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A New Method In Distribution Theory With A Non-Smooth Framework

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A NEW METHOD IN DISTRIBUTION THEORY WITH A NON-SMOOTH FRAMEWORK

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"When I consider thy heavens, the work of thy fingers, the moon and the stars, which thou hast ordained; What is man, that thou art mindful of him? and the son of man, that thou visitest him?" Psalm 8:3-4 (KJV)

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Table of Contents

Abstract

In this work, we present a complete treatment of the theory of thick distributions and its asymptotic expansions. We also present several applications of thick distributions in mathematical physics, function spaces, and measure theory. We also discuss regularization using different surfaces. In the last chapter we present some recent applications of distributions in clarifying the moment terms in the heat kernel expansion, and in explaining the relation between the heat kernel expansion and the cylinder kernel expansion.

Chapter 1
Introduction

1.1 Introduction

My PhD work has been focused on generalized functions and their asymptotic expansions. Besides their intrinsic relation to the theory of function spaces [88], most questions I have been working on with my advisor, Professor Estrada, have a background in spectral geometry and quantum field theory [12, 24, 35, 42, 59].

The Theory of distributions were introduced by Sergei Sobolev and Laurent Schwartz independently in the early 20th century. It is useful in many areas in mathematics, ranging from PDE to number theory. Modern mathematics and physics research suggest a need to consider singularities in the test function space. Several puzzles and paradoxes in the application of distribution theory in quantum field theory, have pointed to a need to develop a theory of distributions where one special point, a "thick" point, was presented. In Physics, the "thick" point, which is essentially a singularity on the test functions, corresponds to the idealization of small bounded solids, where the field equations are singular and where the non-linearities cannot be handled by standard distribution theory. A one-dimensional theory has been built to incorporate jump discontinuities on test functions [35]. I have worked on constructing a systematic theory to deal with discontinuities of test functions in higher dimensions. Because of the fundamental difference between the topology of $\mathbb{R}^n, n > 1$ and $\mathbb{R}$, i.e. $\mathbb{R}^n \setminus \{\mathbf{a}\}$ is connected while $\mathbb{R} \setminus \{a\}$ is not, the theory of thick distributions in higher dimension is very different from that in one dimension. Chapter two will be devoted to the new theory of thick distributions in higher dimensions, where the idea of strong asymptotic expansions near the

singularity of test functions is used as a key ingredient in the construction. This original work is based on a paper I published with my advisor Ricardo Estrada. These new distributions, which can be projected onto the usual distribution space, turned out to have many nice properites. This new theory not only makes objects like Hadamard finite part easier to compute in many cases, but far more importantly, also gives a rigorous mathematical meaning of some otherwise ill-defined notions, such as, for example, $n_i n_j \delta(\mathbf{x})$, where n_i, n_j are unit vectors in $\mathbb{R}^n$ [15]; or, another example, $\delta(\mathbf{x})/|\mathbf{x}|^2$ [58]. Chapter three and four will be devoted to some of the applications in solving certain puzzles in applying the classical distribution theory in quantum field theory, by adopting thick distributions. They are based on two other papers of Professor Estrada and myself. [85, 87].

In Chapter five I will give a full description of the dual space of the space of regulated functions. This original work is another application of thick distributions, in the theory of function spaces and measure theory. This chapter is based on our paper [88]. One class of thick distributions, the "thick delta functions" plays an important role in the dual space of regulated functions. In fact, regulated functions can be viewed as the class of functions whose limit $\lim_{\epsilon\to 0^+} \phi(\mathbf{x}+\epsilon\mathbf{w})$ exists uniformly on the unit sphere $\mathbf{w} \in \mathbb{S}^{n-1}$. We proved in this paper that any continuous linear functional acting on regulated functions on an open set $U \subseteq \mathbb{R}^n$ can be represented as $\mu(\mathbf{x}) = \mu_{cont}(\mathbf{x}) + \sum_{\mathbf{c}\in\mathcal{C}} \left(\xi_{\mathbf{c}}\delta_*(\mathbf{x}-\mathbf{c}) + \gamma_{\mathbf{c}}\delta(\mathbf{x}-\mathbf{c})\right)$, where $\xi_{\mathbf{c}}\delta_*$ are the so called "thick delta functions" in our theory of thick distributions, μ_{cont} is a continuous Radon measure, $\mathcal{C} \subset \mathbb{R}^n$ a countable subset, $\xi_{\mathbf{c}} \in \mathcal{M}(\mathbb{S}^{n-1})$ signed measures, and $\gamma_{\mathbf{c}} \in \mathbb{R}$ constants for $\mathbf{c} \in \mathcal{C}$. Furthermore the norm satisfies $\|\mu\| = \int_{R^n} d|\mu_{cont}| + \sum_{\mathbf{c}\in\mathcal{C}} \left(\|\xi_{\mathbf{c}}\|_{\mathcal{M}(\mathbb{S}^{n-1})} + |\gamma_{\mathbf{c}}|\right)$.

My most recent work on this topic was to develop asymptotic expansions for certain thick distributions. Asymptotic analysis is an old subject and, as distribution

theory, it has found applications in various fields of pure and applied mathematics, physics, engineering. The requirements of modern mathematics, and mathematical physics, have brought the necessity to incorporate ideas from asymptotic analysis to the field of generalized functions, and reciprocally. During the past five decades, numerous definitions of asymptotic behavior for generalized functions have been elaborated and applied to concrete problems in mathematics and mathematical physics.

A thick distribution admits a moment asymptotic expansion if and only if it belongs to the class $\mathcal{K}'_*$. In our recently submitted paper [86] we were able to give a general formula for moment asymptotic expansions. We also computed a few examples $\mathcal{K}'_*$. These results are collected in Chapter Six.

Chapter seven served as an independent chapter. In this chapter, we will consider the surface dependence of the regularization of homogeneous functions in $\mathbb{R}^n$ [90]. It is known that distributions associated to one singular function can be different from each other depending on the regularization method. Suppose $K(\mathbf{x})$ is a singular function, homogeneous of degree $-n$ in $\mathbb{R}^n$; suppose also that, $\Sigma : g(\mathbf{x}) = 1$ is an $n-1$ surface for regularization, then we proved that, the Hadamard finite part $\langle Pf_{\Sigma}(K(\mathbf{x})), \phi(\mathbf{x})\rangle = \int_{g\geq 1} K(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} + \int_{0\leq g\leq 1} K(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{0}))d\mathbf{x}$. Moreover, the difference between two such regularizations is $(\int_{g_1\geq 1, g_2\leq 1} K(\mathbf{x})d\mathbf{x} + \int_{g_1\leq 1, g_2\geq 1} K(\mathbf{x})d\mathbf{x})\delta(\mathbf{x})$. As examples, we recovered the formulae for the second-order generalized derivatives $\frac{\overline{\partial}^2}{\partial x_i \partial x_j}(\frac{1}{r})$ obtained by Hnizdo. Hnizdo suggests [50] that regularizations using a spheroid and a cylinder have applications in general relativity; Farassat [38] showed how this problem has surprising implication in the numerical solution of integral equations of transonic flow.

An application of the distributional asymptotic expansion of spectral functions is to study low frequency asymptotics of various Green functions $G(t, x, y)$ as-

sociated to an elliptic differential operator H. The last chapter, Chapter eight, is based on my recent paper with Professor Stephen Fulling, where we used the asymptotic expansion of the Dirac combs $\sum_{n=1}^{\infty} \delta(x-n)$ and $\sum_{n=1}^{\infty} \delta(x-n^2)$ to clarify a few subtleties in the relationships among heat kernel invariants, eigenvalue distributions, and quantum vacuum energy [46].

There are many open questions and future research directions both in the theory of thick distributions itself and applications of it in pure and applied math, including geometry and number theory. Open questions will be briefly discussed at the end of this chapter.

1.2 Background and Results

This section comes from my research statement. Here it serves as a "map" of my dissertation. In the section of "Results", I listed many of the main results that hopefully could give the reader an overview of my dissertation. One could find detailed proofs and explanations in corresponding chapters.

In order to solve the problem that two evaluations of $\int_0^{\infty} \cos(2kx)\, dx$ in a distributional sense gives us different results, Estrada and Fulling gave a one-dimensional theory of distributions, where one special point, a "thick point", is present [35]. Employing this theory, some puzzles in applications of distribution theory in quantum field theory [14] or in engineering [69, 83] can be solved. For higher dimensions, Blanchet and Faye [12] first proposed to consider functions and generalized functions in $\mathbb{R}^3$ where a finite number of special points are present, in the context of finite parts, pseudo-functions and Hadamard regularization studied by Sellier [76, 75]; their analysis is aimed at the study of the dynamics of point particles in high post-Newtonian approximations of general relativity. Blanchet and Faye [12] also suggested that the special points can correspond to the idealization of small

bounded solids, such as black holes, where the field equations are singular and where the non-linearities cannot be handled by standard distribution theory.

The need to develop a theory of thick distributions near isolated singularities, can also be seen in the work of other authors. Blinder [13] develops a method, to embed $\mathbb{R}^{\mathsf{n}}$ into a space where another mirror image of $\mathbb{R}^{\mathsf{n}}$ is added; Gsponer [49] considers singular points in polar coordinates. Bowen [15] dealt with distributions of the form $n_i n_j \delta(\mathbf{x})$, which he considers to be "nearly meaningless" and which are actually thick distributions; the failure of the product rule reported in [15] is fixed by the theory of thick distributions [85]. Franklin proposed a question towards the well-known formula for distributional second derivative of $1/r$, that can also be answered in the context of thick distributions [40]. Let me present our main results.

1.2.1 Thick test functions and thick distributions

Definition 1. *Let $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ denote the vector space of all smooth functions ϕ defined in $\mathbb{R}^{\mathsf{n}}\setminus\{\mathbf{a}\}$, with compact support K that admit a strong asymptotic expansion of the form*

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\, r^{\mathsf{j}}, \quad \text{as } \mathbf{x}\to\mathbf{0}, \tag{1.2.1}$$

for some $\mathsf{m}\in\mathbb{Z}$, the a_{j} are smooth functions of $\mathbf{w}$, that is, $a_{\mathsf{j}}\in\mathcal{D}(\mathbb{S})$. Strong means that for each $\mathbf{p}\in\mathbb{N}^{\mathsf{n}}$ the asymptotic development of $(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi(\mathbf{x})$ as $\mathbf{x}\to\mathbf{0}$ equals the term-by-term differentiation of $\sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\, r^{\mathsf{j}}$. We call $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ the space of test functions on $\mathbb{R}^{\mathsf{n}}$ with a thick point at $\mathbf{x}=\mathbf{a}$. $\mathcal{D}_{*}(\mathbb{R}^{\mathsf{n}})$ denotes $\mathcal{D}_{*,\mathbf{0}}(\mathbb{R}^{\mathsf{n}})$.

With a proper topology, $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ is a topological vector space. It actually is the inductive limit of some Fréchet spaces, with the usual distribution space $\mathcal{D}(\mathbb{R}^{\mathsf{n}})$ its *closed* subspace. Let me now give the topology:

Definition 2. *Let* m *be a fixed integer and* K *a compact subset of* $\mathbb{R}^{\mathsf{n}}$ *whose interior contains* $\mathbf{a}$. *Let* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m}]}(\mathbb{R}^{\mathsf{n}})$ *denote the space of thick test functions* ϕ *whose expansion (1.2.1) begins at* m*; and let* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m};K]}(\mathbb{R}^{\mathsf{n}})$ *denote the subspace formed by those test functions of* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m}]}(\mathbb{R}^{\mathsf{n}})$ *that vanish in* $\mathbb{R}^{\mathsf{n}}\setminus K$. *The topology of* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m};K]}(\mathbb{R}^{\mathsf{n}})$ *is given by the seminorms* $\left\{\|\quad\|_{\mathsf{q},\mathsf{s}}\right\}_{\mathsf{q}>\mathsf{m},\mathsf{s}\geq 0}$ *defined as*

$$||\phi||_{\mathsf{q},\mathsf{s}}=\sup_{\mathbf{x}-\mathbf{a}\in K}\sup_{|\mathbf{p}|\leq\mathsf{s}} r^{-\mathsf{q}}\left|(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi(\mathbf{a}+\mathbf{x})-\sum_{\mathsf{j}=\mathsf{m}-|\mathbf{p}|}^{\mathsf{q}-1}a_{j,\mathbf{p}}(\mathbf{w})\,r^{j}\right|, \qquad (1.2.2)$$

where $\mathbf{x}=r\mathbf{w},\ \mathbf{p}\in\mathbb{N}^{\mathsf{n}}$*, and* $(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi(\mathbf{a}+\mathbf{x})\sim\sum_{\mathsf{j}=\mathsf{m}-|\mathbf{p}|}^{\infty}a_{\mathsf{j},\mathbf{p}}(\mathbf{w})\,r^{\mathsf{j}}$.

The topology of $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m}]}(\mathbb{R}^{\mathsf{n}})$ *is the inductive limit topology of the* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m};K]}(\mathbb{R}^{\mathsf{n}})$ *as* $K\nearrow\infty$*; and the topology of* $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ *is the inductive limit topology of the* $\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m}]}(\mathbb{R}^{\mathsf{n}})$ *as* $\mathsf{m}\searrow-\infty$.

Standard results on inductive limits [80] yield that a sequence $\{\phi_l\}_{l=0}^{\infty}$ in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ converges to ψ if and only there exists $l_0\geq 0$, an integer m, and a compact set K with $\mathbf{a}$ in its interior, such that $\phi_l\in\mathcal{D}_{*,\mathbf{a}}^{[\mathsf{m};K]}(\mathbb{R}^{\mathsf{n}})$ for $l\geq l_0$ and $||\psi-\phi_l||_{\mathsf{q},\mathsf{s}}\to 0$ as $l\to\infty$ if $\mathsf{q}>\mathsf{m},\mathsf{s}\geq 0$.

Now we have $\mathcal{D}_*(\mathbb{R}^n)$ a topological vector space. Moreover, if ϕ is a standard test function, smooth in $\mathbb{R}^{\mathsf{n}}$ and with compact support, then its Taylor expansion gives us a strong asymptotic expansion, with starting term $a_0=\phi(\mathbf{a})$. Hence there is an inclusion, $i:\mathcal{D}(\mathbb{R}^{\mathsf{n}})\to\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$, making $\mathcal{D}(\mathbb{R}^{\mathsf{n}})$ a *closed* subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$.

Definition 3. *The space of thick distributions on* $\mathbb{R}^n$ *with a thick point at* $\mathbf{x}=\mathbf{a}$ *is the topological dual space of* $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ *equipped with the weak topology. We denote it* $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$.

Let $\pi:\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})\to\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ be the projection operator, then the Hahn-Banach theorem yields:

Theorem 1. Let f be any distribution in $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$, then there exist a thick distribution $g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ such that $\pi(g) = f$.

One class of the most important thick distributions are the thick delta functions, which are closely related to Dirac delta functions.

Definition 4. *(Thick delta functions of degree* q*) Let* $g(\mathbf{w})$ *be a distribution in* $\mathbb{S}$*. Let* C_{n-1} *denotes the surface area of* $(n-1)$ *dimensional unit sphere. The thick delta function of degree* q*, denoted as* $g\delta_*^{[\mathsf{q}]}$*, acts on a thick test function* $\phi(r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w}) r^{\mathsf{j}}$*, as*

$$\left\langle g\delta_*^{[\mathsf{q}]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}}) \times \mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \left\langle g(\mathbf{w}), a_{\mathsf{q}}(\mathbf{w}) \right\rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})} . \tag{1.2.3}$$

$g\delta_*^{[0]}$ *will be denoted as* $g\delta_*$*. If* $g(\mathbf{x}) \equiv 1$*, we obtain the "plain thick delta function"* δ_*,

$$\langle \delta_*, \phi \rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}}) \times \mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \int_{\mathbb{S}} a_0(\mathbf{w}) \, \mathrm{d}\sigma(\mathbf{w}) . \tag{1.2.4}$$

The projection of the plain thick delta function onto $\mathcal{D}'(\mathbb{R}^n)$ is the Dirac delta function δ. In fact, there is a non-trivial relation between thick delta functions and the Laplacian of Dirac delta functions:

$$\pi\left(\delta_*^{[2\mathsf{m}]}\right) = \frac{\Gamma(\mathsf{m}+1/2)\,\Gamma(\mathsf{n}/2)}{\Gamma(\mathsf{m}+\mathsf{n}/2)\,\Gamma(1/2)\,(2\mathsf{m})!} \nabla^{2\mathsf{m}}(\delta), \tag{1.2.5}$$

where $\nabla^2 = \sum_{i=1}^{\mathsf{n}} \partial^2/\partial x_i^2$ is the Laplacian. A more general result for $\pi\left(g\delta_*^{[\mathsf{q}]}\right)$ was also obtained [89].

Notice that there is also an important relation between the finite part distributions $\mathcal{P}f\left(r^{\lambda}\right)$ and $\delta_*^{[\mathsf{q}]}$: The thick distributions $\mathcal{P}f\left(r^{\lambda}\right)$ are analytic functions of λ in the region $\mathbb{C} \setminus \mathbb{Z}$. At the integers $\mathsf{k} \in \mathbb{Z}$ there are simple poles with residues $Res_{\lambda=\mathsf{k}} \mathcal{P}f\left(r^{\lambda}\right) = C_{\mathsf{n}-1}\delta_*^{[-\mathsf{k}-\mathsf{n}]}$. The distribution $\mathcal{P}f\left(r^{\mathsf{k}}\right)$ is the finite part of the analytic function at the pole. This allows us to recover the well known result that, $r^{\lambda} = \pi\left(\mathcal{P}f\left(r^{\lambda}\right)\right)$ is analytic for $\lambda \neq -\mathsf{n}, -\mathsf{n}-1, \ldots$.

1.2.2 Algebraic and Analytic Operators

Algebraic and analytic operators are defined by duality, as naturally expected. As a small example, we have $\left(g\delta_*^{[\mathsf{q}]}\right)(A\mathbf{x}) = g(A\mathbf{w})\,\delta_*^{[\mathsf{q}]}(\mathbf{x})$ for orthogonal transformation A.

We note that the algebra that consists of all multipliers of $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$, denoting $\mathcal{B}_{*,\mathbf{a}}(\mathbb{R}^n)$, is $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$, that is, the space of smooth functions with a "thick" point at $\mathbf{a}$. This is the key to solve certain paradoxes involving multiplication between distributions. We submitted a paper on Bowen's concern [15, 85].

Differential operators are particularly important. Usual distributional derivatives are projections of thick derivatives. The following result allows us to compute derivatives of thick deltas.

Proposition 1. *Let $\frac{\delta g}{\delta x_j}$ be the delta derivative and let $\frac{\partial^*}{\partial x_j}$ denote the distributional derivative in $\mathcal{D}_*$. Then*

$$\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[\mathsf{q}]}\right) = \left(\frac{\delta g}{\delta x_j} - (\mathsf{q}+\mathsf{n})\,n_j g\right)\delta_*^{[\mathsf{q}+1]}, \tag{1.2.6}$$

$$\frac{\partial^{*2}}{\partial x_j \partial x_i}\left(\delta_*^{[\mathsf{q}]}\right) = (\mathsf{q}+\mathsf{n})\left((\mathsf{q}+\mathsf{n}+2)\,n_i n_j - \delta_{ij}\right)\delta_*^{[\mathsf{q}+2]}. \tag{1.2.7}$$

Example 1. *We can compute the Laplacian of the plain thick deltas:*

$$\nabla^2\left(\delta_*^{[\mathsf{q}]}\right) = (\mathsf{q}+\mathsf{n})(\mathsf{q}+2)\,\delta_*^{[\mathsf{q}+2]}, \tag{1.2.8}$$

$$\nabla^{2\mathsf{m}}(\delta_*) = \frac{\Gamma(\mathsf{m}+\mathsf{n}/2)\,\Gamma(1/2)\,(2\mathsf{m})!}{\Gamma(\mathsf{m}+1/2)\,\Gamma(\mathsf{n}/2)}\delta_*^{[2\mathsf{m}]}, \quad \mathsf{m}>0. \tag{1.2.9}$$

Other important formulas are the derivatives of $\mathcal{P}f\left(r^\lambda\right)$. They can be found in [89]. An interesting example is Franklin's question [40, 89]. This can also be found in our paper [87]. We have derived,

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{\mathsf{k}}\right)}{\partial x_i \partial x_j} = \left(\mathsf{k}\delta_{ij} + \mathsf{k}(\mathsf{k}-2)\,n_i n_j\right)\mathcal{P}f\left(r^{\mathsf{k}-2}\right) + \left(\delta_{ij} + 2(\mathsf{k}-1)\,n_i n_j\right)\delta_*^{[-\mathsf{k}-\mathsf{n}+2]}, \quad k \in \mathbb{Z}. \tag{1.2.10}$$

When $\mathsf{n} = 3$ and $\mathsf{k} = -1$, we obtained a formula that answers Franklin's concern. Since $\pi\left(n_i n_j \delta_*\right) = (1/3)\,\delta\left(\mathbf{x}\right)$ in $\mathbb{R}^3$, we reobtain the well-known formula of Frahm [39] by projection.

1.2.3 Asymptotic Expansion

I have also worked on the asymptotic expansions of thick distributions. We recently submitted our paper on this topic to the Studia Mathematica. [86].

Let $\mathcal{K}_{*,\mathbf{a}}\left(\mathbb{R}^n\right) = \left\{\phi \in \varepsilon_{*,\mathbf{a}}\left(\mathbb{R}^n\right) \middle| \exists q \in \mathbb{N}, s.t. D^{\mathbf{k}}\phi\left(\mathbf{x}\right) = O\left(\left|\mathbf{x}\right|^{q-|\mathbf{k}|}\right), as\ \ \left|\mathbf{x}\right| \to \infty\right\}$ be a test function space under suitable topology. Namely, for a fixed q that controls the asymptotic behavior, $\mathcal{K}_{*,\mathbf{a}}^{[m,q]}\left(\mathbb{R}^n\right)$ is a Fréchet space under the semi-norm

$$\left\|\phi\right\|_{q,l,s} = \max\left\{\begin{array}{c} \sup\limits_{|\mathbf{x}|\le 1}\sup\limits_{|\mathbf{p}|\le s} \frac{\left|D^{\mathbf{p}}\phi(\mathbf{a}+\mathbf{x}) - \sum\limits_{j=m-|\mathbf{p}|}^{l-1} a_{j,\mathbf{p}}(\mathbf{w}) r^j\right|}{r^l}, \\ \sup\limits_{|x|\ge 1}\sup\limits_{|\mathbf{p}|\le s} r^{-q+\mathbf{p}} D^{\mathbf{p}}\phi\left(\mathbf{a}+\mathbf{x}\right) \end{array}\right\}$$

This space $\mathcal{K}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ is important for moment asymptotic expansions. We have:

Theorem 2. Suppose $f \in \mathcal{K}'_*\left(\mathbb{R}^n\right)$, then $f\left(\lambda\mathbf{x}\right)$ admits the following asymptotic expansion:

$$f\left(\lambda\mathbf{x}\right) \sim \sum_{k=-\infty}^{\infty} \frac{C_{n-1}\mu_k\left(\mathbf{w}\right)\delta_*^{[k]}}{\lambda^{k+n}},\ \text{as } \lambda \to \infty.$$

C_{n-1} is the surface area of $(n-1)$ unit sphere; such that the moment $\mu_k\left(\mathbf{w}\right)$ is a distribution in $\mathcal{D}'\left(\mathbb{S}^{n-1}\right)$ with

$$\left\langle \mu_k\left(\mathbf{w}\right), a_k\left(\mathbf{w}\right)\right\rangle_{\mathcal{D}'(\mathbb{S}^{n-1})\times\mathcal{D}(\mathbb{S}^{n-1})} = \left\langle f\left(\lambda\mathbf{x}\right), a_k\left(\mathbf{x}/\left|\mathbf{x}\right|\right)\left|\mathbf{x}\right|^k\right\rangle_{\mathcal{K}'_*(\mathbb{R}^n)\times\mathcal{K}_*(\mathbb{R}^n)}.$$

One can project asymptotic expansions of thick distributions onto the usual distribution space $\mathcal{K}'\left(\mathbb{R}^n\right)$, to obtain moment asymptotic expansions [37]. This is helpful because he former is usually easier to compute. We developed a few propositions describing the behavior of such projections [86]. I computed several examples. Let me list one here.

Example 2. *Let $f(\mathbf{x}) = \delta(\mathbb{S}) \in \mathcal{D}'_*(R^n)$ be the delta function on the unit sphere of $\mathbb{R}^n$, i.e., $\langle f, \phi\rangle = \int_{\mathbb{S}^{n-1}} \phi(\mathbf{w})\, d\sigma(\mathbf{w})$. Then*

$$f(\lambda \mathbf{x}) = \frac{1}{\lambda}\delta\left(\mathbb{S}_{1/\lambda}\right) \sim \sum_{k=-\infty}^{\infty} \frac{2\pi^{n/2}\delta_*^{[k]}(\mathbf{x})}{\Gamma(n/2)\lambda^{n+k}},$$

$$\pi(f(\lambda \mathbf{x})) \sim \sum_{k=0}^{\infty} \frac{c_{k,n}\nabla^{2k}(\delta)}{(2k)!\lambda^{n+2k}}, \quad \text{as } \lambda \to \infty.$$

where $c_{k,n} = 2\Gamma(k+1/2)\pi^{(n-1)/2}/\Gamma(k+n/2)$. This agrees with the result in [37], Example 94. Moreover, in general, If $F(\mathbf{x}) = g(\mathbf{w})\delta(\mathbb{S})$, where $g(\mathbf{w})$ is any distribution on the sphere, then we have $\mu_k(\mathbf{w}) = g(\mathbf{w})$, thus

$$F(\lambda \mathbf{x}) \sim \sum_{k=-\infty}^{\infty} \frac{g(\mathbf{w})\, 2\pi^{n/2}\delta_*^{[k]}(\mathbf{x})}{\Gamma(n/2)\lambda^{n+k}}, \quad \text{as } \lambda \to \infty. \tag{1.2.11}$$

The theory of asymptotic expansion of distributions is a very useful tool in many areas, including analytic number theory and spectral geometry [37, 27]. The analysis of the Dirac combs $\sum \phi(n)\delta(x-n)$ leads to several nice results. In the following section I will present one of its applications in spectral theory, based on my work in collaboration with Professor S. Fulling [46].

1.2.4 Spectral Asymptotic Expansions

It was found by Estrada [37], that the moment asymptotic expansion of the Dirac comb is

$$\sum_{n=1}^{\infty} \delta\left(\frac{x}{\epsilon} - n\right) \sim H(x) + \sum_{n=0}^{\infty} \frac{(-1)^n \zeta(-n)\delta^{(n)}(x)}{n!}\epsilon^{n+1} \quad \text{as } \epsilon \to 0. \tag{1.2.12}$$

Applying this result, we were able to clarify that, the "missing" coefficients (which "should" accompany negative powers of λ) addressed by Kolomeisky and other authors in [59] can be identified as those multiplying terms $\delta^{(j)}(\lambda)$ (the moment terms) in the asymptotic analysis of $N'(\lambda)$, where $N(\lambda) = \sum_{n=1}^{\infty} H(\lambda - \lambda_n)$ is the eigenvalue counting function and $N'(\lambda) = \sum_{n=1}^{\infty} \delta(\lambda - \lambda_n)$ is its derivative. In fact, one can

recover the "generalized Weyl expansion" in [59] by applying (1.2.12) on a test function [46].

Moreover, in considering the moment asymptotic expansion of $\sum_{n=1}^{\infty} \delta(\omega^2/\epsilon - n^2)$, one can see the missing terms with half-integer power in the cylinder-kernel expansion. These new, nonlocal specral invariants show up in the Riesz-Cesàro asymptotics of $N(\lambda)$ with respect to $\omega = \sqrt{\lambda}$.

1.2.5 Other Results related to Measure Theory

In [88], we showed the following results between the dual space of the space of regulated functions on $\mathbb{R}^n$ and signed measures.

Theorem 3. Let $\mu \in \mathcal{R}_t'(U)$, the space of regulated functions whose domains are the open set $U \subseteq R^n$. Then there exists a continuous Radon measure in $\mathbb{R}^n$, μ_{cont}; a countable subset $\mathcal{C} \subset \mathbb{R}^n$; signed measures $\xi_{\mathbf{c}} \in \mathcal{M}(\mathbb{S}^{n-1})$ and constants $\gamma_{\mathbf{c}} \in \mathbb{R}$ for $\mathbf{c} \in \mathcal{C}$ such that

$$\mu(\mathbf{x}) = \mu_{cont}(\mathbf{x}) + \sum_{\mathbf{c}\in\mathcal{C}} \left(\xi_{\mathbf{c}}\delta_*(\mathbf{x}-\mathbf{c}) + \gamma_{\mathbf{c}}\delta(\mathbf{x}-\mathbf{c})\right). \tag{1.2.13}$$

Furthermore, the norm of μ in the space $\mathcal{R}_t'(R^n)$ is

$$\|\mu\| = \int_{R^n} d\left|\mu_{cont}\right| + \sum_{\mathbf{c}\in\mathcal{C}} \left(\|\xi_{\mathbf{c}}\|_{\mathcal{M}(\mathbb{S}^{n-1})} + |\gamma_{\mathbf{c}}|\right).$$

In particular, when $n = 1$, our results agree with the work by other researchers [57, 81].

1.2.6 A Simple Application of the Thick Distributional Calculus

Let $n_i = x_i/r$. Let $\frac{\overline{\partial}}{\partial x_i}$ denotes the standard distributional derivative in $\mathcal{D}'$. Bowen [15] was puzzled by his computation, that the product rule does NOT hold:

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right) \neq n_{j_1}n_{j_2}\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right) + \frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right).$$

This should not be too surprising. It happens because $n_{j_1}n_{j_2}$ is not a multiplier of $\mathcal{D}'(\mathbb{R}^3)$. The $n_{j_1}n_{j_2}\overline{\partial}/\partial x_i\left(n_{j_3}/r^2\right)$ is in fact a multiplication between two distributions in $\mathcal{D}'(\mathbb{R}^3)$. So is $\frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right)$.

In the light of thick distributions, $n_{j_1}n_{j_2} \in \mathcal{B}_*(\mathbb{R}^3)$. It is a multiplier of thick distributions. So we should consider the thick distributional derivative $\partial^*/\partial x_i$. In fact, we have

$$\frac{\partial^*}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right) = n_{j_1}n_{j_2}\frac{\partial^*}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right) + \frac{n_{j_3}}{r^2}\frac{\partial^*}{\partial x_i}\left(n_{j_1}n_{j_2}\right).$$

And $\pi\left(\frac{\partial^*}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right)\right) = \frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right)$, which agrees with Bowen's computation.[85]

1.2.7 Regularization using different surfaces

In [90], we discussed regularizations using different surfaces. Let $K(x)$ be a function homogeneous of degree $-n$ which is defined in $\mathbb{R}^n$; $\Sigma : g(\mathbf{x}) = 1$ be an $n-1$ dimensional surface around the origin.

Proposition 2.

$$K(x)\,H(g(x)-\varepsilon) = Pf_{\Sigma}(K(x)) + D\ln\varepsilon + o(1).$$

where $Pf_{\Sigma}(K(x))$ *is the finite part distribution and* $D = \int_{S^{n-1}} K(w)\,d\sigma(w)$ *is independent of* Σ.

We also found that the difference $Pf_{\Sigma_1}(K(x)) - Pf_{\Sigma_2}(K(x))$ is a constant $c(\Sigma_1, \Sigma_2, K)$ times a Dirac delta function. Moreover, as examples, we used four different surfaces to regularize $\left(3x_ix_j - r^2\delta_{ij}\right)/r$ and computed concrete results. Hnizdo suggests [50] that regularizations using a spheroid and a cylinder have applications in general relativity; Farassat [38] showed how this problem has surprising implication in the numerical solution of integral equations of transonic flow.

1.3 Research Objectives and Open questions

1.3.1 Fourier Analysis

A natural next step in researching thick distributions is to work on the Fourier transform of them. This work will link the thick distributional analysis to PDE, Radon transform, non-commutative geometry and will bring it a more "modern" look. We believe that it will be a key step and it will open more doors to future research.

It is found in 1-d that the Fourier transform of thick test functions are those smooth functions that admits the asymptotic formula $\hat{\phi}(u) \sim \frac{c_1}{u} + \frac{c_2}{u^2} + \frac{c_3}{u^3} + ...$ as $|u| \to \infty$. In fact there is a one-to-one correspondence between the space consisting of such functions and the space of thick test functions, and thus a one-to-one correspondence between the corresponding dual, which was denoted $\mathcal{W}'$, and the space of thick distributions. Therefore it is natural to propose the question: does the Fourier transform of thick test functions in higher dimensions admit a similar asymptotic behavior or not?

