3-13-2017

Exact solution of the mean-field plus separable pairing model reexamined

Feng Pan
Liaoning Normal University

Dan Zhou
Liaoning Normal University

Lianrong Dai
Liaoning Normal University

J. P. Draayer
Louisiana State University

Exact solution of the mean-field plus separable pairing model reexamined

Feng Pan, Dan Zhou, Lianrong Dai, and J. P. Draayer

Phys. Rev. C **95**, 034308 — Published 13 March 2017

DOI: 10.1103/PhysRevC.95.034308

Exact solution of the mean-field plus separable pairing model reexamined

Feng Pan*,[1,2] Dan Zhou,[1] Lianrong Dai,[1] and J. P. Draayer[2]

[1]*Department of Physics, Liaoning Normal University, Dalian 116029, China*
[2]*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA*

Exact solution of the nuclear mean-field plus separable pairing model is reexamined. New auxiliary constraints for solving the Bethe ansatz equations of the model are proposed. By using these auxiliary constraints, the Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian with non-degenerate single-particle energies and non-degenerate separable pairing strengths purposed previously is verified. Since the solutions of the model with one- and two-orbit cases are known, verification of the solutions for these two special cases is made. In order to demonstrate structure and features of the solution, the model with three orbits in the ds-shell is taken as a nontrivial example, of which two-pair results and the ground state of the three-pair case are provided explicitly. Since the number of equations involved increases with the number of orbits and pairs, to solve these equations for large number of orbits and pairs seems still difficult.

PACS numbers: 21.60.Fw, 03.65.Fd, 02.20.Qs, 02.30.Ik

I. INTRODUCTION

Pairing has been playing an important role in many branches of physics. In nuclear physics, pairing interaction is considered as one of important types of residual interactions in a nuclear mean-field to describe ground state and low-energy spectroscopic properties of nuclei, such as binding energies, odd-even effects, single-particle occupancies, excitation spectra, and moments of inertia, etc [1, 2]. It has been shown that either spherical or deformed mean-field plus the standard (orbit-independent) pairing interaction can be solved exactly by using the Gaudin-Richardson method [3–5]. The Gaudin-Richardson equations in this case can be solved more easily by using the extended Heine-Stieltjes polynomial approach [6–9]. The deformed and spherical mean-field plus the extended pairing models have also been proposed, which can be solved more easily than the standard pairing model, especially when both the number of valence nucleon pairs and the number of single-particle orbits are large [10, 11].

Exact solution of the separable pairing model with degenerate single-particle energy was proposed in [12], of which the solution is similar to that of the Gaudin-Richardson type for the standard pairing model. The separable pairing model with two non-degenerate orbits was analyzed in [13]. In [14–19], exact solution of a special family of the hyperbolic Richardson-Gaudin models was proposed, of which a special case related to the problem may also be derived based on the simple procedure shown in [20]. General non-degenerate cases were considered previously in [21, 22]. However, the auxiliary constraints used in [21, 22] are awkward and may be too specific though they can be used to provide solutions of the cases presented in [20].

In this work, the Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian with non-degenerate single-particle energies and non-degenerate separable pairing strengths purposed in [21, 22] is verified with the help of a set of new auxiliary constraints for solving the corresponding Bethe ansatz equations. Since solutions of the model for one- and two-orbit cases are well known, verification of the solutions for these two special cases will be made. In order to demonstrate structure and features of the solution, the model with three orbits in the ds-shell will be taken as a nontrivial example, of which two- and three-pair solutions will be provided explicitly.

II. THE MODEL AND ITS GENERAL SOLUTION

The Hamiltonian of the mean-field plus separable pairing model is given as [21, 22]

* The corresponding author's e-mail: daipan@dlut.edu.cn

$$\hat{H} = \sum_{t=1}^{p} \epsilon_{j_t} \hat{N}_{j_t} + \hat{H}_{\rm P} = \sum_{t=1}^{p} \epsilon_{j_t} \hat{N}_{j_t} - G \sum_{1 \le t, t' \le p} c_{j_t} c_{j_{t'}} S^+_{j_t} S^-_{j_{t'}}, \tag{1}$$

where p is the total number of orbits considered, $\{\epsilon_{j_t}\}$ $(t = 1, 2, \cdots, p)$ is a set of single-particle energies generated from any mean-field theory, such as those of the shell model, $\hat{N}_{j_t} = \sum_m a^\dagger_{j_t m} a_{j_t m}$ and $S^+_{j_t} = \sum_{m>0} (-1)^{j_t - m} a^\dagger_{j_t m} a^\dagger_{j_t -m}$, in which $a^\dagger_{j_t m_t}$ $(a_{j_t m})$ is the creation (annihilation) operator for a nucleon with angular momentum quantum number j_t and that of its projection m, G and $\{c_{j_t}\}$ $(t = 1, 2, \cdots, p)$ are the separable pairing interaction parameters, which are all assumed to be real. In order to avoid degeneracy, which will result in no solution from the procedure, $\epsilon_{j_t} \neq \epsilon_{j_{t'}}$ and $c_{j_t} \neq c_{j_{t'}}$ for $1 \le t, t' \le p$ are assumed in this work.

The set of operators $\{S^-_{j_t},\ S^+_{j_t},\ \hat{N}_{j_t}\}$ $(t = 1, 2, \cdots, p)$, where $S^-_{j_t} = (S^+_{j_t})^\dagger$, generate p copies of SU(2) algebra satisfying the commutation relations

$$[\hat{N}_{j_t}/2,\ S^-_{j_{t'}}] = -\delta_{tt'} S^-_{j_t},\ [\hat{N}_{j_t}/2,\ S^+_{j_{t'}}] = \delta_{tt'} S^+_{j_t},\ [S^+_{j_t},\ S^-_{j_{t'}}] = 2\delta_{tt'} S^0_{j_t}, \tag{2}$$

where $S^0_{j_t} = (\hat{N}_{j_t} - \Omega_{j_t})/2$ with $\Omega_{j_t} = j_t + 1/2$.

Let

$$S^+(x_\mu) = \sum_{t=1}^{p} \sum_{i=1}^{q} \frac{a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} c_{j_t} S^+_{j_t} = \sum_{i=1}^{q} a_i(x_\mu) S^+(x_{\mu,i}), \tag{3}$$

where $S^+(x_\mu) \equiv S^+(x_{\mu,1}, \cdots, x_{\mu,q})$, which will be frequently used to simplify the expression, depends on q variables $\{x_{\mu,1}, \cdots, x_{\mu,q}\}$, and

$$S^+(x_{\mu,i}) = \sum_{t=1}^{p} \frac{1}{c^2_{j_t} - x_{\mu,i}} c_{j_t} S^+_{j_t}, \tag{4}$$

