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Exact solution of spherical mean-field plus special orbit-dependent non-separable pairing model with multi non-degenerate j-orbits

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ABSTRACT

Exact solution of spherical mean-field plus a special orbit-dependent non-separable pairing Hamiltonian with multi non-degenerate j-orbits, which is related to two previously known hyperbolic Gaudin models, is explored. It is shown that the Hamiltonian with suitable constraints on the pairing interaction parameters turns to be exactly solvable. The extended one-variable Heine-Stieltjes polynomials associated to the Bethe-Gaudin-Richardson ansatz equations of the solution for any number of pairs k are determined. It is shown that the pair excitation energies can be calculated more easily than those of the separable pairing model studied previously. As examples of the solution, pairing excitation energies with the number of pairs up to the half-filling in the ds-shell with 3 j-orbits and in the pf-shell with 4 j-orbits are presented and compared with those of the mean-field plus the general separable, the special separable, and the standard pairing models. It is shown that the pairing excitation energies of the model are close to those of the mean-field plus special separable pairing or general separable pairing model.

1. Introduction

It is known that spherical or deformed mean-field plus the standard (equal strength) pairing interaction can be solved exactly by using the Bethe-Gaudin-Richardson method [1–3], of which the numerical solution can now be calculated relatively easily by using the extended Heine-Stieltjes polynomial approach [4–7]. Exact solution of the general separable pairing (SP) model with multi degenerate orbits [8] or two non-degenerate j-orbits [9] were also derived. Moreover, it is shown that exact solution to a special separable pairing (SSP) interaction with multi non-degenerate orbits, like the standard pairing model (SPM), can also be calculated relatively easily [10–12], while the general SP case has been analyzed in [13]. However, the solution presented in [13] is very complicated, even for the model with three non-degenerate j-orbits. In this work, based on our recent work on exact solution of the mean-field plus orbit-dependent non-separable pairing model with two non-degenerate orbits [14], it is shown that the spherical mean-field plus orbit-dependent non-separable pairing interaction with multi non-degenerate j-orbits can also be solved relatively easily when the pairing interaction parameters satisfy special constraints.

2. The model and exact solution

The Hamiltonian of a spherical mean-field plus orbit-dependent pairing model with multi non-degenerate j-orbits can be written as

$$\hat{H} = \sum_{t}^{p} \epsilon_t \hat{N}_{j_t} + \hat{H}_{\mathrm{P}} = \sum_{t}^{p} \epsilon_t \hat{N}_{j_t} + \sum_{1 \le t, t' \le p} g_{t,t'} S^{+}_{j_t} S^{-}_{j_{t'}}, \tag{1}$$

where p is the total number of orbits considered above a closed or sub-closed shell, $\{\epsilon_t\}$ $(t = 1, 2, \cdots, p)$ are single-particle energies generated from a mean-field theory, $\hat{N}_j = \sum_m a^{\dagger}_{jm} a_{jm}$, $S^{+}_{j} = \sum_{m>0} (-1)^{j-m} a^{\dagger}_{jm} a^{\dagger}_{j-m}$, in which $a^{\dagger}_{jm}$ (a_{jm}) is the creation (annihilation)

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operator for a valence nucleon with angular momentum quantum number j with projection m, and $\{g_{t,t'}\}$ $(t, t' = 1,\ 2, \cdots, p)$ are all assumed to be real and must be symmetric with $g_{t,t'} = g_{t',t}$. As shown in our recent work on the Hamiltonian with $p = 2$ case [14], exact solution of (1), even with the separable pairing interaction discussed in [13], will be very complicated for $p \geq 3$. In this work, we consider the expansion

$$g_{tt'} = \sum_{\mu,\nu} G_{\mu,\nu} (c_t)^{\mu} (c_{t'})^{\nu}, \tag{2}$$

where $\{c_t\}$ $(t = 1,\ 2, \cdots, p)$ and $G_{\mu,\nu} = G_{\nu,\mu}$ $(\mu, \nu = +1$ or $-1)$ are parameters, with which the pairing interaction term of (1) can be expressed as

$$\hat{H}_{\mathrm{P}} = G_{+,+}\, S^+_1 S^-_1 + G_{+,-}\, S^+_1 S^-_{-1} + G_{-,+}\, S^+_{-1} S^-_1 + G_{-,-}\, S^+_{-1} S^-_{-1}, \tag{3}$$

where

$$S^{\pm}_{\mu} = \sum_{t=1}^{p} (c_t)^{\mu}\, S^{\pm}_{j_t} \tag{4}$$

for $\mu = \pm 1$. The p sets of local operators $\{S^-_{j_t},\ S^+_{j_t},\ \hat{N}_{j_t}\}$ $(t = 1,\ 2, \cdots, p)$, where $S^-_{j_t} = (S^+_{j_t})^{\dagger}$, generate p copies of an SU(2) algebra satisfying the commutation relations $[\hat{N}_{j_t}/2,\ S^-_{j_{t'}}] = -\delta_{tt'} S^-_{j_t}$, $[\hat{N}_{j_t}/2,\ S^+_{j_{t'}}] = \delta_{tt'} S^+_{j_t}$, $[S^+_{j_t},\ S^-_{j_{t'}}] = 2\delta_{tt'} S^0_{j_t}$, where $S^0_{j_t} = (\hat{N}_{j_t} - \Omega_t)/2$ with $\Omega_t = j_t + 1/2$. It will be shown in the following that there is a nontrivial case of (3), with which the Hamiltonian (1) can be solved relatively easily.

