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Exact solution of mean-field plus an extended $T=1$ nuclear pairing Hamiltonian in the seniority-zero symmetric subspace

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ABSTRACT

An extended pairing Hamiltonian that describes multi-pair interactions among isospin $T=1$ and angular momentum $J=0$ neutron–neutron, proton–proton, and neutron–proton pairs in a spherical mean field, such as the spherical shell model, is proposed based on the standard $T=1$ pairing formalism. The advantage of the model lies in the fact that numerical solutions within the seniority-zero symmetric subspace can be obtained more easily and with less computational time than those calculated from the mean-field plus standard $T=1$ pairing model. Thus, large-scale calculations within the seniority-zero symmetric subspace of the model is feasible. As an example of the application, the average neutron–proton interaction in even–even $N \sim Z$ nuclei that can be suitably described in the f_5pg_9 shell is estimated in the present model, with a focus on the role of np-pairing correlations.

1. Introduction

The pairing interaction is known to be very important for mean-field descriptions of ground-state and low-energy properties of nuclei [1,2]. It has been shown that either spherical or deformed mean-field plus the standard (orbit-independent) pairing interaction among angular momentum $J=0$ like-nucleon pairs can be solved exactly by using the Gaudin–Richardson method [3–5]. The deformed and spherical mean-field plus the extended pairing interaction among $J=0$ like-nucleon pairs have also been proposed, which can be solved more easily than the standard pairing model, especially when both the number of like-nucleon pairs and the number of single-particle orbits are large [6,7]. It is also known that the ground-state properties and some properties of low-lying states of a chain of isotopes or isotones can be well described by these exactly solvable models [6–12]. Furthermore, as shown in [13], the extended multi-pairing interaction among like-nucleon pairs [6] can be obtained from the standard pairing interaction with an approximation, in which only the lowest eigenstate and the eigen-energy of the standard pairing interaction are taken into account. Actually, as shown in [13], this part of the standard pairing interaction, expressed as the extended multi-pairing interaction form, plays a dominant role for low-lying states, while the remaining part of the standard pairing interaction is less important to the low-lying states, especially when the number of nucleon pairs is small, which elucidates the origin of the extended pairing interaction. Hence, properties of low-lying states described by the extended pairing model are essentially the same as those described by the standard pairing model.

Extensions to equal strength neutron–neutron (nn), proton–proton (pp), and neutron–proton (np) isospin $T=1$ (charge-independent) pairing interactions has also been formulated [14–18], in which the total isospin T is a conserved quantity. Specifically, it has been shown that the $T=1$ pairing Hamiltonian, which will be called the standard $T=1$ pairing in the following, can be built from generators of the quasi-spin $O(5)$ group. However, a practical algorithm for diagonalizing a model with the $T=1$ pairing interaction in coupled or uncoupled basis of $O(5)$ irreducible representations (irreps) is still lacking. It should also be stated that, similar to the pairing model for like-nucleon pairs, approximate numerical solutions of the mean-field plus standard $T=1$ pairing Hamiltonian can also be obtained by using the BCS or HFB formalism [19–21], while simplified but reasonable exact solutions can be achieved by using an average energy (centroid) of the p orbits (e.g., see [22] for the simplest seniority-zero case). Exact solution of the mean-field plus standard $T=1$ pairing model was considered previously [23,24]. The common feature lies in the fact that a set of coupled multi-variable polynomial equations are involved, in which the order of the polynomials increases with in-

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creasing number of orbits and total number of nucleon-pairs as demonstrated in [25] for applications of [23] for up to three nucleon pairs around the cores of ^{16}O, ^{40}Ca, and ^{56}Ni. Though there is no practical limitations for the application of the exact solution of the standard T = 1 pairing in nuclei, it will be helpful if there is a reasonably simplified model to the problem that can be solved more easily.

2. An extended $T = 1$ pairing model and its exact solution

For a p-orbit system, the standard $T = 1$ pairing Hamiltonian is given by

$$\hat{H}_{\rm SP} = -G \sum_{\mu} A^{\dagger}_{\mu} A_{\mu}, \tag{1}$$

where $G > 0$ is the overall pairing interaction strength,

$$A^{\dagger}_{\mu} = \sum_{i=1}^{p} A^{\dagger}_{\mu}(j_i) = \sum_{i=1}^{p} \sum_{m_i>0} (-)^{j_i-m_i} a^{\dagger}_{j_i,m_i,\,\mu/2} a^{\dagger}_{j_i,-m_i,\,\mu/2}$$

for $\mu = 1$ or -1,

$$A^{\dagger}_{0} = \sum_{i=1}^{p} A^{\dagger}_{0}(j_i) = \sqrt{\frac{1}{2}} \sum_{i=1}^{p} \sum_{m_i>0} (-)^{j_i-m_i} (a^{\dagger}_{j_i,m_i,\,1/2} a^{\dagger}_{j_i,-m_i,\,-1/2} + a^{\dagger}_{j_i,m_i,\,-1/2} a^{\dagger}_{j_i,-m_i,\,1/2}) \tag{2}$$

are nucleon-pair creation operators, in which $a^{\dagger}_{j_i,m_i,\,m_t}$ ($a_{j_i,m_i,\,m_t}$) is the creation (annihilation) operator for a nucleon in the i-th orbit of a mean-field with angular momentum j_i, angular momentum projection m_i, and isospin projection m_t with $m_t = 1/2$ or $-1/2$. As shown in [14–17], $\{A^{\dagger}_{\mu},\ A_{\mu}\}$, together with the number operator of total nucleons $\hat{N} = \sum_{i=1}^{p} \hat{N}_{j_i}$ and the isospin operators $T_{\mu} = \sum_{i=1}^{p} T_{\mu}(j_i)$ ($\mu = +,\,-,\,0$), generate the quasi-spin $O(5)$ algebra, of which the commutation relations can be found, for example, in [23].

