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An exact solution of nuclear spherical mean-field plus orbit-dependent non-separable pairing model with two non-degenerate j-orbits is presented. The extended one-variable Heine-Stieltjes polynomials associated to the Bethe ansatz equations of the solution are determined, of which the sets of the zeros give the solution of the model, and can be determined relatively easily. A comparison of the solution to that of the standard pairing interaction with constant interaction strength among pairs in any orbit is made. It is shown that the overlaps of eigenstates of the model with those of the standard pairing model are always large, especially for the ground and the first excited state. However, the quantum phase crossover in the non-separable pairing model cannot be accounted for by the standard pairing interaction.

Keywords: Non-separable pairing interaction; exact solvable models; Bethe ansatz.

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1. Introduction: Pairing correlations seem evident in various quantum many-body systems. [1–4]. It has been shown that pairing interactions are key to elucidating ground state and low-energy spectroscopic properties of nuclei [5–7]. Though the Bardeen-Cooper-Schrieffer (BCS) [1] and the Hartree-Fock-Bogolyubov (HFB) approximations provide simple and clear pictures [6, 8, 9], tremendous efforts have been made in finding exact solutions to the problem [10–15]. It is known that spherical or deformed mean-field plus the standard (equal strength) pairing interaction can be solved exactly by using the Gaudin-Richardson method [18–20], which can now be solved more easily by using the extended Heine-Stieltjes polynomial approach [21–24]. The separable pairing problem was studied in [25], in which the single-particle energies are all degenerate. The separable pairing interaction with two non-degenerate levels was analyzed in [26], of which solution with multi non-degenerate levels of a special case was given in [27–29], while the general case has been analyzed in [30]. In this work, it will be shown that the orbit-dependent non-separable pairing interaction among valence nucleons over two non-degenerate orbits can also be solved analytically.

2. The model and exact solution: The Hamiltonian of a spherical mean-field plus orbit-dependent non-separable pairing model (NSPM) with two non-degenerate j-orbits can be written as

$$\hat{H} = \sum_{t}^{p} \epsilon_t \hat{N}_{j_t} + \hat{H}_{\rm P} = \sum_{t}^{p} \epsilon_t \hat{N}_{j_t} - \sum_{1\leq t,t'\leq p} g_{t,t'}\, S^+_{j_t} S^-_{j_{t'}}, \tag{1}$$

where $p = 2$ is the total number of orbits considered above a closed or sub-closed shell, $\{\epsilon_t\}$ $(t = 1,\ 2)$ is single-particle energies generated from a mean-field theory with $\epsilon_1 \neq \epsilon_2$, $\hat{N}_j = \sum_m a^\dagger_{jm} a_{jm}$ and $S^+_j = \sum_{m>0} (-1)^{j-m} a^\dagger_{jm} a^\dagger_{j-m}$, in which $a^\dagger_{jm}$ (a_{jm}) is the creation (annihilation) operator for a nucleon with angular momentum quantum number j with projection m, and $\{g_{t,t'}\}$ $(t, t' = 1, 2)$ are the non-separable pairing interaction parameters, which are all assumed to be real and must be symmetric with $g_{1,2} = g_{2,1}$.

The set of local operators $\{S^-_{j_t},\ S^+_{j_t},\ \hat{N}_{j_t}\}$ $(t = 1, 2)$, where $S^-_{j_t} = (S^+_{j_t})^\dagger$, generate two copies of an SU(2) algebra that satisfies the commutation relations $[\hat{N}_{j_t}/2,\ S^-_{j_{t'}}] = -\delta_{tt'} S^-_{j_t}$, $[\hat{N}_{j_t}/2,\ S^+_{j_{t'}}] = \delta_{tt'} S^+_{j_t}$, $[S^+_{j_t},\ S^-_{j_{t'}}] = 2\delta_{tt'} S^0_{j_t}$, where $S^0_{j_t} = (\hat{N}_{j_t} - \Omega_t)/2$ with $\Omega_t = j_t + 1/2$. As adopted in the Gaudin-Richardson approach [18–20] for the standard pairing model (SPM), let

