On Properties of Matroid Connectivity

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ON PROPERTIES OF MATROID CONNECTIVITY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Simon Pfeil
B.S., Southeastern Louisiana University, 2008
M.S., Louisiana State University, 2011
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Abstract

Highly connected matroids are consistently useful in the analysis of matroid structure. Round matroids, in particular, were instrumental in the proof of Rota’s conjecture. Chapter 2 concerns a class of matroids with similar properties to those of round matroids. We provide many useful characterizations of these matroids, and determine explicitly their regular members.

Tutte proved that a 3-connected matroid with every element in a 3-element circuit and a 3-element cocircuit is either a whirl or the cycle matroid of a wheel. This result led to the proof of the 3-connected splitter theorem. More recently, Miller proved that matroids of sufficient size having every pair of elements in a 4-element circuit and a 4-element cocircuit are spikes. This observation simplifies the proof of Rota’s conjecture for $GF(4)$. In Chapters 3 and 4, we investigate matroids having similar restrictions on their small circuits and cocircuits. The main result of each of these chapters is a complete characterization of the matroids therein.
Chapter 1
Introduction

Throughout this dissertation, we follow the conventions of Oxley [5], and we assume that the reader is familiar with the basic concepts of matroid theory, including rank, duality, minors, and connectivity.

We also assume that the reader has an understanding of some basic graph theory. One class of graphs that will arise frequently are the wheel graphs. A wheel, $W_n$, is a graph consisting of a cycle of length $n$, called the rim, and one additional vertex that is adjacent to each other vertex. A whirl, $\mathcal{W}^n$, is a matroid obtained by relaxing the rim of $M(W_n)$. 
Chapter 2
Unbreakable Matroids

2.1 Preliminaries and Equivalent Characterizations

This chapter is devoted to the study of matroids that remain connected upon contracting any flat. Specifically, we call a matroid $M$ unbreakable if $M$ is connected and, for every flat $F$ of $M$, the matroid $M/F$ is also connected. One attractive feature of unbreakable matroids is their many useful equivalent characterizations, presented in Theorem 2.1. One of these characterizations is defined in terms of the local connectivity $\cap_M(S_1, S_2)$, or $\cap(S_1, S_2)$, of two subsets $S_1$ and $S_2$ of $M$, defined by

$$\cap(S_1, S_2) := r(S_1) + r(S_2) - r(S_1 \cup S_2). \tag{2.1.1}$$

Two subsets are called skew if their local connectivity is 0, and if those subsets partition the ground set they form a 1-separation of $M$. One of the characterizations we show is that $M$ is unbreakable if, and only if, $M^*$ has no skew circuits. We say $N = M/e$ is a parallel deletion of $M$ if $e$ is in a 2-circuit of $M$. We say $N$ is a parallel minor of $M$ if $N$ can be obtained from $M$ by a sequence of contractions and parallel deletions. Another characterization shows that $M$ is unbreakable if and only if $M$ does not have $U_{2,2}$ as a parallel minor. Recall that the simplification of a matroid $M$, denoted $\text{si}(M)$, is the matroid obtained from $M$ by deleting all loops and deleting all but one element from each parallel class.

**Theorem 2.1.** The following statements are equivalent for a matroid $M$.

(i) $M$ is unbreakable.

(ii) $M^*$ has no skew circuits.

(iii) Every rank-$(r - 2)$ flat of $M$ is contained in at least three hyperplanes.
(iv) For all $X \subseteq E(M)$, $\text{si}(M/X) \not\sim U_{2,2}$, for all $X \subseteq E(M)$.

(v) $M/F$ is unbreakable for all rank-1 flats $F$ of $M$.

(vi) For every partition $(X,Y)$ of $E(M)$ with $X,Y \neq \emptyset$, if $X'$ is a flat that is properly contained in $X$ and $Y' \subseteq Y$, then $\cap(X',Y') < \cap(X,Y)$.

**Proof.** The structure of the proof is as follows: we shall show that (i) implies (iv), that (iv) implies (iii), that (iii) implies (ii), and that (ii) implies (i). Then we shall show the equivalence of (i) and (v), and lastly the equivalence of (i) and (vi).

To show that (i) implies (iv), let $M$ be unbreakable. A subset $X \subseteq E(M)$ such that $\text{si}(M/X) \nmid U_{2,2}$ cannot exist, since $\text{si}(M/\text{cl}(X)) \nmid \text{si}(M/X)$, and $\text{si}(M/\text{cl}(X))$ is connected since $M$ is unbreakable. Therefore (i) implies (iv).

We show that (iv) implies (iii), by proving the contrapositive. Suppose $F$ is a rank-$(r-2)$ flat of $M$ contained in exactly two hyperplanes $H_1$ and $H_2$. Then $F = H_1 \cap H_2$ and $r(M/F) = 2$. Further, $M/F$ must consist of two disjoint rank-1 flats. The only possibility, then, is that $\text{si}(M/F) = U_{2,2}$. We conclude that (iv) implies (iii).

Now suppose (iii) holds. To show that (ii) holds, let $D_1$ and $D_2$ be cocircuits of $M$, and let $H_i = E(M) - D_i$ for each $i \in \{1,2\}$. Then

\[
\cap_{M^*}(D_1, D_2) = r_{M^*}(D_1) + r_{M^*}(D_2) - r_{M^*}(D_1 \cup D_2)
\]

\[
= |D_1| + |D_2| - 2 - [r_M(E(M) - (D_1 \cup D_2)) + |D_1 \cup D_2| - r(M)]
\]

\[
= |D_1 \cap D_2| - 2 - r_M(H_1 \cap H_2) + r_M(M)
\]

\[
\geq |D_1 \cap D_2| - 2 - (r(M) - 2) + r_M(M)
\]

\[
= |D_1 \cap D_2|.
\]

Since equality holds only when $r_M(H_1 \cap H_2) = r(M) - 2$, we need only argue that, in this case, $|D_1 \cap D_2| \neq 0$. Let $F = H_1 \cap H_2$. Then $F$ is contained in at least three distinct hyperplanes.
by assumption. There must be an element \( e \in E(M) - F \) such that \( \text{cl}(e \cup F) \neq H_i \) for \( i \in \{1, 2\} \). Therefore, \(|D_1 \cap D_2| = |E(M) - (H_1 \cup H_2)| \geq 1\). Thus (iii) implies (ii).

Next, suppose that (ii) holds, but (i) does not. Then \( M \) has a flat \( F \) such that \( M/F \) is not connected. Now, for \( n = r(M) - r(F) \), there are hyperplanes \( H_1, H_2, \ldots, H_n \) of \( M \) such that \( F = \bigcap_{i=1}^{n} H_i \). Note that \( n \neq 1 \), as \( M/H \) is a rank-one loopless matroid and so is connected. Hence, \( n \geq 2 \). Then, if we let \( D_i \) be the corresponding cocircuit complement of each \( H_i \), we get

\[
    M/F = M/[[H_1 \cap H_2 \cap \cdots \cap H_n]
    = M/[(E(M) - D_1) \cap (E(M) - D_2) \cap \cdots \cap (E(M) - D_n)]
    = M/[(E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)]
    = M^\star[E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)]
    = M^\star[(D_1 \cup D_2 \cup \cdots \cup D_n)].
\]

Since \( M/F \) is not connected, we know \( M' = M^\star[(D_1 \cup D_2 \cup \cdots \cup D_n) \) is not connected. Hence, there must be some partition \((S, T)\) of \( M' \) such that \( \lambda_{M'}(S, T) = 0 \). This implies that each \( D_i \) is either contained in \( S \) or contained in \( T \). Therefore, there must be cocircuits \( D_i \) and \( D_j \) for some \( \{i, j\} \subseteq \{1, 2, \ldots, n\} \) such that \( \bigcap_{M^\star}(D_i, D_j) = 0 \), a contradiction. Thus (ii) implies (i).

To show that (i) implies (v), assume \( M \) is unbreakable, and suppose there is a rank-1 flat \( F \) of \( M \) such that \( M/F \) is not unbreakable. Then there must be some flat \( G \) of \( M/F \) such that \( (M/F)/G \) is not connected. This is a contradiction, since \( G \cup F \) is a flat of \( M \), and \( M \) is unbreakable by assumption. Therefore (i) implies (v).

Now assume (v) holds. We shall show that \( M \) is unbreakable. Let \( F \) be a flat of \( M \), and let \( e \in F \). Since \( F \) is closed, we know \( \text{cl}(e) \subseteq F \). Then \( M/F = M/[(\text{cl}(e) \cup (F - \text{cl}(e))] = \)
\((M/\text{cl}(e))/(F - \text{cl}(e))\), which is connected since \(M/\text{cl}(e)\) is unbreakable and has \(F - \text{cl}(e)\) as a flat. Therefore \(M\) is unbreakable, and (v) implies (i).

Next, we show that (i) implies (vi). Assume \(M\) is unbreakable, and suppose \((X, Y)\) partitions \(E(M)\), neither \(X\) nor \(Y\) are empty, \(X'\) is a flat properly contained in \(X\), and \(Y' \subseteq Y\). Suppose \(\cap(X', Y') = \cap(X, Y)\). Then \(\cap(X', Y) = \cap(X, Y)\). Therefore \(r(X') = r(X) - r(X \cup Y) + r(X' \cup Y)\). Now we consider \(M' = M/X'\). Then

\[
\cap_{M'}(X - X', Y) = r_{M'}(X - X') + r_{M'}(Y) - r_{M'}((X - X') \cup Y)
\]

\[
= r_M(X) - r_M(X') + r_{M'}(Y) - (r(M) - r_M(X'))
\]

\[
= r_M(X) + r_M(Y \cup X') - r_M(X') - r(M)
\]

\[
= 0.
\]

Thus, the contraction of \(X'\) yields a matroid that is not connected, a contradiction. Therefore (i) implies (vi).

Now assume (vi) holds, but (i) does not. Then \(M\) has a flat \(F\) such that \(M/F\) is not connected. Let \((X_F, Y_F)\) be a 1-separation of \(M/F\). Consider \((X_F \cup F, Y_F)\), a partition of \(E(M)\). We will show that \(\cap(X_F \cup F, Y_F) = \cap(F, Y_F)\). Observe that

\[
\cap_{M/F}(X_F, Y_F) = r_{M/F}(X_F) + r_{M/F}(Y_F) - r_{M/F}(X_F \cup Y_F)
\]

\[
= r_M(X_F \cup F) - r_M(F) + r_M(Y_F \cup F) - r_M(F) - r_M(M) + r_M(F)
\]

\[
= r_M(X_F \cup F) + r_M(Y_F \cup F) - r_M(M) - r_M(F)
\]

\[
= 0.
\]

Thus

\[
r_M(X_F \cup F) - r(M) = r_M(F) - r_M(Y_F \cup F),
\]
and therefore

\[
\cap_M (X_F \cup F, Y_F) = r_M (X_F \cup F) - r(M) + r_M (Y_F) \\
= r_M (F) - r_M (Y_F \cup F) + r_M (Y_F) \\
= \cap_M (F, Y_F).
\]

As this contradicts \((vi)\), we deduce that \((vi)\) implies \((i)\). We conclude that the theorem holds.

The following corollary is an immediate consequence of part \((v)\) of the last theorem.

**Corollary 2.2.** A loopless parallel minor of an unbreakable matroid is unbreakable.

To close this section, we note the similarity between unbreakable matroids and round matroids. A matroid is called *round* if each of its cocircuits is spanning. Round matroids and unbreakable matroids have related equivalent characterizations, as seen by comparing the following theorem to Theorem 2.1. This yields an immediate corollary that all round matroids are unbreakable.

**Theorem 2.3.** The following statements are equivalent for a matroid \(M\):

\[(i)\] \(M\) is round.

\[(ii)\] \(M\) has no disjoint cocircuits.

\[(iii)\] \(M\) cannot be written as the union of two proper flats.

\[(iv)\] Every cocircuit of \(M\) is spanning.

**Corollary 2.4.** Let \(M\) be a matroid. If \(M\) is round, then \(M\) is unbreakable.
2.2 Classifying Unbreakable Regular Matroids

In order to determine the unbreakable regular matroids, we will first find the unbreakable graphic and cographic matroids, and then apply Seymour’s decomposition theorem for regular matroids.

Before we begin classifying these matroids, we need the following preliminary lemma.

**Lemma 2.5.** If $M$ is an unbreakable matroid and $N$ is a matroid such that $\text{si}(N) \cong M$, then $N$ is unbreakable.

*Proof.* Let $M'$ be such that $\text{si}(M') = M$, and $M' \cong N$. For any flat $F$ of $M$, we have $\text{si}(M' / \text{cl}_{M'}(F)) = M/F$ is connected. Therefore $M'$ is unbreakable, since every flat of $M'$ is the closure in $M'$ of a flat of $M$. Thus $N$ is unbreakable. \[\square\]

We will also use Tutte’s characterization of graphs that are 2-connected but not 3-connected, called *generalized cycles*. Such a graph $G$ can be expressed in parts $G_1, G_2, \ldots, G_n$ such that $n \geq 2$, each $G_i$ is connected, their edge sets partition $E(G)$, each $G_i$ shares exactly two vertices (called *contact vertices*) with $\bigcup_{j \neq i} G_j$, and if each $G_i$ is replaced by an edge joining its contact vertices, the resulting graph is a cycle.

It is not difficult to see that the cycle matroids of the graphs $C_n$ and $K_n$ are unbreakable for all $n > 0$. The following proposition shows that these are essentially the only unbreakable graphic matroids.

**Proposition 2.6.** A graphic matroid $M$ is unbreakable if, and only if, for some $n > 0$, either $\text{si}(M) \cong M(C_n)$ or $\text{si}(M) \cong M(K_n)$.

*Proof.* Let $M$ be a graphic matroid such that $\text{si}(M)$ is isomorphic to $M(C_n)$ or $M(K_n)$. If $F$ is a rank-$k$ flat of $M$, then $\text{si}(M/F)$ is isomorphic to $M(C_{n-k})$ or $M(K_{n-k})$, respectively. As each of the last two matroids is connected, $M$ is unbreakable.
Now, suppose $M$ is an unbreakable graphic matroid, and let $G$ be a connected graph such that $M(G) \cong M$. If $|V(G)| < 3$, then $\text{si}(M) \cong M(K_n)$ for $n = |V(G)|$. Hence, we may assume that $|V(G)| \geq 3$.

Suppose first that $G$ is 3-connected and $\text{si}(M) \not\cong M(K_n)$. Then there are two non-adjacent vertices in $G$, say $v_1$ and $v_2$. Then $G \setminus \{v_1, v_2\}$ is connected, and $\text{si}(M(G/E(G \setminus \{v_1, v_2\}))) \cong U_{2,2}$. Thus $M$ is not unbreakable.

We may now suppose that $G$ is not 3-connected. Assume that $\text{si}(M) \not\cong M(C_n)$. As $M$ is connected, $G$ must be 2-connected. Therefore $G$ is a generalized cycle with parts $G_1, G_2, \ldots, G_n$. One part, say $G_v$, must contain a vertex $v$ such that, if $\{u, w\}$ are the contact vertices of $G_v$, there is a path from $u$ to $w$ not containing $v$. Let $v'$ be a vertex not in $V(G_v)$. Let $S$ be the set of all edges not incident with $v$ or $v'$. Then $\text{si}(M(G/S)) \cong U_{2,2}$. Thus $M$ is not unbreakable.

Concerning unbreakable cographic matroids, we can approach the problem using Theorem 2.1(ii) by considering graphic matroids with no skew circuits. Skew circuits appear in a graph as cycles that share at most one vertex. The following is a theorem of Dirac [1] that determines all 3-connected simple graphs with no two vertex-disjoint cycles. The graphs $K'_3, p$, $K''_3, p$, and $K'''_3, p$ denote $K_3, p$ with one, two, and three additional edges between the vertices of the vertex class of size 3.

**Theorem 2.7.** Every 3-connected graph with no two vertex-disjoint cycles is one of the following graphs:

$W_k$ ($k \geq 4$), $K_5$, $K_5\setminus e$, $K_3, p$, $K'_3, p$, $K''_3, p$, $K'''_3, p$ ($p \geq 3$).

The 3-connected unbreakable cographic matroids must form a subset of the bond matroids of the graphs in the previous theorem. Using this fact, we can find all the unbreakable cographic matroids. It is sufficient to determine those cographic unbreakable matroids that are not also graphic.
Proposition 2.8. Let $M$ be a matroid that is cographic but not graphic. Then $M$ is unbreakable if, and only if, $\text{si}(M) \cong M^*(K_{3,3})$.

Proof. By Theorem 2.1 (ii), the cographic unbreakable matroids are all $M \cong M^*(G)$ such that $M(G)$ has no skew circuits; that is, all cycles of $G$ must share at least two vertices. Therefore, if $G$ is 3-connected, then $G$ must be a graph from the list in Theorem 2.7. The only graph on this list in which all cycles share at least two vertices is $K_{3,3}$. See Figure 2.1 for a demonstration of this fact.

![Figure 2.1](image-url)

Now suppose $G$ is not 3-connected. Then $G$ must be 2-connected, and is therefore a generalized cycle with parts $G_1, G_2, \ldots, G_n$. At least one part of $G$, say $G_k$, must be non-planar. We may assume $G_k$ is chosen to have no pendant edges, and is otherwise maximal.
No part of $G$ besides $G_k$ can contain a cycle; otherwise $M(G)$ would have skew circuits, which would contradict Theorem 2.1(ii). Hence, $G$ is isomorphic to a large cycle where one edge is replaced by the non-planar graph $G_k$. By a repeated application of Theorem 2.1 (v), we may contract all the edges in $E(G) - E(G_k)$ and maintain unbreakability. Hence, $M(G_k)$ is unbreakable. Therefore $G_k \cong K_{3,3}$. Let $\{u, v\}$ be the contact vertices of $G_k$. Then we can find both a path from $u$ to $v$ and a cycle, $C$, in $G_k$ such that they share at most one vertex, as demonstrated in Figure 2.2. Such a path forms a cycle with $E(G) - E(G_k)$, and this cycle is skew with $C$, a contradiction.

\[
\begin{array}{c}
\text{Figure 2.2: A path from } u \text{ to } v \text{ and a cycle sharing at most one vertex with it.}
\end{array}
\]

Using the previous two propositions, along with Seymour’s decomposition theorem for regular matroids [6] restated here, we will be able to find all unbreakable regular matroids. Recall that $R_{10}$ is the unique regular matroid on ten elements that is neither graphic nor cographic.

**Theorem 2.9.** A regular matroid $M$ can be constructed using 1-, 2-, and 3-sums of matroids that are either graphic, cographic, or isomorphic to $R_{10}$, and each matroid used in this construction is isomorphic to a minor of $M$.

We will need a few preliminary lemmas before we prove the main result of this section. We call an element of a matroid free if it is contained in no non-spanning circuits. The first
two lemmas concern the 2-sums of unbreakable matroids, requiring that the basepoint $p$ of a 2-sum be free in both matroids in order to maintain unbreakability.

**Lemma 2.10.** If a matroid $M$ contains a free element, then $M$ is unbreakable.

**Proof.** Let $M$ be a matroid with a free element $p$. Suppose $M$ is not unbreakable. Then $M$ has a flat $F$ such that $M/F$ is not connected. Therefore there are elements $e_1$ and $e_2$ in $M/F$ such that there is no circuit containing both. Observe that $e_1$ and $e_2$ are not loops nor are they parallel, as any loops are contained in $F$, so parallel elements form a circuit. Let $I_F$ be a maximal independent set in $F$. Then $r_M(I_F \cup e_1 \cup e_2) = r_M(I_F) + 2$, and $I_F \cup e_1 \cup e_2$ is independent. Let $B_F$ be a basis of $M$ containing $I_F \cup e_1 \cup e_2$. Then $B_F \cup p$ is a circuit, $C_F$, of $M$ such that $r(F) = |C_F \cap F|$. Hence,

$$r_{M/F}(C_F - F) = r_M(C_F \cup F) - r_M(F) = r(M) - r_M(F),$$

and

$$|C_F - F| = |C_F| - |C_F \cap F| = r(M) + 1 - r_M(F).$$

Therefore $r_{M/F}(C_F - F) = |C_F - F| - 1$, and $C_F - F$ is a circuit of $M/F$ containing both $e_1$ and $e_2$, a contradiction. Thus, $M$ is unbreakable.

\[\square\]

**Lemma 2.11.** The matroid $M \cong (M_1, p) \bigoplus_2 (M_2, p)$ is unbreakable if, and only if, $p$ is a free element in both $M_1$ and $M_2$.

**Proof.** Suppose $(M_1, p) \bigoplus_2 (M_2, p)$ is unbreakable, but $p$ is not a free element of $M_1$. Let $C$ be a non-spanning circuit of $M_1$ containing $p$, and let $F = \text{cl}(C)$. By the definition of 2-sum, the only circuits of $M$ containing elements from both $E(M_1)$ and $E(M_2)$ are those of the form $(C_1 \cup C_2) - p$, where $C_1$ and $C_2$ are circuits of $M_1$ and $M_2$, respectively, that contain $p$. Therefore, in $M/F$, there are no circuits containing elements from both $E(M_1) - F$ and $E(M_2)$; that is, $M/F$ is not connected. This is a contradiction.
Now suppose that $p$ is free in both $M_1$ and $M_2$. By Lemma 2.10, both $M_1$ and $M_2$ are unbreakable. If $M$ is not unbreakable, then there is a flat $F$ of $M$, such that $M/F$ is not connected. Note that $F$ cannot be contained in either of $M_1$ or $M_2$. Therefore, $F = F_1 \cup F_2$, where each $F_i$ is a flat of $M_i$. There must, then, be two elements $e_1$ and $e_2$ of $M/F$ that are not in a circuit together. Note that neither element is a loop. Suppose $e_1 \in E(M_1)$. Then $e_2 \in E(M_2)$ since $M_1$ is unbreakable. As in the previous lemma, we can form a spanning circuit $C_i$ containing $p$ in each $M_i$ such that $|C_i \cap F_i| = r_{M_i}(F_i)$ and $e_i \in C_i$. Then $C = (C_1 \cup C_2) - \{p\}$ is a circuit of $M$, such that $|C - F| = r_{M/F}(C - F) + 1$. Therefore $C - F$ is a circuit of $M/F$ containing both $e_1$ and $e_2$, a contradiction. Thus, $M$ is unbreakable.

The two lemmas that follow describe the utility of the 3-sum. Further information and proofs of these lemmas can be found under Proposition 9.3.5 and Proposition 11.4.14 in [5]. The proof given for the former result actually shows that each of $M_1$ and $M_2$ is a parallel minor of $M$, and the statement here reflects this. The latter result appears as a property of the generalized parallel connection in [5]; however, we restate it here in terms of 3-sums.

**Lemma 2.12.** If a 3-connected matroid $M$ is the 3-sum of binary matroids $M_1$ and $M_2$, then $M$ has parallel minors that are isomorphic to each of $M_1$ and $M_2$.

**Lemma 2.13.** Let $M_1$ and $M_2$ be binary matroids with $E(M_1) \cap E(M_2) = T$, where $M_1|T = M_2|T$ is a triangle. If $e \in E(M_1) - \text{cl}_1(T)$, then $(M_1 \bigoplus_3 M_2)/e = (M_1/e) \bigoplus_3 M_2$.

The following is the main result of this chapter.

**Theorem 2.14.** A regular matroid $M$ is unbreakable if, and only if, $si(M)$ is isomorphic to one of $M(C_n)$, $M(K_n)$, $M^*(K_{3,3})$, or $R_{10}$.

**Proof.** By Propositions 2.6 and 2.8, we only need to show that $R_{10}$ is unbreakable in order to prove that each listed matroid is unbreakable. We know $r(R_{10}) = 5$ and the smallest circuit of $R_{10}$ has 4 elements. If $C_1$ and $C_2$ are circuits of $R_{10}$, then $\cap(C_1, C_2) = r(C_1) + r(C_2) - \dots$
\[ r(C_1 \cap C_2) \geq 3 + 3 - 5 = 1. \] Therefore \( R_{10} \) has no skew circuits and, since \( R_{10} \) is self-dual, by Theorem 2.1(ii), it is unbreakable.

Now let \( M \) be an unbreakable regular matroid. We may assume that \( M \) is simple. Suppose \( M \) is not isomorphic to any of the matroids listed above and that \( |E(M)| \) is a minimum among such matroids. By Theorem 2.9, \( M \) can be obtained by 1-, 2-, and 3-sums of graphic matroids, cographic matroids, and \( R_{10} \). Clearly \( M \) cannot be the result of a 1-sum.

Suppose \( M \) can be decomposed via a 2-sum, say \( M = (M_1, p) \oplus_2 (M_2, p) \). By Lemma 2.11, each \( M_1 \) and \( M_2 \) must have \( p \) as a free element. By Lemma 2.10, having a free element implies that a matroid is unbreakable. Therefore, each \( M_i \) must be a member of the previously determined list of unbreakable matroids. However, the only member from that list having a free element is \( M \) such that \( si(M) \cong C_n \), and the 2-sum of circuits simply yields a circuit. Thus \( M \) does not have a 2-separation. Hence, \( M \) is 3-connected.

Finally, suppose \( M \cong M_1 \oplus_3 M_2 \). By Lemma 2.12, each of \( M_1 \) and \( M_2 \) is isomorphic to a loopless parallel minor of \( M \) and so, by Corollary 2.2, is unbreakable. Therefore, each \( M_i \) must be one of the previously identified unbreakable matroids and must contain a triangle. Hence, the only candidates are \( M_i \) such that \( si(M_i) \cong M(K_n) \) or \( si(M_i) \cong M^*(K_3,3) \), when \( n \geq 4 \). As \( K_4 \) is a minor of each \( K_n \) when \( n \geq 4 \), it suffices to consider \( M(K_4) \oplus_3 M(K_4) \), \( M(K_4) \oplus_3 M^*(K_3,3) \), and \( M^*(K_3,3) \oplus_3 M^*(K_3,3) \). Here we abusing notation since the definition of 3-sum requires that each part have at least seven elements. By \( M(K_4) \oplus_3 M(K_4) \) we mean the graphic matroid \( M(G) \) with \( G \) obtained by identifying the edges of a triangle in two copies of \( K_4 \) and then deleting those edges, and by \( M(K_4) \oplus_3 M^*(K_3,3) \) we mean the matroid whose geometric representation is seen as the first in the sequence in Figure 2.4. Note that \( M(K_4) \oplus_3 M(K_4) \) is graphic and differs from both \( C_n \) and \( K_n \), and so, by Theorem 2.6, is not unbreakable. It is easily checked that the others contain a flat whose contraction produces a matroid that is not connected, as demonstrated in Figures 2.3 and 2.4. The matroids \( M^*(K_3,3) \) has rank 4, so \( M^*(K_3,3) \oplus_3 M^*(K_3,3) \) has rank 5. Figure 2.3
Figure 2.3: A sequence of contractions that disconnects $M(K_{3,3}) \bigoplus_3 M(K_{3,3})$.

gives a representation of the last matroid obtained by combining geometric representations of two copies of $M^*(K_{3,3})$. Figure 2.4 begins with a representation $M^*(K_{3,3}) \bigoplus_3 M(K_4)$, followed by graph pictures. Therefore there are no additional unbreakable regular matroids formed via 3-sum, and our list is complete.

2.3 Unbreakable Representable Matroids

A natural next step in classifying unbreakable matroids is to examine the unbreakable representable matroids. This is, as expected, more difficult than in the previous cases, and, as our results indicate, the variety of unbreakable matroids increases significantly as we begin
to consider larger classes of representable matroids. It is straightforward to determine that $PG(r-1, q)$ and $AG(r-1, q)$, with $r \geq 1$ and $q \geq 2$, are among the unbreakable representable matroids. Unlike in the regular case, these examples are not minimal; that is, the deletion of elements from either of these matroids may produce another unbreakable matroid. To make this notion more precise, we have the following results.

**Theorem 2.15.** Let $S \subseteq E(PG(r-1, q))$. If $|S| \leq q^{r-1} - q^{r-2} - 1$, then $PG(r-1, q) \setminus S$ is unbreakable.

**Proof.** For every $q \geq 3$, $PG(1, q)$ is isomorphic to the $(q+1)$-point line $U_{2,q}$. By Theorem 2.1.(iv), we know that a matroid is unbreakable as long as it has no contraction minor whose simplification is isomorphic to $U_{2,2}$. Also, for $e \in E(PG(r-1, q))$, the matroid $PG(r-1, q)/e$ is isomorphic to $PG(r-2, q)$ with each of its elements replaced by $q$ elements in parallel. If $\{e_1, e_2, \ldots, e_{r-2}\}$ is an independent set in $PG(r-1, q)$, then $PG(r-1, q)/\{e_1, e_2, \ldots, e_{r-2}\}$ is isomorphic to $PG(1, q)$ with every element replaced by $q^{r-2}$ elements in parallel. We can delete $(q-2)q^{r-2}+q^{r-2}-1$ elements from this matroid without the possibility of reducing its rank to 2 if we delete all elements in all but three parallel classes, and then $q^{r-2}-1$ elements from one of the remaining parallel classes. Therefore, we can delete $q^{r-1} - q^{r-2} - 1$ elements from $PG(r-1, q)$ without creating a contraction minor whose simplification is $U_{2,2}$. Thus the desired result holds.

To see that the above bound and those that follow are tight, note that deleting $q^{r-1} - q^{r-2}$ elements is enough to remove all elements except those in two parallel classes of $PG(r-1, q)/\{e_1, e_2, \ldots, e_{r-2}\}$, where $\{e_1, e_2, \ldots, e_{r-2}\}$ is an independent set. That is, we can find a set $S \subseteq E(PG(r-1, q))$ with $|S| = q^{r-1} - q^{r-2}$ such that $si(PG(r-1, q)/S) \cong U_{2,2}$.

**Corollary 2.16.** Let $S \subseteq E(AG(r-1, q))$. If $|S| \leq q^{r-2} - q^{r-3} - 1$, then $AG(r-1, q) \setminus S$ is unbreakable.
Proof. The proof of this is nearly identical to the previous, once we note that $AG(r-1, q) / e \cong PG(r-2, q)$. We omit the details.

In the binary case, we have the following easy corollary, with which we close the chapter.

**Corollary 2.17.** If $M$ is a simple rank-$r$ binary matroid having at least $2^r - 2^{r-1} + 2^{r-2}$ elements, then $M$ is unbreakable.
Chapter 3
Many Triads and 4-circuits

3.1 Introduction and Preliminaries

The study of matroids with many small circuits and cocircuits begins with Tutte’s well-known Wheels-and-Whirls Theorem [7]. The result was originally stated in terms of essential elements of a 3-connected matroid; that is, elements that destroy the 3-connectedness of the matroid both on deletion and on contraction. We present it here in terms of 3-circuits and 3-cocircuits.

Theorem 3.1. Suppose $M$ is a non-empty 3-connected matroid. Every element of $M$ is in both a 3-circuit and a 3-cocircuit if and only if $M$ has rank at least three and is isomorphic to a wheel or a whirl.

This result has been instrumental in the analysis of 3-connected matroids. Seymour’s Splitter Theorem 2.9 is a well-known extension of the last theorem. More recently, Miller [4] proved the following result, which requires conditions similar to those in Tutte’s theorem. A spike is a rank-r matroid $M$ whose ground set $E$ is $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r\}$, and whose circuits consist of the following sets:

(i) all sets of the form $\{x_i, y_i, x_j, y_j\}$ with $1 \leq i < j \leq r$,

(ii) a subset of $\{\{z_1, z_2, \ldots, z_r\} : z_i \in \{x_i, y_i\} \forall i\}$ such that no two members of this subset have more than $r - 2$ common elements, and

(iii) all $(r + 1)$-element subsets of $E$ that contain none of the sets in (i) or (ii).

It should be noted that what we have just defined to be a spike is also known as a tipless spike.
Theorem 3.2. Let $M$ be a matroid in which every pair of elements belongs to a 4-circuit and a 4-cocircuit. If $|E(M)| \geq 13$, then $M$ is a spike.

In this chapter, we continue along a similar line of inquiry by investigating matroids $M$ having the following property:

3.3. A matroid $M$ has property $(P1)$ if, for all $\{e, f\} \in E(M)$, we have:

(i) there exists some 4-circuit $C \in \mathcal{C}(M)$ such that $\{e, f\} \in C$,

(ii) there exists some 3-cocircuit $D \in \mathcal{C}(M^*)$ such that $e \in D$, and

(iii) $M$ is 3-connected.

We will assume throughout this chapter that $M$ has property $(P1)$, and will proceed to determine all such matroids. In order to achieve this, we must first make several observations about the structural consequences of $(P1)$. These lemmas will allow us to determine explicitly the possibilities for $M$ when $|E(M)| \leq 8$. We conclude by showing that, when $M$ is sufficiently large, it belongs to a familiar family of matroids; namely $M \cong M(K_{3,n})$ for some $n \geq 3$. Together, these results prove the following theorem, which is the main result of this chapter.

Theorem 3.4. Suppose $M$ is a 3-connected matroid. If $M$ has every element in a 3-cocircuit and every pair of elements in a 4-circuit, then $M$ is one of the following matroids: $U_{3,5}$, $M(K_4)$, $W^3$, $F_7$, $(F_7^*)^*$, $P_7^*$, and $M(K_3,n)$ for some $n \geq 3$.