$\{x_{\mu,i}\}$ and $\{a_i(x_\mu) \equiv a_i(x_{\mu,1}, \cdots, x_{\mu,q})\}$ $(i = 1,\ 2,\ \cdots, q)$ are two sets of parameters to be determined for a given μ, in which $a_i(x_\mu)$ also depends on the variables $\{x_{\mu,1}, \cdots, x_{\mu,q}\}$. According to the commutation relations given in (2), we have

$$[\sum_t \epsilon_{j_t} \hat{N}_{j_t},\ S^+(x_\mu)] = \sum_{i,t} \frac{2\epsilon_{j_t} a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} c_{j_t} S^+_{j_t}, \tag{5}$$

which, now, is imposed with the following constraints:

$$\sum_{i,t} \frac{2\epsilon_{j_t} a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} c_{j_t} S^+_{j_t} = \sum_t c_{j_t} S^+_{j_t} + \beta(x_\mu) S^+(x_\mu), \tag{6}$$

where $\beta(x_\mu) \equiv \beta(x_{\mu,1}, \cdots, x_{\mu,q})$ may depend on $\{x_{\mu,1}, \cdots, x_{\mu,q}\}$. (6) can be expressed alternatively as

$$\sum_{i=1}^{q} \frac{\left(2\epsilon_{j_t} - \beta(x_\mu)\right) a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} = 1 \quad \text{for} \ \ t = 1,\ 2,\ \cdots,\ p. \tag{7}$$

It is clearly shown that the pairing operator given in (3) is the same as that used in the separable pairing model [20–22] when the constraint (7) is used. Namely,

$$S^+(x_\mu) = \sum_{t=1}^{p} \sum_{i=1}^{q} \frac{a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} c_{j_t} S^+_{j_t} = \sum_{t=1}^{p} \frac{c_{j_t}}{2\epsilon_{j_t} - \beta(x_\mu)} S^+_{j_t}, \tag{8}$$

where $\beta(x_\mu) \equiv \beta(x_{\mu,1}, \cdots, x_{\mu,q})$ is related to eigen-energy of model.

Practically, we can use (7) to get expressions of $a_i(x_\mu)$ with $i = 1, \cdots, q$ and other $p - q \geq 0$ relations among $\beta(x_\mu)$ and $\{x_{\mu,1}, \cdots, x_{\mu,q}\}$ for a given μ. Moreover,

$$[\hat{H}_{\rm P},\ S^+(x_\mu)] = G\sum_{t'=1}^{p} c_{j_{t'}} S^+_{j_{t'}} \sum_{i=1}^{q}\sum_{t=1}^{p} \frac{2S^0_{j_t} a_i(x_\mu)\left(c_{j_t}\right)^2}{c^2_{j_t} - x_{\mu,i}} = G\sum_{t'=1}^{p} c_{j_{t'}} S^+_{j_{t'}} \Lambda_0(x_\mu), \tag{9}$$

where

$$\Lambda_0(x_\mu) = \sum_{i=1}^{q}\sum_{t=1}^{p} \frac{2S^0_{j_t}\left(c_{j_t}\right)^2 a_i(x_\mu)}{c^2_{j_t} - x_{\mu,i}} = \sum_{t=1}^{p} \frac{2S^0_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_\mu)}, \tag{10}$$

and

$$\begin{aligned} S^+(x_\mu, x_\nu) \equiv S^+(x_{\mu,1}, x_{\mu,2}, \cdots, x_{\mu,q}, x_{\nu,1}, x_{\nu,2}, \cdots, x_{\nu,q}) = [[\hat{H}_{\rm P},\ S^+(x_\mu)],\ S^+(x_\nu)] = \\ 2G\sum_{t'} c_{j_{t'}} S^+_{j_{t'}} \sum_{ii'} a_i(x_\mu) a_{i'}(x_\nu) \sum_t \frac{\left(c_{j_t}\right)^2}{(c^2_{j_t} - x_{\mu,i})(c^2_{j_t} - x_{\nu,i'})} c_{j_t} S^+_{j_t} = \\ 2G\sum_{t'} c_{j_{t'}} S^+_{j_{t'}} \sum_{ii'} \frac{a_i(x_\mu) a_{i'}(x_\nu)}{x_{\mu,i} - x_{\nu,i'}} \left(x_{\mu,i} S^+(x_{\mu,i}) - x_{\nu,i'} S^+(x_{\nu,i'})\right). \end{aligned} \tag{11}$$

For two-pair ($k = 2$) case, let

$$F(x_1, x_2) \equiv F(x_{1,1}, x_{1,2}, \cdots, x_{1,q}, x_{2,1}, x_{2,2}, \cdots, x_{2,q}) = \sum_{i'=1}^{q} \frac{a_{i'}(x_2) x_{1,i}}{x_{1,i} - x_{2,i'}} \quad \text{for} \quad i = 1,\ \cdots,\ q. \tag{12}$$

Once (12) are solved, (11) can be rewritten as

$$S^+(x_1, x_2) = 2G\sum_{t'} c_{j_{t'}} S^+_{j_{t'}} \left(F(x_1, x_2) S^+(x_1) + F(x_2, x_1) S^+(x_2)\right). \tag{13}$$

As shown in (11)-(13), $F(x_2, x_1)$ can be obtained from $F(x_1, x_2)$ by permuting $x_{1,i}$ with $x_{2,i}$ for $i = 1, \cdots, q$. However, when $k \geq 3$, (12) will no longer be valid, which will be dealt with shortly.

As can be seen from (10) and (11), $x_{\mu,i} \neq c^2_{j_t}$ for any μ, i, and t, $x_{\mu,i} \neq x_{\nu,i'}$ for given $\mu \neq \nu$, and any i and i', and $\beta(x_\mu) \neq 2\epsilon_{j_t}$ for any t and μ should always be assumed to avoid divergence. Besides the eigen-equation of the model, which provides one constraint to the variable $\{x_{\mu,i}\}$ for a fixed μ, (7) and equations related to (12) for the two-pair case provide $p + q$ equations for a fixed μ, while the total number of unknowns, $\{x_{\mu,i},\ a_i(x_\mu)\}$, $\beta(x_\mu)$ for fixed μ, and $F(x_1, x_2)$ for the two-pair case, is $2q + 2$. Thus, in order to get unique solution to the problem, we need $q = p - 1$, which will be used in the following. Moreover, for a fixed μ, by removing the last equation with $t = p$ in Eq. (7), which may be used to express one of $\{x_{\mu,i}\}$ $(i = 1, 2, \cdots, q)$ in terms of the remaining $q - 1$ variables, the remaining q equations provided by Eq. (7) may be expressed in matrix form with

$$\mathbf{B}\,\mathbf{a} = \mathbf{I}, \tag{14}$$

where the vector $\mathbf{a} = \left(a_1(x_\mu), a_2(x_\mu), \cdots, a_q(x_\mu)\right)^{\rm T}$, in which T denotes the matrix transposition, and $\mathbf{I} = (1, 1, \cdots, 1)^{\rm T}$ with q components, and

$$\mathbf{B} = \begin{pmatrix} \frac{2\epsilon_{j_1} - \beta(x_\mu)}{c^2_{j_1} - x_{\mu,1}} & \frac{2\epsilon_{j_1} - \beta(x_\mu)}{c^2_{j_1} - x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_1} - \beta(x_\mu)}{c^2_{j_1} - x_{\mu,q}} \\ \frac{2\epsilon_{j_2} - \beta(x_\mu)}{c^2_{j_2} - x_{\mu,1}} & \frac{2\epsilon_{j_2} - \beta(x_\mu)}{c^2_{j_2} - x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_2} - \beta(x_\mu)}{c^2_{j_2} - x_{\mu,q}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{2\epsilon_{j_q} - \beta(x_\mu)}{c^2_{j_q} - x_{\mu,1}} & \frac{2\epsilon_{j_q} - \beta(x_\mu)}{c^2_{j_q} - x_{\mu,2}} & \cdots & \frac{2\epsilon_{j_q} - \beta(x_\mu)}{c^2_{j_q} - x_{\mu,q}} \end{pmatrix}. \tag{15}$$

It is obvious that Eq. (14) has unique solution of $\mathbf{a}$ when and only when the matrix $\mathbf{B}$ is nonsingular with Det($\mathbf{B}$) $\neq 0$, which requires $x_{\mu,i} \neq x_{\mu,i'}$ for any μ and $i \neq i'$.