Let $|\rho\rangle$ be the orthonormalized basis vectors of $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$, in which $\rho \equiv \{(\omega_1, \omega_2)\,\beta\,\mathcal{N}\,T\,M_T;\ \eta\}$, where $(\omega_1, \omega_2) = (\Omega - v/2,\ t)$ is an irrep of $O(5)$ occurring in the reduction of the Kronecker product of p copies of $O(5)$ irreps $\otimes_{i=1}^{p}(\omega_{1,i},\ \omega_{2,i})$ of $O_1(5) \otimes \cdots \otimes O_p(5) \downarrow O(5)$, $\Omega = \sum_{i=1}^{p} \Omega_i = \sum_i (j_i + 1/2)$, v is the total seniority number, t is the reduced isospin of unpaired nucleons, β is the branching-multiplicity label needed in the $O(5) \downarrow O_T(3) \otimes O_{\mathcal{N}}(2)$ reduction, T and M_T are quantum number of total isospin and that of its projection, respectively, $\mathcal{N} = \Omega - N/2$ with N being the total number of nucleons, and η stands for a set of other quantum numbers related to the total angular momentum. Thus, $\{|\rho\rangle\}$ is a complete set of basis vectors needed in the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ basis. The standard $T = 1$ pairing interaction Hamiltonian (1) can then be expressed in terms of its complete set of eigenvalues and the corresponding eigenstates as

$$\hat{H}_{\rm SP} = \sum_{\rho} E^{\rho} |\rho\rangle\langle\rho|, \tag{3}$$

where the sum runs over all possible ρ. Since the eigenstates with $v = 0$ are the lowest in eigen-energy, similar to the extended quasi-spin $SU(2)$ pairing interaction [13], only the $v = 0$ sector involved in (3) will be adopted, with the other sectors that lie higher in energy and therefore less important than the $v = 0$ sector neglected as an approximation. To do so, one may equivalently introduce a projected $T = 1$ pairing interaction with

$$\tilde{H}_{\rm SP} = P_{v=0}\, \hat{H}_{\rm SP}\, P_{v=0}, \tag{4}$$

where

$$P_{v=0} = \sum_{\mathcal{N}\,T\,M_T} |(\Omega,\,0)\mathcal{N}\,T\,M_T\rangle\langle(\Omega,\,0)\mathcal{N}\,T\,M_T| \tag{5}$$

is a projection operator, in which the label β can be omitted because the reduction $(\Omega,\,0) \downarrow (\mathcal{N},\,T)$ is branching-multiplicity-free. It can be proven directly that

$$[C_2(O(5)),\ \tilde{H}_{\rm SP}] = 0,\ [\hat{\mathcal{N}},\ \tilde{H}_{\rm SP}] = 0,$$
$$[T_{\mu},\ \tilde{H}_{\rm SP}] = 0 \ \text{for}\ \mu = +,\ -,\ 0 \tag{6}$$

still hold, so that the projected Hamiltonian (4) preserves the $O(5) \supset O_T(3) \otimes O_{\mathcal{N}}(2)$ symmetry.

In the second quantization picture, for given Ω_i ($i = 1, \cdots, p$), the Hamiltonian (4) is

$$\tilde{H}_{\rm SP} = \sum_{n\,T\,M_T} E^{(\Omega,\,0)\,n\,T} \sum_{\rho_1,\cdots,\rho_p,\tilde{\rho}_1,\cdots,\tilde{\rho}_p} F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p} \times F^{n\,T\,M_T}_{\tilde{\rho}_1,\cdots,\tilde{\rho}_p} \prod_{i=1}^{p} K^{-1}_{n_i\,T_i}\, Z^{(n_i\,0)}_{T_i\,M_{T,\,i}} [\mathbf{A}^{\dagger}(j_i)] \times \prod_{i'=1}^{p} K^{-1}_{\tilde{n}_{i'}\,\tilde{T}_{i'}}\, Z^{(\tilde{n}_{i'}\,0)}_{\tilde{T}_{i'}\,\tilde{M}_{T,\,i'}} [\mathbf{A}(j_{i'})], \tag{7}$$

where $E^{(\Omega,\,0)\,n\,T} = -\frac{G_{\rm ext}}{2}\,(n(2\Omega + 3 - n) - T(T+1))$, in which n is the total number of nucleon-pairs, while the overall pairing strength G of the standard $T = 1$ pairing interaction is replaced by $G_{\rm ext}$, the additional quantum numbers η_i can be omitted in this case with $\rho_i \equiv \{n_i\,T_i\,M_{T,i}\}$ and $\tilde{\rho}_i \equiv \{\tilde{n}_i\,\tilde{T}_i\,\tilde{M}_{T,i}\}$, in which n_i or $\tilde{n}_i$ is the number of nucleon-pairs in the i-th orbit, $Z^{(n\,0)}_{TM_T}[\mathbf{A}^{\dagger}]$ is the polynomial of $\{A^{\dagger}_{\mu}\}$ given by [18]