Question 1. *What is the Fourier transform of the thick test functions $\mathcal{S}_*\left(\mathbb{R}^n\right)$? Is the Fourier transform an isomorphism between the spaces $\mathcal{S}'_*\left(\mathbb{R}^n\right)$ and $\mathcal{W}'\left(\mathbb{R}^n\right)$ as it is in 1-d?*

Another challenge is to give the Fourier transform of several thick distributions. Estrada and Fulling computed the Fourier transform of a thick delta function in 1-d. They found that $\mathcal{F}\left\{\widetilde{\delta}\left(x\right);u\right\} = \widetilde{1}$, where $\left\langle \widetilde{1}, \phi\left(u\right)\right\rangle = \int_{-1}^{1}\phi\left(u\right)du + \int_{|u|>1}\left(\phi\left(u\right) - \frac{c_1}{u}\right)du$. What is the case in higher dimensions then? In fact, it seems that there should be a relation between the Fourier transform of a higher dimensional thick delta function and the Wodzicki residue in noncommutative geometry. [27]

Question 2. $\mathcal{F}\{\delta_*(\mathbf{x});\mathbf{u}\} = ?$

The Fourier transform can be seen in the light of moment asymptotic expansions. In one dimension, $\mathcal{F}\{\phi(x);\lambda\}$ is to evaluate the distribution $\exp(i\lambda x)$ on ϕ. Using the moment asymptotic expansion of $\exp(i\lambda x)$, one can find an asymptotic expansion of $\mathcal{F}\{\phi(x);\lambda\}$ [37]. This should also apply in higher dimensions. Can we come up with a new idea with respect to the relation between Fourier transform and moment asymptotic expansion in the spirit of higher dimensional thick distributions?

1.3.2 More on asymptotic expansions

Although we have finished some research on the asymptotic expansion of thick distributions, there is more to be done. We know that the moment expansion allows us to asymptotically expand a distribution to a series of delta functions. Thus evaluating certain moment expansions on test functions gives us asymptotic series that could be of interest [37, 86]. This technique has been used in [37] on usual distributions. We can generalize this technique on larger spaces $\mathcal{K}_*$ and $\mathcal{K}_*^{'}$, which means we now have more asymptotic series that we can put our hands on.

1.3.3 Number Theory

Generalized functions are used in analytic number theory. Our work in the asymptotic expansions of distributions is related to the Riemann Zeta function, the explicit formulas, and so on. It seems there are much research could be done in this direction on unrestricted partitions and certain series considered by Ramanujan. [20, 37, 27] I hope to consider them in greater details.

1.3.4 Other Topics

The asymptotic expansions of distributions can be used to research the Weyl expansion of the eigenvalue counting function. There are relations to the complex

dimensional fractal geometry in this direction that may lead to interesting results. [62, 37]

There should be a connection between thick distributions and the Radon transform, which should be more clear after discussing the Fourier transform of thick distributions. This is also a possibility for further research.

Chapter 2
Theory of Thick Distributions

2.1 Introduction

A one-dimensional theory of distributions where one special point, a *thick point,* is present was given in [35]. Employing this theory, several puzzles, apparent paradoxes in the applications of distribution theory in quantum field theory [14] or in engineering [69, 83] can be solved. Our aim in this chapter is to give a corresponding theory in dimensions $n \geq 2$, which is a theoretical foundation of the following few chapters.

It must be said that the theory of thick distributions in higher dimesions is *very different* from that in one dimension, and thus our methods and results are not just a simple extension of those of [35]. Indeed, if $\mathbf{a} \in \mathbb{R}^n$, the topology of $\mathbb{R}^n \setminus \{\mathbf{a}\}$, $n \geq 2$, is quite unlike that of $\mathbb{R} \setminus \{a\}$ for $a \in \mathbb{R}$, since the latter space is disconnected, consisting of two unrelated rays, while the former is connected, all directions of approach to the point $\mathbf{a}$ are related, and such behavior imposses strong restrictions on the singularities [22]. In one variable, the derivative of a function with a jump discontinuity at a point may also have a jump discontinuity there, but such situation is not to be expected in higher dimensions, since derivatives of functions with a jump type singularity at a point will have, in general, derivatives that tend to infinity at the point. Therefore, we define test functions as those functions that are smooth away from the thick point but which have *strong* asymptotic expansions of the form

$$\phi(\mathbf{a} + r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j, \tag{2.1.1}$$

as $\mathbf{x} \to \mathbf{a}$ for some $m \in \mathbb{Z}$; strong means that (2.1.1) can be differentiated of any order. In general if the expansion of ϕ starts at m, then that of $\partial\phi/\partial x_j$ will start at $m-1$, and more generally, that of $D\phi$, where D is a differential operator of degree p, starts at $m-p$; therefore our space of test functions contains functions with developments of the type (2.1.1) for *any* integer $m \in \mathbb{Z}$. In one variable [35] it is enough to consider test functions whose expansion starts at $m=0$, but that approach does not work in dimensions $n \geq 2$.

The plan of this chapter is as follows. In Section 2.2 we review the δ–derivatives, that allow us to consider derivatives of functions and distributions defined *only* on a smooth hypersurface Σ of $\mathbb{R}^n$ with respect to the outside variables, those of $\mathbb{R}^n$ [28, 31]; we shall need δ–derivatives on the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$ in order to develop our calculus of thick distributions. In Section 2.3 we define the thick test functions and construct the topology of the corresponding space; we also show how the expansion of $\partial\phi/\partial x_j$ and other derivatives can be obtained from (2.1.1). Then in Section 2.4 we define the space of standard thick distributions, $\mathcal{D}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$, and give several examples, including finite part distributions and thick deltas; other spaces, such as the tempered thick distributions $\mathcal{S}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ are considered in Section 2.5. Algebraic and analytic operations are studied in Section 2.6; two important computations, the derivatives of thick deltas and the derivatives of powers of $|\mathbf{x}-\mathbf{a}|$ are considered in Sections 2.7 and 2.8.

2.2 Surface Derivatives

We shall need to consider the differentiation of functions and distributions defined only on a smooth hypersurface Σ of $\mathbb{R}^n$. Naturally, if $(v_\alpha)_{1\leq\alpha\leq n-1}$ is a local Gaussian coordinate system and f is defined on Σ then one may consider the derivatives $\partial f/\partial v_\alpha$, $1 \leq \alpha \leq n-1$. However, it is many times convenient and necessary to consider derivatives with respect to the variables $(x_j)_{1\leq j\leq n}$ of the surrounding space

$\mathbb{R}^n$. The $\delta-$derivatives [28, 31] are defined as follows. Suppose f is a smooth function defined in Σ and let F be any smooth extension of f to an open neighborhood of Σ in $\mathbb{R}^n$; the derivatives $\partial F/\partial x_j$ will exist, but their restriction to Σ will depend not only on f but also on the extension employed. However, it can be shown that the formulas

$$\frac{\delta f}{\delta x_j} = \left.\left(\frac{\partial F}{\partial x_j} - n_j \frac{dF}{dn}\right)\right|_{\Sigma}, \tag{2.2.1}$$

where $\mathbf{n} = (n_j)$ is the normal unit vector to Σ and where $dF/dn = n_k \partial F/\partial x_k$ is the derivative of F in the normal direction, define derivatives $\delta f/\delta x_j$, $1 \leq j \leq n$, that depend *only* on f and not on the extension. In fact, it can be shown that for any local Gaussian coordinate system $(v_\alpha)_{1\leq\alpha\leq n-1}$,

$$\frac{\delta f}{\delta x_j} = g^{\alpha\beta} \frac{\partial f}{\partial v_\alpha} \frac{\partial x_j}{\partial v_\beta}, \tag{2.2.2}$$

where $\left(g^{\alpha\beta}\right)$ is the first fundamental form of the surface.

Suppose now that the surface is $\mathbb{S}$, the unit sphere in $\mathbb{R}^n$. Let f be a smooth function defined in $\mathbb{S}$, that is, $f(\mathbf{w})$ is defined if $\mathbf{w} \in \mathbb{R}^n$ satisfies $|\mathbf{w}| = 1$. Observe that the expressions $\partial f/\partial x_j$ are not defined and, likewise, if $\mathbf{w} = (w_j)_{1\leq j\leq n}$ the expressions $\partial f/\partial w_j$ do not make sense either; the derivatives that are always defined and that one should consider are the $\delta f/\delta x_j$, $1 \leq j \leq n$. While (2.2.1) can be applied for any extension F of f, the fact that our surface is $\mathbb{S}$ allows us to consider some rather natural extensions. In particular, there is an extension to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ that is homogeneous of degree 0, namely,

$$F_0(\mathbf{x}) = f\left(\frac{\mathbf{x}}{r}\right), \tag{2.2.3}$$

where $r = |\mathbf{x}|$. Since $dF_0/dn = 0$ we obtain

$$\frac{\delta f}{\delta x_j} = \left.\frac{\partial F_0}{\partial x_j}\right|_{\mathbb{S}}. \tag{2.2.4}$$

Alternatively, if we use polar coordinates, $\mathbf{x} = r\mathbf{w}$, so that $F_0(\mathbf{x}) = f(\mathbf{w})$, then since $\partial F_0/\partial x_j$ is homogeneous of degree -1,

$$\frac{\partial F_0}{\partial x_j} = \frac{1}{r}\frac{\delta f}{\delta x_j}\,, \quad \text{if } \mathbf{x} \neq \mathbf{0}\,. \tag{2.2.5}$$

Actually in the more general case when $\Sigma = \rho\mathbb{S}$ is a sphere of radius ρ and center at the origin and F_0 is the extension of f that is homogeneous of degree 0, then $\delta f/\delta x_j$ is the restriction to Σ of $\partial F_0/\partial x_j$.

The matrix $\mu = (\mu_{ij})_{1\le i,j\le n}$, where $\mu_{ij} = \delta n_i/\delta x_j$, plays an important role in the study of distributions on a surface Σ. If $\Sigma = \rho\mathbb{S}$ then $\mathbf{n} = \mathbf{x}/\rho$, and thus

$$\mu_{ij} = \rho^{-1}\left(\delta_{ij} - n_i n_j\right)\,. \tag{2.2.6}$$

Observe that $\mu_{ij} = \mu_{ji}$, an identity that holds in any surface. The trace of μ, namely $\omega = \mu_{jj}$ is $n-1$ times the mean curvature of Σ; for a sphere of radius ρ (2.2.6) yields that ω is constant, and equal to $(n-1)\,\rho^{-1}$.

The differential operators are initially defined if f is a smooth function defined on Σ, but we can also define them when f is a distribution. We can do this in two ways: we can use the fact that smooth functions are dense in the space of distributions on Σ or we can use duality. The duality procedure works as follows. If $f \in \mathcal{D}'(\Sigma)$ then we define $\delta f/\delta x_j$ by its action on a test function $\phi \in \mathcal{D}(\Sigma)$ as

$$\left\langle \frac{\delta f}{\delta x_j}, \phi \right\rangle = -\left\langle f, \frac{\delta^T \phi}{\delta x_j} \right\rangle\,, \tag{2.2.7}$$

where $\delta^T/\delta x_j$ is the formal adjoint operator.

In general the operators $\delta/\delta x_j$ and $\delta^T/\delta x_j$ do not coincide. For instance, $\delta f/\delta x_j$ is tangent to Σ, that is, $n_j \delta f/\delta x_j = 0$, but in general $n_j \delta^T f/\delta x_j$ does not vanish; also, if c is a constant, then $\delta c/\delta x_j = 0$, while $\delta^T c/\delta x_j \neq 0$. Observe that (2.2.2) implies that $\delta/\delta x_j$ satisfies the usual product rule,

$$\frac{\delta(\phi\psi)}{\delta x_j} = \phi\frac{\delta\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi\,, \tag{2.2.8}$$

but $\delta^T/\delta x_j$ satisfies the alternative rule

$$\frac{\delta^T(\phi\psi)}{\delta x_j} = \phi\frac{\delta^T\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi\,, \tag{2.2.9}$$

since

$$\begin{aligned}\left\langle \zeta, \frac{\delta^T(\phi\psi)}{\delta x_j}\right\rangle &= -\left\langle \frac{\delta\zeta}{\delta x_j}, \phi\psi\right\rangle \\ &= -\left\langle \phi\frac{\delta\zeta}{\delta x_j}, \psi\right\rangle \\ &= -\left\langle \frac{\delta(\phi\zeta)}{\delta x_j} - \frac{\delta\phi}{\delta x_j}\zeta, \psi\right\rangle \\ &= \left\langle \zeta, \phi\frac{\delta^T\psi}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\psi\right\rangle\,.\end{aligned}$$

If we now apply (2.2.9) with $\psi = 1$ we obtain

$$\frac{\delta^T\phi}{\delta x_j} = \frac{\delta^T(\phi\cdot 1)}{\delta x_j} = \phi\frac{\delta^T 1}{\delta x_j} + \frac{\delta\phi}{\delta x_j}\,,$$

or

$$\frac{\delta^T\phi}{\delta x_j} = \frac{\delta\phi}{\delta x_j} + c_j\phi\,, \tag{2.2.10}$$

where $c_j = \delta^T 1/\delta x_j$. It is not hard to see that (c_j) has to be normal to the surface, $c_j = cn_j$ for some c. Observe now that since $(\delta\phi/\delta x_j)$ is tangent to Σ we obtain that $n_j\delta\phi/\delta x_j = 0$, and thus, by duality,

$$\frac{\delta^T(n_j\phi)}{\delta x_j} = 0\,. \tag{2.2.11}$$

Since $\delta^T n_i/\delta x_j = \mu_{ij} + cn_in_j$, we thus obtain $\omega + c = 0$, where $\omega = \mu_{jj}$. Hence

$$\frac{\delta^T\phi}{\delta x_j} = \frac{\delta\phi}{\delta x_j} - \omega n_j\phi\,. \tag{2.2.12}$$

When $\Sigma = \mathbb{S}$, then $\omega = n-1$ and thus

$$\frac{\delta^T\phi}{\delta x_j} = \frac{\delta\phi}{\delta x_j} - (n-1)\,n_j\phi\,, \qquad \frac{\delta^T n_i}{\delta x_j} = \delta_{ij} - nn_in_j\,. \tag{2.2.13}$$

The operators $\delta/\delta x_j$ and $\delta/\delta x_i$ do not commute, in general, but the expression

$$D_{ij}^2(\phi) = \frac{\delta}{\delta x_i}\left(\frac{\delta\phi}{\delta x_j}\right) - \mu_{jk} n_i \frac{\delta\phi}{\delta x_k}\,, \tag{2.2.14}$$

satisfies $D_{ij}^2 = D_{ji}^2$. When $\Sigma = \mathbb{S}$, then $\mu_{ij} = \delta_{ij} - n_i n_j$, so that

$$D_{ij}^2(\phi) = \frac{\delta}{\delta x_i}\left(\frac{\delta\phi}{\delta x_j}\right) - n_i \frac{\delta\phi}{\delta x_j}\,. \tag{2.2.15}$$

2.3 Space of Test Functions on $\mathbb{R}^n$ with a Thick Point

If $\mathbf{a}$ is a fixed point of $\mathbb{R}^n$, then the space of test functions with a thick point at $\mathbf{x} = \mathbf{a}$ is defined as follows.

Definition 5. *Let $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ denote the vector space of all smooth functions ϕ defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, with support of the form $K \setminus \{\mathbf{a}\}$, where K is compact in $\mathbb{R}^n$, that admit a strong asymptotic expansion of the form*

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j, \quad \textit{as } \mathbf{x} \to \mathbf{0}\,. \tag{2.3.1}$$

where m is an integer (positive or negative), and where the a_j are smooth functions of $\mathbf{w}$, that is, $a_j \in \mathcal{D}(\mathbb{S})$.

Observe that we require the asymptotic development of $\phi(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ to be "strong". This means that for any differentiation operator $(\partial/\partial\mathbf{x})^{\mathbf{p}} = (\partial^{p_1} ... \partial^{p_n}) / \partial x_1^{p_1} ... \partial x_n^{p_n}$, the asymptotic development of $(\partial/\partial\mathbf{x})^{\mathbf{p}} \phi(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ exists and is equal to the term-by-term differentiation of $\sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j$. Observe that saying that the expansion exists as $\mathbf{x} \to \mathbf{0}$ is the same as saying that it exits as $r \to 0$, uniformly with respect to $\mathbf{w}$.

We call $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ the space of test functions on $\mathbb{R}^n$ with a thick point located at $\mathbf{x} = \mathbf{a}$. It is sometimes convenient to take $\mathbf{a} = \mathbf{0}$; we denote $\mathcal{D}_{*,\mathbf{0}}(\mathbb{R}^n)$ as $\mathcal{D}_*(\mathbb{R}^n)$.

Observe that if ϕ is a standard test function, namely, smooth in all $\mathbb{R}^n$ and with compact support, then it has a Taylor expansion,

$$\phi(\mathbf{a}+r\mathbf{w}) \sim a_0 + \sum_{j=1}^{\infty} a_j(\mathbf{w})\, r^j. \tag{2.3.2}$$

where a_0 is just the real number $\phi(\mathbf{a})$. Hence $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$; we denote by

$$i : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n) \ , \tag{2.3.3}$$

the inclusion map. In fact, with the topology constructed in Definition 7, $\mathcal{D}(\mathbb{R}^n)$ is not only a subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ but actually a *closed* subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$.

In order to define the topology of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ we need to introduce several auxiliary spaces.

Definition 6. *Let m be a fixed integer. The subspace $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$, consists of those test functions ϕ whose expansion (2.3.1) begins at m. For a fixed compact K whose interior contains $\mathbf{a}$, $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ is the subspace formed by those test functions of $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ that vanish in $\mathbb{R}^n \setminus K$.*

We now give the topology of the space of thick test functions.

Definition 7. *Let m be a fixed integer and K a compact subset of $\mathbb{R}^n$ whose interior contains $\mathbf{a}$. The topology of $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ is given by the seminorms $\left\{\|\|_{q,s}\right\}_{q>m,s\geq 0}$ defined as*

$$\|\phi\|_{q,s} = \sup_{\mathbf{x}-\mathbf{a}\in K} \sup_{|\mathbf{p}|\leq s} \frac{\left|\dfrac{\partial^{\mathbf{p}}\phi}{\partial \mathbf{x}}(\mathbf{a}+\mathbf{x}) - \displaystyle\sum_{j=m-|\mathbf{p}|}^{q-1} a_{j,\mathbf{p}}(\mathbf{w})\, r^j\right|}{r^q}, \tag{2.3.4}$$

where $\mathbf{x} = r\mathbf{w}$ and

$$\frac{\partial^{\mathbf{p}}\phi}{\partial \mathbf{x}}(\mathbf{a}+\mathbf{x}) \sim \sum_{j=m-|\mathbf{p}|}^{\infty} a_{j,\mathbf{p}}(\mathbf{w})\, r^j. \tag{2.3.5}$$

The topology of $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ as $K \nearrow \infty$.

The topology of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{D}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ as $m \searrow -\infty$.

A sequence $\{\phi_l\}_{l=0}^{\infty}$ in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ converges to ψ if and only there exists $l_0 \geq 0$, an integer m, and a compact set K with $\mathbf{a}$ in its interior, such that $\phi_l \in \mathcal{D}_{*,\mathbf{a}}^{[m;K]}(\mathbb{R}^n)$ for $l \geq l_0$ and $||\psi - \phi_l||_{q,s} \to 0$ as $l \to \infty$ if $q > m, s \geq 0$. Notice that if $\{\phi_l\}_{l=0}^{\infty}$ converges to ψ in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ then ϕ_l and the corresponding derivatives, converge uniformly to ψ and its derivatives in any set of the form $\mathbb{R}^n \setminus B$, where B is a ball with center at $\mathbf{a}$; in fact, $r^{|\mathbf{p}|-m}(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi_l$ converges uniformly to $r^{|\mathbf{p}|-m}(\partial/\partial\mathbf{x})^{\mathbf{p}}\psi$ over all $\mathbb{R}^n$. Furthermore, if $\{a_j^l\}$ are the coefficients of the expansion of ϕ_l and $\{b_j\}$ are those for ψ, then $a_j^l \to b_j$ in the space $\mathcal{D}(\mathbb{S})$ for each $j \geq m$.

2.3.1 The expansion of $(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi$

Notice that the definition of the space $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ requires $(\partial/\partial\mathbf{x})^{\mathbf{p}}\phi(\mathbf{x})$ to have an asymptotic expansion equal to the term-by-term differentiation of $\sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j$, which yields an expansion of the form (2.3.5). Actually the coefficients $a_{j,\mathbf{p}}(\mathbf{w})$ can be computed for given $\mathbf{p}$, as we now show. Let us start with the expansion of $\partial\phi/\partial x_i$ when

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j, \quad \text{as } r \to 0\,. \tag{2.3.6}$$

Each function a_j belongs to $\mathcal{D}(\mathbb{S})$, so that it is defined on $\mathbb{S}$ only; the expression $\partial a_j/\partial x_i$ does not make sense, of course. If, on the other hand, $A_j(\mathbf{x}) = a_j(\mathbf{w})$, then $\partial A_j/\partial x_i$ is well defined and actually according to (2.2.5), it equals $r^{-1}\delta a_j/\delta x_i$.

Therefore, since $\partial r^j/\partial x_i = jw_i r^{j-1} = jn_i r^{j-1}$,

$$\begin{aligned}\frac{\partial}{\partial x_i}\left(a_j\left(\mathbf{w}\right)r^j\right) &= \frac{\partial}{\partial x_i}\left(A_j r^j\right)\\ &= \frac{\partial A_j}{\partial x_i}r^j + A_j\frac{\partial r^j}{\partial x_i}\\ &= \left(\frac{\delta a_j}{\delta x_i} + ja_j n_i\right)r^{j-1},\end{aligned}$$

and we obtain

$$\frac{\partial \phi}{\partial x_i}\left(\mathbf{a}+r\mathbf{w}\right) \sim \sum_{j=m-1}^{\infty}\left(\frac{\delta a_{j+1}}{\delta x_i} + (j+1)\, a_{j+1}n_i\right)r^j, \quad \text{as } r \to 0\,. \tag{2.3.7}$$

Iteration of formula (2.3.7) and the use of (2.2.15) yield, in turn, the expansion

$$\begin{aligned}&\frac{\partial^2 \phi}{\partial x_i \partial x_k}\left(\mathbf{a}+r\mathbf{w}\right)\\ &\sim \sum_{j=m-2}^{\infty}\left(D_{ik}^2 a_{j+2} + (j+2)\left(\frac{\delta a_{j+2}}{\delta x_i}n_k + \right.\right.\\ &\qquad\qquad \left.\left.\frac{\delta a_{j+2}}{\delta x_k}n_i\right) + (j+2)\left(\delta_{ik} + jn_i n_k\right)a_{j+2}\right)r^j,\end{aligned} \tag{2.3.8}$$

as $r \to 0$, where the operator D_{ik}^2 is defined in (2.2.15).

Notice that the space $\mathcal{D}_{*,\mathbf{a}}^{[m]}\left(\mathbb{R}^n\right)$ is *not* closed with respect to differentiation, since in fact, if $\phi \in \mathcal{D}_{*,\mathbf{a}}^{[m]}\left(\mathbb{R}^n\right)$ then $\partial\phi/\partial x_i \in \mathcal{D}_{*,\mathbf{a}}^{[m-1]}\left(\mathbb{R}^n\right)$.

2.4 Space of Distributions on $\mathbb{R}^n$ with a Thick Point

We can now consider distributions in a space with one thick point.

Definition 8. *The space of distributions on $\mathbb{R}^n$ with a thick point at $\mathbf{x} = \mathbf{a}$ is the dual space of $\mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$. We denote it $\mathcal{D}_{*,\mathbf{a}}'\left(\mathbb{R}^n\right)$, or just as $\mathcal{D}_{*}'\left(\mathbb{R}^n\right)$ when $\mathbf{a} = \mathbf{0}$.*

We shall call the elements of $\mathcal{D}_{*,\mathbf{a}}'\left(\mathbb{R}^n\right)$ "thick distributions."

Let

$$\pi : \mathcal{D}_{*,\mathbf{a}}'\left(\mathbb{R}^n\right) \to \mathcal{D}'\left(\mathbb{R}^n\right), \tag{2.4.1}$$

be the projection operator, dual of the inclusion (2.3.3). Since $\mathcal{D}(\mathbb{R}^n)$ is closed in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$, the Hahn-Banach theorem immediatly yields the following extension result.

Theorem 4. Let f be any distribution in $\mathcal{D}'(\mathbb{R}^n)$, then there exist thick distributions $g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ such that $\pi(g) = f$.

Naturally, if $f \in \mathcal{D}'(\mathbb{R}^n)$ then there are infinitely many thick distributions g with $\pi(g) = f$. In some cases there is a canonical way to construct such a g, but no general extension procedure exists, as follows from the ideas of [34]. We could think of this situation as follows: If $g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$, then knowing $\pi(g)$ gives us a lot of information about g, but not enough to know g completely.

It is well known that any locally integrable function f defined in $\mathbb{R}^n$ yields a distribution, usually denoted by the same notation f, by the prescription

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,, \qquad \phi \in \mathcal{D}(\mathbb{R}^n)\,. \tag{2.4.2}$$

If $\mathbf{a} \notin \operatorname{supp} f$, that is, if $f(\mathbf{x}) = 0$ for $|\mathbf{x} - \mathbf{a}| < \varepsilon$ for some $\varepsilon > 0$, then (2.4.2) will also work in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$; however, if $\mathbf{a} \in \operatorname{supp} f$ then, in general, the integral $\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}$ would be divergent and thus a thick distribution that one could call "f" cannot be defined in a canonical way. Nevertheless, it is possible in many cases to define a "finite part" distribution $\mathcal{P}f\,(f(\mathbf{x}))$ which is the canonical thick distribution corresponding to f.

Let us recall at this point that the finite part of the limit of $F(\varepsilon)$ as $\varepsilon \to 0^+$ exists and equals A if we can write $F(\varepsilon) = F_{\text{fin}}(\varepsilon) + F_{\text{infin}}(\varepsilon)$, where the *infinite* part $F_{\text{infin}}(\varepsilon)$ is a linear combination of functions of the type $\varepsilon^{-p}\ln^q \varepsilon$, where $p > 0$ or $p = 0$ and $q > 0$, and where the *finite* part $F_{\text{fin}}(\varepsilon)$ is a function whose limit as $\varepsilon \to 0^+$ is A [37, Section 2.3].

Definition 9. *Let f be a locally integrable function defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$. The thick distribution $\mathcal{P}f\,(f(\mathbf{x}))$ is defined as*

$$\langle \mathcal{P}f\,(f(\mathbf{x})),\phi(\mathbf{x})\rangle = \text{F.p.}\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x}\,,\qquad \phi\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)\,, \tag{2.4.3}$$

provided that the finite part integrals exist for all $\phi\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$.

Since the integral $\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x}$ could be divergent at $\mathbf{x}=\mathbf{a}$, the finite part of the integral is defined as

$$\text{F.p.}\int_{\mathbb{R}^n} f(\mathbf{x})\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x} = \text{F.p.}\lim_{\varepsilon\to 0^+}\int_{|\mathbf{x}-\mathbf{a}|\geq\varepsilon} f(\mathbf{x})\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x}\,. \tag{2.4.4}$$

We must emphasize that for a given function f the finite part of the above limit may or may not exist, and thus, in general, $\mathcal{P}f\,(f(\mathbf{x}))$ is not well defined for *all* such functions f. Fortunately, as we shall show, $\mathcal{P}f\,(f(\mathbf{x}))$ is defined in many important and interesting cases.

It should be clear that, more generally, $\mathcal{P}f\,(f(\mathbf{x}))$ can be constructed if f is a distribution in $\mathbb{R}^n\setminus\{\mathbf{a}\}$ such that the product $H(|\mathbf{x}-\mathbf{a}|-\varepsilon)\,f(\mathbf{x})$ is defined for all $\varepsilon>0$, where H is the Heaviside function, as

$$\langle \mathcal{P}f\,(f(\mathbf{x})),\phi(\mathbf{x})\rangle = \text{F.p.}\lim_{\varepsilon\to 0^+}\langle H(|\mathbf{x}-\mathbf{a}|-\varepsilon)\,f(\mathbf{x}),\phi(\mathbf{x})\rangle\,, \tag{2.4.5}$$

provided that the finite part of the limits exist for all $\phi\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$.

Example 3. *If $\lambda\in\mathbb{C}$ then we shall see that $\mathcal{P}f\left(r^\lambda\right)$ is a well defined thick distribution of $\mathcal{D}'_*(\mathbb{R}^n)$. Actually, more generally, $\mathcal{P}f\left(|\mathbf{x}-\mathbf{a}|^\lambda\right)$ is a thick distribution of $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$. Indeed, one needs to consider the finite part of the integral $\int_{\mathbb{R}^n}|\mathbf{x}-\mathbf{a}|^\lambda\,\phi(\mathbf{x})\;\mathrm{d}\mathbf{x}$ for any $\phi\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$. Explicitly, since*

$$\text{F.p.}\int_0^A r^\alpha\mathrm{d}r = \frac{A^{\alpha+1}}{\alpha+1}\,,\qquad \alpha\neq -1\,,\quad \text{F.p.}\int_0^A r^{-1}\mathrm{d}r = \log A\,,$$

we obtain

$$\left\langle \mathcal{P}f\left(\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\right),\phi\left(\mathbf{x}\right)\right\rangle = \text{F.p.}\int_{\mathbb{R}^n}\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}$$

$$=\int_{\left|\mathbf{x}-\mathbf{a}\right|\geq A}\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}$$

$$+\int_{\left|\mathbf{x}-\mathbf{a}\right|<A}\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\left(\phi\left(\mathbf{x}\right)-\sum_{j\leq-\Re e\,\lambda-n-1}a_j\left(\mathbf{w}\right)\left|\mathbf{x}-\mathbf{a}\right|^{j}\right)\mathrm{d}\mathbf{x}$$

$$+\,\text{F.p.}\int_{\left|\mathbf{x}-\mathbf{a}\right|<A}\sum_{j\leq-\Re e\,\lambda-n-1}a_j\left(\mathbf{w}\right)\left|\mathbf{x}-\mathbf{a}\right|^{\lambda+j}\,\mathrm{d}\mathbf{x}\,,$$

so that if $\lambda\notin\mathbb{Z}$ *then*

$$\left\langle \mathcal{P}f\left(\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\right),\phi\left(\mathbf{x}\right)\right\rangle = \int_{\left|\mathbf{x}-\mathbf{a}\right|\geq A}\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}$$

$$+\int_{\left|\mathbf{x}-\mathbf{a}\right|<A}\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}\left(\phi\left(\mathbf{x}\right)-\sum_{j\leq-\Re e\,\lambda-n-1}a_j\left(\mathbf{w}\right)\left|\mathbf{x}-\mathbf{a}\right|^{j}\right)\mathrm{d}\mathbf{x}$$

$$+\sum_{j\leq-\Re e\,\lambda-n-1}\left(\int_{\mathbb{S}}a_j\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right)\right)\frac{A^{\lambda+j+n}}{\lambda+j+n}\,, \tag{2.4.6}$$

while if $\lambda=k\in\mathbb{Z}$ *then*

$$\left\langle \mathcal{P}f\left(\left|\mathbf{x}-\mathbf{a}\right|^{k}\right),\phi\left(\mathbf{x}\right)\right\rangle = \int_{\left|\mathbf{x}-\mathbf{a}\right|\geq A}\left|\mathbf{x}-\mathbf{a}\right|^{k}\phi\left(\mathbf{x}\right)\,\mathrm{d}\mathbf{x}$$

$$+\int_{\left|\mathbf{x}-\mathbf{a}\right|<A}\left|\mathbf{x}-\mathbf{a}\right|^{k}\left(\phi\left(\mathbf{x}\right)-\sum_{j\leq-k-n-1}a_j\left(\mathbf{w}\right)\left|\mathbf{x}-\mathbf{a}\right|^{j}\right)\mathrm{d}\mathbf{x}$$

$$+\sum_{j<-k-n-1}\left(\int_{\mathbb{S}}a_j\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right)\right)\frac{A^{k+j+n}}{k+j+n}$$

$$+\left(\int_{\mathbb{S}}a_{-k-n}\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right)\right)\log A\,. \tag{2.4.7}$$

Formulas (2.4.6) and (2.4.7) hold for any $A>0$*. The finite part is needed for all* λ *in the space of thick distributions* $\mathcal{D}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right);$ *on the other hand,* $\left|\mathbf{x}-\mathbf{a}\right|^{\lambda}$ *is a regular distribution of* $\mathcal{D}'\left(\mathbb{R}^n\right)$ *for* $\Re e\,\lambda>-n$*, and the finite part integral is only needed for* $\Re e\,\lambda\leq-n$*.*

Using the ideas of the previous example one can show that if $\psi\in\mathcal{E}_{*,\mathbf{a}}\left(\mathbb{R}^n\right),$ that is, if ψ is somooth in $\mathbb{R}^n\setminus\{\mathbf{a}\}$, and near $\mathbf{x}=\mathbf{a}$ the function ψ has a strong

expansion of the form (2.3.1), then the finite part $\mathcal{P}f\left(\psi\left(\mathbf{x}\right)\right)$ exists as an element of $\mathcal{D}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$. In particular, when ψ is smooth in all of $\mathbb{R}^n$, $\psi \in \mathcal{E}\left(\mathbb{R}^n\right)$, then $\mathcal{P}f\left(\psi\left(\mathbf{x}\right)\right) \in \mathcal{D}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$; notice that the *finite part is always needed if there is a thick point,* even if $\pi\left(\mathcal{P}f\left(\psi\left(\mathbf{x}\right)\right)\right) = \psi\left(\mathbf{x}\right)$ in the space $\mathcal{D}'\left(\mathbb{R}^n\right)$ of standard distributions, so that no finite part is needed there.