As adopted in the separable pairing model [8,12], let

$$S^+(x) = \sum_{t=1}^{p} \frac{1}{c_t^2 - x}\, c_t\, S^+_{j_t}, \tag{5}$$

where x is the spectral parameter to be determined. According to the commutation relations of the generators of the p copies of the SU(2) algebra, (5) can be expressed as

$$[\sum_t \epsilon_t \hat{N}_{j_t},\ S^+(x)] = \sum_t \frac{2\epsilon_t}{c_t^2 - x}\, c_t\, S^+_{j_t} = \alpha(x)\, S^+(x) + \beta(x)\, S^+_1 + \gamma(x)\, S^+_{-1}, \tag{6}$$

where $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ are independent functions of x to be determined. For $p = 3$, $S^+(x)$ given in (5) is a trinomial in $S^+_{j_t}$ with 3 independent terms. Hence, (6) requires

$$\begin{aligned}
\alpha(x) &= \frac{2\left(c_1^2(c_2^2 - c_3^2)(x - c_2^2)(x - c_3^2)\,\epsilon_1 - c_2^2(c_1^2 - c_3^2)(x - c_1^2)(x - c_3^2)\,\epsilon_2 + c_3^2(c_1^2 - c_2^2)(x - c_1^2)(x - c_2^2)\,\epsilon_3\right)}{(c_1^2 - c_2^2)(c_1^2 - c_3^2)(c_2^2 - c_3^2)\, x},\\
\beta(x) &= \beta = \frac{2\, c_1^2 c_2^2(\epsilon_1 - \epsilon_2) + 2\, c_1^2 c_3^2(\epsilon_3 - \epsilon_1) + 2\, c_2^2 c_3^2(\epsilon_2 - \epsilon_3)}{(c_1^2 - c_2^2)(c_1^2 - c_3^2)(c_2^2 - c_3^2)},\\
\gamma(x) &= \gamma/x = \frac{2\, c_1^2 c_2^2 c_3^2\left(c_1^2(\epsilon_2 - \epsilon_3) + c_3^2(\epsilon_1 - \epsilon_2) + c_2^2(\epsilon_3 - \epsilon_1)\right)}{(c_1^2 - c_2^2)(c_1^2 - c_3^2)(c_2^2 - c_3^2)\, x}
\end{aligned} \tag{7}$$

for $p = 3$, where $c_1 \neq c_2 \neq c_3$ is assumed. (7) is obtained after comparing the coefficients of $S^+_{j_t}$ for given t on both sides of (6). The expression used in (6) is similar to that used in our recent work on the non-separable pairing problem with two non-degenerate j-orbits [14]. In general, the parameters $\{\epsilon_t\}$ and $\{c_t\}$ $(t = 1,\ 2, \cdots, p)$ must satisfy the constraints

$$\epsilon_t = \frac{1}{2}\left(u_0 + u_2 (c_t)^2 + u_{-2}(c_t)^{-2}\right) \tag{8}$$

for $t = 1, 2, \cdots, p$. It is obvious that the parameters u_0, u_2, u_{-2} can be determined by given $\{\epsilon_t\}$ and $\{c_t\}$ $(t = 1,\ 2, 3)$, namely by the first three constraints shown in (8). When $p \geq 4$, the other $p - 3$ parameters $c_4, c_5, \cdots, c_p$ will no longer be free, which should also satisfy the constraints shown in (8). The constrains (8) is similar to the SSP case shown in [10–12], in which the single-particle energies $\{\epsilon_t\}$ and the pairing interaction parameters $\{c_t\}$ satisfy the simple linear relation with $\epsilon_t = a_1 + a_2 c_t^2$ for $1 \leq t \leq p$, where a_1 and a_2 are free parameters introduced in [10–12]. By using (8), $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ in (6) can be expressed as

$$\alpha(x) = u_0 + u_2\, x - \frac{u_{-2}}{x},\quad \beta = u_2,\quad \gamma(x) = -\frac{u_{-2}}{x} \tag{9}$$

for $p \geq 3$.

Similar to the Bethe-Gaudin-Richardson ansatz used for solving the SPM [1–3], the k-pair eigenvectors of (1) can be written as

$$|\zeta,\ k;\ JM\rangle = \prod_{\rho=1}^{k} S^+(x^{(\zeta)}_{\rho})|JM\rangle, \tag{10}$$

where ζ labels the ζ-th set of solution $\{x_1^{(\zeta)}, \cdots, x_k^{(\zeta)}\}$. If the seniority number of the t-th orbit is ν_t, the pairing vacuum states of the p orbits are denoted as $|\nu_t\eta_t J_t M_t\rangle$ satisfying $S^-_{j_t}|\nu_{j_t}\eta_t J_t M_t\rangle = 0$, where J_t and M_t are the angular momentum quantum number and that of its third component, respectively, and η_t is the multiplicity label needed to distinguish different possible ways of ν_t particles coupled to the angular momentum J_t. Thus, a pairing vacuum state of the system with the total seniority number $\nu = \sum_{t=1}^{p} \nu_t$ and the total angular momentum J can be expressed as $|JM\rangle \equiv |\nu_1\eta_1, \nu_2\eta_2, \cdots, \nu_p\eta_p; (J_1 \otimes J_2 \otimes \cdots \otimes J_p), \rho, JM\rangle$, where ρ is the outer-multiplicity label needed for the coupling of $J_1 \otimes J_2 \otimes \cdots \otimes J_p \downarrow J$. Thus, $|JM\rangle$ satisfies $S^-_{j_t}|JM\rangle = 0$ for $t = 1, 2, \cdots, p$, which is used in (10).