$$Z^{(n\,0)}_{TM_T}[\mathbf{A}^{\dagger}] = \left[\frac{2^{T+M_T}(2T+1)!!(T+M_T)!(T-M_T)!T!}{(n-T)!!(n+T+1)!!(2T)!}\right]^{\frac{1}{2}} \times \left(2\,A^{\dagger}_1\,A^{\dagger}_{-1} - A^{\dagger 2}_0\right)^{\frac{n-T}{2}} \times \sum_{x={\rm Max}[0,M_T]}^{[(T+M_T)/2]} \frac{A^{\dagger\,x}_1 A^{\dagger\,T+M_T-2x}_0 A^{\dagger\,x-M_T}_{-1}}{2^x(x-M_T)!x!(T+M_T-2x)!}, \tag{8}$$

where x should be positive integer, and $[y]$ denotes the integer part of y.

$$K^{-1}_{n\,T} = \left[\frac{2^{\frac{1}{2}(n-T)}(\Omega - (n+T)/2)!(2\Omega + 1 - n + T)!!}{\Omega_i!(2\Omega+1)!!}\right]^{\frac{1}{2}}, \tag{9}$$

and

$$F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p} = \langle\rho_1, \cdots, \rho_p|(\Omega,\,0)\mathcal{N}\,T\,M_T\rangle \tag{10}$$

is the $O(5) \supset (O_T(3) \supset O_T(2)) \otimes O_{\mathcal{N}}(2)$ multi-coupling coefficient. According to the vector coherent state theory [18], in general, the overlap $F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}$ can be expressed as

$$F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}=\frac{K^{-1}_{n\,T}}{\prod_{i=1}^{p}K^{-1}_{n_i\,T_i}}\times$$
$$\left\langle\begin{matrix}(n_1,\,0) & \cdots & (n_p,\,0)\\ T_1\,M_{T,\,1} & \cdots & T_p\,M_{T,\,p}\end{matrix}\,\middle|\,\begin{matrix}(n,\,0)\\ T,\,M_T\end{matrix}\right\rangle=$$
$$\frac{K^{-1}_{n\,T}}{\prod_{i=1}^{p}K^{-1}_{n_i\,T_i}}\langle 0|\prod_{i=1}^{p}Z^{(n_i\,0)}_{T_i\,M_{T,\,i}}[\mathbf{b}(j_i)]Z^{(n\,0)}_{T\,M_T}[\mathbf{b}^\dagger]|0\rangle, \tag{11}$$

where $\left\langle\begin{matrix}(n_1,\,0) & \cdots & (n_p,\,0)\\ T_1\,M_{T,\,1} & \cdots & T_p\,M_{T,\,p}\end{matrix}\,\middle|\,\begin{matrix}(n,\,0)\\ T,\,M_T\end{matrix}\right\rangle$ is the $U_T(3)\supset O_T(3)\supset O_T(2)$ multi-coupling coefficient, $Z^{(n\,0)}_{T\,M_T}[\mathbf{b}^\dagger]$ is a polynomial of $\{b^\dagger_\mu\}$, which is of the same form as that shown in (8) with the replacement: $A^\dagger_\mu\Rightarrow b^\dagger_\mu$, $b^\dagger_\mu=\sum_{i=1}^{p}b^\dagger_\mu(j_i)$, and $\{b^\dagger_\mu(j_i),\,b_\mu(j_i)\}$ $(i=1,\cdots,p)$ are p copies of boson creation and annihilation operators satisfying $[b_\mu(j_i),\ b^\dagger_{\mu'}(j_{i'})]=\delta_{ii'}\delta_{\mu\mu'}$. The expression (11) is extremely useful since the boson-calculus can be mapped to the differential form with $b^\dagger_\mu(j_i)\Rightarrow z_{i,\,\mu}$ and $b_\mu(j_i)\Rightarrow\partial/\partial z_{i,\,\mu}$, which can then be calculated by using symbolic computation tools, such as Maple or Mathematica. Since $b^\dagger_\mu=\sum_{i=1}^{p}b^\dagger_\mu(j_i)$ is a symmetric function of $\{b^\dagger_\mu(j_i)\}$, the $U_T(3)\supset O_T(3)\supset O_T(2)$ multi-coupling coefficient involved in (11) can effectively be simplified as

$$\left\langle\begin{matrix}(n_1,\,0) & \cdots & (n_p,\,0)\\ T_1\,M_{T,\,1} & \cdots & T_p\,M_{T,\,p}\end{matrix}\,\middle|\,\begin{matrix}(n,\,0)\\ T,\,M_T\end{matrix}\right\rangle=$$
$$\frac{K^{-1}_{n\,T}}{\prod_{i=1}^{p}K^{-1}_{n_i\,T_i}}\langle 0|\prod_{i=1}^{p}Z^{(n_i\,0)}_{T_i\,M_{T,\,i}}[\mathbf{b}]Z^{(n\,0)}_{T\,M_T}[\mathbf{b}^\dagger]|0\rangle, \tag{12}$$

which greatly simplifies the computation process in comparison to the expression used in (11). It can be verified after the expansion that (7) not only contains the original standard $T=1$ pairing interaction $\hat{H}_{\rm SP}=-G_{\rm ext}\sum_\mu A^\dagger_\mu A_\mu$ among nucleon-pairs, but also contains multi-body $T=1$ pairing interactions with the overall pairing interaction strength $G_{\rm ext}$ up to infinite order when $\Omega\to\infty$ similar to the extended quasi-spin $SU(2)$ pairing interaction among like-nucleon pairs [6,7].