In the following, since the formalism for even-odd systems is similar, we focus on the seniority zero cases for simplicity. Let $|0\rangle$ be the pairing vacuum state satisfying $S^-_{j_t}|0\rangle = 0\ \forall\ t$. A k-pair eigenstate of (1) may be expressed as

$$|\zeta,\ k\rangle = S^+(x_1^{(\zeta)})S^+(x_2^{(\zeta)})\cdots S^+(x_k^{(\zeta)})|0\rangle, \tag{16}$$

where ζ is an additional quantum number introduced to label the ζ-th excitation state, the explicit operator form of $S^+(x_\mu^{(\zeta)})$ is still given by (8), which was also used in [20–22]. Using (6), (9), and (11), we can directly check that

$$\begin{gathered}\textstyle\sum_t \epsilon_{j_t}\hat{N}_{j_t}|\zeta,\ k\rangle = \\ \left(\textstyle\sum_t c_{j_t}S^+_{j_t} + \beta(x_1^{(\zeta)})S^+(x_1^{(\zeta)})\right)S^+(x_2^{(\zeta)})\cdots S^+(x_k^{(\zeta)})|0\rangle + \\ \cdots + S^+(x_1^{(\zeta)})\cdots S^+(x_{k-1}^{(\zeta)})\left(\textstyle\sum_t c_{j_t}S^+_{j_t} + \beta(x_k^{(\zeta)})S^+(x_k^{(\zeta)})\right)|0\rangle\end{gathered} \tag{17}$$

and

$$\begin{gathered}\hat{H}_{\rm P}|\zeta,\ k\rangle = G\textstyle\sum_{t'} c_{j_{t'}}S^+_{j_{t'}}\left(\bar{\Lambda}_0(x_1^{(\zeta)})S^+(x_2^{(\zeta)})\cdots S^+(x_k^{(\zeta)}) + \right. \\ \left.\cdots + S^+(x_1^{(\zeta)})\cdots S^+(x_{k-1}^{(\zeta)})\bar{\Lambda}_0(x_k^{(\zeta)})\right)|0\rangle \\ +\left(S^+(x_1^{(\zeta)}, x_2^{(\zeta)})S^+(x_3^{(\zeta)})\cdots S^+(x_k^{(\zeta)}) + S^+(x_1^{(\zeta)}, x_3^{(\zeta)})S^+(x_2^{(\zeta)})S^+(x_4^{(\zeta)})\cdots S^+(x_k^{(\zeta)})\right. \\ +\cdots + S^+(x_1^{(\zeta)}, x_k^{(\zeta)})S^+(x_2^{(\zeta)})\cdots S^+(x_{k-1}^{(\zeta)}) + \\ \cdots + S^+(x_k^{(\zeta)}, x_1^{(\zeta)})S^+(x_2^{(\zeta)})\cdots S^+(x_{k-1}^{(\zeta)}) + S^+(x_k^{(\zeta)}, x_2^{(\zeta)})S^+(x_3^{(\zeta)})\cdots S^+(x_{k-1}^{(\zeta)}) + \\ \left.\cdots + S^+(x_k^{(\zeta)}, x_{k-1}^{(\zeta)})S^+(x_2^{(\zeta)})\cdots S^+(x_{k-2}^{(\zeta)})\right)|0\rangle,\end{gathered} \tag{18}$$

where

$$\bar{\Lambda}_0(x_\mu^{(\zeta)}) = -\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{\Omega_{j_t}\left(c_{j_t}\right)^2 a_i(x_\mu^{(\zeta)})}{c^2_{j_t} - x^{(\zeta)}_{\mu,i}} = -\sum_{t=1}^{p}\frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_\mu^{(\zeta)})}. \tag{19}$$

Using (17) and (18), one can prove that the eigen-equation $\hat{H}|\zeta,k\rangle = E_k^{(\zeta)}|\zeta,k\rangle$ is fulfilled if and only if

$$\sum_{i=1}^{q}\sum_{t=1}^{p}\frac{\Omega_{j_t}\left(c_{j_t}\right)^2 a_i(x_\mu^{(\zeta)})}{c^2_{j_t} - x^{(\zeta)}_{\mu,i}} - 2W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)}) = \frac{1}{G} \quad \text{for}\ \ \mu = 1, 2, \cdots, k, \tag{20}$$

or

$$\sum_{t=1}^{p}\frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_\mu^{(\zeta)})} - 2W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)}) = \frac{1}{G} \quad \text{for}\ \ \mu = 1, 2, \cdots, k, \tag{21}$$

where

$$W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)}) = \sum_{\nu\neq\mu}\sum_{i'=1}^{q}\frac{a_{i'}(x_\mu^{(\zeta)})x^{(\zeta)}_{\nu,i}}{x^{(\zeta)}_{\nu,i} - x^{(\zeta)}_{\mu,i'}} \quad \text{for}\ \ i = 1, 2, \cdots, q, \tag{22}$$

of which each term for fixed ν in the sum is the same as that shown in (12). When $k = 2$, (22) becomes (12). However, every term for fixed ν in the sum of (22) dependents on $\{x^{(\zeta)}_{\nu,i}\}$ with $\nu \neq \mu$, which is different from that in the Gaudin-Richardosn solution of the standard pairing model [3, 4], and must be considered together to be solved as shown in (22). In addition, it is obvious that $W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)})$ for a fixed μ is symmetric with respect to any permutation among