$$S^+(x) = \sum_{t}^{2} \frac{1}{2\epsilon_t - x} S^+_{j_t}, \tag{2}$$

where x is the spectral parameter to be determined. According to the commutation relations of the generators of the two copies of the SU(2) algebra, we have

$$[\sum_t \epsilon_t \hat{N}_{j_t},\ S^+(x)] = \sum_t \frac{2\epsilon_t}{2\epsilon_t - x} S^+_{j_t} = S^+ + x\, S^+(x), \tag{3}$$

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where $S^+ = \sum_t S^+_{j_t}$, and

$$[\hat{H}_{\rm P},\ S^+(x)] = \sum_{t',t} g_{t',t} S^+_{j_{t'}} \frac{2S^0_{j_t}}{2\epsilon_t - x}, \tag{4}$$

$$[[\hat{H}_{\rm P},\ S^+(x)],\ S^+(y)] = 2\sum_{t',t} g_{t',t}\frac{1}{(2\epsilon_t - x)(2\epsilon_t - y)} S^+_{j_{t'}} S^+_{j_t}. \tag{5}$$

The k-pair eigenvectors of (1) can be still written as the Gaudin-Richardson form with

$$|\zeta,\ k; JM\rangle = \prod_{\rho}^{k} S^+(x^{(\zeta)}_\rho)|JM\rangle, \tag{6}$$

where ζ labels the ζ-th set of solution $\{x^{(\zeta)}_1, \cdots, x^{(\zeta)}_k\}$. If the seniority number of the t-th orbit is ν_t, the pairing vacuum states of these two orbits are denoted as $|\nu_t\eta_t J_t M_t\rangle$ satisfying $S^-_{j_t}|\nu_{j_t}\eta_t J_t M_t\rangle = 0$, where J_t and M_t are the angular momentum quantum number and that of its third component, respectively, and η_t is the multiplicity label needed to distinguish different possible ways of ν_t particles coupled to the angular momentum J_t. Thus, a pairing vacuum state of a two j-orbit system with the total seniority number $\nu = \nu_1 + \nu_2$ and the total angular momentum J can be expressed as $|JM\rangle \equiv |\nu_1\eta_1, \nu_2\eta_2; (J_1 \otimes J_2)JM\rangle$. Thus, $|JM\rangle$ satisfies $S^-_{j_t}|JM\rangle = 0$ for $t = 1, 2$, which is used in (6).

To solve the eigen-equation of (1) with ansatz (6), one can calculate commutators of $\hat{H}$ with the pairing operators $S^+(x^{(\zeta)}_\rho)$ as was done in Richardson's work on solving the SPM [19, 20]. Since (1) only contains one- and two-body interaction terms, the q-time commutators $[\cdots[\hat{H}, S^+(x^{(\zeta)}_{\rho_1})], \cdots, S^+(x^{(\zeta)}_{\rho_{q-1}})], S^+(x^{(\zeta)}_{\rho_q})]$ vanish when $q \geq 3$. Namely, one only needs to calculate single and double commutators of $\hat{H}$ with the operators $S^+(x^{(\zeta)}_\rho)$. Since we use the pairing operator (2) to construct the eigen-vectors (6), the commutator of the one-body mean-field term of (1) with $S^+(x^{(\zeta)}_\rho)$ is given by (3), while (4) can be expressed in terms of the collective operators $S^+(x)$ and S^+ appearing on the right-hand-side of (3) when the commutator is applied to the vacuum state $|JM\rangle$ with

$$[\hat{H}_{\rm P},\ S^+(x)]|JM\rangle = \sum_{t',t} g_{t',t} S^+_{j_{t'}} \frac{2S^0_{j_t}}{2\epsilon_t - x}|JM\rangle = (\alpha(x)\, S^+ + \beta(x)\, S^+(x))|JM\rangle. \tag{7}$$