By using the procedure similar to that provided in [6,7], it can be shown that a spherical mean-field plus the extended pairing Hamiltonian

$$\hat{H}_{\rm ext}=\sum_{i=1}^{p}\epsilon_{j_i}\hat{N}_{j_i}+\tilde{H}_{\rm SP}, \tag{13}$$

where ϵ_{j_i} $(i=1,2,\cdots,p)$ are single-particle energies generated from any mean-field, is exactly solvable within the seniority-zero symmetric subspace, namely, with $v_i=0\ \forall\ i$, and for a given number of nucleon-pairs n with $T=n-2k$ for $k=0,1,\cdots,[n/2]$, where $[r]$ is the integer part of r, which is sufficient to describe ground state of most even–even and odd–odd nuclei. However, direct diagonalization of (13) in the general case seems not to be simple. Moreover, since (7) is still invariant under the $O_T(3)$ transformation, eigenvalues of (13) should also be independent of the quantum number M_T.

Similar to [6,7], in the seniority-zero case, the eigenstate of (13), up to a normalization constant, may be written as

$$|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle=\sum_{\rho_1,\cdots,\rho_p}\frac{F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}}{2\sum_{i=1}^{p}\epsilon_{j_i}n_i-E^{(\zeta_{n\,T})}_{n\,T}}\times$$
$$\prod_{i'=1}^{p}K^{-1}_{n_{i'}\,T_{i'}}\,Z^{(n_{i'}\,0)}_{T_{i'}\,M_{T,i'}}[\mathbf{A}^\dagger(j_{i'})]|0\rangle, \tag{14}$$

where $\mathcal{N}=\Omega-n$, $E^{(\zeta_{n\,T})}_{n\,T}$ is an eigenvalue of $\hat{H}_{\rm ext}$, in which $\zeta_{n\,T}$ labels the ζ-th excitation state for fixed quantum numbers n and T. It should be noted that the eigenstate expressed in (14) is only valid with $T=n-2k$ for $k=0,1,2,\cdots,[n/2]$ because the overlap $F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}$ is involved in (14), which is zero for other T values, namely, the solution is within the seniority-zero symmetric subspace.

One can directly check that

$$\sum_{i=1}^{p}\epsilon_{j_i}\hat{N}_{j_i}|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle=E^{(\zeta)}_{n\,T}|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle+$$
$$\sum_{\rho_1,\cdots,\rho_p}F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}\prod_{i'=1}^{p}K^{-1}_{n_{i'}\,T_{i'}}\ Z^{(n_{i'}\,0)}_{T_{i'}\,M_{T,i'}}[\mathbf{A}^\dagger(j_{i'})]|0\rangle \tag{15}$$

and

$$\tilde{H}_{\rm SP}|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle=$$
$$E^{(\Omega,0)\,n\,T}\sum_{\rho_1,\cdots,\rho_p}\frac{(F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p})^2}{2\sum_{i=1}^{p}\epsilon_{j_i}n_i-E^{(\zeta_{n\,T})}_{n\,T}}\times$$
$$\sum_{\tilde{\rho}_1,\cdots,\tilde{\rho}_p}F^{n\,T\,M_T}_{\tilde{\rho}_1,\cdots,\tilde{\rho}_p}\prod_{i'=1}^{p}K^{-1}_{n_{i'}\,\tilde{T}_{i'}}\ Z^{(n_{i'}\,0)}_{\tilde{T}_{i'}\,\tilde{M}_{T,i'}}[\mathbf{A}^\dagger(j_{i'})]|0\rangle. \tag{16}$$

Thus, the eigen-equation $\hat{H}_{\rm ext}|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle=E^{(\zeta_{n\,T})}_{n\,T}|\zeta_{n\,T},\mathcal{N}\,T\,M_T\rangle$ results in the following equation in determining the eigenvalue $E^{(\zeta_{n\,T})}_{n\,T}$:

$$1-\frac{G_{\rm ext}}{2}\left(n(2\Omega+3-n)-T(T+1)\right)\times$$
$$\sum_{\rho_1,\cdots,\rho_p}\frac{\left(F^{n\,T\,M_T}_{\rho_1,\cdots,\rho_p}\right)^2}{2\sum_{i=1}^{p}\epsilon_{j_i}n_i-E^{(\zeta_{n\,T})}_{n\,T}}=0\ \text{ for }\ n\neq 0, \tag{17}$$