$\{x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)}\}$, which is similar to that in the Gaudin-Richardosn solution of the standard pairing model. The corresponding eigen-energy is given by

$$E_k^{(\zeta)} = \sum_{\mu=1}^{k} \beta(x_\mu^{(\zeta)}). \tag{23}$$

Since $\{a_i(x_\mu^{(\zeta)})\}$ $(i = 1, 2, \cdots, q)$ and $\{x_{\mu,q=p-1}^{(\zeta)}\}$ are expressed in terms of $\{\beta(x_\mu^{(\zeta)})\}$ and $\{x_{\mu,i}^{(\zeta)}\}$ $(i = 1, 2, \cdots, q-1)$ according to (7), q equations given by (22) provide expressions of $x_{\mu,i}^{(\zeta)}$ $(i = 1, 2, \cdots, q-1)$ and the final expression of $W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)})$ for a fixed μ. When $p \geq 3$, for a fixed μ, we use (7) to get solution of $\{a_1(x_\mu^{(\zeta)}), \cdots, a_{q=p-1}(x_\mu^{(\zeta)})\}$ and $x_{\mu,p-1}^{(\zeta)}$, and use (22) to get that of $W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)})$ for given μ and $\{x_{\mu,1}^{(\zeta)}, \cdots, x_{\mu,p-2}^{(\zeta)}\}$. While $\beta(x_\mu^{(\zeta)})$ is determined by the Bethe ansatz equations (20) or (21). Though (20) or (21) looks quite similar to the Bethe ansatz equations for the standard pairing case, the term $W(x_\mu^{(\zeta)}; x_1^{(\zeta)}, \cdots, x_{\mu-1}^{(\zeta)}, x_{\mu+1}^{(\zeta)}, \cdots, x_k^{(\zeta)})$ involved for a fixed μ should be determined by q equations given in (22). Therefore, solutions of (20) or (21) can not be obtained easily as those in the standard pairing case shown in [7, 8].

III. SOME EXPLICIT EXAMPLES

In the following, we provide solutions of the model for some special cases and compare them with those known previously and display the procedure for some nontrivial cases with $p = 3$, which demonstrates that the procedure indeed works for the model with non-degenerate single-particle energies $\{\epsilon_{j_t}\}$ and non-degenerate separable pairing strengths $\{c_{j_t}\}$.

(1) $p = 1$ case: For this case, we may set $a_1(x) = 1$ in (7) because it only changes the normalization factor of the eigenstates of the model. Thus, we have

$$\beta(x) = 2\epsilon_{j_1} - \left(c_{j_1}\right)^2 + x. \tag{24}$$

By using (12), (20) can be written as

$$\frac{\Omega_{j_1}\left(c_{j_1}\right)^2}{c_{j_1}^2 - x_\mu} + \sum_{\nu\neq\mu} \frac{2x_\nu}{x_\mu - x_\nu} = \frac{1}{G} \quad \text{for } \mu = 1, 2, \cdots k, \tag{25}$$

where the subscript $i(=1)$ is omitted with $\{x_{\mu,i=1} \equiv x_\mu\}$. For any k, up to permutations among the k components of the root, there is only one set of solution of Eq. (25). Though root of (25) may be complex when $k \geq 2$, without solving (25), one can evaluate the eigen-energy according to (23) from the following procedures: Summing (25) over μ, one has

$$\sum_{\mu=1}^{k} \frac{\Omega_{j_1}\left(c_{j_1}\right)^2}{c_{j_1}^2 - x_\mu} = \frac{k}{G} + k(k-1), \tag{26}$$

while multiplying (25) by x_μ and then summing over μ, one gets

$$\sum_{\mu=1}^{k} \frac{\Omega_{j_1} x_\mu \left(c_{j_1}\right)^2}{c_{j_1}^2 - x_\mu} = \left(c_{j_1}\right)^2 \sum_{\mu=1}^{k} \frac{\Omega_{j_1}\left(c_{j_1}\right)^2}{c_{j_1}^2 - x_\mu} - \Omega_{j_1}\left(c_{j_1}\right)^2 k = \frac{1}{G}\sum_{\mu=1}^{k} x_\mu. \tag{27}$$

Combining (26) and (27), one obtains

$$\sum_{\mu=1}^{k} x_\mu = \left(c_{j_1}\right)^2 G\left(k^2 - k + \frac{k}{G} - \Omega_{j_1} k\right). \tag{28}$$

On the other hand, substituting (24) into (23), we have

$$E_k^{(\zeta)} = 2\epsilon_{j_1}k - \left(c_{j_1}\right)^2 k + \sum_{\mu=1}^{k} x_\mu. \tag{29}$$

Finally, substituting (28) into (29), we get

$$E_k^{(\zeta)} = 2\epsilon_{j_1}k + G\left(c_{j_1}\right)^2\left(k^2 - k - \Omega_{j_1}k\right), \tag{30}$$

which is exactly the same as that given by the well-known Racah quasi-spin formalism for the standard pairing model with modified pairing strength $\tilde{G} = \left(c_{j_1}\right)^2 G$.

(2) $p = 2$ case: For this case $q = 1$. It should be assumed that $c_{j_1} \neq c_{j_2}$ and $\epsilon_{j_1} \neq \epsilon_{j_2}$. Hence, (7) provides

$$a_1(x) \equiv a = \frac{\left(c_{j_1}\right)^2 - \left(c_{j_2}\right)^2}{2\epsilon_{j_1} - 2\epsilon_{j_2}}, \quad \beta(x) = \frac{2\epsilon_{j_2}\left(c_{j_1}\right)^2 - 2\epsilon_{j_1}\left(c_{j_2}\right)^2}{\left(c_{j_1}\right)^2 - \left(c_{j_2}\right)^2} + x\frac{2\epsilon_{j_1} - 2\epsilon_{j_2}}{\left(c_{j_1}\right)^2 - \left(c_{j_2}\right)^2}. \tag{31}$$

Eq. (20) is simply given by

$$\sum_{t=1}^{p} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{c_{j_t}^2 - x_\mu^{(\zeta)}} + \sum_{\nu\neq\mu} \frac{2x_\nu^{(\zeta)}}{x_\mu^{(\zeta)} - x_\nu^{(\zeta)}} = \frac{1}{a\,G} \quad \text{for } \mu = 1, 2, \cdots k, \tag{32}$$

where the subscript $i(= 1)$ is also omitted with $\{x_{\mu,i=1} \equiv x_\mu\}$.

Moreover, as shown in [18–20], when

$$\left(c_{j_t}\right)^2 = g_1\epsilon_{j_t} + g_2 \tag{33}$$

for $t = 1, 2, \cdots, p$, where g_1 and g_2 are two constants, the Hamiltonian (1) for any p in this case is exactly solvable. For the $p = 2$ case, since t can only be taken as 1 or 2, (33) provides unique solution of the two parameters g_1 and g_2 for the $p = 2$ case with

$$g_1 = 2a, \quad g_2 = \frac{\left(c_{j_2}\right)^2\epsilon_{j_1} - \left(c_{j_1}\right)^2\epsilon_{j_2}}{\epsilon_{j_1} - \epsilon_{j_2}}. \tag{34}$$

As shown in [20] for the $p = 2$ case, the Bethe ansatz equations are

$$\sum_{t=1}^{p} \frac{\left(c_{j_t}\right)^2\Omega_{j_t}}{2\epsilon_{j_t} - z_i^{(\zeta)}} + \sum_{l\neq i} \frac{g_1 z_l^{(\zeta)} + 2g_2}{z_i^{(\zeta)} - z_l^{(\zeta)}} = 1/G \quad \text{for} \quad i = 1, 2, \cdots k, \tag{35}$$

with ζ-th eigen-energy given by

$$E_k^{(\zeta)} = \sum_{i=1}^{k} z_i^{(\zeta)}, \tag{36}$$

of which the corresponding eigenstate is expressed as

$$|\zeta,\ k\rangle = S^+(z_1^{(\zeta)})\cdots S^+(z_k^{(\zeta)})|0\rangle, \tag{37}$$

where

$$S^+(z) = \sum_{t=1}^{p} \frac{1}{2\epsilon_{j_t} - z} c_{j_t} S^+_{j_t}. \tag{38}$$

By substituting $z_l = 2(x_l - g_2)/g_1$ for $l = 1, \cdots, k$ into (35)-(37), (35)-(37) become (20), (23), and (16), respectively, for which the constraints given in (33) should be used. Thus, it is shown that the results for the $p = 1$ and $p = 2$ cases obtained from the procedure proposed in this work are consistent with those obtained previously.