To solve the eigen-equation of (1) with the ansatz (10), one can calculate commutators of $\hat{H}$ with the pairing operators $S^+(x_\rho^{(\zeta)})$ as was done in Richardson's work on solving the SPM [2,3]. Since (1) only contains one- and two-body interaction terms, the q-time commutators $[\cdots[\hat{H}, S^+(x_{\rho_1}^{(\zeta)})], \cdots, S^+(x_{\rho_{q-1}}^{(\zeta)})], S^+(x_{\rho_q}^{(\zeta)})]$ vanish when $q \geq 3$. Namely, one only needs to calculate single and double commutators of $\hat{H}$ with the operators $S^+(x_\rho^{(\zeta)})$:

$$[\hat{H}_{\rm P},\ S^+(x)] = -S_1^+\left(G_{+,+}\sum_j \frac{c_j^2(2S_j^0)}{c_j^2 - x} + G_{+,-}\sum_j \frac{(2S_j^0)}{c_j^2 - x}\right) - S_{-1}^+\left(G_{-,+}\sum_j \frac{c_j^2(2S_j^0)}{c_j^2 - x} + G_{-,-}\sum_j \frac{(2S_j^0)}{c_j^2 - x}\right), \tag{11}$$

$$\begin{aligned}[][[\hat{H}_{\rm P},\ S^+(x)],\ S^+(y)] = &-S_1^+\left(G_{+,+}\left(\frac{2x}{x-y}S^+(x) + \frac{2y}{y-x}S^+(y)\right) + G_{+,-}\left(\frac{2}{x-y}S^+(x) + \frac{2}{y-x}S^+(y)\right)\right) - \\ &S_{-1}^+\left(G_{-,+}\left(\frac{2x}{x-y}S^+(x) + \frac{2y}{y-x}S^+(y)\right) + G_{-,-}\left(\frac{2}{x-y}S^+(x) + \frac{2}{y-x}S^+(y)\right)\right).\end{aligned} \tag{12}$$

Using Eqs. (6), (11), and (12), one can directly check that

$$\sum_t \epsilon_t \hat{N}_{j_t}|\zeta,\ k;\ JM\rangle = \sum_i^k \left(\beta\, S_1^+ + \gamma(x_i^{(\zeta)})\, S_{-1}^+\right) \prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle + \left(\sum_i^k \alpha(x_i^{(\zeta)}) + \sum_{t=1}^p \epsilon_t \nu_t\right)\prod_\rho^k S^+(x_\rho^{(\zeta)})|JM\rangle \tag{13}$$

and

$$\begin{aligned}\hat{H}_{\rm P}|\zeta,\ k;\ JM\rangle = &-\sum_i^k \left(S_1^+(G_{+,+}\,\Lambda_1(x_i^{(\zeta)}) + G_{+,-}\,\Lambda_0(x_i^{(\zeta)})) + S_{-1}^+(G_{-,+}\,\Lambda_1(x_i^{(\zeta)}) + G_{-,-}\,\Lambda_0(x_i^{(\zeta)}))\right) \prod_{\rho(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle - \\ &S_1^+ \sum_i^k \sum_{i'\,(\neq i)}^k (G_{+,+}\frac{2x_{i'}^{(\zeta)}}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}} + G_{+,-}\frac{2}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}}) \prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle - \\ &S_{-1}^+ \sum_i^k \sum_{i'\,(\neq i)}^k (G_{-,+}\frac{2x_{i'}^{(\zeta)}}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}} + G_{-,-}\frac{2}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}}) \prod_{\rho\,(\neq i)}^k S^+(x_\rho^{(\zeta)})|JM\rangle,\end{aligned} \tag{14}$$

where

$$\Lambda_\mu(x) = -\sum_t \frac{\Omega_t (c_t^2)^\mu}{c_t^2 - x}. \tag{15}$$

In establishing the eigen-equation $\hat{H}|\zeta, k;\ JM\rangle = E_k^{(\zeta)}|\zeta, k;\ JM\rangle$ based on (13) and (14), one gets the eigenvalue $E_k^{(\zeta)}$ with

$$E_k^{(\zeta)} = \sum_{t=1}^p \epsilon_t \nu_t + \sum_{i=1}^k \alpha(x_i^{(\zeta)}), \tag{16}$$

where $\sum_{t=1}^p \epsilon_t \nu_t$ is contributed from particles in the pairing vacuum, and $2k$ equations for k variables $\{x_i^{(\zeta)}\}$:

$$\beta - G_{+,+}\,F(x_i^{(\zeta)}) - G_{+,-}\,V(x_i^{(\zeta)}) = 0 \quad \text{for } i = 1,\ 2,\ \cdots,\ k, \tag{17}$$

$$\frac{\gamma}{x_i^{(\zeta)}} - G_{-,+}\,F(x_i^{(\zeta)}) - G_{-,-}\,V(x_i^{(\zeta)}) = 0 \quad \text{for } i = 1,\ 2,\ \cdots,\ k, \tag{18}$$

where

$$F(x_i^{(\zeta)}) = \Lambda_1(x_i^{(\zeta)}) + \sum_{i'\,(\neq i)}^k \frac{2x_{i'}^{(\zeta)}}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}},\quad V(x_i^{(\zeta)}) = \Lambda_0(x_i^{(\zeta)}) + \sum_{i'\,(\neq i)}^k \frac{2}{x_{i'}^{(\zeta)} - x_i^{(\zeta)}}. \tag{19}$$

Since β and γ are all nonzero in general, the Hamiltonian (1) with $G_{+,+} \neq 0$, $G_{+,-} \neq 0$, $G_{-,+} \neq 0$, and $G_{-,-} \neq 0$ can not be diagonalized by using the ansatz (10) because (17) and (18) can not be simultaneously satisfied in this case. Nevertheless, there are two special cases for nonzero $\{c_t\}$ $(t = 1,\ 2,\ \cdots,\ p)$ that can be solved exactly. One is $\gamma = 0$ case corresponding to $u_{-2} = 0$, while another is $\beta = 0$ case corresponding to $u_2 = 0$. The former case with $G_{+,+} \neq 0$, $G_{+,-} = G_{-,+} = G_{-,-} = 0$ is just the special separable pairing (SSP) case discussed in [10–12], while the latter case with $G_{-,-} \neq 0$, $G_{+,-} = G_{-,+} = G_{+,+} = 0$ is equivalent to the former one with the replacements: $c_t \to c_t^{-1}$ for $t = 1,\ 2,\ \cdots,\ p$.