and $E^{(\zeta_{0,0}=1)}_{0\,0}=0$ for $n=0$. Once (17) is solved, in which $E^{(\zeta_{n\,T})}_{n\,T}$ is the only variable of the equation, one obtains excitation energies $E^{(\zeta_{n\,T})}_{n\,T}$ and the corresponding eigenstates (14). When all single-particle energies are degenerate with $\epsilon_{j_i}=\epsilon\ \forall\ i$, there is only a unique solution of (17) with

$$E^{(\zeta_{n\,T}=1)}_{n\,T}=2\epsilon\,n-\frac{G_{\rm ext}}{2}\left(n(2\Omega+3-n)-T(T+1)\right), \tag{18}$$

which is exactly the ground-state eigenvalue of the standard isovector pairing model with degenerate single-particle energies [17] for given number of pairs n and isospin T. The corresponding normalized ground state is given by

$$|(\Omega,\,0)\mathcal{N}\,T\,M_T\rangle=K^{-1}_{n\,T}\,Z^{(n\,0)}_{T\,M_T}[\mathbf{A}^\dagger]|0\rangle. \tag{19}$$

As shown in (8), besides the T pairs with the third projection of the isospin M_T, (19) involves a condensate of $J=0$ and $T=0$ quartet $A^\dagger\cdot A^\dagger$. Therefore, it is obvious that the ground state of the extended isovector pairing model with degenerate single-particle energies is dominated by the α-like quartets when $T=0$ for even n cases and $T=1$ for odd n cases as shown in (8) and concluded in [26]. In addition, as shown recently in [27,28], the α-like quartets are also very important ingredients in the ground state of isovector pairing Hamiltonians with non-degenerate single-particle energies. As shown in (14), the ground state of the extended model with non-degenerate single-particle energies is a superposition of $\prod_{i=1}^{p}Z^{(n_i\,0)}_{T_i\,M_{T,i}}[\mathbf{A}^\dagger(j_i)]$, in which the $J=0$ and $T=0$ quartets $A^\dagger(j_i)\cdot A^\dagger(j_i)$ are also important. Since the ground state is symmetric with respect to the orbit permutations, it can be expected that the overlap of the ground state with an approximate one with $J = 0$ and $T = 0$ quartet condensate should also be significant. Due to the complicated multi-pair coupling structure of (14), a quantitative analysis of the $J = 0$ and $T = 0$ quartet content in the ground state of the model with non-degenerate single-particle energies may be made in our future work.

In order to show the difference and similarity of the mean-field plus standard $T = 1$ pairing model (SP) and the mean-field plus extended $T = 1$ pairing model (EXT), energy levels and eigenstates of the EXT Hamiltonian (13) were compared to those of the SP one [23] with the number of pairs $n \leq 3$ and other quantum numbers to be the same within the seniority-zero symmetric subspace. In our analysis, we take $G_{\rm ext} = G = 1.0$ MeV, and consider $p = 4$ orbitals with single-particle energies to be those in the f_5pg_9-shell deduced in [29] with $\epsilon_{3/2} = 0.000$ MeV, $\epsilon_{5/2} = 1.1193$ MeV, $\epsilon_{1/2} = 1.9892$ MeV, and $\epsilon_{9/2} = 3.5663$ MeV, where a constant $\epsilon_0 = -9.828$ MeV has been subtracted from each original single-particle energy provided in [29]. All level energies of the two models in the seniority-zero symmetric subspace for $n \leq 2$ and the $T = 3$ case for $n = 3$ are shown in Table 1. The $T = 1$ case for $n = 3$ is not provided because to solve the related equations for this case given in [23] is not easy, which may be analyzed in our future work.

It can be verified that the number of pairing excitation states for given n and T in the two models within the seniority-zero symmetric subspace is exactly the same, but there are obvious difference of the level energies from the two models when $n \geq 2$, especially in excited levels, because the pairing interaction term of the two models is different when $n \geq 2$. In the extended $T = 1$ pairing Hamiltonian there are high order terms involved, while there is only two-body term in the standard $T = 1$ pairing one. As shown in Table 1, the eigenstates and the corresponding eigenenergies of the two models are exactly the same when $n = 1$ and $T = 1$ if $G_{\rm ext} = G$ is taken, which is understandable because the high order terms vanish except the standard two-body one when the extended $T = 1$ pairing Hamiltonian is applied to an one-pair state. Since the pairing interaction of the two models is different, an eigenstate of the one model can be expanded in terms of those of the other. However, our calculation shows that the overlap of the lowest eigenstate of the two models is significant, where the overlap is defined by

$$\mathrm{O}(n, T, \zeta) = |\langle \zeta, n\, T\, M_T | \zeta, n\, T\, M_T \rangle_{\rm SP}| \tag{20}$$

for given n, T, M_T with $\zeta = 1, 2, \cdots$, where $|\zeta, n\, T\, M_T\rangle_{\rm SP}$ is the corresponding eigenstate of the mean-field plus standard $T = 1$ model, which is greater than 70% in the lowest eigenstate for given n and T, but is typically smaller for increasing excitation energy and the number of pairs, while overlaps for some higher-lying states are also significant. Though at present we do not know which model is better in describing excited states with 8 MeV higher in excitation energy than that of the ground state of $N \sim Z$ nuclei in this region, the ground state of the two models is basically similar in nature. The ground-state energy difference can be diminished by adjusting the pairing interaction strength. Therefore, it is expected that the extended $T = 1$ pairing model (EXT), like the standard $T = 1$ pairing model (SP), can be used to describe ground state of $N \sim Z$ nuclei with similar fitting quality.