After solving the above binomial equations of the local operators $S^+_{j_1}$ and $S^+_{j_2}$, one obtains

$$\begin{aligned}\alpha(x) &= -\tfrac{(x-2\epsilon_2)\left((x-2\epsilon_1)g_{1,1}-(x-2\epsilon_2)g_{1,2}\right)(\Omega_1-\nu_1)+(x-2\epsilon_1)\left((x-2\epsilon_1)g_{1,2}-(x-2\epsilon_2)g_{2,2}\right)(\Omega_2-\nu_2)}{2(x-2\epsilon_1)(x-2\epsilon_2)(\epsilon_1-\epsilon_2)},\\ \beta(x) &= -\tfrac{(x-2\epsilon_2)(g_{1,1}-g_{1,2})(\Omega_1-\nu_1)+(x-2\epsilon_1)(g_{1,2}-g_{2,2})(\Omega_2-\nu_2)}{2(\epsilon_1-\epsilon_2)},\end{aligned} \tag{8}$$

where the condition $g_{2,1} = g_{1,2}$ is used. It is obvious that (8) also assumes $\epsilon_1 \neq \epsilon_2$, which is valid for non-degenerate cases. It is clear that the expression shown on the right-hand-side of (7) is impossible when the number of orbits $p \geq 3$. For the standard pairing interaction with $g_{t,t'} = G\ \forall\ t, t'$, (7) becomes the commutators shown in Richardson's work [19, 20] with $\beta(x) = 0$. Similarly, the double commutator $S^+(x,y) = [[\hat{H}_{\rm P},\ S^+(x)],\ S^+(y)]$ given in (5) is a homogenous binomial of degree 2 in $S^+_{j_t}$ with $t = 1, 2$ for the two j-orbit case, which, therefore, can be expressed in terms of 3 independent terms. Hence, similar to the commutators shown in the SPM, one can write (5) as

$$S^+(x,y) = 2\sum_{t',t} g_{t',t}\,\frac{1}{(2\epsilon_t - x)(2\epsilon_t - y)} S^+_{j_{t'}} S^+_{j_t} = a(x,y)\, S^+S^+(x) + b(x,y)\, S^+S^+(y) + c(x,y)\, S^+(x)S^+(y), \tag{9}$$

which expressed in terms of S^+, $S^+(x)$, and $S^+(y)$ is only possible for two j-orbit case. For a system with p j-orbits, $p(p+1)/2$ terms are needed on the right-hand-side of (9). For example, six terms on the right-hand-side of (9) are needed for the three j-orbit case. Hence, though it is possible to solve a multi j-orbit system by using this procedure, the results will be very complicated with p variables for a two-pair state. After comparing the coefficients of $S^+_{j_t}S^+_{j_{t'}}$ with the same t and t' on both sides of (9), one gets

$$\begin{aligned} a(x,y) &= \tfrac{1}{2(x-y)(\epsilon_1-\epsilon_2)^2} F(x,y),\ \ b(x,y) = a(y,x),\\ c(x,y) &= \tfrac{1}{2(\epsilon_1-\epsilon_2)^2}\,\big((x+y)(2\epsilon_2(g_{1,2}-g_{1,1}) + 2\epsilon_1(g_{1,2}-g_{2,2})) +\\ &\quad x\,y\,(g_{1,1}+g_{2,2}-2g_{1,2}) + 4\epsilon_2^2(g_{1,1}-g_{1,2}) + 4\epsilon_1^2(g_{2,2}-g_{1,2})\big),\end{aligned} \tag{10}$$

where

$$F(x,y) = x\,(2\epsilon_1(g_{1,1}-g_{1,2}) + 2\epsilon_2(g_{2,2}-g_{1,2})) + y\,(2\epsilon_2(g_{1,1}-g_{1,2}) + 2\epsilon_1(g_{2,2}-g_{1,2})) + x\,y\,(2g_{1,2}-g_{1,1}-g_{2,2}) + 4g_{1,2}(\epsilon_1^2+\epsilon_2^2) - 4\epsilon_1\epsilon_2(g_{1,1}+g_{2,2}), \tag{11}$$

and $c(x,y)$ is obviously symmetric in x and y.