Suppose $g\left(\mathbf{w}\right)$ is a distribution in $\mathbb{S}$. Let us now define the "thick delta function" $g\delta_* \in \mathcal{D}'_*\left(\mathbb{R}^n\right)$. Let ϕ be a test function in $\mathcal{D}_*\left(\mathbb{R}^n\right)$, so that by definition ϕ could be asymptotically expanded as $a_{-m}\left(\mathbf{w}\right)r^{-m} + \cdots + a_0\left(\mathbf{w}\right) + a_1\left(\mathbf{w}\right)r + \cdots$ as $r = |\mathbf{x}| \to 0^+$. Then $g\delta_*$ is given by

$$\left\langle g\delta_*, \phi\right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} := \frac{1}{C_{n-1}}\left\langle g\left(\mathbf{w}\right), a_0\left(\mathbf{w}\right)\right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} ,$$

where C_{n-1} is the surface area of $\mathbb{S}$, the unit sphere in $\mathbb{R}^n$. If g is locally integrable in $\mathbb{S}$, then

$$\left\langle g\delta_*, \phi\right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C_{n-1}}\int_{\mathbb{S}} g\left(\mathbf{w}\right)a_0\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right) . \tag{2.4.8}$$

We sometimes use the notations $g\left(\mathbf{w}\right)\delta_*$ or $g\left(\mathbf{w}\right)\delta_*\left(\mathbf{x}\right)$ to denote the thick delta $g\delta_*$.

In particular, if $g\left(\mathbf{x}\right) \equiv 1$, then we obtain the "plain thick delta function" $\delta_* = g\delta_*$, given as

$$\left\langle \delta_*, \phi\right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C_{n-1}}\int_{\mathbb{S}} a_0\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right) . \tag{2.4.9}$$

Now let us consider the projection $\pi\left(g\delta_*\right)$ of the thick delta $g\delta_*$ onto $\mathcal{D}'\left(\mathbb{R}^n\right)$. As pointed out before, a test function $\psi \in \mathcal{D}\left(\mathbb{R}^n\right)$ can be asymptotically expanded as $\psi\left(r\mathbf{w}\right) \sim a_0 + \sum_{j=1}^{\infty} a_i\left(\mathbf{w}\right)r^i$, where the negative index terms of the expansion

are 0 and the 0−th term a_0 is just a constant, namely, $\psi(\mathbf{0})$. Therefore

$$\begin{aligned}\langle \pi(g\delta_*), \psi\rangle &= \langle g\delta_*, i(\psi)\rangle \\ &= \frac{1}{C_{n-1}}\langle g(\mathbf{w}), a_0\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \\ &= \frac{\psi(\mathbf{0})}{C_{n-1}}\langle g(\mathbf{w}), 1\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \ ,\end{aligned} \tag{2.4.10}$$

and this yields the following result.

Proposition 3. *If $g \in \mathcal{D}'(\mathbb{S})$ then*

$$\pi(g(\mathbf{w})\,\delta_*(\mathbf{x})) = I_g\delta(\mathbf{x}) \ , \tag{2.4.11}$$

where the constant I_g is given by

$$\begin{aligned}I_g &= \frac{1}{C_{n-1}}\langle g(\mathbf{w}), 1\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \\ &= \frac{1}{C_{n-1}}\int_{\mathbb{S}} g(\mathbf{w})\ \mathrm{d}\sigma(\mathbf{w}) \ ,\end{aligned} \tag{2.4.12}$$

the second expression being valid in case g is locally integrable.

In particular, since $I_1 = 1$, the projection of the plain thick delta function δ_* is no other than the usual delta function in $\mathcal{D}'(\mathbb{R}^n)$,

$$\pi(\delta_*) = \delta \, . \tag{2.4.13}$$

In fact, the notion of thick delta functions can be generalized to a much broader range of the distributions in $\mathcal{D}'_*(\mathbb{R}^n)$, the thick delta functions of degree q, so that $g\delta_*$ is the special case when $q = 0$. We have the following definition.

Definition 10. *(Thick delta functions of degree q) Let $g(\mathbf{w})$ is a distribution in $\mathbb{S}$. The thick delta function of degree q, denoted as $g\delta_*^{[q]}$, or as $g(\mathbf{w})\,\delta_*^{[q]}$, acts on a thick test function $\phi(\mathbf{x})$ as*

$$\left\langle g\delta_*^{[q]}, \phi\right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C_{n-1}}\langle g(\mathbf{w}), a_q(\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \ , \tag{2.4.14}$$

where $\phi(r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j,$ *as* $r \to 0^+$.

If g is locally integrable function in $\mathbb{S}$, then

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C_{n-1}} \int_{\mathbb{S}} g(\mathbf{w})\, a_q(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{2.4.15}$$

Observe that $g\delta_*$ is a thick delta function of degree 0, namely, $g\delta_*^{[0]}$. However, for the sake of simplicity, we shall use the notation $g\delta_*$ instead of $g\delta_*^{[0]}$ whenever is possible.

Notice, also that

$$\pi\left(g\delta_*^{[q]}\right) = 0\,, \qquad \text{whenever } q < 0\,. \tag{2.4.16}$$

The projection of the thick deltas for $q > 0$ is more interesting; observe, in particular, that $\pi\left(\delta_*^{[1]}\right) = 0$, but $\pi\left(C_{n-1}\delta(\mathbf{w}-\mathbf{e}_k)\,\delta_*^{[1]}\right) = -\partial\delta(\mathbf{x})/\partial x_k$ if $\mathbf{e}_k$ is the k-th unit vector. Furthermore,

$$\pi\left(\delta_*^{[2]}\right) = \frac{1}{2n}\nabla^2\delta(\mathbf{x})\ , \tag{2.4.17}$$

where $\nabla^2 = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplacian. More generally, we have the following result.

Proposition 4. *If* $g \in \mathcal{D}'(\mathbb{S})$ *and* $q \geq 0$ *then*

$$\pi\left(g\delta_*^{[q]}\right) = \frac{(-1)^q}{C_{n-1}} \sum_{j_1+\cdots+j_n=q} \frac{\left\langle g(\mathbf{w}), \mathbf{w}^{(j_1,\ldots,j_n)}\right\rangle}{j_1!\cdots j_n!} \mathbf{D}^{(j_1,\ldots,j_n)}\delta(\mathbf{x})\ . \tag{2.4.18}$$

Proof. Indeed, if $\psi \in \mathcal{D}(\mathbb{R}^n)$ then

$$\psi(\mathbf{x}) \sim \sum_{q=0}^{\infty} a_q(\mathbf{w})\, r^q, \quad \text{as } r = |\mathbf{x}| \to 0\,, \tag{2.4.19}$$

where

$$a_q(\mathbf{w}) = \sum_{\mathbf{j}\in\mathbb{N}^n,\, |\mathbf{j}|=q} \frac{\mathbf{D}^{\mathbf{j}}\psi(\mathbf{0})\ \mathbf{w}^{\mathbf{j}}}{\mathbf{j}!}\,, \tag{2.4.20}$$

and the usual notations for $\mathbf{j}=(\mathbf{j_1},\ldots,\mathbf{j_n})\in\mathbb{N}^n$,

$$\mathbf{j}!=j_1!\cdots j_n!\,,\quad |\mathbf{j}|=j_1+\cdots+j_n\,,\quad \mathbf{D}^{\mathbf{j}}=\frac{\partial^q}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}}\,, \tag{2.4.21}$$

are employed. Formula (2.4.18) follows at once. $\square$

There is an important relation between the finite part distributions $\mathcal{P}f\left(r^\lambda\right)$ and the thick delta functions $\delta_*^{[q]}$, as follows from the formulas (2.4.6) and (2.4.7).

Proposition 5. *The thick distributions $\mathcal{P}f\left(r^\lambda\right)$ are analytic functions of λ in the region $\mathbb{C}\setminus\mathbb{Z}$. There are simple poles at all of the integers $k\in\mathbb{Z}$ with residues*

$$\operatorname*{Res}_{\lambda=k}\mathcal{P}f\left(r^\lambda\right)=C_{n-1}\delta_*^{[-k-n]}. \tag{2.4.22}$$

The distribution $\mathcal{P}f\left(r^k\right)$ is the finite part of the analytic function[1] *at the pole, namely,*

$$\mathcal{P}f\left(r^k\right)=\lim_{\lambda\to k}\left(\mathcal{P}f\left(r^\lambda\right)-\frac{C_{n-1}\delta_*^{[-k-n]}}{\lambda-k}\right). \tag{2.4.23}$$

Formula (2.4.16) allows us to recover the well known result that $r^\lambda=\pi\left(\mathcal{P}f\left(r^\lambda\right)\right)$, the usual distribution of $\mathcal{D}'\left(\mathbb{R}^n\right)$ is analytic for $\lambda\neq -n,\,-n-1,\,-n-2,\ldots$ since the residues at the poles $-n+1,\,-n+2,\,-n+3,\ldots$ vanish.

Example 4. *In [11] Blanchet and Faye called the function of three variables and one parameter, ${}_\varepsilon\delta\left(\mathbf{x}\right)=\left(\varepsilon\left(1-\varepsilon\right)/4\pi\right)\mathcal{P}f\left(\left|\mathbf{x}\right|^{\varepsilon-3}\right)$ the* Riesz delta function; *if $\varepsilon>0$ then ${}_\varepsilon\delta$ is locally integrable and the work of M. Riesz on multidimensional fractional integration [72] gives that in $\mathcal{D}'\left(\mathbb{R}^3\right)$, $\lim_{\varepsilon\to0^+}\left({}_\varepsilon\delta\left(\mathbf{x}\right)\right)=\delta\left(\mathbf{x}\right)$. It is proved in [11, Lemma 2] that in $\mathcal{D}'_*\left(\mathbb{R}^3\right)$ one actually has*

$$\lim_{\varepsilon\to0^+}\left({}_\varepsilon\delta\left(\mathbf{x}\right)\right)=\delta_*\,. \tag{2.4.24}$$

[1] If $g\left(\lambda\right)$ is analytic for $0<\left|\lambda-\lambda_0\right|<\rho$ and there is a simple pole with residue $a=\operatorname{Res}_{\lambda=\lambda_0}g\left(\lambda\right)$ at $\lambda=\lambda_0$, then the finite part of the analytic function g at λ_0 is given by the limit $\lim_{\lambda\to\lambda_0}\left(g\left(\lambda\right)-a\left(\lambda-\lambda_0\right)^{-1}\right)$.

More generally, in $\mathcal{D}'_*(\mathbb{R}^n)$, use of (2.4.23) yields

$$\lim_{\varepsilon\to 0^+}\left({}_{\varepsilon}\widetilde{\delta}(\mathbf{x})\right)=\delta_*\,, \tag{2.4.25}$$

where

$${}_{\varepsilon}\widetilde{\delta}(\mathbf{x})=\frac{\varepsilon}{C_{n-1}}\mathcal{P}f\left(|\mathbf{x}|^{\varepsilon-n}\right)\,, \tag{2.4.26}$$

since $\varepsilon\mathcal{P}f\left(|\mathbf{x}|^{\varepsilon-n}\right)=\varepsilon(\varepsilon^{-1}C_{n-1}\delta_*+\mathcal{P}f\left(r^{-n}\right)+o\left(\varepsilon\right))=C_{n-1}\delta_*+o\left(1\right)$ *as* $\varepsilon\to 0.$

2.5 Other Spaces with Thick Points

The ideas of the previous sections can be applied to construct other spaces of test functions and of distributions with a thick point.

Let us illustrate this procedure with the spaces $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\mathcal{E}'_{*,\mathbf{a}}(\mathbb{R}^n)$. A test function of $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ is a smooth function ϕ, defined in $\mathbb{R}^n\setminus\{\mathbf{a}\}$, that has the strong asymptotic development (2.3.1) as $|\mathbf{x}-\mathbf{a}|\to 0$. Alternatively, $\phi\in\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ if and only if $\phi=\phi_1+\phi_2$, where $\phi_1\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\phi_2\in\mathcal{E}(\mathbb{R}^n)$. The topology of $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ is constructed in the obvious way and the dual space is $\mathcal{E}'_{*,\mathbf{a}}(\mathbb{R}^n)$.

We can then define $\mathcal{S}'_{*,\mathbf{a}}(\mathbb{R}^n)$, the space of tempered distributions with a thick point at $\mathbf{x}=\mathbf{a}$, and its corresponding space of test functions $\mathcal{S}_{*,\mathbf{a}}(\mathbb{R}^n)$. We can also define the spaces $\mathcal{K}_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\mathcal{K}'_{*,\mathbf{a}}(\mathbb{R}^n)$.

Moreover, if $\mathbf{a}\in U$, where U is an open set of $\mathbb{R}^n$ then one can consider, for instance, the spaces $\mathcal{D}_{*,\mathbf{a}}(U)$ and $\mathcal{D}'_{*,\mathbf{a}}(U)$.

2.6 Algebraic and Analytic Operations in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$

Naturally, we define the algebraic and analytic operations in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ in the same way they are defined for the usual distributions, namely, by duality.

2.6.1 Basic Definitions

Let $f,g\in\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\phi(\mathbf{x})\in\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ be a thick test function. Then the sum $f+g$ is given as

$$\langle f+g,\phi\rangle=\langle f,\phi\rangle+\langle g,\phi\rangle\,, \tag{2.6.1}$$

while if $\lambda \in \mathbb{C}$ then $\lambda f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ is given as

$$\langle \lambda f, \phi \rangle = \lambda \langle f, \phi \rangle \ . \tag{2.6.2}$$

Translations are handled by the formula

$$\langle f(\mathbf{x}+\mathbf{c}), \phi(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \phi(\mathbf{x}-\mathbf{c}) \rangle \ , \tag{2.6.3}$$

where $\mathbf{c} \in \mathbb{R}^n$. Here $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ while the translation $f(\mathbf{x}+\mathbf{c})$ belongs to $\mathcal{D}'_{*,\mathbf{a}-\mathbf{c}}(\mathbb{R}^n)$; naturally $\phi \in \mathcal{D}_{*,\mathbf{a}-\mathbf{c}}(\mathbb{R}^n)$.

Observe that any distribution g of the space $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ can be written as $g(\mathbf{x}) = f(\mathbf{x}-\mathbf{a})$ for some $f \in \mathcal{D}'_{*}(\mathbb{R}^n)$, and this justifies studying most results in $\mathcal{D}'_{*}(\mathbb{R}^n)$ only.

Linear changes of variables are as follows. Let A be a non-singular $n\times n$ matrix. If $f \in \mathcal{D}'_{*}(\mathbb{R}^n)$ then $f(A\mathbf{x})$ is defined as

$$\langle f(A\mathbf{x}), \phi(\mathbf{x}) \rangle = \frac{1}{|\det A|} \left\langle f(\mathbf{x}), \phi\left(A^{-1}\mathbf{x}\right) \right\rangle \ , \tag{2.6.4}$$

as in the space $\mathcal{D}'(\mathbb{R}^n)$ of usual distributions. In particular, $f(-\mathbf{x})$ is defined as

$$\langle f(-\mathbf{x}), \phi(\mathbf{x}) \rangle = \langle f(\mathbf{x}), \phi(-\mathbf{x}) \rangle \ . \tag{2.6.5}$$

Example 5. *Let us consider the action of a linear change in the thick delta functions of degree q. If $\phi \in \mathcal{D}_{*}(\mathbb{R}^n)$ with expansion*

$$\phi(\mathbf{x}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w}) r^j, \quad \text{as } r \to 0^+, \tag{2.6.6}$$

where $\mathbf{x} = r\mathbf{w}$, $r = |\mathbf{x}|$, $|\mathbf{w}| = 1$,then

$$\phi\left(A^{-1}\mathbf{x}\right) \sim \sum_{j=m}^{\infty} \widetilde{a}_j(\mathbf{w}) r^j, \tag{2.6.7}$$

where

$$\widetilde{a}_j(\mathbf{w}) = a_j\left(\frac{A^{-1}\mathbf{w}}{|A^{-1}\mathbf{w}|}\right)\left|A^{-1}\mathbf{w}\right|^j . \tag{2.6.8}$$

Therefore, if $g \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ *is homogeneous of degree* 0*, then*

$$g\delta_*^{[q]}(A\mathbf{x}) = g_{A,q}\delta_*^{[q]}(\mathbf{x}) , \tag{2.6.9}$$

where if $a \in \mathcal{D}(\mathbb{S})$

$$\langle g_{A,q}(\mathbf{w}), a(\mathbf{w})\rangle = \frac{1}{|\det A|}\left\langle g(\mathbf{w}), a\left(\frac{A^{-1}\mathbf{w}}{|A^{-1}\mathbf{w}|}\right)\left|A^{-1}\mathbf{w}\right|^q\right\rangle . \tag{2.6.10}$$

If $|A^{-1}\mathbf{w}| = 1$ *for all* $\mathbf{w} \in \mathbb{S}$*, in particular if* A *is a rotation, then*

$$(g\delta_*^{[q]})(A\mathbf{x}) = g(A\mathbf{w})\,\delta_*^{[q]}(\mathbf{x}) , \tag{2.6.11}$$

in particular

$$(g\delta_*^{[q]})(-\mathbf{x}) = g(-\mathbf{w})\,\delta_*^{[q]}(\mathbf{x}) . \tag{2.6.12}$$

On the other hand, if $A = tI$*,* $t > 0$*, a diagonal matrix, so that* $A\mathbf{x} = t\mathbf{x}$*, then we obtain*

$$(g\delta_*^{[q]})(t\mathbf{x}) = t^{-n-q}g(\mathbf{w})\,\delta_*^{[q]}(\mathbf{x}) . \tag{2.6.13}$$

Example 6. *Let us now consider the linear change* $A\mathbf{x} = t\mathbf{x}$*,* $t > 0$*, on the thick distribution* $\mathcal{P}f\left(r^\lambda\right)$*. If we employ formula (2.4.6) we immediately obtain that*

$$\mathcal{P}f\left(|t\mathbf{x}|^\lambda\right) = t^\lambda\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) , \qquad \lambda \notin \mathbb{Z} . \tag{2.6.14}$$

The corresponding transformation formula for $\mathcal{P}f\left(r^k\right)$*,* $k \in \mathbb{Z}$*, can likewise be derived, with a little more effort, from (2.4.7); or we may use (2.4.23) as follows,*

$$\begin{aligned}
\mathcal{P}f\left(|t\mathbf{x}|^k\right) &= \lim_{\lambda\to k}\left(\mathcal{P}f\left(|t\mathbf{x}|^\lambda\right) - \frac{C_{n-1}\delta_*^{[-k-n]}(t\mathbf{x})}{\lambda-k}\right) \\
&= \lim_{\lambda\to k}\left(t^\lambda\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) - t^k\frac{C_{n-1}\delta_*^{[-k-n]}(t\mathbf{x})}{\lambda-k}\right) \\
&= t^k\lim_{\lambda\to k}\left(\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) - \frac{C_{n-1}\delta_*^{[-k-n]}(t\mathbf{x})}{\lambda-k}\right) + \lim_{\lambda\to k}\left(t^\lambda - t^k\right)\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) \\
&= t^k\mathcal{P}f\left(|\mathbf{x}|^k\right) + \lim_{\lambda\to k}\left(t^\lambda - t^k\right)\mathcal{P}f\left(|\mathbf{x}|^\lambda\right),
\end{aligned}$$

but $\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) = C_{n-1}\delta_*^{[-k-n]}/(\lambda-k) + o(1)$ *as* $\lambda\to k$*, so that*

$$\lim_{\lambda\to k}\left(t^\lambda - t^k\right)\mathcal{P}f\left(|\mathbf{x}|^\lambda\right) = \lim_{\lambda\to k}\frac{\left(t^\lambda - t^k\right)}{\lambda-k}C_{n-1}\delta_*^{[-k-n]} = t^k\log t\, C_{n-1}\delta_*^{[-k-n]},$$

and we obtain

$$\mathcal{P}f\left(|t\mathbf{x}|^k\right) = t^k\mathcal{P}f\left(|\mathbf{x}|^k\right) + t^k\log t\, C_{n-1}\delta_*^{[-k-n]}. \tag{2.6.15}$$

If we consider the projection of these results onto $\mathcal{D}'(\mathbb{R}^n)$*, recalling that* $\pi\left(\mathcal{P}f\left(r^\lambda\right)\right) = r_+^\lambda$ *for* $\lambda \neq -n, -n-2, -n-4, \ldots$*, we obtain*

$$(tr)_+^\lambda = t^\lambda (r)_+^\lambda, \tag{2.6.16}$$

from (2.6.14) for $\lambda \in \mathbb{C}\setminus\mathbb{Z}$ *and from (2.6.15) for* $\lambda\in\mathbb{Z}$*,* $\lambda \neq -n, -n-2, -n-4, \ldots$*, because* $\pi\left(\delta_*^{[q]}\right) = 0$ *if* $q< 0$ *or* $q > 0$ *is odd. On the other hand, for* $k = -n, -n-2, -n-4, \ldots$*, we have* $\pi\left(\mathcal{P}f\left(r^k\right)\right) = \mathcal{P}f\left(r^k\right)$ *and thus (2.6.15) yields, when the particular case (1.2.5) of (**??**) is taken into account,*

$$\mathcal{P}f\left(|t\mathbf{x}|^k\right) = t^k\mathcal{P}f\left(|\mathbf{x}|^k\right) + \frac{t^k\log t\, c_{m,n}\nabla^{2m}\delta(\mathbf{x})}{(2m)!}, \tag{2.6.17}$$

where $2m = -k-n$ *and where [30, 32],*

$$c_{m,n} = \frac{2\Gamma(m+1/2)\,\pi^{(n-1)/2}}{\Gamma(m+n/2)} = \int_{\mathbb{S}} w_i^{2m}\,\mathrm{d}\sigma(\mathbf{w}). \tag{2.6.18}$$

2.6.2 Multiplication

The space of multipliers for a space of test functions and for its dual space are the same, their Moyal algebra.

Definition 11. *Let $\rho \in \mathcal{B}$, the space of multipliers of a space of test functions $\mathcal{A}$, that is, $\rho\phi \in \mathcal{A}$, $\forall \phi \in \mathcal{A}$. Then if $f \in \mathcal{A}'$ the multiplication $\rho f \in \mathcal{A}'$ is given by*

$$\langle \rho f, \phi \rangle = \langle f, \rho\phi \rangle \ . \tag{2.6.19}$$

The space $\mathcal{B}$ is the Moyal algebra of $\mathcal{A}$ and of $\mathcal{A}'$.

Thick distributions can be multiplied by certain multipliers, functions that are smooth away from the thick point, and that behave like test functions near the thick point. Indeed, it is not hard to see that the Moyal algebra of $\mathcal{D}_{*,\mathbf{a}}$, the set of functions ψ, defined in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, such that $\psi\phi \in \mathcal{D}_{*,\mathbf{a}}$, for any $\phi \in \mathcal{D}_{*,\mathbf{a}}$, is precisely $\mathcal{E}_{*,\mathbf{a}}$, the set of all smooth functions in $\mathbb{R}^n \setminus \{\mathbf{a}\}$, that behave like thick test functions at $\mathbf{x} = \mathbf{a}$. On the other hand, the Moyal algebra of the spaces $\mathcal{S}$ and $\mathcal{S}'$ is the space $\mathcal{O}_M$ [54] so that the space of multipliers of $\mathcal{S}_{*,\mathbf{a}}$ and $\mathcal{S}'_{*,\mathbf{a}}$ is the space $(\mathcal{O}_M)_{*,\mathbf{a}}$.

Example 7. *The function r^k is a multiplier of $\mathcal{D}'_*(\mathbb{R}^n)$ for any $k \in \mathbb{Z}$. In particular, the multiplication $r^k \delta_*^{[q]}$ is defined for any $q \in \mathbb{Z}$, and a simple computation yields the useful formula*

$$r^k \delta_*^{[q]} = \delta_*^{[q-k]}. \tag{2.6.20}$$

Observe that also for any $\lambda \in \mathbb{C}$,

$$r^k \mathcal{P}f\left(r^\lambda\right) = \mathcal{P}f\left(r^{\lambda+k}\right) \ . \tag{2.6.21}$$

Example 8. *If* ψ_0 *is a* smooth *homogeneous function of degree* 0, *defined in* $\mathbb{R}^n \setminus \{\mathbf{0}\}$, *then it is a multiplier of* $\mathcal{D}_*'(\mathbb{R}^n)$ *and thus the product* $\psi_0 \delta_*^{[q]}$ *is well defined. Suppose* $\psi_0(r\mathbf{w}) = \psi(\mathbf{w})$, *where* $\psi \in \mathcal{D}(\mathbb{S})$; *then* $\psi \delta_*^{[q]}$, *is a thick delta of order* q, *as defined in Definition 10. Then*

$$\psi_0 \delta_*^{[q]} = \psi \delta_*^{[q]} . \tag{2.6.22}$$

That the two definitions coincide is good, of course, since ψ_0 *and* ψ *are basically the same function. It should be clear that the smoothness of* ψ_0 *plays no role in this analysis and that one can define the product,* in an extended sense, *of any distribution* g_0, *homogeneous of degree* 0 *and defined in* $\mathbb{R}^n \setminus \{\mathbf{0}\}$, *and* $\delta_*^{[q]}$; *this is actually related to the extended products of distributions shown to exist in [53] when the wave front sets of the factors are disjoint.*

Example 9. *We can also consider a smooth homogeneous function* ψ_k *of degree* k, *so that* $\psi_k(r\mathbf{w}) = r^k \psi(\mathbf{w})$. *Here we obtain* $\psi_k \delta_*^{[q]} = \psi \delta_*^{[q-k]}$.

Given a smooth function $\psi \in \mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ then we can associate two related but different objects to it. On the one hand, we can consider the finite part distribution $\mathcal{P}f(\psi) \in \mathcal{D}_{*,\mathbf{a}}'(\mathbb{R}^n)$, while on the other we can consider the multiplication operator $f \mapsto \psi f$, from $\mathcal{D}_{*,\mathbf{a}}'(\mathbb{R}^n)$ to itself, $m_\psi : \mathcal{D}_{*,\mathbf{a}}'(\mathbb{R}^n) \longrightarrow \mathcal{D}_{*,\mathbf{a}}'(\mathbb{R}^n)$.

Observe that

$$m_\psi m_\chi = m_{\psi\chi} , \tag{2.6.23}$$

and, generalizing (2.6.21),

$$m_\psi(\mathcal{P}f(\chi)) = \psi \mathcal{P}f(\chi) = \mathcal{P}f(\psi\chi) . \tag{2.6.24}$$

However, the expression $\mathcal{P}f(\psi)\mathcal{P}f(\chi)$ is *not* defined.

In the study of thick points in one variable [35] one also considers "projection multiplicaiton operators," given as follows.

Definition 12. *If* $\rho \in \mathcal{B}_{*,\mathbf{a}}$, *then*

$$\begin{aligned} M_\rho : \mathcal{D} &\to \mathcal{D}_{*,\mathbf{a}} \\ \phi &\mapsto \rho\phi \end{aligned} \ . \tag{2.6.25}$$

By duality, the corresponding multiplication operator is defined as

$$\begin{aligned} M'_\rho : \mathcal{D}'_{*,\mathbf{a}} &\to \mathcal{D}' \\ f &\mapsto \rho f \end{aligned} \ . \tag{2.6.26}$$

Notice that

$$\pi\left(\rho f\right) = M'_\rho\left(f\right) \ . \tag{2.6.27}$$

2.6.3 Derivatives of Thick Distributions

The derivatives of thick distributions are defined in much the same way as the usual distributional derivatives, that is, by duality.

Definition 13. *If* $f \in \mathcal{D}'_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ *then its thick distributional derivative* $\partial^* f/\partial x_j$ *is defined as*

$$\left\langle \frac{\partial^* f}{\partial x_j}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_j} \right\rangle , \qquad \phi \in \mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right) \ . \tag{2.6.28}$$

This definition makes sense, of course, because if $\phi \in \mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ then $\partial\phi/\partial x_j$ also belongs to $\mathcal{D}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$. The spaces $\mathcal{D}^{[m]}_{*,\mathbf{a}}\left(\mathbb{R}^n\right)$ are *not* closed under differentiation, and thus they are not an adequate test function space when $n \geq 2$, but, interestingly, in one variable one may just consider the space $\mathcal{D}^{[0]}_{*,\mathbf{a}}\left(\mathbb{R}\right)$, which is closed under derivatives [35].

Notice the notation. Thick distributional derivatives are denoted as $\partial^*/\partial x_j$ while ordinary derivatives are denoted as $\partial/\partial x_j$. We also follow the convention introduced by the late Professor Farassat [38] of denoting distributional derivatives with a bar, namely as $\overline{\partial}/\partial x_j$.

If $\mathbf{p} \in \mathbb{N}^n$ is a multi-index then we will also consider the operator $(\partial^*/\partial \mathbf{x})^{\mathbf{p}}$,

$$\left(\frac{\partial^*}{\partial \mathbf{x}}\right)^{\mathbf{p}} = \frac{\partial^{*p_1}...\partial^{*p_n}}{\partial x_1^{p_1}...\partial x_n^{p_n}},$$

so that if $|\mathbf{p}| = p_1 + \cdots + p_n$,

$$\left\langle \left(\frac{\partial^*}{\partial \mathbf{x}}\right)^{\mathbf{p}} f, \phi \right\rangle = (-1)^{|\mathbf{p}|} \left\langle f, \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\mathbf{p}} \phi \right\rangle = (-1)^{|\mathbf{p}|} \left\langle f, \frac{\partial^{*p_1}...\partial^{*p_n}\phi}{\partial x_1^{p_1}...\partial x_n^{p_n}} \right\rangle.$$

Let ϕ be an ordinary test function, that is, $\phi \in \mathcal{D}(\mathbb{R}^n)$, let $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$, and let $g = \pi(f)$. Then

$$\begin{aligned}\left\langle \frac{\partial^* f}{\partial x_j}, i(\phi) \right\rangle &= -\left\langle f, \frac{\partial i(\phi)}{\partial x_j} \right\rangle = -\left\langle f, i\left(\frac{\partial \phi}{\partial x_j}\right) \right\rangle \\ &= -\left\langle \pi(f), \frac{\partial \phi}{\partial x_j} \right\rangle = \left\langle \frac{\overline{\partial} g}{\partial x_j}, \phi \right\rangle,\end{aligned}$$

and we obtain the following lemma.

Proposition 6. *Let $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$. Then*

$$\pi\left(\frac{\partial^* f}{\partial x_j}\right) = \frac{\overline{\partial}\, \pi(f)}{\partial x_j}. \tag{2.6.29}$$

Formula (**??**) has an interesting consequence, as we shall explain next.

Example 10. *Let $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$; if $\pi(f) = 0$ (which is perhaps easy to see), then Proposition 6 yields $\pi(\partial^* f/\partial x_j) = 0$ (which is perhaps harder to see). Consider, for instance, $f = g\delta_*^{[-1]}$, a thick delta of order -1; that $\pi\left(g\delta_*^{[-1]}\right) = 0$ is obvious, but the formula $\pi\left((\delta g/\delta x_j - (n-1)\, n_j g)\, \delta_*^{[0]}\right) = 0$, that follows from (1.2.6), is not evident. In fact, even a particular case, such as $g = n_i$, which gives $\delta g/\delta x_j - (n-1)\, n_j g = \delta_{ij} - n n_i n_j$, yields an interesting formula, namely, $\pi\left(n_i n_j \delta_*^{[0]}\right) = \delta_{ij}/n$.*

However, a warning is in order. *Indeed, consider the projection* $\pi_q : \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n) \longrightarrow (\mathcal{D}^{[q]}_{*,\mathbf{a}}(\mathbb{R}^n))'$; *then*

$$\pi_q(f) = 0 \nRightarrow \pi_q\left(\frac{\partial^* f}{\partial x_j}\right) = 0\,. \tag{2.6.30}$$

For example if $f = g\delta_*^{[-1]}$ *then, of course,* $\pi_0(f) = 0$, *but, in general,* $\pi_0(\partial^* f/\partial x_j) \neq 0$.

Partial derivative operators are linear, of course, and they satisfy the product rule

$$\frac{\partial^*(\psi f)}{\partial x_j} = \frac{\partial \psi}{\partial x_j} f + \psi \frac{\partial^* f}{\partial x_j}\,, \tag{2.6.31}$$

if f is a thick distribution and ψ is a multiplier.

In general $\mathcal{P}f(\partial\psi/\partial x_j)$ and $\partial\mathcal{P}f(\psi)/\partial x_j$, even if both exist, *will not be equal.* We shall consider the case when $\psi = r^\lambda$ in detail later on.

2.7 Derivatives of Thick Deltas

In this section we shall compute the first order derivatives of thick deltas of any order.