3. An example of application

As an example of an application, we use the exact solution of the EXT within the seniority-zero symmetric subspace to estimate np-pairing contribution in even–even $N \sim Z$ nuclei suitably

Table 1
Eigen-energies of the mean-field plus extended $T = 1$ pairing model (EXT) and those of the standard $T = 1$ pairing model (SP) obtained by using the exact solution provided in [23] in the seniority-zero symmetric subspace with single-particle energies of the f_5pg_9-shell and $G_{\rm ext} = G = 1.0$ MeV for the number of pairs $n \leq 3$ (see text), where "–" denotes that the corresponding level does not exist. The overlap of the corresponding eigenstates of the two models is defined by (20).

	n	T	$\zeta=1$	$\zeta=2$	$\zeta=3$	$\zeta=4$	$\zeta=5$	$\zeta=6$	$\zeta=7$	$\zeta=8$	$\zeta=9$	$\zeta=10$	$\zeta=11$	$\zeta=12$	$\zeta=13$	$\zeta=14$	$\zeta=15$
SP	1	1	−7.53623	0.867038	3.56528	5.45351	–	–	–	–	–	–	–	–	–	–	–
EXT			−7.53623	0.867038	3.56528	5.45351	–	–	–	–	–	–	–	–	–	–	–
Overlap			100%	100%	100%	100%											
SP	2	2	−12.9216	−4.1491	−2.2039	−0.63758	1.4522	4.5081	6.2417	8.3344	11.1669	–	–	–	–	–	–
EXT			−12.3311	0.16425	2.9793	4.1425	5.0180	6.4203	8.1939	10.7869	13.4172	–	–	–	–	–	–
Overlap			98.90%	66.39%	55.06%	27.21%	9.90%	42.91%	79.54%	85.34%	89.96%	–	–	–	–	–	–
SP	2	0	−16.1364	−7.84463	−4.89564	−2.8597	1.8126	4.3968	6.35437	6.96743	9.12783	10.8254	–	–	–	–	–
EXT			−15.3366	0.337164	2.89732	4.09608	5.18967	6.42473	7.86375	8.29938	10.7813	13.1952	–	–	–	–	–
Overlap			70.32%	63.28%	27.80%	11.38%	12.34%	44.17%	61.06%	15.28%	82.34%	16.26%					
SP	3	3	−16.1532	−6.7326	−5.7113	−4.5175	0.50469	1.0709	2.4890	3.7915	5.5207	6.2377	6.9312	8.8449	11.7519	13.5953	17.4442
EXT			−15.1038	2.4473	4.0157	4.81954	6.44238	6.7633	7.3955	8.5280	10.3652	11.3204	12.5754	13.8132	15.6437	18.0005	21.0409
Overlap			97.36%	50.09%	31.44%	33.59%	42.42%	12.92%	30.19%	36.80%	13.46%	10.97%	21.84%	52.06%	68.58%	79.81%	84.49%

to be described in the f_5pg_9-shell outside the ^{56}Ni core with the single-particle energies shown above. In our calculation, interaction between the core and valence nucleon-pairs is neglected. For even–even $N \sim Z$ nuclei, the average np-interaction energy defined as [31–33]

$$\delta V_{\rm pn}^{\rm ee}(A = Z + N) \equiv \delta V_{\rm pn}^{\rm ee}(Z, N) = \frac{1}{4}(B(Z, N) + B(Z-2, N-2) - B(Z, N-2) - B(Z-2, N)), \quad (21)$$

where $B(Z, N)$ is the binding energy of the even–even nucleus, is used to estimate the np-interaction, which is considered to be the np-pairing contribution approximately, in the even–even $N \sim Z$ nuclei.

Since ^{56}Ni is taken to be the core, the binding energy of a nucleus considered is defined as

$$B(28 + N_\pi, 28 + N_\nu) = B(28, 28) + E_{\rm C}(28, 28) - E_{\rm C}(28 + N_\pi, 28 + N_\nu) - E_{\rm sym}(28 + N_\pi, 28 + N_\nu) + (N_\pi + N_\nu)E_0 - E^{(1)}_{(N_\pi + N_\nu)/2}, \quad (22)$$

where

$$E_{\rm C}(Z, N) = 0.7173\,\frac{Z(Z-1)}{A^{1/3}}(1 - Z^{-2/3})\ {\rm MeV} \quad (23)$$

and, with $I = |N - Z|/A$,

$$E_{\rm sym}(Z, N) = \frac{29.2876}{A}|N - Z|^2 \times (1 + \frac{2 - |I|}{2 + |I|A} - \frac{1.4492}{A^{1/3}})\ {\rm MeV}, \quad (24)$$