Using Eqs. (3) ,(7), and (9), one can directly check that

$$\sum_t \epsilon_t\,\hat{N}_{j_t}|\zeta,\ k;JM\rangle = \sum_i^k S^+\prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle + \sum_i^k x_i^{(\zeta)}\prod_\rho^k S^+(x_\rho^{(\zeta)})|JM\rangle \tag{12}$$

and

$$\hat{H}_{\rm P}|\zeta,\ k;JM\rangle = \sum_i^k \alpha(x_i^{(\zeta)})\,S^+\prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle + \sum_i^k \beta(x_i^{(\zeta)})\prod_\rho^k S^+(x_\rho^{(\zeta)})|JM\rangle + \sum_i^k\sum_{i'\,(\neq i)}^k a(x_{i'}^{(\zeta)},x_i^{(\zeta)})\,S^+\prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle + \sum_i^k\sum_{i'=i+1}^k c(x_i^{(\zeta)},x_{i'}^{(\zeta)})\prod_\rho^k S^+(x_\rho^{(\zeta)})|JM\rangle. \tag{13}$$

With (12) and (13), one can prove that the eigen-equation $\hat{H}|\zeta,k;JM\rangle = E_k^{(\zeta)}|\zeta,k;JM\rangle$ is fulfilled if and only if

$$1 + \alpha(x_i^{(\zeta)}) + \sum_{i'\,(\neq i)}^{k} a(x_{i'}^{(\zeta)},x_i^{(\zeta)}) = 0 \ \text{ for } \ i = 1,2,\cdots k, \tag{14}$$

with the corresponding eigen-energy

$$E_k^{(\zeta)} = \sum_{t=1}^p \epsilon_t\nu_t + \sum_{i=1}^k\left(x_i^{(\zeta)} + \beta(x_i^{(\zeta)}) + \sum_{i'=i+1}^k c(x_i^{(\zeta)},x_{i'}^{(\zeta)})\right) = \sum_{t=1}^p \epsilon_t\nu_t + \left(\frac{(g_{1,1}-g_{1,2})(\Omega_1-\nu_1)\epsilon_2}{\epsilon_1-\epsilon_2} + \frac{(g_{1,2}-g_{2,2})(\Omega_2-\nu_2)\epsilon_1}{\epsilon_1-\epsilon_2} + \frac{\epsilon_1^2(g_{2,2}-g_{1,2})+\epsilon_2^2(g_{1,1}-g_{1,2})}{(\epsilon_1-\epsilon_2)^2}(k-1)\right)k + \left(1 - \frac{(g_{1,1}-g_{1,2})(\Omega_1-\nu_1)}{2\epsilon_1-2\epsilon_2} - \frac{(g_{1,2}-g_{2,2})(\Omega_2-\nu_2)}{2\epsilon_1-2\epsilon_2} + \frac{\epsilon_2(g_{1,2}-g_{1,1})+\epsilon_1(g_{1,2}-g_{2,2})}{(\epsilon_1-\epsilon_2)^2}(k-1)\right)\sum_{i=1}^k x_i^{(\zeta)} + \frac{g_{1,1}+g_{2,2}-2g_{1,2}}{4(\epsilon_1-\epsilon_2)^2}\left((\sum_{i=1}^k x_i^{(\zeta)})^2 - \sum_{i=1}^k (x_i^{(\zeta)})^2\right), \tag{15}$$