One property of matroids that we will exploit repeatedly is the restriction on circuit-cocircuit intersection, commonly referred to as orthogonality.

Theorem 3.5. If $C \in \mathcal{C}(M)$ and $D \in \mathcal{C}(M^*)$, then $|C \cap D| \neq 1$.

We shall need the following useful theorem of Lucas 3.6, which uses weak maps where, for two matroid $M_1$ and $M_2$ on the same set, the latter is a weak-map image of the former if every set that is independent in $M_2$ is also independent in $M_1$. 

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Theorem 3.6. Let $M_2$ be the weak-map image of a binary matroid $M_1$ and suppose that $r(M_2) = r(M_1)$. Then $M_2$ is binary, and if $M_2 \neq M_1$, then $M_2$ is disconnected.

3.2 Structure Lemmas

Our ability to determine $M$ explicitly will rely heavily on being able to determine the arrangement of the 3-cocircuits of $M$. The mindful reader will note the approach taken here, as this chapter is something of a warm-up to Chapter 4. We first prove that should $M$ have any 3-cocircuits that meet in two elements $M$ must be $U_{3,5}$.

Proposition 3.7. There exist 3-cocircuits $D_1$ and $D_2$ of $M$ such that $|D_1 \cap D_2| = 2$ if and only if $M \cong U_{3,5}$. Moreover, if $|E(M)| \leq 5$, then $M \cong U_{3,5}$.

Proof. If $M$ is $U_{3,5}$, then it certainly has a pair of 3-cocircuits meeting in two elements. Now, suppose $M$ has two cocircuits such $D_1$ and $D_2$. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_2, x_4\}$. Note that $D_1 \cup D_2$ is a 4-point line in $M^*$, and, therefore, any circuit meeting $D_1 \cup D_2$ must do so in at least three elements by orthogonality.

If $|E(M)| = 4$, then $M \cong U_{2,4}$, a contradiction since $M$ must have a 4-circuit. Hence, we may assume $|E(M)| \geq 5$. By (P2), we have a 4-circuit $C_1$ containing $\{x_1, x_5\}$. Without loss of generality, $C_1 = \{x_1, x_2, x_3, x_5\}$. Similarly, there is a 4-circuit $C_2$ containing $\{x_4, x_5\}$. Without loss of generality, we may assume $C_2 = \{x_1, x_2, x_4, x_5\}$. Then $r(C_1 \cup C_2) = 3$. Therefore $\lambda(C_1 \cup C_2) = r(C_1 \cup C_2) + r^*(C_1 \cup C_2) - |C_1 \cup C_2| \leq 3 + 2 - 4 = 1$. This implies $|E(M)| \leq 5$, since $M$ is 3-connected. Thus $M$ must be the 5-point plane $U_{3,5}$.

In order to see that $U_{3,5}$ is the only possibility when $|E(M)| = 5$, we only need to determine what happens when such an $M$ has no two 3-cocircuits meeting in two elements. In this case, we get cocircuits $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$ without loss of generality. Circuit elimination on this pair indicates there is a cocircuit contained in $\{x_2, x_3, x_4, x_5\}$. This
A cocircuit cannot have 3 elements without contradicting our assumption. Further, $M$ cannot have a cocircuit of size 4 since $r \ast (M) \leq 2$. This contradiction completes the proof.

\begin{center}
\begin{tikzpicture}
\foreach \x in {1,2,3,4}
\node (v\x) at (360*\x/4-90:1) {$\bullet$};
\end{tikzpicture}
\end{center}

Figure 3.1: The matroid $U_{3,5}$

This result yields the following useful corollary concerning triangles in $M$.

**Corollary 3.8.** If $|E(M)| \neq 6$, then $M$ contains no triangles.

**Proof.** Proposition 3.7 handles the case in which $|E(M)| \leq 5$.

Let $E(M) = \{x_1, x_2, \ldots, x_n\}$ for some $n \geq 7$, and suppose $T = \{x_1, x_2, x_3\}$ is a triangle in $M$. By (P1) there is a 3-cocircuit $D_1$ containing $x_1$. By orthogonality, $|D_1 \cap T| > 1$, and by the 3-connectedness of $M$ we know $T \neq D_1$. Therefore, we may assume that $D_1 = \{x_1, x_2, x_4\}$. Similarly, there is a 3-cocircuit $D_2$ containing $x_3$. Without loss of generality, $x_1 \in D_2$, and by Proposition 3.7 we may assume $D_2 = \{x_1, x_3, x_5\}$. Now, (P1) guarantees a 4-circuit $C$ containing $\{x_2, x_3\}$. As $T \not\subset C$, we must have $C = \{x_2, x_3, x_4, x_5\}$ by orthogonality. However, this means $\lambda(\{x_1, x_2, x_3, x_4, x_5\}) \leq 3 + 3 - 5 = 1$, a contradiction.

The next proposition addresses the case when $M$ has two disjoint 3-cocircuits. Specifically, we prove that two such 3-cocircuits must be locally isomorphic to $M(K_{2,3})$. This structure is the foundation for the main result, and will feature heavily in its proof.

**Proposition 3.9.** If $M$ has two disjoint 3-cocircuits $D_1$ and $D_2$, then $M|(D_1 \cup D_2) \cong M(K_{2,3})$. 

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. As $M$ is 3-connected, $n \geq 7$. By (P1), there exists a 4-circuit $C_1$ containing $\{x_1, x_4\}$. By orthogonality, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$. Similarly, there is a 4-circuit $C_2$ containing $\{x_3, x_6\}$. By symmetry, we may assume $C_2 = \{x_1, x_3, x_4, x_6\}$. Lastly, there must be a 4-circuit $C_3$ containing $\{x_2, x_6\}$. Note that

Claim 3.9.1. $C_3$ must not meet either $C_1$ or $C_2$ in three elements.

Assume it does; that is, without loss of generality, $|C_1 \cap C_3| = 3$. Then $C_1 \cup C_3$ is a 5-point plane, in which there exists a 4-circuit meeting one of $D_1$ or $D_2$ in exactly one element. This contradiction proves the claim.

Therefore, neither $x_1$ nor $x_4$ can be in $C_3$, and we get $C_3 = \{x_2, x_3, x_5, x_6\}$. We will now apply Theorem 3.6 in order to complete the proof.

First, note that $r(\overline{(D_1 \cup D_2)}) = 4$, as each 3-cocircuit is an independent hyperplane. Next, let $K_{2,3}$ be labeled as in Figure 3.2 and suppose $M(K_{2,3})$ inherits the edge labels. Evidently, the identity map $i : E(M(K_{2,3})) \to E(M)$ is a weak map, and, since $M$ is 3-connected, it must be that $M \cong M(K_{2,3})$.

Figure 3.2: The graph $K_{2,3}$

Our final observation about the structure of $M$ is that $M$ must have three pairwise-disjoint 3-cocircuits when $|E(M)| \geq 9$. We build up to this in three steps. First, we prove a
preliminary lemma and some subsequent corollaries revealing the necessary restrictions on the interaction between the 4-circuits and 3-cocircuits of $M$. The lemma indicates that there are no 3-cocircuits contained in 4-circuits. Following that, we prove that $M$ is guaranteed to have at least two disjoint 3-cocircuits when $|E(M)| \geq 9$. We then extend this result to ensure three pairwise-disjoint 3-cocircuits, and, further, prove that they produce a local $M|M(K_{3,3})$-structure.

**Lemma 3.10.** If $|E(M)| \geq 9$ and $M$ has $C$ as a 4-circuit and $D$ as a 3-cocircuit, then $D \not\subseteq C$.

**Proof.** Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$ for some $n \geq 9$, and suppose we have a 3-cocircuit $D_1 = \{x_1, x_2, x_3\}$ and a 4-circuit $C_1 = D_1 \cup \{x_4\}$. By (P1), we are guaranteed a 3-cocircuit $D_2$ containing $x_4$. By orthogonality, $D_2$ must contain a second element of $C_1$, and by Proposition 3.7 it has at most one element in common with $D_1$. Hence, we may assume $D_2 = \{x_1, x_4, x_5\}$.

Now, there is a 4-circuit $C_2$ containing $\{x_2, x_5\}$. By orthogonality, either $x_1 \in C_2$, or $C_2 = \{x_2, x_3, x_4, x_5\}$. Note, however, that a second 4-circuit contained in $\{x_1, x_2, x_3, x_4, x_5\}$ means

$$\lambda(\{x_1, x_2, x_3, x_4, x_5\}) = r(\{x_1, x_2, x_3, x_4, x_5\}) + r^*(\{x_1, x_2, x_3, x_4, x_5\}) - |\{x_1, x_2, x_3, x_4, x_5\}| \leq 3 + 3 - 5 = 1.$$ 

This implies that $x_1 \in C_2$, and, further, that $C_2 = \{x_1, x_2, x_5, x_6\}$.

Next, consider a 4-circuit $C_3$ containing $\{x_3, x_5\}$. By orthogonality, $x_1 \in C_3$, and by the above argument we know $C_3 \not\subseteq \{x_1, x_2, x_3, x_4, x_5\}$. Therefore, either $C_3 = \{x_1, x_3, x_5, x_6\}$, or $C_3 = \{x_1, x_3, x_5, x_7\}$. We show next that

**Claim 3.10.1.** $C_3 = \{x_1, x_3, x_5, x_7\}$.

Suppose not, that $C_3 = \{x_1, x_3, x_5, x_6\}$. Then $\{x_1, x_2, x_3, x_5, x_6\}$ is a 5-point plane, implying $\{x_1, x_2, x_5, x_6\}$ is a circuit. This violates orthogonality with $D_2$, a contradiction.

Finally, consider a 3-cocircuit $D_3$ containing $x_6$. By orthogonality with $C_2$, we have $\{x_1, x_2, x_5\} \cap D_3 \neq \emptyset$. We will show that the inclusion of any of $x_1, x_2$, or $x_5$ in $D_3$ produces a contradic-
tion. If $x_1 \in D_3$, then this forces $D_3 = \{x_1, x_3, x_6\}$ by orthogonality with $C_1$ and $C_3$, and this contradicts Proposition 3.7. Therefore, $x_1 \not\in D_3$. If $x_2 \in D_3$, then $D_3 = \{x_2, x_4, x_6\}$ by orthogonality with $C_1$. However, now $\lambda(\{x_1, x_2, \ldots, x_6\}) \leq 4 + 3 - 6 = 1$, a contradiction. Therefore, $x_2 \not\in D_3$. Lastly, suppose $x_5 \in D_3$. By orthogonality with $C_3$, one of $x_3$ and $x_7$ must be in $D_3$. The inclusion of $x_3$ contradicts orthogonality with $C_1$, so $D_3 = \{x_5, x_6, x_7\}$. But now, $\lambda(\{x_1, x_2, \ldots, x_7\}) \leq 4 + 4 - 7 = 1$, a contradiction. Thus there is no 3-cocircuit containing $x_6$, and this contradiction proves the lemma.

\[ \square \]

**Corollary 3.11.** If $|E(M)| \geq 9$, then $M$ contains no 5-point planes.

*Proof.* Let $S$ be a 5-point plane in $M$ and suppose $e \in S$. Then by (P1) there must be a 3-cocircuit containing $e$, and, by orthogonality, that cocircuit must be contained in $S$, a contradiction to the last lemma. \[ \square \]

**Corollary 3.12.** If $|E(M)| \geq 9$, then $M$ contains no 4-point colines.

*Proof.* Let $S$ be a 4-point coline in $M$ and suppose $\{x_1, x_2\} \subseteq S$. Then by (P1) there must be a 4-circuit containing $\{x_1, x_2\}$, and, by orthogonality, that circuit must contain at least one additional element of $S$, a contradiction. \[ \square \]

**Lemma 3.13.** If $|E(M)| \geq 9$, then $M$ has two disjoint 3-cocircuits.

*Proof.* Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Let $D_1$ and $D_2$ be distinct 3-cocircuits of $M$. By Proposition 3.7, $|D_1 \cap D_2| \leq 1$, so we may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$. We demonstrate that

**Claim 3.13.1.** there is a third 3-cocircuit containing $x_1$.

Suppose not. By (P1) there is a 3-cocircuit $D_3$ containing $x_6$. By assumption, $D_3$ meets each of $D_1$ and $D_2$. Then, $D_3 = \{x_2, x_4, x_6\}$, without loss of generality. But (P1) further guarantees a 3-cocircuit containing $x_7$. By the pigeonhole principle, such a cocircuit cannot...
meet each of $D_1$, $D_2$, and $D_3$ without using an element shared by two of them, thus proving the claim.

Therefore, we may assume that $D_3 = \{x_1, x_6, x_7\}$. Now, consider a 3-cocircuit $D_4$ containing $x_8$. In order to meet each of $D_1$, $D_2$, and $D_3$, it must be that $x_1 \in D_4$, and so we may assume $D_4 = \{x_1, x_8, x_9\}$. However, $(P1)$ guarantees a 4-circuit $C$ containing $\{x_1, x_2\}$. By orthogonality, $C$ must contain a second element from each of $D_2$, $D_3$, and $D_4$, implying $|C| = 5$. This contradiction proves the lemma.

\[\Box\]

**Proposition 3.14.** If $|E(M)| \geq 9$, then $M$ contains three pairwise-disjoint 3-cocircuits.

**Proof.** Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. By Lemma 3.13, we get disjoint 3-cocircuits $D_1$ and $D_2$. We may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. By Proposition 3.9, $M|(D_1 \cup D_2) \cong M(K_{2,3})$. Therefore we get circuits $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$, without loss of generality. Any additional 3-cocircuit must meet $D_1 \cup D_2$, and, by orthogonality, it must do so in one of the series pairs of $M|(D_1 \cup D_2)$. Therefore, if $D_3$ is a 3-cocircuit containing $x_7$, we may assume $D_3 = \{x_1, x_4, x_7\}$.

Similarly, if $D_4$ is a 3-cocircuit containing $x_8$, we may assume $D_4 = \{x_2, x_5, x_8\}$, since $D_4$ must not contain $\{x_1, x_4\}$ by Proposition 3.7. Finally, if $D_5$ is a 3-cocircuit containing $x_9$, it must be that $D_5 = \{x_3, x_6, x_9\}$. But then $D_3$, $D_4$, and $D_5$ are disjoint, a contradiction.

\[\Box\]

**Lemma 3.15.** Let $D_1 = \{x_1, x_2, x_3\}$, $D_2 = \{x_4, x_5, x_6\}$, and $D_3 = \{x_7, x_8, x_9\}$ be cocircuits of $M$. Suppose the sets of 4-circuits contained in $M|(D_1 \cup D_2)$ and $M|(D_1 \cup D_3)$ are

\[
\{\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}\} \text{ and } \{\{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_7, x_9\}, \{x_2, x_3, x_8, x_9\}\}.
\]

Then the 4-circuits contained in $M|(D_2 \cup D_3)$ are $\{\{x_4, x_5, x_7, x_8\}, \{x_4, x_6, x_7, x_9\}, \{x_5, x_6, x_8, x_9\}\}$, and $M|(D_1 \cup D_2 \cup D_3) = M(K_{3,3})$, where the vertex bonds of $K_{3,3}$ are $D_1$, $D_2$, $D_3$, $\{x_1, x_4, x_7\}$, $\{x_2, x_5, x_8\}$, and $\{x_3, x_6, x_9\}$.  

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Figure 3.3: The graph $K_{3,3}$ provides the underlying structure to three disjoint 3-cocircuits.

**Proof.** In order to prove $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,3})$, we will recruit Theorem 3.6. To that end, we must determine the circuits in $M|(D_1 \cup D_2 \cup D_3)$. Specifically, we will find the nine 4-circuits and six 6-circuits that exist in $M(K_{3,3})$. See Figure 3.3 for reference.

By Lemma 3.9, we know that $M$ restricted to each pair of disjoint 3-cocircuits is isomorphic to $M(K_{2,3})$. Hence, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$ are the circuits in $M|(D_1 \cup D_2)$. Further, we may assume that $C_4 = \{x_1, x_2, x_7, x_8\}$, $C_5 = \{x_1, x_3, x_7, x_9\}$, and $C_6 = \{x_2, x_3, x_8, x_9\}$ are the 4-circuits in $M|(D_1 \cup D_3)$, without loss of generality. The 4-circuits in $M|(D_2 \cup D_3)$ can then be determined by circuit elimination. First, take the circuit $C_7 \subseteq (C_1 \cup C_4) - x_1$. By orthogonality, $C_7 = \{x_4, x_5, x_7, x_8\}$. Similarly, the circuit $C_8 \subseteq (C_2 \cup C_5) - x_1$ must be $C_8 = \{x_4, x_6, x_7, x_9\}$. Lastly, the circuit $C_9 \subseteq (C_3 \cup C_6) - x_2$ must be $C_9 = \{x_5, x_6, x_8, x_9\}$. The six 6-circuits are, by orthogonality, precisely the following sets: $(C_1 \cup C_8) - x_4$, $(C_1 \cup C_9) - x_5$, $(C_2 \cup C_7) - x_4$, $(C_2 \cup C_9) - x_6$, $(C_3 \cup C_7) - x_5$, and $(C_3 \cup C_8) - x_6$.

Now, let $K_{3,3}$ be labeled as in Figure 3.3, and have $M(K_{3,3})$ inherit the edge labels. Then, the identity map $i : E(M(K_{3,3})) \rightarrow E(M)$ is a weak map, and, since $r(M) = r(M \setminus D_1) + 1 = r(M(K_{2,3})) + 1 = r(M(K_{3,3}))$, Theorem 3.6 indicates $M \cong M(K_{3,3})$. \[\square\]
3.3 When $M$ Has Few Elements

Here will shall determine all matroids with property $(P1)$ having fewer than nine elements. Part of the job is done, by Proposition 3.7. We first show that, if $M$ has disjoint 3-cocircuits, then it must be the matroid $P_7^*$. Afterwards, we will find a couple of matroids on six elements, and a couple of matroids on seven elements. There are no matroids on eight elements that have property $(P1)$.

**Proposition 3.16.** If $|E(M)| \leq 8$ and $M$ has two disjoint 3-cocircuits, then $M \cong P_7^*$.

**Proof.** The result is immediate if $|E(M)| \leq 5$.

Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose that $M$ has a pair of disjoint 3-cocircuits $D_1$ and $D_2$. Let $M|(D_1 \cup D_2)$ be labeled as in Figure 3.2. We take the proof in three cases, first ruling out the 6- and 8-element cases. If $|E(M)| = 6$, then by Lemma 3.9, $M \cong M(K_{2,3})$. This matroid is not 3-connected, contradicting $(P1)$.

Assume, next, that $|E(M)| = 8$. By $(P1)$, there is a 3-cocircuit $D_3$ containing $x_7$. As $M$ has only eight elements, $D_3$ meets $D_1 \cup D_2$ and, by orthogonality, must do so in a series pair of $M|(D_1 \cup D_2)$. Without loss of generality, $D_3 = \{x_1, x_4, x_7\}$. Similarly, a 3-cocircuit $D_4$ containing $x_8$ must meet $D_1 \cup D_2$ in a series pair, and that pair cannot be $\{x_1, x_4\}$ by Proposition 3.7. Now, there is a 4-circuit $C$ containing $\{x_3, x_7\}$. By orthogonality, either $x_1 \in C$, or $C = \{x_2, x_3, x_4, x_7\}$. The latter case is out by orthogonality with $D_4$, so suppose $x_1 \in C$. Each of the remaining elements in $E(M) - \{x_1, x_3, x_7\}$ are in some cocircuit disjoint from $\{x_1, x_3, x_7\}$. Thus, $C$ cannot include any additional elements without violating orthogonality.

Now, we may assume $|E(M)| = 7$. We begin as in the previous case. By $(P2)$, $x_7$ is in a 3-cocircuit $D_3$. Then, by orthogonality, $D_3$ meets $D_1 \cup D_2$ in a series pair of $M|(D_1 \cup D_2)$. Without loss of generality, $D_3 = \{x_1, x_4, x_7\}$. There must be a circuit $C_1 \{x_1, x_7\}$. By orthogonality, this is forced to be $C_1 = \{x_1, x_2, x_3, x_7\}$. Hence, $x_7 \in \text{cl}(D_1)$, and it follows that $r(M) = 4$. Therefore, the complement of each 4-circuit of $M$ is a 3-cocircuit, and the
3-cocircuits align as in Figure 3.4. This structure admits no further 3-cocircuits, and since \( r^*(M) = 3 \), we have determined the full list of cocircuits of \( M \). Thus \( M \cong P_7^* \), and the lemma is proved.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\end{array}
\]

Figure 3.4: The matroid \( P_7 \).

For the remainder of the section we need only consider matroids having no two disjoint 3-cocircuits. With this last proposition, we will have determined all matroids having property \( (P1) \) on fewer than nine elements.

**Proposition 3.17.** If \( 6 \leq |E(M)| \leq 8 \) and \( M \) has no two disjoint 3-cocircuits, then \( M \) is one of the following matroids: \( M(K_4) \), \( \mathcal{W}^3 \), \( F_7^* \), and \( (F_7^-)^* \).

**Proof.** Let \( E(M) = \{x_1, x_2, \ldots, x_n\} \) for some \( n \in \{6, 7, 8\} \). Without loss of generality, we may assume \( D_1 = \{x_1, x_2, x_3\} \) and \( D_2 = \{x_1, x_4, x_5\} \) are cocircuits. We proceed in cases by the size of \( M \).

If \( |E(M)| = 6 \), then, without loss of generality, we may assume a cocircuit \( D_3 \) containing \( x_6 \) is \( \{x_2, x_4, x_6\} \). Now \( r(M) = r^*(M) = 3 \), so any the complement of any 3-cocircuit of \( M \) is a triangle. Should \( D_1, D_2, \) and \( D_3 \) be the complete list of 3-cocircuits of \( M \), then, evidently, \( M \cong \mathcal{W}^3 \). However, this structure also admits \( \{x_3, x_5, x_6\} \) as a cocircuit without violating \( (P1) \). In this case, \( M \cong M(K_4) \).

Suppose \( |E(M)| = 7 \). In this case, either \( r(M) = 3 \), or \( r(M) = 4 \). In the former case, hyperplanes of \( M \) will have rank 2. Therefore, the complement of each 3-cocircuit is a 4-
point line; but, if both \( \{x_4, x_5, x_6, x_7\} \) and \( \{x_2, x_3, x_6, x_7\} \) are 4-point lines, then their union is a 6-point line, a contradiction. Hence \( r(M) = 3 \).

Now, all 4-circuits of \( M \) are hyperplanes. We get circuits \( C_1 = E(M) - D_1 \) and \( C_2 = E(M) - D_2 \) immediately. By \( (P1) \) we are guaranteed a 4-circuit \( C_3 \) containing \( \{x_1, x_5\} \). By orthogonality, it suffices to assume \( C_3 = \{x_1, x_3, x_5, x_7\} \). Hence, \( D_3 = \{x_2, x_4, x_6\} \) is a cocircuit. Similarly, a 4-circuit \( C_4 \) containing \( \{x_2, x_3\} \) can be assumed to be \( C_4 = \{x_2, x_4, x_5, x_6\} \), without loss of generality. This yields the cocircuit \( D_4 = \{x_1, x_2, x_7\} \). Further, a 4-circuit \( C_5 \) containing \( \{x_3, x_7\} \) is either \( \{x_3, x_4, x_6, x_7\} \) or \( \{x_2, x_3, x_5, x_7\} \). These are symmetric, so we may assume \( C_5 = \{x_3, x_4, x_6, x_7\} \) and that \( D_5 = \{x_1, x_2, x_5\} \) is a cocircuit. The only pair of elements not yet in some 4-circuit is \( \{x_5, x_7\} \). Suppose \( C_6 \) is the 4-circuit containing them. Again, this circuit must be either \( \{x_1, x_4, x_5, x_7\} \) or \( \{x_2, x_3, x_5, x_7\} \), and these sets are symmetric. Thus, we may let \( C_6 = \{x_1, x_4, x_5, x_7\} \). Such a matroid \( M \) having \( \mathcal{C}(M) = \{C_1, C_2, C_3, C_4, C_5, C_6\} \) is isomorphic to \( (F_7^-)^* \). It is possible, however, that \( M \) admits a seventh 4-circuit, \( C_7 = \{x_2, x_3, x_5, x_7\} \). This produces the matroid \( F_7^* \), and concludes this case.

Lastly, we assume \( |E(M)| = 8 \). Here we will first prove that \( M \) has three 3-cocircuits meeting in a shared element. If not, then, without loss of generality, a 3-cocircuit \( D_3 \) containing \( x_6 \) is \( D_3 = \{x_2, x_4, x_6\} \). In order to meet each of the previous 3-cocircuits but not use any element already shared between them, a cocircuit \( D_4 \) containing \( x_7 \) must be \( \{x_3, x_5, x_7\} \). But \( (P1) \) guarantees a 3-cocircuit containing \( x_8 \), and all other elements of \( M \) are already in two 3-cocircuits.

Therefore, we may assume that a 3-cocircuit \( D_3 \) containing \( x_6 \) is \( D_3 = \{x_1, x_6, x_7\} \). But now, a 3-cocircuit \( D_4 \) containing \( x_8 \) again leads to contradiction, as, in order to meet all other 3-cocircuits, it must be that \( x_1 \in D_4 \). Therefore, \( D_4 \) is forced to meet one of \( D_1, D_2, \) and \( D_3 \) in two elements, contradicting Proposition 3.7. Thus there are no matroids on eight elements, and the proof is complete.
3.4 The Main Result

All that remains is to determine $M$ when $|E(M)| \geq 10$. We first prove that such an $M$ can be partitioned into 3-cocircuits. Using these disjoint cocircuits, we will be able to complete an induction argument to prove the final component of the main result.

**Lemma 3.18.** If $|E(M)| \geq 9$, then $E(M)$ can be partitioned into $D_1 \cup D_2 \cup \cdots \cup D_n$, where each $D_i$ is a 3-cocircuit.

**Proof.** If $|E(M)| = 9$, then by Proposition 3.14 we are done.

Now suppose $|E(M)| > 9$, and let $S = \{D_1, D_2, \ldots, D_n\}$ be the largest collection of pairwise-disjoint 3-cocircuits of $M$. Let $e$ be an element not in any $D_i$, for $i \in \{1, 2, \ldots, n\}$. By (P1), there is a 3-cocircuit $D_e$ containing $e$. As $D_e$ must meet some cocircuit in $S$, we may assume that $D_e \cap D_1 \neq \emptyset$. By Lemma 3.9, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \ldots, n\}$. By orthogonality, $D_e$ must meet each of these local $M(K_{2,3})$’s in a series pair, a contradiction. Thus, the lemma is proved.

**Proposition 3.19.** If $|E(M)| \geq 9$, then $M \cong M(K_{3,n})$ for some $n \geq 3$.

**Proof.** By Lemma 3.18, for some $n \geq 3$, there is a partition of $E(M)$ into 3-cocircuits $D_1, D_2, \ldots, D_n$ where $D_i = \{x_i, y_i, z_i\}$ for all $i$. By Proposition 3.9, as $M|(D_1 \cup D_2) \cong M(K_{2,3})$
when \( i \neq j \), we can assume that \( M \) has \( \{x_1, x_2, y_1, y_2\}, \{x_1, x_2, z_1, z_2\}, \) and \( \{y_1, y_2, z_1, z_2\} \) as circuits. By repeatedly applying Lemma 3.15, we can assume that \( M \) has \( \{x_i, x_j, y_i, y_j\}, \{x_i, x_j, z_i, z_j\}, \) and \( \{y_i, y_j, z_i, z_j\} \) as circuits for all \( 1 \leq i < j < n \). We prove by induction on \( k \) that, for \( 3 \leq k \leq n \), with \( K_{3,k} \) labelled so that its vertex bonds are \( D_1, D_2, \ldots, D_k \), \( \{x_1, x_2, \ldots, x_k\}, \{y_1, y_2, \ldots, y_k\}, \) and \( \{z_1, z_2, \ldots, z_k\} \), we have \( M| (D_1 \cup D_2 \cup \cdots \cup D_n) = M(K_{3,k}) \).

By Lemma 3.15, this is true when \( k = 3 \). Assume it is true for \( k < m \), and let \( k = m \geq 3 \). Suppose \( M| (D_1 \cup D_2 \cup \cdots \cup D_{m-1}) \neq M(K_{3,m}) \). Let \( Z \) be a minimal set that is independent in one of \( M| (D_1 \cup D_2 \cup \cdots \cup D_m) \) and \( M(K_{3,m}) \) and dependent in the other. Then \( Z \) is independent in one matroid, say \( M_I \), and a circuit in the other, say \( M_C \).

As \( M| (D_1 \cup D_2 \cup \cdots \cup D_{m-1}) = M(K_{3,m-1}) \), it follows by the induction assumption that \( Z \) meets \( D_m \). By symmetry, \( Z \) meets each of \( D_1, D_2, \ldots, D_{m-1} \). As \( D_1, D_2, \ldots, D_m \) are cocircuits of each \( M| (D_1 \cup D_2 \cup \cdots \cup D_m) \) and \( M(K_{3,m}) \), it follows, by orthogonality in \( M_C \), that \( Z \) meets each of \( D_1, D_2, \ldots, D_{m-1} \) in at least two elements. Therefore \( |Z| \geq 2m \). But

\[
 r(M| (D_1 \cup D_2 \cup \cdots \cup D_m)) = r(M(K_{3,m-1})) + 1 = m + 2.
\]

Hence \( |Z| \leq m + 3 \). Thus \( 2m \leq |Z| \leq m + 2 \), so \( m \leq 2 \); a contradiction. The result follows by induction.

Upon combining the previous propositions, we get the main result.

**Theorem 3.20.** Suppose \( M \) is a non-empty 3-connected matroid. If \( M \) has every element in a 3-cocircuit and every pair of elements in a 4-circuit, then \( M \) is one of the following matroids: \( U_{3,5} \), \( M(K_4) \), \( W^3 \), \( F_7 \), \( (F_7^-)^* \), \( P_7^\ast \), and \( M(K_{3,n}) \) for some \( n \geq 3 \).
Chapter 4
Many 4-cocircuits and 4-circuits

4.1 Introduction and Preliminaries

This chapter continues the line of inquiry from Chapter 3. We turn our focus now to matroids $M$ having the following property:

4.1. A matroid $M$ has property ($P_2$) if, for all distinct elements $e$ and $f$ of $M$:

(i) there exists some 4-circuit $C \in \mathcal{C}(M)$ such that $\{e, f\} \subseteq C$,

(ii) there exists some 4-cocircuit $D \in \mathcal{C}(M^*)$ such that $e \in D$, and

(iii) $M$ is 4-connected.

We will assume throughout this chapter that $M$ has property ($P_2$), and will proceed to determine all such matroids. The necessary preliminaries are the same as in Chapter 3, and we will take a similar approach with the arguments. As before, we will begin by making several observations about the structural consequences of ($P_2$), and then proceed to determine explicitly the possibilities for $M$ when $|E(M)|$ is small. More precisely, we show that $M$ is either $U_{3,6}$, one of 21 paving matroids with eight elements, one of 10 matroids with nine elements, the matroid $R_{10}$, a matroid on twelve elements, a matroid on fourteen elements, or $M$ has at least sixteen elements and $M \cong M(K_4,n)$.

4.2 Structure Lemmas

The first proposition of this section addresses when $M$ has two 4-cocircuits that meet in three elements, and shows that this structure only occurs in one case.
Proposition 4.2. There exist 4-cocircuits $D_1$ and $D_2$ of $M$ such that $|D_1 \cap D_2| = 3$ if and only if $M \cong U_{3,6}$.

Proof. If $M \cong U_{3,6}$, it certainly has such a pair of 4-cocircuits. Now, suppose $M$ contains two such 4-cocircuits. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_3, x_5\}$. Note that $D_1 \cup D_2$ is a 5-point plane in $M^*$, and, therefore, any circuit meeting $D_1 \cup D_2$ must do so in at least 3 elements, otherwise it will violate orthogonality.

We will first show that $D_1 \cup D_2$ must contain a circuit. Consider a 4-circuit, $C_1$, containing $\{x_1, x_2\}$. If $C_1$ is not contained in $D_1 \cup D_2$, then $C_1 = \{x_1, x_2, x_3, x_6\}$, without loss of generality. There must also be a 4-circuit, $C_2$, containing $\{x_4, x_6\}$. Clearly, $C_2 \subseteq D_1 \cup D_2 \cup \{x_6\}$. Then, by circuit elimination, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{x_6\} \subseteq D_1 \cup D_2$, as desired.

Let $Z$ be a 4-circuit that is contained in $D_1 \cup D_2$. Then $Z$ is also a cocircuit, so $r(Z) + r^*(Z) - |Z| = 3 + 3 - 4 = 2$. Hence $|E(M) - Z| = 2$, so $|E(M)| = 6$. As $M$ has no circuits of size less than three, it follows that $M \cong U_{3,6}$.

The next proposition states that having two disjoint 4-cocircuits in $M$ ensures a local $K_{2,4}$-structure. We later use this local structure as a basis for the induction argument proving that, when $M$ has a sufficient number of elements, it must be isomorphic to $M(K_{4,n})$ for some natural number $n \geq 4$.