(3) $p = 3$ case: For this case $q = 2$. $\epsilon_{j_1} \neq \epsilon_{j_2} \neq \epsilon_{j_3}$ and $c_{j_1} \neq c_{j_2} \neq c_{j_3}$ should also be assumed. To explicitly demonstrate the solutions, we take the ds-shell with 3 orbitals $0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$, of which the single-particle energies are provided in [23], while the values of the parameters $\{c_j\}$ and the overall pairing strength G provided in [20] are used for this example, which are shown in Table I. As is known, one-pair ($k = 1$) solution of the model for any p can be obtained easily by using $S^+(\beta)$ given in the rightmost expression of (8) directly, of which β is the only variable in the solution. Thus, the $k = 1$ trivial case will not be discussed. It should be stated that the condition (33), which is sufficient to be used for $p = 2$ case, is not needed for $p \geq 3$ cases according to this procedure. When $p = 3$, the nontrivial cases are those with $k \geq 2$.

TABLE I: The single-particle energies ϵ_{j_t} (in MeV) for the ds-shell deduced from [23], the parameters $\{c_{j_t}\}$, and G (in MeV) are taken from [20], where $j_1 = 5/2$, $j_2 = 1/2$, and $j_3 = 3/2$, and the overlaps $\rho(\zeta, k) = |\langle \zeta, k | \zeta, k\rangle_{\rm FMD}|$ for the $k = 2$ case and the ground state of the $k = 3$ case, where $|\zeta, k\rangle$ is the ζ-th k-pair excitation state given by (16), and $|\zeta, k\rangle_{\rm FMD}$ is that obtained from the full matrix diagonalization within the ds-shell subspace shown in [20].

$\epsilon_{j_1} = -3.70$	$\epsilon_{j_2} = -2.92$	$\epsilon_{j_3} = 1.90$	$c_{j_1} = 0.99583$	$c_{j_2} = -0.06334$	$c_{j_3} = 0.06562$	$G = 0.945$
$\rho(1,2) = 0.99849$	$\rho(2,2) = 0.99625$	$\rho(3,2) = 0.99884$	$\rho(4,2) = 0.99931$	$\rho(5,2) = 0.999997$	$\rho(1,3) = 0.978308$	

For the $k = 2$ case, according to the procedure, we may use (7) to get $a_1(\beta(x_\mu), x_{\mu,1})$, $a_2(\beta(x_\mu), x_{\mu,1})$, and $x_{\mu,2}$, which can be expressed as

$$a_1(\beta(x_\mu), x_{\mu,1}) = \frac{x_{\mu,1}^3 - 0.999995 x_{\mu,1}^2 + 0.00826599 x_{\mu,1} - 0.0000171316}{\beta(x_\mu) x_{\mu,1}^2 + 7.39946 x_{\mu,1}^2 - 0.0087071 \beta(x_\mu) x_{\mu,1} - 0.0633624 x_{\mu,1} - 0.00003\, \beta(x_\mu)^2 - 0.0000425 \beta(x_\mu) + 0.000804},$$

$$a_2(\beta(x_\mu), x_{\mu,1}) =$$

$$\frac{0.00003\, \beta(x_\mu)^3 + \beta(x_\mu)^2 (0.004197 - 0.98694 x_{\mu,1}) + \beta(x_\mu)(0.00754 - 1.904 x_{\mu,1} - 12.758 x_{\mu,1}^2) - 0.09289 + 22.4168 x_{\mu,1} - 59.2576 x_{\mu,1}^2 - 17.6893 x_{\mu,1}^3}{\beta(x_\mu)^3(-0.0044 + x_{\mu,1}) + \beta(x_\mu)^2(-0.038 + 7.55 x_{\mu,1} + 433.7 x_{\mu,1}^2 - 33206.8 x_{\mu,1}^3) + \beta(x_\mu)(0.072 - 34.58 x_{\mu,1} + 6365.3 x_{\mu,1}^2 - 491425 x_{\mu,1}^3) + 0.846 - 264.2 x_{\mu,1} + 23353.4 x_{\mu,1}^2 - 1818140 x_{\mu,1}^3},$$

$$x_{\mu,2} = \frac{-0.00003\, \beta(x_\mu)^2 - \beta(x_\mu)(0.0000425 + 0.0043536 x_{\mu,1}) - 0.03168 x_{\mu,1} + 0.000804}{\beta(x_\mu)(0.004354 - x_{\mu,1}) - 7.39946 x_{\mu,1} + 0.03168}. \tag{39}$$

It should be stated that (39) is valid for any μ and k. On the other hand, Eq. (12) or (22) provides two sets of solutions, of which one set involves $x_{\mu,i} = x_{\mu,i'}$ for any μ and $1 \leq i \neq i' \leq 2$ for this case. This solution violates the nonsingular condition Det($\mathbf{B}$) $\neq 0$ with the maitrx $\mathbf{B}$ given by (15), and should be discarded. The other set of solutions for $x_{\mu,1}$, $W(x_1; x_2) = F(x_2, x_1)$, and $W(x_2; x_1) = F(x_1, x_2)$ for this case obtained from the constraints shown in (12) or (22) are given as

$$x_{1,1} = \frac{x_{2,1}\, x_{2,2}(-x_{1,2}\, a_1(\beta(x_2), x_{2,1}) + x_{2,2}\, a_1(\beta(x_2), x_{2,1}) - x_{1,2}\, a_2(\beta(x_2), x_{2,1}) + x_{2,1}\, a_2(\beta(x_2), x_{2,1}))}{-x_{1,2} x_{2,1} a_1(\beta(x_2), x_{2,1}) + x_{2,1} x_{2,2} a_1(\beta(x_2), x_{2,1}) - x_{1,2} x_{2,2} a_2(\beta(x_2), x_{2,1}) + x_{2,1} x_{2,2} a_2(\beta(x_2), x_{2,1})},$$

$$x_{2,1} = \frac{x_{1,1}\, x_{1,2}(x_{1,2}\, a_1(\beta(x_1), x_{1,1})) - x_{2,2}\, a_1(\beta(x_1), x_{1,1}) + x_{1,1}\, a_2(\beta(x_1), x_{1,1}) - x_{2,2}\, a_2(\beta(x_1), x(1,1)))}{x_{1,1} x_{1,2} a_1(\beta(x_1), x_{1,1}) - x_{1,1} x_{2,2} a_1(\beta(x_1), x_{1,1}) + x_{1,1} x_{1,2} a_2(\beta(x_1), x_{1,1}) - x_{1,2} x_{2,2} a_2(\beta(x_1), x_{1,1})}, \tag{40}$$