Proposition 7. *Let* $g \in \mathcal{D}'(\mathbb{S})$. *Then*

$$\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[q]}\right) = \left(\frac{\delta g}{\delta x_j} - (q+n)\, n_j g\right)\delta_*^{[q+1]}. \tag{2.7.1}$$

Proof. Let $\phi \in \mathcal{D}_*(\mathbb{R}^n)$, with expansion $\phi(\mathbf{x}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j$, as $\mathbf{x} \to 0$. Then, employing (2.3.7) we obtain,

$$\begin{aligned}\left\langle \frac{\partial^*}{\partial x_j}\left(g\delta_*^{[q]}\right), \phi\right\rangle &= -\left\langle g\delta_*^{[q]}, \frac{\partial\phi}{\partial x_j}\right\rangle \\ &= -\frac{1}{C_{n-1}}\left\langle g, \frac{\delta a_{q+1}}{\delta x_j} + (q+1)\, n_j a_{q+1}\right\rangle\,,\end{aligned}$$

but (2.2.13) yields

$$\begin{aligned}-\left\langle g, \frac{\delta a_{q+1}}{\delta x_j} + (q+1)\, a_{q+1}\right\rangle &= \left\langle \frac{\delta^T g}{\delta x_j} - (q+1)\, n_j g, a_{q+1}\right\rangle \\ &= \left\langle \frac{\delta g}{\delta x_j} - (q+n)\, n_j g, a_{q+1}\right\rangle\,,\end{aligned}$$

and (2.7.1) follows. □

Observe, in particular, the formula

$$\frac{\partial^*}{\partial x_j}\left(\delta_*^{[q]}\right) = -\left(q+n\right) n_j \delta_*^{[q+1]}, \tag{2.7.2}$$

for the derivatives of plain thick deltas. If g_0 is a smooth homogeneous function of degree 0 defined in $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and $g_0\left(r\mathbf{w}\right) = g\left(\mathbf{w}\right)$, $g \in \mathcal{D}'\left(\mathbb{S}\right)$, then (2.2.5) yields that $\partial g_0/\partial x_j = r^{-1}\delta g/\delta x_j$. Then we have the alternative computation

$$\begin{aligned}
\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[q]}\right) &= \frac{\partial^*}{\partial x_j}\left(g_0\delta_*^{[q]}\right) \\
&= \frac{\partial g_0}{\partial x_j}\delta_*^{[q]} + g_0\frac{\partial^*}{\partial x_j}\left(\delta_*^{[q]}\right) \\
&= r^{-1}\frac{\delta g}{\delta x_j}\delta_*^{[q]} - g_0\left(q+n\right) n_j\delta_*^{[q+1]} \\
&= \frac{\delta g}{\delta x_j}\delta_*^{[q+1]} - \left(q+n\right) g n_j \delta_*^{[q+1]},
\end{aligned}$$

that gives another proof of (2.7.1)[2].

Example 11. *We can compute the Laplacian of the plain thick deltas as follows,*

$$\begin{aligned}
\nabla^2\left(\delta_*^{[q]}\right) &= \frac{\partial^*}{\partial x_j}\left(\frac{\partial^*}{\partial x_j}\left(\delta_*^{[q]}\right)\right) \\
&= -\frac{\partial^*}{\partial x_j}\left(\left(q+n\right) n_j\delta_*^{[q+1]}\right) \\
&= -\left(q+n\right)\left\{\frac{\delta n_j}{\delta x_j} - \left(q+n+1\right)\right\}\delta_*^{[q+2]},
\end{aligned}$$

but (2.2.6) yields $\delta n_j/\delta x_j = \delta_{jj} - n_j n_j = n-1$, so that

$$\nabla^2\left(\delta_*^{[q]}\right) = \left(q+n\right)\left(q+2\right)\delta_*^{[q+2]}. \tag{2.7.3}$$

In particular, if $m > 0$,

$$\nabla^{2m}\left(\delta_*\right) = \frac{\Gamma\left(m+n/2\right)\Gamma\left(1/2\right)\left(2m\right)!}{\Gamma\left(m+1/2\right)\Gamma\left(n/2\right)}\delta_*^{[2m]}. \tag{2.7.4}$$

[2]This proof is for g smooth, and thus a continuity argument is needed if $g \in \mathcal{D}'\left(\mathbb{S}\right)$.

If we now consider the projection of this identity onto $\mathcal{D}'(\mathbb{R}^n)$ and recall that $\pi(\delta_*) = \delta$, we obtain the following special case of (2.4.18),

$$\pi\left(\delta_*^{[2m]}\right) = \frac{\Gamma(m+1/2)\,\Gamma(n/2)}{\Gamma(m+n/2)\,\Gamma(1/2)\,(2m)!}\nabla^{2m}(\delta)\ . \tag{2.7.5}$$

Formula (2.7.3) also yields that $\nabla^2\left(\delta_*^{[-2]}\right) = 0$ and $\nabla^2\left(\delta_*^{[-n]}\right) = 0$.

Notice that since $\delta n_i/\delta x_j = \delta_{ij} - n_i n_j$, we have, more generally than (2.7.3),

$$\frac{\partial^{*2}}{\partial x_j \partial x_i}\left(\delta_*^{[q]}\right) = (q+n)\left((q+n+2)\, n_i n_j - \delta_{ij}\right)\delta_*^{[q+2]}. \tag{2.7.6}$$

2.8 Partial Derivatives of $\mathcal{P}f\left(r^\lambda\right)$

Another important set of formulas we want to discuss are the derivatives of $\mathcal{P}f\left(r^\lambda\right)$.

Theorem 5. If $\lambda \in \mathbb{C}\setminus\mathbb{Z}$, then

$$\frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^\lambda\right)\right) = \lambda x_j \mathcal{P}f\left(r^{\lambda-2}\right) = \lambda w_j \mathcal{P}f\left(r^{\lambda-1}\right)\ , \tag{2.8.1}$$

while if $k \in \mathbb{Z}$,

$$\frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^k\right)\right) = k x_j \mathcal{P}f\left(r^{k-2}\right) + C_{n-1} n_j \delta_*^{[-k-n+1]}. \tag{2.8.2}$$

In order to prove this theorem we need a couple of lemmas, which have an interest of their own.

Lemma 1. *Let $\phi \in \mathcal{D}_*(\mathbb{R}^n)$, then for any $\varepsilon > 0$, there exists a $\psi \in \mathcal{D}(\mathbb{R}^n)$, such that $\phi(\mathbf{x}) = \psi(\mathbf{x})$ whenever $|\mathbf{x}| \geq \varepsilon$.*

Proof. For any $\varepsilon > 0$, we can find a smooth function in $\mathbb{R}^n$, ρ, such that $\rho(\mathbf{x}) = 1$ if $|\mathbf{x}| \geq \varepsilon$ while $\rho(\mathbf{x}) = 0$ if $|\mathbf{x}| < \varepsilon/2$. Then we can take $\psi = \rho\phi$. □

Let us denote by $\mathbb{S}_\varepsilon = \varepsilon\mathbb{S}$ the sphere of radius ε and center at the origin.

Lemma 2. *Let g_0 be a smooth function, homogeneous of degree* 0 *in* $\mathbb{R}^n \setminus \{\mathbf{0}\}$. *If* $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, *then*

$$\text{F.p.}\lim_{\varepsilon\to 0} r^{\lambda} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right) = 0\,, \tag{2.8.3}$$

while if $k \in \mathbb{Z}$,

$$\text{F.p.}\lim_{\varepsilon\to 0} r^{k} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right) = C_{n-1} g_0 \delta_*^{[1-n-k]}, \tag{2.8.4}$$

in the space $\mathcal{D}'_*\left(\mathbb{R}^n\right)$.

Proof. Let $\phi \in \mathcal{D}_*\left(\mathbb{R}^n\right)$, with expansion $\phi\left(\mathbf{x}\right) \sim \sum_{j=m}^{\infty} a_j\left(\mathbf{w}\right) r^j$, as $\mathbf{x} \to 0$. Then

$$\begin{aligned}\left\langle r^{\lambda} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right), \phi\left(\mathbf{x}\right)\right\rangle &= \int_{\mathbb{S}_{\varepsilon}} \varepsilon^{\lambda} g_0\left(\mathbf{w}\right) \phi\left(\mathbf{w}\right)\, \mathrm{d}\sigma_{\varepsilon}\left(\mathbf{w}\right) \\ &= \int_{\mathbb{S}} \varepsilon^{\lambda+n-1} g_0\left(\mathbf{w}\right) \phi\left(\varepsilon\mathbf{w}\right)\, \mathrm{d}\sigma\left(\mathbf{w}\right)\,,\end{aligned}$$

so that

$$\left\langle r^{\lambda} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right), \phi\left(\mathbf{x}\right)\right\rangle \sim \sum_{j=m}^{\infty} \left(\int_{\mathbb{S}} g_0\left(\mathbf{w}\right) a_j\left(\mathbf{w}\right)\, \mathrm{d}\sigma\left(\mathbf{w}\right)\right) \varepsilon^{\lambda+n-1+j}, \tag{2.8.5}$$

as $\varepsilon \to 0$. The finite part of the limit of $\left\langle r^{\lambda} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right), \phi\left(\mathbf{x}\right)\right\rangle$ is equal to the coefficient of ε^0; if $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, this coefficient is 0, since none of the exponents in (**??**) is an integer, while if $\lambda = k \in \mathbb{Z}$, we should take $j = 1 - k - n$, and this yields

$$\begin{aligned}\text{F.p.}\lim_{\varepsilon\to 0} \left\langle r^{k} g_0 \delta\left(\mathbb{S}_{\varepsilon}\right), \phi\left(\mathbf{x}\right)\right\rangle &= \int_{\mathbb{S}} g_0\left(\mathbf{w}\right) a_{1-k-n}\left(\mathbf{w}\right)\, \mathrm{d}\sigma\left(\mathbf{w}\right) \\ &= \left\langle C_{n-1} g_0 \delta_*^{[1-n-k]}, \phi\right\rangle\,,\end{aligned}$$

as required. □

Now we are ready to give a proof of the Theorem 5.

Proof of Theorem 5. Let $\phi \in \mathcal{D}_*(\mathbb{R}^n)$. Then

$$\begin{aligned}\left\langle \frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^\lambda\right)\right), \phi\right\rangle &= -\left\langle \mathcal{P}f\left(r^\lambda\right), \frac{\partial \phi}{\partial x_j}\right\rangle \\ &= -\mathrm{F.p.}\lim_{\varepsilon \to 0} \int_{|\mathbf{x}| \geq \varepsilon} r^\lambda \frac{\partial \phi}{\partial x_j}\, \mathrm{d}\mathbf{x} \\ &= -\mathrm{F.p.}\lim_{\varepsilon \to 0}\left\langle H(r-\varepsilon) r^\lambda, \frac{\partial \phi}{\partial x_j}\right\rangle .\end{aligned}$$

By the Lemma 1, there exists $\psi \in \mathcal{D}(\mathbb{R}^n)$, such that $\phi(\mathbf{x}) = \psi(\mathbf{x})$ for $|\mathbf{x}| \geq \varepsilon$, and thus

$$\begin{aligned}-\left\langle H(r-\varepsilon) r^\lambda, \frac{\partial \phi}{\partial x_j}\right\rangle &= -\int_{|\mathbf{x}| \geq \varepsilon} r^\lambda \frac{\partial \psi}{\partial x_j}\, \mathrm{d}\mathbf{x} \\ &= \left\langle \frac{\overline{\partial}}{\partial x_j}\left(H(r-\varepsilon) r^\lambda\right), \psi\right\rangle \\ &= \left\langle \frac{\overline{\partial}}{\partial x_j}\left(H(r-\varepsilon) r^\lambda\right), \phi\right\rangle .\end{aligned}$$

The usual distributional derivative of $H(r-\varepsilon) r^\lambda$ is given by [58]

$$\frac{\overline{\partial}}{\partial x_j}\left(H(r-\varepsilon) r^\lambda\right) = \lambda x_j r^{\lambda-2} H(r-\varepsilon) + r^\lambda n_j \delta(\mathbb{S}_\varepsilon) ,$$

so that

$$\begin{aligned}\left\langle \frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^\lambda\right)\right), \phi\right\rangle &= \mathrm{F.p.}\lim_{\varepsilon \to 0}\left\langle \frac{\overline{\partial}}{\partial x_j}\left(H(r-\varepsilon) r^\lambda\right), \phi\right\rangle \\ &= \mathrm{F.p.}\lim_{\varepsilon \to 0}\left\langle \lambda x_j r^{\lambda-2} H(r-\varepsilon) + r^\lambda n_j \delta(\mathbb{S}_\varepsilon), \phi\right\rangle \\ &= \left\langle \lambda x_j \mathcal{P}f\left(r^\lambda\right), \phi\right\rangle + \mathrm{F.p.}\lim_{\varepsilon \to 0}\left\langle r^\lambda n_j \delta(\mathbb{S}_\varepsilon), \phi\right\rangle ,\end{aligned}$$

and Theorem 5 is obtained by applying the Lemma 2. □

Example 12. *In* $\mathbb{R}^3$, $\partial^* \mathcal{P}f(r^{-1}) / \partial x_j$ *is given by*

$$\frac{\partial^* \mathcal{P}f(r^{-1})}{\partial x_j} = -x_j \mathcal{P}f\left(r^{-3}\right) + 4\pi n_j \delta_*^{[-1]}. \tag{2.8.6}$$

This is very similar to the usual distributional derivative of $1/r$ *except for the extra term* $4\pi n_j \delta_*^{[-1]}$. *Of course,* $\pi\left(4\pi n_j \delta_*^{[-1]}\right) = 0$, *so that we recover the well known*

formula $\overline{\partial}\left(r^{-1}\right)/\partial x_j = -x_j/r^3$. *We also have that* $\pi_0\left(\partial^* r^{-1}/\partial x_j\right) = -x_j/r^3$, *but as we mentioned in Example 10, especially (2.6.30), this is a rather incomplete result.*

Example 13. *If we apply the projection operator to (2.8.1) and (2.8.2), we obtain the formulas for the partial derivatives of* $\mathcal{P}f\left(r^\lambda\right)$ *in* $\mathcal{D}'\left(\mathbb{R}^n\right)$. *Since (2.7.2) yields that* $\pi\left(n_j\delta_*^{[q]}\right) = 0$ *unless* $q = 2m+1$, $m \geq 0$, *in which case*

$$\pi\left(n_j\delta_*^{[2m+1]}\right) = \frac{-\Gamma\left(m+1/2\right)\Gamma\left(n/2\right)}{\left(2m+n\right)\Gamma\left(m+n/2\right)\Gamma\left(1/2\right)\left(2m\right)!}\frac{\overline{\partial}}{\partial x_j}\nabla^{2m}\delta\,, \tag{2.8.7}$$

we obtain $\overline{\partial}/\partial x_j\left(\mathcal{P}f\left(r^\lambda\right)\right) = \lambda x_j \mathcal{P}f\left(r^{\lambda-2}\right)$ *unless* $\lambda = -n, -n-2, -n-4, \ldots$. *If* $\lambda = -p = -n-2m$,

$$\frac{\overline{\partial}}{\partial x_j}\left(\mathcal{P}f\left(\frac{1}{r^p}\right)\right) = -px_j\mathcal{P}f\left(\frac{1}{r^{p+2}}\right) - \frac{c_{m,n}}{(2m)!p}\frac{\overline{\partial}}{\partial x_j}\nabla^{2m}\delta\,, \tag{2.8.8}$$

where the constant $c_{m,n}$ *is given by (2.6.18), since we have the identity* $C_{n-1} = c_{0,n} = 2\Gamma\left(1/2\right)\pi^{(n-1)/2}/\Gamma\left(n/2\right)$; *this formula agrees with the known derivative formulas [32, eqn. (3.16)], [36, eqn. (3)].*

Let us now discuss the second-order thick derivatives of r^λ.

Theorem 6. If $\lambda \in \mathbb{C}\setminus\mathbb{Z}$, then

$$\frac{\partial^{*2}\mathcal{P}f\left(r^\lambda\right)}{\partial x_i\partial x_j} = \left(\lambda\delta_{ij} + \lambda\left(\lambda-2\right)n_in_j\right)\mathcal{P}f\left(r^{\lambda-2}\right)\,. \tag{2.8.9}$$

If $\lambda = k \in \mathbb{Z}$, then

$$\frac{\partial^{*2}\mathcal{P}f\left(r^k\right)}{\partial x_i\partial x_j} = \left(k\delta_{ij} + k\left(k-2\right)n_in_j\right)\mathcal{P}f\left(r^{k-2}\right) + \left(\delta_{ij} + 2\left(k-1\right)n_in_j\right)\delta_*^{[-k-n+2]}. \tag{2.8.10}$$

Proof. We shall prove (2.8.10), (2.8.9) being easier. We have

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{k}\right)}{\partial x_i\partial x_j}=\frac{\partial^{*}}{\partial x_i}\left\{kx_j\mathcal{P}f\left(r^{k-2}\right)+C_{n-1}n_j\delta_{*}^{[-k-n+1]}\right\}$$

$$=k\delta_{ij}\mathcal{P}f\left(r^{k-2}\right)+kx_j\left\{(k-2)x_i\mathcal{P}f\left(r^{k-4}\right)+C_{n-1}n_i\delta_{*}^{[-k+2-n+1]}\right\}$$
$$+C_{n-1}\frac{\partial^{*}}{\partial x_i}\left(n_j\delta_{*}^{[-k-n+1]}\right)\ ,$$

but

$$\frac{\partial^{*}}{\partial x_i}\left(n_j\delta_{*}^{[-k-n+1]}\right)=\left(\frac{\delta n_j}{\delta x_i}-(n-k-n+1)n_in_j\right)\delta_{*}^{[-k-n+2]}$$
$$=\left(\delta_{ij}+(k-2)n_in_j\right)\delta_{*}^{[-k-n+2]},$$

so that

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{k}\right)}{\partial x_i\partial x_j}=k\delta_{ij}\mathcal{P}f\left(r^{k-2}\right)+k\left(k-2\right)x_ix_j\mathcal{P}f\left(r^{k-4}\right)$$
$$+kx_j\left(C_{n-1}n_i\delta_{*}^{[-k-n+3]}\right)+C_{n-1}\left(\delta_{ij}-(k-2)n_in_j\right)\delta_{*}^{[-k-n+2]},$$

and (2.8.10) follows since $x_j\delta_{*}^{[q]}=n_j\delta_{*}^{[q-1]}$. □

We would like to point out that the right side of (2.8.9), and in a similar fashion the first part of (2.8.10), can be rewritten alternatively as

$$\lambda\delta_{ij}\mathcal{P}f\left(r^{\lambda-2}\right)+\lambda\left(\lambda-2\right)x_ix_j\mathcal{P}f\left(r^{\lambda-4}\right)\ ,$$

or as

$$\left(\lambda\delta_{ij}r^{2}+\lambda\left(\lambda-2\right)x_ix_j\right)\mathcal{P}f\left(r^{\lambda-4}\right)\ .$$

Example 14. *When $n=3$ and $k=-1$ we obtain*

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i\partial x_j}=\left(3x_ix_j-\delta_{ij}r^{2}\right)\mathcal{P}f\left(r^{-5}\right)+4\pi\left(\delta_{ij}-4n_in_j\right)\delta_{*}\,. \qquad (2.8.11)$$

Since $\pi\left(n_i n_j \delta_*\right) = (1/3)\,\delta\left(\mathbf{x}\right)$ *in* $\mathbb{R}^3$*, we obtain the well known formula of Frahm [39]*

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta\left(\mathbf{x}\right) , \tag{2.8.12}$$

when we apply the projection operator π *to (2.8.11).*

In his article [40], Franklin gave another formula for the second order partial derivatives of $1/r$*, a formula that in our notation would read as* $\pi_0\left(\partial^{*2}\mathcal{P}f\left(r^{-1}\right)/\partial x_i \partial x_j\right) = \left(3x_i x_j - \delta_{ij} r^2\right)\mathcal{P}f\left(r^{-5}\right) - 4\pi n_i n_j \delta_*$*, and which does not agree with (2.8.12) because it does not include the derivative term* $\partial^*/\partial x_i (C_{n-1} n_j \delta_*^{[-1]})$*; see the Example 10, especially (2.6.30).*

Example 15. *We also have the following important special case of the Theorem 6, namely,*

$$\nabla^2 \mathcal{P}f\left(r^\lambda\right) = \lambda\left(n + \lambda - 2\right)\mathcal{P}f\left(r^{\lambda-2}\right) , \tag{2.8.13}$$

if $\lambda \in \mathbb{C}\setminus\mathbb{Z}$*, while if* $\lambda = k \in \mathbb{Z}$*, then*

$$\nabla^2 \mathcal{P}f\left(r^k\right) = k\left(n+k-2\right)\mathcal{P}f\left(r^{k-2}\right) + \left(n+2k-2\right)\delta_*^{[-k-n+2]}. \tag{2.8.14}$$

Chapter 3
An extension of Frahm's Formula

3.1 Introduction

In this chapter we will give a detailed investigation of the last example from the previous chapter. The derivatives of $1/r$ has been given great considerations, because the inverse-square field is very important in classic field theories when considering a point source. The potential of an inverse-square field usually involves an $1/r$ term. And the field equations typically involves a derivative term with respect to the field. For example,

$$\partial_i E_i = 4\pi\rho(\mathbf{r}) \tag{3.1.1}$$

Where E_i corresponds to the electric field provided by a point charge q.

$$E_i = qn_i/r^2 \tag{3.1.2}$$

Because $1/r$ (r is the radial coordinate) has a singular point at the origin, we cannot treat its derivative there as a usual derivative, but a "distributional derivative". In classical distribution theory, the second order derivative of $1/r$ has been presented in the following form,

$$\frac{\partial^2(1/r)}{\partial x_i \partial x_j} = \frac{3x_i x_j - \delta_{ij} r^2}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta(r) \tag{3.1.3}$$

One can find a derivation in Frahm's paper [39]. And similar results were presented in textbooks [56], [58].

Yet some discussions were arose. One of them were from Franklin [40].

In [40] the equation

$$\frac{\partial^2(1/r)}{\partial x_i \partial x_j} = \frac{3x_i x_j - \delta_{ij} r^2}{r^5} - 4\pi\frac{x_i x_j}{r^2}\delta(\mathbf{x}) \tag{3.1.4}$$

is proposed to correct the familiar textbook formula 3.1.3

Although Franklin pointed out a few acute observation in his paper [40], his formula was not completely correct in the sense that, in his proof, several of the so called "spaces of distributions" were mixed up. In fact, in the usual distribution space, the formula 3.1.3 is correct. Which means, rigorously speaking, if we "integrate" $\frac{\partial^2 (1/r)}{\partial x_i \partial x_j}$ times a compactly supported smooth function ϕ over the whole space, it gives us the same result as if we integrate $\frac{3x_i x_j - \delta_{ij} r^2}{r^5} - \left(\frac{4\pi}{3}\right) \delta_{ij} \delta(r)$ times this same smooth function ϕ over the whole space. In most of the important cases in physics, any such ϕ involved are compactly supported and smooth.

However, what if ϕ is not smooth? Indeed, [40] pointed out that in Frahm's proof, the identity

$$\oint d\Omega \frac{x_i x_j}{R^2} = \frac{4\pi}{3} \delta_{ij} \tag{3.1.5}$$

is used and can be used only when multiplied by "an arbitrary smooth function". In fact, such a ϕ is called a "test function".

In the previous chapter, we have given a systematically construction of a larger space, which is called "thick distribution space". So that in the thick distribution space, an extended formula of $\frac{\partial^2 (1/r)}{\partial x_i \partial x_j}$ is obtained. Roughly speaking, the extended formula can be applied even if the above "test function" is not smooth.

In this chapter, we will further explain this formula in details. All the computations can be found in the the appendix. We will show that Frahm's formula is indeed the correct formula when one considers the special case that ϕ is a compactly supported smooth test function.

3.2 Proof

We know that δ function is a distribution that, in 3-dimensional space,

$$\int_{\mathbb{R}^3} \delta(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = \phi(\mathbf{0}) \tag{3.2.1}$$

for any compactly supported smooth test function f.

So if 3.1.3 holds, it must satisfy

$$\int_{\mathbb{R}^3} \partial_i \partial_j \left(\frac{1}{r}\right) \phi\left(\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^3} \left[\frac{3x_i x_j - \delta_{ij} r^2}{r^5} - \left(\frac{4\pi}{3}\right) \delta_{ij} \delta\left(r\right)\right] \phi\left(\mathbf{x}\right) d\mathbf{x} \tag{3.2.2}$$

In fact,

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_i \partial_j \left(1/r\right) \phi\left(\mathbf{x}\right) d\mathbf{x} &= \int_{\mathbb{R}^3} \partial_i (\partial_j \left(1/r\right) \phi(\mathbf{x})) d\mathbf{x} - \int_{\mathbb{R}^3} \partial_j \left(1/r\right) \partial_i \phi(\mathbf{x}) d\mathbf{x} \\
&= 0 - \int_{\mathbb{R}^3} \partial_j \left(1/r\right) \partial_i \phi(\mathbf{x}) d\mathbf{x} \\
&= - \int_{\mathbb{R}^3} \partial_j ((1/r) \partial_i \phi(\mathbf{x})) d\mathbf{x} + \int_{\mathbb{R}^3} (1/r) \partial_i \partial_j \phi(\mathbf{x}) d\mathbf{x} \\
&= \int_{\mathbb{R}^3} (1/r) \partial_i \partial_j \phi(\mathbf{x}) d\mathbf{x}.
\end{aligned}$$

So equation (3.2.2) becomes

$$\int_{\mathbb{R}^3} \left[\frac{3x_i x_j - \delta_{ij} r^2}{r^5} - \left(\frac{4\pi}{3}\right) \delta_{ij} \delta\left(r\right)\right] f\left(\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^3} (1/r) \partial_i \partial_j f(\mathbf{x}) d\mathbf{x}. \tag{3.2.3}$$

It's been proved that [58] if ϕ is a compactly supported smooth function, equation 3.2.3 indeed holds.

Yet if $\phi\left(\mathbf{x}\right)$ is not smooth at the origin, e.g., $f\left(x\right) = x_i/r$,, then it's hard to talk about $\partial_i \partial_j f(\mathbf{x})$. One then has to use the "δ-derivatives", which was briefly reviewed in the previous chapter. Thus the above equation 3.2.3 or 3.1.3 is no longer valid. $f\left(r, \theta, \phi\right) = \cos\theta$ is another example that is not well-defined at the origin, In fact, δ function does not make sense in equation (3.2.1) if f does not have a definite value at the origin [**?**]

Let's go back to $\partial_i \partial_j (1/r)$. Though ϕ might not be smooth, it should have a "thick" limit at the origin, namely, $\lim_{\varepsilon \to 0^+} \phi\left([35] \varepsilon \mathbf{w}\right) = a_0\left(\mathbf{w}\right)$ exists for each $\mathbf{w} \in \mathbb{S}$, where $\mathbb{S} = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ is the unit sphere. This $a_0\left(\mathbf{w}\right)$ actually corresponds to the first term of the Asymptotic expansion of ϕ.

Now let us give a proof of the extended formula [89]

$$\frac{\partial^{*2}\left(r^{-1}\right)}{\partial x_i \partial x_j} = \left(3x_i x_j - \delta_{ij} r^2\right) \mathcal{P}f\left(r^{-5}\right) + 4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_* \,, \tag{3.2.4}$$

that applies to thick test functions ϕ. Here ∂^* means in the distributional thick sense. The δ_* is a *thick delta,* defined for g integrable on $\mathbb{S}$ as

$$\langle g(\mathbf{w})\,\delta_*, \phi(\mathbf{x})\rangle = \frac{1}{4\pi}\int_{\mathbb{S}} g(\mathbf{w})\, a_0(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ , \tag{3.2.5}$$

$\mathrm{d}\sigma(\mathbf{w})$ being the Lebesgue measure on the sphere and $a_0(\mathbf{w})$ the thick value of ϕ at the origin. The distribution $(3x_i x_j - \delta_{ij} r^2)\,\mathcal{P}f\,(r^{-5})$ is given by the spherical *Hadamard finite part* limit

$$\left\langle \mathcal{P}f\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right), \phi(\mathbf{x})\right\rangle = \mathrm{F.p.}\lim_{\varepsilon\to 0^+}\int_{|\mathbf{x}|\geq\varepsilon}\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right)\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

One needs to consider the finite part because when ϕ is a thick test function then the ordinary limit does not have to exist. See [37] for more on finite parts. This terms gives a same result of $\dfrac{3x_i x_j - \delta_{ij} r^2}{r^5} - \left(\dfrac{4\pi}{3}\right)$ in equation (3.1.3) if ϕ is smooth.

We shall now present an *explicit* computation of $\int_{\mathbb{R}^3}(\partial^2\phi/\partial x_i\partial x_j) r^{-1}\mathrm{d}\mathbf{x}$, when ϕ is a thick test function where the variables can be separated, $\phi(\mathbf{x}) = \rho(r)\,\varphi(\mathbf{w})$, $\mathbf{x} = r\mathbf{w}$, $r = |\mathbf{x}|$. This computation should convince the reader that (3.2.4) is correct. Actually, as we explain later, once the separated variables case is known, it is easy to see that (3.2.4) holds in general.

Technical conditions are as follows. The function φ is of class C^2 on $\mathbb{S}$ while the function of one variable ρ is a standard test function, namely, it is smooth and with compact support. Notice that ϕ is of class C^2 everywhere except at the origin, where ϕ has the thick limit $a_0(\mathbf{w}) = \lim_{\varepsilon\to 0^+}\phi(\varepsilon\mathbf{w}) = \rho(0)\,\varphi(\mathbf{w})$. Observe first that,

$$\frac{\partial\phi}{\partial x_i} = \frac{\partial(\rho(r)\,\varphi(\mathbf{w}))}{\partial x_i} = \rho'(r)\, n_i \varphi(\mathbf{w}) + \frac{\rho(r)}{r}\frac{\delta\varphi}{\delta x_i}\,.$$

where $n_i = x_i/r$ are the components of $\mathbf{n}$, the unit tangent vector on the sphere $\mathbb{S}$ and where $\delta\varphi/\delta x_i$ is the *delta derivative* of φ with respect to x_i, as explained in the previous chapter. (If φ is defined only on $\mathbb{S}$ and φ_0 is the extension to $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{0}\}$ that is homogeneous of degree 0, namely, $\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}/r)$, where $r = |\mathbf{x}|$, then $\dfrac{\delta\varphi}{\delta x_j} = \left.\dfrac{\partial\varphi_0}{\partial x_j}\right|_{\mathbb{S}}$.) Similarly, taking derivatives again, we have

$$\frac{\partial^2\phi}{\partial x_i \partial x_j} = \rho'' n_i n_j \varphi + \frac{\rho'}{r}\frac{\delta(n_i\varphi)}{\delta x_j} + \frac{\rho'}{r} n_j \frac{\delta\varphi}{\delta x_i} + \frac{\rho}{r^2}\frac{\delta}{\delta x_j}\left(\frac{\delta\varphi}{\delta x_i}\right) - \frac{\delta\varphi}{\delta x_i}\frac{\rho}{r^2} n_j \,.$$

Therefore $\displaystyle\int \frac{1}{r}\frac{\partial^2\phi}{\partial x_i\partial x_j} d\mathbf{x}$ is the sum of the five terms

$$\begin{aligned}
&\int \frac{1}{r}\rho'' n_i n_j \varphi d\mathbf{x} + \int \frac{1}{r}\frac{\rho'}{r}\frac{\delta(n_i\varphi)}{\delta x_j} d\mathbf{x} \\
&+ \int \frac{1}{r}\frac{\rho'}{r} n_j \frac{\delta\varphi}{\delta x_i} d\mathbf{x} + \int \frac{1}{r}\frac{\rho}{r^2}\frac{\delta}{\delta x_j}\left(\frac{\delta\varphi}{\delta x_i}\right) d\mathbf{x} - \int \frac{1}{r}, \frac{\delta\varphi}{\delta x_i}\frac{\rho}{r^2} n_j d\mathbf{x}.
\end{aligned}$$

In standard notation of functional analysis, it can be written as

$$\begin{aligned}
\left\langle \frac{1}{r}, \frac{\partial^2\phi}{\partial x_i\partial x_j} \right\rangle &= \left\langle \frac{1}{r}, \rho'' n_i n_j \varphi \right\rangle + \left\langle \frac{1}{r}, \frac{\rho'}{r}\frac{\delta(n_i\varphi)}{\delta x_j} \right\rangle + \left\langle \frac{1}{r}, \frac{\rho'}{r} n_j \frac{\delta\varphi}{\delta x_i} \right\rangle \\
&+ \left\langle \frac{1}{r}, \frac{\rho}{r^2}\frac{\delta}{\delta x_j}\left(\frac{\delta\varphi}{\delta x_i}\right) \right\rangle - \left\langle \frac{1}{r}, \frac{\delta\varphi}{\delta x_i}\frac{\rho}{r^2} n_j \right\rangle \,.
\end{aligned}$$

The detailed computation of each of these five terms are given in the appendix. The results are

$$\left\langle \frac{1}{r}, \rho'' n_i n_j \varphi \right\rangle = 4\pi \left\langle n_i n_j \delta_*(\mathbf{x}), \phi(\mathbf{x}) \right\rangle_{\mathbb{R}^3}$$

$$\left\langle \frac{1}{r}, \frac{\rho'}{r}\frac{\delta(n_i\varphi)}{\delta x_j} \right\rangle = 4\pi \left\langle -2 n_i n_j \delta_*(\mathbf{x}), \phi(\mathbf{x}) \right\rangle_{\mathbb{R}^3} \,,$$

$$\left\langle \frac{1}{r}, \frac{\rho}{r^2}\frac{\delta}{\delta x_j}\left(\frac{\delta\varphi}{\delta x_i}\right) \right\rangle = -2 \left\langle \frac{\delta_{ij} r^2 - 3x_i x_j}{r^5}, \phi(\mathbf{x}) \right\rangle_{\mathbb{R}^3} \,,$$

$$\left\langle \frac{1}{r}, \frac{\rho'}{r} n_j \frac{\delta\varphi}{\delta x_i} \right\rangle = 4\pi \left\langle (\delta_{ij} - 3n_i n_j)\, \delta_*(\mathbf{x}), \phi(\mathbf{x}) \right\rangle_{\mathbb{R}^3} \,,$$

$$-\left\langle \frac{1}{r}, \frac{\delta\varphi}{\delta x_i}\frac{\rho}{r^2} n_j \right\rangle = \left\langle \frac{\delta_{ij} r^2 - 3x_i x_j}{r^5}, \phi \right\rangle_{\mathbb{R}^3} \,,$$

Combining all above equations, we have (3.2.4), namely

$$\left\langle \frac{\partial^{*2}\left(r^{-1}\right)}{\partial x_i \partial x_j}, \phi \right\rangle = \left\langle \frac{3x_i x_j - \delta_{ij} r^2}{r^5} + 4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*\left(\mathbf{x}\right), \phi \right\rangle_{\mathbb{R}^3} . \tag{3.2.6}$$

Let us now show that (3.2.6) holds if ϕ is a quite general thick test function, in the following sense. Suppose that the integrals in (3.2.6), or, what is the same, in (**??**) are absolutely convergent at infinity, and suppose that can write $\phi\left(\mathbf{x}\right) = \widetilde{\phi}\left(r, \mathbf{w}\right)$, where $\widetilde{\phi}$ is a function of class C^2 of the variables $r \in (-\varepsilon, \infty)$, for some $\varepsilon > 0$, and $\mathbf{w} \in \mathbb{S}$. The thick limit at the origin is then $a_0\left(\mathbf{w}\right) = \widetilde{\phi}\left(0, \mathbf{w}\right)$. To see that (3.2.6) holds for this ϕ, we choose a standard test function of one variable ρ with $\rho\left(0\right) = 1$, and write $\phi = \phi_1 + \phi_2$, where $\phi_1\left(\mathbf{x}\right) = \rho\left(r\right) a_0\left(\mathbf{w}\right)$ and $\phi_2 = \phi - \phi_1$; then (3.2.6) holds for ϕ_1 because the variables are separated, while ϕ_2 is actually continuous at the origin, where it vanishes, and clearly the Frahm formulas (3.1.3) are valid for ϕ_2. One could check that this reduces to equation (3.1.3) if ϕ is smooth.