The proof of this requires three preliminary lemmas, which each rule out a particular configuration of 4-circuits that might occur between the disjoint 4-cocircuits. In each of these lemmas, $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ are disjoint 4-cocircuits of $M$. Observe that orthogonality and the 4-connectedness of $M$ implies that every 4-circuit contained in $X \cup Y$ meets each of $X$ and $Y$ in exactly two elements.
Lemma 4.3. If $C_1$ and $C_2$ are distinct 4-circuits contained in $X \cup Y$ such that $|C_1 \cap C_2 \cap X| \geq 1$, then $|C_1 \cap C_2 \cap X| = 1$.

Proof. Clearly $|C_1 \cap C_2 \cap X| \neq 4$. If $|C_1 \cap C_2 \cap X| = 3$, then each of $C_1$ and $C_2$ has only one element from $Y$, which contradicts their orthogonality with $Y$. Therefore $|C_1 \cap C_2 \cap X| \neq 3$.

Now suppose $|C_1 \cap C_2 \cap X| = 2$. Therefore, either $|C_1 \cap C_2| = 3$, or $|C_1 \cap C_2| = 2$. If $|C_1 \cap C_2| = 3$, then, without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_1, y_3\}$. By circuit elimination, there is a circuit contained in $\{x_1, y_1, y_2, y_3\}$, and such a circuit will violate either the 4-connectedness of $M$, or orthogonality. If $|C_1 \cap C_2| = 2$, then, without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_3, y_4\}$. This implies, by circuit elimination, that there is a circuit contained in $\{x_1, y_1, y_2, y_3, y_4\}$. Such a circuit must contain $x_1$, and thus will contradict orthogonality with the cocircuit $X$. 

The last lemma shows that pairs of elements from $X$ or from $Y$ will occur at most once among the 4-circuits contained within $X \cup Y$.

We must now determine how those pairs of elements from each 4-cocircuit match up within the 4-circuits that contain them. We achieve this in the following two results. The next lemma indicates that three circuits contained in $X \cup Y$ cannot cover $X$ unless they also cover $Y$.

Lemma 4.4. If $C_1$, $C_2$, and $C_3$ are distinct 4-circuits of $M$ such that $C_1 \cup C_2 \cup C_3 \subseteq X \cup Y$ and $C_1 \cup C_2 \cup C_3 \supseteq X$, then $C_1 \cup C_2 \cup C_3 \supseteq Y$.

Proof. Suppose not. Then, by Lemma 4.3, we may assume that $C_1 \cap Y = \{y_2, y_3\}$, $C_2 \cap Y = \{y_1, y_3\}$, and $C_3 \cap Y = \{y_1, y_2\}$. Without loss of generality, we may assume that $C_1 \cap X = \{x_1, x_2\}$ and $C_2 \cap X = \{x_1, x_3\}$. Then $\{x_1, y_1, y_2, y_3\}$ spans $X$. As $X$ is independent by the 4-connectedness of $M$, we have that $X$ spans $X \cup Y$. This implies that, for any $y \in Y$, there is a circuit containing $y$ and contained in $X \cup \{y\}$. This contradicts orthogonality, and thus the lemma is proved. 

□
Lemma 4.5. If $C_1$ and $C_2$ are distinct 4-circuits in $X \cup Y$ such that $|C_1 \cap C_2| \geq 1$, then $|C_1 \cap C_2| = 2$.

Proof. Assume the lemma fails. As $|C_1 \cap C_2| = |C_1 \cap C_2 \cap X| + |C_1 \cap C_2 \cap Y|$, it follows by Lemma 4.3 and symmetry that we may assume $|C_1 \cap C_2 \cap X| = 1$ and $|C_1 \cap C_2 \cap Y| = 0$. Without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$ and $C_2 = \{x_1, x_3, y_3, y_4\}$. By Lemma 4.4, any further 4-circuit contained in $X \cup Y$ must contain $x_4$. Property $(P2)$ guarantees the existence of a 4-circuit $C_3$ containing $\{x_2, y_3\}$. By Lemma 4.3, we have $C_3 = \{x_2, x_4, y_1, y_3\}$, without loss of generality. Now $(P2)$ similarly guarantees a 4-circuit $C_4$ containing $\{x_2, y_4\}$. As noted above, it must be that $x_4 \in C_4$, but this contradicts Lemma 4.3. Thus, no such $C_4$ exists, and our assertion holds. □

The three previous lemmas combined yield the following result.

Proposition 4.6. Let $M$ be a matroid with property $(P2)$. If $M$ has $X$ and $Y$ as disjoint 4-cocircuits, then $M|(X \cup Y) \cong M(K_{2,4})$.

Proof. There is a 4-circuit, $C_1$, in $M$ containing $x_1$ and $y_1$. Without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$. Similarly, there is a 4-circuit, $C_2$, in $M$ containing $x_1$ and $y_3$. Without loss of generality, $C_2 = \{x_1, x_3, y_1, y_3\}$, since, by Lemma 4.5, $C_1$ and $C_2$ intersect in exactly two elements. There is a 4-circuit, $C_3$ containing $x_1$ and $y_4$. By Lemmas 4.3 and 4.5, $C_3 = \{x_1, x_4, y_1, y_4\}$.

A fourth 4-circuit, $C_4$, may be found containing $x_2$ and $y_3$. Since $C_4$ meets $C_1$ in $x_2$, we have $x_1 \not\in C_4$ by Lemma 4.3. Therefore by Lemma 4.5, since $C_4$ meets $C_2$ in $y_3$, we have $y_1 \not\in C_4$, so $x_3 \in C_4$, and, similarly, since $C_4$ meets $C_1$ in $x_2$, we have $y_2 \in C_4$. Hence $C_4 = \{x_2, x_3, y_2, y_3\}$. Similarly, the 4-circuit containing $x_2$ and $y_4$ must be $C_5 = \{x_2, x_4, y_2, y_4\}$, and the final 4-circuit, one containing $x_3$ and $y_4$, must be $C_6 = \{x_3, x_4, y_3, y_4\}$.

To see that $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$, first observe that, since every 2-element subset of each of $X$ and $Y$ is in one of $C_1, C_2, \ldots, C_6$, Lemma 4.3 implies that $M|(X \cup Y)$ has
no other 4-circuits. Clearly $r(X \cup Y) = 5$. Suppose $C \in \mathcal{C}(M|(X \cup Y)) - \{C_1, C_2, \ldots, C_6\}$. If $|C| = 6$, then $C$ contains some $C_i$; a contradiction. Therefore, $|C| = 5$. To maintain orthogonality, $C$ must be comprised of two elements from one of $X$ and $Y$, and three elements from the other. To avoid containing one of the six 4-circuits, we may assume that $C = \{x_1, x_2, y_2, y_3, y_4\}$. Then, $\text{cl}(\{x_1, y_2, y_3, y_4\}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, so $r(X \cup Y) = 4$, a contradiction.

With the structure of the circuits in $\mathcal{C}(M|(X \cup Y))$ determined, we are now able to show that $M|(X \cup Y) \cong M(K_{2,4})$. First note that $r(M|(X \cup Y)) = 5$. Then, with $K_{2,4}$ labeled as in Figure 4.1 and $M(K_{2,4})$ inheriting the edge labels, the map $\phi : E(M|(X \cup Y)) \to E(M(K_{2,4}))$, given by $\phi(x_i) = a_i$ and $\phi(y_i) = b_i$, is an isomorphism. Thus, by Theorem 3.6, we have $M|(X \cup Y) \cong M(K_{2,4})$, and the proposition holds.

![Figure 4.1: The graph $K_{2,4}$.](image)

The third proposition of this section proves that the desirable $K_{2,4}$-structure must be present when $|E(M)| \geq 11$. The proof of this requires five preliminary results which restrict the ways in which 4-cocircuits may intersect. The first of these shows that three 4-cocircuits cannot pairwise meet in a common element unless those 4-cocircuits cover the matroid.

**Lemma 4.7.** If $D_1, D_2,$ and $D_3$ are 4-cocircuits of $M$ such that $|D_1 \cap D_2 \cap D_3| = 1$ and $|D_i \cap D_j| = 1$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, then $E(M) = D_1 \cup D_2 \cup D_3$. 

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Proof. Suppose $D_1, D_2$, and $D_3$ as above, and yet $E(M) - (D_1 \cup D_2 \cup D_3) \neq \emptyset$. Take $e \in E(M) - (D_1 \cup D_2 \cup D_3)$, and $x \in D_1 \cap D_2 \cap D_3$. There is a 4-circuit, $C$, containing $\{e, x\}$, and $C$ must contain at least two elements from each $D_i$ by orthogonality. This forces $C$ to have at least five elements, a contradiction.

Building on the previous lemma, we demonstrate that $M$ must have two 4-cocircuits that meet in two elements when $|E(M)| \geq 11$ and $M$ has no disjoint 4-cocircuits. Then, we show that the union of two such 4-cocircuits meets every other 4-cocircuit in at least two elements.

**Lemma 4.8.** If $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 11$, then there exist 4-cocircuits $D_1$ and $D_2$ of $M$ such that $|D_1 \cap D_2| = 2$.

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 11$. The element $x_1$ is in a 4-cocircuit; without loss of generality, that cocircuit is $D_1 = \{x_1, x_2, x_3, x_4\}$. We also have a 4-cocircuit, $D_2$, that contains $x_5$ and meets $D_1$. By Proposition 4.2, $|D_2 \cap D_1| = 1$. Without loss of generality, $D_2 = \{x_1, x_5, x_6, x_7\}$. Similarly, there is a 4-cocircuit, $D_3$, that contains $x_8$ and meets both $D_1$ and $D_2$ in a single element. By Lemma 4.7, we know $x_1 \notin D_3$. Therefore, without loss of generality, $D_3 = \{x_2, x_5, x_8, x_9\}$. Lastly, there is a 4-cocircuit, $D_4$, containing $x_{10}$ and meeting each of $D_1$, $D_2$, and $D_3$ in a single element. Lemma 4.7 forces $D_4 = \{x_3, x_6, x_8, x_{10}\}$, without loss of generality. Then, the 4-cocircuit containing $x_{11}$ must contain $\{x_4, x_7, x_9, x_{11}\}$. Thus $D_4$ and $D_5$ are disjoint, a contradiction.

Note that the next three lemmas only require $M$ to have at least 10 elements.

**Lemma 4.9.** Suppose $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$. If $|D_1 \cap D_2| = 2$, then $|D_3 \cap (D_1 \cup D_2)| \geq 2$.

Proof. Suppose not. Then $|D_3 \cap (D_1 \cup D_2)| = 1$, and, more specifically, $|D_1 \cap D_2 \cap D_3| = 1$. Let $\{e\} = D_1 \cap D_2 \cap D_3$. By circuit elimination, there is a cocircuit $D_4 \subseteq (D_1 \cup D_2) - \{e\}$. As $D_3 \cap D_4 = \emptyset$, we see that $|D_4| \neq 4$. Therefore, $D_4 = (D_1 \cup D_2) - \{e\}$.
As $|E(M)| \geq 10$, we have $|E(M) - (D_1 \cup D_2 \cup D_3)| \geq 1$. Let $f \in E(M) - (D_1 \cup D_2 \cup D_3)$, and let $C$ be a 4-circuit containing \{e, f\}. To preserve orthogonality, $C$ must contain an element $g \in D_3 - \{e\}$ and an element $h \in (D_1 \cap D_2) - \{e\}$. But then $C = \{e, f, g, h\}$, and $|C \cap D_4| = 1$. This contradicts the orthogonality of $C$ and $D_4$, proving the lemma.

Concerning 4-cocircuits that meet in two elements, we may now prove that the two shared elements do not appear together in any other 4-cocircuits.

**Lemma 4.10.** Suppose $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let $D_1$, $D_2$, and $D_3$ be distinct 4-cocircuits of $M$. If $D_1 \cap D_2 = \{x_1, x_2\}$, then $\{x_1, x_2\} \not\subseteq D_3$.

**Proof.** Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 10$. Without loss of generality, $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_2, x_7, x_8\}$. Using circuit elimination on each pair in $\{D_1, D_2, D_3\}$ and eliminating $x_2$, we find that each of $\{x_1, x_3, x_4, x_5, x_6\}$, $\{x_1, x_3, x_4, x_7, x_8\}$, and $\{x_1, x_5, x_6, x_7, x_8\}$ contains a cocircuit. Each of those cocircuits must contain $x_1$, otherwise we get two disjoint 4-cocircuits. Further, each of these 5-element sets is, in fact, a cocircuit, as any of their 4-element subsets containing $x_1$ meets another 4-cocircuit in three elements. We will refer to these 5-cocircuits as $D_5, D_6$, and $D_7$, respectively.

Consider a 4-circuit, $C_1$, containing $\{x_1, x_9\}$. By considering the intersection of $C_1$ with each of $D_1, D_2$, and $D_3$, we see that $x_2 \in C_1$. However, $C_1$ only meets each of $D_5, D_6$, and $D_7$ in a single element. By orthogonality, $C_1$ must have a second element in common with each of them. However, $C_1$ has only one additional element, and there is no single element other than $x_1$ that these 5-cocircuits have in common. This contradiction proves the lemma.

This last preliminary lemma prohibits a particular configuration of 4-cocircuits.

**Lemma 4.11.** Suppose $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$. If $|D_1 \cap D_2 \cap D_3| = 1$, then $|D_i \cap D_j| = 1$ for some pair $\{i, j\} \subseteq \{1, 2, 3\}$.
Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 10$. By combining Proposition 4.2 and Lemma 4.10, we may let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_3, x_5, x_7\}$, without loss of generality.

Consider a 4-circuit, $C_1$, containing $\{x_8, x_9\}$. If $C_1$ meets $D_1 \cup D_2 \cup D_3$, then it must do so in at least three elements to avoid an orthogonality contradiction; therefore, $C_1 \cap (D_1 \cup D_2 \cup D_3) = \emptyset$. We may assume $C_1 = \{x_8, x_9, x_{10}, x_{11}\}$.

Now consider a 4-cocircuit, $D_4$, containing $x_8$. Without loss of generality, $x_9 \in D_4$. By assumption, $D_4$ meets each of $D_1$, $D_2$, and $D_3$, and by Lemma 4.9, $D_4$ must contain two elements from each of $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$. If $x_1 \in D_4$, then, by Lemma 4.10, none of $x_2, x_3, x_4, x_5$, or $x_6$ is in $D_4$, a contradiction. Therefore, without loss of generality, $D_4 = \{x_2, x_3, x_8, x_9\}$.

Finally, consider a 4-circuit, $C_2$, containing $\{x_4, x_{10}\}$. It must meet $D_1 - \{x_4\}$ to avoid an orthogonality contradiction. If $x_1 \not\in C_2$, then $x_2 \in C_2$, without loss of generality. This means $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_4 \neq \emptyset$. Since $D_2 - \{x_2\}$ has no element in common with $D_4 - \{x_2\}$, this is a contradiction to orthogonality. Thus $x_1 \in C_2$, and $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_3 \neq \emptyset$. Thus, $C_2 = \{x_1, x_4, x_5, x_{10}\}$. Similarly, a 4-circuit, $C_3$, containing $\{x_4, x_{11}\}$ must be $C_3 = \{x_1, x_4, x_5, x_{11}\}$. Then, $C_2 \cup C_3$ is a 5-point plane, and $\{x_4, x_5, x_{10}, x_{11}\}$ is a circuit that meets $D_1$ in a single element. This contradiction proves the lemma. 

With those lemmas proved, we are ready to show that $M$ must have two disjoint 4-cocircuits when it has at least 11 elements. This is the final proposition needed before proving our first main theorem. In the proof that follows, we will frequently refer to the way in which a 4-cocircuit meets two other 4-cocircuits that share two elements. For convenience, we introduce the following terminology.

4.12. Let $D_1$, $D_2$, and $D_3$ be 4-cocircuits of $M$, and suppose $|D_1 \cap D_2| = 2$. With respect to $(D_1, D_2)$, we say that $D_3$ is of:
(i) type-1 if $|D_3 \cap D_1 \cap D_2| = 1$, and $|D_3 \cap (D_1 - D_2)| = 1$, and $|D_3 \cap (D_2 - D_1)| = 0$;

(ii) type-2 if $D_3 \cap D_1 \cap D_2 = \emptyset$, and $|D_3 \cap D_1| = |D_3 \cap D_2| = 1$; and

(iii) type-3 if $D_3 \cap D_1 \cap D_2 = \emptyset$, and $|D_3 \cap D_1| = 2$, and $|D_3 \cap D_2| = 1$.

Note that type-2 intersections are symmetric; therefore, we will denote this intersection by $\{D_1, D_2\}$-type-2. There will be occasions in which it is sufficient to specify that $D_3$ is either $(D_1, D_2)$-type-$i$ or $(D_2, D_1)$-type-$i$, for $i \in \{1, 3\}$. In these instances, we will say that $D_3$ is $\{D_1, D_2\}$-type-$i$. The previous lemmas ensure that any 4-cocircuit not contained in such $D_1$ and $D_2$ must be one of the above types when $M$ has no two disjoint 4-cocircuits and $|E(M)| \geq 11$. We prove this in the following lemma.

**Lemma 4.13.** Suppose $M$ has no two disjoint 4-cocircuits, $|E(M)| \geq 10$, and let $D_1$ and $D_2$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. If $D_3$ is a 4-cocircuit of $M$ such that $D_3 \not\subseteq D_1 \cup D_2$, then $D_3$ is $\{D_1, D_2\}$-type-$i$, for exactly one $i \in \{1, 2, 3\}$.

**Proof.** By Lemma 4.10, either $D_1 \cap D_2 \cap D_3 = \emptyset$ or $|D_1 \cap D_2 \cap D_3| = 1$. 

Figure 4.2: Set diagrams of the structures described in 4.12.
Suppose, first, that \( |D_1 \cap D_2 \cap D_3| = 1 \). In this case, \( |D_3 \cap (D_1 \cup D_2)| \geq 2 \) by Lemma 4.9, so we may assume \( D_3 \cap (D_1 - D_2) \neq \emptyset \). By Lemma 4.11, we know \( |D_i \cap D_j| = 1 \) for some \( \{i, j\} \subseteq \{1, 2, 3\} \), and, since \( |D_1 \cap D_2| = 2 \) and \( |D_1 \cap D_3| = 2 \), it must be that \( |D_2 \cap D_3| = 1 \). Therefore, \( |D_3 \cap (D_2 - D_1)| = 0 \), and \( D_3 \) is \((D_1, D_2)\)-type-1.

Suppose, instead, that \( D_1 \cap D_2 \cap D_3 = \emptyset \). As \( M \) has no two disjoint 4-cocircuits, we have \( D_1 \cap D_3 \neq \emptyset \) and \( D_2 \cap D_3 \neq \emptyset \). By Lemma 4.11, it cannot be that \( |D_1 \cap D_3| = |D_2 \cap D_3| = 2 \), so either \( |D_1 \cap D_3| = |D_2 \cap D_3| = 1 \), or \( |D_1 \cap D_3| = 2 \) and \( |D_2 \cap D_3| = 1 \), without loss of generality. These cases yield the \( \{D_1, D_2\}\)-type-2 and \((D_1, D_2)\)-type-3 configurations, respectively. \(\square\)

Now that we have narrowed down the possible configurations of 4-cocircuit intersections, we will systematically prove that each of these configurations cannot occur when \( |E(M)| \geq 11 \) unless \( M \) has two disjoint 4-cocircuits. This proof is very technical, and will be divided into three parts, with each part addressing one of the 4-cocircuit configurations. Parts one and two will be considered lemmas, and the final part will be the central proposition of this section.

Throughout, we assume that \( E(M) = \{x_1, x_2, \ldots, x_n\} \), and \( D_1 \) and \( D_2 \) are 4-cocircuits of \( M \) such that \( D_1 = \{x_1, x_2, x_3, x_4\} \) and \( D_2 = \{x_1, x_2, x_5, x_6\} \).

**Lemma 4.14.** Suppose \( M \) has no two disjoint 4-cocircuits, \( |E(M)| \geq 11 \), and let \( D_1 \) and \( D_2 \) be 4-cocircuits of \( M \) such that \( |D_1 \cap D_2| = 2 \). If \( D_3 \) is another 4-cocircuit of \( M \), then \( D_3 \) is not \( \{D_1, D_2\}\)-type-2.

**Proof.** Suppose \( D_3 \) is \( \{D_1, D_2\}\)-type-2. Without loss of generality, \( D_3 = \{x_3, x_5, x_7, x_8\} \). By \((P2)\), we are guaranteed a 4-cocircuit \( D_4 \) containing \( x_9 \). Before determining the rest of the elements in \( D_4 \), we will prove that

**Claim 4.14.1.** \( D_4 \) and all further 4-cocircuits must be \( \{D_1, D_2\}\)-type-1.

If \( D_4 \) is not \( \{D_1, D_2\}\)-type-1, then, by Lemma 4.12, it must be either \( \{D_1, D_2\}\)-type-2 or \( \{D_1, D_2\}\)-type-3. We treat the second case first.

**Claim 4.14.1.1.** \( D_4 \) is not \( \{D_1, D_2\}\)-type-3.
Assume the contrary. Then either $D_4 = \{x_3, x_4, x_5, x_9\}$ or $D_4 = \{x_3, x_4, x_6, x_9\}$. In the former case we have $|D_3 \cap D_4| = 2$ and $|D_2 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 4.9. Similarly, in the latter case, we have $|D_1 \cap D_4| = 2$ and $|D_3 \cap (D_1 \cup D_4)| < 2$, again contradicting Lemma 4.9. This completes the argument, proving that $D_4$ and all further 4-cocircuits in Claim 4.14.1 cannot be $\{D_1, D_2\}$-type-3.

Claim 4.14.1.2. $D_4$ is not $\{D_1, D_2\}$-type-2.

Assume, instead, that $D_4$ is $\{D_1, D_2\}$-type-2. Then $|D_4 \cap \{x_3, x_5\}| \leq 1$; otherwise, if $\{x_3, x_5\} \subseteq D_4$, then $|D_1 \cap (D_3 \cup D_4)| < 2$ and Lemma 4.9 provides a contradiction.

Suppose $|D_4 \cap \{x_3, x_5\}| = 1$. Then, without loss of generality, $x_3 \in D_4$. Since $D_4$ is $\{D_1, D_2\}$-type-2, this implies $x_6 \in D_4$. Further, one of $x_7$ or $x_8$ must be in $D_4$, otherwise $|D_1 \cap D_3 \cap D_4| = 1$ and Lemma 4.7 provides a contradiction. Hence, without loss of generality, $D_4 = \{x_3, x_6, x_7, x_9\}$, but now $|D_3 \cap D_4| = 2$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, a contradiction to Lemma 4.9.

We now know that $D_4$ avoids $\{x_3, x_5\}$. This means we may assume that $D_4 = \{x_4, x_6, x_7, x_9\}$. By (P2), we have a 4-cocircuit $D_5$ containing $x_{10}$. By 4.14.1.1, $D_5$ cannot be $\{D_1, D_2\}$-type-3. If $D_5$ is $\{D_1, D_2\}$-type-2, then, by the above analysis, $\{x_4, x_6\} \subseteq D_5$ and $\{x_7, x_8\} \cap D_5 \neq \emptyset$. If $D_5 = \{x_4, x_6, x_7, x_{10}\}$, then $|D_4 \cap D_5| = 3$ and Proposition 4.2 provides a contradiction. If $D_5 = \{x_4, x_6, x_8, x_{10}\}$, then $|D_1 \cap (D_4 \cup D_5)|$ and Lemma 4.9 provides a contradiction. Therefore $D_5$ must be $\{D_1, D_2\}$-type-1. By symmetry, demonstrated in Figure 4.3, we may assume that $D_5$ is $(D_1, D_2)$-type-1.

![Figure 4.3: Set diagram of the symmetry in 4.14.1.2.](image)

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Hence, without loss of generality, \( \{x_1, x_3\} \subseteq D_5 \). Further, the remaining element of \( D_5 \) must be either \( x_7 \) or \( x_9 \), otherwise \( D_4 \cap D_5 = \emptyset \). Each of these leads to contradiction. If \( D_5 = \{x_3, x_5, x_7, x_{10}\} \), then \( |D_5 \cap D_3| = 2 \) and \( |D_4 \cap (D_5 \cup D_3)| < 2 \), contradicting Lemma 4.9. Similarly, if \( D_5 = \{x_3, x_5, x_9, x_{10}\} \), then \( |D_5 \cap D_1| = 2 \) and \( |D_3 \cap (D_5 \cup D_1)| < 2 \), again contradicting Lemma 4.9. This completes the proof of Claim 4.14.1.

The following claim is an immediate corollary of the previous claim. In fact, it is merely a generalized restatement presented here for ease of reference.

**Claim 4.14.2.** If \( D_i, D_j, D_k, \) and \( D_l \) are 4-cocircuits of \( M \) such that \( |D_i \cap D_j| = 2 \) and \( D_k \) is \( \{D_i, D_j\}\)-type-2, then \( D_l \) is \( \{D_i, D_j\}\)-type-1.

By 4.14.1, \( D_4 \) is \( \{D_1, D_2\}\)-type-1. We may assume without loss of generality that \( D_4 \) is \( (D_1, D_2)\)-type-1; that is, \( D_4 \) meets \( \{x_3, x_4\} \) but avoids \( \{x_5, x_6\} \). It follows that, since \( |D_1 \cap D_4| = 2 \), we have \( D_4 \cap \{x_7, x_8\} \neq \emptyset \), otherwise \( |D_3 \cap (D_1 \cup D_4)| < 2 \), contradicting Lemma 4.9. Hence, either \( D_4 = \{x_1, x_3, x_7, x_9\} \) or \( D_4 = \{x_1, x_4, x_7, x_9\} \), without loss of generality. We will first show that

**Claim 4.14.3.** \( D_4 \neq \{x_1, x_3, x_7, x_9\} \).

Assume the contrary, in which case \( D_1, D_2, D_3, \) and \( D_4 \) are as in Figure 4.4. Now \( |D_3 \cap D_4| = 2 \), and \( D_2 \) is \( (D_3, D_4)\)-type-2. Therefore Claim 4.14.2 implies that all further 4-cocircuits must be \( \{D_3, D_4\}\)-type-1. Moreover, just as \( D_4 \) necessarily meets \( \{x_7, x_8\} \), so must all further cocircuits meet both \( \{x_7, x_8\} \) and \( \{x_2, x_6\} \). This is because the structure given by \( D_1, D_2, \) and \( D_3 \) that forced one of \( x_7 \) and \( x_8 \) in \( D_4 \) is now present in \( D_1, D_2, \) and \( D_4 \), as demonstrated in the symmetry about a vertical line through \( x_5 \) in Figure 4.4.

By \((P2)\), there is a 4-cocircuit \( D_5 \) containing \( x_{10} \). In order to be both \( \{D_1, D_2\}\)-type-1 and \( \{D_3, D_4\}\)-type-1, \( D_5 \) must contain exactly one element from each of the following: \( D_1 \cap D_2 = \{x_1, x_2\} \), \( D_1 \Delta D_2 = \{x_3, x_4, x_5, x_6\} \), \( D_3 \cap D_4 = \{x_3, x_7\} \), and \( D_3 \Delta D_4 = \{x_1, x_5, x_8, x_9\} \).
Figure 4.4: Set diagram of the structure of cocircuits in Claim 4.14.3.

Note that we are not asserting that either $D_1 \triangle D_4$ or $D_3 \triangle D_4$ is a cocircuit. Because $D_1 \cap D_4 = \{x_1, x_3\}$, Lemma 4.10 implies $\{x_1, x_3\} \not\subseteq D_5$.

**Claim 4.14.3.1.** $x_1 \in D_5$.

Assume the contrary. By the symmetry demonstrated in Figure 4.4, we may assume that $x_3 \not\in D_5$. This implies $\{x_2, x_7\} \subseteq D_5$, and therefore $\{x_5\} = (D_1 \triangle D_2) \cap (D_3 \triangle D_4) \subseteq D_5$. Hence $D_5 = \{x_2, x_5, x_7, x_{10}\}$. Note that the vertical symmetry in Figure 4.4 still holds with the inclusion of $D_5$. Now we have two new pairs of 4-cocircuits that meet in two elements; namely, $\{D_2, D_5\}$ and $\{D_3, D_5\}$. Further, $D_4$ is $\{D_2, D_5\}$-type-2, $D_1$ is $\{D_3, D_5\}$-type-2, and $D_5$ is $\{D_1, D_4\}$-type-2. Therefore, by Claim 4.14.2, any further 4-cocircuits must also contain exactly one element from each intersection and from each symmetric difference of these pairs.

By property ($P_2$), there is a 4-cocircuit $D_6$ containing $x_{11}$. By Lemma 4.10, if $x_1 \in D_6$, then $\{x_2\} = (D_1 \cap D_2) - x_1 \not\subseteq D_6$ and $\{x_3\} = (D_1 \cap D_4) - x_1 \not\subseteq D_6$. This implies $\{x_5, x_7\} \subseteq D_6$, since $\{x_5\} = (D_2 \cap D_3) - x_2$ and $\{x_7\} = (D_3 \cap D_4) - x_3$. This, however, is a contradiction by Lemma 4.10, since $\{x_5, x_7\} = D_3 \cap D_5$. Thus $x_1 \not\in D_6$, and, by the aforementioned symmetry, this implies $x_3 \not\in D_6$. However, then $D_6$ does not contain an element from $D_1 \cap D_4$. This contradicts Claim 4.14.2, and this case cannot arise.

By the symmetry noted above, we may assume that one of $x_1$ or $x_3$ is in every further 4-cocircuit of $M$, otherwise we get a contradiction as in 4.14.3.1. Therefore, we may assume $x_1 \in D_5$. Hence, $x_7 \in D_5$ by Claim 4.14.2, since $\{x_7\} = (D_3 \cap D_4) - x_3$. Note that $x_5 \not\in D_5$.
by Lemma 4.13 with respect to \( \{D_3, D_4\} \). Therefore, the remaining element of \( D_5 \) must be one of \( x_4 \) and \( x_6 \), as \( D_5 \cap (D_1 \triangle D_2) \neq \emptyset \).

Assume, first, that \( x_4 \in D_5 \). In this case, \(|D_1 \cap D_4| = 2\) and \( D_3 \) is \( \{D_1, D_4\} \)-type-2. Given \((P2)\), there must be a 4-cocircuit \( D_6 \) containing \( x_{11} \). By Claim 4.14.2, \( D_6 \) must contain exactly one element from each of \( \{x_1, x_2\} = D_1 \cap D_2 \), \( \{x_1, x_4\} = D_1 \cap D_5 \), and \( \{x_3, x_7\} = D_3 \cap D_4 \), as each of these pairs has a 4-cocircuit that is type-2 with respect to them. Further, by Claim 4.14.3.1, we know one of \( x_1 \) and \( x_3 \) is in \( D_6 \). If \( x_1 \notin D_6 \), then \( \{x_2, x_3, x_4\} \subseteq D_6 \), and Proposition 4.2 provides a contradiction. If \( x_1 \in D_6 \), then \( \{x_2, x_3, x_4\} \cap D_6 = \emptyset \), and so \( x_7 \in D_6 \). But \( \{x_1, x_7\} = D_4 \cap D_5 \), and so we contradict Lemma 4.10. Therefore \( x_4 \notin D_5 \).

This implies that \( D_5 = \{x_1, x_6, x_7, x_{10}\} \). Again, take a 4-cocircuit \( D_6 \) containing \( x_{11} \), and consider its remaining elements. We know that one of \( x_1 \) and \( x_3 \) is in \( D_6 \). If \( x_1 \in D_6 \), then by the previous argument concerning \( D_5 \), now applied to \( D_6 \), we get \( D_6 = \{x_1, x_6, x_7, x_{11}\} \), which contradicts Proposition 4.2. Therefore, \( x_1 \notin D_6 \), and \( x_3 \in D_6 \). Observe that \(|D_2 \cap D_5| = 2\), and \( D_3 \) is \( \{D_2, D_5\} \)-type-2. By Claim 4.14.2, \( D_6 \) must contain exactly one element from each of \( \{x_1, x_2\} = D_1 \cap D_2 \), \( \{x_1, x_6\} = D_2 \cap D_5 \), and \( \{x_1, x_5, x_8, x_9\} \). This is impossible, as \( D_6 \) has only four elements. Thus this case cannot arise and Claim 4.14.3 holds.

Restating Claim 4.14.3 more generally, we have the following. This is seen by replacing \( D_1, D_2, D_3, \) and \( D_4 \) with \( D_i, D_j, D_k, \) and \( D_l \).

**Claim 4.14.4.** If \( D_i, D_j, D_k, \) and \( D_l \) are 4-cocircuits of \( M \) such that \(|D_i \cap D_j| = 2\) and \( D_k \) is \( \{D_i, D_j\} \)-type-2, then \( D_l \cap D_i \cap D_k = \emptyset \) and \( D_l \cap D_j \cap D_k = \emptyset \).