where $\sum_{t=1}^p \epsilon_t\nu_t$ is contributed from particles in the pairing vacuum. One can easily check that, when $g_{t,t'} = G\ \forall\ t,t'$, $\alpha(x) = -G\sum_t(\Omega_t-\nu_t)/(2\epsilon_t - x)$, $a(x,y) = 2G/(x-y)$, $\beta(x) = c(x,y) = 0$, with which (14) and (15) become the Bethe ansatz equations and the corresponding eigen-energy of the SPM with $E_k^{(\zeta)} = \sum_{i=1}^k x_i^{(\zeta)}$ known previously [19, 20]. Thus, the solution provided by (6), (14), and (15) include the standard and separable pairing models with two non-degenerate j-orbits as special cases, though the form of the eigenstates shown in (6) for the separable pairing case with $g_{t,t'} = g_t g_{t'}$, where g_t ($t = 1,\cdots,p$) is a set of real parameters, looks quite different from that used previously [26–29]. It should be pointed out that (14) also implies $g_{1,2} \neq 0$. There will no solution of (14) when $g_{1,2} = 0$. Actually, similar to the case with no pairing interaction, a product of the single-particle states is an eigen-state of (1) when $g_{1,2} = 0$. Hence, $g_{1,2} \neq 0$ is assumed.

According to the Heine-Stieltjes correspondence [21, 22], zeros $\{x_i^{(\zeta)}\}$ of the extended Heine-Stieltjes polynomials $y_k(x)$ of degree k are roots of Eq. (14), which should satisfy the following second-order Fuchsian equation:

$$A(x)y_k''(x) + B(x,k)y_k'(x) - V(x,k)y_k(x) = 0. \tag{16}$$

Here,

$$A(x) = \frac{1}{2}(x^2F_{12} + x\,(F_1+F_2) + F_0)\prod_{t=1}^{2}(2\epsilon_t - x) \tag{17}$$

is a polynomial of degree 4, in which

$$F_1 = \frac{\epsilon_1(g_{1,1}-g_{1,2})+\epsilon_2(g_{2,2}-g_{1,2})}{(\epsilon_1-\epsilon_2)^2},\ F_2 = \frac{\epsilon_2(g_{1,1}-g_{1,2})+\epsilon_1(g_{2,2}-g_{1,2})}{(\epsilon_1-\epsilon_2)^2},\ F_{12} = \frac{2g_{1,2}-g_{1,1}-g_{2,2}}{2(\epsilon_1-\epsilon_2)^2},\ F_0 = \frac{2g_{1,2}(\epsilon_1^2+\epsilon_2^2)-2\epsilon_1\epsilon_2(g_{1,1}+g_{2,2})}{(\epsilon_1-\epsilon_2)^2}, \tag{18}$$

the polynomial $B(x,k)$ of degree 3 is given as

$$B(x,k)/A(x) = \frac{2}{x^2F_{12} + x\,(F_1+F_2) + F_0}\left(\sum_{t=1}^{2}\frac{\alpha_t^{(1)} + \alpha_t^{(2)}\,x}{2\epsilon_t - x} - (F_1 + F_{12}\,x)\,(k-1) - 1\right), \tag{19}$$

where

$$\alpha_1^{(1)} = \frac{(\epsilon_1 g_{1,1} - \epsilon_2 g_{1,2})(\Omega_1 - \nu_1)}{\epsilon_1 - \epsilon_2},\ \alpha_1^{(2)} = \frac{(g_{1,2} - g_{1,1})(\Omega_1 - \nu_1)}{2\epsilon_1 - 2\epsilon_2},$$
$$\alpha_2^{(1)} = \frac{(\epsilon_1 g_{1,2} - \epsilon_2 g_{2,2})(\Omega_2 - \nu_2)}{\epsilon_1 - \epsilon_2},\ \alpha_2^{(2)} = \frac{(g_{2,2} - g_{1,2})(\Omega_2 - \nu_2)}{2\epsilon_1 - 2\epsilon_2}, \tag{20}$$

and $V(x)$ is a Van Vleck polynomial of degree 2, which is determined according to Eq. (16). Therefore, the polynomial approach for the SPM proposed in [22, 23] applies to the this case as well. For given the number of pairs k, k zeros $\{x_i^{(\zeta)}\}$ of $y(x)$ provides a solution of (14) with the corresponding eigen-energy given by (15).