Since

$$\Lambda_1(x) = -\sum_t \frac{\Omega_t\, c_t^2}{c_t^2 - x} = -\Omega + x\,\Lambda_0(x), \tag{20}$$

where $\Omega = \sum_t \Omega_t$, and

$$\sum_{i'\,(\neq i)}^{k} \frac{2x_{i'}}{x_{i'} - x_i} = 2(k-1) + x_i \sum_{i'\,(\neq i)}^{k} \frac{2}{x_{i'} - x_i}, \tag{21}$$

we have

$$F(x_i^{(\zeta)}) = 2(k-1) - \Omega + x_i^{(\zeta)}\, V(x_i^{(\zeta)}). \tag{22}$$

When $\beta \neq 0$ and $\gamma \neq 0$, consistent solution of (17) and (18) only exists when $G_{+,-} = G_{-,+} = 0$ and

$$\frac{\beta - G_{+,+}\,(2(k-1) - \Omega)}{G_{+,+}} = \frac{\gamma}{G_{-,-}}. \tag{23}$$

When (23) is satisfied, Eqs. (17) and (18) coalesce into k equations in determining the k variables

$$\frac{\gamma}{G_{-,-}} - x_i^{(\zeta)}\, V(x_i^{(\zeta)}) = 0 \quad \text{for } i = 1,\, 2,\, \cdots,\, k. \tag{24}$$

For this case, the pairing interaction matrix elements (2) can be expressed in terms of $p+2$ parameters $\{c_1, c_2, \cdots, c_p\}$, $G_{+,+}$, and $G_{-,-}$ as

$$g_{tt'} = G_{+,+}\, c_t\, c_{t'} + \frac{G_{-,-}}{c_t\, c_{t'}} \tag{25}$$

with the constraints (8) and (23), which are obviously non-separable in general, and will be called the special non-separable pairing (SNSP) in the following. It is obvious that the SNSP is dependent on the number of pairs k due to the constraint (23). Though there are only 1 condition (23) for the $p+2$ parameters $\{c_i\}$, $G_{+,+}$, and $G_{-,-}$, there are only 3 parameters adjustable when $p = 3$ because the two-body pairing interaction matrix elements $\{g_{t,t'}\}$ shown in (25) are independent of one of nonzero c_i $(i = 1, 2, 3)$ due to the condition (23) in this case, for which $c_3 = 1$ is taken, while c_1, c_2, and $G_{+,+}$ are adjustable parameters for the SNSP. When $p \geq 4$, $p-3$ parameters c_i $(i = 4, 5, \cdots, p)$ are restricted by (8). In this case, due to the constraints (8) and (23), there are 4 parameters, c_1, c_2, c_3, and $G_{+,+}$ adjustable for the SNSP. Once the condition (23) is satisfied, the Hamiltonian (1) is relatively easily solvable with the pairing interaction parameters shown in (25). Due to the fact that the related expressions are lengthy, only numerical examples for the *ds*-shell and *pf*-shell will be shown in the next section.

In addition, as the example for the *ds*-shell shown in the next section, complex values may be taken for the parameters $\{c_i\}$ and $G_{+,+}$ in order to get a best fit to experimentally determined effective two-body pairing interaction matrix elements. Though these parameters are complex with $G_{+,+} = G^*_{-,-}$, the resultant two-body pairing interaction matrix elements $\{g_{t,t'}\}$ obtained according to (25) are still real and symmetric. However, the pairing operators $S^\pm = \sum_t c_t S_t^\pm$ used in (4) do not satisfy the hermitian conjugation relation in general, namely $S^- \neq (S^+)^\dagger$ when the parameters $\{c_t\}$ are complex.

According to the Heine-Stieltjes correspondence [4,5], zeros $\{x_i^{(\zeta)}\}$ of the extended Heine-Stieltjes polynomials $y_k(x)$ of degree k related to the roots of (17) or (18) should satisfy the following second-order Fuchsian equation:

$$A(x)\, y_k''(x) + B(x,k)\, y_k'(x) - V(x,k)\, y_k(x) = 0. \tag{26}$$

Here

$$A(x) = x \prod_{t=1}^{p} (c_t^2 - x) \tag{27}$$

is a polynomial of degree $p+1$, the polynomial $B(x,k)$ of degree p is given as

$$B(x,k)/A(x) = \frac{\gamma}{x\, G_{-,-}} + \sum_{t=1}^{p} \frac{\Omega_t}{c_t^2 - x}, \tag{28}$$

and $V(x)$ is a Van Vleck polynomial of degree $p-1$ determined according to Eq. (26) for each case. Therefore, the polynomial approach for the SPM proposed in [5,6] applies to the SNSP as well. For given number of pairs k, k zeros $\{x_i^{(\zeta)}\}$ of $y_k(x)$ in each case give a solution of (24) with the corresponding eigen-energy shown in (16).