The only remaining case is when \( D_4 = \{x_1, x_4, x_7, x_9\} \). We illustrate this arrangement of cocircuits in Figure 4.5. Now we have \(|D_1 \cap D_4| = 2\) and \( D_3 \) is \( \{D_1, D_4\} \)-type-2. Therefore, by Claim 4.14.2, further 4-cocircuits must be both \( \{D_1, D_2\} \)-type-1 and \( \{D_1, D_4\} \)-type-1.

Let \( D_5 \) be a 4-cocircuit containing \( x_{10} \). We first prove that \( x_1 \in D_5 \), via the following claim.
Figure 4.5: Set diagram of remaining possible structure of cocircuits.

**Claim 4.14.5.** There is a 4-cocircuit containing $x_1$ and some element not in $\{x_1, x_2, \ldots, x_9\}$.

Suppose not. Then $\{x_2, x_4\} \subseteq D_5$. Further, as $M$ has no disjoint 4-cocircuits, $D_5 \cap D_3 \neq \emptyset$. Since $D_5$ is both $\{D_1, D_2\}$-type-1 and $\{D_1, D_4\}$-type-1, we have $\{x_3, x_5, x_7\} \cap D_5 = \emptyset$. Therefore, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. By (P2), we have a 4-cocircuit $D_6$ containing $x_{11}$. However, by the same reasoning, we get $\{x_2, x_4\} \subseteq D_6$. This is a contradiction to Lemma 4.10, since $D_1 \cap D_5 = \{x_2, x_4\}$. Thus the claim holds.

We may assume, then, that $x_1 \in D_5$. Now,

**Claim 4.14.6.** exactly one element from each $\{x_3, x_5, x_6\}$ and $\{x_3, x_7, x_9\}$ is in $D_5$.

This is because $D_5$ must be either type-1, type-2, or type-3 with respect to the pairs $\{D_1, D_2\}$ and $\{D_1, D_4\}$, by Lemma 4.13.

**Claim 4.14.7.** Any 4-cocircuit containing an element not in $\{x_1, x_2, \ldots, x_9\}$ does not contain both $x_1$ and $x_3$.

If $x_3 \in D_5$, then $\{x_2, x_4, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$ by Claim 4.14.2. In this case, $|D_1 \cap D_5| = 2$, which mandates that $D_3$ is one $\{D_1, D_5\}$-type-1, $\{D_1, D_5\}$-type-2, or $\{D_1, D_5\}$-type-3, by Lemma 4.12. The only possibility is that $D_3$ is $(D_5, D_1)$-type-1, which requires $x_8 \in D_5$. Therefore, $D_5 = \{x_1, x_3, x_8, x_{10}\}$. Now (P2) guarantees a 4-cocircuit $D_6$ containing $x_{11}$. Note
that $|D_3 \cap D_5| = 2$ and $D_4$ is $\{D_3, D_5\}$-type-2, so, in addition to the previous restrictions dictated by Claim 4.14.2, namely, that $D_6$ must contain exactly one element from each intersection and symmetric difference of the pairs $\{D_1, D_2\}$ and $\{D_1, D_4\}$, we have that $D_6$ must contain exactly one element from each $\{x_3, x_8\} = D_3 \cap D_5$ and $\{x_1, x_5, x_7, x_{10}\} = D_3 \triangle D_5$. Therefore, we must have $x_1 \in D_6$, which precludes any of $x_2, x_4, x_5, x_7$, or $x_{10}$ from being members of $D_6$. This forces $x_3 \in D_6$, otherwise $D_6$ will miss either $D_1 \triangle D_2$ or $D_3 \cap D_5$, but this contradicts Lemma 4.10, since $D_1 \cap D_5 = \{x_1, x_3\}$. Thus $x_3 \notin D_5$. This proves the claim by the left-right symmetry between $x_1$ and $x_3$, pictured in Figure 4.5.

Since $D_5$ must be both $\{D_1, D_2\}$-type-1 and $\{D_1, D_4\}$-type-1, but does not contain $x_3$, it must contain exactly one element from each $\{x_5, x_6\}$ and $\{x_7, x_9\}$, by 4.14.6. Further, $D_5$ must contain at least one of $x_5$ or $x_7$, otherwise it is disjoint from $D_3$. By the symmetry about the vertical line through $x_1$ in Figure 4.5, we may assume $x_5 \in D_5$. Then $|D_2 \cap D_5| = 2$, and $D_3$ must be one of $\{D_2, D_5\}$-type-1, $\{D_2, D_5\}$-type-2, or $\{D_2, D_5\}$-type-3, forcing $x_7$ in $D_5$. Now, (P2) guarantees a 4-cocircuit $D_6$ with $x_{11}$, but, by symmetry, a similar argument as the one just applied to $D_5$ forces $\{x_5, x_7\} \subseteq D_6$. This contradicts Lemma 4.10, as $D_3 \cap D_5 = \{x_3, x_5\}$. Thus we are unable to find viable cocircuits to cover the elements of $M$, and the lemma is proved.

\[\square\]

**Lemma 4.15.** Suppose $M$ has no two disjoint 4-cocircuits, $|E(M)| \geq 11$, and let $D_1$ and $D_2$ be 4-cocircuits of $M$ such that $|D_1 \cap D_2| = 2$. If $D_3$ is another 4-cocircuit of $M$, then $D_3$ is not $\{D_1, D_2\}$-type-3.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose the contrary, that $D_3$ is $\{D_1, D_2\}$-type-3. Without loss of generality, we may assume $D_3 = \{x_3, x_4, x_5, x_7\}$. Note that $D_2$ is $(D_1, D_3)$-type-3.

**Claim 4.15.1.** There are no further $\{D_1, D_i\}$-type-3 4-cocircuits, for $i \in \{2, 3\}$.
By symmetry, it suffices to show that no further 4-cocircuits are \((D_1, D_2)\)-type-3 or \((D_2, D_1)\)-type-3. By \((P2)\), we have a 4-cocircuit \(D_4\) containing \(x_8\). If \(D_4\) is \((D_1, D_2)\)-type-3, then \(\{x_3, x_4\} \subseteq D_4\), contradicting Lemma 4.10.

Suppose, then, that \(D_4\) is \((D_2, D_1)\)-type-3. Without loss of generality, \(D_4 = \{x_3, x_5, x_6, x_8\}\), as depicted in Figure 4.6.

![Figure 4.6: Set diagram of the 4-cocircuits in Claim 4.15.1.](image)

By \((P2)\), there must be a 4-cocircuit \(D_5\) containing \(x_9\). By Lemma 4.10, \(D_5\) cannot be \(\{D_1, D_2\}\)-type-3. Therefore, by Lemma 4.12 and Lemma 4.14, \(D_5\) must be of \(\{D_1, D_2\}\)-type-1. By symmetry, we may assume \(x_1 \in D_5\) and, further, that \(D_5\) contains exactly one element from \(\{x_3, x_4\}\) and is disjoint from \(\{x_5, x_6\}\). If \(x_4 \in D_5\), then \(x_8\) must also be in \(D_5\), otherwise \(D_4 \cap D_5 = \emptyset\). However, in this case, \(D_5\) is \(\{D_3, D_4\}\)-type-2 and Lemma 4.14 provides a contradiction.

Therefore, we may assume that \(x_3 \in D_5\). Now \(x_8 \in D_5\), otherwise \(D_5\) is \(\{D_2, D_4\}\)-type-2. Hence \(D_5 = \{x_1, x_3, x_8, x_9\}\). As \(M\) has \((P2)\), there must be a 4-cocircuit \(D_6\) containing \(x_{10}\). By similar reasoning to above, \(D_6\) must contain either \(x_1\) or \(x_2\), and either \(\{x_3, x_8\}\) or \(\{x_5, x_7\}\). Lemma 4.10 indicates that \(\{x_5, x_7\} \subseteq D_6\). There must be a further 4-cocircuit \(D_7\) that contains \(\{x_{11}\}\). However, now \(D_7\) cannot contain either \(\{x_3, x_8\}\) or \(\{x_5, x_7\}\) without violating Lemma 4.10. This contradiction completes the proof of Claim 4.15.1.

Observe that Claim 4.15.1 can be restated more generally, as follows:
Claim 4.15.2. Given 4-circuits \(D_i, D_j, D_k,\) and \(D_l\) such that \(|D_i \cap D_j| = |D_i \cap D_k| = 2\), if \(D_k\) is \((D_i, D_j)\)-type-3, then \(D_l\) is neither \(\{D_i, D_j\}\)-type-3 nor \(\{D_i, D_k\}\)-type-3.

Now, every 4-cocircuit containing an element not in \(\{x_1, x_2, \ldots, x_7\}\) must be both \(\{D_1, D_2\}\)-type-1 and \(\{D_1, D_3\}\)-type-1. We restrict this even further by proving the following:

Claim 4.15.3. All 4-cocircuits containing an element not in \(\{x_1, x_2, \ldots, x_7\}\) must be both \((D_1, D_2)\)-type-1 and \((D_1, D_3)\)-type-1.

![Figure 4.7: A set diagram of the 4-cocircuits structure after Claim 4.15.3.](image)

Note that any 4-cocircuit that is \((D_1, D_2)\)-type-1 is also \((D_1, D_3)\)-type-1. Let \(D_4\) be a 4-cocircuit containing \(x_8\), and suppose \(D_4\) is neither \((D_1, D_2)\)-type-1 nor \((D_1, D_3)\)-type-1. If \(x_5 \in D_4\), then \(D_4 = \{x_1, x_3, x_5, x_8\}\), without loss of generality. However, now \(|D_1 \cap D_2 \cap D_4| = 1\) and \(|D_1 \cap D_2| = |D_1 \cap D_4| = |D_2 \cap D_4| = 2\), and Lemma 4.11 provides a contradiction.

We now know that, \(x_5 \notin D_4\), and so \(\{x_6, x_7\} \subseteq D_4\). By assumption, \(D_4\) must meet \(D_1\). Without loss of generality, we may assume \(x_1 \in D_4\), but this gives a contradiction to Lemma 4.14, as now \(D_4\) is \(\{D_1, D_3\}\)-type-2. Thus Claim 4.15.3 is proved.

By \((P2)\), there is a 4-cocircuit \(D_5\) containing \(x_{10}\). We know now that \(D_5\) must be both \((D_1, D_2)\)-type-1 and \((D_1, D_3)\)-type-1. This implies that \(D_5\) meets each of \(\{x_1, x_2\}\) and \(\{x_3, x_4\}\) in exactly one element, and is disjoint from \(\{x_5, x_6, x_7\}\). By Lemma 4.10, we know \(\{x_1, x_3\} \nsubseteq D_5\), and so, by symmetry, we have that \(D_5\) contains either \(\{x_1, x_4\}\) or \(\{x_2, x_4\}\).
If \( \{x_1, x_4\} \subseteq D_5 \), then \( \{x_8, x_9\} \cap D_5 \neq \emptyset \), otherwise \( |D_2 \cap D_4 \cap D_5| = 1 \) and \( |D_2 \cap D_4| = |D_2 \cap D_5| = |D_4 \cap D_5| = 1 \), and Lemma 4.7 provides a contradiction. Therefore, we may assume that \( D_5 = \{x_1, x_4, x_8, x_{10}\} \); however, now \( |D_1 \cap D_4 \cap D_5| = 1 \) and \( |D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2 \), and Lemma 4.11 provides a contradiction.

Therefore, it must be that \( \{x_2, x_4\} \subseteq D_5 \). Now, \( D_5 \cap \{x_8, x_9\} \neq \emptyset \), otherwise \( D_4 \) and \( D_5 \) are disjoint. Hence, \( D_5 = \{x_2, x_4, x_8, x_{10}\} \), without loss of generality.

There must be a 4-cocircuit \( D_6 \) containing \( \{x_{11}\} \). As the restrictions on \( D_5 \) apply similarly to \( D_6 \), we may immediately conclude that \( \{x_2, x_4\} \subseteq D_6 \), which contradicts Lemma 4.10. Thus, no such \( D_6 \) may exist, and the proof of the lemma is complete. 

\[ \square \]

**Proposition 4.16.** If \( |E(M)| \geq 11 \), then \( M \) has two disjoint 4-cocircuits.

**Proof.** Let \( E(M) = \{x_1, x_2, \ldots, x_n\} \), and suppose that \( M \) has no two disjoint 4-cocircuits. By Lemma 4.8, we have 4-cocircuits \( D_1 \) and \( D_2 \) meeting in two elements. Without loss of generality, let \( D_1 = \{x_1, x_2, x_3, x_4\} \) and \( D_2 = \{x_1, x_2, x_5, x_6\} \). Lemma 4.14 and Lemma 4.15 indicate that all further 4-cocircuits of \( M \) must be \( \{D_1, D_2\}\)-type-1. By \( (P2) \), there is a 4-cocircuit \( D_3 \) containing \( x_7 \). Without loss of generality, \( D_3 = \{x_1, x_3, x_7, x_8\} \). Since \( |D_1 \cap D_3| = 2 \), further 4-cocircuits must also be \( \{D_1, D_3\}\)-type-1. By \( (P2) \), there is a 4-cocircuit \( D_4 \) containing \( x_9 \).

**Claim 4.16.1.** \( D_4 \) is not both \( \{D_2, D_1\}\)-type-1 and \( \{D_3, D_1\}\)-type-1.

If \( D_4 \) is both \( \{D_2, D_1\}\)-type-1 and \( \{D_3, D_1\}\)-type-1, then \( \{x_5, x_7\} \subseteq D_4 \), without loss of generality. As \( D_4 \) must also contain an element from both \( D_1 \cap D_2 \) and \( D_1 \cap D_3 \), we have \( D_4 = \{x_1, x_5, x_7, x_9\} \). See Figure 4.8 for reference.

By \( (P2) \), there is a 4-cocircuit \( D_5 \) containing \( x_{10} \) which must be \( \{D_1, D_2\}\)-type-1, \( \{D_1, D_3\}\)-type-1, \( \{D_2, D_4\}\)-type-1, and \( \{D_3, D_4\}\)-type-1. This forces \( x_1 \) into \( D_5 \), and further requires either \( \{x_4, x_9\} \) or \( \{x_6, x_8\} \) to be contained in \( D_5 \). These two possibilities are equivalent by symmetry, observed by rotating the second configuration in Figure 4.8, so we may assume
Figure 4.8: A set diagram of the 4-cocircuits in Claim 4.16.1.

\[ D_5 = \{x_1, x_4, x_9, x_{10}\}. \text{But now } |D_2 \cap D_3 \cap D_5| = 1 \text{ and } |D_2 \cap D_3| = |D_2 \cap D_5| = |D_3 \cap D_5| = 1. \]

Since \(|E(M)| > 10\), this is a contradiction by Lemma 4.7. This proves Claim 4.16.1.

Now we may assume that \(D_4\) is \((D_1, D_2)\)-type-1. Since \(D_4\) is \(\{D_1, D_3\}\)-type-1, it must contain exactly one element from \(D_1 \cap D_3 = \{x_1, x_3\}\). If \(x_1 \in D_4\), then \(x_4 \in D_4\), and we may assume \(D_4 = \{x_1, x_4, x_9, x_{10}\}\); however, now \(|D_2 \cap D_3 \cap D_4| = 1\) and \(|D_2 \cap D_3| = |D_2 \cap D_4| = |D_3 \cap D_4| = 1\), and so Lemma 4.7 provides a contradiction.

Hence, \(x_1 \notin D_4\). There \(D_4\) must contain both \(x_2\) and \(x_3\) and no other elements of \(D_1 \cup D_2 \cup D_3\). Without loss of generality, \(D_4 = \{x_2, x_3, x_9, x_{10}\}\). There must be a 4-cocircuit \(D_6\) containing \(x_{11}\). As the above analysis applies to \(D_6\), it follows that \(\{x_2, x_3\} \subseteq D_6\). This is a contradiction by Lemma 4.10, as \(D_1 \cap D_5 = \{x_2, x_3\}\). We have eliminated all possible types of 4-cocircuit intersection with \(D_1 \cup D_2\). Thus there must be at least two disjoint 4-cocircuits in \(M\), and the proof is complete.

\(\square\)

We conclude this section by extending the previous proposition to a statement for matroids on thirteen elements. This result comes in three parts. First, we show that three pairwise-disjoint 4-cocircuits form a local \(M(K_{3,4})\)-structure. Then, we determine that, when \(M\) has exactly twelve elements, there is a specific configuration of 4-cocircuits that arises. Finally, we prove that, when \(M\) has at least thirteen elements, it must also have three pairwise-disjoint 4-cocircuits.
Lemma 4.17. If $D_1$, $D_2$, and $D_3$ are pairwise-disjoint 4-cocircuits of $M$, then $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$.

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. By Proposition 4.6, we know $M|(D_i \cup D_j) \cong M(K_{2,4})$, for $\{i, j\} \subseteq \{1, 2, 3\}$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, $D_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$, and $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. The elements in these pairs are always found together in circuits contained in $D_1 \cup D_2$. The elements of $D_3$ appear together in pairs in 4-circuits contained in $D_1 \cup D_3$ and $D_2 \cup D_3$, again by the $M(K_{2,4})$ structure given by Proposition 4.6. We will show that these pairs correspond with the pairs that form the 4-circuits of $D_1 \cup D_2$; that is, we show that if $x_i$ and $x_j$ always appear together in the 4-circuits contained in $D_1 \cup D_2$, and $x_i$ and $x_k$ always appear together in the 4-circuits contained in $D_1 \cup D_3$, then $x_j$ and $x_k$ always appear together in the 4-circuits contained in $D_2 \cup D_3$. Without loss of generality, suppose $\{x_1, x_9\}$, $\{x_2, x_{10}\}$, $\{x_3, x_{11}\}$, and $\{x_4, x_{12}\}$ always appear together in the 4-circuits contained in $D_1 \cup D_3$, and compare this with Figure 4.9.

![Figure 4.9: $K_{3,4}$ with labeled edges.](image)

By circuit elimination, there is a circuit, $C$, contained in $(\{x_1, x_2, x_5, x_6\} \cup \{x_1, x_2, x_9, x_{10}\}) - \{x_1\}$. By orthogonality, $C = \{x_5, x_6, x_9, x_{10}\}$. By repeating this argument, we find that the pairs $\{x_5, x_9\}$, $\{x_6, x_{10}\}$, $\{x_7, x_{11}\}$, and $\{x_8, x_{12}\}$ always appear together in 4-circuits contained in $D_2 \cup D_3$. 

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Consider $M' = M(K_{3,4})$ on the ground set $\{x_1, x_2, \ldots, x_{12}\} = X$, say, where $K_{3,4}$ is labeled as in Figure 4.9. Given that $M|X$ is connected, Theorem 3.6 indicates that if $r(M') = r(M|X)$ and the identity map is a weak map from $M'$ to $M|X$, then $M' = M|X$.

First we check the rank. Evidently, $r(M|X) = r_M(X - D_1) + 1 = 6 = r(M')$, as desired. We will show that each circuit of $M'$ is a dependent set in $M|X$. The $4$-circuits are identical, so all that remains is to check the $6$-circuits. Let $C'$ be a $6$-circuit of $M'$. Without loss of generality, we may assume $C' = \{x_1, x_3, x_5, x_6, x_{10}, x_{11}\}$. We know that $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_2, x_3, x_{10}, x_{11}\}$ are circuits in $M|X$. Therefore, there is a circuit in $M|X$ contained in $(C_1 \cup C_2) - x_2$, and so $\{x_1, x_3, x_5, x_6, x_{10}, x_{11}\}$ is dependent in $M|X$. Thus the identity map is a weak map from $M'$ to $M|X$. Hence $M' = M|X$.  

The configuration of $4$-cocircuits described in the following lemma is depicted in Figure 4.10. Note that elements in $4$-cocircuits are contained in an oval, and elements in local series pair are connected by a green line segment.

**Lemma 4.18.** If $|E(M)| = 12$, then $M$ has four $4$-cocircuits $D_1, D_2, D_3$, and $D_4$ such that $D_1 \cap D_2 = D_3 \cap D_4 = \emptyset$ and $|D_i \cap D_j| = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. We know $M$ has two disjoint $4$-cocircuits, $D_1$ and $D_2$ by Proposition 4.16. Moreover, $M|(D_1 \cup D_2) \cong M(K_{2,4})$ by Proposition 4.6. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $\{x_1, x_5\}, \{x_2, x_6\}, \{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. The elements in these pairs are always found together in circuits contained in $D_1 \cup D_2$. By assumption, $x_9$ is in a $4$-cocircuit, say $D_3$. If $D_3$ is disjoint from both $D_1$ and $D_2$, then, by Lemma 4.17, we know $M \cong M(K_{3,4})$. This is a contradiction, as $M(K_{3,4})$ is not $4$-connected. Therefore, we may assume that $D_3$ meets $D_1 \cup D_2$. By orthogonality with the $4$-circuits contained in $M|(D_1 \cup D_2)$, we see that $D_3 \cap (D_1 \cup D_2)$ must be one of the aforementioned pairs. Without loss of generality, $D_3 = \{x_1, x_5, x_9, x_{10}\}$. Let $D_4$ be a $4$-cocircuit containing $x_{11}$. As with $D_3$, we know that $D_4$
must meet $D_1 \cup D_2$ in one of the aforementioned series pairs. Therefore, if $D_3 \cup D_4 = \emptyset$, then we are done. Hence, assume the contrary.

Suppose $x_1 \in D_4$. Then we may assume that $D_4 = \{x_1, x_5, x_{11}, x_{12}\}$. Now $(D_3 \cup D_4) - x_1$ contains a cocircuit and, by orthogonality, this cocircuit avoids $x_5$. Hence $\{x_9, x_{10}, x_{11}, x_{12}\}$ is a cocircuit that is disjoint from both $D_1$ and $D_2$; a contradiction.

We now know that $x_1 \notin D_4$. Then, without loss of generality, $D_4 = \{x_2, x_6, x_9, x_{11}\}$. Similarly, if $D_5$ is a 4-cocircuit containing $x_{12}$, then $D_5$ is either $\{x_3, x_7, x_9, x_{12}\}$ or $\{x_3, x_7, x_{10}, x_{12}\}$. In the second case, $D_4 \cap D_5 = \emptyset$, so, by symmetry, this contradicts the assumption that $D_3$ and $D_4$ are not disjoint. We deduce that $D_5 = \{x_3, x_7, x_9, x_{12}\}$. Consider a 4-circuit $C$ containing $\{x_4, x_9\}$. To avoid an orthogonality contradiction, $C$ must meet each of $\{x_1, x_5, x_{10}\}$, $\{x_2, x_6, x_{11}\}$, and $\{x_3, x_7, x_{12}\}$. This contradiction completes the proof. 

The proof of the following proposition is similar to that of the preceding lemma.

**Proposition 4.19.** If $|E(M)| \geq 13$, then $M$ has three pairwise-disjoint 4-cocircuits.

**Proof.** Assume first the $M$ has no three pairwise-disjoint 4-cocircuits. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. We know $M$ has two disjoint 4-cocircuits, $D_1$ and $D_2$ by Proposition 4.16. Moreover, $M|(D_1 \cup D_2) \cong M(K_{2,4})$ by Proposition 4.6. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$.
be the series pairs in \( M(D_1 \cup D_2) \). The elements in these pairs are always found together in circuits contained in \( D_1 \cup D_2 \). By assumption, \( x_9 \) is in a 4-cocircuit, say \( D_3 \). As \( D_3 \) meets \( D_1 \cup D_2 \), then by orthogonality with the 4-circuits contained in \( M(D_1 \cup D_2) \), we see that \( D_3 \cap (D_1 \cup D_2) \) must be one of the aforementioned pairs. Without loss of generality, \( D_3 = \{x_1, x_5, x_9, x_{10}\} \). Let \( D_4 \) be a 4-cocircuit containing \( x_{11} \).

**Claim 4.19.1.** \( D_3 \cap D_4 \neq \emptyset \).

Suppose \( D_3 \cap D_4 = \emptyset \). Then, without loss of generality, \( D_4 = \{x_2, x_6, x_{11}, x_{12}\} \). Now \( M(D_3 \cup D_4) \cong M(K_{2,4}) \). The pairs \( \{x_1, x_2\} \) and \( \{x_5, x_6\} \) always appear together in the circuits contained in \( D_3 \cup D_4 \). Without loss of generality, so do \( \{x_9, x_{11}\} \) and \( \{x_{10}, x_{12}\} \). This is the configuration depicted in Figure 4.10.

A 4-cocircuit \( D_5 \) containing \( x_{13} \) must meet both \( D_1 \cup D_2 \) and \( D_3 \cup D_4 \). By orthogonality, the 4-circuits contained in \( M(D_1 \cup D_2) \) and \( M(D_3 \cup D_4) \) imply that \( D_5 \) meets \( D_1 \cup D_2 \) in \( \{x_1, x_5\}, \{x_2, x_6\}, \{x_3, x_7\}, \) or \( \{x_4, x_8\} \), and \( D_5 \) meets \( D_3 \cup D_4 \) in \( \{x_1, x_2\}, \{x_5, x_6\}, \{x_9, x_{11}\}, \) or \( \{x_{10}, x_{12}\} \). In each case, the first two possibilities cannot arise, otherwise \( M \) has \( \{x_1, x_2, x_5, x_6\} \) as both a circuit and a cocircuit, a contradiction to the 4-connectedness of \( M \). Hence \( |D_5| \geq 5 \), a contradiction. Thus Claim 4.19.1 holds.

We may assume, then, that \( D_3 \) meets \( D_4 \) and any further 4-cocircuits of \( M \). Suppose \( x_1 \in D_4 \). Then we may assume that \( D_4 = \{x_1, x_5, x_{11}, x_{12}\} \). Now \( (D_3 \cup D_4) - x_1 \) contains a cocircuit and, by orthogonality, this cocircuit avoids \( x_5 \). Hence \( \{x_9, x_{10}, x_{11}, x_{12}\} \) is a cocircuit that is disjoint from both \( D_1 \) and \( D_2 \); a contradiction.

We now know that \( x_1 \notin D_4 \). Then, without loss of generality, \( D_4 = \{x_2, x_6, x_9, x_{11}\} \). Similarly, if \( D_5 \) is a 4-cocircuit containing \( x_{12} \), then \( D_5 \) is either \( \{x_3, x_7, x_9, x_{12}\} \) or \( \{x_3, x_7, x_{10}, x_{12}\} \). In the second case, \( D_4 \cap D_5 = \emptyset \), so, by symmetry, we have a contradiction to Claim 4.19.1. We deduce that \( D_5 = \{x_3, x_7, x_9, x_{12}\} \). Consider a 4-circuit \( C \) containing \( \{x_4, x_9\} \). To avoid an orthogonality contradiction, \( C \) must meet each of \( \{x_1, x_5, x_{10}\} \),
\{x_2, x_6, x_{11}\}, and \{x_3, x_7, x_{12}\}. This is impossible, as |C| = 4. This contradiction completes the proof. \qed

4.3 When M Has Exactly Eight Elements

Throughout this section, we assume that |E(M)| = 8. The bulk of the examples on at most ten elements come from this case; as such, the analysis here is somewhat tedious. While we proceed in a more traditional manner, we concede that these results may possibly be checked by exhaustive computer search. In order to facilitate our analysis, we restrict M as follows. Observe that M must not have two disjoint 4-cocircuits, as Lemma 4.6 implies such an M must be isomorphic to M(K_{2,4}), a matroid that is not 4-connected. First, we show that r(M) = 4. Then, after proving a quick technical lemma, we treat the case when every 4-circuit of M meets every other in a single element. Finally, we address the remaining matroids in order of the maximum number of 4-circuits that may contain a particular element.

Lemma 4.20. If |E(M)| = 8, then r(M) = 4.

Proof. By (P2), we know that M has a 4-cocircuit. That cocircuit must be independent, otherwise M is not 4-connected. Similarly, the 4-circuits of M must be coindependent. Since r(M) + r^*(M) = |E(M)|, it follows that r(M) = 4. \qed

A consequence of the previous lemma is that the complement of every 4-circuit is a 4-cocircuit, and vice versa. Further, when coupled with the following result of Hartmanis [?], this observation guarantees that the objects we find in the main propositions of this section are actually matroids. Let k and m be integers with k > 1 and m > 0. Given a set E, we call a set \(T = \{T_1, T_2, \ldots, T_k\}\) an \(m\)-partition of E if each \(T_i\) is a subset of E with at least \(m\) elements, and each \(m\)-element subset of E is contained in a unique member of \(T\).

Proposition 4.21. If \(T\) is an \(m\)-partition \(\{T_1, T_2, \ldots, T_k\}\) of a set E, then \(T\) is the set of hyperplanes of a paving matroid of rank \(m + 1\) on E.
The following lemma, while easy, is quite useful. It states two obvious restrictions to the structure of 4-circuits of $M$. An important consequence of this lemma is that, whenever two 4-circuits $C_1$ and $C_2$ meet in a single element, every other 4-circuit containing that element must also contain the one element not in $C_1 \cup C_2$. This will frequently be referred to as the forced inclusion of an element; and, a sequence of forced inclusions will be called a \textit{chain}.

**Lemma 4.22.** Let $C_1$, $C_2$, and $C_3$ be distinct 4-circuits of $M$, and suppose $C_1 \cap C_2 = \{e\}$. Then $|C_3 \cap C_i| = 2$ for some $i \in \{1, 2\}$. Further, if $e \in C_3$, then $|C_3 \cap C_i| = 2$ for both $i \in \{1, 2\}$.

**Proof.** For the first part, note that $|C_3 \cap (C_1 \cup C_2)| = 3$ since $|C_1 \cup C_2| = 7$. Hence the assertion follows by Proposition 4.2. That proposition also yields the second part. \hfill \Box

**Corollary 4.23.** Let $C_1$, $C_2$, and $C_3$ be distinct 4-circuits of $M$. If $C_1 \cap C_2 = \{e\}$ and $e \in C_3$, then $E(M) - (C_1 \cup C_2) = \{f\} \in C_3$.

We split the work of the main theorem of this section into two propositions. In the first proposition that follows, we examine matroids in which every 4-circuit meets another 4-circuit in a single element. We handle the remaining cases in the subsequent proposition. The list of 4-circuits of the matroids in these propositions are compiled in Figures 4.11 and 4.16 for reference.

**Proposition 4.24.** If, for every 4-circuit $C$ of $M$, there is a 4-circuit $C'$ such that $|C \cap C'| = 1$, then $M$ is one of the following matroids: $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,3^+}$, $M_{8,4}$, $M_{8,4^+}$, $M_{8,5}$, $M_{8,6}$.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_8\}$, and suppose $C_1$ is a 4-circuit of $M$. Without loss of generality, $C_1 = \{x_1, x_2, x_3, x_4\}$. By assumption, $C_1$ meets another 4-circuit in a single element. Let $C_2$ be such a 4-circuit; then, without loss of generality, $C_2 = \{x_1, x_5, x_6, x_7\}$. There is a 4-circuit, $C_3$, containing $\{x_1, x_8\}$. By Lemma 4.22, we may assume $C_3 = \{x_1, x_2, x_5, x_8\}$.

We will refer to such a configuration of three 4-circuits as a \textit{two-flap} configuration. The
The 4-circuits of $M$

<table>
<thead>
<tr>
<th>$M$</th>
<th>The 4-circuits of $M$</th>
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<tbody>
<tr>
<td>$M_{8,1}$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_3, x_4, x_5, x_8},$</td>
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<td>${x_2, x_3, x_6, x_7}, {x_2, x_4, x_5, x_6}, {x_1, x_4, x_7, x_8}, {x_1, x_3, x_6, x_8}$</td>
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<tr>
<td>$M_{8,2}$</td>
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<td>${x_2, x_3, x_6, x_7}, {x_2, x_4, x_5, x_6}, {x_1, x_3, x_6, x_8}, {x_4, x_6, x_7, x_8}$</td>
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<td>$M_{8,3}$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_3, x_4, x_5, x_8},$</td>
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<tr>
<td>$M_{8,3}^+$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_3, x_4, x_5, x_8},$</td>
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<td>${x_2, x_3, x_6, x_7}, {x_1, x_4, x_6, x_8}, {x_2, x_4, x_7, x_8}, {x_1, x_3, x_7, x_8}$</td>
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<tr>
<td>$M_{8,4}$</td>
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<td>${x_2, x_3, x_6, x_7}, {x_4, x_6, x_7, x_8}$</td>
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<tr>
<td>$M_{8,4}^+$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_3, x_4, x_5, x_8},$</td>
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<td>$M_{8,5}$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_2, x_3, x_6, x_7},$</td>
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<tr>
<td>$M_{8,6}$</td>
<td>${x_1, x_2, x_3, x_4}, {x_1, x_5, x_6, x_7}, {x_1, x_2, x_5, x_8}, {x_2, x_3, x_6, x_7},$</td>
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<td>${x_3, x_4, x_5, x_6}, {x_2, x_4, x_7, x_8}, {x_1, x_3, x_6, x_8}$</td>
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Figure 4.11: The 4-circuits of the matroids in Proposition 4.24.

two-flap configuration here is depicted in Figure 4.12. By circuit elimination, we get four possible circuits $C_4 \subseteq (C_1 \cup C_3) - \{x_1\}, C_5 \subseteq (C_1 \cup C_3) - \{x_2\}, C_6 \subseteq (C_2 \cup C_3) - \{x_1\}$, and $C_7 \subseteq (C_2 \cup C_3) - \{x_3\}$.

Figure 4.12: The two-flap configuration.