are the Coulomb and symmetry energy [34], respectively, where A is the mass number $A = N + Z$, N_π and N_ν are the number of valence protons and neutrons, respectively, E_0 is the average binding energy per valence nucleon in the f_5pg_9-shell, which is almost a constant, and $E^{(1)}_n$ with $n = (N_\pi + N_\nu)/2$ is the lowest eigen-energy calculated from the mean-field plus extended $T = 1$ pairing model. The I correction term introduced in the symmetry energy (24) approximately describes the Wigner effect [34]. Using (22), we fit the even–even $N = Z$ and $N = Z \pm 2$ nuclei with mass number $A = 58$–80 in this region. The constant E_0 was chosen such that the extended pairing interaction strength $G_{\rm ext} \sim 1.0$ MeV for the $n = 1$ case is comparable to the orbit-dependent pairing interaction parameters of the $J = 0$ and $T = 1$ pairing interactions determined in [29] for the f_5pg_9-shell. Hence, we set $E_0 = 7.5$ MeV which is very close to the empirical binding energy per particle in nuclei. The results for the even–even $N \sim Z$ nuclei concerned are provided in Table 2. The pairing interaction strength $G_{\rm ext}$ can be adjusted accurately to fit the binding energy of even-n nuclei with isospin $T = 0$ at the ground state. The deviation occurs in fitting the binding energy of odd-n nuclei with a $T = 1$ ground state because there is a less than 0.475 MeV difference in the actual pairing energy contribution to the binding energies of the mirror nuclei due to a small isospin asymmetry. The calculated average np-interaction energies for mass number $A = 60$–80 in comparison to the corresponding experimental data are shown in Fig. 1.

It should be noted that the isovector pairing energy contribution to the total binding energy considered in this work is different from that considered previously [27,28,35,36]. In [27,28], the Coulomb, symmetry, and Winer energy contribution are not considered, with which the charge-independent pairing interaction strength G follows the simple mass-dependent law with

Table 2
The pairing interaction strength $G_{\rm ext}$ (in MeV) for some even–even $N \sim Z$ nuclei with valence nucleons confined to the f_5pg_9-shell deduced from the mean-field plus extended $T = 1$ pairing Hamiltonian (13) according to (22), where n is the number of valence nucleon-pairs in the corresponding nucleus, $E^{(1)}_n$ (in MeV) is the lowest eigen-energy of the mean-field plus extended $T = 1$ Hamiltonian (13) with total isospin $T = 0$ for the even–even $N = Z$ nuclei or $T = 1$ for the even–even $N = Z \pm 2$ nuclei, $B_{\rm Th}$ (in MeV) is the binding energy calculated according to (22), and the experimental binding energy $B_{\rm Exp}$ (in MeV) of these nuclei is taken from [30].

Nucleus	n	$E^{(1)}_n$	$G_{\rm ext}$	$B_{\rm Th}$	$B_{\rm Exp}$
$^{58}_{28}$Ni$_{30}$	1	−8.6317	1.1060	506.498	506.459
$^{58}_{30}$Zn$_{28}$	1	−8.6317	1.1060	486.878	486.962
$^{60}_{30}$Zn$_{30}$	2	−17.5138	0.9871	514.982	514.982
$^{62}_{30}$Zn$_{32}$	3	−26.4140	1.2010	537.989	538.119
$^{62}_{32}$Ge$_{30}$	3	−26.4140	1.2010	517.384	517.266
$^{64}_{32}$Ge$_{32}$	4	−35.7222	1.2376	545.844	545.844
$^{66}_{32}$Ge$_{34}$	5	−44.6005	1.3290	569.045	569.279
$^{66}_{34}$Se$_{32}$	5	−44.6005	1.3290	547.477	547.470
$^{68}_{34}$Se$_{34}$	6	−54.4656	1.3900	576.439	576.439
$^{70}_{34}$Se$_{36}$	7	−63.8158	1.4974	600.312	600.322
$^{70}_{36}$Kr$_{34}$	7	−63.8158	1.4974	577.801	577.780
$^{72}_{36}$Kr$_{36}$	8	−73.8718	1.5756	606.911	606.911
$^{74}_{36}$Kr$_{38}$	9	−83.8873	1.7100	631.636	631.445
$^{74}_{38}$Sr$_{36}$	9	−83.8873	1.7100	608.199	608.354
$^{76}_{38}$Sr$_{38}$	10	−94.6019	1.8182	637.936	637.936
$^{78}_{38}$Sr$_{40}$	11	−105.2193	1.9905	663.475	663.000
$^{78}_{40}$Zr$_{38}$	11	−105.2193	1.9905	639.130	639.600
$^{80}_{40}$Zr$_{40}$	12	−117.047	2.1450	669.920	669.920

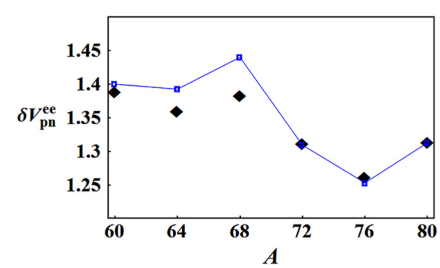

Fig. 1. (Color online.) $\delta V_{\rm pn}^{\rm ee}$ values (in MeV) derived from binding energies of even–even $N = Z$ and $N = Z \pm 2$ nuclei with mass number $A = 60 + 4k$ for $k = 0, 1, \cdots, 5$, where the solid diamonds are the experimental data [30], and the open squares, linked with the solid line to guide the eye, are the results calculated according to the mass formula (22) using the mean-field plus extended $T = 1$ Hamiltonian (13).