In the past a few decades, the exact solvability of Gaudin-type models was studied extensively [1,3,15,16]. It can be shown that the Hamiltonian (1) with the SNSP (3) under the parameter constraints $G_{+,-} = G_{-,+} = 0$ is related to the hyperbolic Gaudin models studied in [10,17–21]. In order to establish a link of the SNSP to the hyperbolic solutions, the single-particle energies $\{\epsilon_t\}$ of the model are expanded in terms of $\{c_t\}$ for $t = 1, 2, \cdots, p$ as shown in (8). It should be noted that the expansion coefficients may be complex when $\{c_t\}$ are complex in order to keep $\{\epsilon_t\}$ to be real. Then, we can use the invariants of the hyperbolic or the non-skew symmetric solution to construct the SNSP Hamiltonian as

Table 1
The single-particle energies ϵ_j (in MeV), the pairing interaction matrix elements $G_{t,t'}$ (in MeV) in the $\{S_j^+\}$ basis of the *ds*-shell [22], and the parameters of the model, where $g_{t,t'}$ (in MeV) are the fitting results obtained by using the parameters c_1, c_2, and $G_{+,+}$ for given k with $G_{-,-} = G_{+,+}^*$ provided in the corresponding row of the table, and $\mathrm{i} = \sqrt{-1}$.

The *ds*-shell [22]	$\epsilon_1 = \epsilon_{1/2} = -2.92$ $\epsilon_2 = \epsilon_{5/2} = -3.70$ $\epsilon_3 = \epsilon_{3/2} = 1.90$ $G_{1,1} = -1.075$ $G_{2,2} = -0.728$ $G_{3,3} = -0.410$ $G_{1,2} = 0.121$ $G_{2,3} = -0.355$ $G_{1,3} = 0.000$
$k = 1$	$g_{1,1} = -0.765$ $g_{2,2} = -0.580$ $g_{3,3} = -0.340$ $g_{1,2} = 0.677$ $g_{2,3} = -1.038$ $g_{1,3} = 1.015$ $c_1 = -0.544 + 0.839\,\mathrm{i}$ $c_2 = -0.443 - 0.896\,\mathrm{i}$ $G_{+,+} = (-0.17 - 0.495\,\mathrm{i})$ MeV
$k = 2$	$g_{1,1} = -0.710$ $g_{2,2} = -0.580$ $g_{3,3} = -0.340$ $g_{1,2} = 0.649$ $g_{2,3} = -0.835$ $g_{1,3} = 0.812$ $c_1 = -0.646 + 0.763\,\mathrm{i}$ $c_2 = -0.551 - 0.835\,\mathrm{i}$ $G_{+,+} = (-0.17 - 0.388\,\mathrm{i})$ MeV
$k = 3$	$g_{1,1} = -0.573$ $g_{2,2} = -0.580$ $g_{3,3} = -0.340$ $g_{1,2} = 0.577$ $g_{2,3} = -0.508$ $g_{1,3} = 0.493$ $c_1 = -0.925 + 0.379\,\mathrm{i}$ $c_2 = 0.905 - 0.426\,\mathrm{i}$ $G_{+,+} = (-0.17 - 0.236\,\mathrm{i})$ MeV

$$\hat{H} = u_0 \sum_t S_t^0 + \lambda_+ \sum_t c_t^2 R_t + C = \sum_t^p \epsilon_t \hat{N}_{j_t} - G_{+,+} S_1^+ S_1^- - G_{-,-} S_{-1}^+ S_{-1}^-, \tag{29}$$

where $S_t^0 = \frac{1}{2}(\hat{N}_{j_t} - \Omega_t/2)$, C is the constant related to a linear combination of the Casimir operators of the local SU(2) algebras generated by $\{S_t^+, S_t^-, S_t^0\}$, and

$$R_t = u_2 R_t^{(+)} + \frac{\lambda_-}{\lambda_+} u_{-2} (c_t)^{-4} R_t^{(-)}, \tag{30}$$

in which

$$\lambda_\pm = \frac{1}{1 + 2\gamma_\pm(1-k) + \gamma_\pm \Omega} \tag{31}$$

for the seniority-zero case, and

$$R_t^{(\pm)} = S_t^0 - 2\gamma_\pm \sum_{t'(\neq t)} \left(\frac{X_{tt'}^{(\pm)}}{2} (S_t^+ S_{t'}^- + S_t^- S_{t'}^+) + Z_{tt'}^{(\pm)} S_t^0 S_{t'}^0 \right) \tag{32}$$

with

$$X_{tt'}^{(\pm)} = \frac{2\,(c_t)^{\pm 1} (c_{t'})^{\pm 1}}{(c_t)^{\pm 2} - (c_{t'})^{\pm 2}}, \quad Z_{tt'}^{(\pm)} = \frac{(c_t)^{\pm 2} + (c_{t'})^{\pm 2}}{(c_t)^{\pm 2} - (c_{t'})^{\pm 2}}. \tag{33}$$

As is known from previous works [10,17–20], the two sets of the operators, $\{R_t^{(+)}\}$ and $\{R_t^{(-)}\}$, are commutative with $[R_t^{(+)}, R_{t'}^{(+)}] = 0$ and $[R_t^{(-)}, R_{t'}^{(-)}] = 0$ for $1 \le t, t' \le p$. It can be easily verified that $[R_t^{(+)}, R_{t'}^{(-)}] = 0$ and thus $\{R_t\}$ are commutative if and only if $\gamma_- = -\gamma_+$ due to the fact that $X_{tt'}^{(+)} = -X_{tt'}^{(-)}$ and $Z_{tt'}^{(+)} = -Z_{tt'}^{(-)}$, which thus requires the constraint shown in (23) with $G_{+,+} = 2\lambda_+ u_2 \gamma_+$, $G_{-,-} = 2\lambda_- u_{-2}\gamma_-$. Hence, the exact solvability of the SNSP Hamiltonian (29) is related to the two hyperbolic solutions, of which one is constructed by using the parameters $\{c_t\}$ with the invariants $\{R_t^{(+)}\}$, while the other one is by $\{(c_t)^{-1}\}$ with the invariants $\{R_t^{(-)}\}$. As is clearly shown in [17], these two solutions are related to two SSP Hamiltonians separately. While the SNSP Hamiltonian (29) is the combination of the two with the constraint (23), which is obviously non-trivial, and has not been noticed previously.