**Claim 4.24.1.** There is at most one 4-circuit produced by circuit elimination on 4-circuits in a two-flap configuration.
Suppose not. Then \( C_4 = C_5 = \{x_3, x_4, x_5, x_8\} \), and \( C_6 = C_7 = \{x_2, x_6, x_7, x_8\} \). Observe that \( C_1 \cap C_2 = \{x_1\}, E(M) - (C_1 \cup C_2) = \{x_8\} \), \( C_4 \cap C_6 = \{x_8\} \), and \( E(M) - (C_4 \cup C_6) = \{x_1\} \). By Corollary 4.23, any further 4-circuit containing \( x_1 \) must contain \( x_8 \), and vice versa. Similarly, since \( C_1 \cap C_6 = \{x_2\} \), \( E(M) - (C_1 \cup C_6) = \{x_5\} \), \( C_2 \cap C_4 = \{x_5\} \), and \( E(M) - (C_2 \cup C_4) = \{x_2\} \), it follows by Corollary 4.23, that any further 4-circuit containing \( x_2 \) must contain \( x_5 \), and vice versa. But, by assumption, there must be a 4-circuit meeting \( C_3 \) in one element. This contradiction proves the claim.

Now, whenever three 4-circuits meet in a two-flap configuration, we know that circuit elimination on the pairs sharing two elements produces at most one 4-circuit. We will investigate these cases separately.

**Case 4.24.2.** Suppose there exists a two-flap configuration that produces an additional 4-circuit via circuit elimination.

We may assume \( C_1, C_2, \) and \( C_3 \) produce one such 4-circuit, say \( C_4 = \{x_3, x_4, x_5, x_8\} \). As before, Corollary 4.23 applied to pairs \( \{C_1, C_2\} \) and \( \{C_2, C_4\} \) implies that further 4-circuits containing \( x_1 \) must contain \( x_8 \), and those containing \( x_5 \) must also contain \( x_2 \). Observe the symmetry between \( x_2 \) and \( x_8 \) and between \( x_3 \) and \( x_4 \), evident in Figure 4.13. We know there must be a 4-circuit, \( C_5 \), meeting \( C_3 \) in a single element. The forced inclusions noted

![Figure 4.13: The 4-circuits in Case 4.24.2.](image-url)
above indicate that $C_5$ must meet $C_3$ in either $x_2$ or $x_8$. Observe that the permutation $(x_1, x_5)(x_2, x_8)$ is an automorphism of $M$, and so we may assume $x_2 \in C_5$. The remaining elements of $C_5$ come from $E(M) – C_3 = \{x_3, x_4, x_6, x_7\}$. Since $E(M) – C_1 = \{x_5, x_6, x_7, x_8\}$ is a cocircuit, if $C_5$ contains one of $x_6$ or $x_7$, then it must contain the other to avoid an orthogonality contradiction. Therefore, $C_5 = \{x_2, x_3, x_6, x_7\}$, without loss of generality. Note that $C_5$ meets each of $C_3$ and $C_4$ in a single element. Corollary 4.23 applied to these pairs forces further 4-circuits containing $x_2$ to contain $x_4$ and those containing $x_3$ to contain $x_1$.

There must be a 4-circuit $C_6$ containing $x_4$ and $x_6$. We take two cases.

**Subcase 4.24.2.1.** Assume $C_6$ does not contain $x_8$.

Then $x_5 \in C_6$, otherwise $|C_6 \cap (E(M) – C_1)| = 1$, a contradiction by orthogonality. Further, $x_2 \in C_6$, otherwise it is orthogonal with one of $E(M) – C_2$ or $E(M) – C_4$. Therefore, $C_6 = \{x_2, x_4, x_5, x_6\}$.

There must be a 4-circuit $C_7$ meeting $C_6$ in one element. By the forced inclusions, such a 4-circuit must contain either $x_4$ or $x_6$. We treat these cases separately, and each case will yield one matroid. We know that these, and all further examples, are matroids because the 4-circuits together with every 3-set that is in no 4-circuit form a 3-partition of $E(M)$, since no two such sets meet in more than two elements. This means that these sets are the hyperplanes of a paving matroid on $E(M)$, by Proposition 4.21.

**Subcase 4.24.2.1.1.** Suppose $x_4 \in C_7$.

As $(C_7 – \{x_4\}) \subseteq (E(M) – C_6) = \{x_1, x_3, x_7, x_8\}$, and, by forced inclusions, if $x_3 \in C_7$, then so must $x_1$ and $x_8$ be, we get that $C_7$ is either $\{x_1, x_3, x_4, x_8\}$ or $\{x_1, x_4, x_7, x_8\}$. However, the former set violates orthogonality with $E(M) – C_1$, so $C_7 = \{x_1, x_4, x_7, x_8\}$. With this, we get a long chain of forced inclusion of elements: if a further 4-circuit contains $x_7$, then it must contain $x_5$, and in turn must contain $x_2$, then $x_4$, then $x_3$, then $x_1$, and finally $x_8$. Thus, the only possible additional 4-circuits are $\{x_1, x_3, x_4, x_8\}$ or $\{x_1, x_3, x_6, x_8\}$. Further,
both of these cannot be 4-circuits by Lemma 4.2. In this case, as \( x_6 \) and \( x_8 \) do not yet appear together in a 4-circuit, there must be a 4-circuit \( C_8 = \{x_1, x_3, x_6, x_8\} \). This being the final possible 4-circuit of \( M \), we conclude this subcase having determined our first matroid, which we call \( M_{8,1} \).

**Subcase 4.24.2.1.2.** We may now assume \( x_4 \notin C_7 \).

Therefore \( C_7 \) contains \( x_6 \). As before, \( (C_7 - \{x_6\}) \subseteq (E(M) - C_6) = \{x_1, x_3, x_7, x_8\} \). If \( x_7 \in C_7 \), then \( x_3 \notin C_7 \) because otherwise \(|(E(M) - C_5) \cap C_7| = 1 \). That implies \( C_7 = \{x_1, x_6, x_7, x_8\} \), a contradiction since now \(|(E(M) - C_2) \cap C_7| = 1 \). Therefore, \( x_7 \notin C_7 \), and so \( C_7 = \{x_1, x_3, x_6, x_8\} \). In this case, both pairs \( \{x_4, x_7\} \) and \( \{x_7, x_8\} \) are not in an identified 4-circuit. Consider a 4-circuit \( C_8 \) containing \( \{x_4, x_7\} \). It must be that \( x_8 \in C_8 \), otherwise \( C_8 \) cannot contain two elements from each of the following cocircuits: \( E(M) - C_1 = \{x_5, x_6, x_7, x_8\} \), \( E(M) - C_2 = \{x_2, x_3, x_4, x_8\} \), \( E(M) - C_5 = \{x_1, x_4, x_5, x_8\} \), and \( E(M) - C_6 = \{x_1, x_3, x_7, x_8\} \). Then, we have \( x_3 \) and \( x_5 \) not in \( C_8 \), by applying Lemma 4.2 to \( C_8 \) and \( C_4 \). Therefore, \( C_8 \) is one of \( \{x_1, x_4, x_7, x_8\} \), \( \{x_2, x_4, x_7, x_8\} \), or \( \{x_4, x_6, x_7, x_8\} \). If \( C_8 \) is either the first or second set, the resulting matroid is isomorphic to \( M_{8,1} \); in the first case identically, and in the second case via the automorphism of \( M \) given by the permutation \((x_1, x_2)(x_3, x_5)(x_4, x_8)\). Therefore, we may assume \( C_8 = \{x_4, x_6, x_7, x_8\} \). The forced inclusions determined by Corollary 4.23 are: containing \( x_1 \) forces \( x_8 \) which forces \( x_3 \) which forces \( x_1 \), and containing \( x_2 \) forces \( x_4 \) which forces \( x_5 \) which forces \( x_2 \), and finally \( x_6 \) forces the inclusion of \( x_7 \). Using this, we construct a short list of possible additional 4-circuits, all of which contradict orthogonality with some 4-cocircuit. Thus we have found a single matroid, which we call \( M_{8,2} \).

Now case 4.24.2.1 is closed, and we may assume that \( \{x_2, x_4, x_5, x_6\} \) is not a circuit. We return to \( C_6 \), which must now include \( x_8 \). The remaining element of \( C_6 \) must come from \( (E(M) - C_4) - \{x_6\} = \{x_1, x_2\} \), in order to avoid an orthogonality contradiction.
However, $x_1$ and $x_2$ are symmetric under the automorphism given by $(x_1, x_2)(x_3, x_5)(x_4, x_8)$. Therefore, $C_6$ is either $\{x_1, x_4, x_6, x_8\}$ or $\{x_4, x_6, x_7, x_8\}$, without loss of generality.

**Subcase 4.24.2.2.** Suppose $C_6 = \{x_1, x_4, x_6, x_8\}$.

In this case, there is not yet a 4-circuit containing $x_4$ and $x_7$. Consider such a 4-circuit, and call it $C_7$. It is useful to consider the forced inclusions dictated by Corollary 4.23. Using this, we know that if $C_7$ contains $x_3$, it must also contain $x_1$, and consequently $x_8$; therefore, $x_3 \notin C_7$. Additionally, if $x_1 \in C_7$, then $C_7 = \{x_1, x_4, x_7, x_8\}$, and $|C_7 \cap C_8| = 3$, contradicting Lemma 4.2; therefore, $x_1 \notin C_7$. Similarly, if $C_7$ contains $x_6$, it must contain $x_5$ and then $x_2$, and then $x_4$, so $x_6 \notin C_7$. Further, if we suppose that $x_5 \in C_7$, then $C_7 = \{x_2, x_4, x_5, x_7\}$, which yields a matroid isomorphic to that considered in case 4.24.2.1, under the automorphism of $M$ given by the permutation $(x_6, x_7)$. Therefore, we may assume $x_5 \notin C_7$.

Thus $C_7 = \{x_2, x_4, x_7, x_8\}$. This extends the chains of forced inclusions. Since $|C_7 \cap C_2| = 1$, we get that a further 4-circuit containing $x_7$ must contain $x_3$. Therefore, the only possible additional 4-circuits allowed by the chains are $\{x_1, x_3, x_7, x_8\}$, $\{x_1, x_3, x_4, x_8\}$, $\{x_1, x_2, x_4, x_8\}$, $\{x_2, x_4, x_5, x_8\}$, and $\{x_2, x_4, x_5, x_6\}$. Each of these has a prohibitive intersection of size one with some 4-circuit, save $\{x_1, x_3, x_7, x_8\}$ and $\{x_2, x_4, x_5, x_6\}$. The latter set is out by assumption. Thus our analysis in this case produces two matroids: $M_{8,3}$ having 4-circuits $\{C_1, C_2, \ldots, C_7\}$, and $M_{8,3}^+$ having 4-circuits $\{C_1, C_2, \ldots, C_7\} \cup \{x_1, x_3, x_7, x_8\}$.

**Subcase 4.24.2.3.** Suppose $C_6 = \{x_4, x_6, x_7, x_8\}$.

There are no pairs of elements not in a 4-circuit. Therefore, we get a matroid $M_{8,4}$ having 4-circuits $\{C_1, C_2, \ldots, C_6\}$. This structure may admit additional 4-circuits, but such are subject to the following forced inclusions determined by Corollary 4.23: containing $x_1$ implies $x_8$, which implies $x_3$, which implies $x_1$, and containing $x_2$ implies $x_4$, which implies $x_5$, which implies $x_2$. Hence, any additional 4-circuits must be one of $\{x_1, x_3, x_6, x_8\}$, $\{x_1, x_3, x_7, x_8\}$, $\{x_2, x_4, x_5, x_6\}$, or $\{x_2, x_4, x_5, x_7\}$. The inclusion of any one of these sets as
a 4-circuit produces an isomorphic matroid, as $x_6$ and $x_7$ are clones in $M_{8,4}$, and the permutation $(x_1, x_2)(x_3, x_5)(x_4, x_8)$ gives rise to an automorphism of $M$. Thus we get a second matroid, $M_{8,4,+}$ which has 4-circuits $\{C_1, C_2, \ldots, C_6\} \cup \{x_1, x_3, x_6, x_8\}$. This exhausts the case in which we assume a 4-circuit from the circuit elimination on $C_1$ and $C_3$.

**Case 4.24.3.** Circuit elimination on 4-circuits in a two-flaps configuration produces no additional 4-circuits.

As before, there is a 4-circuit, say $C_4$, meeting $C_3$ in a single element. That element cannot be $x_1$ by Corollary 4.23, as $\{x_1\} = C_1 \cap C_2$. Also, elements $x_2$ and $x_5$ are symmetric under the automorphism of $M$ given by the permutation $(x_2, x_5)(x_3, x_6)(x_4, x_7)$. Therefore, it suffices to assume $C_3 \cap C_4$ is either $\{x_2\}$ or $\{x_8\}$.

**Subcase 4.24.3.1.** Suppose $C_3 \cap C_4 = \{x_2\}$.

Then $C_4 - \{x_2\} \subseteq \left( E(M) - C_3 \right) = \{x_3, x_4, x_6, x_7\}$. Since $E(M) - C_1 = \{x_5, x_6, x_7, x_8\}$ is a cocircuit, if one of $x_6$ and $x_7$ is in $C_4$, then they both are. Therefore, $\{x_6, x_7\} \subseteq C_4$. Further, $x_3$ and $x_4$ are indistinguishable as they only appear thus far in the same 4-circuits, so without loss of generality we may assume $C_4 = \{x_2, x_3, x_6, x_7\}$.

![Figure 4.14: The configuration of 4-circuits in Case 4.24.3.1.](image)
Now consider a 4-circuit $C_5$ containing \{x_3, x_5\}. Note that $x_4$ and $x_8$ are symmetric by the automorphism of $M$ given by the permutation $(x_1, x_2)(x_3, x_5)(x_4, x_8)$. If $x_4 \not\in C_5$, then, in order to avoid an orthogonality contradiction with $E(M) - C_4$, one of $x_1$ or $x_8$ must be in $C_5$. But, if $x_1 \in C_5$, then $x_8$ must also be in $C_5$ by Corollary 4.23. Therefore, in either case, $x_8 \in C_5$.

Therefore, without loss of generality, $x_4 \in C_5$. Hence, so must be one of $x_6$, $x_7$, or $x_8$ to avoid an orthogonality contradiction with $E(M) - C_1$. If $x_8 \in C_5$, then $C_5 = \{x_3, x_4, x_5, x_8\}$, which is a contradiction, as $C_5 \subseteq (C_1 \cup C_3) - \{x_1\}$. Therefore, as $x_6$ and $x_7$ are indistinguishable, we may assume $C_5 = \{x_3, x_4, x_5, x_6\}$.

Next, we consider a 4-circuit $C_6$ containing \{x_4, x_7\}. If $x_8 \not\in C_6$, then, in order to avoid an orthogonality contradiction, $C_6$ must contain both $x_2$ and $x_5$; therefore, $C_6 = \{x_2, x_4, x_5, x_7\}$. There must be some 4-circuit $C_7$ meeting $C_6$ in a single element, by assumption. This element cannot be either $x_2$ or $x_5$, as these force the inclusion of $x_4$ and $x_7$, respectively. Observe that the permutation $(x_2, x_5)(x_3, x_6)(x_4, x_7)$ is an automorphism of $M$, so, without loss of generality, $x_4 \in C_7$. The rest of the elements of $C_7$ come from $E(M) - C_6 = \{x_1, x_3, x_6, x_8\}$. If $x_1 \not\in C_7$, then $|C_7 \cap (E(M) - C_5)| = 1$, a contradiction. Therefore $x_1 \in C_7$. By Corollary 4.23, this forces $x_8 \in C_7$. Then, in order to avoid an orthogonality contradiction with $E(M) - C_1$, it must be that $x_6 \in C_7$. Therefore, $C_7 = \{x_1, x_4, x_6, x_8\}$. However, now $C_4 \cap C_7 = \{x_6\}$, and $C_6 \subseteq (C_4 \cup C_5) - \{x_6\}$, a contradiction. Thus $x_8 \in C_6$.

By the most recently cited automorphism, we may assume $C_6$ to be one of the following three sets, without loss of generality: \{x_1, x_4, x_7, x_8\}, \{x_2, x_4, x_7, x_8\}, or \{x_3, x_4, x_7, x_8\}.

**Subcase 4.24.3.1.1.** Suppose $C_6 = \{x_1, x_4, x_7, x_8\}$.

Now, since $C_4 \cap C_6 = \{x_7\}$ and $C_5 \cap C_6 = \{x_4\}$, further 4-circuits meeting \{x_5, x_7\} or \{x_3, x_4\} must contain both elements of that subset. Consider a 4-circuit $C_7$ containing $x_3$ and $x_8$. If $x_7 \not\in C_7$, then neither is $x_5$. In order to avoid an orthogonality contradiction with $E(M) - C_1$, it must be that $x_6 \in C_7$. Now, neither $x_2$ nor $x_4$ may be elements of $C_7$, and so
\[ C_7 = \{ x_1, x_3, x_6, x_8 \} \] 

There must be a 4-circuit, \( C_8 \), that meets \( C_7 \) in a single element. They cannot share \( x_1 \), as that forces \( x_8 \), and \( x_3 \) and \( x_6 \) are isomorphic under the automorphism noted previously. Therefore, without loss of generality, either \( x_3 \) or \( x_8 \) are in \( C_8 \), with its other elements coming from \( E(M) - C_7 = \{ x_2, x_4, x_5, x_7 \} \). However, the forced inclusions prevent \( C_8 \) from having only three elements from \( E(M) - C_7 \), a contradiction. Therefore \( x_7 \in C_7 \), and \( C_7 = \{ x_3, x_5, x_7, x_8 \} \). In this case, there remains an undetermined 4-circuit, \( C_8 \), containing \( \{ x_6, x_8 \} \). Note that \( x_3 \) and \( x_6 \) are symmetric in circuits \( C_1, C_2, \ldots, C_6 \). Therefore, determining the elements of \( C_8 \) is comparable to determining \( C_7 \). Either \( x_2 \in C_8 \), or we get a 4-circuit that cannot meet any other in a single element. Therefore, \( C_8 = \{ x_2, x_4, x_6, x_8 \} \). There are no possible additional 4-circuits, as these eight determine, by Corollary 4.23, that any further 4-circuits must contain elements together in the following pairs: \( \{ x_1, x_8 \} \), \( \{ x_2, x_4 \} \), \( \{ x_3, x_6 \} \), and \( \{ x_5, x_7 \} \). One may quickly check that any 4-circuit containing any two of these pairs violates orthogonality with some 4-cocircuit of \( M \). Thus this case determines a unique matroid, having 4-circuits \( \{ C_1, C_2, \ldots, C_8 \} \), which we label \( M_{8,5} \).

**Subcase 4.24.3.1.2.** Suppose \( C_6 = \{ x_2, x_4, x_7, x_8 \} \).

Here \( C_2 \cap C_6 = \{ x_7 \} \) and \( C_5 \cap C_6 = \{ x_4 \} \). This gives two chains: inclusion of \( x_2 \) implies \( x_4 \), which implies \( x_1 \), which implies \( x_8 \), and inclusion of \( x_5 \) implies \( x_7 \), which implies \( x_3 \). Consider, again, a 4-circuit \( C_7 \) containing \( x_3 \) and \( x_8 \). By the chains, \( x_2 \notin C_7 \), and if \( x_4 \in C_7 \), then \( C_7 = \{ x_1, x_3, x_4, x_8 \} \) which meets \( E(M) - C_1 \) in a single element, a contradiction. Therefore \( x_4 \notin C_7 \).

Consider when \( x_5 \in C_7 \). The chains noted above indicate that having \( x_5 \) forces the inclusion of \( x_7 \), which forces \( x_3 \). Therefore \( C_7 = \{ x_3, x_5, x_7, x_8 \} \). In this case, there must be a 4-circuit \( C_8 \) containing the pair \( \{ x_6, x_8 \} \). Since \( C_1 \cap C_7 = \{ x_3 \} \), further 4-circuits containing \( x_3 \) must contain \( x_6 \). In light of these chains, \( C_8 \) must be one of \( \{ x_3, x_6, x_7, x_8 \} \), \( \{ x_1, x_4, x_6, x_8 \} \), or \( \{ x_1, x_3, x_6, x_8 \} \). The first of these is out by orthogonality with \( E(M) - C_7 \), and the third is out as \( C_7 \subseteq (C_2 \cup \{ x_1, x_3, x_6, x_8 \}) - \{ x_1 \} \). Now suppose \( C_8 = \{ x_1, x_4, x_6, x_8 \} \). In this case we
get a matroid. This choice gives two further forced inclusions: \( C_4 \cap C_8 = \{ x_6 \} \), so \( x_6 \) forces \( x_5 \), and \( C_7 \cap C_8 = \{ x_8 \} \), so \( x_8 \) forces \( x_2 \). Therefore, the only possible additional 4-circuits are \( \{ x_1, x_2, x_4, x_8 \} \) and \( \{ x_3, x_5, x_6, x_7 \} \), each of these leading to an orthogonality contradiction. Thus we get one matroid, with 4-circuits \( \{ C_1, C_2, \ldots, C_8 \} \). However, this matroid is isomorphic to \( M_{8,6} \) via the automorphism given by the permutation \((x_1, x_2)(x_3, x_5)(x_4, x_8)(x_6, x_7)\).

We may assume, then, that \( x_5 \notin C_7 \). In this case, if \( x_6 \notin C_7 \), then \( C_7 = \{ x_1, x_3, x_7, x_8 \} \), which is a contradiction as \( C_6 \subseteq (C_1 \cup \{ x_1, x_3, x_7, x_8 \}) - \{ x_1 \} \). Therefore, \( x_6 \in C_7 \). Then, in order to avoid an orthogonality contradiction with either \( E(M) - C_4 = \{ x_1, x_4, x_5, x_8 \} \) or \( \{ x_1, x_2, x_7, x_8 \} \), it must be that \( C_7 = \{ x_1, x_3, x_6, x_8 \} \). We get the forced inclusion \( x_8 \) implies \( x_5 \), so the only possible additional 4-circuits are \( \{ x_3, x_5, x_7, x_8 \} \) and \( \{ x_3, x_5, x_6, x_7 \} \). These both lead to contradictions: in the first case, \( C_2 \subseteq (C_7 \cup \{ x_3, x_5, x_7, x_8 \}) - \{ x_3 \} \), and in the second, \(|(E(M) - C_2) \cap \{ x_3, x_5, x_6, x_7 \}| = 1 \). Thus we get one matroid, with 4-circuits \( \{ C_1, C_2, \ldots, C_7 \} \), which we call \( M_{8,6} \).

**Subcase 4.24.3.1.3.** Suppose \( C_6 = \{ x_3, x_4, x_7, x_8 \} \).

In this case we get the following chains: \( x_5 \) implies \( x_7 \), which implies \( x_2 \), which implies \( x_4 \), and \( x_1 \) implies \( x_8 \), which implies \( x_6 \). With this in mind, consider a 4-circuit \( C_7 \) containing \( \{ x_6, x_8 \} \). By the chains of forced inclusion, it is clear that neither \( x_5 \) nor \( x_7 \) may be in \( C_7 \). Additionally, if \( x_1 \notin C_7 \), then, to avoid an orthogonality contradiction with \( E(M) - C_3 \) or \( E(M) - C_6 \), it must be that \( C_7 = \{ x_2, x_4, x_6, x_8 \} \). But then \( C_4 \subseteq (C_6 \cup C_7) - \{ x_8 \} \), a contradiction. Therefore, \( x_1 \in C_7 \). The forced inclusions prove that \( x_2 \notin C_7 \), and so \( C_7 \) is one of \( \{ x_1, x_3, x_6, x_8 \} \) or \( \{ x_1, x_4, x_6, x_8 \} \). In the first case, there is no 4-circuit that meets \( \{ x_1, x_3, x_6, x_8 \} \) in a single element. Such a 4-circuit would have to contain either \( x_3 \) or \( x_6 \), with the rest of its elements coming from \( E(M) - \{ x_1, x_3, x_6, x_8 \} = \{ x_2, x_4, x_5, x_7 \} \). By the chains of forced inclusion, these 4-circuits would necessarily be \( \{ x_2, x_3, x_4, x_7 \} \) and \( \{ x_2, x_4, x_6, x_7 \} \), respectively, both of which are out by orthogonality with \( E(M) - C_4 \). Therefore, the only possibility remaining is that \( C_7 = \{ x_1, x_4, x_6, x_8 \} \). Here we get a new forced inclusion: \( x_6 \) forces
This creates a long chain of forced inclusions, which dictate that any additional 4-circuit must be either \( \{x_2, x_3, x_4, x_7\} \), or \( \{x_2, x_4, x_5, x_7\} \). The first of these is out by orthogonality with \( E(M) - C_1 \). In the second case, \( \{x_2, x_4, x_5, x_7\} \subseteq (C_4 \cup C_5) - \{x_6\} \), a contradiction. Thus, this case provides one matroid, with 4-circuits \( \{C_1, C_2, \ldots, C_7\} \). However, this matroid is isomorphic to \( M_{8,6} \), via the automorphism given by \( (x_1, x_2, x_7, x_5, x_8, x_4, x_3, x_6) \).

**Subcase 4.24.3.2.** Suppose \( C_3 \cap C_4 = \{x_8\} \).

Our initial assumptions in Case 4.24.3.1 produced the configuration in Figure 4.14. As we have now exhausted the possible matroids that arise from that configuration, we may from now on assume the configuration in Figure 4.14 is disallowed. We will refer such an arrangement of 4-circuits as the forbidden configuration. With only \( C_1, C_2, \) and \( C_3 \) determined, the elements \( x_3 \) and \( x_4 \) are indistinguishable, as are \( x_6 \) and \( x_7 \). As \( C_4 \) must contain a second element from each of \( E(M) - C_1 \) and \( E(M) - C_2 \) in order to avoid an orthogonality contradiction, we may assume, without loss of generality, that \( \{x_3, x_6\} \subseteq C_4 \). The fourth element of \( C_4 \) must be either \( x_4 \) or \( x_7 \), and these are symmetric choices, as the permutation \( (x_2, x_5)(x_3, x_6)(x_4, x_7) \) is an automorphism of \( M \). Therefore, it suffices to assume \( C_4 = \{x_3, x_4, x_6, x_8\} \). Note that \( x_3 \) and \( x_4 \) remain symmetric. Now, there must be a 4-circuit \( C_5 \) containing \( x_2 \) and \( x_7 \).

**Subcase 4.24.3.2.1.** Suppose \( x_8 \in C_5 \).

To avoid an orthogonality contradiction with \( E(M) - C_3 \), we must have one of \( x_3, x_4, \) and \( x_6 \) in \( C_5 \). If \( x_6 \in C_5 \), then \( C_1, C_3, C_4, \) and \( C_5 \) form the forbidden configuration, a contradiction. Therefore, we may assume \( C_5 = \{x_2, x_3, x_7, x_8\} \), without loss of generality. Note that the single-element intersections of the known 4-circuits give the following chains: inclusion of \( x_1 \) implies \( x_8 \), which implies \( x_7 \), which implies \( x_4 \), and the inclusion of \( x_6 \) implies \( x_2 \). Now, there must be an 4-circuit, \( C_6 \), containing \( \{x_2, x_6\} \). By the forced inclusions, neither \( x_1 \) nor \( x_8 \) is in \( C_6 \). If \( x_7 \in C_6 \), then \( C_6 = \{x_2, x_4, x_6, x_7\} \), which forms the forbidden configuration with \( C_2 \).
$C_3$, and $C_5$. Therefore, $x_7 \not\in C_6$. To avoid an orthogonality contradiction with $E(M) - C_4$, it must be that $x_5 \in C_6$. Therefore, $C_6$ is either $\{x_2, x_4, x_5, x_6\}$ or $\{x_2, x_3, x_5, x_6\}$. Both cases lead to a contradiction. In the first case, $C_3$, $C_4$, $C_5$, and $C_6$ form the forbidden configuration. In the second case, there must be a 4-circuit, $C_7$, meeting $C_6$ in a single element. That element cannot be $x_6$ by forced inclusion of $x_2$. Therefore, $C_7$ contains one of $x_2$, $x_3$, or $x_5$, with its remaining elements coming from $E(M) - C_6 = \{x_1, x_4, x_7, x_8\}$. By the long chain of forced inclusions, this means $C_7$ is one of $\{x_2, x_4, x_7, x_8\}$, $\{x_3, x_4, x_7, x_8\}$, or $\{x_4, x_5, x_7, x_8\}$. The first two choices give contradictions to orthogonality with $E(M) - C_5$, and the final choice, together with $C_2$, $C_5$, and $C_6$ creates the forbidden configuration. Thus there are no viable matroids when $x_8 \in C_5$. We have now reduced to that following:

**Subcase 4.24.3.2.2.** $x_8 \not\in C_5$.

Now, one element from each of $\{x_3, x_4\}$ and $\{x_5, x_6\}$ must be in $C_5$, otherwise $C_5$ meets each of $E(M) - C_2$ and $E(M) - C_1$, respectively, in a single element. We may assume $x_3 \in C_5$ without loss of generality, and so $C_5$ is either $\{x_2, x_3, x_5, x_7\}$ or $\{x_2, x_3, x_6, x_7\}$. In the latter case, $C_1$, $C_2$, $C_3$, and $C_5$ form the forbidden configuration. Therefore, we need only consider the former case. This extends one chain of forced inclusions, with $x_3$ implying $x_1$, which implies $x_8$, which implies $x_7$. There must be a 4-circuit $C_6$ containing $\{x_2, x_6\}$. The noted chain gives $x_3 \not\in C_6$ and $x_1 \not\in C_6$. If $x_4 \not\in C_6$, then $C_6 = \{x_2, x_6, x_7, x_8\}$, otherwise it violates orthogonality with one of $E(M) - C_2$ or $E(M) - C_3$. But, then $C_1$, $C_3$, $C_4$, and $C_6$ form the forbidden configuration. Therefore, $x_4 \in C_6$, and $C_6$ is either $\{x_2, x_4, x_5, x_6\}$ or $\{x_2, x_4, x_6, x_7\}$. In the latter case, $C_1$, $C_2$, $C_3$, and $C_6$ form the forbidden configuration, so assume $C_6 = \{x_2, x_4, x_5, x_6\}$. There must be a 4-circuit $C_7$ that meets $C_6$ in a single element, and that element cannot be $x_6$, as $x_6$ forces $x_2$ as well. By the long chain of forced inclusions in this case, $C_7$ must be one of $\{x_1, x_2, x_7, x_8\}$, $\{x_1, x_4, x_7, x_8\}$, or $\{x_1, x_5, x_7, x_8\}$. The first of these violates orthogonality with $E(M) - C_3$, and the third with $E(M) - C_2$. This leaves the possibility that $C_7 = \{x_1, x_4, x_7, x_8\}$, but then $C_2$, $C_4$, $C_6$, and $C_7$ form the forbidden
configuration. Thus, there are no matroids possible in this case. This concludes the analysis when we assume every 4-circuit meets some other in a single element, and we have found eight matroids.

To see that these matroids are all unique, we perform the following analysis. First, we distinguish the matroids using a count of their 4-circuits. Next, we assign an 8-tuple \((w_1, w_2, \ldots, w_8)\) to each matroid, where \(w_i\) is the number of distinct 4-circuits of \(M\) containing \(x_i\). This is sufficient to determine the uniqueness of each matroid, as summarized in Figure 4.15.

<table>
<thead>
<tr>
<th>Matroid</th>
<th>4-Circuit #</th>
<th>Element Weights</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_{8,1})</td>
<td>8</td>
<td>((5, 4, 4, 4, 4, 4, 3, 4))</td>
<td>The elements with weights 3 and 5 appear together twice in 4-circuits.</td>
</tr>
<tr>
<td>(M_{8,2})</td>
<td>8</td>
<td>((4, 4, 4, 4, 5, 3, 4))</td>
<td>The elements with weights 3 and 5 appear together three times in 4-circuits.</td>
</tr>
<tr>
<td>(M_{8,3})</td>
<td>7</td>
<td>((4, 4, 3, 3, 3, 3, 4))</td>
<td>Two elements with weight 4 appear in 4-circuits without any other elements of weight 4.</td>
</tr>
<tr>
<td>(M_{8,3+})</td>
<td>8</td>
<td>((5, 4, 4, 4, 3, 4, 5))</td>
<td>There are two element with weight 5.</td>
</tr>
<tr>
<td>(M_{8,4})</td>
<td>6</td>
<td>((3, 3, 3, 3, 3, 3, 3))</td>
<td>This matroid only has six 4-circuits.</td>
</tr>
<tr>
<td>(M_{8,4+})</td>
<td>7</td>
<td>((4, 3, 4, 3, 3, 4, 4))</td>
<td>All elements with weight 4 appear together in a 4-circuit.</td>
</tr>
<tr>
<td>(M_{8,5})</td>
<td>8</td>
<td>((4, 4, 4, 4, 4, 4))</td>
<td>All elements have weight 4.</td>
</tr>
<tr>
<td>(M_{8,6})</td>
<td>7</td>
<td>((4, 4, 4, 3, 3, 3, 3))</td>
<td>Only one element with weight 4 appears in a 4-circuit without any other elements of weight 4.</td>
</tr>
</tbody>
</table>

Figure 4.15: Evidence for the uniqueness of each matroid determined in Proposition 4.24.