3.3 Appendix

We now compute each of these five terms. We have

$$\left\langle \frac{1}{r}, \rho'' n_i n_j \varphi \right\rangle = \int_0^\infty r\rho''\left(r\right)\,\mathrm{d}r \int_{\mathbb{S}} n_i n_j \varphi \,\mathrm{d}\sigma\left(\mathbf{w}\right) = -\left\langle n_i n_j, \varphi\right\rangle_{\mathbb{S}} \int_0^\infty \rho'\left(r\right)\,\mathrm{d}r$$
$$= \rho\left(0\right)\left\langle n_i n_j, \varphi\right\rangle_{\mathbb{S}} = 4\pi\left\langle n_i n_j \delta_*\left(\mathbf{x}\right), \phi\left(\mathbf{x}\right)\right\rangle_{\mathbb{R}^3} .$$

Next,

$$\left\langle \frac{1}{r}, \frac{\rho'}{r}\frac{\delta\left(n_i\varphi\right)}{\delta x_j} \right\rangle = -\rho\left(0\right)\left\langle 1, \frac{\delta}{\delta x_j}\left(n_i\varphi\right)\right\rangle_{\mathbb{S}} = \rho\left(0\right)\left\langle n_i \frac{\delta^T\left(1\right)}{\delta x_j}, \varphi\right\rangle_{\mathbb{S}}$$
$$= -2\rho\left(0\right)\left\langle n_i n_j, \varphi\right\rangle_{\mathbb{S}} = 4\pi\left\langle -2n_i n_j \delta_*\left(\mathbf{x}\right), \phi\left(\mathbf{x}\right)\right\rangle_{\mathbb{R}^3} ,$$

where $\delta^T/\delta x_j$ is the adjoint of $\delta/\delta x_j$, so that [89], $\dfrac{\delta^T\phi}{\delta x_j} = \dfrac{\delta\phi}{\delta x_j} - 2n_j\phi$, and $\dfrac{\delta^T n_i}{\delta x_j} = \delta_{ij} - 3n_i n_j$. Similarly,

$$\left\langle \frac{1}{r}, \frac{\rho}{r^2}\frac{\delta}{\delta x_j}\left(\frac{\delta\varphi}{\delta x_i}\right)\right\rangle = \int_0^\infty \frac{\rho}{r}\,\mathrm{d}r \left\langle \frac{\delta^T}{\delta x_i}\left(\frac{\delta^T 1}{\delta x_j}\right), \varphi\right\rangle_{\mathbb{S}} = \int_0^\infty \frac{\rho}{r}\,\mathrm{d}r\left\langle \frac{\delta^T\left(-2n_j\right)}{\delta x_i}, \varphi\right\rangle_{\mathbb{S}}$$
$$= -2\int_0^\infty \frac{\rho}{r}\,\mathrm{d}r\left\langle \delta_{ij} - 3n_i n_j, \varphi\right\rangle_{\mathbb{S}} = -2\left\langle \frac{\delta_{ij} r^2 - 3x_i x_j}{r^5}, \phi\right\rangle_{\mathbb{R}^3} ,$$

$$\left\langle \frac{1}{r}, \frac{\rho'}{r} n_j \frac{\delta\varphi}{\delta x_i} \right\rangle = \int_0^\infty \rho'(r)\ \mathrm{d}r \left\langle 1, n_j \frac{\delta\varphi}{\delta x_i} \right\rangle_{\mathbb{S}} = \rho(0) \left\langle \frac{\delta^T n_j}{\delta x_i}, \varphi \right\rangle_{\mathbb{S}}$$
$$= \rho(0) \left\langle \delta_{ij} - 3n_i n_j, \varphi \right\rangle_{\mathbb{S}} = 4\pi \left\langle (\delta_{ij} - 3n_i n_j)\, \delta_*(\mathbf{x}), \phi(\mathbf{x}) \right\rangle_{\mathbb{R}^3}\ ,$$

$$-\left\langle \frac{1}{r}, \frac{\delta\varphi}{\delta x_i} \frac{\rho}{r^2} n_j \right\rangle = -\left\langle 1, n_j \frac{\delta\varphi}{\delta x_i} \right\rangle_S \int_0^\infty \frac{\rho(r)}{r}\,\mathrm{d}r = \int_0^\infty \frac{\rho}{r}\,\mathrm{d}r \left\langle \frac{\delta^T n_j}{\delta x_i}, \varphi \right\rangle_{\mathbb{S}}$$
$$= \int_0^\infty \frac{\rho}{r}\,\mathrm{d}r \left\langle \delta_{ij} - 3n_i n_j, \varphi \right\rangle_{\mathbb{S}} = \left\langle \frac{\delta_{ij} r^2 - 3x_i x_j}{r^5}, \phi \right\rangle_{\mathbb{R}^3}\ ,$$

Chapter 4
Thick Distributional Calculus

4.1 Introduction

Now let me present another very important application of the thick distributions, the one with regarding to the ”product rule” failure in the multiplication between distributions. The theory of Colombeau algebras has provided a complete treatment for the multiplication between distributions **??**. What we present here is a different treatment that is not aimed to completely solve the problems arising from the product between distributions, but it is very practical in solving quite a few important puzzles and it points out a direction of solving such problems. I have to comment that our theory is a different approach to the one of Colombeau's and thus gives a different interpretation of certain multiplication problems. This chapter will be dedicated to examples of such an application, and the problem in Section 4.4 arises from applying distribution theory in point source field equations.

Section 4.2.

Our first application, given in Section 4.3, is the computation of the distributional derivatives of homogeneous distributions in $\mathbb{R}^n$ by first computing the thick distributional derivatives and then projecting onto the space of standard distributions. Our analysis makes several delicate points quite clear.

Next, in Section 4.4, we consider an application to point source fields. In [15], Bowen computed the derivative of the distribution

$$g_{j_1,\ldots,j_k}(\mathbf{x}) = \frac{n_{j_1}\cdots n_{j_k}}{r^2}\,, \tag{4.1.1}$$

of $\mathcal{D}'(\mathbb{R}^3)$, where $r = |\mathbf{x}|$ and $\mathbf{n} = (n_i)$ is the unit normal vector to a sphere centered at the origin, that is, $n_i = x_i/r$. His result can be written as[1]

$$\frac{\overline{\partial}}{\partial x_i} g_{j_1,\ldots,j_k} = \left\{ \sum_{q=1}^{k} \delta_{ij_q} \frac{n_{j_1} \cdots n_{j_k}}{n_{j_q}} - (k+2)\, n_i n_{j_1} \cdots n_{j_k} \right\} \frac{1}{r^3} + A\delta(\mathbf{x}) \ , \qquad (4.1.2)$$

where $n_i n_{j_1} \cdots n_{j_k} = n_1^a n_2^b n_3^c$, and $A = 0$ if a, b, or c is odd, while

$$A = \frac{2\Gamma((a+1)/2)\,\Gamma((b+1)/2)\,\Gamma((c+1)/2)}{\Gamma((a+b+c+3)/2)} , \qquad (4.1.3)$$

if the three exponents are even. Interestingly, he observes that if one tries to compute this formula by induction, employing the product rule for derivatives, the result obtained is *wrong.* In this article we show that one can actually apply the product rule in the space of thick distributions, obtaining (4.1.2) by induction; furthermore, our analysis shows *why* the wrong result is obtained when applying the product rule in [15].

Finally in Section 4.5 we show how the thick distributional calculus allows one to avoid mistakes in the computation of higher order derivatives of thick distributions of order 0.

4.2 Review

Let me rewrite a few results introduced in Chapter 1, which will be used in this chapter. Recall that if ϕ is a standard test function, namely, smooth in all $\mathbb{R}^n$ and with compact support, then it has a Taylor expansion,

$$\phi(\mathbf{a} + r\mathbf{w}) \sim a_0 + \sum_{j=1}^{\infty} a_j(\mathbf{w})\, r^j . \qquad (4.2.1)$$

where a_0 is just the real number $\phi(\mathbf{a})$. Hence $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$; we denote by

$$i : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n) \ , \qquad (4.2.2)$$

[1]Following the notation introduced by the late Professor Farassat [38], we shall denote distributional derivatives with an overbar.

the inclusion map. In fact, with the topology constructed in Definition 7, $\mathcal{D}(\mathbb{R}^n)$ is not only a subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ but actually a *closed* subspace of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$. We denote by

$$\Pi : \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n) , \tag{4.2.3}$$

the projection operator, dual of the inclusion (4.2.2).

Let $g(\mathbf{w})$ is a distribution in $\mathbb{S}$. The thick delta function of degree q, denoted as $g\delta_*^{[q]}$, or as $g(\mathbf{w})\,\delta_*^{[q]}$, acts on a thick test function $\phi(\mathbf{x})$ as

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C}\left\langle g(\mathbf{w}), a_q(\mathbf{w})\right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} , \tag{4.2.4}$$

where $\phi(r\mathbf{w}) \sim \sum_{j=m}^{\infty} a_j(\mathbf{w})\, r^j$, as $r \to 0^+$, and where

$$C = \frac{2\pi^{n/2}}{\Gamma(n/2)} , \tag{4.2.5}$$

is the surface area of the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$.. If g is locally integrable function in $\mathbb{S}$, then

$$\left\langle g\delta_*^{[q]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^n)\times\mathcal{D}_*(\mathbb{R}^n)} = \frac{1}{C}\int_{\mathbb{S}} g(\mathbf{w})\, a_q(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) . \tag{4.2.6}$$

Thick deltas of order 0 are called just thick deltas, and we shall use the notation $g\delta_*$ instead of $g\delta_*^{[0]}$

Let $g \in \mathcal{D}'(\mathbb{S})$. Then

$$\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[q]}\right) = \left(\frac{\delta g}{\delta x_j} - (q+n)\, n_j g\right)\delta_*^{[q+1]}. \tag{4.2.7}$$

Here $\delta g/\delta x_j$ is the $\delta-$derivative of g, which we.have defined in Chapter 1. [**?**] The $\delta-$derivatives can be applied to functions and distributions defined only on a smooth hypersurface Σ of $\mathbb{R}^n$. Suppose now that the surface is $\mathbb{S}$, the unit sphere in $\mathbb{R}^n$. Let f be a smooth function defined in $\mathbb{S}$, that is, $f(\mathbf{w})$ is defined if $\mathbf{w} \in \mathbb{R}^n$ satisfies $|\mathbf{w}| = 1$. Observe that the expressions $\partial f/\partial x_j$ are not well defined and, likewise, if $\mathbf{w} = (w_j)_{1\le j\le n}$ the expressions $\partial f/\partial w_j$ do not make sense either; the

derivatives that are always defined and that one should consider are the $\delta f/\delta x_j$, $1 \leq j \leq n$. Let F_0 be the extension of f to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ that is homogeneous of degree 0, namely, $F_0(\mathbf{x}) = f(\mathbf{x}/r)$ where $r = |\mathbf{x}|$. Then [?]

$$\frac{\delta f}{\delta x_j} = \left. \frac{\partial F_0}{\partial x_j} \right|_{\mathbb{S}} . \tag{4.2.8}$$

Also, if we use polar coordinates, $\mathbf{x} = r\mathbf{w}$, so that $F_0(\mathbf{x}) = f(\mathbf{w})$, then $\partial F_0/\partial x_j$ is homogeneous of degree -1, and actually

$$\frac{\partial F_0}{\partial x_j} = \frac{1}{r} \frac{\delta f}{\delta x_j}, \quad \text{if } \mathbf{x} \neq \mathbf{0}\,. \tag{4.2.9}$$

The matrix $\mu = (\mu_{ij})_{1 \leq i,j \leq n}$, where $\mu_{ij} = \delta n_i/\delta x_j$, plays an important role in the study of distributions on a surface Σ. If $\Sigma = \mathbb{S}$ then

$$\mu_{ij} = \frac{\delta n_i}{\delta x_j} = \delta_{ij} - n_i n_j\,. \tag{4.2.10}$$

Observe that $\mu_{ij} = \mu_{ji}$, an identity that holds in any surface.

The differential operators $\delta f/\delta x_j$ are initially defined if f is a smooth function defined on Σ, but we can also define them when f is a distribution. We can do this if we use the fact that smooth functions are dense in the space of distributions on Σ.

4.3 The thick distribution $\mathcal{P}f(1)$

Let us consider one of the simplest functions, namely, the function 1, defined in $\mathbb{R}^n$. Naturally this function is locally integrable, and thus it defines a regular distribution, also denoted as 1, and the ordinary derivatives and the distributional derivatives both coincide and give the value 0. On the other hand, 1 does not automatically give an element of $\mathcal{D}'_*(\mathbb{R}^n)$ since if $\phi \in \mathcal{D}_*(\mathbb{R}^n)$ the integral $\int_{\mathbb{R}^n} \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$ could be divergent, and thus we consider the *spherical* finite part[2] thick distribution

[2] If instead of removing balls of radius ε solids of other shapes are removed one obtains a different thick distribution.

$\mathcal{P}f\,(1)$ given as

$$\langle \mathcal{P}f\,(1)\,,\phi\rangle = \text{F.p.}\int_{\mathbb{R}^n}\phi\,(\mathbf{x})\;\mathrm{d}\mathbf{x} = \text{F.p.}\lim_{\varepsilon\to 0^+}\int_{|\mathbf{x}|\geq\varepsilon}\phi\,(\mathbf{x})\;\mathrm{d}\mathbf{x}\,. \tag{4.3.1}$$

The derivatives of $\mathcal{P}f\,(1)$ do not vanish, since actually we have the following formula [**?**].

Lemma 3. *In* $\mathcal{D}'_*\,(\mathbb{R}^n)$,

$$\frac{\partial^*}{\partial x_i}\,(\mathcal{P}f\,(1)) = Cn_i\delta_*^{[-n+1]}, \tag{4.3.2}$$

where C *is given by (4.2.5).*

Proof. One can find a proof of a more general statement in [**?**], but in this simpler case the proof can be written as follows,

$$\begin{aligned}\left\langle \frac{\partial^*}{\partial x_i}\,(\mathcal{P}f\,(1))\,,\phi\right\rangle &= -\left\langle \mathcal{P}f\,(1)\,,\frac{\partial\phi}{\partial x_i}\right\rangle\\ &= -\text{F.p.}\lim_{\varepsilon\to 0^+}\int_{|\mathbf{x}|\geq\varepsilon}\frac{\partial\phi}{\partial x_i}\,\mathrm{d}\mathbf{x}\\ &= \text{F.p.}\lim_{\varepsilon\to 0^+}\int_{\varepsilon\mathbb{S}^{n-1}} n_i\phi\,\mathrm{d}\sigma\,,\end{aligned}$$

so that if $\phi\in\mathcal{D}_*\,(\mathbb{R}^n)$ has the expansion $\phi\,(\mathbf{x})\sim\sum_{j=m}^{\infty}a_j\,(\mathbf{w})\,r^j$, as $\mathbf{x}\to\mathbf{0}$, then

$$\int_{\varepsilon\mathbb{S}^{n-1}} n_i\phi\,\mathrm{d}\sigma\sim\sum_{j=m}^{\infty}\left(\int_{\mathbb{S}} n_i a_j\,(\mathbf{w})\;\mathrm{d}\sigma\,(\mathbf{w})\right)\varepsilon^{n-1+j},$$

as $\varepsilon\to 0^+$. The finite part of the limit is equal to the coefficient of ε^0, thus

$$\begin{aligned}\text{F.p.}\lim_{\varepsilon\to 0}\int_{\varepsilon\mathbb{S}^{n-1}} n_i\phi\,\mathrm{d}\sigma &= \int_{\mathbb{S}} n_i a_{1-n}\,(\mathbf{w})\;\mathrm{d}\sigma\,(\mathbf{w})\\ &= \left\langle Cn_i\delta_*^{[1-n]},\phi\right\rangle\,,\end{aligned}$$

as required. $\square$

If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ is a multiplier of $\mathcal{D}_* (\mathbb{R}^n)$, then we define, in a similar way, the thick distribution $\mathcal{P}f (\psi) \in \mathcal{D}'_* (\mathbb{R}^n)$, then the definition of multiplication and of finite parts immediately implies the useful formula

$$\mathcal{P}f (\psi) = \psi \mathcal{P}f (1) , \tag{4.3.3}$$

which, by the definition of distributional derivatives, will give the thick distributional derivative of $\mathcal{P}f (\psi)$ as

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f (\psi)) = \frac{\partial \psi}{\partial x_i} \mathcal{P}f (1) + \psi \frac{\partial^*}{\partial x_i} (\mathcal{P}f (1)) ,$$

so that we obtain the ensuing formula.

Proposition 8. *If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ then*

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f (\psi)) = \mathcal{P}f \left(\frac{\partial \psi}{\partial x_i} \right) + C n_i \psi \delta_*^{[1-n]}. \tag{4.3.4}$$

Notice that, in general, the term $C n_i \psi \delta_*^{[1-n]}$ is *not* a thick delta of order $1-n$. In fact, let us simply consider the case when $\psi = r^k \psi_0 (\mathbf{x})$ is homogeneous of order $k \in \mathbb{Z}$.We know $r^k \delta_*^{[q]} = \delta_*^{[q-k]}$**??**, thus we obtain the following particular case of (4.3.4), where now the term $C n_i \psi_0 \delta_*^{[1-n-k]}$ *is* a thick delta of order $1-n-k$.

Proposition 9. *If $\psi \in \mathcal{E}_* (\mathbb{R}^n)$ is homogeneous of order $k \in \mathbb{Z}$, then*

$$\frac{\partial^*}{\partial x_i} (\mathcal{P}f (\psi)) = \mathcal{P}f \left(\frac{\partial \psi}{\partial x_i} \right) + C n_i \psi_0 \delta_*^{[1-n-k]} , \tag{4.3.5}$$

where $\psi_0 (\mathbf{x}) = |\mathbf{x}|^{-k} \psi (\mathbf{x})$.

If we now apply the projection Π onto the usual distribution space $\mathcal{D}' (\mathbb{R}^n)$, we obtain the formula for the distributional derivatives of homogeneous distributions. Observe first that if $k > -n$ then ψ is integrable at the origin, and thus ψ is a regular distribution and $\Pi (\mathcal{P}f (\psi)) = \psi$. If $k \leq -n$ then $\Pi (\mathcal{P}f (\psi)) = \mathcal{P}f (\psi)$,

since in that case the integral $\int_{\mathbb{R}^n} \psi(\mathbf{x}) \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$ would be divergent, in general, if $\phi \in \mathcal{D}(\mathbb{R}^n)$. A particularly interesting case is when $k = -n$. Suppose ψ is homogeneous of degree $-n$ and suppose

$$\int_{\mathbb{S}} \psi(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) = 0\,, \tag{4.3.6}$$

then the *principal value* of the integral

$$\text{p.v.} \int_{\mathbb{R}^n} \psi(\mathbf{x}) \phi(\mathbf{x})\, \mathrm{d}\mathbf{x} = \lim_{\varepsilon \to 0^+} \int_{|\mathbf{x}| \geq \varepsilon} \psi(\mathbf{x}) \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \tag{4.3.7}$$

actually exists for each $\phi \in \mathcal{D}(\mathbb{R}^n)$, so that $\mathcal{P}f(\psi) = \text{p.v.}(\psi)$, the principal value distribution[3]. Condition (4.3.6) holds whenever $\psi = \partial\xi/\partial x_j$ for some ξ homogeneous of order $-n+1$.The following theorem is proved by R. Estrada.

Proposition 10. *Let ψ be homogeneous of order $k \in \mathbb{Z}$ in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then, in $\mathcal{D}'(\mathbb{R}^n)$ the distributional derivative $\overline{\partial}\psi/\partial x_i$ is given as follows:*

$$\frac{\overline{\partial}\psi}{\partial x_i} = \frac{\partial\psi}{\partial x_i}\,, \qquad k > 1 - n\,, \tag{4.3.8}$$

equality of regular distributions;

$$\frac{\overline{\partial}\psi}{\partial x_i} = \text{p.v.}\left(\frac{\partial\psi}{\partial x_i}\right) + A\delta(\mathbf{x})\,, \qquad k = 1 - n\,, \tag{4.3.9}$$

where $A = \int_{\mathbb{S}} n_i\psi_0(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) = \langle n_i\psi_0, 1\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}$, while

$$\frac{\overline{\partial}\psi}{\partial x_i} = \mathcal{P}f\left(\frac{\partial\psi}{\partial x_i}\right) + D(\mathbf{x})\,, \qquad k < 1 - n\,, \tag{4.3.10}$$

where $D(\mathbf{x})$ is a homogeneous distribution of order $k-1$ concentrated at the origin and given by

$$D(\mathbf{x}) = (-1)^{-k-n+1} \sum_{j_1+\cdots+j_n = -k-n+1} \frac{\langle n_i\psi_0, \mathbf{w}^{j_1,\ldots,j_n}\rangle}{j_1!\cdots j_n!} \mathbf{D}^{j_1,\ldots,j_n}\delta(\mathbf{x})\,. \tag{4.3.11}$$

[3] Let Σ be a closed surface in $\mathbb{R}^n$ that encloses the origin. We describe Σ by an equation of the form $g(\mathbf{x}) = 1$, where $g(\mathbf{x})$ is continuous in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and homogeneous of degree 1. Then $\langle \mathcal{R}_\Sigma(\psi(\mathbf{x})), \phi(\mathbf{x})\rangle = \lim_{\varepsilon\to 0} \int_{g(\mathbf{x})\geqslant\varepsilon} \psi(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}$, defines another regularization of ψ, but in general $\mathcal{R}_\Sigma(\psi(\mathbf{x})) \neq \text{p.v.}(\psi(\mathbf{x}))$ [**?**], a fact observed by Farassat [38], who indicated its importance in numerical computations, and studied by several authors [**?**, **?**].

Proof. It follows from (4.3.5) if we observe [**?**, Prop. 4.7] that if $g \in \mathcal{D}'(\mathbb{S})$ then

$$\Pi\left(g\delta_*^{[q]}\right) = \frac{(-1)^q}{C} \sum_{j_1+\cdots+j_n=q} \frac{\langle g(\mathbf{w}), \mathbf{w}^{j_1,\ldots,j_n}\rangle}{j_1!\cdots j_n!} \mathbf{D}^{j_1,\ldots,j_n}\delta(\mathbf{x}) , \tag{4.3.12}$$

and, in particular,

$$\Pi(g\delta_*) = \frac{1}{C}\langle g(\mathbf{w}), 1\rangle \delta(\mathbf{x}) , \tag{4.3.13}$$

if $q = 0$. □

Our next task is to compute the second order thick derivatives of homogeneous distributions. Indeed, if ψ is homogeneous of degree k then we can iterate the formula (4.3.5) to obtain

$$\begin{aligned} \frac{\partial^{*2}}{\partial x_i \partial x_j}(\mathcal{P}f(\psi)) &= \frac{\partial^*}{\partial x_i}\left(\mathcal{P}f\left(\frac{\partial\psi}{\partial x_j}\right) + Cn_j\psi_0\delta_*^{[1-n-k]}\right) \\ &= \mathcal{P}f\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right) + Cn_j\xi_0\delta_*^{[2-n-k]} + \frac{\partial^*}{\partial x_i}\left(Cn_j\psi_0\delta_*^{[1-n-k]}\right) , \end{aligned} \tag{4.3.14}$$

where $\xi = \partial\psi/\partial x_j$ is homogeneous of degree $k-1$ and $\xi_0(\mathbf{x}) = |\mathbf{x}|^{1-k}\xi(\mathbf{x})$ is the associated homogeneous of degree 0 function. Use of (4.2.7) and (4.2.9) allows us to write

$$\begin{aligned} \frac{\partial^*}{\partial x_i}\left(Cn_j\psi_0\delta_*^{[1-n-k]}\right) &= C\left(\frac{\delta}{\delta x_i}(n_j\psi_0) + (k-1)n_in_j\psi_0\right)\delta_*^{[2-n-k]} \\ &= C\left((\delta_{ij} - n_in_j)\psi_0 + n_j\frac{\delta\psi_0}{\delta x_i} + (k-1)n_in_j\psi_0\right)\delta_*^{[2-n-k]} \\ &= C\left((\delta_{ij} + (k-2)n_in_j)\psi_0 + n_j\frac{\delta\psi_0}{\delta x_j}\right)\delta_*^{[2-n-k]} , \end{aligned} \tag{4.3.15}$$

while the equation $\psi = r^k\psi_0$ yields $\partial\psi/\partial x_j = r^{k-1}\{kn_j\psi_0 + \delta\psi_0/\delta x_j\}$, so that

$$\xi_0 = kn_j\psi_0 + \frac{\delta\psi_0}{\delta x_j} . \tag{4.3.16}$$

Collecting terms we thus obtain the following formula.

Proposition 11. *If $\psi \in \mathcal{E}_*(\mathbb{R}^n)$ is homogeneous of order $k \in \mathbb{Z}$, then*

$$\frac{\partial^{*2}}{\partial x_i \partial x_j}\left(\mathcal{P}f\left(\psi\right)\right) = \mathcal{P}f\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right) \tag{4.3.17}$$
$$+ C\left(\left(\delta_{ij} + 2\left(k-1\right)n_i n_j\right)\psi_0 + n_j \frac{\delta \psi_0}{\delta x_j} + n_i \frac{\delta \psi_0}{\delta x_j}\right)\delta_*^{[2-n-k]} .$$

where $\psi_0\left(\mathbf{x}\right) = \left|\mathbf{x}\right|^{-k}\psi\left(\mathbf{x}\right)$.

We would like to observe that while ψ_0 has been supposed smooth, a continuity argument immediately gives that ψ_0 could be any distribution of $\mathcal{D}'\left(\mathbb{R}^n \setminus \{\mathbf{0}\}\right)$ that is homogeneous of degree 0.

4.4 Bowen's formula

If we apply formula (4.3.9) to the function $\psi = n_{j_1}\cdots n_{j_k}/r^2$, which is homogeneous of degree -2 in $\mathbb{R}^3$ we obtain at once that

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}\cdots n_{j_k}}{r^2}\right) = \tag{4.4.1}$$
$$\text{p.v.}\left(\left\{\sum_{q=1}^{k}\delta_{ij_q}\frac{n_{j_1}\cdots n_{j_k}}{n_{j_q}} - (k+2)\,n_i n_{j_1}\cdots n_{j_k}\right\}\frac{1}{r^3}\right) + A\delta\left(\mathbf{x}\right)\ ,$$

where

$$A = \int_{\mathbb{S}} n_i n_{j_1}\cdots n_{j_k}\,\mathrm{d}\sigma\left(\mathbf{w}\right)\ . \tag{4.4.2}$$

This integral was computed in [30, (3.13)], the result being

$$A = \frac{2\Gamma\left(\left(a+1\right)/2\right)\Gamma\left(\left(b+1\right)/2\right)\Gamma\left(\left(c+1\right)/2\right)}{\Gamma\left(\left(a+b+c+3\right)/2\right)}\ , \tag{4.4.3}$$

if $n_i n_{j_1}\cdots n_{j_k} = n_1^a n_2^b n_3^c$, and a, b, or c are even, while $A = 0$ if any exponent is odd. Bowen [15, Eqn. (A5)] also computes the integral, and obtains a different but equivalent expression; in particular, his formula for $k = 3$ reads as

$$A = \frac{4\pi}{15}\left(\delta_{ij_1}\delta_{j_2 j_3} + \delta_{ij_2}\delta_{j_1 j_3} + \delta_{ij_3}\delta_{j_1 j_2}\right)\ , \tag{4.4.4}$$

so that (4.4.3) or (4.4.4) would yield that if (a, b, c) is a permutation of $(2, 2, 0)$ then $A = 4\pi/15$ while if a permutation of $(4, 0, 0)$ then $A = 4\pi/5$.

Our main aim is to point out why the product rule for derivatives, as employed in [15] does not produce the correct result. Indeed, if we use [15, Eqn. (16)] written as[4]

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}}{r^2}\right) = \text{p.v.}\left(\frac{\delta_{ij_1} - 3n_i n_{j_i}}{r^3}\right) + \frac{4\pi}{3}\delta_{ij_1}\delta\left(\mathbf{x}\right) , \tag{4.4.5}$$

and then try to proceed as in [15, Eqn. (18)],

$$\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right) \text{“?} = \text{?”}\ n_{j_1}n_{j_2}\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right) + \frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right) . \tag{4.4.6}$$

Thus (4.4.5) and the formula

$$\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right) = \frac{\delta_{ij_1}n_{j_2} + \delta_{ij_2}n_{j_1} - 2n_i n_{j_1}n_{j_2}}{r} , \tag{4.4.7}$$

give

$$n_{j_1}n_{j_2}\frac{\overline{\partial}}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right) + \frac{n_{j_3}}{r^2}\frac{\overline{\partial}}{\partial x_i}\left(n_{j_1}n_{j_2}\right) = \text{“Normal”} + \text{“Src”} , \tag{4.4.8}$$

where

$$\text{“Normal”} = \text{p.v.}\left(\frac{\delta_{ij_1}n_{j_2}n_{j_3} + \delta_{ij_2}n_{j_1}n_{j_3} + \delta_{ij_3}n_{j_1}n_{j_2} - 5n_i n_{j_1}n_{j_2}n_{j_3}}{r^3}\right) , \tag{4.4.9}$$

coincides with the first term of (4.4.1) while

$$\text{“Src”} = \frac{4\pi}{3}\delta_{ij_3}n_{j_1}n_{j_2}\delta\left(\mathbf{x}\right) . \tag{4.4.10}$$

The right hand side of (4.4.10) is not a well defined distribution, of course, but Bowen suggested that we treat it as what we now call the projection of a thick distribution, that is, as

$$\text{“Src”} = \Pi\left(\frac{4\pi}{3}\delta_{ij_3}n_{j_1}n_{j_2}\delta_*\right) = \frac{4\pi}{9}\delta_{ij_3}\delta_{j_1j_2}\delta\left(\mathbf{x}\right) , \tag{4.4.11}$$

since $\Pi\left(n_{j_1}n_{j_2}\delta_*\right) = (1/3)\,\delta_{j_1j_2}\delta\left(\mathbf{x}\right)$ [**?**, Example 5.10]. In order to compare with (4.4.1) and (4.4.4) we observe that by symmetry the same result would be obtained

[4] We shall employ our notation, not the original one of [15].

if j_3 and j_1, or j_3 and j_2, are exchanged, so that if in the term "Src" we do these exchanges, add the results and divide by 3, we would get

$$\text{"SrcSym"} = \frac{4\pi}{27}\left(\delta_{ij_1}\delta_{j_2j_3} + \delta_{ij_2}\delta_{j_1j_3} + \delta_{ij_3}\delta_{j_1j_2}\right)\delta(\mathbf{x}) \ , \tag{4.4.12}$$

and thus the symmetric version of the (4.4.8) is "Normal"+"SrcSym", which of course is different from (4.4.1) since the coefficient in (4.4.4) is $4\pi/15$, while that in (4.4.12) is $4\pi/27$. Therefore, the relation "¿=?" in (4.4.6) cannot be replaced by $=$.

Hence the product rule for derivatives fails in this case. *The question is why?* Indeed, when computing the right side of (4.4.6), that is, the left side of (4.4.8), we found just one irregular product, namely $n_{j_1}n_{j_2}\delta(\mathbf{x})$, but using the average value $(1/3)\,\delta_{j_1j_2}\delta(\mathbf{x})$ seems quite reasonable.