$$W(x_2; x_1) = \frac{x_{1,2}\,(-x_{2,2} a_1(\beta(x_2), x_{2,1}) + x_{1,2} a_1(\beta(x_2), x_{2,1}) + x_{1,2} a_2(\beta(x_2), x_{2,1}) - x_{2,1} a_2(\beta(x_2), x_{2,1}))}{(x_{1,2} - x_{2,1})(x_{1,2} - x_{2,2})},$$

$$W(x_1; x_2) = \frac{x_{2,2}\,(-x_{1,2}\, a_1(\beta(x_1), x_{1,1}) + x_{2,2}\, a_1(\beta(x_1), x_{1,1}) - x_{1,1}\, a_2(\beta(x_1), x_{1,1}) + x_{2,2}\, a_2(\beta(x_1), x_{1,1}))}{(x_{2,2} - x_{1,1})(x_{2,2} - x_{1,2})}. \tag{41}$$

It can easily be verified that $x_{\mu,1}$ or $W(x_\mu; x_\nu)$ can be obtained from $x_{\nu,1}$ or $W(x_\nu; x_\mu)$ by permuting $x_{\mu,i}$ with $x_{\nu,i}$ in the expressions. While Eq. (21) is simply given by

$$\sum_{t=1}^{3} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_1^{(\zeta)})} - 2W(x_1^{(\zeta)}; x_2^{(\zeta)}) = \frac{1}{G},$$
$$\sum_{t=1}^{3} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_2^{(\zeta)})} - 2W(x_2^{(\zeta)}; x_1^{(\zeta)}) = \frac{1}{G}. \tag{42}$$

By substituting (39) into (40) and (41), and then by substituting (41) into (42), (40) and (42) provide 4 equations for $x_{1,1}^{(\zeta)}$, $x_{2,1}^{(\zeta)}$, $\beta(x_1^{(\zeta)})$, and $\beta(x_2^{(\zeta)})$, which can be solved numerically.

TABLE II: Two-pair solutions of the spherical mean-field plus separable pairing model with parameters shown in Table I, in which the excitation energies of the model, $E_{k=2}^{(\zeta)}$ (in MeV), the corresponding expansion coefficients, and the spectral parameters are provided, where $x_{\mu,i}$ is dimensionless, the unit of $a_i(x_\mu)$ is MeV^{-1}, the unit of $\beta(x_\mu)$ is MeV, and $I = \sqrt{-1}$.

The excitation energy	$x_{1,1}^{(\zeta)}$	$x_{1,2}^{(\zeta)}$	$a_1(x_1^{(\zeta)})$	$a_2(x_1^{(\zeta)})$	$x_{2,1}^{(\zeta)}$	$x_{2,2}^{(\zeta)}$	$a_1(x_2^{(\zeta)})$	$a_2(x_2^{(\zeta)})$	$\beta(x_1^{(\zeta)})$	$\beta(x_2^{(\zeta)})$
$E_{k=2}^{(\zeta=1)} = -18.557$	$0.00427 - 0.008I$	$-0.0036 - 0.006I$	$0.0245 + 0.38I$	$-0.07 + 0.068I$	$-0.00393 - 0.007I$	$0.0473 - 0.0351I$	$0.0293 + 0.028I$	$0.163 - 0.12I$	$-7.216 + 2.116I$	$-11.341 - 2.116I$
$E_{k=2}^{(\zeta=2)} = -16.051$	0.002	0.262	0.0025	0.258	0.0047	0.00466	−6.509	5.8752	−10.209	−5.842
$E_{k=2}^{(\zeta=3)} = -6.421$	0.00014	0.00435	−0.081	−0.00071	0.00015	0.1538	0.0084	0.2908	3.7925	−10.213
$E_{k=2}^{(\zeta=4)} = -2.046$	0.00465	0.00388	−0.073	−0.01495	0.00862	0.00404	−0.594	−0.03658	3.794	−5.8407
$E_{k=2}^{(\zeta=5)} = 7.593$	$0.02249 - 0.001286I$	$0.04329 - 0.00004I$	$-0.079 - 0.0073I$	$-0.009 + 0.0072I$	$0.003 + 0.00077I$	$0.0043276 + 0.000057I$	$-0.073 + 0.0124I$	$-0.0149 - 0.01232I$	$3.796 + 0.0021I$	$3.797 - 0.0021I$

Table II shows all pairing excitation energies of the model in the ds-shell with $k = 2$. According to the equations (39)-(42), we use FindRoot provided by Wolfram Mathematica to search for possible roots $x_{1,1}^{(\zeta)}$, $x_{2,1}^{(\zeta)}$, $\beta(x_1^{(\zeta)})$, and $\beta(x_2^{(\zeta)})$ of (40) and (42). Then, we use the resultants to verify whether they indeed satisfy (40) and (42) due to the fact that the resultants are iteratively obtained by Mathematica approximately, of which the accuracy cannot always be guaranteed. A better algorithm is needed when one implements the procedure to solve large p and k cases. We found that FindRoot of Mathematica frequently provides one of solutions of the Bethe ansatz equations given in (21) with one of $\{x_{\mu,i}\}$ being zero $x_{\mu,i} = 0$ or at one of the poles of (20), namely $x_{\mu,i} = c_j^2$, with the corresponding expansion coefficient $a_i(x_\mu) = 0$, which are not solutions for our purpose and should be discarded. For the results provided in Table II, (40) and (42) are valid with errors about 10^{-3}. Though we cannot prove the completeness of the solutions of (40) and (42) at present, our numerical calculation shows that there are indeed 5 solutions given in Table II, which are in one-to-one correspondence to those obtained by full matrix diagonalization shown in [20]. Since eigenvalues of the Hamiltonian should be real, the constraint with $\sum_{\mu=1}^{k} \beta(x_\mu)$ being real should be helpful. It should be stated that there may be many different solutions of $\{x_{\mu,i}\}$ resulting in the same set of $\beta(x_\mu)$. However, one only needs to choose one set of $\{x_{\mu,i}\}$ to get the spectral parameters $\{\beta_\mu = \beta(x_\mu)\}$ if the final results of $\{\beta_\mu\}$ up to permutations among different μ are the same because eigenstates and eigenenergies of the model only depends on $\{\beta_\mu\}$, which are all the same with any permutation among different μ. In addition, similar to that occurring in the original Richardson-Gaudin solution to the standard pairing model, there is also S_k permutation symmetry among k components of the roots. Namely, if $\{x_{1,i}, x_{2,i}, \cdots, x_{k,i}\}$ is a solution, any permutation among $\{x_{1,i}, x_{2,i}, \cdots, x_{k,i}\}$ is also a solution, among which we only need to choose one set of the solutions because they result in the same eigenstate and the corresponding eigen-energy of the model. The ground-state solution of the $k = 2$ case shown in Table II generated by Wolfram Mathematica version 9.0 is provided in [25].