From here on, we change our tack. We may now assume \(M\) has at least one 4-circuit that does not meet any other in a single element. Our strategy in the following proof will be to progressively limit the number of 4-circuits that may contain a specific element of \(M\).
Proposition 4.25. If $M$ contains a 4-circuit $C$ such that $|C \cap C'| \neq 1$ for all 4-circuits $C' \in \mathcal{C}(M)$, then $M$ is one of the following matroids: $M_{8,7}$, $M_{8,7+}$, $M_{8,8a}$, $M_{8,8b}$, $M_{8,9a}$, $M_{8,9b}$, $M_{8,9b+}$, $M_{8,10}$, $M_{8,10+}$, $M_{8,10++}$, $M_{8,11}$, $M_{8,12}$, and $F_7^+$. 

<table>
<thead>
<tr>
<th>$M$</th>
<th>The 4-circuits of $M$</th>
</tr>
</thead>
</table>
| $M_{8,7}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_3, x_6, x_8\}, \{x_1, x_4, x_5, x_8\}, \{x_2, x_4, x_6, x_7\}$            |
| $M_{8,7+}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_3, x_6, x_8\}, \{x_1, x_4, x_5, x_8\}, \{x_2, x_4, x_6, x_7\}$            |
| $M_{8,8a}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_3, x_6, x_8\}, \{x_2, x_4, x_5, x_8\}, \{x_3, x_4, x_6, x_7\}$            |
| $M_{8,8b}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_3, x_6, x_8\}, \{x_2, x_4, x_5, x_8\}, \{x_2, x_4, x_6, x_7\}$            |
| $M_{8,9a}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_5, x_8\}, \{x_2, x_3, x_6, x_8\}, \{x_2, x_4, x_6, x_7\}$            |
| $M_{8,9b}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_5, x_8\}, \{x_2, x_3, x_6, x_8\}, \{x_3, x_4, x_6, x_7\}$            |
| $M_{8,9b+}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_5, x_8\}, \{x_2, x_3, x_6, x_8\}, \{x_3, x_4, x_6, x_7\}, \{x_2, x_4, x_5, x_7\}$ |
| $M_{8,10}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_6, x_8\}, \{x_3, x_4, x_5, x_8\}, \{x_3, x_4, x_6, x_7\}$            |
| $M_{8,10+}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_6, x_8\}, \{x_3, x_4, x_5, x_8\}, \{x_3, x_4, x_6, x_7\}, \{x_2, x_3, x_6, x_8\}$ |
| $M_{8,10++}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_6, x_8\}, \{x_3, x_4, x_5, x_8\}, \{x_3, x_4, x_6, x_7\}, \{x_2, x_3, x_6, x_8\}$ |
| $M_{8,11}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_5, x_7\},$  
|       | $\{x_1, x_4, x_6, x_8\}, \{x_3, x_4, x_5, x_8\}, \{x_2, x_3, x_6, x_7\}, \{x_2, x_4, x_5, x_7\}$ |
| $M_{8,12}$ | $\{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_5, x_6\}, \{x_1, x_3, x_5, x_7\}, \{x_1, x_4, x_5, x_8\},$  
|       | $\{x_2, x_3, x_7, x_8\}, \{x_2, x_4, x_6, x_7\}, \{x_3, x_4, x_6, x_8\}$            |

Figure 4.16: The 4-circuits of the matroids in Proposition 4.25.

Proof. Let $C_s$ be a 4-circuit of $M$ that meets no other 4-circuit in a single element, and let $C_s = \{x_1, x_2, x_3, x_4\}$. Note that this implies that every other 4-circuit of $M$ meets $C_s$ in
exactly two elements, as \( E(M) - C_* \) is a cocircuit. We may assume that, of all the elements in \( C_* \), the element \( x_1 \) is contained in the most 4-circuits. Let \( C_1, C_2, \ldots, C_n \) be the list of 4-circuits distinct from \( C_* \) that contain \( x_1 \). For each \( i \) and \( j \) in \( \{1, 2, \ldots, n\} \), if \( C_i \cap C_* = C_j \cap C_* \), then either \( i = j \), or \( C_i \cap C_j \cap (E(M) - C_*) = \emptyset \). Further, as each \( C_i \) contains \( x_1 \), it must be that \( C_i \cap (E(M) - C_*) \neq C_j \cap (E(M) - C_*) \) when \( i \neq j \). We divide the work that follows into cases determined by the maximum value of \( n \).

**Case 4.25.1. Suppose** \( n \geq 6 \).

We may assume \( C_1 = \{x_1, x_2, x_5, x_6\} \) and \( C_2 = \{x_1, x_2, x_7, x_8\} \), without loss of generality. No further 4-circuits may contain both \( x_1 \) and \( x_2 \) without meeting one of \( C_1, C_2, \) or \( C_3 \) in at least three elements, which violates orthogonality as the complements of 4-circuits are 4-cocircuits. Therefore, we may assume that \( x_3 \in C_3 \). Without loss of generality, \( C_3 = \{x_1, x_3, x_5, x_7\} \), and so we may assume \( C_4 = \{x_1, x_3, x_6, x_8\} \). The only possible additional 4-circuits containing \( x_1 \) must also contain \( x_4 \), so, as before and without loss of generality, we may assume \( C_5 = \{x_1, x_4, x_5, x_8\} \) and \( C_6 = \{x_1, x_4, x_6, x_7\} \). It is clear there can be no more 4-circuits containing \( x_1 \). Indeed, there can be no further 4-circuits, as every other 4-element set meeting \( C_* \) in two elements either shares three elements with one of \( C_1, C_2, \ldots, C_6 \), or is disjoint from them, violating orthogonality and 4-connectivity, respectively. The matroid in this case is recognizable as the unique free coextenstion of the Fano matroid. Thus, the maximum number of additional 4-circuits containing \( x_1 \) is six, and there is one example when \( n \geq 6 \).

**Case 4.25.2. Suppose** \( n = 5 \).

As in Case 4.25.1, we may assume, without loss of generality, that the pairs \( \{x_1, x_2\} \) and \( \{x_1, x_3\} \) appear twice in other 4-circuits, and the pair \( \{x_1, x_4\} \) appears once. We are free, then, to assume that \( C_1 = \{x_1, x_2, x_5, x_6\} \), \( C_2 = \{x_1, x_2, x_7, x_8\} \), \( C_3 = \{x_1, x_3, x_5, x_7\} \), \( C_4 = \{x_1, x_3, x_6, x_8\} \), and \( C_5 = \{x_1, x_4, x_5, x_8\} \), as before. Now, however, there is not yet a
4-circuit containing \( \{x_4, x_6\} \). Let \( D_1 \) be such a 4-circuit. Without loss of generality, \( x_2 \in D_1 \), as \( x_2 \) and \( x_3 \) are symmetric via the automorphism given by the permutation \((x_2, x_3)(x_5, x_8)\).

Proceeding, \( x_5 \not\in D_1 \), otherwise \( |D_1 \cap (E(M) - C_1)| = 1 \), and \( x_8 \not\in D_1 \), otherwise \( D_1 \) and \( C_3 \) are disjoint. This implies \( D_1 = \{x_2, x_4, x_6, x_7\} \). We label this matroid \( M_{8,7} \), having 4-circuits \( C_1, C_2, C_3, C_4, C_5 \), and \( D_1 \).

It is possible that this structure permits an additional 4-circuit. Such a circuit, call it \( D_2 \), must contain a pair from \( \{x_2, x_3, x_4\} \) and from \( \{x_5, x_6, x_7, x_8\} \). If \( \{x_2, x_4\} \subseteq D_2 \), then the only possible pairs from \( \{x_5, x_6, x_7, x_8\} \) are those which do not already appear in a 4-circuit containing either \( x_2 \) or \( x_4 \). That leaves \( \{x_5, x_7\} \) and \( \{x_6, x_8\} \); however, \( \{x_2, x_4, x_5, x_7\} \) and \( \{x_2, x_4, x_6, x_8\} \) are disjoint from \( E(M) - C_4 \) and \( E(M) - C_3 \), respectively. This is a contradiction, and so \( \{x_2, x_4\} \not\subseteq D_2 \). The case when \( \{x_3, x_4\} \subseteq D_2 \) is similar. Here, the only viable pairs from \( \{x_5, x_6, x_7, x_8\} \) are \( \{x_5, x_6\} \) and \( \{x_7, x_8\} \), each of which leads to a connectivity contradiction, as before. Therefore \( \{x_3, x_4\} \not\subseteq D_2 \). The last case has \( \{x_2, x_3\} \subseteq D_2 \). Again, the possible remaining elements of \( D_2 \) are either \( \{x_5, x_8\} \) or \( \{x_6, x_7\} \). The latter
choice gives $D_2 = \{x_2, x_3, x_6, x_7\}$ which is disjoint from $C_5$, a contradiction. However, there is no problem with $D_2 = \{x_2, x_3, x_5, x_8\}$. Thus we have a second matroid in this case, which we call $M_{8,7^+}$, and which has 4-circuits $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $D_1$, and $D_2$.

**Case 4.25.3.** Suppose $n = 4$.

We may now assume that an element of $C_*$ is contained in at most four other 4-circuits. These circuits meet $C_*$ in one of $\{x_1, x_2\}$, $\{x_1, x_3\}$, or $\{x_1, x_4\}$. There are two non-isomorphic ways this may happen: either two pairs are used twice, and one pair not at all; or one pair is used twice, and the others are used once. We treat these in cases.

**Subcase 4.25.3.1.** Assume two pairs are used twice, and one pair not at all.

We may assume the pairs in question are $\{x_1, x_2\}$ and $\{x_1, x_3\}$. Without loss of generality, we get $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_2, x_7, x_8\}$, $C_3 = \{x_1, x_3, x_5, x_7\}$, and $C_4 = \{x_1, x_3, x_6, x_8\}$. There must be a 4-circuit, say $D_1$ containing $x_4$ and $x_5$. As $D_1$ must meet $C_*$ in two elements, and as $x_1$ cannot be in $D_1$ by assumption, we may assume without loss of generality that $x_2 \in D_1$. Now, $x_6 \not\in D_1$ otherwise we obtain an orthogonality contradiction with $E(M) - C_1$; and $x_7 \not\in D_1$ since $\{x_2, x_4, x_5, x_7\} = E(M) - C_3$ is a cocircuit. Therefore, $D_1 = \{x_2, x_4, x_5, x_8\}$. There must also be a 4-circuit, $D_2$, containing $\{x_4, x_6\}$. From this we get two cases.

**Subcase 4.25.3.1.1.** Suppose $x_3 \in D_2$.

In order to avoid an orthogonality contradiction with $E(M) - C_4$, one of $x_5$ and $x_7$ must be in $D_2$; however, $\{x_3, x_4, x_5, x_6\} = E(M) - C_2$, so $x_7 \not\in D_2$. Now $D_2 = \{x_5, x_6, x_7, x_8\}$, and there are no further 4-circuits possible, as every pair from $\{x_5, x_6, x_7, x_8\}$ has appeared with one of $x_2$, $x_3$, or $x_4$ in one of the known 4-circuits. Therefore we get one matroid, which we denote $M_{8,8^a}$.

**Subcase 4.25.3.1.2.** Suppose $x_3 \not\in D_2$. 

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This forces $x_7 \in D_2$, otherwise $|D_2 \cap (E(M) - D_1)| = 1$, and so $D_2 = \{x_2, x_4, x_6, x_7\}$. Again, there are no further 4-circuits possible. An additional 4-circuit would necessarily include $x_3$ and $x_4$, and the only pair from $E(M) - C_1$ not appearing in a known 4-circuit with either of those elements is $\{x_5, x_6\}$. But $\{x_3, x_4, x_5, x_6\} = E(M) - C_2$, a contradiction. Thus we get a second matroid from these cases, which we call $M_{8,8b}$.

**Subcase 4.25.3.2. Assume one pair is used twice and the others are used once**

Without loss of generality, suppose both $C_1$ and $C_2$ contain $\{x_1, x_2\}$, while $C_3$ and $C_4$ contain $\{x_1, x_3\}$ and $\{x_1, x_4\}$, respectively. It suffices to assume $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_1, x_2, x_7, x_8\}$. From here there are two possibilities: either $|C_3 \cap C_4| = 2$ or $|C_3 \cap C_4| = 1$.

**Subcase 4.25.3.2.1. $|C_3 \cap C_4| = 2$.**

We may assume $C_3 = \{x_1, x_3, x_5, x_7\}$ and $C_4 = \{x_1, x_4, x_5, x_8\}$. In this case, consider a 4-circuit, $D_1$, containing $x_3$ and $x_6$.

Suppose $x_2$ is in $D_1$. Then $D_1$ must contain one of $x_7$ and $x_8$ in order to avoid an orthogonality contradiction with $E(M) - C_1$. Since $\{x_2, x_3, x_6, x_7\} = E(M) - C_4$, we get $D_1 = \{x_2, x_3, x_6, x_8\}$. Consider a 4-circuit $D_2$ containing $x_4$ and $x_6$. Such a circuit must contain one of $x_5$ and $x_7$ to avoid an orthogonality contradiction with $E(M) - D_1$. However, $x_5 \notin D_2$, as then neither $x_2$ nor $x_3$ may be members of $D_2$ without violating orthogonality with some 4-cocircuit of $M$. Therefore, $D_2$ is one of $\{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_6, x_7\}$. If we allow $D_2$ to be the former set, we get a matroid that admits no further 4-circuits, which we call $M_{8,9a}$. If we allow $D_2$ to be the latter set, we get a second matroid which we call $M_{8,9b}$. This, however, does admit one further 4-circuit. It is possible that $\{x_2, x_4, x_5, x_7\}$ is a circuit in addition to those of $M_{8,9b}$ without producing contradictions. We call this third matroid in this case $M_{8,9b^+}$. 

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Now assume that \( x_2 \) is not in \( D_1 \). Then \( D_1 = \{x_3, x_4, x_6, x_7\} \) without loss of generality, as \( x_7 \) and \( x_8 \) are symmetric given only \( C_4, C_1, C_2, C_3, \) and \( C_4 \). Now, there must be a 4-circuit, \( D_2 \), containing \( \{x_3, x_8\} \). In order to avoid an orthogonality contradiction with one of \( E(M) - C_4 \) or \( E(M) - D_1 \), such a circuit must contain \( x_2 \). Further, \( D_2 \) must also contain one of \( x_5 \) and \( x_6 \), otherwise it violates orthogonality with \( E(M) - C_2 \). But \( \{x_2, x_3, x_6, x_8\} \) was considered in the previous case, so \( D_2 = \{x_2, x_3, x_5, x_6\} \). Now, there yet must be a 4-circuit containing \( x_6 \) and \( x_8 \). Such a circuit cannot contain either \( \{x_2, x_3\} \) or \( \{x_3, x_4\} \), otherwise we get an orthogonality contradiction. Further, \( \{x_2, x_4, x_6, x_8\} = E(M) - C_3 \), and cannot be a circuit. Thus \( M_{8,9a}, M_{8,9b}, \) and \( M_{8,9c} \) are the only matroids determined by this case.

**Subcase 4.25.3.2.2.** \(|C_3 \cap C_4| = 1| \).

We may assume \( C_3 = \{x_1, x_3, x_5, x_7\} \) and \( C_4 = \{x_1, x_4, x_6, x_8\} \). If \( x_3 \) and \( x_4 \) do not appear together in an additional 4-circuit, then, by the case restriction since \( x_2 \) already appears twice with \( x_1 \), there are at most two more 4-circuits in \( M \), one containing \( \{x_2, x_3\} \) and the other containing \( \{x_2, x_4\} \). Now, there must be a 4-circuit containing \( \{x_4, x_7\} \), and also a 4-circuit containing \( \{x_6, x_7\} \), so it must be that \( \{x_2, x_4, x_6, x_7\} \) is a circuit. But, there must also be a 4-circuit containing \( \{x_4, x_5\} \), a contradiction.

Therefore there must be some 4-circuit, say \( D_1 \), containing \( \{x_3, x_4\} \). We may assume that \( x_5 \in D_1 \), as any two elements from \( \{x_5, x_6, x_7, x_8\} \) are symmetric, since both permutations \((x_3, x_4)(x_5, x_6)(x_7, x_8)\) and \((x_5, x_7)(x_6, x_8)\) are automorphisms of \( M \). This implies that \( D_1 = \{x_3, x_4, x_5, x_8\} \), as \( D_1 \) having either \( x_6 \) or \( x_7 \) produces a contradiction to connectivity or orthogonality, respectively. It is possible that \( x_3 \) and \( x_4 \) appear together again in some 4-circuit, \( D_2 \).

In that case, the only possibility is that \( D_2 = \{x_3, x_4, x_6, x_7\} \). This collection of circuits satisfies all our assumptions, and therefore gives a matroid, which we label \( M_{8,11} \). However, this structure also admits additional 4-circuits. A further 4-circuit, say \( D_3 \), must contain \( x_2 \) and one of \( x_3 \) or \( x_4 \). Within \( M_{8,10} \), these last two elements are symmetric under the
automorphism given by the permutation \((x_3, x_4)(x_5, x_8)(x_6, x_7)\), so we may assume \(x_3 \in D_3\). Then \(D_3 = \{x_2, x_3, x_6, x_8\}\). The inclusion of this circuit produces a second matroid in this case, which we label \(M_{8,10^+}\). Further, this structure admits yet another 4-circuit, which must be \(D_4 = \{x_2, x_4, x_5, x_7\}\). This third example we label \(M_{8,10^{++}}\).

We may assume, then, that \(x_3\) and \(x_4\) do not appear in another 4-circuit outside of \(C_\ast\) and \(D_1\). There must still be 4-circuits containing \(\{x_3, x_6\}\) and \(\{x_4, x_7\}\). Let these circuits be \(D_2\) and \(D_3\), respectively. In this case, \(x_2\) is in each of these 4-circuits. In order to avoid an orthogonality contradiction with \(E(M) - C_2\), one of \(x_5\) and \(x_6\) must be in \(D_2\); and, in order to avoid an orthogonality contradiction with \(E(M) - C_3\), one of \(x_6\) and \(x_8\) must be in \(D_2\). Therefore, \(D_2 = \{x_2, x_3, x_6, x_7\}\). Similar reasoning indicates \(D_3 = \{x_2, x_4, x_5, x_7\}\). This collection of 4-circuits yields a matroid which permits no additional 4-circuits. We label this \(M_{8,11}\).

**Case 4.25.4.** Suppose \(n \leq 3\).

With this, we may assume that each element of \(C_\ast\) appears in at most three other 4-circuits, and that \(x_1\) attains that maximum. This is the last major case, as in order for all the two-element subsets of \(\{x_5, x_6, x_7, x_8\}\) to appear in a 4-circuit of \(M\), at least six 4-circuits are required. We approach this last case in three phases of restricting the structure of the 4-circuits. First, we rule out the case when \(x_1\) appears twice with the same element from \(C_\ast\) in two other 4-circuits. Next, we consider when \(x_1\) appears with a certain element of \(E(M) - C_\ast\) in three distinct 4-circuits, a case which produces one example. The final case yields no additional matroids, and concludes the search for eight-element matroids.

**Subcase 4.25.4.1.** \(x_1\) appears twice with the same element of \(C_\ast\) in two other 4-circuits.

Suppose, without loss of generality, that \(x_1\) and \(x_2\) appear together in two 4-circuits in addition to \(C_\ast\). It suffices to assume these are \(C_1 = \{x_1, x_2, x_5, x_6\}\) and \(C_2 = \{x_1, x_2, x_7, x_8\}\). As \(x_1\) is in one more 4-circuit, say \(C_3\), we may assume \(\{x_1, x_3, x_5\} \subseteq C_3\), as \(x_3\) and \(x_4\)
are symmetric, as are the elements of $E(M) - C_*$. Then $C_3 = \{x_1, x_3, x_5, x_7\}$ without loss of generality. Now, there is a 4-circuit, $C_4$, containing $\{x_6, x_8\}$. The remaining elements of $C_4$ come from $\{x_2, x_3, x_4\}$, and cannot be $\{x_2, x_4\}$, as $\{x_2, x_4, x_6, x_8\} = E(M) - C_3$. If $C_4 = \{x_2, x_3, x_6, x_8\}$, then all additional 4-circuits must contain $x_3$ and $x_4$, and $x_3$ may appear only once more. But the pairs $\{x_5, x_8\}$ and $\{x_6, x_7\}$ have yet to appear in a 4-circuit, a contradiction. Therefore $C_4 = \{x_3, x_4, x_6, x_8\}$. Further 4-circuits must contain either $\{x_2, x_4\}$ or $\{x_3, x_4\}$, and each of these may be used once. Therefore, the remaining circuits are either $\{x_2, x_4, x_5, x_8\}$ and $\{x_3, x_4, x_6, x_7\}$, or $\{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_5, x_8\}$; each of these possibilities gives an orthogonality contradiction with $E(M) - C_4$. Thus $x_1$ cannot appear in two 4-circuits outside of $C_*$ with the same element from $C_*$. 

**Subcase 4.25.4.2.** $x_1$ appears with a fixed element of $E(M) - C_*$ in three distinct 4-circuits.

Without loss of generality, we may assume $C_1$, $C_2$, and $C_3$ each contain $\{x_1, x_5\}$. This implies $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_3, x_5, x_7\}$, and $C_3 = \{x_1, x_4, x_5, x_8\}$. There must be a 4-circuit, $C_4$, with $\{x_7, x_8\}$. This cannot be $\{x_3, x_4, x_7, x_8\} = E(M) - C_1$. Further, since the permutation $(x_3, x_4)(x_7, x_8)$ is an automorphism of $M$, we see that $x_3$ and $x_4$ are symmetric. Hence, we may assume $C_4 = \{x_2, x_3, x_7, x_8\}$ without loss of generality. There must also be 4-circuits $C_5$ and $C_6$ containing $\{x_6, x_7\}$ and $\{x_6, x_8\}$, respectively. By the restriction on pairs of elements in this case, this forces $C_5 = \{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_6, x_8\}$. This collection of circuits satisfies all conditions on $M$. We label this matroid $M_{8,12}$. 

**Subcase 4.25.4.3.** Elements of $C_*$ appear in a 4-circuit with an element $E(M) - C_*$ at most twice.

Without loss of generality, we may assume the 4-circuits containing $x_1$ in this case are $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_3, x_5, x_7\}$, and $C_3 = \{x_1, x_4, x_6, x_8\}$. The pairs $\{x_5, x_8\}$, $\{x_6, x_7\}$, and $\{x_7, x_8\}$ must all appear in 4-circuits. The remaining elements of those 4-circuits come from $\{x_2, x_3, x_4\}$, with each pair of these occurring exactly once. Of these, if
we consider the 4-circuit $C_4$ containing $\{x_3, x_4\}$, then we find that $C_4 = \{x_3, x_4, x_5, x_8\}$, as $\{x_3, x_4, x_7, x_8\} = E(M) - C_1$, and the pairs $\{x_5, x_8\}$ and $\{x_6, x_7\}$ are symmetric under the automorphism given by $(x_3, x_4)(x_5, x_6)(x_7, x_8)$. Now, $x_3$ must appear in a 4-circuit with $x_6$, so $C_5 = \{x_2, x_3, x_6, x_7\}$ must be a circuit. This implies $\{x_2, x_4, x_7, x_8\}$ is a circuit, but this is a contradiction, as then $x_4$ appears in three 4-circuits with $x_8$. Thus there are no matroids in this case, and our analysis of the eight-element matroids is complete.

To see that these matroids are all unique, we perform the following analysis, as in Proposition 4.24. Note that, by the structure of the cases in this argument, we need only be concerned with the matroids coming from Case 4.25.3. First, we distinguish the matroids using a count of their 4-circuits. Next, we assign an 8-tuple $(w_1, w_2, \ldots, w_8)$ to each matroid, where $w_i$ is the number of distinct 4-circuits of $M$ containing $x_i$. This is sufficient to determine the uniqueness of each matroid, as summarized in Figure 4.18.

<table>
<thead>
<tr>
<th>Matroid</th>
<th>4-Circuit #</th>
<th>Element Weights</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{8,10a}$</td>
<td>7</td>
<td>$(5,5,3,3,3,3,2,2)$</td>
<td>There are no elements with weight 4.</td>
</tr>
<tr>
<td>$M_{8,10b}$</td>
<td>7</td>
<td>$(5,4,4,3,3,3,2,2)$</td>
<td>There are elements of weight 4 and 2.</td>
</tr>
<tr>
<td>$M_{8,10b}^+$</td>
<td>8</td>
<td>$(5,5,4,4,3,3,2,2)$</td>
<td>There is an element of weight 2.</td>
</tr>
<tr>
<td>$M_{8,11}$</td>
<td>7</td>
<td>$(5,3,4,3,3,3,3,3)$</td>
<td>There are elements of weight 4, but none of weight 2.</td>
</tr>
<tr>
<td>$M_{8,11}^+$</td>
<td>8</td>
<td>$(5,4,5,3,3,3,4,3)$</td>
<td>The elements of weight 5 appear together only twice.</td>
</tr>
<tr>
<td>$M_{8,11}^{++}$</td>
<td>9</td>
<td>$(5,5,5,5,4,4,4,4)$</td>
<td>This matroid has nine 4-circuits.</td>
</tr>
<tr>
<td>$M_{8,12}$</td>
<td>8</td>
<td>$(5,5,4,4,4,3,4,3)$</td>
<td>The elements of weight 5 appear together three times.</td>
</tr>
</tbody>
</table>

Figure 4.18: Evidence for the uniqueness of each matroid determined in Proposition 4.25.
4.4 When $M$ Has Exactly Nine Elements

The second major small-element case is when $|E(M)| = 9$. We first show that such a matroid must have rank 4, and then that it cannot have two disjoint 4-cocircuits. Finally, we determine all such matroids explicitly. In both the proof of Lemma 4.26 and Proposition 4.28, we will abuse the structure of the complements of 4-cocircuits of $M$. Specifically, we demonstrate that the corank of $M$ restricted to the complement of a 4-cocircuit must be small, and thus narrow our search considerably. This technique will be utilized again in the ten-element case.

**Lemma 4.26.** If $|E(M)| = 9$, then $r(M) = 4$.

*Proof.* Suppose not. If $r(M) \leq 3$, then $r(M) = 3$ since $M$ has 4-circuits. Further, each 4-element set must be a circuit, so $M \cong U_{3,9}$, which has no 4-cocircuit. Therefore $r(M) \geq 5$. Moreover, $r(M) = 5$, since $M$ has 4-cocircuits and $r(M) \leq 6$ by dual reasoning to the above argument.

Let $E(M) = \{x_1, x_2, \ldots, x_9\}$, and consider a 4-cocircuit $D_1 = \{x_6, x_7, x_8, x_9\}$. If $X = E(M) - D_1$, then $r^*(M|X) = 1$. Further, the smallest cocircuits of $(M|X)^*$ have four elements. The only possibilities for $(M|X)^*$ are $U_{1,5}$ and $U_{1,4} \bigoplus U_{0,1}$, so we proceed in two cases.

First, assume that $(M|X)^* \cong U_{1,5}$. Then $X$ contains no 4-circuits of $M$. Therefore, every circuit of $M$ meets $D_1$, and must do so in at least two elements. There are $\binom{6}{2}$ distinct pairs of elements, each of which must be in some 4-circuit. These each meet one of the six distinct pairs of elements from $D_1$, and therefore some pair of elements from $D_1$ is used in at least two 4-circuits. Let $C_1$ and $C_2$ be those 4-circuits, and suppose, without loss of generality, that $C_1 \cap C_2 = \{x_6, x_7\}$. Then there is a circuit contained in $C_1 \cup C_2 - x_6$. This circuit cannot contain $x_7$, otherwise it violates orthogonality, and thus it is a 4-circuit contained in $X$, a contradiction. This implies $(M|X)^* \not\cong U_{1,5}$.

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Now suppose \((M|X)^* \cong U_{1,4} \oplus U_{0,1}\). In this case, \(M|X\) contains exactly one 4-circuit which we label \(C_1 = \{x_1, x_2, x_3, x_4\}\), without loss of generality. There are 4-circuits containing \(\{x_1, x_5\}, \{x_2, x_5\}, \{x_3, x_5\}, \text{ and } \{x_4, x_5\}\), and each such 4-circuit meets \(D_1\) in two elements. Therefore, two of these 4-circuits must share one element from \(D_1\). Without loss of generality, say \(C_2 = \{x_1, x_5, x_6, x_7\}\) and \(C_3 = \{x_2, x_5, x_6, x_8\}\). Let \(S = C_2 \cup C_3\). Then \(\lambda(S) = r(S) + r^*(S) - |S| \leq 4 + 4 - 6 = 2\), a contradiction. Thus \(r(M) \neq 5\), and so the lemma is proved. \(\square\)

**Lemma 4.27.** If \(|E(M)| = 9\), then \(M\) has no two disjoint 4-cocircuits.

**Proof.** Suppose not, letting \(D_1\) and \(D_2\) be a pair of disjoint 4-cocircuits. Let \(\{e\} = E(M) - (D_1 \cup D_2)\). If \(x \in D_1\), then there is a 4-circuit, \(C_1\), containing \(\{e, x\}\). This circuit must have at least two elements from \(D_1\), and so must be disjoint from \(D_2\). Let \(\{y\} = D_1 - C_1\). Then there is a 4-circuit, \(C_2\), containing \(\{e, y\}\). As with \(C_1\), we must have \(|D_1 \cap C_2| = 3\). Then, there exists a circuit \(C_3 \subseteq (C_1 \cup C_2) - e = D_1\), a contradiction. \(\square\)

The matroids in the statement of the following proposition are defined throughout the proof. As in the previous section, we can be assured that these are indeed matroids because the 4-circuits together with every 3-set that is in no 4-circuit form a 3-partition of \(E(M)\), since no two such sets meet in more than two elements. This means that these sets are the hyperplanes of a paving matroid on \(E(M)\), by Proposition 4.21.

**Proposition 4.28.** Suppose \(M\) is a 4-connected matroid. If \(M\) has every element in a 4-cocircuit and every pair of elements in a 4-circuit, and \(|E(M)| = 9\), then \(M\) is one of the following matroids: \(M_{9,1}, M_{9,1a}, M_{9,1b}, M_{9,2}, M_{9,3}, M_{9,3^+}, M_{9,4}, M_{9,4^+}, M_{9,5}, M_{9,6}\).

**Proof.** Let \(E(M) = \{x_1, x_2, \ldots, x_9\}\), and consider a 4-cocircuit \(D_1 = \{x_6, x_7, x_8, x_9\}\). If \(X_1 = E(M) - D_1\), then \(r^*(M|X_1) = 2\). Further, the smallest cocircuits of \((M|X_1)^*\) have four elements. This means that the only possibility for \((M|X_1)^*\) is \(U_{2,5}\). Therefore, every subset of \(X_1\) with four elements is a circuit of \(M\), and so \(M|X_1 \cong U_{3,5}\). This is true for every five-element hyperplane of \(M\). With that in mind, consider a 4-cocircuit, \(D_2\), containing \(x_1\). This
cocircuit must contain two additional elements from \( X_1 \) in order to avoid an orthogonality contradiction, and must also contain at least one element from \( D_1 \) by Lemma 4.27. Therefore, \( D_2 = \{x_1, x_2, x_3, x_6\} \), without loss of generality. If we let \( X_2 = E(M) - D_2 \), then \( M|X_2 \cong U_{3,5} \). There must also be a 4-cocircuit, \( D_3 \), containing \( x_4 \). As before, this cocircuit must contain three elements from \( X_1 \), and also three elements from \( X_2 \). Further, it must have at least one element from each \( D_1 \) and \( D_2 \). Therefore, without loss of generality, \( D_3 = \{x_1, x_4, x_5, x_7\} \). Again, if we let \( X_3 = E(M) - D_3 \), then \( X_3 \cong U_{3,5} \). Observe the symmetry between \( x_1, x_6 \), and \( x_7 \).

It should be noted that there may be no further 4-cocircuits of \( M \), as there is no 4-element set that meets each of \( X_1, X_2, \) and \( X_3 \) in three elements. Therefore, all further hyperplanes of \( M \) have either three or four elements, which implies that all additional 4-circuits are hyperplanes and their complements cocircuits.

**Case 4.28.1.** Suppose there is a 4-circuit, \( C_1 \), containing \( \{x_1, x_6, x_7\} \).

Without loss of generality, \( C_1 = \{x_1, x_2, x_6, x_7\} \). Note that no other 4-circuit may contain \( \{x_1, x_6, x_7\} \), or otherwise we get another local \( U_{3,5} \), and, in turn, another 4-cocircuit. However, every additional 4-circuit must contain two of \( x_1, x_6, \) and \( x_7 \), in order to avoid an orthogonality contradiction with one of \( D_1, D_2, \) or \( D_3 \). In this case, consider a 4-circuit, \( C_2 \), containing \( x_3 \) and \( x_7 \). In order to avoid an orthogonality contradiction with one of \( D_1, D_2, \) or \( D_3 \), this circuit must contain either \( x_1 \) or \( x_6 \). These elements are symmetric under the automorphism given by the permutation \( (x_1, x_6)(x_4, x_8)(x_5, x_9) \), so we may assume \( x_1 \in C_2 \).