In order to see what went wrong let us compute $\overline{\partial}/\partial x_i\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$ by computing the thick derivative $\partial^*/\partial x_i\mathcal{P}f\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$, applying the product rule for thick derivatives, and then taking the projection π of this. We have,

$$\begin{aligned}\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_1}n_{j_2}n_{j_3}}{r^2}\right) &= \frac{\partial^*}{\partial x_i}\left[n_{j_1}n_{j_2}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right)\right] \\ &= n_{j_1}n_{j_2}\frac{\partial^*}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right) + \frac{\partial\left(n_{j_1}n_{j_2}\right)}{\partial x_i}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right) \ ,\end{aligned}$$

and taking (4.3.5) into account, we obtain

$$\begin{aligned} n_{j_1}n_{j_2}&\left\{\mathcal{P}f\left(\frac{\delta_{ij_3} - 3n_in_{j_3}}{r^3}\right) + 4\pi n_{j_3}n_i\delta_*\right\} \\ &+ \frac{\delta_{ij_1}n_{j_2} + \delta_{ij_2}n_{j_1} - 2n_in_{j_1}n_{j_2}}{r}\mathcal{P}f\left(\frac{n_{j_3}}{r^2}\right) \ ,\end{aligned}$$

that is, $\partial^*/\partial x_i\mathcal{P}f\left(n_{j_1}n_{j_2}n_{j_3}/r^2\right)$ equals

$$\begin{aligned}&\mathcal{P}f\left(\frac{\delta_{ij_1}n_{j_2}n_{j_3} + \delta_{ij_2}n_{j_1}n_{j_3} + \delta_{ij_3}n_{j_1}n_{j_2} - 5n_in_{j_1}n_{j_2}n_{j_3}}{r^3}\right) \\ &\qquad + 4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_* \, .\end{aligned} \tag{4.4.13}$$

Applying the projection operator Π we obtain that the $\mathcal{P}f$ becomes a p.v., so that the term "Normal" given by (4.4.9) is obtained, while (4.3.13) yields that the projection of thick delta is exactly $A\delta(\mathbf{x})$ where $A = \int_{\mathbb{S}} n_i n_{j_1} n_{j_2} n_{j_3} \, d\sigma(\mathbf{w})$, that is, the *correct* term

$$\frac{4\pi}{15}\left(\delta_{ij_1}\delta_{j_2 j_3} + \delta_{ij_2}\delta_{j_1 j_3} + \delta_{ij_3}\delta_{j_1 j_2}\right)\delta(\mathbf{x}) \ .$$

The reason we now obtain the correct result is while it is true that $\Pi(n_{j_1}n_{j_2}\delta_*) = (1/3)\,\delta_{j_1 j_2}\delta(\mathbf{x})$ and that $\Pi(n_{j_3}n_i\delta_*) = (1/3)\,\delta_{ij_3}\delta(\mathbf{x})$, it is *not* true that $\Pi(4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_*)$ can be obtained as $4\pi(1/3)\,\delta_{ij_3}\Pi(n_{j_1}n_{j_2}\delta_*)$ nor as $4\pi(1/3)\,\delta_{j_1 j_2}\Pi(n_{j_3}n_i\delta_*)$, and actually not even the symmetrization of such results, given by (4.4.12), works. Put in simple terms, it is not true that the average of a product is the product of the averages!

One can, alternatively, compute $\partial^*/\partial x_i \mathcal{P}f(n_{j_1}n_{j_2}n_{j_3}/r^2)$ as

$$\frac{\partial}{\partial x_i}\left(\frac{n_{j_3}}{r^2}\right)\mathcal{P}f(n_{j_1}n_{j_2}) + \left(\frac{n_{j_3}}{r^2}\right)\frac{\partial^*}{\partial x_i}\mathcal{P}f(n_{j_1}n_{j_2}) \ , \tag{4.4.14}$$

since

$$\frac{\partial^*}{\partial x_i}\mathcal{P}f(n_{j_1}n_{j_2}) = \mathcal{P}f\left(\frac{\delta_{ij_1}n_{j_2} + \delta_{ij_2}n_{j_1} - 2n_i n_{j_1}n_{j_2}}{r}\right) + 4\pi n_{j_1}n_{j_2}n_i\delta_*^{[-2]} \,. \tag{4.4.15}$$

Here the thick delta term in (4.4.14) is $4\pi(n_{j_3}/r^2)\,n_{j_1}n_{j_2}n_i\delta_*^{[-2]}$, which becomes, as it should, $4\pi n_{j_1}n_{j_2}n_{j_3}n_i\delta_*$.

Complications in the use of the product rule for derivatives in one variable were considered in [35] when analysing the formula [83]

$$\frac{d}{dx}\left(H^n(x)\right) = nH^{n-1}(x)\,\delta(x) \ , \tag{4.4.16}$$

where H is the Heaviside function; see also [69].

4.5 Higher order derivatives

We now consider the computation of higher order derivatives in the space $\left(\mathcal{D}_*^{[0]}(\mathbb{R}^n)\right)'$. If $f \in \mathcal{D}'_*(\mathbb{R}^n)$ then, of course, the thick derivative $\partial^* f/\partial x_i$ is defined by duality, that is,

$$\left\langle \frac{\partial^* f}{\partial x_i}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_i} \right\rangle , \tag{4.5.1}$$

for $\phi \in \mathcal{D}_*(\mathbb{R}^n)$. Suppose now that $\mathcal{A}$ is a subspace of $\mathcal{D}_*(\mathbb{R}^n)$ that has a topology such that the imbedding $i : \mathcal{A} \hookrightarrow \mathcal{D}_*(\mathbb{R}^n)$ is continuous; then the transpose $i^T : \mathcal{D}'_*(\mathbb{R}^n) \to \mathcal{A}'$ is just the restriction operator $\Pi_{\mathcal{A}}$. If $\mathcal{A}$ is closed under the differentiation operators[5], then we can also define the derivative of any $f \in \mathcal{A}'$, say $\partial_{\mathcal{A}} f/\partial x_i$, by employing (4.5.1) for $\phi \in \mathcal{A}$. Then

$$\Pi_{\mathcal{A}}\left(\frac{\partial^* f}{\partial x_i}\right) = \frac{\partial_{\mathcal{A}}}{\partial x_i}\left(\Pi_{\mathcal{A}}(f)\right) , \tag{4.5.2}$$

for any thick distribution $f \in \mathcal{D}'_*(\mathbb{R}^n)$. In the particular case when $\mathcal{A} = \mathcal{D}(\mathbb{R}^n)$ then $\partial_{\mathcal{A}} f/\partial x_i = \overline{\partial} f/\partial x_i$, the usual distributional derivative, and thus (4.5.2) becomes [**?**, Eqn. (5.22)],

$$\Pi\left(\frac{\partial^* f}{\partial x_i}\right) = \frac{\overline{\partial}\Pi(f)}{\partial x_i} . \tag{4.5.3}$$

What this means is that one can use thick distributional derivatives to compute $\partial_{\mathcal{A}} f/\partial x_i$, as we have already done to compute distributional derivatives.

When $\mathcal{A}$ is not closed under the differentiation operators then $\partial_{\mathcal{A}} f/\partial x_i$ cannot be defined by (4.5.1) if $f \in \mathcal{A}'$ since in general $\partial\phi/\partial x_i$ does not belong to $\mathcal{A}$ and thus the right side of (4.5.1) is not defined. However, if $f \in \mathcal{A}'$ has a *canonical* extension $\widetilde{f} \in \mathcal{D}'_*(\mathbb{R}^n)$ then we could define $\partial_{\mathcal{A}} f/\partial x_i$ as $\Pi_{\mathcal{A}}\left(\partial^* \widetilde{f}/\partial x_i\right)$. This applies, in particular when $\mathcal{A} = \mathcal{D}_*^{[0]}(\mathbb{R}^n)$: if $f \in \left(\mathcal{D}_*^{[0]}(\mathbb{R}^n)\right)'$ then $\partial_0^* f/\partial x_i = \partial_{\mathcal{A}} f/\partial x_i$ cannot be defined, in general, but if f has a canonical extension $\widetilde{f} \in \mathcal{D}'_*(\mathbb{R}^n)$ then $\partial_0^* f/\partial x_i$ is understood as $\Pi_{\mathcal{D}_*^{[0]}(\mathbb{R}^n)}\left(\partial^* \widetilde{f}/\partial x_i\right)$.

[5]The space $\mathcal{A}'$ would be a space of (thick) distributions in the sense of Zemanian [91].

Our aim is to point out that, in general, if $P = RS$ is the product of two differential operators with constant coefficients, then while, with obvious notations, $P^* = R^* S^*$, $P_{\mathcal{A}} = R_{\mathcal{A}} S_{\mathcal{A}}$, if $\mathcal{A}$ is closed under differential operators, and $\overline{P} = \overline{R}\,\overline{S}$, it is *not* true that $P_0^* = R_0^* S_0^*$. Therefore the space $\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^n\right)\right)'$ is not a convenient framework to generalize distributions to thick distributions; the whole $\mathcal{D}'_*\left(\mathbb{R}^n\right)$ is needed if we want a theory that includes the possibility of differentiation.

Example 16. *Let us consider the second order derivatives of the distribution* $\mathcal{P}f\left(1\right)$. *Formula (4.3.17) yields*

$$\frac{\partial^{*2}}{\partial x_i \partial x_j}\left(\mathcal{P}f\left(1\right)\right) = C\left(\delta_{ij} - 2n_i n_j\right)\delta_*^{[-n+2]}. \tag{4.5.4}$$

In particular, in $\mathbb{R}^2$, $\partial^{*2}/\partial x_i \partial x_j\left(\mathcal{P}f\left(1\right)\right) = 2\pi\left(\delta_{ij} - 2n_i n_j\right)\delta_*$. *If we consider the function* 1 *as an element of* $\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^2\right)\right)'$ *then it has the canonical extension* $\mathcal{P}f\left(1\right) \in \mathcal{D}'_*\left(\mathbb{R}^2\right)$ *and so*

$$\frac{\partial_0^*\left(1\right)}{\partial x_j} = \Pi_{\mathcal{D}_*^{[0]}\left(\mathbb{R}^2\right)}\left(2\pi n_j \delta_*^{[-1]}\right) = 0\,,$$

and consequently,

$$\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*\left(1\right)}{\partial x_j}\right) = \frac{\partial_0^*}{\partial x_i}\left(0\right) = 0 \neq 2\pi\left(\delta_{ij} - 2n_i n_j\right)\delta_* = \frac{\partial^{*2}\left(1\right)}{\partial x_i \partial x_j}\,. \tag{4.5.5}$$

Observe that $\Pi\left(2\pi\left(\delta_{ij} - 2n_i n_j\right)\delta_*\right) = 0$, *but observe also that this means very little.*

Example 17. *It was obtained in [89, Thm. 7.6] that in* $\mathcal{D}'_*\left(\mathbb{R}^3\right)$

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i \partial x_j} = \left(3x_i x_j - \delta_{ij} r^2\right)\mathcal{P}f\left(r^{-5}\right) + 4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*\,. \tag{4.5.6}$$

Since $\Pi\left(n_i n_j \delta_*\right) = \left(1/3\right)\delta_{ij}\delta\left(\mathbf{x}\right)$ *in* $\mathbb{R}^3$, *this yields the well known formula of Frahm [39]*

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \text{p.v.}\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta\left(\mathbf{x}\right)\,. \tag{4.5.7}$$

We also immediately obtain that

$$\frac{\partial_0^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i \partial x_j} = \mathcal{P}f\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) + 4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*\,, \tag{4.5.8}$$

a formula that can also be proved by other methods [87]. On the other hand, in [40] one can find the computation of

$$\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*}{\partial x_j}\left(\frac{1}{r}\right)\right) = \mathcal{P}f\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) - 4\pi n_i n_j \delta_*\,. \tag{4.5.9}$$

The fact that $\frac{\partial_0^*}{\partial x_i}\left(\frac{\partial_0^*}{\partial x_j}\right) \neq \frac{\partial_0^{*2}}{\partial x_i \partial x_j}$ *is obvious in the Example 16, but it is harder to see it in cases like this one*[6]*. Observe that the projection of both* $4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*$ *and of* $-4\pi n_i n_j \delta_*$ *onto* $\mathcal{D}'\left(\mathbb{R}^3\right)$ *is given by* $-\left(4\pi/3\right)\delta_{ij}\delta\left(\mathbf{x}\right)$*, but this does not mean that they are equal; observe also that one needs the finite part in (4.5.8) and in (4.5.9) since the principal value, as used in (4.5.7), exists in* $\mathcal{D}'\left(\mathbb{R}^3\right)$ *but not in* $\left(\mathcal{D}_*^{[0]}\left(\mathbb{R}^3\right)\right)'$.

[6] That the two results are different is overlooked in [40].

Chapter 5

Dual Space Of The Space Of Regulated Functions

5.1 Introduction

In this chapter, we will discuss the space of regulated functions in several variable. There is a strong relation to the space of thick distributions.

5.2 Spaces in Several Variables

We first study spaces of regulated functions in several variables [22].

Let $\mathcal{X}$ be a topological vector space of functions defined in the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$. For example, $\mathcal{X}$ could be $C(\mathbb{S})$, the continuous functions on $\mathbb{S}$, or the Lebesgue spaces $L^p(\mathbb{S})$, or $\mathcal{B}_s(\mathbb{S})$, the space of bounded functions in $\mathbb{S}$ with the topology of uniform convergence, or $\mathcal{B}_w(\mathbb{S})$, the space of bounded functions in $\mathbb{S}$ with the topology of pointwise convergence.

Definition 14. *Let ϕ be a function defined in a region U of $\mathbb{R}^n$. We say that ϕ is $\mathcal{X}-$regulated if for each $\mathbf{x}_0 \in U$ there exists $r(\mathbf{x}_0) > 0$, such that if $0 < \varepsilon < r(\mathbf{x}_0)$, then the function $\phi(\mathbf{x}_0 + \varepsilon\mathbf{w})$, $\mathbf{w} \in \mathbb{S}$, belongs to $\mathcal{X}$, and the limit*

$$\phi_{\mathbf{x}_0}(\mathbf{w}) = \lim_{\varepsilon \to 0^+} \phi(\mathbf{x}_0 + \varepsilon\mathbf{w}) \ , \tag{5.2.1}$$

exists in $\mathcal{X}$.

If the function $\phi_{\mathbf{x}_0}(\mathbf{w})$, $\mathbf{w} \in \mathbb{S}$, is not a constant, then we call it the *thick limit* of f at the point $\mathbf{x}_0$. Otherwise if $\phi_{\mathbf{x}_0}(\mathbf{w}) \equiv \phi_{\mathbf{x}_0}$,a constant, then we call it an *ordinary limit* at the point $\mathbf{x}_0$; if $\phi(\mathbf{x}_0) = \phi_{\mathbf{x}_0}$ then ϕ would be continuous, in some sense that depends on $\mathcal{X}$, at the point $\mathbf{x}_0$ and naturally, in many cases, one can redefine the function ϕ at $\mathbf{x}_0$ to make it continuous at $\mathbf{x}_0$ if the ordinary limit exists.

Observe that when $n = 1$ then $\mathbb{S} = \{-1, 1\}$, so that the space of all functions defined in $\mathbb{S}$ is basically $\mathbb{R}^{\mathbb{S}} \simeq \mathbb{R}^2$. Since all Haussdorf vector topologies in $\mathbb{R}^2$ are equivalent, there is just one space $\mathcal{X}$ in this case, and for this space, $\mathcal{X}-$regulated means regulated, namely, it means that both limits from the right and the left exist at each point.

We shall consider $\mathcal{X}-$regulated functions where $\mathcal{X} = \mathcal{B}_s(\mathbb{S})$, the space of bounded functions in $\mathbb{S}$ with the supremum norm.

Definition 15. *Let U be an open bounded set in $\mathbb{R}^n$. The space of regulated functions $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ is the space of all $\mathcal{B}_s(\mathbb{S})-$regulated functions defined in all $\mathbb{R}^n$, that vanish in $\mathbb{R}^n \setminus \overline{U}$, and such that the thick limits $\phi_{\mathbf{x}_0}$ are continuous functions in $\mathbb{S}$ for each $\mathbf{x}_0 \in \mathbb{R}^n$.*

Observe that if $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$ then for a given ε the function $\phi(\mathbf{x}_0 + \varepsilon\mathbf{w})$ may or may not be continuous, but the limit of such functions as $\varepsilon \to 0$, $\phi_{\mathbf{x}_0}$ must be, that is, $\phi_{\mathbf{x}_0} \in C(\mathbb{S})$. It is known that if the thick limits are continuous, then the set of points where $\phi_{\mathbf{x}_0}$ is not constant is countable at the most [22].

Definition 16. *A regulated function $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$ is normalized if*

$$\phi(\mathbf{x}_0) = \frac{1}{c_n} \int_{\mathbb{S}} \phi_{\mathbf{x}_0}(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}), \tag{5.2.2}$$

for each $\mathbf{x}_0 \in \mathbb{R}^n$, where $c_n = \int_{\mathbb{S}} \mathrm{d}\sigma(\mathbf{w})$ is the $(n-1)-$measure of the unit sphere $\mathbb{S}$. The space $\mathcal{R}[\overline{U}]$ is the subspace of $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ formed by the normalized regulated functions.

We shall now establish that $\mathcal{R}[\overline{U}]$ and $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ are Banach spaces with the supremum norm.

Proposition 12. *Any function $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$ is bounded in $\mathbb{R}^n$.*

Proof. If ϕ were not bounded, we would be able to find a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of points where $|\phi(\mathbf{x}_n)| \geq n$. Clearly all the points $\mathbf{x}_n$ belong to the compact space $\overline{U}$, and thus the sequence has a convergent subsequence, which for simplicity we may assume is $\{\mathbf{x}_n\}_{n=1}^{\infty}$ itself, $\mathbf{x}_n \to \mathbf{x}^*$. Hence for any $M, r > 0$, $\exists$ $\mathbf{x}_m$, *s.t.*, $0 < |\mathbf{x}_m - \mathbf{x}^*| < r$ and $|\phi(\mathbf{x}_m)| > M$.

However, ϕ is $\mathcal{B}_s(\mathbb{S})$ $-$regulated at $\mathbf{x}^*$, and thus there exists $r_0 > 0$ such that

$$\max_{\mathbf{w}\in\mathbb{S}} |\phi_{\mathbf{x}^*}(\mathbf{w}) - \phi(\mathbf{x}^* + \varepsilon\mathbf{w})| \leq 1\,, \tag{5.2.3}$$

for $0 < \varepsilon < r_0$. But (5.2.3) implies that $|\phi(\mathbf{x})| \leq \|\phi_{\mathbf{x}^*}\|_{\sup} + 1$ for $0 < |\mathbf{x} - \mathbf{x}^*| < r_0$. □

It is interesting that if $\mathcal{X} \neq \mathcal{B}_s(\mathbb{S})$ then $\mathcal{X}-$regulated functions do not have to be bounded. For example, if $p < \infty$, there are functions that is $L^p(\mathbb{S})$ $-$regulated, but unbounded in each neighborhood of the origin.

It follows from the Proposition 12 that $\mathcal{R}_{\mathrm{t}}[\overline{U}] \subset \mathcal{B}_s(\mathbb{R}^n)$. Therefore with the supremum norm $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ becomes a normed space.

Proposition 13. *$\mathcal{R}_{\mathrm{t}}[\overline{U}]$ is closed in $\mathcal{B}_s(\mathbb{R}^n)$, and hence a Banach space.*

Proof. Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ that converges uniformly in $\mathbb{R}^n$ to $\phi \in \mathcal{B}_s(\mathbb{R}^n)$; we need to show that $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$. Since it is clear that ϕ vanishes outside of $\overline{U}$, what we need to show is that ϕ is $\mathcal{B}_s(\mathbb{S})$ $-$regulated and that $\phi_{\mathbf{x}_0} \in C(\mathbb{S})$ for each $\mathbf{x}_0 \in \mathbb{R}^n$.

Fix $\mathbf{x}_0 \in \mathbb{R}^n$. Let $\eta > 0$. There exists $n_0 \in \mathbb{N}$ such that $\|\phi - \phi_n\|_{\sup} < \eta/3$ for $n \geq n_0$. There exists $r_0 > 0$ such that if $0 < \varepsilon_1, \varepsilon_2 < r_0$ then $|\phi_{n_0}(\mathbf{x}_0 + \varepsilon_1\mathbf{w}) - \phi_{n_0}(\mathbf{x}_0 + \varepsilon_2\mathbf{w})| < \eta/3$. Hence

$$\limsup_{\varepsilon_1,\varepsilon_2\to 0^+} \left(\max_{\mathbf{w}\in\mathbb{S}} |\phi(\mathbf{x}_0 + \varepsilon_1\mathbf{w}) - \phi(\mathbf{x}_0 + \varepsilon_2\mathbf{w})| \right) \leq \eta\,, \tag{5.2.4}$$

and it follows that $\phi_{\mathbf{x}_0}$, the limit of $\phi(\mathbf{x}_0 + \varepsilon\mathbf{w})$ as $\varepsilon \to 0^+$, exists in $\mathcal{B}_s(\mathbb{S})$. It is easy to see that $(\phi_n)_{\mathbf{x}_0}$ converges to $\phi_{\mathbf{x}_0}$ in $\mathcal{B}_s(\mathbb{S})$, and the continuity of the $(\phi_n)_{\mathbf{x}_0}$ yields that $\phi_{\mathbf{x}_0} \in C(\mathbb{S})$. □

We can define a normalization operation in $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ as follows.

Definition 17. *Let $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$. The function $\psi = N(\phi)$ is the normalization of ϕ, given as*

$$\psi(\mathbf{x}) = \frac{1}{c_n}\int_{\mathbb{S}} \phi_{\mathbf{x}}(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{5.2.5}$$

The normalization of a regulated function of the space $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ is also regulated, and in fact the normalized function coincides with the original function save on a countable set.

Proposition 14. *If $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$ then $\psi = N(\phi)$ is $\mathcal{B}_s(\mathbb{S})-$regulated, and actually $\psi_{\mathbf{x}} = \phi_{\mathbf{x}}$ at each $\mathbf{x} \in \mathbb{R}^n$. The operator N satisfies $N^2 = N$ and is a projection from $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ to $\mathcal{R}[\overline{U}]$.*

Proof. Fix $\mathbf{x}_0 \in \mathbb{R}^n$. Let $\eta > 0$. There exists $\delta > 0$ such that if $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{S}$ satisfy $|\mathbf{w}_1 - \mathbf{w}_2| < \delta$, then $|\phi_{\mathbf{x}_0}(\mathbf{w}_1) - \phi_{\mathbf{x}_0}(\mathbf{w}_2)| < \eta/2$, and there exists r_0 such that if $0 < \varepsilon < r_0$ then $|\phi_{\mathbf{x}_0}(\mathbf{w}) - \phi(\mathbf{x}_0 + \varepsilon\mathbf{w})| < \eta/2$ for all $\mathbf{w} \in \mathbb{S}$. Hence

$$|\phi_{\mathbf{x}_0}(\mathbf{w}) - \phi(\mathbf{x}_0 + \varepsilon\mathbf{v})| < \eta \qquad \text{if } |\mathbf{w} - \mathbf{v}| < \delta \text{ and } 0 < \varepsilon < r_0\,. \tag{5.2.6}$$

It follows that if $\rho > 0$ is small enough, then for any $\mathbf{w} \in \mathbb{S}$,

$$\left|\phi_{\mathbf{x}_0}(\mathbf{w}) - \frac{1}{c_n}\int_{\mathbb{S}} \phi(\mathbf{x}_0 + \varepsilon\mathbf{w} + \rho\mathbf{v})\, \mathrm{d}\sigma(\mathbf{v})\right| < \eta\,, \tag{5.2.7}$$

and since $\int_{\mathbb{S}} \phi(\mathbf{x} + \rho\mathbf{v})\, \mathrm{d}\sigma(\mathbf{v}) \to c_n\psi(\mathbf{x})$ as $\rho \to 0^+$ for any $\mathbf{x}$,

$$|\phi_{\mathbf{x}_0}(\mathbf{w}) - \psi(\mathbf{x}_0 + \varepsilon\mathbf{w})| \leq \eta \qquad \text{for } 0 < \varepsilon < r_0\,.$$

We thus obtain that ψ is $\mathcal{B}_s(\mathbb{S})-$regulated and $\psi_{\mathbf{x}_0} = \phi_{\mathbf{x}_0}$.

The fact that N is a projection from $\mathcal{R}_{\mathrm{t}}[\overline{U}]$ to $\mathcal{R}[\overline{U}]$ that satisfies $N^2 = N$ is now clear. $\square$

The Lemma 14 allows to write

$$\mathcal{R}_{\mathrm{t}}[\overline{U}] = \mathcal{R}[\overline{U}] \oplus \mathcal{R}_{\mathrm{n}}[\overline{U}] \,, \tag{5.2.8}$$

where $\mathcal{R}_{\mathrm{n}}[\overline{U}] = \operatorname{Ker} N$. A function $\phi \in \mathcal{R}_{\mathrm{t}}[\overline{U}]$ belongs to $\mathcal{R}_{\mathrm{n}}[\overline{U}]$ if and only if $\phi_{\mathbf{x}} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. It is not hard to see that $\phi \in \mathcal{R}_{\mathrm{n}}\left[\overline{U}\right]$ if and only if there exists a sequence of different points of U, $\{\mathbf{x}_n\}_{n=1}^{\infty}$, such that $\phi(\mathbf{x}) = 0$ if $\mathbf{x} \neq \mathbf{x}_n$ for all n, and $\lim_{n\to\infty} \phi(\mathbf{x}_n) = 0$.

If $\pi : \mathcal{B}_{\mathrm{meas}}[\overline{U}] \to L^{\infty}[\overline{U}]$ is the projection, then we have that $\pi\left(\mathcal{R}_{\mathrm{n}}[\overline{U}]\right) = \{0\}$, and thus

$$\pi\left(\mathcal{R}_{\mathrm{t}}[\overline{U}]\right) = \pi\left(\mathcal{R}[\overline{U}]\right) \,. \tag{5.2.9}$$

Therefore, the two Banach spaces $\mathcal{R}[\overline{U}]$ and the subspace $\pi\left(\mathcal{R}_{\mathrm{t}}[\overline{U}]\right)$ of $L^{\infty}[\overline{U}]$ are isometric.

5.3 The dual spaces $\mathcal{R}_t'[\overline{U}]$ and $\mathcal{R}'[\overline{U}]$

We shall now describe the dual spaces $\mathcal{R}_t'[\overline{U}]$.and $\mathcal{R}'[\overline{U}]$. If μ belongs to the dual space of $\mathcal{R}[\overline{U}]$ or of $\mathcal{R}_{\mathrm{t}}[\overline{U}]$, we shall denote the evaluation of μ on a regulated function as $\langle \mu, \phi \rangle$, or as $\langle \mu(\mathbf{x}), \phi(\mathbf{x}) \rangle$ when we want to clearly indicate the variable of evaluation. We will show that the elements of this dual space $\mathcal{R}'[\overline{U}]$ are measures with a sum of "thick delta functions". In what follows let us denote by $\mathcal{M}(X) = (C(X))'$ the space of Radon measures on a compact space X.

First let us consider $\mathcal{R}_t\left([\overline{U}];F\right)$, the set of regulated functions that are continuous in $\overline{U}\backslash F$,where F is a finite set. of points. Respectively, we denote $\mathcal{R}\left([\overline{U}];F\right)$ as normalized regulated functions.

If $F = \emptyset$, then $\mathcal{R}_t\left([\overline{U}];\emptyset\right)$ is a closed subspace of $C[\overline{U}]$.It is the space of continuous functions in $[\overline{U}]$ whose value is 0 on the boundary.

Hence, if $\mu_\emptyset \in \mathcal{R}_t\left([\overline{U}];\emptyset\right)$, then $\mu_\emptyset$ gives a Radon measure in $[\overline{U}]$, which can be written as

$$\mu_\emptyset = \mu_{\text{cont}} + \mu_{\text{discr}} \, , \tag{5.3.1}$$

$$\mu_{\text{discr}}\left(\mathbf{x}\right) = \sum_{n=1}^{\infty} \gamma_n \delta\left(\mathbf{x} - \mathbf{x}_n\right) \, , \tag{5.3.2}$$

where μ_{cont} is the continuous part, sum of an absolutely continuous part and a continuous singular part, and where μ_{discr} is the discrete part, a sum of Dirac delta functions at some points $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of $[\overline{U}]$ that satisfies $\sum_{n=1}^{\infty} |\gamma_n| < \infty$, since actually

$$\|\mu_\emptyset\| = \int_U \mathrm{d}\left|\mu_{\text{cont}}\right| + \sum_{n=1}^{\infty} |\gamma_n| \, . \tag{5.3.3}$$

Here $\|\mu_\emptyset\| = \sup\left\{\langle \mu_\emptyset, \phi\rangle : \phi \in \mathcal{R}_t\left([\overline{U}];\emptyset\right), \ \ \|\phi\|_{\sup} = 1\right\}$ is the norm of $\mu_\emptyset$ in the space $\mathcal{R}_t'\left([\overline{U}];\emptyset\right)$.

We shall now consider the spaces $\mathcal{R}_t'\left([\overline{U}];F\right)$ .when $F \neq \emptyset$. Let us introduce the idea of *thick* delta functions.

Definition 18. *Let* $\mathbf{c} \in \overline{U}$. *Let* $\xi \in \mathcal{M}\left(\mathbb{S}^{n-1}\right)$. *The formula*

$$\left\langle \xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right), \phi\left(\mathbf{x}\right)\right\rangle_{\mathcal{R}_t''[\overline{U}]\times\mathcal{R}_t[\overline{U}]} = \left\langle \xi\left(\mathbf{w}\right), \phi_{\mathbf{c}}\left(\mathbf{w}\right)\right\rangle_{\mathcal{M}(\mathbb{S})\times C(\mathbb{S})} \, , \tag{5.3.4}$$

defines an element of $\mathcal{R}_t'[\overline{U}]$ *which we call a thick delta function at* $\mathbf{x} = \mathbf{c}$, *and which we denote as* $\xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right)$ *or as* $\xi\delta_*\left(\mathbf{x}-\mathbf{c}\right)$.

Before going to our first main result in this section, let's first prove two short lemmas:

Lemma 4. *If* $f \in \mathcal{R}_t[\overline{U}]$, *let* $g : \mathbb{R} \to \mathbb{R}$ *be continuous, then* $g \circ f$ *is also a regulated function.*

Proof. This is trivial since $g \circ \phi_{\mathbf{x}_0}\left(\mathbf{w}\right)$ is continuous. □

Lemma 5. *Let* f_1, f_2 *be two regulated funcitons, then* $Max\left(f_1, f_2\right)$ *and* $Min\left(f_1, f_2\right)$ *are also regulated.*

Proof. Since $Max\left(f_1,f_2\right) = f_2+\frac{f_1-f_2+|f_1-f_2|}{2}$,a composition of addition and absolute value functions which are continuous. Similarly for $Min\left(f_1, f_2\right) = f_2+\frac{f_1-f_2-|f_1-f_2|}{2}$. □

Proposition 15. *Let* $\mu \in \mathcal{R}_t'[\overline{U}]$ *with* $\operatorname{supp}\mu = \{\mathbf{c}\}$. *Then there exists* $\xi \in \mathcal{M}\left(\mathbb{S}^{n-1}\right), \gamma \in \mathbb{R}$ *such that*

$$\mu\left(\mathbf{x}\right) = \xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right) + .\gamma\delta\left(\mathbf{x}-\mathbf{c}\right) \tag{5.3.5}$$

Proof. Let μ be a measure on $\overline{U}$, supp$\mu = \{c\}$ .Since $\mathcal{R}_{\mathrm{t}}[\overline{U}] = \mathcal{R}[\overline{U}] \oplus \mathcal{R}_{\mathrm{n}}[\overline{U}]$, μ can be decomposed as $\mu_1 + \mu_2$, where$\mu_1 \in \mathcal{R}'[\overline{U}]$ and $\mu_2 \in \mathcal{R}_{\mathrm{n}}'[\overline{U}]$.

As described above, $\varphi \in \mathcal{R}_{\mathrm{n}}\left[\overline{U}\right]$ if and only if there exists a sequence of different points of U, $\{\mathbf{x}_n\}_{n=1}^{\infty}$, such that $\varphi\left(\mathbf{x}\right) = 0$ if $\mathbf{x} \neq \mathbf{x}_n$ for all n, and $\lim_{n\to\infty}\varphi\left(\mathbf{x}_n\right) = 0$. Hence $\mu_2 = \gamma\delta\left(\mathbf{x}-\mathbf{c}\right)$ for some real number γ.

For $\mu_{1,}$ let's define a functional

$$\begin{aligned} \xi_0 : \left\{\phi_c : \phi \in \mathcal{R}_t\left[\overline{U}\right]\right\} &\to \mathbb{R} \\ \phi_c &\longmapsto \langle\mu_1, \phi\rangle \end{aligned} \tag{5.3.6}$$

Note $\left\{\phi_c : \phi \in \mathcal{R}_t\left(\overline{U}\right)\right\} := E$ is a linear subspace of $C\left(\mathbb{S}^{n-1}\right)$.

First we show that ξ_0 is well-defined. Namely, if $\phi_c = \psi_c$ for different ϕ and ψ, then $\langle\xi_0, \phi_c\rangle = \langle\mu, \phi\rangle = \langle\mu, \psi\rangle = \langle\xi_0, \psi_c\rangle$.

Now we want to show: there exists a $\xi \in \mathcal{M}\left(\mathbb{S}^{n-1}\right)$, s.t. $\xi|_E = \xi_{0.}$ Thus $\mu\left(\mathbf{x}\right) = \xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right)$.

In order to use Hahn-Banach theorem, we need to show ξ_0 is a continuous linear functional. The linearity is guarteened by the linearity of μ. So we only need to prove continuity, or rather, boundedness:

Let $\phi \in \mathcal{R}\left(\overline{U}\right)$, $\phi_c = \alpha \in E$.