3. A simple analysis of the model: To demonstrate the use of the solution, the validity of the SPM is analyzed, of which only one overall pairing interaction strength can be adjusted. We consider 5 pairs in the NSPM with $\epsilon_1 = 1$ MeV and $\epsilon_2 = 2$ MeV, $j_1 = 19/2$ and $j_2 = 21/2$, with which each orbit can accommodate 5 pairs. The on-site pairing interaction parameters $g_{1,1} = g_{2,2} = 1$ MeV are fixed. We calculated the pair excitation energies of the NSPM for serval values of $g_{1,2} = g$, which are presented in Table I. Then, the overall pairing interaction strength of the SPM is adjusted according to the ground-state energy of the NSPM for each case. Though pairing excitation energies of the SPM are about 2 MeV different from the corresponding ones of the NSPM, as shown in Table I, the overlap-square of the NSPM with the corresponding one of the SPM, $\eta(\zeta) = |\langle\zeta|\zeta\rangle_{\rm SP}|^2$, is always greater than 94% calculated in this way, where $|\zeta\rangle \equiv |\zeta, k = 5;\, 0\, 0\rangle$ is obtained according to (6) for each case, while $|\zeta\rangle_{\rm SP}$ is the corresponding eigen-state of the SPM. The results of the overlaps show that the SPM seem a good approximation to the NSPM. In fact, with the increasing of the pairing interaction strength g of nucleon pairs from different orbits, the system undergoes a phase crossover from localized normal phase mainly determined by the pure mean-field and the on-site pairing interaction strengths $g_{t,t}$ $(t = 1, 2)$ among nucleon pairs within the same orbits to the delocalized superconducting phase, for which there are a few effective order parameters. Here we calculate the occupation probability of nucleon pairs in the j_1 orbit at the ζ-th excited state defined by

$$\rho(j_1, \zeta) = \frac{1}{k}\langle\zeta|S^+_{j_1}\frac{\partial}{\partial S^+_{j_1}}|\zeta\rangle \tag{21}$$

for $\zeta = 1$ and $\zeta = 2$. As clearly shown in Fig. 1, the ground-state (the first excited state) occupation probability of the NSPM decreases (increases) with the increasing of g noticeably around $g \sim 0.05$–0.1 MeV, and there is a crossing point around $g \sim 0.21$ MeV. However, the occupation probability of the ground-state is always a little smaller than that of the first excited state in the SPM, which is opposite to the result of the NSPM when g is smaller than the value of the crossing point. They gradually decrease with the increasing of g with the overall pairing interaction strength fitted to the ground-state energy of the NSPM, and become close to those of the NSPM in the strong g limit. Therefore, the SPM is a good approximation to the NSPM only when the pairing interaction among nucleon pairs in different orbits is sufficiently strong. Nevertheless, the SPM cannot account for the actual quantum phase crossover when the pairing interaction strengths of different orbits are relatively weaker and differ from those of the same orbits as required, for example, in the ds- and fp-shell nuclei [31, 32]. Moreover, the on-site pairing interaction strengths $g_{t,t}$ can also change the actual ordering of the single-particle energies. For example, when $g_{2,2}$ is sufficiently greater than $g_{1,1}$, the ground state of the system may be dominated by the nucleon pairs of the j_2-orbit though ϵ_2 is greater than ϵ_1, which may be used to elucidate the inversion of the single-particle energy ordering of a shell model. Obviously, these phase transition associated issues cannot be described by the SPM, for which the NSPM should be adopted.