As \( C_2 \) must contain another element of \( D_1 \), we may assume \( C_2 = \{x_1, x_3, x_7, x_8\} \), as \( x_8 \) and \( x_9 \) are symmetric.

Next, consider a 4-circuit, \( C_3 \), containing \( x_1 \) and \( x_9 \). Suppose \( x_7 \in C_3 \). In order to avoid an orthogonality contradiction with \( D_2 \), one of \( x_2 \) and \( x_3 \) must be in \( C_3 \), but these lead to an orthogonality contradiction with \( E(M) - C_1 \) or \( E(M) - C_2 \), respectively. Therefore \( x_7 \notin C_3 \), and one of \( x_4 \) and \( x_5 \) must be in \( C_3 \) in order to avoid an orthogonality contradiction with
These elements are symmetric, so we may assume \( x_4 \in C_3 \). Further, as the only element shared by \( D_1 \) and \( D_2 \) is \( x_6 \), it must be that \( C_3 = \{x_1, x_4, x_6, x_9\} \) by orthogonality.

Now there is only one pair of elements not yet in a 4-circuit, and that is \( \{x_5, x_6\} \). Let \( C_4 \) be 4-circuit containing \( \{x_5, x_6\} \). As with the previous 4-circuits, \( C_4 \) must contain one of \( x_1 \) and \( x_7 \).

**Subcase 4.28.1.1.** Suppose \( x_1 \in C_4 \).

In order to avoid a contradiction, \( C_4 \) must contain another element from each of \( D_1 \), \( E(M) \setminus C_1 \), and \( E(M) \setminus C_3 \). Therefore \( C_4 = \{x_1, x_5, x_6, x_8\} \). A matroid with this collection of circuits satisfies all our assumptions, and thus we get our first example, which we label \( M_{9,1} \). It is possible, though, that additional 4-circuits exist as well as those noted above. Any such additional 4-circuit cannot contain \( x_1 \), as every four-element set with \( x_1 \) and one of \( x_6 \) and \( x_7 \) will meet some cocircuit in a single element. Now, if \( C_5 \) is another 4-circuit, it must be that \( \{x_6, x_7\} \subseteq C_5 \). Now, \( x_2 \notin C_5 \) because of orthogonality with \( E(M) \setminus C_1 \), and so \( x_3 \in C_5 \), otherwise \( C_5 \) violates orthogonality with \( E(M) \setminus D_2 \). The final element of \( C_5 \) must be either \( x_4 \) or \( x_5 \). Each of these gives rise to a distinct matroid. Let \( M_{9,1a} \) be the matroid in which \( C_5 = \{x_3, x_4, x_6, x_7\} \), and let \( M_{9,1b} \) be the matroid in which \( C_5 = \{x_3, x_5, x_6, x_7\} \). Neither of these matroids permits any additional 4-circuits.

We may now assume \( x_1 \notin C_4 \).

**Subcase 4.28.1.2.** \( x_1 \notin C_4 \).

In this case, \( C_4 \) must have another element from each of \( D_2 \) and \( E(M) \setminus C_1 \). The only element they share is \( x_3 \), so \( C_4 = \{x_3, x_5, x_6, x_7\} \). The inclusion of this circuit produces a matroid that satisfies our assumptions, and we label it \( M_{9,2} \). Unlike the previous case, this structure admits no further 4-circuits. We know this matroid is distinct from \( M_{9,1} \); because, \( x_1 \) is in four 4-circuits of \( M_{9,1} \) not contained in \( X_1 \), \( X_2 \), or \( X_3 \), but each of \( x_1 \), \( x_6 \), or \( x_7 \) only appears three times in such 4-circuits of \( M_{9,2} \). This completes the analysis of Case 4.28.1.
Case 4.28.2. $x_1$, $x_6$, and $x_7$ do not appear together in a 4-circuit.

Let $C_1$ be a 4-circuit containing $x_1$ and $x_6$. We may assume $C_1 = \{x_1, x_4, x_6, x_8\}$, without loss of generality.

Subcase 4.28.2.1. $x_1$ and $x_8$ appear together in another 4-circuit.

Let $C_2$ be an additional 4-circuit that contains $\{x_1, x_8\}$. Then $x_6 \notin C_2$, which means $C_2$ contains $x_7$. One of $x_2$ and $x_3$ must be in $C_2$, and so, as these elements are symmetric, we may assume $C_2 = \{x_1, x_2, x_7, x_8\}$. Next, consider a 4-circuit, $C_3$, containing $x_1$ and $x_9$. As $x_6$ and $x_7$ are symmetric under the automorphism given by the permutation $(x_2, x_4)(x_3, x_5)(x_6, x_7)$, we may assume $x_6 \in C_3$. In order to avoid an orthogonality contradiction with $D_3$ or $E(M) - C_1$, the final element of $C_3$ must be $x_5$, so $C_3 = \{x_1, x_5, x_6, x_9\}$. There must also be a 4-circuit, $C_4$, containing $\{x_3, x_7\}$. Either $x_1$ or $x_6$ is also on $C_4$, and this produces two cases.

Subcase 4.28.2.1.1. Suppose $x_1 \in C_4$.

Then $C_4 = \{x_1, x_3, x_7, x_9\}$, by orthogonality with one of $D_1$ or $E(M) - C_2$. In this case, the pair $\{x_6, x_7\}$ has yet to appear in a 4-circuit. Let $C_5$ be that circuit. To avoid an orthogonality contradiction, $C_5$ must have one element from each $\{x_2, x_3\}$ and $\{x_4, x_5\}$. Note that $x_2$ and $x_3$ are symmetric under an automorphism given by the permutation $(x_2, x_3)(x_4, x_5)(x_6, x_7)$. This automorphism also swaps $x_4$ and $x_5$, giving two distinct cases: either $C_5 = \{x_2, x_4, x_6, x_7\}$ or $C_5 = \{x_2, x_5, x_6, x_7\}$. Both cases satisfy our assumptions for $M$, and thus yield matroids which we label $M_{9,3}$ and $M_{9,4}$, respectively. Further, these sets of circuits permit one additional 4-circuit in each case. If we let $M_{9,3}^+$ be the matroid with all the 4-circuits of $M_{9,3}$ and also $\{x_3, x_5, x_6, x_7\}$, we find another example. Similarly, we get a fourth example from a matroid with all the 4-circuits $M_{9,4}$ together with $\{x_3, x_4, x_6, x_7\}$, which we label $M_{9,4}^+$. Evidently, $M_{9,4}^+$ is distinct from $M_{9,3}^+$, as they have all 4-circuits in common except one. No further 4-circuits may be added to these latter examples without contradicting either connectivity or orthogonality.
Subcase 4.28.2.1.2. Suppose $x_1 \notin C_4$.

This implies that $C_4$ contains $x_6$. Now, $C_4$ must also contain one of $x_4$ and $x_5$ in order to avoid an orthogonality contradiction with $D_3$. These elements are not symmetric; however, the two choices of $C_4$ produce sets of 4-circuits that are symmetric under the automorphism given by the permutation $(x_1, x_6)(x_2, x_3)(x_4, x_8)(x_5, x_9)$. Therefore, we may assume $C_4 = \{x_3, x_4, x_6, x_7\}$. With this, we have a matroid satisfying all assumptions, which we label $M_{9,5}$. This structure admits one further possible 4-circuit. Note that such a 4-circuit, say $C_5$, must not contain $\{x_1, x_6\}$, as, in order to avoid an orthogonality contradiction with $D_1$, such a circuit must also contain one of $x_8$ and $x_9$, but then would meet either $C_1$ or $C_3$ in three elements. Also, $C_5$ cannot contain $\{x_1, x_7\}$, as then $C_5 = \{x_1, x_3, x_7, x_9\}$ is forced, and we addressed this circuit in the previous case. Therefore, $x_1 \notin C_5$, and $C_5$ must contain both $x_6$ and $x_7$. As with $C_4$, this implies that one element from each $\{x_2, x_3\}$ and $\{x_4, x_5\}$ is in $C_5$. Therefore, since $C_5$ does not have three common elements with $C_4$, we get $C_5 = \{x_2, x_5, x_6, x_7\}$. The addition of this 4-circuit to the set of circuits produces a second matroid satisfying our hypotheses; however, the resulting matroid is isomorphic to $M_{9,3}$ under the automorphism given by the permutation $(1, 6)(4, 9)(5, 8)$. Therefore, we close this case having found one additional matroid.

Subcase 4.28.2.2. $x_1$ and $x_8$ do not appear together again in any further 4-circuits of $M$.

We may generalize this assumption and say that each pair of elements, one from $\{x_1, x_6, x_7\}$ and the other from $E(M) - \{x_1, x_6, x_7\}$, appears at most once in a 4-circuit outside of the local $U_{3,5}$ structures of $M$. As a reminder to the reader, we still have $C_1 = \{x_1, x_4, x_6, x_8\}$ in this case. Let $C_2$ be a 4-circuit containing $\{x_2, x_7\}$. It must be that one of $x_1$ and $x_6$ is in $C_2$, and these elements are symmetric. Therefore, we may assume $x_1 \in C_2$. In order to not contradict orthogonality, one of $x_8$ and $x_9$ must be in $C_2$. By our case assumption, $C_2$ must contain $x_9$. Therefore, $C_2 = \{x_1, x_2, x_7, x_9\}$. There must also be a 4-circuit, $C_3$, containing
\{x_5, x_6\}. Again, in order to avoid an orthogonality contradiction with \(D_1\), one element from \{x_7, x_8, x_9\} must be in \(C_3\). Therefore it must be that \(x_7 \in C_3\), since, if not, then \(x_1 \in C_3\), in which case \(C_3\) may contain neither \(x_8\) nor \(x_9\) by the case assumption. Further, in order to avoid contradicting orthogonality with \(D_2\), one of \(x_2\) and \(x_3\) must be in \(C_3\). Again, by our case assumption, this implies \(C_3 = \{x_3, x_5, x_6, x_7\}\), since \(x_2\) and \(x_7\) appear together in \(C_2\).

This gives a set of 4-circuits that satisfy all our assumptions. Label the matroid with these 4-circuits \(M_{9,6}\). It is easy to see that our condition on this case prohibits the addition of any further 4-circuits to this list. Thus our analysis of 9-element matroids is complete.

\[\square\]

4.5 When \(M\) Has Exactly Ten Elements

This section closely resembles the nine-element case in the organization of its arguments. We begin by determining the rank of a ten-element matroid with property \((P2)\), and proceed to show that it cannot have two disjoint 4-cocircuits. We then restrict the structure of the complements of 4-cocircuits, and finally prove that the only matroid with property \((P2)\) on ten elements is the well-known \(R_{10}\).

**Lemma 4.29.** If \(|E(M)| = 10\), then \(r(M) = 5\).

*Proof.* Clearly \(4 \leq r(M) \leq 6\), as \(M\) is 4-connected.

If \(r(M) = 4\), then the complement of any 4-cocircuit is a 6-point plane. There must be a 4-cocircuit using an element of that plane. Such a cocircuit must be contained in that plane in order to avoid an orthogonality contradiction. But then that 4-cocircuit is also a 4-circuit, a contradiction. Thus \(r(M) \neq 4\).

The case in which \(r(M) = 6\) leads to contradiction by a similar dual argument. \[\square\]

**Lemma 4.30.** If \(|E(M)| = 10\), then \(M\) has no two disjoint 4-cocircuits.
Proof. Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$. Suppose the lemma fails, and let $D_1$ and $D_2$ be disjoint 4-cocircuits of $M$. We may assume $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$. By Proposition 4.6, we know $M|(D_1 \cup D_2) \cong M(K_{2,4})$. Without loss of generality, let $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the pairs that appear together in the 4-circuits of $M|(D_1 \cup D_2)$.

The rank of each 6-element set comprised of three of the aforementioned pairs is 4, therefore making it a hyperplane. This gives us cocircuits $D_3 = \{x_1, x_5, x_9, x_{10}\}$, $D_4 = \{x_2, x_6, x_9, x_{10}\}$, $D_5 = \{x_3, x_7, x_9, x_{10}\}$, and $D_6 = \{x_4, x_8, x_9, x_{10}\}$. Consider a 4-circuit, $C$, containing $x_1$ and $x_9$. In order to avoid an orthogonality contradiction with one of these four 4-cocircuits, it must be that $x_{10} \in C$. In order to avoid a similar contradiction with $D_1$, we may assume, without loss of generality, that $x_2 \in C$. Circuit elimination on $D_5$ and $D_6$ indicates that there is a cocircuit contained in $\{x_3, x_4, x_5, x_7, x_8\} = (D_5 \cup D_6) - x_6$. This cannot contain $x_5$, otherwise we get an orthogonality contradiction with $C$. But then $\{x_3, x_4, x_7, x_8\}$ is both a circuit and a cocircuit, contradicting the 4-connectivity of $M$. Thus $M$ has no two disjoint 4-cocircuits.

In a simple matroid, we say that a point is *doubled* if that element is replaced by two elements in parallel.

**Lemma 4.31.** Suppose $|E(M)| = 10$. If $X$ is the complement of a 4-cocircuit of $M$, then $(M|X)^* \cong T^2$, where $T^2$ is the matroid $U_{2,3}$ with every point doubled.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$, and consider a 4-cocircuit $D_1 = \{x_7, x_8, x_9, x_{10}\}$. If $X_1 = E(M) - D_1$, then by Lemma 4.29 we have $r^*(M|X_1) = 2$. Further, the smallest cocircuits of $(M|X_1)^*$ have four elements. As possibilities for $(M|X_1)^*$, there are five rank-2 6-element matroids with cocircuits having at least 4 elements: $U_{2,6}$, $U_{2,5} \oplus U_{0,1}$, $U_{2,5}$ with one point doubled, $U_{2,4}$ with two points doubled, and $T^2$. We address these cases in order.

**Case 4.31.1.** Suppose $(M|X_1)^* \cong U_{2,6}$. 

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Now there are no 4-circuits contained in \( X_1 \). Every pair of elements of \( X_1 \) is in some 4-circuit. Each such 4-circuit must contain two elements of \( D_1 \) to avoid a contradiction to orthogonality. There are 15 distinct pairs in \( X_1 \) and only 6 distinct pairs in \( D_1 \). Therefore there are two 4-circuits, say \( C_1 \) and \( C_2 \) such that \( C_1 \cap D_1 = C_2 \cap D_1 \). Let \( e \) be one of the elements in \( C_1 \cap D_1 \). Then the circuit contained in \( (C_1 \cup C_2) - e \) is fully contained in \( X \), a contradiction. Therefore \( (M|X_1)^* \not\cong U_{2,6} \).

**Case 4.31.2.** Suppose \( (M|X_1)^* \cong U_{2,5} \oplus U_{0,1} \).

Now \( M|X_1 \cong U_{3,5} \oplus U_{1,1} \). Suppose \( x_1 \) is the element corresponding to the \( U_{1,1} \)-component. There must be a 4-cocircuit, \( D_2 \), containing \( x_1 \). This cocircuit must contain some element of \( D_1 \) by Lemma 4.30, but must also contain some element of \( X_1 - x_1 \), by Lemma 4.2. But every 4-element subset of \( X_1 - x_1 \) is a circuit, so, in order to avoid an orthogonality contradiction, any cocircuit meeting \( X_1 - x_1 \) must do so in at least three elements. This requires \( D_2 \) to have at least five elements, a contradiction. Therefore \( (M|X_1)^* \not\cong U_{2,5} \oplus U_{0,1} \).

**Case 4.31.3.** Suppose \( (M|X_1)^* \) is isomorphic to \( U_{2,5} \) with one point doubled.

In this case, \( (M|X_1) \cong U_{3,5} \oplus_2 U_{1,3} \). Suppose \( \{x_1, x_2, x_3, x_4\} \) corresponds to the \( U_{3,4} \)-component of \( M|X_1 \); then \( \{x_5, x_6\} \) corresponds to the \( U_{1,2} \)-component. There is a 4-circuit containing \( \{x_i, x_j\} \) for every pair with \( i \in \{1, 2, 3, 4\} \), and \( j \in \{5, 6\} \). These 4-circuits cannot be contained in \( X_1 \), and so must have two elements from \( D_1 \). There are eight such pairs from \( X_1 \), and only six distinct pairs of elements from \( D_1 \), so, again, some pair from \( D_1 \) must occur twice in these 4-circuits. This leads to a contradiction as in the first case.

**Case 4.31.4.** Suppose \( (M|X_1)^* \) be isomorphic to \( U_{2,4} \) with two points doubled.

This is the most lengthy case, and will require several subcases of analysis. We know \( M|X_1 \) is isomorphic to \( U_{2,4} \) with two copies of \( U_{2,3} \) 2-summed at different points. Thus, there are exactly two 4-circuits contained in \( X_1 \), and these share two elements. Without loss of
generality, we may assume those circuits are $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_3, x_4, x_5, x_6\}$.

Consider a 4-cocircuit, $D_2$, containing $x_5$.

**Subcase 4.31.4.1.** Assume $x_5$ and $x_6$ do not appear together in a 4-cocircuit.

If $x_6 \notin D_2$, then $D_2$ must have one element from each of $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Without loss of generality, say $\{x_1, x_3\} \subseteq D_2$. By Lemma 4.30, $D_1$ and $D_2$ must share an element; therefore, we may assume $D_2 = \{x_1, x_3, x_5, x_7\}$. Consider, then, the 4-circuits $C_3$ and $C_4$ containing $\{x_1, x_4\}$ and $\{x_2, x_3\}$, respectively. These circuits are not contained in $X_1$, and therefore must have two elements from $D_1$. In order to avoid an orthogonality contradiction with $D_2$, both $C_3$ and $C_4$ must contain $x_7$. Therefore, we may assume $C_3 = \{x_1, x_4, x_7, x_8\}$ and $C_4 = \{x_2, x_3, x_7, x_9\}$. Then, consider a 4-circuit, $C_5$, containing $x_2$ and $x_4$. Again, $C_5$ must have two elements from $D_2$, and, evidently, $x_7 \notin C_5$. If $C_5 = \{x_2, x_4, x_8, x_9\}$, then $r(\{x_1, x_2, x_3, x_4, x_7, x_8, x_9\}) = 4$, and $M$ has a 3-cocircuit, a contradiction. Therefore, without loss of generality, $C_5 = \{x_2, x_4, x_8, x_9\}$. In this case, $r(C_3 \cup C_5) = 4$, and $\{x_3, x_5, x_6, x_9\} = E(M) - (C_3 \cup C_5)$ is a cocircuit containing both $x_5$ and $x_6$, a contradiction. We now know that we may assume that

**Subcase 4.31.4.2.** $D_2$ contains $\{x_5, x_6\}$.

From here we are able to systematically determine all circuits and cocircuits of $M$ until we arrive at a contradiction. Circuit elimination on $C_1$ and $C_2$ indicates that both $C' = \{x_1, x_2, x_3, x_4, x_5\}$ and $C'' = \{x_1, x_2, x_3, x_4, x_6\}$ are circuits, so $D_2$ must contain an element from $\{x_1, x_2, x_3, x_4\}$. Without loss of generality, we may assume $D_2 = \{x_1, x_5, x_6, x_7\}$. We now suppose $C_3$ and $C_4$ are 4-circuits containing $\{x_1, x_3\}$ and $\{x_1, x_4\}$, respectively. Each of these must contain $x_7$, otherwise we get an orthogonality contradiction. Therefore, it suffices to let $C_3 = \{x_1, x_3, x_7, x_8\}$ and $C_4 = \{x_1, x_4, x_7, x_9\}$. Then $r(C_3 \cup C_4) = 4$, and $D_3 = \{x_2, x_5, x_6, x_{10}\} = E(M) - (C_3 \cup C_4)$ is a cocircuit. Circuit elimination on $D_2$ and $D_3$ produces an additional 4-cocircuit, $D_4 = \{x_1, x_2, x_7, x_{10}\}$. We also get a 4-circuit $C_5 \subseteq
(C_3 \cup C_4) - x_1$. This circuit cannot contain $x_7$, otherwise it violates orthogonality with $D_2$, so $C_5 = \{x_3, x_4, x_8, x_9\}$.

Now we may determine 4-circuits $C_6$ and $C_7$ containing $\{x_2, x_3\}$ and $\{x_2, x_4\}$, respectively. Neither of these may contain $x_7$, otherwise a pair of elements from $D_1$ is shared by at least two 4-circuits and we may find an extra 4-circuit in $X_1$, a contradiction as in the previous cases. Therefore, $x_{10}$ is in both of these circuits. This presents two possibilities: either $x_8 \in C_6$ and $x_9 \in C_7$, or vice versa. In the former case, $r(C_3 \cup C_6) = 4 = r(C_4 \cup C_7)$, so 
\[\{x_4, x_5, x_6, x_9\} = E(M) - (C_3 \cup C_6)\] and 
\[\{x_3, x_5, x_6, x_8\} = E(M) - (C_4 \cup C_7)\] are cocircuits.
This is a contradiction to Lemma 4.10, as $\{x_5, x_6\} = D_2 \cap D_3$. Therefore, we get $C_6 = \{x_2, x_3, x_9, x_{10}\}$ and $C_7 = \{x_2, x_4, x_8, x_{10}\}$.

Consider, now, a 4-cocircuit $D_5$ containing $x_3$. If $x_1 \not\in D_5$, we may assume $x_2 \not\in D_5$, by the symmetry of these elements under the automorphism given by the permutation $(x_1, x_2)(x_7, x_{10})(x_8, x_9)$. In this case, $D_5$ must contain an element from each of $\{x_4, x_5\}$, $\{x_4, x_6\}$, $\{x_7, x_8\}$, and $\{x_9, x_{10}\}$ in order to avoid an orthogonality contradiction with $C'$, $C''$, $C_3$, and $C_6$, respectively. Therefore $x_4 \in D_5$. Also, as $M$ has no two disjoint 4-cocircuits, $D_5$ must contain one of $x_7$ and $x_{10}$ so as to meet $D_4$. This forces $D_5 = \{x_3, x_4, x_7, x_{10}\}$.

The last 4-circuit we will determine is $C_8$, containing $\{x_5, x_8\}$. If $x_{10} \in C_8$, then $C_8 = \{x_1, x_5, x_8, x_{10}\}$, to avoid orthogonality contradictions with $D_2$ and $D_4$. But then, circuit elimination with $C_7$ and $C_8$ forces $\{x_1, x_2, x_4, x_5\} \subseteq X_1$ to be a circuit, a contradiction. Therefore, it must be that $x_{10} \not\in C_8$. Note, also, that $x_2 \not\in C_8$, as then $C_8$ cannot avoid meeting one of $D_1$, $D_2$, or $D_5$ in a single element. Therefore, in order to avoid an orthogonality contradiction with $D_3$, we have $x_6 \in C_8$. This forces $x_9 \in C_8$, otherwise we get a similar contradiction with one of $D_1$ or $D_5$. Therefore $C_8 = \{x_5, x_6, x_8, x_{9}\}$. Then, $r(C_1 \cup C_8) = 4$, which implies $\{x_3, x_4, x_7, x_{10}\}$ is a cocircuit. This contradicts Lemma 4.10, as $\{x_7, x_{10}\} = D_1 \cap D_4$. Thus $(M|X_1)^* \not\cong T^2$, as desired. \[\square\]
We now have all the tools necessary to determine the lone matroid on ten elements.

**Proposition 4.32.** Suppose $M$ is a 4-connected matroid. If $M$ has every element in a 4-cocircuit and every pair of elements in a 4-circuit, and $|E(M)| = 10$, then $M \cong R_{10}$.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$, and suppose $D_1 = \{x_7, x_8, x_9, x_{10}\}$ is a cocircuit of $M$. Then, by Lemma 4.31, $(M\setminus D_1)^* \cong T^2$, and we get circuits $C_1 = \{x_1, x_2, x_3, x_4\}$, $C_2 = \{x_1, x_2, x_5, x_6\}$, and $C_3 = \{x_3, x_4, x_5, x_6\}$, without loss of generality. Further, a 4-cocircuit, $D_2$, containing $x_1$ may be assumed to be $D_2 = \{x_1, x_2, x_7, x_8\}$ by orthogonality. Then $(M\setminus D_2)^* \cong T^2$. The elements $x_9$ and $x_{10}$ either appear together or not at all in all the 4-circuits contained in $M\setminus D_2$. One of these 4-circuits is $C_3$. If we let $C_4$ and $C_5$ be the other two 4-circuits, we get two possibilities: either $C_4 = \{x_3, x_4, x_9, x_{10}\}$ and $C_5 = \{x_5, x_6, x_9, x_{10}\}$, or, without loss of generality, $C_4 = \{x_3, x_5, x_9, x_{10}\}$ and $C_5 = \{x_4, x_6, x_9, x_{10}\}$. In the first case, $r(C_1 \cup C_4) = 4$, so $\{x_5, x_6, x_7, x_8\} = E(M) - (C_1 \cup C_4)$ is a cocircuit, a contradiction to Lemma 4.10 as $\{x_7, x_8\} = D_1 \cap D_2$.

![Figure 4.19: A forbidden configuration of 4-circuits when |E(M)| = 10.](image)

Therefore, we get the circuits in the latter case, and rule out the previous configuration, depicted in Figure 4.19 in all further instances when two 4-cocircuits share two elements. That is, we need only consider matroids that do not have the as a restriction the matroid depicted in Figure 4.19. Consider a 4-cocircuit, $D_3$, containing $x_3$.

**Claim 4.32.1.** $M$ has a 4-cocircuit containing $\{x_3, x_4\}$
Suppose not. The elements $x_4$ and $x_6$ are symmetric under the automorphism given by the permutation $(x_1, x_9)(x_2, x_{10})(x_4, x_6)$, so we assume further that there is no 4-cocircuit containing $\{x_3, x_6\}$. Then, in order to avoid an orthogonality contradiction with $C_3$, we get $x_5 \in D_3$. Additionally, $D_3$ must have one element from each $\{x_1, x_2\}$ and $\{x_9, x_0\}$ to avoid an orthogonality contradiction with $C_1$ and $C_4$, respectively. The elements within each of these pairs are symmetric, so we may assume $D_3 = \{x_1, x_3, x_5, x_9\}$. Then $(M \setminus D_3)^* \cong T^2$.

In order to avoid orthogonality contradictions with $D_1$ and $D_2$, the 4-circuits contained in $M \setminus D_3$ must always contain two elements of $\{x_2, x_7, x_8\}$ and $\{x_7, x_8, x_{10}\}$. Therefore, without loss of generality, we get circuits $C_6 = \{x_2, x_4, x_7, x_{10}\}$, $C_7 = \{x_2, x_6, x_8, x_{10}\}$, and $C_8 = \{x_4, x_6, x_7, x_8\}$. But then, $r(C_2 \cup C_7) = 4$, and $\{x_3, x_4, x_7, x_9\} = E(M) - (C_2 \cup C_7)$ is a 4-cocircuit containing $\{x_3, x_4\}$, a contradiction. This proves our claim.

It must be, then, that there is a 4-cocircuit containing both $x_3$ and $x_4$. We may suppose $D_3$ is such a cocircuit. In order to avoid an orthogonality contradiction with $C_4$ or $C_5$, there must be a second element of each $C_4$ and $C_5$ in $D_3$. We argue that

Claim 4.32.2. $x_9 \in D_3$.

Suppose $x_5 \in D_3$. Then, as $M$ has no disjoint 4-cocircuits, $x_7 \in D_3$. But then $D_3$ only meets $C_5$ in one element, a contradiction. A similar argument shows $x_6 \not\in D_3$. Hence, we may assume $x_9 \in D_3$. The final element of $D_3$ must come from $D_2$, and cannot be either $x_1$ or $x_2$, otherwise $D_3$ violates orthogonality with $C_2$. Therefore, without loss of generality, $D_3 = \{x_3, x_4, x_7, x_9\}$.

This implies $(M \setminus D_3)^* \cong T^2$. Since $D_1$ and $D_3$ share two elements, we can argue as before on $D_1$ and $D_2$. Therefore, we may assume, without loss of generality, that the 4-circuits contained in $M \setminus D_3$ are $C_6 = \{x_1, x_5, x_8, x_{10}\}$, $C_7 = \{x_2, x_6, x_8, x_9\}$, and $C_2$. Now we know $r(C_5 \cup C_6) = 4$, so $D_4 = \{x_2, x_3, x_6, x_7\} = E(M) - (C_5 \cup C_6)$ is a 4-cocircuit. Similarly, $D_5 = \{x_1, x_4, x_5, x_7\} = E(M) - (C_4 \cup C_7)$ is a cocircuit. In turn, these force 4-circuits
$C_8 = \{x_1, x_4, x_8, x_9\}$ and $C_9 = \{x_2, x_3, x_8, x_9\}$ in the local $T^2$-structure of their complements. Then, $r(C_8 \cup C_9) = 4$, so $D_6 = \{x_5, x_6, x_7, x_{10}\}$.

Note that there is not yet a 4-circuit containing $x_1$ and $x_7$. Such a 4-circuit, say $C_{10}$, must contain a second element from each of $D_1, D_3, D_4$, and $D_6$, in order to avoid an orthogonality contradiction. Noting the automorphism given by the permutation $(x_3, x_6)(x_4, x_5)(x_9, x_{10})$, we may assume $C_{10} = \{x_1, x_3, x_7, x_{10}\}$. This last 4-circuit will allow us to determine a hyperplane, which determines a 4-cocircuit, which determines a local $T^2$, which, in turn, determines a further 4-circuit, which then allows this process to repeat until all 4-circuits and 4-cocircuits of $M$ are determined. These are all determined explicitly, and no further assumptions are necessary. In the list that follows, we maintain the convention that those sets labeled $D_i$ represent cocircuits, while those labeled $C_i$ represent circuits. We list these in the sequence that they may be determined, without further comment: $D_7 = \{x_5, x_6, x_8, x_9\}$, $C_{11} = \{x_2, x_4, x_7, x_{10}\}$, $D_8 = \{x_2, x_4, x_6, x_9\}$, $C_{12} = \{x_1, x_6, x_7, x_9\}$, $D_9 = \{x_2, x_4, x_6, x_9\}$, $C_{13} = \{x_3, x_5, x_7, x_8\}$, $D_{10} = \{x_1, x_3, x_6, x_8\}$, $C_{14} = \{x_2, x_5, x_7, x_9\}$, $D_{11} = \{x_1, x_3, x_5, x_9\}$, $C_{15} = \{x_4, x_6, x_7, x_8\}$. We also get $D_{12} = \{x_2, x_3, x_5, x_{10}\}$, $D_{13} = \{x_1, x_2, x_9, x_{10}\}$, $D_{14} = \{x_1, x_4, x_6, x_{10}\}$, and $D_{15} = \{x_3, x_4, x_8, x_{10}\}$.

\[
A = \begin{pmatrix}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

Figure 4.20: The matrix $A$.

This provides a complete list of all 4-circuits and 4-cocircuits of $M$. Let $M' = M(A)$ for the matrix $A$ in Figure 4.20. Evidently, $M' \cong R_{10}$. Let $\phi : E(M') \to E(M)$ be a map given by $\phi(y_i) = x_i$. Then $\phi$ is a weak map, and an application of Theorem 3.6 concludes our proof. \qed
4.6 When $M$ Has More Than 10 and Fewer Than 16 elements

In this section, we find only two examples: one when $M$ has 12 elements and the other when $M$ has 14 elements. We show, first, that $M$ cannot have exactly 11, 13, or 15 elements.

**Proposition 4.33.** If $M$ has property (P2), then $|E(M)| \neq 11$.

**Proof.** Let $E(M) = \{x_1, x_2, \ldots, x_{11}\}$. By Proposition 4.16, we know $M$ has two disjoint 4-cocircuits. We may assume $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$ are those cocircuits. By Proposition 4.6, we have $M|(D_1 \cup D_2) \cong M(K_{2,4})$. We may assume that $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ are the series pairs in $M|(D_1 \cup D_2)$. By (P2), there is a 4-cocircuit $D_3$ that contains $x_9$. By orthogonality, $|D_3 \cap (D_1 \cup D_2)| = 2$, and so we may assume $D_3 = \{x_1, x_5, x_9, x_{11}\}$. Similarly, there is a 4-cocircuit $D_4$ containing $x_{10}$. By Proposition 4.2, we know $\{x_1, x_5\} \not\subseteq D_4$; therefore $D_4 = \{x_2, x_6, x_{10}, x_{11}\}$. Now the basic structure of the 4-cocircuits has been determined, and is depicted in Figure 4.21. Next, we will show that

**Claim 4.33.1.** $r(M) = 5$.

Since $M|(D_1 \cup D_2) \cong M(K_{2,4})$ and $r(M(K_{2,4})) = 5$, we know that $r(M) \geq 5$. If $r(M) > 5$, then $D_1 \cup D_2$ is contained in a hyperplane $H$ of $M$. But then $M$ has a cocircuit of size $|E(M) - H| \leq 3$, a contradiction to 4-connectivity. Therefore $r(M) = 5$.

