes U(1)$ as determined above, one can construct matrix representations of $O(7)\supset O(5)\otimes U_3(1)$ in a similar way, where the generator of $U_3(1)$ is $\frac{1}{2}(b_3^\dagger b_3 - b_{-3}^\dagger b_{-3})$, with which one can construct $O(7)\supset O(3)$ basis vectors from those of $O(7)\supset O(5)\otimes U_3(1)$. A similar procedure for identical fermion systems is also possible because anti-symmetric irreps of $Sp(2j+1)$ in the reduction $Sp(2j+1)\downarrow Sp(2j-1)\otimes U(1)$ are also simply reducible, and this in turn suggests that one can establish a similar recursive procedure to construct basis vectors of $Sp(2j+1)\supset O(3)$ in terms of those of $Sp(2j+1)\downarrow Sp(2j-1)\otimes U(1)$. The related work is in progress.

Acknowledgments

One of the authors (PF) is grateful to Faculty of Science, The University of Queensland, for support through an Ethel Raybould Visiting Fellowship. Support from U.S. National Science Foundation (OCI-0904874), Southeastern Universities Research Association, Natural Science Foundation of China (11175078), Australian Research Council (DP110103434), Doctoral Program Foundation of the State Education Ministry of China (20102136110002), and LSU–LNNU joint research program (9961) is acknowledged.

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