TABLE I: Excited level energies $E^{(\zeta)}$ (in MeV) of the NSPM and the overlap-square $\eta(\zeta) = |\langle\zeta|\zeta\rangle_{\rm SP}|^2$ of the pairing excited states with the corresponding ones of the SPM for $k = 5$ pairs over $j_1 = 19/2$ and $j_2 = 21/2$ orbits with single-particle energies $\epsilon_1 = 1$ MeV, $\epsilon_2 = 2$ MeV, and $g_{1,1} = g_{2,2} = 1$ MeV, where $g_{1,2} = g$, and $\delta g = g - g_{1,1}$ (in MeV). The overall pairing strength in the SPM is adjusted to reproduce the same ground-state energy of the NSPM for each case, with which the corresponding overlap $\eta(\zeta)$ is obtained.

		0_1^+	0_2^+	0_3^+	0_4^+	0_5^+	0_6^+
$\delta g = -0.50$	$E^{(\zeta)}$	−48.95	−36.66	−26.23	−17.30	−9.90	−4.96
	$\eta(\zeta)$	99.600%	98.989%	97.968%	96.600%	95.370%	94.548%
$\delta g = -0.25$	$E^{(\zeta)}$	−59.35	−42.61	−28.02	−15.10	−3.86	4.93
	$\eta(\zeta)$	99.949%	99.870%	99.756%	99.653%	99.714%	99.714%
δg=0.25	$E^{(\zeta)}$	−80.28	−54.81	−31.93	−10.96	8.34	25.66
	$\eta(\zeta)$	99.977%	99.946%	99.905%	99.883%	99.9287%	99.929%
δg=0.50	$E^{(\zeta)}$	−90.78	−60.98	−33.94	−8.91	14.49	36.12
	$\eta(\zeta)$	99.934%	99.842%	99.728%	99.674%	99.806%	99.818%

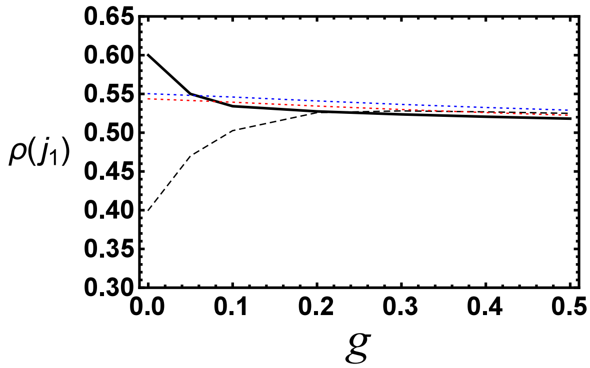

FIG. 1: (Color online) The occupation probability of nucleon pairs in the j_1-orbit at the ζ-th excited state for $\zeta = 1$ and $\zeta = 2$ as a function of $g_{12} = g$ (in MeV) with other model parameters the same as those shown in the caption of Table I, where the solid curve represents the occupation probability at the ground state ($\zeta = 1$) of the NSPM, the dashed curve is that of the first excited state ($\zeta = 2$) of the NSPM, and the dotted lines from bottom (Red) to the top (Blue) are that of the ground-state and the first excited state, respectively, in the SPM.

4. Summary: In this work, it is shown that the nuclear spherical mean-field plus orbit-dependent non-separable pairing model with two non-degenerate j-orbits, like the standard and separable pairing models, is also exactly solvable. The solution of the model by using the Bethe ansatz method is presented. The extended one-variable Heine-Stieltjes polynomials associated to the Bethe ansatz equations of the solution are determined. As the use of the solution, a comparison of the solution to that of the standard pairing interaction with constant interaction strength among pairs in any orbit is made via a concrete example. It is shown that the overlaps of eigenstates of the model with those of the standard pairing model are always large, especially for the ground and the first excited state. However, the quantum phase crossover in the non-separable pairing model cannot be accounted for by the standard pairing interaction, for which the NSPM should be adopted.

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