![Figure 4.21: A set diagram of the 4-cocircuits in $M$.](image-url)
We need two additional 4-circuits in order to produce a contradiction. By \((P2)\), we have a 4-circuit \(C_1\) containing \(\{x_9, x_{11}\}\). By orthogonality, \(|C_1 \cap D_4| = 2\), so we may assume \(x_6 \in C_1\). Similarly, \(|C_1 \cap D_2| = 2\), so without loss of generality, either \(x_5 \in C_1\) or \(x_7 \in C_1\). We assert that, possibly with some relabeling,

**Claim 4.33.2.** \(\{x_6, x_7, x_9, x_{11}\}\) is a circuit.

Note that, by symmetry, we are satisfied to find a 4-cocircuit containing \(x_{11}\) that meets both \(\{x_9, x_{10}\}\) and \(\{x_3, x_4, x_7, x_8\}\). Therefore, \(C_1 = \{x_5, x_6, x_9, x_{11}\}\). By circuit elimination with \(\{x_1, x_2, x_5, x_6\}\), we get that \(C_2 = \{x_1, x_2, x_9, x_{11}\}\) is a circuit. Similarly, there is a 4-circuit \(C_3\) containing \(\{x_{10}, x_{11}\}\). Applying the same reasoning as before, we may assume that \(C_3 = \{x_5, x_6, x_{10}, x_{11}\}\), without loss of generality. But now there must be a circuit \(C_4 \subseteq (C_1 \cup C_3) - x_5 = \{x_6, x_9, x_{10}, x_{11}\}\). By orthogonality, \(x_6 \notin C_4\), but then \(|C_4| = 3\), a contradiction. Therefore the claim holds.

Now we may assume \(C_1 = \{x_6, x_7, x_9, x_{11}\}\) is a circuit. Given that \(D_3\) is a cocircuit, we have \(r(M \setminus D_3) = 4\). Therefore, since \(\{x_2, x_6, x_7, x_8\}\) is independent in \(M\) by orthogonality, it must be that \(\{x_2, x_6, x_7, x_8\}\) spans \(M \setminus D_3\). Hence there is a circuit \(C_2\) contained in \(\{x_2, x_6, x_7, x_8\} \cup \{x_{10}\}\) that must contain \(x_{10}\). By orthogonality, \(x_2 \notin C_2\), so \(C_2 = \{x_6, x_7, x_8, x_{10}\}\).

By circuit elimination, there is a cocircuit \(D' \subseteq (D_1 \cup D_4) - \{x_2\} = \{x_1, x_3, x_4, x_6, x_{10}, x_{11}\}\). We know \(|D'| \geq 4\); therefore, \(D'\) meets \(\{x_1, x_3, x_4, x_6\}\). Given the circuits in \(M \setminus (D_1 \cup D_2)\), it must be that \(\{x_1, x_3, x_4, x_6\} \subseteq D'\), by orthogonality. Further, since \(x_6\) is in \(D'\), so too must \(x_{10}\) and \(x_{11}\) be, by orthogonality with \(C_2\) and \(C_1\), respectively. Hence \(D' = \{x_1, x_3, x_4, x_6, x_{10}, x_{11}\}\), and so \(E(M) - D' = \{x_2, x_5, x_7, x_8, x_9\}\) is a circuit hyperplane. This circuit-hyperplane violates orthogonality with \(D_1\), and this contradiction proves the proposition.

\(\square\)

**Proposition 4.34.** If \(M\) has property \((P2)\), then \(|E(M)| \neq 13\) and \(|E(M)| \neq 15\).
Proof. In both cases, we may assume $M$ has three pairwise-disjoint 4-cocircuits, $D_1$, $D_2$, and $D_3$, forming a local $K_{3,4}$-structure, by Proposition 4.19 and Lemma 4.17. Let the elements these sets be $\{x_1, x_2, x_3, x_4\}$, $\{x_5, x_6, x_7, x_8\}$, and $\{x_9, x_{10}, x_{11}, x_{12}\}$, respectively. We may assume the circuits of $M\mid (D_1 \cup D_2 \cup D_3)$ are as they appear in Figure 4.22.

![Figure 4.22: The graph $K_{3,4}$ provides structure to $M\mid (D_1 \cup D_2 \cup D_3)$](image)

Assume $|E(M)| = 13$, and let $x_{13}$ be the element of $M$ not in $D_1$, $D_2$, or $D_3$. Then $x_{13}$ is in a 4-cocircuit, $D_4$, which must meet each of $D_1$, $D_2$, and $D_3$. Without loss of generality, $D_4 = \{x_1, x_5, x_9, x_{13}\}$. Consider a 4-circuit, $C_1$, containing $\{x_2, x_{13}\}$. This must contain a second element from each of $D_1$ and $D_4$, and is therefore disjoint from $D_2$ and $D_3$. Therefore, it suffices to assume $C_1 = \{x_1, x_2, x_3, x_{13}\}$. Similarly, a 4-circuit, $C_2$, containing $\{x_4, x_{13}\}$ must contain $x_1$ and one of $\{x_2, x_3\}$. But, then $D_1 \subseteq (C_1 \cup C_2) - \{x_{13}\}$ contains a circuit, a contradiction. Hence, $|E(M)| \neq 13$.

Assume, then, that $|E(M)| = 15$. Now, we have three elements not in $D_1$, $D_2$, or $D_3$, call them $x_{13}$, $x_{14}$, and $x_{15}$. Each of these is in a 4-cocircuit, which we may assume are $D_4 = \{x_1, x_5, x_9, x_{13}\}$, $D_5 = \{x_2, x_6, x_{10}, x_{14}\}$, and $D_4 = \{x_3, x_7, x_{11}, x_{15}\}$, respectively. Note that this implies $M\mid (D_4 \cup D_5 \cup D_6) \cong M(K_{3,4})$. Consider a 4-circuit, $C_1$, containing $x_4$ and $x_{13}$. In order to avoid an orthogonality contradiction, this circuit must contain a second element from each $D_1$ and $D_4$, and is therefore disjoint from all other 4-cocircuits. But every
element of $M$, save $x_1$, is in some other 4-cocircuit, a contradiction. Thus $|E(M)| \neq 15$, proving the proposition. 

Next, we introduce matroids on 12 and 14 elements, which we call $M_{12} \cong M(P)$ and $M_{14} \cong M(Q)$. The matrix entries are over $GF(4)$, where every element is its additive inverse and $\alpha^2 + \alpha + 1 = 0$. We proceed to prove that these are the unique matroids of their respective sizes with property $(P2)$. The proofs for each proposition are similar, although the 14-element case is more lengthy.

Next, we introduce matroids on 12 and 14 elements, which we call $M_{12} \cong M(P)$ and $M_{14} \cong M(Q)$. The matrix entries are over $GF(4)$, where every element is its additive inverse and $\alpha^2 + \alpha + 1 = 0$. We proceed to prove that these are the unique matroids of their respective sizes with property $(P2)$. The proofs for each proposition are similar, although the 14-element case is more lengthy.

\begin{figure}[h]
\centering
\begin{equation*}
P = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_9 & x_6 & x_7 & x_8 & x_{10} & x_{11} & x_{12} \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \alpha & 1 & \alpha \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{equation*}
\caption{The matrix $P$.}
\end{figure}

\begin{figure}[h]
\centering
\begin{equation*}
Q = \begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_9 & x_6 & x_7 & x_8 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \alpha & \alpha \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \alpha^2 & \alpha \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{equation*}
\caption{The matrix $Q$.}
\end{figure}

**Proposition 4.35.** Let $|E(M)| = 12$. Then $M$ is a 4-connected matroid in which every element is in a 4-cocircuit and every pair of elements in a 4-circuit if and only if $M \cong M_{12}$.

**Proof.** Clearly, $r(M(P)) = 5$. It is straightforward to verify that $M(P)$ is 4-connected and has property $(P2)$.

Now, suppose that $M$ satisfies the given conditions. By Corollary ??, $M$ has two pairs of disjoint 4-cocircuits, and $M$ restricted to either pair is isomorphic to $M(K_{2,4})$ by Proposition 4.6. Let $E(M) = \{x_1, x_2, \ldots, x_{12}\}$; then, without loss of generality, $M$ has cocircuits $D_1 =$
Without loss of generality, the circuits contained in $M|((D_1 \cup D_2)$ and $M|((D_3 \cup D_4)$ are given by Figure 4.25.

![Figure 4.25: The underlying $K_{2,4}$ structure in $M|((D_1 \cup D_2)$ and $M|((D_3 \cup D_4)$](image)

We first prove that

**Claim 4.35.1.** $r(M) = 5$.

Consider the set $S = E(M) - \{x_3, x_4, x_7, x_8\}$. As $\{x_3, x_4, x_7, x_8\}$ is a circuit, $S$ is a co-hyperplane. Further, $\text{cl}(\{x_1, x_2, x_5, x_9, x_{10}\}) = S$, so $r(S) \leq 5$. Since $M$ is 4-connected, $3 \leq \lambda_M(S) = r(S) + r^*(S) - |S| \leq 5 + r^*(S) - 8$. Therefore, $r^*(S) \geq 6$ and $r(M^*) \geq 7$, so $r(M) \leq 5$. Clearly $r(M) \geq 4$, so we must only prove $r(M) \neq 4$.

If $r(M) = 4$, then $B = \{x_3, x_7, x_9, x_{11}\}$ is a basis, as it cannot be a circuit by orthogonality. Consider the fundamental circuit $C(x_1, B)$. Neither $x_7$ nor $x_{11}$ may be in $C(x_1, B)$, as there are no other elements from $D_2$ or $D_4$. Therefore, $|C(x_1, B)| \leq 3$, a contradiction. Thus $r(M) = 5$.

There are eight elements that appear in only one of the given 4-cocircuits; namely, the members of $X = \{x_3, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$. Consider a 5-element subset of those elements that meets every given 4-cocircuit at least once. Note that such a subset must contain two elements from one 4-cocircuit, and one element from each other 4-cocircuit. Therefore, such a set must be a basis, by orthogonality. Every fundamental circuit that is given by one
of these bases together with an element \( e \) from \( E(M) - X \) must either be a 4-circuit or a 5-circuit. The number of elements in the fundamental circuit depends on whether \( e \) is contained in the same 4-cocircuit from which the basis has two elements; if so, then we get a 4-circuit, and if not, then it must be a 5-circuit. For example, if we choose \( B' = \{x_3, x_4, x_7, x_9, x_{11}\} \subseteq X \) to be our basis, then \( C(x_1, B') = \{x_1, x_3, x_4, x_9\} \), while \( C(x_5, B') = \{x_3, x_4, x_5, x_7, x_{11}\} \). It should be noted that some of the 4-circuits determined in this way intersect in 3 elements. Therefore, we get the following 5-point planes in \( M \): \( \{x_1, x_3, x_4, x_9, x_{10}\} \), \( \{x_5, x_7, x_8, x_9, x_{10}\} \), \( \{x_2, x_3, x_4, x_{11}, x_{12}\} \), and \( \{x_6, x_7, x_8, x_{11}, x_{12}\} \). We call the 12-element matroid with these 4-circuits \( M_{12} \), and proceed to prove its uniqueness.

Suppose there is some other 12-element matroid, say \( M' \), with every element in a 4-cocircuit and every pair of elements in a 4-circuit, and let \( M' \) share ground sets with \( M_{12} \). The circuits and cocircuits mentioned above are forced, so \( M' \) and \( M_{12} \) agree on those. As \( M' \not\cong M_{12} \), there must be a minimal subset, \( T \), such that \( T \) is independent in one and dependent in the other. Since \( r(M') = r(M_{12}) = 5 \), it must be that \( 4 \leq |T| \leq 5 \). If \( |T| = 4 \), then it must be independent in \( M_{12} \). The only 4-element independent sets in \( M_{12} \) meet at least one 4-cocircuit in a single element. Therefore, \( |T| = 5 \).

In this case, \( T \) is a circuit in one of \( M' \) or \( M_{12} \), and a basis in the other. A 5-circuit cannot be a subset of \( E(M) - X \), so, without loss of generality, \( x_1 \in T \). We prove next that

**Claim 4.35.2.** \( x_2 \in T \).

Suppose not. We may further assume that \( x_5 \not\in T \), as the permutation \((x_2, x_5)(x_3, x_9)(x_4, x_{10})\) is an automorphism of \( M \). Now, \( T \) must contain one element from each \( \{x_3, x_4\} \) and \( \{x_9, x_{10}\} \), in order to avoid an orthogonality contradiction with \( D_1 \) or \( D_3 \). These pairs are symmetric under the automorphism given by the permutation \((x_3, x_4)(x_7, x_8)(x_9, x_{10})(x_{11}, x_{12})\), so, without loss of generality, \( \{x_1, x_3, x_9\} \subseteq T \). As \( T \) is a basis in one matroid, it must meet every cocircuit of that matroid, and, as it is a circuit in the other matroid, it must do so in at least two elements. However, it does not yet
have an element from either $D_2$ or $D_4$, and there are only two undetermined elements; a contradiction.

Therefore, $x_2 \in T$. By similar reasoning, $T$ must also contain one of $x_5$ or $x_6$. These elements are symmetric under the permutation $(x_1, x_2)(x_5, x_6)(x_9, x_{11})(x_{10}, x_{12})$, so we may assume $x_5 \in T$. In order to avoid an orthogonality contradiction with one of $D_2$ and $D_4$, an element from each $\{x_6, x_7, x_8\}$ and $\{x_6, x_{11}, x_{12}\}$ must be in $T$. This element cannot be $x_6$, as $\{x_1, x_2, x_5, x_6\}$ is a circuit. Therefore, without loss of generality, $T = \{x_1, x_2, x_5, x_7, x_{11}\}$. But, by circuit elimination, there is a circuit $C \subseteq (T \cup \{x_1, x_2, x_5, x_6\}) - \{x_2\} = \{x_1, x_5, x_6, x_7, x_{11}\}$. Clearly $x_1 \not\in C$, as otherwise $|C \cap D_1| = 1$, but then $x_5 \not\in C$, otherwise $|C \cap D_3| = 1$. Therefore $|C| \leq 3$, a contradiction. Thus no such $T$ exists, and $M' \cong M_{12}$. Thus $M_{12} \cong M(P)$.

Proposition 4.36. Let $|E(M)| = 14$. Then $M$ is a 4-connected matroid in which every element is in a 4-cocircuit and every pair of elements in a 4-circuit if and only if $M \cong M_{14}$.

Proof. Clearly, $r(M(Q)) = 6$. It is straightforward to verify that $M(Q)$ is 4-connected and has property $(P2)$.

Now, suppose that $M$ satisfies the given conditions. By Proposition 4.19, $M$ has three pairwise-disjoint 4-cocircuits $D_1$, $D_2$, and $D_3$. By Lemma 4.17, we know that $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$. Let $E(M) = \{x_1, x_2, \ldots, x_{14}\}$. Without loss of generality, we may assume that $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $D_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$, and the circuits contained in $D_1 \cup D_2 \cup D_3$ are given by Figure 4.22.

Therefore, without loss of generality, we may assume there are 4-cocircuits containing $x_{13}$ and $x_{14}$ which are given by $D_4 = \{x_1, x_5, x_9, x_{13}\}$ and $D_5 = \{x_2, x_6, x_{10}, x_{14}\}$, respectively. As $D_4$ and $D_5$ are disjoint, $M|(D_4 \cup D_5) \cong M(K_{2,4})$, and, in order to avoid violating orthogonality, the pairs $\{x_1, x_2\}$, $\{x_5, x_6\}$, $\{x_9, x_{10}\}$, and $\{x_{13}, x_{14}\}$ always appear together in the 4-circuits contained in $D_4 \cup D_5$, as depicted in Figure 4.26.
Figure 4.26: This $K_{2,4}$ determines the circuits in $M|(D_4 \cup D_5)$.

We prove that

**Claim 4.36.1.** $r(M) = 6$.

Consider $X = (E(M) - \{x_3, x_4, x_7, x_8\})$. Observe that $\text{cl}(\{x_1, x_2, x_5, x_9, x_{11}, x_{13}\}) = X$, so $r(X) \leq 6$. As $M$ is 4-connected, $3 \leq \lambda_M(X) = r(X) + r^*(X) - |X| \leq r^*(X) - 4$, so $r^*(X) \geq 7$. As $X$ is a cohyperplane, this implies $r^*(M) \geq 8$, so $r(M) \leq 6$. Now consider a set, $Y$, with one element from each 4-cocircuit, such that none of the elements is in more than one 4-cocircuit. Clearly, $|Y| = 5$, and $Y$ is independent. Therefore $r(M) \leq 4$. If $r(M) = 5$, then $Y$ is a basis of $M$. In this case, consider the fundamental circuit $C(x_1, Y)$. Such a circuit must not have elements from $D_2, D_3,$ or $D_5$, otherwise it will violate orthogonality. Therefore $|C(x_1, Y)| \leq 3$, a contradiction. Thus $r(M) = 6$.

We now prove that this structure allows for exactly one matroid on 14 elements. Before we begin, note the following six 4-circuits that are, without loss of generality, necessarily in any 14-element matroid having property $(P2)$: $\{x_i, x_{i+2}, x_{i+3}, x_{13}\}$ and $\{x_j, x_{j+1}, x_{j+2}, x_{14}\}$ for each $i \in \{1, 5, 9\}$ and $j \in \{2, 6, 10\}$. To see that these must exist, consider a 4-circuit $C'$ containing $\{x_{i+2}, x_{13}\}$, and let $D' \in \{D_1, D_2, D_3\}$ be the 4-cocircuit that contains $x_{i+2}$. In order to avoid an orthogonality contradiction with $D_4$ or $D'$, we know $x_1 \in C'$. Then, the last element of $C'$ must also come from $D'$, and it cannot be $x_j$ by orthogonality. Therefore, $C' = \{x_i, x_{i+2}, x_{i+3}, x_{13}\}$. The other case holds similarly: simply swap $x_1$ with $x_2$, and replace
(x_i, x_{i+2}, x_{i+3}, x_{13}) by (x_j, x_{j+1}, x_{j+2}, x_{14}) in the above argument. With this in mind, suppose there are two such matroids, and call them M_{14} and M'. If M_{14} \not\cong M', then there is a minimal set T that is independent in one and dependent in the other. Therefore, 4 \leq |T| \leq 6. We treat each possibility in a separate case.

**Case 4.36.2.** Suppose |T| = 4.

As T is a circuit in one matroid, it must contain at least two elements from each 4-cocircuit it meets. Suppose one of \{x_1, x_2, x_5, x_6, x_9, x_{10}\} is not in T. Then neither x_{13} nor x_{14} are in T, and so T \subseteq \{x_3, x_4, x_7, x_8, x_{11}, x_{12}\}. However, we have accounted for all such 4-circuits in both matroids. Therefore, without loss of generality, x_1 \in T. In this case, T must also contain one element from each \{x_2, x_3, x_4\} and \{x_5, x_9, x_{13}\}. We can prove that

**Claim 4.36.2.1.** x_{13} \not\in T.

Suppose x_{13} \in T. If x_2 \in T as well, then T = \{x_1, x_2, x_{13}, x_{14}\}, which is a circuit in both M_{14} and M'. Therefore, x_2 \not\in T, in which case T = \{x_1, x_3, x_4, x_{13}\}. As noted above, this is a circuit in both matroids.

Therefore, x_{13} \not\in T, and, without loss of generality, x_5 \in T. In this case, T must contain an element from each of \{x_2, x_3, x_4\} and \{x_6, x_7, x_8\}, but then T \subseteq D_1 \cup D_2, and all such 4-circuits are forced. Thus |T| \neq 4.

**Case 4.36.3.** Suppose |T| = 5.

Again, T is a circuit in one matroid, and we may assume x_1 \in T.

**Subcase 4.36.3.1.** Suppose x_5 \in T.

In this case, T must contain an element from both \{x_2, x_3, x_4\} and \{x_6, x_7, x_8\} in order to avoid an orthogonality contradiction in whichever matroid T is a circuit.

We show next that x_2 is not in T. Assume the contrary. Then T also contains an element from \{x_6, x_{10}, x_{14}\}. This element cannot be x_6, as \{x_1, x_2, x_5, x_6\} is a circuit in both matroids.
Further, \( x_{10} \not\in T \), as then \( T \) must contain an element from \( \{x_9, x_{11}, x_{12}\} \). Then, without loss of generality, \( T = \{x_1, x_2, x_5, x_7, x_{14}\} \). In whichever matroid \( T \) is a circuit, the rank of \( T \) is 4. However, in both matroids \( \text{cl}(T) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{13}, x_{14}\} \), which is a hyperplane. As both matroids have rank 6, neither may have a rank-4 hyperplane, a contradiction.

We deduce that \( x_2 \) is not in \( T \), and, by symmetry, \( x_6 \) is not in \( T \). Hence, without loss of generality, we may assume \( x_3 \in T \). As \( \{x_1, x_3, x_5, x_7\} \) is a circuit in both matroids, \( x_7 \not\in T \), and so \( x_8 \in T \). The fifth element of \( T \) cannot come from \( D_3 \) or \( D_5 \), and \( x_4 \not\in T \) as \( \{x_1, x_4, x_5, x_8\} \) is a circuit in both matroids, so \( T = \{x_1, x_3, x_5, x_8, x_{13}\} \). As \( T \) is independent in one matroid, \( \text{cl}(T) \) is a hyperplane of that matroid. However, \( T \subseteq (E(M_{14}) - D_3) \cap (E(M_{14}) - D_5) \); that is, \( T \) is contained in the intersection of two hyperplanes. Thus \( r(T) \leq 4 \), a contradiction. We now know that

**Subcase 4.36.3.2.** \( x_5 \not\in T \)

By symmetry, we also have \( x_9 \not\in T \). In order to avoid an orthogonality contradiction with \( D_4 \), this implies \( x_{13} \in T \). There must be an element from \( \{x_2, x_3, x_4\} \) in \( T \).

We show next that \( x_2 \) is not in \( T \). Assume the contrary. Then \( T \) needs an element from \( \{x_6, x_{10}, x_{14}\} \). Suppose \( x_6 \in T \). In this case, \( T \) must contain an element from \( \{x_5, x_7, x_8\} \), and, since \( x_7 \) and \( x_8 \) are symmetric, we may assume \( T = \{x_1, x_2, x_6, x_7, x_{13}\} \). However, this is symmetric to the previous case in which \( T \) was equal to \( \{x_1, x_2, x_5, x_7, x_{14}\} \), under the automorphism given by the permutation \((x_1, x_2)(x_5, x_6)(x_9, x_{10})(x_{13}, x_{14})\). Therefore, if \( x_2 \in T \), then \( x_6 \not\in T \), and, by symmetry, neither is \( x_{10} \). It must be that \( x_{14} \in T \). But then \( \{x_1, x_2, x_{13}, x_{14}\} \), a circuit on both matroids, is a subset of \( T \), a contradiction. We conclude that \( x_2 \) is not in \( T \).

This implies, without loss of generality, that \( x_3 \in T \). Now, neither \( x_6 \) nor \( x_{10} \) may be in \( T \), as they are each contained in two 4-cocircuits disjoint from \( T \). Similarly, \( x_{14} \not\in T \), as then one of \( x_6 \) or \( x_{10} \) would be forced in order to avoid an orthogonality contradiction with \( D_5 \). Therefore,
the remaining members of $T$ must be either \( \{x_7, x_8\} \) or \( \{x_{11}, x_{12}\} \). These are symmetric under the automorphism given by the permutation \((x_5, x_9)(x_6, x_{10})(x_7, x_{11})(x_8, x_{12})\), so it suffices to assume $T = \{x_1, x_3, x_7, x_8, x_{13}\}$. However, as both $M_{14}$ and $M'$ contain circuits \( \{x_1, x_3, x_5, x_7\} \) and \( \{x_5, x_7, x_8, x_{13}\} \), we may see, by circuit elimination on these excluding $x_5$, that this $T$ is a circuit in both matroids. This last contradiction proves that $|T| \neq 5$.

We are left with one possibility.

**Case 4.36.4.** Suppose $|T| = 6$.

In this case, $T$ is a circuit in one of $M_{14}$ and $M'$, and a basis in the other. By similar reasoning to the previous cases, we may assume $x_1 \in T$. We will first show that

**Claim 4.36.4.1.** $T \cap \{x_5, x_9\} \neq \emptyset$.

If neither $x_5$ nor $x_9$ is in $T$, then $T$ must contain $x_{13}$. Furthermore, $T$ must contain a second element from $D_1$. As $T$ is a basis in one matroid, it must meet all the cocircuits of that matroid; specifically, it must meet both $D_2$ and $D_3$. However, as $T$ is a circuit in the other matroid, it must meet both $D_2$ and $D_3$ in at least two elements. Since $D_2$ and $D_3$ are disjoint, and there are only three unaccounted-for elements of $T$ in this case, this is a contradiction.

Therefore, $T$ must contain one of $x_5$ and $x_9$. Without loss of generality, say $x_5 \in T$. Now $T$ needs a second element from each $D_1$ and $D_2$.

**Claim 4.36.4.2.** $T \cap \{x_2, x_6\} \neq \emptyset$.

If neither $x_2$ nor $x_6$ is in $T$, then, without loss of generality, \( \{x_1, x_3, x_5, x_8\} \subseteq T \). In this case, $T$ must still meet both $D_3$ and $D_5$ in at least two elements, and $|D_3 \cap D_5| = 1$, a contradiction.

Now $T$ must contain one of $x_2$ and $x_6$. These are symmetric under the automorphism given by the permutation \((x_1, x_5)(x_2, x_6)(x_3, x_7)(x_4, x_8)\), so we may assume that $x_2 \in T$. As \( \{x_1, x_2, x_5, x_6\} \) is a circuit in both matroids, we may further assume that $x_7 \in T$, and
in order to avoid an orthogonality contradiction with $D_2$. Now $T$ must have a second element from $D_5$, and must meet $D_3$ in two elements. This forces $x_{10} \in T$, and the final element of $T$ is either $x_{11}$ or $x_{12}$. If $T = \{x_1, x_2, x_5, x_7, x_{10}, x_{11}\}$, then consider that the sets $\{x_1, x_2, x_5, x_6\}$ and $\{x_6, x_7, x_{10}, x_{11}\}$ are circuits in both matroids. Therefore $T = (\{x_1, x_2, x_5, x_6\} \cup \{x_6, x_7, x_{10}, x_{11}\}) - \{x_6\}$ is a circuit in both matroids, a contradiction. This brings us to the final possibility for $T$ in this case: $T = \{x_1, x_2, x_5, x_7, x_{10}, x_{12}\}$. By inspection of its closure, we see that $\text{cl}(T)$ is the entire matroid in both cases. Thus $T$ is a basis in both matroids, and $M_{14}$ is unique. Thus $M_{14} \cong M(Q)$.

4.7 The Main Result

This section will conclude our analysis of matroids having property $(P_2)$. Once the size of $M$ is at least 16, the matroids that satisfy our conditions all fall into one family; that is, $M \cong M(K_{4,n})$ for some $n \geq 4$. The proof of this is a straightforward application of induction on the number of elements, requiring two quick preceding lemmas.

**Lemma 4.37.** Let $M|X \cong M(K_{2,4})$ and $D$ be a 4-cocircuit of $M$ meeting $X$. Then either $D$ contains exactly one element from each of the four series pairs in $M|X$, or $D$ meets $X$ in a series pair of $M|X$.

**Proof.** Suppose not. Let $\{x_i, y_i\}$ for $i \in \{1, 2, 3, 4\}$ be the series pairs of $M|X$. Since $M$ is 4-connected, $D \cap X \neq \{x_i, y_i, x_j, y_j\}$ for every $\{i, j\} \subseteq \{1, 2, 3, 4\}$. Therefore, $D$ must meet some series pair in exactly one element, and another series pair not at all. Without loss of generality, $D \cap \{x_1, y_2\} = \{x_1\}$ and $D \cap \{x_2, y_2\} = \emptyset$. But $\{x_1, x_2, y_1, y_2\}$ is a circuit. This contradiction completes the proof of the lemma.

**Lemma 4.38.** If $|E(M)| \geq 16$, then $M$ has four pairwise-disjoint 4-cocircuits. Further, $|E(M)| = 4n$ for some $n \geq 4$, and $M$ may be partitioned into 4-cocircuits.
Proof. By Proposition 4.19, $M$ has three pairwise-disjoint 4-cocircuits, $D_1$, $D_2$, and $D_3$, forming a local $K_{3,4}$ structure. Let the elements in these sets be $\{x_1, x_2, x_3, x_4\}$, $\{x_5, x_6, x_7, x_8\}$, and $\{x_9, x_{10}, x_{11}, x_{12}\}$, respectively. Then, the circuits contained in $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$ are given by Figure 4.22.

Let $x_{13}$, $x_{14}$, $x_{15}$, and $x_{16}$ be distinct elements of $E(M) - (D_1 \cup D_2 \cup D_3)$. Each of these elements is in a 4-cocircuit. We may assume that each of these 4-cocircuits contains at least one element from $D_1 \cup D_2 \cup D_3$. As $M | (D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$, orthogonality forces each of these 4-cocircuits to contain three elements of $D_1 \cup D_2 \cup D_3$. Moreover, by Lemma 4.37, these three elements form a triad in $M | (D_1 \cup D_2 \cup D_3)$. It follows that two such 4-cocircuits are disjoint, otherwise they are forced to share three elements, a contradiction to Proposition 4.2. Therefore, $M$ has four disjoint 4-cocircuits.

It is clear, then, that when $|E(M)| = 16$, there is a partition of $E(M)$ into 4-cocircuits. Suppose $|E(M)| > 16$, and that $M$ cannot be partitioned into 4-cocircuits. Let $\{D_1, D_2, \ldots, D_k\}$ be a maximum-sized set of pairwise-disjoint 4-cocircuits of $M$. Then $k \geq 4$. Let $e$ be an element of $E(M) - (D_1 \cup D_2 \cup \cdots \cup D_k)$ and $D$ be a 4-cocircuit containing $e$. Then $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) \neq \emptyset$. But, by Lemma 4.37, $D$ must contain at least four elements from $D_1 \cup D_2 \cup D_3 \cup D_4$. This contradiction completes the proof of the lemma. 

\begin{proposition}
If $|E(M)| \geq 16$, then $M \cong M(K_{4,n})$ for some $n \geq 4$.
\end{proposition}

\begin{proof}
We argue by induction on $n$. Suppose $n = 4$ and let $D_1$, $D_2$, $D_3$, and $D_4$ be pairwise-disjoint 4-cocircuits of $M$. By Proposition 4.19, the restriction of $M$ to any three of these is isomorphic to $M(K_{3,4})$. Therefore, if $\phi : E(M(K_{4,4})) \to E(M)$ is a bijection that maps the 4-circuits and 4-cocircuits of $M(K_{4,4})$ to those of $M$, then $\phi$ is a weak map. By Theorem 3.6, this means $M(K_{4,4}) \cong M$.

Now, suppose that $|E(M)| = 4i$ implies $M \cong M(K_{4,i})$ for all $4 \leq i \leq m - 1$, and consider $M$ such that $|E(M)| = 4m$. If $M \not\cong M(K_{4,m})$, then there is a minimal set $Z$ that
is independent in one of these matroids and is a circuit in the other. As such, if $Z$ has one element from a 4-cocircuit of $M$, then it has at least two. Suppose $Z \cap D = \emptyset$ for some 4-cocircuit $D$ in the cocircuit partition of $E(M)$. Then $Z \subseteq M \setminus D$. But $M \setminus D$ has property $(P2)$ and $|E(M \setminus D)| \geq 16$, so, by the induction hypothesis, $Z$ must meet each of the 4-cocircuits that partitions $M$. As there are $m$ 4-cocircuits in the partition, we have $|Z| \geq 2m$. Also note that $r(M) = r(M \setminus D) + 1 = r(M(K_{4,m} - 1)) + 1 = m + 3$. Since $Z$ is assumed to be a circuit in one $M$ or $M(K_{4,m})$, we get $2m \leq |Z| \leq m + 4$. This inequality fails if $m > 4$. Thus the proposition follows by induction.
Putting it all together, we get the main theorem of this chapter.

**Theorem 4.40.** Suppose $M$ is a 4-connected matroid. If $M$ has every element in a 4-cocircuit and every pair of elements in a 4-circuit, then $M$ is one of the following matroids: $U_{3,6}$, $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,3}^+$, $M_{8,4}$, $M_{8,4}^+$, $M_{8,5}$, $M_{8,6}$, $M_{8,7}$, $M_{8,7}^+$, $M_{8,8}$, $M_{8,8}^+$, $M_{8,9}$, $M_{8,9}^+$, $M_{8,9}^+$, $M_{8,10}$, $M_{8,10}^+$, $M_{8,10}^{++}$, $M_{8,11}$, $M_{8,12}$, $F_7^+$, $M_{9,1}$, $M_{9,1}^+$, $M_{9,2}$, $M_{9,3}$, $M_{9,3}^+$, $M_{9,4}$, $M_{9,4}^+$, $M_{9,5}$, $M_{9,6}$, $R_{10}$, $M_{12}$, $M_{14}$ or $M(K_4,n)$ for some $n \geq 4$. 
References


Vita

Simon Pfeil was born and raised in Louisiana. He finished his undergraduate degree in mathematics at Southeastern Louisiana University in 2008. In August 2009, he came to Louisiana State University to pursue graduate studies in mathematics. He earned a Master of Science degree in mathematics from Louisiana State University in 2011. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2016.