For any $\varepsilon > 0$,

$\exists\ \psi' \in \mathcal{R}_t\left(\overline{U}\right),$

$$\psi\left(\mathbf{x}\right)=\begin{cases}\phi\left(\mathbf{x}\right) & \text{if } \left|\phi\left(\mathbf{x}\right)\right|\leq\left\|\alpha\right\|_{\sup}+\varepsilon\\ \left\|\alpha\right\|_{\sup}+\varepsilon & \text{if } \phi\left(\mathbf{x}\right)\geq\left\|\alpha\right\|_{\sup}+\varepsilon\\ -\left(\left\|\alpha\right\|_{\sup}+\varepsilon\right) & \text{if } \phi\left(\mathbf{x}\right)\leq-\left(\left\|\alpha\right\|_{\sup}+\varepsilon\right)\end{cases} \tag{5.3.7}$$

ψ' *is* a regulated function because $Max\left(f_1,f_2\right), Min\left(f_1,f_2\right)$ are regulated functions for two regulated functions f_1, f_2 by the above lemma.

Let $\psi = N\left(\psi'\right)$, thus $\psi_c = \phi_c$.

Moreover, $\left|\langle\xi_0,\alpha\rangle\right| \leq \left\|\mu\right\| \left\|\psi\right\|_{\sup} \leq \left\|\mu\right\| \left(\left\|\alpha\right\|_{\sup}+\varepsilon\right).$

Since ε is arbitrary,

$$\langle\xi_0,\alpha\rangle \leq \left\|\mu\right\| \left\|\alpha\right\|_{\sup} \tag{5.3.8}$$

i.e. ξ_0 is bounded.

Hence $\mu_1 = \xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right)$ for some $\xi \in \mathcal{M}\left(\mathbb{S}^{n-1}\right)$ by the Hahn-Banach Theorem. □

In fact, if the support point c is not on the boundary ∂U, ξ could be uniquely found as in the following way:

Proposition 16. *Let* $\mu_1 \in \mathcal{R}'[\overline{U}]$ *with* $\operatorname{supp}\mu_1 = \{\mathbf{c}\} \in U$. *Then there exists* $\xi \in \mathcal{M}\left(\mathbb{S}^{n-1}\right)$ *such that*

$$\mu_1\left(\mathbf{x}\right) = \xi\left(\mathbf{w}\right)\delta_{\mathbf{w}}\left(\mathbf{x}-\mathbf{c}\right). \tag{5.3.9}$$

Proof. Let $\rho \in \mathcal{R}[\overline{U}]$ be a continuous function in $\mathbb{R}^n$ that satisfies $\rho\left(\mathbf{c}\right)=1$ and $0 \leq \rho\left(\mathbf{x}\right) \leq 1$ in all $\mathbb{R}^n$. If $\alpha \in C\left(\mathbb{S}\right)$, let $A\left(\alpha\right) \in \mathcal{R}[\overline{U}]$ be the function given by

$$A\left(\alpha\right)\left(\mathbf{x}\right)=\begin{cases}\rho\left(\mathbf{x}\right)\alpha\left(\frac{\mathbf{x}-\mathbf{c}}{\left|\mathbf{x}-\mathbf{c}\right|}\right) & x\neq c\\ \alpha & x=c\end{cases}. \tag{5.3.10}$$

Then $A : C(\mathbb{S}) \to \mathcal{R}[\overline{U}]$ is a continuous linear operator of norm 1.Hence the linear functional ξ defined by $\langle \xi, \alpha \rangle = \langle \mu_1, A(\alpha) \rangle$ is continuous, $\xi \in \mathcal{M}(\mathbb{S})$.

If $\phi \in \mathcal{R}[\overline{U}]$ we can write $\phi = A(\phi_{\mathbf{c}}) + \psi$, where $\psi_{\mathbf{c}} = 0$, and it follows that

$$\langle \mu_1, \phi \rangle = \langle \mu_1, A(\phi_{\mathbf{c}}) \rangle + \langle \mu_1, \psi \rangle = \langle \mu_1, A(\phi_{\mathbf{c}}) \rangle = \langle \xi, \phi_{\mathbf{c}} \rangle \ ,$$

so that $\mu_1(\mathbf{x}) = \xi(\mathbf{w}) \delta_{\mathbf{w}}(\mathbf{x} - \mathbf{c})$. □

Similarly, if $\mu \in \mathcal{R}'_t\left([\overline{U}]; F\right)$, where $F = G \cup \{\mathbf{c}\}$, is a finite set and the restriction of μ to $\mathcal{R}_t\left([\overline{U}]; G\right)$ vanishes, then μ could be expressed as:

$$\begin{aligned} \mu(\mathbf{x}) &= \mu_1 + \mu_2 \\ &= \xi(\mathbf{w}) \delta_{\mathbf{w}}(\mathbf{x} - \mathbf{c}) + \gamma \delta(\mathbf{x} - \mathbf{c}) \end{aligned} \tag{5.3.11}$$

Where $\mu_1 \in \mathcal{R}'[\overline{U}]$ could be expressed by a thick delta function.

Now we obtain the following:

Proposition 17. *Let $\mu \in \mathcal{R}'_t\left([\overline{U}]; F\right)$, where F is a finite set of points, then*

$$\mu = \mu_{conti} + \sum_{n=1}^{\infty} \gamma_n \delta(\mathbf{x} - \mathbf{x}_n) + \sum_{c \in F} \xi_c \delta_*(\mathbf{x} - \mathbf{c}) \tag{5.3.12}$$

Where $\xi_c \in C'(\mathbb{S})$, and $\xi_c \delta_*(\mathbf{x} - \mathbf{c})$ is a thick delta as defined in 5.3.4.

As mentioned before, it is known that if the thick limits are continuous, then the set of points where $\phi_{\mathbf{x}_0}$ is not constant is countable at the most [22].Hence we can characterize $\mathcal{R}'_t\left([\overline{U}]\right)$ now.

Let us use two different section to discuss $\mathcal{R}'\left([\overline{U}]\right)$ and then $\mathcal{R}'_t\left([\overline{U}]\right)$ so things could be more clear.

5.3.1 The dual space $\mathcal{R}'[\overline{U}]$

By 16 and 5.3.1, it is clear that similar as 17, we have the following in $\mathcal{R}'\left[\overline{U}\right]$:

Proposition 18. *Let $\mu \in \mathcal{R}'\left([\overline{U}]; F\right)$, where F is a finite set of points, then*

$$\mu = \mu_{conti} + \sum_{\substack{n=1 \\ c \notin F}}^{\infty} \gamma_n \delta\left(\mathbf{x} - \mathbf{x}_n\right) + \sum_{c \in F} \xi_c \delta_*\left(\mathbf{x} - \mathbf{c}\right) \tag{5.3.13}$$

Where $\xi_c \in C'\left(\mathbb{S}\right)$, and $\xi_c \delta_*\left(\mathbf{x} - \mathbf{c}\right)$ is a thick delta as defined in 5.3.4.

Now we claim:

Proposition 19. *The union of the spaces $\mathcal{R}\left([\overline{U}]; F\right)$ is dense in $\mathcal{R}[\overline{U}]$, hence $\mathcal{R}\left([\overline{U}]; F\right)$ is dense in $\mathcal{R}[\overline{U}]$.i.e.*

$$\mathcal{R}[\overline{U}] = \overline{\cup \mathcal{R}\left([\overline{U}]; F\right)} \tag{5.3.14}$$

Proof. Suppose $\phi \in \mathcal{R}_t\left([\overline{U}]\right)$, let us find a sequence $\{\phi_n\} \subseteq \cup \mathcal{R}_t\left([\overline{U}]; F\right)$,s.t. $\phi_n \to \phi$ uniformly.

First define the "variation function":

$$v_f\left(\mathbf{x}\right) = \lim_{\varepsilon \to 0} Var\left(f; B_{\mathbf{x}}\left(\varepsilon\right)\right) \tag{5.3.15}$$

$$v_f^*\left(\mathbf{x}\right) = \lim_{\varepsilon \to 0} Var\left(f; B_{\mathbf{x}}^*\left(\varepsilon\right)\right) \tag{5.3.16}$$

Where $B_{\mathbf{x}}^*\left(\varepsilon\right)$ is a punctured ball with radius ε.

Note $0 \leq v_f^*\left(\mathbf{x}\right) \leq v_f\left(\mathbf{x}\right)$

Also note if $\phi \in \mathcal{R}_t[\overline{U}]$, then $v_\phi^*\left(\mathbf{x}\right) = Var\left(\phi_{\mathbf{x}}; \mathbb{S}^{n-1}\right)$.

Claim: If $\phi \in \mathcal{R}_t[\overline{U}]$,then $\lim_{\mathbf{z} \to \mathbf{x}} v_\phi\left(\mathbf{z}\right) = 0$ for any $\mathbf{x} \in \overline{U}$. Hence this is automatically true for $\phi \in \mathcal{R}[\overline{U}]. \subset \mathcal{R}_t[\overline{U}]$.

Proof. Let x be fixed. Let $\varepsilon > 0$, then there exists an $r_0 > 0$, s.t.

$\left|\phi_x\left(\mathbf{w}\right) - \phi\left(\mathbf{x} + r\mathbf{w}\right)\right| < \frac{\varepsilon}{3}$, $0 < r < r_0, \forall \mathbf{w} \in \mathbb{S}^{n-1}$

But ϕ_x is continuous. So $\exists\ \delta > 0, s.t.$

if $\left|\mathbf{w}_1 - \mathbf{w}_2\right| < \delta$, then $\left|\phi_x\left(\mathbf{w}_1\right) - \phi_x\left(\mathbf{w}_2\right)\right| < \varepsilon/3$.

If $\mathbf{z} \in B_{\mathbf{x}}^*\left(r_0\right)$, and $\eta > 0$ is small, then for any $\mathbf{y}_1, \mathbf{y}_2 \in B_z\left(\eta\right)$,we have

$$|\phi(\mathbf{y}_1) - \phi(\mathbf{y}_2)| \le |\phi(\mathbf{y}_1) - \phi_{\mathbf{x}}(\mathbf{w}_1)| + |\phi(\mathbf{w}_1) - \phi_x(\mathbf{w}_2)| + |\phi(\mathbf{w}_2) - \phi_{\mathbf{x}}(\mathbf{y}_2)|$$
$$< \varepsilon. \qquad \square$$

Let ε_0 be fixed, consider $E = \{\mathbf{x} : v_\phi(\mathbf{x}) \ge \varepsilon_0/3\}$ .By the above property, we know E is finite.

First consider a simpler case where no boundary point is in E. Hence for any two different points $\mathbf{x}_1, \mathbf{x}_2 \in E$, there are real numbers $r_1 < r_1'; r_2 < r_2'$, such that closed balls centered at $\mathbf{x}_{1,}\mathbf{x}_2$ with radius r_1', r_2' do not intersect. i.e. $\overline{B_{x_1}(r_1')} \cap \overline{B_{\mathbf{x}_2}(r_2')} = \emptyset$.

Moreover, it is known that there are continuous functions $f_i, i = 1, 2, s.t.$

$$f_i(x) = \begin{cases} 1 & x \in \overline{B_{\mathbf{x}_i}(r_i)} \\ 0 & x \notin \overline{B_{\mathbf{x}_i}(r_i')} \end{cases}$$

Now let

$$\phi_E(\mathbf{x}) = \begin{cases} \sum\limits_{\mathbf{x}_i \in E} f_i(\mathbf{x})\, \phi_{x_i}(\mathbf{w}) = \sum\limits_{\mathbf{x}_i \in E} f_i(\mathbf{x})\, \phi_{\mathbf{x}_i}(\mathbf{x}/r), & \text{when } \mathbf{x} \ne \mathbf{x}_i \\ \phi(\mathbf{x}) & \text{when } \mathbf{x} = \mathbf{x}_i \end{cases} \tag{5.3.17}$$

where r is the distance between points $\mathbf{x}$ and $\mathbf{x}_i$.

Clearly, $\phi_E(\mathbf{x})$ is continuous.

We can write $\psi = \phi - \phi_E$. Notice ψ has no thick limit at $\mathbf{x}_i \in E$. Thus also $v_\psi(\mathbf{x}) < \varepsilon_0/3$,where $v_\psi(\mathbf{x})$ is the variation function defined above.

Now let's approximate ψ with $g = \psi * \rho$, the convolution.

Let $\rho \in C(\mathbb{R}^n)$, be a continuous function on $\mathbb{R}^n$ s.t. $\rho(\mathbf{x}) = 0$ when $|\mathbf{x}| > \delta$. and $\int \rho(\mathbf{x})\, d\mathbf{x} = 1$.

Then $g = \psi * \rho \in C(\mathbb{R}^n)$, being a continuous funciton.

Moreover, for $\forall \mathbf{x} \in \overline{U}$,

$$\begin{aligned}
&|g(\mathbf{x}) - \psi(\mathbf{x})| \\
&= \left| \int \psi(\mathbf{x} - \mathbf{z}) \rho(\mathbf{z}) \, d\mathbf{z} - \psi(\mathbf{x}) \right| \\
&= \left| \int \psi(\mathbf{x} - \mathbf{z}) \rho(\mathbf{z}) \, d\mathbf{z} - \int \psi(\mathbf{x}) \rho(\mathbf{z}) \, d\mathbf{z} \right| \\
&\leq \int |\psi(\mathbf{x} - \mathbf{z}) - \psi(\mathbf{x})| \rho(\mathbf{z}) \, d\mathbf{z} \leq \varepsilon_0/3 \int \rho(\mathbf{z}) \, d\mathbf{z} = \varepsilon_0/3 \qquad (5.3.18)
\end{aligned}$$

We know $|\psi(\mathbf{x})| = 0$ on ∂U. Let $L = \{\mathbf{x} | \psi(\mathbf{x}) \geq \varepsilon_0/3\}$.

Take a continuous function h that is 1 on $\overline{L}$, 0 on $\mathbb{R}^n \backslash U$, thus $gh \in C\left(\overline{U}\right)$.

$|gh - g|_{\sup} = \sup\left\{\left|gh(\mathbf{x}) - g(\mathbf{x})\right| \mathbf{x} \in \mathbb{R} \backslash \overline{L}\right|\right\} \leq 2\varepsilon_0/3$.

In conclusion, let ε_0 be given, then there exists $\phi_{e_0} = \phi_E + gh \in \underset{F}{\cup} \mathcal{R}\left(\left[\overline{U}\right]; F\right)$, *s.t.*

$$|\phi - \phi_{\varepsilon_0}|_{\sup} = |\phi - \phi_E - gh|_{\sup} = |\psi - gh|_{\sup} \leq |\psi - g|_{\sup} + |g - gh|_{\sup} < \varepsilon_0. \qquad (5.3.19)$$

The only difficulty with the boundary is when we are trying to find ϕ_E as above, the domain might not be that of $\overline{U}$.

Now suppose there are n boundary points in E, denote $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$.Denotes E$\backslash\{\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n\} := E'$

Denote $\varepsilon_1 = \varepsilon_0/5$, and $\varepsilon_2 = \varepsilon_0/3$ be given. For each $\mathbf{c}_i$, consider the sets $H = \{\mathbf{w} \in \mathbb{S}^{n-1} | \phi_{c_i}(\mathbf{w}) \leq \varepsilon_1\}$ and $K = \{\mathbf{w} \in \mathbb{S}^{n-1} | \phi_{c_i}(\mathbf{w}) \geq \varepsilon_2\}$.

We know there are $\rho_{c_i} \in C(\mathbb{S}^{n-1})$, s.t. $\rho_{c_i} = \begin{cases} 1 & \text{in K} \\ 0 & \text{in H} \end{cases}$

Let $\widetilde{\phi}_{c_i} = \rho_{c_i} \phi_{c_i}$.

Also, by definition of regulated functions, there are $\delta_i > 0$, s.t. $|\phi_{c_i}(w) - \phi(c_i + rw) < \varepsilon_1|$ for $r \leq \delta_i$, any $w \in \mathbb{S}^{n-1}$. Now pick up a $\delta_i' < \delta_i$

We can find a function l_i for each $\mathbf{c}_i$, s.t.

$$l_i(x) = \begin{cases} 1 & x \in \overline{B_{c_i}(\delta_i')} \\ 0 & x \notin B_{\mathbf{c}_i}(\delta_i) \end{cases} \tag{5.3.20}$$

Now let

$$\widetilde{\phi}(\mathbf{c}_i + r\mathbf{w}) = \begin{cases} \phi(\mathbf{c}_i) & \text{when } r = 0 \\ l_i(\mathbf{x})\,\widetilde{\phi}_{\mathbf{c}_i}(\mathbf{x}/r) & \text{when } r \neq 0 \end{cases} \tag{5.3.21}$$

Thus for each $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \in B_{c_i}(\delta_i')$, if $\phi(\mathbf{x}) = 0$, then $|\phi_{c_i}(\mathbf{w}) - \phi(\mathbf{x})| < \varepsilon_1$ implies $|\phi_{c_i}(\sqsupseteq)| < \varepsilon_1$,i.e.$\mathbf{w} \in H$. Hence $\widetilde{\phi}(\mathbf{c}_i + r\mathbf{w}) = \rho_{c_i}\phi_{c_i} = 0$.

Now let $\varphi = \sum_{c_i} .\widetilde{\phi}(\mathbf{c}_i + r\mathbf{w})$.

$\phi - \varphi$ then beccomes a normalized regulated function in $\overline{U}$ with no thick limits on the boundary s.t. $v_\phi(x) \geq \varepsilon_0/3$. In fact, $v_{\phi-\varphi} < \varepsilon_2 = \varepsilon_0/3$.So we can construct $\phi_{E'}$ as in the first case. Namely, $\phi = \varphi + \phi_{E'} + \psi$, where ψ can be approximated by gh similar as above.

Hence, $\underset{F}{\cup}\mathcal{R}\left(\left[\overline{U}\right];F\right)$ is dense in $\mathcal{R}[\overline{U}]$ □

Thus each such μ has associated a family of elements $\{\mu_F\}$ of the duals of the spaces $\mathcal{R}\left(\left[\overline{U}\right];F\right)$, for F finite, obtained by restriction, and this family determines μ uniquely.

Proposition 20. *Let $\mu \in \mathcal{R}'[\overline{U}]$, then*

$$\mu = \mu_{conti} + \sum_{c \in C} \xi_c \delta_* (\mathbf{x} - \mathbf{c}) \tag{5.3.22}$$

where C is at most countable.

5.3.2 The dual space $\mathcal{R}_t'[\overline{U}]$

Theorem 7. Let $\mu \in \mathcal{R}_t'[\overline{U}]$, then

$$\mu = \mu_{conti} + \sum_{c \in C} \xi_c \delta_* (\mathbf{x} - \mathbf{c}) + \sum_{n=1}^{\infty} \gamma_n \delta(\mathbf{x} - \mathbf{x}_n) \tag{5.3.23}$$

where C is at most countable.

Proof. This theorem is directly from 17 and 20. $\square$

5.4 Spaces of Regulated Functions in One Variable

In order to consider Banach spaces of regulated functions, we need to first discuss a simple but annoying situation. Indeed, shall we consider regulated functions as functions defined at all points?, or, shall we consider their equivalence class in the almost everywhere sense? The problem is that when one uses the supremum norm then functions equal to zero a.e. do not have null length, as is the case in the spaces L^p for $p < \infty$.

Let $\mathcal{B}_{\mathrm{meas}}[a,b]$ be the space of bounded, Lebesgue measurable functions defined in $[a,b]$, with the supremum norm, and let $L^\infty[a,b]$ be the usual Lebesgue space of equivalence classes of bounded measurable functions, equal a.e., with the essential supremum norm. Then there is a natural projection $\pi : \mathcal{B}_{\mathrm{meas}}[a,b] \to L^\infty[a,b]$, but the two spaces are not the same. The characteristic function of a set of null measure, χ_Z, has norm 1 in $\mathcal{B}_{\mathrm{meas}}[a,b]$, even though $\pi(\chi_Z) = 0$.

If ϕ is a regulated function of one variable, defined for $a \leq x \leq b$, we shall assume that it is actually defined in the whole real line $\mathbb{R}$, and that $\phi(x) = 0$ if $x \notin [a,b]$. We say that ϕ is a *normalized* regulated function if

$$\phi(x) = \frac{1}{2}\left(\phi(x+0) + \phi(x-0)\right), \tag{5.4.1}$$

for each $x \in \mathbb{R}$. Naturally one may normalize in other ways, by asking, for instance, continuity from the right, or perhaps continuity from the left, but as we shall see the normalization used is not so relevant for our analysis.

Definition 19. *If $a < b$, the space of normalized regulated functions in $[a,b]$ is denoted as $\mathcal{R}[a,b]$. The space of all regulated functions in $[a,b]$ is denoted as $\mathcal{R}_{\mathrm{t}}[a,b]$. The norm of $\phi \in \mathcal{R}_{\mathrm{t}}[a,b]$ is the supremum norm,*

$$\|\phi\|_{\sup} = \sup\left\{|\phi(x)| : a \leq x \leq b\right\}, \tag{5.4.2}$$

so that $\mathcal{R}_{\mathrm{t}}[a,b]$ *is a Banach space and* $\mathcal{R}[a,b]$ *is a closed subspace, and, consequently, also a Banach space.*

The space $\mathcal{R}_{\mathrm{t}}[a,b]$ can be decomposed as $\mathcal{R}[a,b] \oplus \mathcal{R}_{\mathrm{n}}[a,b]$, where $\phi \in \mathcal{R}_{\mathrm{n}}[a,b]$ if ϕ is regulated and equal to zero almost everywhere. Actually it is easy to see that $\phi \in \mathcal{R}_{\mathrm{n}}[a,b]$ if and only if there exists a sequence of different points of $[a,b]$, $\{x_n\}_{n=1}^{\infty}$, such that $\phi(x) = 0$ if $x \neq x_n$ for all n, and $\lim_{n\to\infty} \phi(x_n) = 0$.

If $\pi : \mathcal{B}_{\mathrm{meas}}[a,b] \to L^{\infty}[a,b]$ is the projection, then we have that $\pi(\mathcal{R}_{\mathrm{n}}[a,b]) = \{0\}$, and thus

$$\pi(\mathcal{R}_{\mathrm{t}}[a,b]) = \pi(\mathcal{R}[a,b]) . \tag{5.4.3}$$

Therefore, the two Banach spaces $\mathcal{R}[a,b]$ and the subspace $\pi(\mathcal{R}_{\mathrm{t}}[a,b])$ of $L^{\infty}[a,b]$ are isometric. This means that in a sense $\mathcal{R}[a,b]$ "is" the space of regulated functions in the *context* of $L^{\infty}[a,b]$. Naturally this would also hold for any other fixed normalization convention, not just (5.4.1).

Let F be a finite set, $F \subset (a,b)$; denote by $\mathcal{R}_{\mathrm{t}}([a,b];F)$ [respectively $\mathcal{R}([a,b];F)$] the set of regulated functions that are continuous in $(a,b)\setminus F$ [respectively normalized regulated functions]. Observe that if $F_1 \subset F_2$, then $\mathcal{R}_{\mathrm{t}}([a,b];F_1) \subset \mathcal{R}_{\mathrm{t}}([a,b];F_2)$ and $\mathcal{R}([a,b];F_1) \subset \mathcal{R}([a,b];F_2)$, the inclusions being isometries. Observe also that the union of the spaces $\mathcal{R}_{\mathrm{t}}([a,b];F)$ for F finite [respectively $\mathcal{R}([a,b];F)$] is dense in $\mathcal{R}_{\mathrm{t}}[a,b]$ [respectively in $\mathcal{R}[a,b]$]. Therefore $\mathcal{R}_{\mathrm{t}}[a,b]$ is the inductive limit of the system of Banach spaces $\{\mathcal{R}_{\mathrm{t}}([a,b];F)\}$ as $F \nearrow$, and similarly $\mathcal{R}[a,b]$ is the inductive limit of the system of Banach spaces $\{\mathcal{R}([a,b];F)\}$ as $F \nearrow$.

5.5 Dual Spaces in One Variable

5.5.1 The dual space $\mathcal{R}'[a,b]$

Definition 20. *Let $c \in [a,b)$. Then the right sided Dirac delta function at $x = c$ is the element $\delta_+(x-c) = \delta(x-(c+0))$ of $\mathcal{R}'[a,b]$ given by*

$$\langle \delta_+(x-c), \phi(x) \rangle = \phi(c+0) = \lim_{x \to c^+} \phi(x) \ . \tag{5.5.1}$$

Similarly, if $c \in (a,b]$, the left sided Dirac delta function at $x = c$ is the element $\delta_-(x-c) = \delta(x-(c-0))$ of $\mathcal{R}'[a,b]$ given by

$$\langle \delta_-(x-c), \phi(x) \rangle = \phi(c-0) = \lim_{x \to c^-} \phi(x) \ . \tag{5.5.2}$$

A linear combination of the one sided deltas, $\gamma_+\delta_+(x-c) + \gamma_-\delta_-(x-c)$, is called a thick delta function at $x = c$.

Observe that if a point c is one of the endpoints, a or b, then the Dirac delta functions $\delta(x-a)$ and $\delta(x-b)$, are actually a multiple of the one sided deltas in the space $\mathcal{R}'[a,b]$,

$$\delta(x-a) = \frac{1}{2}\delta_+(x-a) \ , \qquad \delta(x-b) = \frac{1}{2}\delta_-(x-b) \ . \tag{5.5.3}$$

Remark 1. *The projection of a thick delta function $\gamma_+\delta_+(x-c) + \gamma_-\delta_-(x-c)$ to the usual space of Radon measures, $(C[a,b])'$, is given by $(\gamma_+ + \gamma_-)\delta(x-c)$, a standard delta at $x = c$. This projection might vanish even if the thick delta is not zero: The test functions of $C[a,b]$ do not have the ability to detect thick deltas, while the test functions of $\mathcal{R}[a,b]$ do have this ability.*

The following characterizations of thick delta functions are useful.

Proposition 21. *Let $\mu \in \mathcal{R}'([a,b];F)$, where $F = G \cup \{c\}$ is a finite set. Suppose that the restriction of μ to $\mathcal{R}([a,b];G)$ vanishes. Then μ is a thick delta function at $x = c$.*

Proof. Let ϕ_+ and ϕ_- be elements of $\mathcal{R}([a,b];F)$ such that $\lim_{x\to c^+}\phi_+(x)=1$, $\lim_{x\to c^-}\phi_+(x)=0$, $\lim_{x\to c^+}\phi_-(x)=0$, and $\lim_{x\to c^-}\phi_-(x)=1$. If $\phi\in\mathcal{R}([a,b];F)$, then

$$\phi(x)=\left(\lim_{x\to c^+}\phi(x)\right)\phi_+(x)+\left(\lim_{x\to c^-}\phi(x)\right)\phi_-(x)+\psi(x)\ , \tag{5.5.4}$$

where $\psi\in\mathcal{R}([a,b];G)$. Let $\gamma_\pm=\langle\mu,\phi_\pm\rangle$. Since $\langle\mu,\psi\rangle=0$ we obtain that

$$\begin{aligned}\langle\mu,\phi\rangle&=\gamma_+\left(\lim_{x\to c^+}\phi(x)\right)+\gamma_-\left(\lim_{x\to c^-}\phi(x)\right)\\&=\langle\gamma_+\delta_+(x-c)+\gamma_-\delta_-(x-c),\phi(x)\rangle\ ,\end{aligned}$$

so that $\mu(x)=\gamma_+\delta_+(x-c)+\gamma_-\delta_-(x-c)$, as required. □

The next result can be proved in a similar way. Notice that the notion of support for an element $\mu\in\mathcal{R}'[a,b]$ is the standard one, namely, $\operatorname{supp}\mu\subset K$, where K is a closed subset of $[a,b]$ if $\langle\mu,\phi\rangle=0$ for $\phi\in\mathcal{R}[a,b]$ with $\phi(x)=0$ for all $x\in K$.

Proposition 22. *Let $\mu\in\mathcal{R}'[a,b]$ with $\operatorname{supp}\mu=\{c\}$. Then μ is a thick delta function at $x=c$.*

The space $\mathcal{R}([a,b];\{c\})$ is naturally isomorphic to $C[a,c]\times C[c,b]$. Therefore, if $\mu\in\mathcal{R}'[a,b]$ then $\mu_{\{c\}}$ is basically equal to $\mu_\emptyset$ plus a possible thick delta at $x=c$,

$$\begin{aligned}\mu_{\{c\}}(x)&=\mu_{\text{cont}}(x)\\&\quad+\sum_{n=1,x_n\neq c}^{\infty}\gamma_n\delta(x-x_n)+\gamma_+(c)\delta_+(x-c)+\gamma_-(c)\delta_-(x-c)\ ,\end{aligned}\tag{5.5.5}$$

where

$$\gamma_+(c)+\gamma_-(c)=\gamma_n\,,\quad \text{if } c=x_n\,,$$

$$\gamma_+(c)+\gamma_-(c)=0\,,\quad \text{if } c\neq x_n\,,\quad \forall n\,. \tag{5.5.6}$$

It is convenient to set $\gamma_+(a) = \gamma_n$ if $a = x_n$ and $\gamma_+(a) = 0$ if $a \neq x_n$ for all n; and $\gamma_-(a) = 0$ always. Similarly, $\gamma_-(b) = \gamma_n$ if $b = x_n$ and $\gamma_-(b) = 0$ if $b \neq x_n$ for all n; and $\gamma_+(b) = 0$ always.

A knowledge of $\mu_\emptyset$ is not enough to know $\mu_{\{c\}}$. If $\mu_\emptyset$ has a delta function at $x = c$ then $\mu_{\{c\}}$ has a thick delta at $x = c$, but there might be points c with thick delta functions but no ordinary delta functions.

On the other hand, knowledge of $\mu_{\{c\}}$ for all $c \in (a, b)$ allows us to find a formula for μ_F for each finite set $F \subset (a, b)$, namely,

$$\begin{aligned}\mu_F(x) &= \mu_{\mathrm{cont}}(x) \\ &+ \sum_{n=1, x_n \notin F}^{\infty} \gamma_n \delta(x - x_n) + \sum_{c \in F} (\gamma_+(c)\, \delta_+(x - c) + \gamma_-(c)\, \delta_-(x - c)) .\end{aligned} \tag{5.5.7}$$

Moreover, the family $\{\mu_F\}_{F \text{ finite}}$ determines μ uniquely. Hence μ determines the continuous measure μ_{cont} and the two functions γ_+ and γ_- and conversely, μ_{cont}, γ_+, and γ_- determine μ uniquely.

We shall now show that the functions γ_+ and γ_- vanish outside of a countable set.

Lemma 6. *The set $\mathfrak{S} = \{c \in [a, b] : \gamma_+(c) \neq 0$ or $\gamma_-(c) \neq 0\}$ is countable and*

$$\sum_{c \in \mathfrak{S}} (|\gamma_+(c)| + |\gamma_-(c)|) < \infty . \tag{5.5.8}$$

Proof. The result will be obtained if we show that

$$\sum_{c \in F} (|\gamma_+(c)| + |\gamma_-(c)|) \leq \|\mu\| , \tag{5.5.9}$$

for each finite set F, where $\|\mu\|$ is the norm of μ in the space $\mathcal{R}'[a, b]$. Indeed, let $\|\mu_F\|$ be the norm of μ_F in the space $\mathcal{R}'([a, b]; F)$. Writing $F = \{c_j : 1 \leq j \leq m\}$, where $a = c_0 < c_1 < \cdots < c_m < c_{m+1} = b$, we have that $\mathcal{R}([a, b]; F) \simeq$

$\prod_{j=0}^{m} C\left[c_j, c_{j+1}\right]$, and hence

$$\|\mu_F\| = \int_a^b \mathrm{d}\left|\mu_{\mathrm{cont}}\right| + \sum_{n=1, x_n \notin F}^{\infty} |\gamma_n| + \sum_{c \in F} \left(\left|\gamma_+ (c)\right| + \left|\gamma_- (c)\right|\right). \tag{5.5.10}$$

But $\|\mu_F\| \leq \|\mu\|$, and thus (5.5.9) follows. $\square$

We then obtain the ensuing formula.

Theorem 8. Let $\mu \in \mathcal{R}'[a, b]$. Then there exists a continuous Radon measure with support in $[a, b]$, μ_{cont}, and two functions γ_+ and γ_-, defined in $\mathbb{R}$, and that vanish outside of a countable subset $\mathfrak{S}$ of $[a, b]$, such that

$$\mu(x) = \mu_{\mathrm{cont}}(x) + \sum_{c \in \mathfrak{S}} \left(\gamma_+ (c)\, \delta_+ (x - c) + \gamma_- (c)\, \delta_- (x - c)\right). \tag{5.5.11}$$

Furthermore, the norm of μ in the space $\mathcal{R}'[a, b]$ is

$$\|\mu\| = \int_a^b \mathrm{d}\left|\mu_{\mathrm{cont}}\right| + \sum_{c \in \mathfrak{S}} \left(\left|\gamma_+ (c)\right| + \left|\gamma_- (c)\right|\right). \tag{5.5.12}$$

Proof. Let $\widetilde{\mu}(x) = \mu_{\mathrm{cont}}(x) + \sum_{c \in \mathfrak{S}} (\gamma_+ (c)\, \delta_+ (x - c) + \gamma_- (c)\, \delta_- (x - c))$. Employing the Lemma 6 we obtain that the series converges in $\mathcal{R}'[a, b]$, so that $\widetilde{\mu} \in \mathcal{R}'[a, b]$. Formula (5.5.7) yields that for each finite set F we have that $\mu = \widetilde{\mu}$ in the subspace $\mathcal{R}([a, b]; F)$ of $\mathcal{R}[a, b]$. But since $\bigcup_{F \text{ finite}} \mathcal{R}([a, b]; F)$ is dense in $\mathcal{R}[a, b]$, we obtain that $\mu = \widetilde{\mu}$ in $\mathcal{R}[a, b]$; this proves (5.5.11). Formula (5.5.12) follows because the right side of this equation is equal to $\|\widetilde{\mu}\|$. $\square$

5.5.2 The dual space $\mathcal{R}_{\mathrm{t}}'[a, b]$

We shall now consider the dual space of the space $\mathcal{R}_{\mathrm{t}}[a, b]$. Let us first observe that there are two types of Dirac delta functions in $\mathcal{R}_{\mathrm{t}}'[a, b]$. On the one hand we have the thick deltas, as in the space $\mathcal{R}'[a, b]$, but we also have the standard deltas,

$$\langle \delta(x - c), \phi(x) \rangle = \phi(c), \tag{5.5.13}$$

defined for any $c \in [a, b]$. The three deltas functions $\delta(x-c)$, $\delta_+(x-c)$, and $\delta_-(x-c)$ are linearly independent in $\mathcal{R}'_{\mathrm{t}}[a, b]$, while, of course, $\delta(x-c) = (1/2)\,\delta_+(x-c) + (1/2)\,\delta_-(x-c)$ in $\mathcal{R}'[a, b]$, and so they are linearly dependent in this latter space.