Finally, we present the solution for the $k = 3$ ground state of the model, which shows general features of the procedure for cases with arbitrary p and k as well. Similar to the $k = 2$ case, the expansion coefficients $a_1(\beta(x_\mu), x_{\mu,1})$, $a_2(\beta(x_\mu), x_{\mu,1})$, and $x_{\mu,2}$ for $\mu = 1,\ 2,\ 3$ are still given by (39). However, (22) in this case provides rather complicate expressions of non-degenerate $x_{\mu,1}$ for $\mu = 1, 2, 3$, of which only equations in determining them are given. Specifically, (22) involves 6 equations with

$$W(x_1; x_2, x_3) = \frac{x_{3,1}a_1(\beta(x_1), x_1)}{x_{3,1} - x_{1,1}} + \frac{x_{2,1}a_1(\beta(x_1), x_1)}{x_{2,1} - x_{1,1}} + \frac{x_{3,1}a_2(\beta(x_1), x_1))}{x_{3,1} - x_{1,2}} + \frac{x_{2,1}a_2(\beta(x_1), x_1)}{x_{2,1} - x_{1,2}},$$

$$
\begin{aligned}
W(x_2;x_1,x_3) &= \frac{x_{1,1}a_1(\beta(x_2),x_2)}{x_{1,1}-x_{2,1}} + \frac{x_{3,1}a_1(\beta(x_2),x_2)}{x_{3,1}-x_{2,1}} + \frac{x_{1,1}a_2(\beta(x_2),x_2))}{x_{1,1}-x_{2,2}} + \frac{x_{3,1}a_2(\beta(x_2),x_2)}{x_{3,1}-x_{2,2}},\\
W(x_3;x_1,x_2) &= \frac{x_{1,1}a_1(\beta(x_3),x_3)}{x_{1,1}-x_{3,1}} + \frac{x_{2,1}a_1(\beta(x_3),x_3)}{x_{2,1}-x_{3,1}} + \frac{x_{1,1}a_2(\beta(x_3),x_3))}{x_{1,1}-x_{3,2}} + \frac{x_{2,1}a_2(\beta(x_3),x_3)}{x_{2,1}-x_{3,2}},\\
W(x_1;x_2,x_3) &= \frac{x_{3,2}a_1(\beta(x_1),x_1)}{x_{3,2}-x_{1,1}} + \frac{x_{2,2}a_1(\beta(x_1),x_1)}{x_{2,2}-x_{1,1}} + \frac{x_{3,2}a_2(\beta(x_1),x_1))}{x_{3,2}-x_{1,2}} + \frac{x_{2,2}a_2(\beta(x_1),x_1)}{x_{2,2}-x_{1,2}},\\
W(x_2;x_1,x_3) &= \frac{x_{1,2}a_1(\beta(x_2),x_2)}{x_{1,2}-x_{2,1}} + \frac{x_{3,2}a_1(\beta(x_2),x_2)}{x_{3,2}-x_{2,1}} + \frac{x_{1,2}a_2(\beta(x_2),x_2))}{x_{1,2}-x_{2,2}} + \frac{x_{3,2}a_2(\beta(x_2),x_2)}{x_{3,2}-x_{2,2}},\\
W(x_3;x_1,x_2) &= \frac{x_{1,2}a_1(\beta(x_3),x_3)}{x_{1,2}-x_{3,1}} + \frac{x_{2,2}a_1(\beta(x_3),x_3)}{x_{2,2}-x_{3,1}} + \frac{x_{1,2}a_2(\beta(x_3),x_3))}{x_{1,2}-x_{3,2}} + \frac{x_{2,2}a_2(\beta(x_3),x_3)}{x_{2,2}-x_{3,2}},
\end{aligned}
\tag{43}
$$

of which the first 3 of them may be used as the expressions of $W(x_\mu; x_\nu, x_{\nu'})$ with $\mu \neq \nu \neq \nu'$, while the last 3 equations given in (43) are used to determine $x_{\mu,1}$ ($\mu = 1, 2, 3$).

The Bethe ansatz equations (21) can then be expressed as

$$
\begin{aligned}
\sum_{t=1}^{3} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_1^{(\zeta)})} - 2W(x_1;x_2,x_3) &= \frac{1}{G},\\
\sum_{t=1}^{3} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_2^{(\zeta)})} - 2W(x_2;x_1,x_3) &= \frac{1}{G},\\
\sum_{t=1}^{3} \frac{\Omega_{j_t}\left(c_{j_t}\right)^2}{2\epsilon_{j_t} - \beta(x_3^{(\zeta)})} - 2W(x_3;x_1,x_2) &= \frac{1}{G}.
\end{aligned}
\tag{44}
$$

In solving (44), one should keep in mind that

$$
\sum_{i=1}^{2} \frac{a_i(x_\mu^{(\zeta)})}{c_{j_t}^2 - x_{\mu,i}^{(\zeta)}} = \frac{1}{2\epsilon_{j_t} - \beta(x_\mu^{(\zeta)})}
\tag{45}
$$

for any t should also be satisfied for $\mu = 1,\ 2$, and 3 as required by the constraints shown in (7), which may be used to check the final results. Similar to the $k = 2$ case, by substituting (39) with $\mu = 1,\ 2,\ 3$ into (43), and then by substituting the first 3 equations of (43) into (44), (44) and the last 3 equations of (43) provide 6 equations for $x_{\mu,1}^{(\zeta)}$ and $\beta(x_\mu^{(\zeta)})$ for $\mu = 1, 2, 3$, which may be solved numerically. For the results provided in Table III, (43), (44), and (45) are valid with errors about 10^{-3}, 10^{-3}, and 10^{-14}, respectively. The results shown in Table III generated by Wolfram Mathematica version 9.0 is also provided in [25].

TABLE III: The same as Table II, but for the three-pair solution of the ground state of the model.

	$x_{1,1}$	$x_{1,2}$	$x_{2,1}$	$x_{2,2}$	$x_{3,1}$	$x_{3,2}$	$\beta(x_1)$	$\beta(x_2)$	$\beta(x_3)$
$E_{k=3}^{(\zeta=1)} = -25.03$	$0.0123 + 0.0114I$	$-0.0168 + 0.0158I$	$-0.0067 + 0.0096I$	$0.0168 + 0.0042I$	$-0.0061 + 0.03146I$	$0.0103 + 0.01836I$	$-6.9986 + 2.6736I$	$-5.3945 - 1.8138I$	$-12.6369 - 0.8598I$
	$-0.1658 + 0.0580I$	$0.1209 + 0.3093I$	$-0.2145 - 0.0507I$	$-0.0593 - 0.1901I$	$0.2098 - 0.2089I$	$-0.0270 + 0.1694I$			
	$a_1(x_1)$	$a_2(x_1)$	$a_1(x_2)$	$a_2(x_2)$	$a_1(x_3)$	$a_2(x_3)$			