$G \sim 24/A$ MeV shown in [27,28]. In [35], besides the pure symmetry energy contribution, the Wigner energy contribution to the binding energy was also considered, but the Coulomb energy is treated as a constant for a given isobaric chain, with which the simple mass-dependent law becomes $G \sim 12/A^{3/4}$ MeV as shown in [35]. While, besides Wigner energy contribution, Coulomb and symmetry energy are treated to be a constant in [36], with which a similar mass-dependent law with $G \sim 13.9/A^{3/4}$ MeV was obtained. In (22), however, both the Coulomb and symmetry energy are mass number dependent, in which the Wigner energy contribution is also included approximately. As shown in Table 2, the total binding energy of each nucleus considered is fitted much more accurately with the inclusion of the Coulomb and symmetry energies in (22). We observe that the inclusion of the Coulomb and symmetry energies in the binding energy (22) affects the mass-dependent law of the pairing strength $G_{\rm ext}$. The fitting quality of (22) for the 18 nuclei shown in Table 2 is measured by

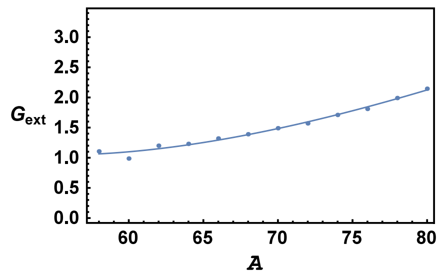

Fig. 2. (Color online.) The extended isovector pairing interaction strength G_{ext} (in MeV) fitted by a quadratic function of the mass number A for $A = 58$–80, from which we get $G_{ext} = 4.262 - 0.1308A + 0.0013A^2$ MeV (solid line), where the solid dots are obtained from the exact fitting to the binding energy shown in Table 2.

$\chi^2 = \sum_{i=1}^{18}(B_{\mathrm{Exp,\,i}} - B_{\mathrm{Th,\,i}})^2/17$ with $\chi^2 = 0.0354$ (MeV)^2, where B_{Exp} and B_{Th} are experimental binding energy and the corresponding result of this theory, respectively. As shown in Fig. 1, $\delta V_{\mathrm{pn}}^{\mathrm{ee}}(N, Z)$ with $A = N + Z = 60 + 4k$ for $k = 0, 1, \cdots, 5$ calculated from the mean-field plus extended $T = 1$ pairing model is in good agreement with the experimental data. It should be stated that the average np-pairing interaction energy $\delta V_{\mathrm{pn}}^{\mathrm{ee}}$ mainly depends on the pairing interaction strength G_{ext} of the four nuclei involved, and is independent of the constant E_0, while the Coulomb and symmetry energy terms used in (22) also affect $\delta V_{\mathrm{pn}}^{\mathrm{ee}}$ a little. As stated previously, deviation from the experimental data, especially for $A = 64$ and 68, is mainly due to the small isospin asymmetry in the mirror nuclei concerned. Generally speaking, G_{ext} is a smooth function of the number of pairs n, which slightly increases with increasing number of nucleon-pairs, except a small decrease for the $n = 2$ case. The mass-dependent law of the extended isovector pairing interaction strength G_{ext} can then be obtained by using a polynomial of A fitting to G_{ext}. The fitting result of a quadratic function of the mass number A with $A = 58$–80 is shown in Fig. 2. Though G_{ext} seems fitted by the quadratic function of the mass number A quite well, there will be more than 0.5 MeV deviation in the average np-pairing interaction energy $\delta V_{\mathrm{pn}}^{\mathrm{ee}}$ if the degree of the polynomial $G_{ext}(A)$ is less 8, which is quite similar to the situation shown in [35,36], where only a reasonable overall agreement of the mass differences with experimental data can be provided with the simple mass-dependent law $G \propto A^{-3/4}$ MeV, indicating the average np-pairing interaction energy is very sensitive to the isovector pairing interaction strength. The difference of the mass dependent law of G_{ext} from that of the standard isovector pairing indicates the two models are quite different for $n \geq 3$. A small decrease in G_{ext} for the $n = 2$ case is mainly due to the two models are quite the same for $n \leq 2$.

4. Conclusions

In this work, an extended pairing Hamiltonian that describes multi-pair interactions among isospin $T = 1$ and angular momentum $J = 0$ nn-, pp-, and np-pairs in a spherical mean-field, such as the spherical shell model, is proposed based on the standard $T = 1$ pairing formalism. Since only a single-variable polynomial equation is involved, large-scale calculations within the seniority-zero symmetric subspace of the model should be feasible, though the calculation of the example provided is only within the f_5pg_9-shell. By using the exact solution within the seniority-zero symmetric subspace of the model, the average np-pairing interaction energies in even–even $N \sim Z$ nuclei with mass number $A = 58$–80 are estimated. Due to the special Bethe ansatz used, only symmetric solution of the model with seniority zero can be solved exactly, while other solutions of the model still need to be obtained from direct diagonalizations, in which the matrix elements of the Hamiltonian needed are considerably more complicated than those of the standard $T = 1$ pairing Hamiltonian. Therefore, an algorithm based on the quasi-spin $O(5)$ algebra is in demand for investigating $N \sim Z$ nuclei systematically, which will be a part of our future work. A detailed investigation of the Wigner energy contribution to the binding energy and a detailed comparison of the extended with the standard isovector pairing model can then be made.

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