3. Model application to the *ds*- and *pf*-shell

To demonstrate the solution of the SNSP, we consider two examples. One is the *ds*-shell with 3 orbitals $0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$, of which the single-particle energies are provided in [22]. The effective pairing interaction matrix elements $\{G_{t,t'}\}$ in the $\{S_{j_t}^+\}$ basis with $\hat{H}_{\mathrm{P}} = \sum_{jj'} G_{t,t'} S_{j_t}^+ S_{j_{t'}}^-$ for this case are obtained from the $J = 0$ and $T = 1$ two-body matrix elements of the SDPF-NR interaction shown in [22], where $j_1 = 1/2$, $j_2 = 5/2$, and $j_3 = 3/2$ are assigned, and the mass scaling factor $(A/18)^{1/3}$ of the two-body matrix elements is not included in our analysis. Another is the *pf*-shell with 4 orbitals $1p_{3/2}$, $1p_{1/2}$, $0f_{7/2}$, and $0f_{5/2}$, of which the single-particle energies are provided in [23]. The effective pairing interaction matrix elements $\{G_{t,t'}\}$ in the $\{S_{j_t}^+\}$ basis with $\hat{H}_{\mathrm{P}} = \sum_{jj'} G_{t,t'} S_{j_t}^+ S_{j_{t'}}^-$ for this case are obtained from the $J = 0$ and $T = 1$ two-body matrix elements of the GXPF1 interaction shown in [23], where $j_1 = 7/2$, $j_2 = 3/2$, $j_3 = 1/2$, and $j_4 = 5/2$ are assigned, and the mass scaling factor $(A/42)^{0.3}$ for the *pf*-shell is not included. For both examples, only the seniority-zero cases up to the half-filling are shown.

For the *ds*-shell case, because $\{g_{t,t'}\}$ obtained from the SNSP (25) are independent of one of nonzero c_i $(i = 1, 2, 3)$ due to (23), only c_1, c_2, and $G_{+,+}$ will be taken as fitting parameters, while $G_{-,-}$ is determined according to (23). The fitting results of $\{g_{t,t'}\}$ will be independent of nonzero c_3 in this case. Hence, $c_3 = 1$ is taken in our calculation. Thus, one can use 3 of the 6 pairing interaction matrix elements $G_{t,t'}$ in the $\{S_{j_t}^+\}$ basis shown in Table 1 to get c_1, c_2, and $G_{+,+}$. Similar procedure is taken for the *pf*-shell case, for which the parameters c_1, c_2, c_3, and $G_{+,+}$ are adjustable with c_4 fixed by the constraint (8). Since the four orbital *pf*-shell case is complicated, c_1, c_2, c_3, and $G_{+,+}$ are taken to be real parameters in the fitting. Due to the special constraint (23), besides the fact that $g_{t,t'}$ are dependent on the number of pairs k, the resultant pairing interaction matrix elements $\{g_{t,t'}\}$ for given k are certainly different from the original ones,

Table 2
The single-particle energies ϵ_j (in MeV), the pairing interaction matrix elements $G_{t,t'}$ (in MeV) in the $\{S_j^+\}$ basis of the pf-shell [23], and the parameters of the model, where $g_{t,t'}$ (in MeV) are the fitting results obtained by using the real parameters c_1, c_2, c_3, and $G_{+,+}$ for given k with c_4 determined by (8) and $G_{-,-}$ given by (23) shown in the corresponding row of the table.

The pf-shell [23]	$\epsilon_1=\epsilon_{7/2}=-8.624$	$\epsilon_2=\epsilon_{3/2}=-5.6793$	$\epsilon_4=\epsilon_{1/2}=-4.137$	$\epsilon_3=\epsilon_{5/2}=-1.3829$		
	$G_{11}=-1.21295$	$G_{12}=-0.50728$	$G_{13}=-0.3800$	$G_{14}=-0.79859$	$G_{22}=-1.1165$	
	$G_{23}=-2.11114$	$G_{24}=-1.01711$	$G_{33}=-0.89380$	$G_{34}=-0.934499$	$G_{44}=-0.8054$	
$k=1$	$g_{11}=-1.0000$	$g_{12}=-0.9482$	$g_{13}=-0.9184$	$g_{14}=-0.8587$	$g_{22}=-0.8966$	
	$g_{23}=-0.8669$	$g_{24}=-0.8074$	$g_{33}=-0.8373$	$g_{34}=-0.7779$	$g_{44}=-0.7187$	
	$c_1=1.3798$	$c_2=1.3597$	$c_3=1.3482$	$c_4=1.3255$	$G_{+,+}=-1.1914$	$G_{-,-}=2.4146$
$k=2$	$g_{11}=-1.0000$	$g_{12}=-0.9193$	$g_{13}=-0.8551$	$g_{14}=-0.6637$	$g_{22}=-0.8045$	
	$g_{23}=-0.7404$	$g_{24}=-0.5492$	$g_{33}=-0.6763$	$g_{34}=-0.4852$	$g_{44}=-0.2943$	
	$c_1=1.3774$	$c_2=1.3597$	$c_3=1.3500$	$c_4=1.3211$	$G_{+,+}=-2.6164$	$G_{-,-}=7.4554$
$k=3$	$g_{11}=-1.0000$	$g_{12}=-0.8955$	$g_{13}=-0.8258$	$g_{14}=-0.5777$	$g_{22}=-0.7911$	
	$g_{23}=-0.7214$	$g_{24}=-0.4735$	$g_{33}=-0.6519$	$g_{34}=-0.4041$	$g_{44}=-0.1565$	
	$c_1=1.3740$	$c_2=1.3598$	$c_3=1.3503$	$c_4=1.3171$	$G_{+,+}=-2.9137$	$G_{-,-}=8.4978$
$k=4$	$g_{11}=-0.8960$	$g_{12}=-0.7986$	$g_{13}=-0.7335$	$g_{14}=-0.4788$	$g_{22}=-0.7013$	
	$g_{23}=-0.6362$	$g_{24}=-0.3818$	$g_{33}=-0.5712$	$g_{34}=-0.3168$	$g_{44}=-0.0625$	
	$c_1=1.3741$	$c_2=1.3600$	$c_3=1.3504$	$c_4=1.3146$	$G_{+,+}=-2.7358$	$G_{-,-}=8.0621$
$k=5$	$g_{11}=-0.5350$	$g_{12}=-0.4745$	$g_{13}=-0.4341$	$g_{14}=-0.2734$	$g_{22}=-0.4141$	
	$g_{23}=-0.3737$	$g_{24}=-0.2131$	$g_{33}=-0.3334$	$g_{34}=-0.1728$	$g_{44}=-0.01240$	
	$c_1=1.3741$	$c_2=1.3597$	$c_3=1.3504$	$c_4=1.3134$	$G_{+,+}=-1.6774$	$G_{-,-}=4.9704$