In comparison to the exact numerical results obtained from the progressive diagonalization scheme for this case provided in [20], which is equivalent to the full matrix diagonalization within the ds-shell subspace, it is shown that the eigen-energies obtained from the procedure proposed in this work are very close to those shown in [20] with errors about 0.002MeV for the $k = 2$ case shown in Table II, while the ground state energy of the $k = 3$ case shown in Table III is exactly the same as that given in [20]. Since eigenstates of the model should be sensitive to the results of solution, we also calculated overlaps $\rho(\zeta,k) = |\langle\zeta,k|\zeta,k\rangle_{\rm FMD}|$, where $|\zeta,k\rangle$ is the ζ-th k-pair excitation state obtained in this work, and $|\zeta,k\rangle_{\rm FMD}$ is that obtained from the full matrix diagonalization within the ds-shell subspace shown in [20], which are also shown in the last row of Table I. It can be seen from Table I that the overlaps for the $k = 2$ case are all greater than 99.63%, while it is 97.83% for the $k = 3$ ground state. The overlap for the $k = 3$ ground state is not perfect mainly due to the fact that the errors in the roots of (44) and the last 3 equations of (43) obtained by FindRoot of Mathematica seem still significant, which are difficult to be reduced with the increasing of p and k, especially when the roots are complex. Therefore, a better numerical algorithm for solving equations (7), (20) or (21), and (22) is needed. Anyway, our analysis indicates that the results obtained from the procedure shown in this work are indeed reliable.

IV. CONCLUSIONS

Exact solution of the nuclear mean-field plus separable pairing model is reexamined. The suitable auxiliary constraints for solving the Bethe ansatz equations of the model are proposed. The Bethe ansatz form of eigenvectors of the mean-field plus separable pairing Hamiltonian purposed in [21, 22] is verified with these new auxiliary constraints. Specifically, when $p \geq 3$, we need to solve $p \times k$ auxiliary equations given by (7) and another $p \times k$ equations given by (20) or (21) and (22) to get $(p-1) \times k$ variables $\{x_{\mu,1}, x_{\mu,2}, \cdots, x_{\mu,p-1}\}$ and another $(p-1) \times k$ variables $\{a_1(x_\mu), a_2(x_\mu), \cdots, a_{p-1}(x_\mu)\}$, together with k variables $\beta(x_\mu)$ and another k variables $W(x_\mu; x_1, \cdots, x_{\mu-1}, x_{\mu+1}, \cdots, x_k)$, for $\mu = 1, 2, \cdots, k$. Once the k variables $\beta(x_\mu)$ $(\mu = 1, \cdots, k)$ are obtained, they can then be used to get k-pair eigenstates (16) and the corresponding eigen-energies (23). It clearly shows that the number of equations involved equals exactly to the number of unknowns, which ensures the uniqueness of the solution. Besides the solution of the model for one- and two-orbit cases, to demonstrate structure and features of the solution, the model with three orbits ($p = 3$) in the ds-shell is taken as a nontrivial example, of which two-pair results and the ground state of the three-pair case are provided explicitly, which are in one-to-one correspondence to the results obtained from the full matrix diagonalization in the ds-shell subspace provided in [20]. Though only some $p = 3$ cases are presented, the formulism shown in (7), (20) or (21), and (22) applies to any p and k as well, which seems valid for any p and k with non-degenerate single-particle energies and non-degenerate separable pairing strengths, though we can not provide a rigorous proof of the completeness for general case at present. Since eigenvalues of the Hamiltonian should always be real, the constraint with $\sum_{\mu=1}^{k} \beta(x_\mu)$ being real may be used in finding solutions of the model. Since the number of equations involved increases with the number of orbits and pairs, to solve these equations for large number of orbits and pairs seems still difficult. As shown in [20], the progressive diagonalization method [24] seems more practical to solve the problem.

Acknowledgments

Support from the National Natural Science Foundation of China (11375080 and 11675071), the U. S. National Science Foundation (OCI-0904874 and ACI-1516338), U. S. Department of Energy (DE-SC0005248), the Southeastern Universities Research Association, the China-U. S. Theory Institute for Physics with Exotic Nuclei (CUSTIPEN) (DE-SC0009971), and the LSU–LNNU joint research program (9961) is acknowledged.

[1] P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer Verleg, Berlin, 1980).
[2] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958); S. T. Belyaev, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **31**, 11 (1959).
[3] M. Gaudin, J. Physique 37, 1087 (1976).
[4] R. W. Richardson, Phys. Lett. **3**, 277 (1963); **5**, 82 (1963); R. W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964); **52**, 253 (1964).
[5] J. Dukelsky, S. Pittel, and G. Sierra, Rev. Mod. Phys. **76**, 643 (2004).
[6] F. Pan, L. Bao, L. Zhai, X. Cui, and J. P. Draayer, J. Phys. A: Math. Theor. **44**, 395305 (2011).
[7] X. Guan, K. D. Launey, M. Xie, L. Bao, F. Pan, J. P. Draayer, Phys. Rev. C **86**, 024313 (2012).
[8] X. Guan, K. D. Launey, M. Xie, L. Bao, F. Pan, J. P. Draayer, Comp. Phys. Commun. **185**, 2714 (2014).
[9] C. Qi and T. Chen, Phys. Rev. C **92**, 051304(R) (2015).
[10] F. Pan, V. G. Gueorguiev, and J. P. Draayer, Phys. Rev. Lett. **92**, 112503 (2004).
[11] F. Pan, X. Ding, K. D. Launey, H. Li, X. Xu, J. P. Draayer, Nucl. Phys. A **947**, 234 (2016).
[12] F. Pan, J. P. Draayer, and W. E. Ormand, Phys. Lett. B **422**, 1 (1998).
[13] A. B. Balantekin and Y. Pehlivan, Phys. Rev. C **76**, 051001(R) (2007).
[14] T. Skrypnyk, J. Math. Phys. **50**, 033504 (2009).

[15] T. Skrypnyk, Nucl. Phys. B **806**, 504 (2009).

[16] T. Skrypnyk, J. Phys. A: Math. Theor. **42**, 472004 (2009).

[17] S. M. A. Rombouts, J. Dukelsky, G. Ortiz, Phys. Rev. B **82**, (2010) 224510.

[18] A. Birrell, P. S. Isaac, J. Links, Inverse Problems **28**, 035008 (2012).

[19] M. Van Raemdonck, S. De Baerdemacker, D. Van Neck, Phys. Rev. B **89**, 155136 (2014).

[20] L. Dai, F. Pan, J. P. Draayer, Nucl. Phys. A **957**, 51 (2017).

[21] F. Pan and J. P. Draayer, Phys. Lett. B **442**, 7 (1998).

[22] F. Pan and J. P. Draayer, Ann. Phys. (N. Y.) **271**, 120 (1999).

[23] F. Nowacki and A. Poves, Phys. Rev. C **79**, 014310 (2009).

[24] F. Pan, M. X. Xie, X. Guan, L. R. Dai, and J. P. Draayer, Phys. Rev. C **80**, 044306 (2009).

[25] See Supplemental Material at [URL will be inserted by publisher] containing two files written in Wolfram Mathematica Version 9.0 with file name [k=2 Ground-state.nb] for the ground state of the two-pair case and [k=3 Ground-state.nb] for the ground state of the three-pair case.