Table 3
Comparison of pairing excitation energies $E_k^{(\zeta)}$ (in MeV) of the seniority-zero case obtained from the SNSP with those of the SP for a given number of pairs k up to the half-filling, where $g_{t,t}$ in the SNSP are adjusted to $E_{k=1}^{(\zeta)}$ of the SP indicated by *, while $g_{1,1}$ for given k in the SNSP is then adjusted to ground-state energy $E_k^{(\zeta=1)}$ of the SP indicated by *, the overall strength g in the SPM and the pairing parameters in the SSP are adjusted to produce the ground-state energy $E_{k=1}^{(\zeta=1)}$ of the SP indicated by *.

	$k=1$				$k=2$				$k=3$			
	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM
$\zeta=1$	−10.22*	−10.22*	−10.22	−10.22*	−18.56	−18.56*	−18.82	−18.89	−25.03	−25.03*	−25.81	−25.97
$\zeta=2$	−5.84	−6.16	−6.09	−6.12	−16.05	−15.18	−15.44	−15.39	−24.38	−22.83	−23.26	−23.21
$\zeta=3$	3.79	3.76	3.72	2.72	−6.42	−6.16	−6.48	−7.20	−14.76	−14.56	−15.08	−15.86
$\zeta=4$					−2.05	−2.78	−2.37	−3.53	−12.25	−11.81	−11.72	−12.76
$\zeta=5$					7.59	7.33	7.52	6.49	−2.62	−2.30	−2.65	−3.25
$\zeta=6$									1.75	0.92	1.42	0.26

$\{G_{t,t'}\}$. So we adjust the diagonal matrix elements $g_{t,t}$ to fit the one-pair ground-state eigen-energy of (1) to that with the original pairing interaction matrix elements $\{G_{t,t'}\}$. When $k \geq 2$, $g_{1,1}$ for given k is adjusted to the ground-state energy $E_k^{(\zeta=1)}$ of the SP. The fitting results $\{g_{t,t'}\}$ for given k, together with $\{c_t\}$, $G_{+,+}$, and $G_{-,-}$, are shown in Table 1 for the ds-shell and Table 2 for the pf-shell, respectively.

Since the SP model results [12,13] are very close to those obtained from the exact diagonalization of the mean-field plus non-separable pairing model with the effective pairing matrix elements $\{G_{t,t'}\}$ in the ds-shell [22] and those in the pf-shell [23], respectively, the pairing excitation spectrum thus determined by the SNSP is compared with that generated by the same mean-field plus separable pairing (SP) with $g_{tt'} = G_t G_{t'}$ and the special separable pairing (SSP) for a given number of pairs k up to the half-filling, of which the results in the ds- and pf-shell were provided in [12]. The results of the SPM with $g_{tt'} = g$, of which the exact solution is well known [2,3,5–7], are also shown for comparison. The difference of the SPM and SSP results shown in Table 3 from those provided in [12] lies in the fact that the SPM and the SSP parameters were fitted to the ground-state energy of the SP for a given k in [12], which are also used for the pf-shell case shown in Table 4, while the SPM and the SSP parameters used for Table 3 are always adjusted to the one-pair ($k=1$) ground-state energy of the SP. It is clearly shown in Tables 3 and 4 that, for any k up to the half-filling, the SNSP results, similar to those of the SSP, are close to the corresponding ones of the SP, but the SSP results are closer to those of the SP for this case, simply because they are all separable. It can be observed from Tables 1 and 2 that, in comparison to $G_{tt'}$, the diagonal matrix elements g_{tt} are smaller. The deviations of $g_{tt'}$ from the corresponding $G_{tt'}$ are mainly due to the constraint (23).

4. Summary

In this work, it is shown that there is the SNSP case of the spherical mean-field plus orbit-dependent pairing model with multi non-degenerate j-orbits related to two previously known hyperbolic Gaudin models, which, like the standard, the previously proposed special separable [10–12], and two-orbit non-separable [14] pairing models, is relatively easily solvable than the general separable pairing model shown in [13]. Similar to the special separable pairing [10–12], there are 3 or 4 independent parameters adjustable in the pairing interaction matrix due to the special constraints for $p=3$ or $p \geq 4$ cases, where p is the number of non-degenerate j-orbits considered, which, however, is non-separable in general. The solution of the SNSP case of the model is derived by using the Bethe-Gaudin-Richardson method. In the analysis, the SNSP model is applied to describe the ds-shell with 3 orbitals $0d_{5/2}$, $1s_{1/2}$, and $0d_{3/2}$, and the pf-shell with 4 orbitals $1p_{3/2}$, $1p_{1/2}$, $0f_{7/2}$, and $0f_{5/2}$, of which the single-particle energies and the pairing interaction matrix elements fitted are extracted from the SDPF-NR interaction shown in [22] and the GXPF1 interaction shown in [23], respectively. It is shown that the pairing excitation energies calculated are close to the corresponding ones of the model with the general separable pairing interaction [13]. It is

Table 4
Comparison of pairing excitation energies $E_k^{(\zeta)}$ (in MeV) of the seniority-zero case obtained from the SNSP with those of the SP for a given number of pairs k up to the half-filling, where $g_{t,t}$ in the SNSP are adjusted to $E_{k=1}^{(\zeta)}$ of the SP indicated by *, while $g_{1,1}$ for given k in the SNSP is then adjusted to ground-state energy $E_k^{(\zeta=1)}$ of the SP indicated by *, the overall strength g in the SPM and the pairing parameters in the SSP are all adjusted to produce the ground-state energy $E_k^{(\zeta=1)}$ of the SP indicated by *.

	$k=1$				$k=2$				$k=3$				$k=4$				$k=5$			
	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM	SP	SNSP	SSP	SPM
$\zeta=1$	−23.10	−23.10*	−23.10*	−23.10*	−43.31	−43.31*	−43.31*	−43.31*	−60.67	−60.67*	−60.67*	−60.67*	−75.13	−75.13*	−75.13*	−75.13*	−87.59	−87.59*	−87.59*	−87.59*
$\zeta=2$	−11.45	−12.59	−12.46	−12.75	−34.45	−34.31	−34.36	−34.43	−54.58	−53.55	−53.57	−53.50	−71.86	−70.32	−70.31	−70.19	−83.99	−82.10	−82.17	−82.32
$\zeta=3$	−8.29	−8.74	−8.67	−8.82	−31.36	−30.82	−30.84	−30.71	−51.57	−50.31	−50.26	−49.96	−68.92	−67.08	−67.01	−66.64	−83.54	−81.88	−81.82	−81.56
$\zeta=4$	−2.66	−3.78	−3.50	−4.13	−25.95	−25.75	−26.17	−26.43	−46.17	−45.78	−45.81	−45.83	−65.79	−63.52	−63.54	−63.50	−80.61	−77.96	−77.94	−77.76
$\zeta=5$					−22.80	−23.87	−23.98	−24.40	−45.72	−44.85	−44.97	−45.08	−63.54	−62.07	−62.62	−63.56	−78.38	−75.75	−75.86	−75.58
$\zeta=6$					−19.73	−20.91	−21.01	−21.36	−42.72	−41.98	−42.15	−42.12	−62.85	−60.77	−60.76	−60.53	−77.47	−74.97	−75.18	−75.49
$\zeta=7$					−14.33	−15.60	−16.18	−17.02	−37.32	−37.26	−37.53	−37.93	−57.45	−55.50	−56.22	−58.39	−77.11	−73.15	−73.17	−72.94
$\zeta=8$					−11.18	−11.90	−12.52	−13.42	−34.22	−33.04	−34.09	−34.50	−54.43	−52.83	−52.94	−53.15	−75.02	−71.92	−72.13	−71.89
$\zeta=9$					−5.69	−6.11	−6.82	−7.68	−31.09	−32.14	−32.67	−33.29	−54.00	−52.16	−52.91	−53.04	−74.52	−71.90	−71.91	−71.75
$\zeta=10$									−28.73	−28.36	−28.61	−28.91	−48.93	−47.22	−48.12	−48.60	−72.05	−68.02	−68.22	−67.86
$\zeta=11$									−25.69	−27.23	−27.73	−28.68	−48.59	−46.81	−47.55	−47.54	−71.40	−67.85	−67.92	−67.80
$\zeta=12$									−22.62	−23.48	−24.86	−25.98	−45.59	−44.41	−45.39	−45.98	−68.95	−66.23	−66.47	−66.68
$\zeta=13$									−17.14	−18.56	−19.18	−20.22	−40.09	−38.82	−39.86	−40.33	−65.97	−65.59	−65.97	−66.30
$\zeta=14$									−13.98	−14.08	−15.57	−16.69	−37.00	−35.33	−36.43	−37.74	−65.46	−62.69	−63.02	−62.81
$\zeta=15$									−8.42	−8.74	−9.25	−9.91	−33.98	−35.22	−36.43	−36.93	−62.77	−61.19	−61.36	−61.57
$\zeta=16$													−31.42	−29.73	−30.63	−31.72	−62.57	−58.61	−59.07	−59.25
$\zeta=17$													−28.50	−29.29	−30.29	−30.30	−59.97	−58.20	−58.84	−58.92
$\zeta=18$													−25.43	−26.34	−27.76	−28.98	−57.16	−55.87	−56.40	−56.52
$\zeta=19$													−19.86	−20.51	−21.45	−22.18	−54.02	−54.38	−54.88	−55.89
$\zeta=20$													−16.71	−16.86	−17.87	−18.64	−50.98	−51.94	−52.70	−53.03
$\zeta=21$																	−45.61	−48.70	−49.32	−49.91
$\zeta=22$																	−42.53	−44.60	−45.57	−46.37

shown that the SNSP solution can be derived more easily than that of the general separable pairing case [13] by using a similar method. The extended one-variable Heine-Stieltjes polynomials associated to the corresponding Bethe-Gaudin-Richardson ansatz equations of the solution for any number of pairs k are determined. Anyway, based on the Gaudin-Richardson solution of the standard paring model, an alternative approach to the exact solution of the spherical mean-field plus non-separable pairing model with multi non-degenerate j-orbits is proposed, of which applications to realistic nuclear systems with multi non-degenerate j-orbits will be made in our future work.

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