Proposition 23. *Let $\mu \in \mathcal{R}'_{\mathrm{t}}[a, b]$ with $\operatorname{supp}\mu = \{c\}$. Then there are numbers $\gamma_\pm$ and λ such that*

$$\mu(x) = \lambda\delta(x-c) + \gamma_+\delta_+(x-c) + \gamma_-\delta_-(x-c)\ . \tag{5.5.14}$$

Similarly, (5.5.14) must hold if $\mu \in \mathcal{R}'_{\mathrm{t}}([a, b]\,; F)$, where $F = G \cup \{c\}$ is a finite set and the restriction of μ to $\mathcal{R}_{\mathrm{t}}([a, b]\,; G)$ vanishes.

Let us recall that $\mathcal{R}_{\mathrm{t}}[a, b] = \mathcal{R}[a, b] \oplus \mathcal{R}_{\mathrm{n}}[a, b]$. A regulated function is continuous except perhaps at a countable set, so a function $\phi \in \mathcal{R}_{\mathrm{n}}[a, b]$ vanishes outside of a countable set $\mathfrak{Y} \subset [a, b]$. If we describe $\mathfrak{Y}$ by a sequence of *different* points $\{x_n\}_{n=1}^{\infty}$, then $\phi(x_n) \to 0$ as $n \to \infty$. Therefore, all elements ν_{n} of $\mathcal{R}'_{\mathrm{n}}[a, b]$ are of the form

$$\nu_0(x) = \sum_{c \in \mathfrak{Y}} \lambda(c)\,\delta(x-c)\ , \tag{5.5.15}$$

for some countable set $\mathfrak{Y}$, and constants $\lambda(c)$ that satisfy $\sum_{c\in\mathfrak{Y}} |\lambda(c)| < \infty$; the norm of ν_{n} in $\mathcal{R}'_{\mathrm{n}}[a, b]$ is given as $\|\nu_{\mathrm{n}}\| = \sum_{c\in\mathfrak{Y}} |\lambda(c)|$.

Let $\nu \in \mathcal{R}'_{\mathrm{t}}[a, b]$. Let μ be its restriction to $\mathcal{R}[a, b]$ and let ν_{n} be its restriction to $\mathcal{R}_{\mathrm{n}}[a, b]$. Then, combining formulas (5.5.11) and (5.5.15), we obtain the formula for ν.

Theorem 9. Let $\nu \in \mathcal{R}'_{\mathrm{t}}[a, b]$. Then there exists a continuous Radon measure with support in $[a, b]$, μ_{cont}, and three functions γ_+, γ_-, and λ, defined in $\mathbb{R}$, and that vanish outside of a countable subset $\mathfrak{S}$ of $[a, b]$, such that

$$\begin{aligned}\nu(x) &= \mu_{\mathrm{cont}}(x) \\ &\quad + \sum_{c\in\mathfrak{S}} \left(\gamma_+(c)\,\delta_+(x-c) + \gamma_-(c)\,\delta_-(x-c) + \lambda(c)\,\delta(x-c)\right).\end{aligned} \tag{5.5.16}$$

Moreover, the norm of ν in the space $\mathcal{R}_{\mathrm{t}}'[a,b]$ is

$$\|\nu\| = \int_a^b \mathrm{d}\,|\mu_{\mathrm{cont}}| + \sum_{c\in\mathfrak{S}} (|\gamma_+(c)| + |\gamma_-(c)| + |\lambda(c)|) \,. \tag{5.5.17}$$

5.6 Spaces over Open Intervals

We shall now consider the dual spaces of several spaces of regulated functions defined on an open interval (a,b). Observe that in such a case the lateral limits exist at each interior point of (a,b), but not necessarily at the endpoints. Notice also that $|a|$ or b can be infinite.

The spaces $\mathcal{R}(a,b)$ and $\mathcal{R}_{\mathrm{t}}(a,b)$ are the analogous of $C(a,b)$. A function ϕ belongs to $\mathcal{R}_{\mathrm{t}}(a,b)$ if it is regulated over (a,b), without any restriction on its behavior at the endpoints; it belongs to $\mathcal{R}(a,b)$ if it is a normalized regulated function in (a,b). The spaces $\mathcal{R}(a,b)$ and $\mathcal{R}_{\mathrm{t}}(a,b)$ are, like $C(a,b)$, topological vector spaces, but not Banach spaces: Convergence means uniform convergence over each compact subset of (a,b). The dual space $(C(a,b))'$ is the space of measures with compact support in (a,b), and thus the dual spaces of $\mathcal{R}(a,b)$ and $\mathcal{R}_{\mathrm{t}}(a,b)$ are formed by measures with thick deltas and compact support. If $\mu \in \mathcal{R}'[c,d]$, and $[c,d] \subset (a,b)$, then μ has a natural extension to $\mathcal{R}'(a,b)$, which we shall denote with the same notation, μ; similarly the elements of $\mathcal{R}_{\mathrm{t}}'[c,d]$ have a natural extension to $\mathcal{R}_{\mathrm{t}}'(a,b)$.

Theorem 10. Let $\mu \in \mathcal{R}'(a,b)$. Then there exists a closed interval $[c,d] \subset (a,b)$ such that $\mu \in \mathcal{R}'[c,d]$. The form of μ, a measure with thick delta functions, is given in the Theorem 8.

Let $\nu \in \mathcal{R}_{\mathrm{t}}'(a,b)$. Then there exists a closed interval $[c,d] \subset (a,b)$ such that $\nu \in \mathcal{R}_{\mathrm{t}}'[c,d]$. The form of ν is given in the Theorem 9.

The space $C_0(\mathbb{R})$ consists of the continuous functions on the real line that vanish at infinity, a Banach space with the supremum norm. Similarly we can define the

spaces $\mathcal{R}_0(\mathbb{R})$, the space of normalized regulated functions on the real line that have limit zero at infinity, and $\mathcal{R}_{\mathrm{t},0}(\mathbb{R})$, the space of all regulated functions on the real line that have limit zero at infinity; both are Banach spaces with the supremum norm. The dual space $(C_0(\mathbb{R}))'$ is the space of measures of total finite variation in $\mathbb{R}$, and we obtain similar results for the corresponding spaces of regulated functions.

Theorem 11. Let $\mu \in \mathcal{R}_0'(\mathbb{R})$. Then there exists a continuous Radon measure in $\mathbb{R}$, μ_{cont}, and two functions γ_+ and γ_-, defined in $\mathbb{R}$, and that vanish outside of a countable subset $\mathfrak{S}$, such that

$$\mu(x) = \mu_{\mathrm{cont}}(x) + \sum_{c\in\mathfrak{S}} (\gamma_+(c)\,\delta_+(x-c) + \gamma_-(c)\,\delta_-(x-c)) . \tag{5.6.1}$$

Furthermore, the norm of μ in the space $\mathcal{R}_0'(\mathbb{R})$ is

$$\|\mu\| = \int_{-\infty}^{\infty} \mathrm{d}\,|\mu_{\mathrm{cont}}| + \sum_{c\in\mathfrak{S}} (|\gamma_+(c)| + |\gamma_-(c)|) . \tag{5.6.2}$$

Let $\nu \in \mathcal{R}_{\mathrm{t},0}'(\mathbb{R})$. Then there exists a continuous Radon measure in $\mathbb{R}$, μ_{cont}, and three functions γ_+, γ_-, and λ, defined in $\mathbb{R}$, and that vanish outside of a countable subset $\mathfrak{S}$, such that

$$\begin{aligned} \nu(x) &= \mu_{\mathrm{cont}}(x) \\ &\quad + \sum_{c\in\mathfrak{S}} (\gamma_+(c)\,\delta_+(x-c) + \gamma_-(c)\,\delta_-(x-c) + \lambda(c)\,\delta(x-c)) . \end{aligned} \tag{5.6.3}$$

Moreover, the norm of ν in the space $\mathcal{R}_{\mathrm{t},0}'(\mathbb{R})$ is

$$\|\nu\| = \int_{-\infty}^{\infty} \mathrm{d}\,|\mu_{\mathrm{cont}}| + \sum_{c\in\mathfrak{S}} (|\gamma_+(c)| + |\gamma_-(c)| + |\lambda(c)|) . \tag{5.6.4}$$

The spaces $\mathcal{R}_{\mathsf{c}}(a,b)$ and $\mathcal{R}_{\mathrm{t},\mathsf{c}}(a,b)$ are the subspaces of $\mathcal{R}[a,b]$ and $\mathcal{R}_{\mathrm{t}}[a,b]$, respectively, formed by the functions with compact support in (a,b). As with $C_{\mathsf{c}}(a,b)$, we give them the topology of the inductive limit of the spaces $\mathcal{R}[c,d]$ (or correspondingly $\mathcal{R}_{\mathrm{t}}[a,b]$) for $[c,d] \subset (a,b)$ as $[c,d] \nearrow$. The elements of the

dual spaces $\mathcal{R}_{\mathsf{c}}'(a,b)$ and $\mathcal{R}_{\mathrm{t},\mathsf{c}}'(a,b)$ are measures with thick delta function over (a,b) with finite variation over each compact subinterval $[c,d]$ of (a,b), but whose variation over the whole interval (a,b) might be infinite.

Chapter 6

Asymptotic Expansions of Thick Distributions

6.1 Introduction

The theory of asymptotic expansions of generalized functions was probably initiated by the work of Łojasiewicz [65], and has been studied by several authors in the recent years [37, 70, 71, 84]. These studies have provided a deep understanding of the local properties of distributions and of their behavior at infinity and have helped us understand many aspects of tauberian theory; applications in several areas, particularly in number theory, geometry, and mathematical physics have been developed.

In this chapter we will present a theory of the asymptotic expansion of *thick distributions.*

The main result in the approach of [37] is the *moment asymptotic expansion,* that says that if g is a distribution defined in the whole $\mathbb{R}^n$ that decays very fast at infinity[1] then $g(\lambda\mathbf{x})$ has the asymptotic expansion $\sum_{j=0}^{\infty}\sum_{|\mathbf{k}|=j}(-1)^j\mu_{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\delta(\mathbf{x})/(\mathbf{k}!\lambda^{n+j})$ as $\lambda\to\infty$, where the constants $\mu_{\mathbf{k}}$ are the moments of g, $\mu_{\mathbf{k}}=\left\langle g(\mathbf{x}),\mathbf{x}^{\mathbf{k}}\right\rangle$. In this article we prove that thick distributions f that show rapid decay at infinity, in the sense that $f\in\mathcal{K}'_{*}(\mathbb{R}^n)$, satisfy a generalized moment expansion, namely,

$$f(\lambda\mathbf{x})\sim\sum_{k=-\infty}^{\infty}\frac{2\pi^{n/2}\mu_k(\mathbf{w})\,\delta_*^{[k]}}{\Gamma(n/2)\,\lambda^{k+n}},\ \text{as}\ \lambda\to\infty\,, \tag{6.1.1}$$

where the $\mu_k\in\mathcal{D}'(\mathbb{S})$ are the moment *functions* of f, and where the $\delta_*^{[k]}$ are thick delta functions as defined in Definition 10.

[1]Technically rapid decay means $f\in\mathcal{K}'(\mathbb{R}^n)$.

6.2 Other spaces of thick distributions

We call $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ the space of *standard* test functions on $\mathbb{R}^n$ with a thick point located at $\mathbf{x}=\mathbf{a}$. We denote $\mathcal{D}_{*,\mathbf{0}}(\mathbb{R}^n)$ as $\mathcal{D}_{*}(\mathbb{R}^n)$. The space of *standard* distributions on $\mathbb{R}^n$ with a thick point at $\mathbf{x}=\mathbf{a}$ is the dual space of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$. We denote it $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$, or just as $\mathcal{D}'_{*}(\mathbb{R}^n)$ when $\mathbf{a}=\mathbf{0}$.

6.2.1 The spaces $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\mathcal{E}'_{*,\mathbf{a}}(\mathbb{R}^n)$

The space $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the space of multipliers of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ [89]. Its elements are smooth functions defined in $\mathbb{R}^n\setminus\{\mathbf{a}\}$ that behave like elements of $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^n)$ near $\mathbf{x}=\mathbf{a}$ but without any restriction at infinity. No topology for the space $\mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)$ was considered in [89]; with the topology given in the Definition **??** then the dual space $\mathcal{E}'_{*,\mathbf{a}}(\mathbb{R}^n)$ is the subspace of $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^n)$ formed by those thick distributions whose support, a subset of $\mathbb{R}^n\setminus\{\mathbf{a}\}$, is of the form $K\setminus\{\mathbf{a}\}$ where K is compact. For a fixed m the family of seminorms $\left\{\| \quad \|_{q,s,K}\right\}$ defined in (1.2.2), for $q>m, s\geq 0$, and K a compact set of $\mathbb{R}^n$ that contains $\mathbf{a}$, form a basis of seminorms for the topology of $\mathcal{E}^{[m]}_{*,\mathbf{a}}(\mathbb{R}^n)$.

6.2.2 The spaces $\mathcal{K}_{*,\mathbf{a}}(\mathbb{R}^n)$ and $\mathcal{K}'_{*,\mathbf{a}}(\mathbb{R}^n)$

Now let us discuss the asymptotic expansions of thick distributions. The following spaces are very important in the asymptotic expansion of distributions. After introducing them, in the next section I will present our main results.

As convention, if $\mathbf{x}\in\mathbb{R}^n$, then $\mathbf{x}^{\mathbf{k}}=x_1^{k_1}...x_n^{k_n}$; $\mathbf{k}!=k_1!...k_n!$; $\mathbf{D}^{\mathbf{k}}\phi(\mathbf{x})=\partial_1^{k_1}...\partial_n^{k_n}\phi(\mathbf{x})$ in Cartesian coordinate.

Definition 21. $\mathcal{K}(\mathbb{R}^n)=\left\{\phi\in C^\infty(\mathbb{R}^n)\,\middle|\,\exists q\in\mathbb{N}, s.t. D^{\mathbf{k}}\phi(\mathbf{x})=O\left(|\mathbf{x}|^{q-|\mathbf{k}|}\right), as\ \ |\mathbf{x}|\to\infty\right\}.$

Definition 22. $\mathcal{K}_{*,\mathbf{a}}(\mathbb{R}^n)=\left\{\phi\in \mathcal{E}_{*,\mathbf{a}}(\mathbb{R}^n)\,\middle|\,\exists q\in\mathbb{N}, s.t. D^{\mathbf{k}}\phi(\mathbf{x})=O\left(|\mathbf{x}|^{q-|\mathbf{k}|}\right), as\ \ |\mathbf{x}|\to\infty\right\}.$
We denote $\mathcal{K}_{*,\mathbf{0}}(\mathbb{R}^n)=\mathcal{K}_{*}(\mathbb{R}^n)$

$\mathcal{K}_{*,\mathbf{a}}^{[m,q]}(\mathbb{R}^n)$ is a Frechet space under the semi-norm .

$$\|\phi\|_{q,l,s} = \max\left\{\sup_{|\mathbf{x}|\leq 1}\sup_{|\mathbf{p}|\leq s}\frac{\left|D^{\mathbf{p}}\phi(\mathbf{a}+\mathbf{x}) - \sum_{j=m-|\mathbf{p}|}^{l-1} a_{j,\mathbf{p}}(\mathbf{w})\, r^j\right|}{r^l}, \sup_{|x|\geq 1}\sup_{|\mathbf{p}|\leq s} r^{-q+\mathbf{p}} D^{\mathbf{p}}\phi(\mathbf{a}+\mathbf{x})\right\}$$

The topology of $\mathcal{K}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{K}_{*,\mathbf{a}}^{[m,q]}(\mathbb{R}^n)$ as $q \nearrow \infty$.

The topology of $\mathcal{K}_{*,\mathbf{a}}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{K}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ as $m \searrow -\infty$.

Recall that [37] $f \in \mathcal{K}'(\mathbb{R}^n)$ iff $f(\lambda\mathbf{x})$ admits the moment asymptotic expansion:

$$f(\lambda\mathbf{x}) \sim \sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}\mu_{\mathbf{k}} D^{\mathbf{k}}\delta(\mathbf{x})}{\lambda^{|\mathbf{k}|+n}\mathbf{k}!}, \text{ as } \lambda \to \infty. \tag{6.2.1}$$

where $\mu_{\mathbf{m}} = \langle f(\mathbf{x}), \mathbf{x}^{\mathbf{m}}\rangle$.

6.3 The moment asymptotic expansion

If the distribution f belongs to $\mathcal{K}'(\mathbb{R}^n)$ then its *moments* are the numbers

$$\mu_{\mathbf{k}} = \left\langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\right\rangle, \tag{6.3.1}$$

for $\mathbf{k} \in \mathbb{N}^n$. For thick distributions we employ the following definition.

Definition 23. *Let $f \in \mathcal{K}'_*(\mathbb{R}^n)$. Then its moments are the distributions $\mu_k \in \mathcal{D}'(\mathbb{S})$ defined as*

$$\langle \mu_k(\mathbf{w}), \varphi(\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} = \left\langle f(\mathbf{x}), |\mathbf{x}|^{\mathbf{k}}\varphi(\mathbf{x}/|\mathbf{x}|)\right\rangle_{\mathcal{K}'_*(\mathbb{R}^n)\times\mathcal{K}_*(\mathbb{R}^n)}, \tag{6.3.2}$$

for $k \in \mathbb{Z}$.

Observe that if $f \in \mathcal{K}'_*(\mathbb{R}^n)$ is a thick distribution and $f_0 = \Pi(f) \in \mathcal{K}'(\mathbb{R}^n)$ is the usual distribution obtained by projection, then the moments $\mu_{\mathbf{k}}$ of f_0 are related to the moment functions $\mu_k(\mathbf{w})$ of f by the formula

$$\mu_{\mathbf{k}} = \left\langle \mu_k(\mathbf{w}), \mathbf{w}^{\mathbf{k}}\right\rangle, \tag{6.3.3}$$

where $k = |\mathbf{k}|$.

Any distribution of $\mathcal{K}'(\mathbb{R}^n)$ satisfies the moment asymptotic expansion [37]. We now prove that the thick distributions of $\mathcal{K}'_*(\mathbb{R}^n)$ also satisfy a generalized moment asymptotic expansion.

6.3.1 The main theorem

Recall the definition of a "thick delta function" $\mu\delta_*^{[k]} \in \mathcal{D}'_*(\mathbb{R}^n)$ is given by:

Definition 24. *If $\mu \in \mathcal{D}'(\mathbb{S}^{n-1})$, i.e. a distribution on the n-1 dimensional unit sphere, then*

$$\left\langle \mu\delta_*^{[k]}, \phi \right\rangle = \frac{1}{C_{n-1}} \langle \mu, a_k \rangle$$

given any thick test function ϕ admits an asymptotic expansion $\sum\limits_{i=m}^{\infty} a_i r^i$ at the origin.

We have the following theorem:

Theorem 12. Suppose $f \in \mathcal{K}'_*(\mathbb{R}^n)$, then $f(\lambda\mathbf{x})$ admits the following asymptotic expansion:

$$f(\lambda\mathbf{x}) \sim \sum_{k=-\infty}^{\infty} \frac{C_{n-1}\mu_k(\mathbf{w})\,\delta_*^{[k]}}{\lambda^{k+n}}, \text{ as } \lambda \to \infty.$$

where the moment $\mu_k(\mathbf{w})$ is a distribution in $\mathcal{D}'(\mathbb{S}^{n-1})$ s.t.

$$\langle \mu_k(\mathbf{w}), a_k(\mathbf{w}) \rangle_{\mathcal{D}'(\mathbb{S}^{n-1}) \times \mathcal{D}(\mathbb{S}^{n-1})} = \left\langle f(\lambda\mathbf{x}), a_k(\mathbf{x}/|\mathbf{x}|)\,|\mathbf{x}|^k \right\rangle_{\mathcal{K}'_*(\mathbb{R}^n) \times \mathcal{K}_*(\mathbb{R}^n)}$$

. for $a(\mathbf{w}) \in \mathcal{D}(\mathbb{S}^{n-1})$. And C_{n-1} is the surface area of $n-1$ dimensional sphere.

Observe that $a(\mathbf{x}/|\mathbf{x}|)\,|\mathbf{x}|^k$ is in $\mathcal{K}_*(\mathbb{R}^n)$, so $\mu_k(\mathbf{w})$is well-defined.

This means that if $\phi \in \mathcal{K}_*(\mathbb{R}^n), \phi \in \mathcal{K}_*^{[m]}(\mathbb{R}^n)$, then for each $M \geq m$,

$$\langle f(\lambda\mathbf{x}), \phi(\mathbf{x}) \rangle = \sum_{k=m}^{M} \left\langle \frac{C_{n-1}\mu_k(\mathbf{w})\,\delta_*^{[k]}(\mathbf{x})}{\lambda^{k+n}}, \phi(\mathbf{x}) \right\rangle + o\left(\frac{1}{\lambda^{M+n}}\right),$$

as $\lambda \to \infty$.

Proof. Let us denote $C_t = \left\{ \phi \sim \sum_{i=m}^{\infty} a_i r^i \,|\phi \in \mathcal{K}_* (\mathbb{R}^n), a_k (\mathbf{w}) = 0 \text{ if } k \le t \right\}$, we claim:

Suppose $\phi \in \mathcal{K}_{*,\mathbf{a}}^{[m,q]} (\mathbb{R}^n)$, if $\exists t$, such that.$t \le q$ and $\phi \in C_t$, then

$$\left\| \phi \left(\frac{\mathbf{x}}{\lambda} \right) \right\|_{q,l,s} \le \frac{K}{\lambda^t}$$

for some constant K. In this proof we use r to denote $|\mathbf{x}|$.

In fact, observe that

$$|\phi (\mathbf{x})| \le r^t K \qquad \text{when } r \le 1$$

$$|\phi (\mathbf{x})| \le r^q K \qquad \text{when } r \ge 1$$

Hence

$$\left\| \phi \left(\frac{\mathbf{x}}{\lambda} \right) \right\|_{q,l,0} \le \frac{K}{\lambda^t}$$

Now if $s \neq 0$, then

$$\| \phi (\mathbf{x}/\lambda) \|_{q,l,s} = \sup_{|\mathbf{p}| \le s} \left(1/\lambda^{|\mathbf{p}|} \right) \| D^{\mathbf{p}} \phi (\mathbf{y}) \|_{q-|\mathbf{p}|,l,0}$$

where $\mathbf{y} = \mathbf{x}/\lambda$ and the derivative $D^{\mathbf{p}} \phi (\mathbf{y})$ is a derivative with respect to $\mathbf{y}$. Notice $D^{\mathbf{p}} \phi (\mathbf{x}) \in C_{t-|\mathbf{p}|}$, hence $\| \phi (\mathbf{x}/\lambda) \|_{q,l,s} \le K'/\lambda^t$ for some K'.

So if $\phi \in \mathcal{K}_{*,\mathbf{a}}^{[m,q]} (\mathbb{R}^n)$, then

$$\begin{aligned} \langle f (\lambda \mathbf{x}), \phi \rangle &= \frac{1}{\lambda^n} \left\langle f (\mathbf{x}), \phi \left(\frac{\mathbf{x}}{\lambda} \right) \right\rangle \\ &= \sum_{k=m}^{q-1} \frac{1}{\lambda^{k+n}} \langle \mu_k (\mathbf{w}), a_k (\mathbf{w}) \rangle + R_q (\lambda) \end{aligned}$$

Where $R_q (\lambda)$is

$$\begin{aligned} R_q (\lambda) &= \left\langle f (\lambda \mathbf{x}), \phi (\mathbf{x}) - \sum_{k=m}^{q-1} a_k r^k \right\rangle \\ &= \langle f (\lambda \mathbf{x}), P_q (\mathbf{x}) \rangle \\ &= \frac{1}{\lambda^n} \left\langle f (\mathbf{x}), P_q \left(\frac{\mathbf{x}}{\lambda} \right) \right\rangle . \end{aligned}$$

Observe $P_q(\mathbf{x}) = \phi(\mathbf{x}) - \sum_{k=m}^{q-1} a_k r^k \in C_q$, so by the proof above, for every seminorm $\|P_q(\mathbf{x})\|_{q,l,s}$, there is some K, such that $\|P_q(\mathbf{x}/\lambda)\|_{q,l,s} \le K/\lambda^q$.

We know that $f(\mathbf{x})$ is continuous. Hence

$$|R_q(\lambda)| = O\left(\frac{1}{\lambda^{q+n}}\right).$$

Because the topology of $\mathcal{K}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$ is the inductive limit topology of the $\mathcal{K}_{*,\mathbf{a}}^{[m,q]}(\mathbb{R}^n)$ as $q \nearrow \infty$, we have, for $\phi \in \mathcal{K}_{*,\mathbf{a}}^{[m]}(\mathbb{R}^n)$,

$$\langle f(\lambda\mathbf{x}), \phi(\mathbf{x})\rangle = \sum_{k=m}^{M} \left\langle \frac{C_{n-1}\mu_k(\mathbf{w})\,\delta_*^{[k]}(\mathbf{x})}{\lambda^{k+n}}, \phi(\mathbf{x})\right\rangle + o\left(\frac{1}{\lambda^{M+n}}\right).$$

□

Since $\mathcal{K}(\mathbb{R}^n)$ is a closed proper subspace of $\mathcal{K}_*(\mathbb{R}^n)$, we can consider the projection operator $\pi : \mathcal{K}_*'(\mathbb{R}^n) \to \mathcal{K}'(\mathbb{R}^n)$, dual to the inclusion $\mathcal{K}(\mathbb{R}^n) \hookrightarrow \mathcal{K}_*(\mathbb{R}^n)$.

We know

$$\pi\left(\mu(\mathbf{w})\,\delta_*^{[k]}\right) = \frac{(-1)^k}{C_{n-1}} \sum_{|\mathbf{j}|=k} \frac{\langle \mu(\mathbf{w}), \mathbf{w}^{\mathbf{j}}\rangle}{\mathbf{j}!} \mathbf{D}^{\mathbf{j}}\delta(\mathbf{x}).$$

In particular, if μ is a constant, we have

$$\pi\left(\mu\delta_*^{[2k]}\right) = \frac{\mu\Gamma(k+1/2)\,\Gamma(n/2)}{\Gamma(k+n/2)\,\Gamma(1/2)\,(2k)!}\nabla^{2k}(\delta).$$

where $\nabla^{2m} = (\nabla^2)^m$ is the $m-th$ power of the Laplacian.

So we have the following corollary[37, Thm. 4.3.1].:

Corollary 1. *If $g(x) \in \mathcal{K}'(\mathbb{R}^n)$, with moments $\mu_{\mathbf{k}}$, then*

$$g(\lambda\mathbf{x}) \sim \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} (-1)^k \frac{\mu_{\mathbf{k}}\mathbf{D}^{\mathbf{k}}\delta(\mathbf{x})}{\mathbf{k}!\lambda^{n+k}}, \quad \textit{as } \lambda \to \infty. \tag{6.3.4}$$

6.3.2 Expansion of radial distributions

The moment asymptotic expansion takes a rather simple form when applied to radial thick distributions, as we now explain. A radial distribution is one that is

invariant with respect to rotations; if $f \in \mathcal{D}'_*(\mathbb{R}^n)$ is radial then $f(\mathbf{x}) = f_1(|\mathbf{x}|)$, where $f_1 \in \mathcal{D}'_*(\mathbb{R})$ has support in $[0, \infty)$, but f_1 is not unique [?].

Lemma 7. *Let $f \in \mathcal{K}'_*(\mathbb{R}^n)$ be radial, $f(\mathbf{x}) = f_1(|\mathbf{x}|)$, where $f_1 \in \mathcal{K}'_*(\mathbb{R})$ has support in $[0, \infty)$. Then the moment functions of f, $\mu_k^{\{n\}} \in \mathcal{D}'(\mathbb{S})$ are constants, given by*

$$\mu_k^{\{n\}} = \mu_{k+n-1}^{\{1\}}, \tag{6.3.5}$$

where $\mu_k^{\{1\}} = \left\langle f_1(r), |r|^k \right\rangle$.

Proof. Indeed, if $\varphi \in \mathcal{D}(\mathbb{S})$ then $\left\langle \mu_k^{\{n\}}(\mathbf{w}), \varphi(\mathbf{w}) \right\rangle$ is given as

$$\left\langle f(\mathbf{x}), |\mathbf{x}|^k \varphi(\mathbf{x}/|\mathbf{x}|) \right\rangle = \left\langle f_1(r), |r|^{k+n-1} \right\rangle_r \langle 1, \varphi(\mathbf{w}) \rangle_{\mathbf{w}} = \left\langle \mu_{k+n-1}^{\{1\}}, \varphi(\mathbf{w}) \right\rangle,$$

as required. □

If we employ the Theorem 12 we obtain the ensuing asymptotic expansion of radial thick distributions.

Corollary 2. *If $f \in \mathcal{K}'_*(\mathbb{R}^n)$ is radial, then*

$$f(\lambda \mathbf{x}) \sim \sum_{k=-\infty}^{\infty} \frac{C_{n-1} \mu_{k+n-1}^{\{1\}} \delta_*^{[k]}}{\lambda^{k+n}}, \quad \text{as } \lambda \to \infty, \tag{6.3.6}$$

where the constants $\mu_k^{\{n\}}$ are given in (6.3.5).

Let us now apply the projection operator $\Pi : \mathcal{K}'_*(\mathbb{R}^n) \longrightarrow \mathcal{K}'(\mathbb{R}^n)$, and recall [?] that $\Pi\left(\delta_*^{[j]}\right) = 0$ unless $j = 2k \geq 0$, while

$$\Pi\left(\delta_*^{[2k]}\right) = \frac{\Gamma(k+1/2)\,\Gamma(n/2)}{\Gamma(k+n/2)\,(2k)!\sqrt{\pi}} \nabla^{2k} \delta(\mathbf{x}). \tag{6.3.7}$$

where $\nabla^{2k} = (\nabla^2)^k$ is the k-th power of the Laplacian. We thus obtain the following form of the moment asymptotic expansion.

Corollary 3. *If $g(x) \in \mathcal{K}'(\mathbb{R}^n)$ is radial, $g(\mathbf{x}) = g_1(|\mathbf{x}|)$, $g_1 \in \mathcal{K}'(\mathbb{R})$ with support in $[0,\infty)$, then its moments are of the form*

$$\mu_{\mathbf{k}} = \mu^{\{1\}}_{|\mathbf{k}|+n-1}\,, \tag{6.3.8}$$

where $\left\{\mu^{\{1\}}_k\right\}_{k=0}^{\infty}$ are the moments of g_1. Furthermore,

$$g(\lambda\mathbf{x}) \sim \sum_{k=0}^{\infty} \frac{\nu_k}{\lambda^{2k+n}} \nabla^{2k}\delta(\mathbf{x})\ , \ \text{as } \lambda \to \infty\,, \tag{6.3.9}$$

where

$$\nu_k = \frac{2\Gamma(k+1/2)\,\pi^{(n-1)/2}}{\Gamma(k+n/2)\,(2k)!}\mu^{\{1\}}_{k+n-1}\,. \tag{6.3.10}$$

6.4 Illustrations

Let us compute several examples to illustrate our formulas.

Example 18. *Let $f = \delta(\mathbb{S}) \in \mathcal{E}'_*(\mathbb{R}^n)$ be the delta function on the unit sphere of $\mathbb{R}^n$, that is, $\langle f, \phi\rangle = \int_{\mathbb{S}} \phi(\mathbf{w})\,\mathrm{d}\sigma(\mathbf{w})$. Observe that $\langle \mu_k(\mathbf{w}), a(\mathbf{w})\rangle$ equals*

$$\left\langle \delta(\mathbb{S}), a(\mathbf{w})\, r^k\right\rangle = \int_{\mathbb{S}} a(\mathbf{w})\,\mathrm{d}\sigma(\mathbf{w}) = \langle 1, a(\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}\ ,$$

so that

$$\mu_k(\mathbf{w}) = 1\,, \qquad \text{for all } k\,. \tag{6.4.1}$$

Denote by $\mathbb{S}_t = t\mathbb{S}$ the sphere of center $\mathbf{0}$ and radius t. Then, $f(\lambda\mathbf{x}) = (1/\lambda)\delta\left(\mathbb{S}_{1/\lambda}\right)$, so that the generalized moment asymptotic expansion yields

$$\delta\left(\mathbb{S}_{1/\lambda}\right) \sim \sum_{k=-\infty}^{\infty} \frac{2\pi^{n/2}\delta_*^{[k]}(\mathbf{x})}{\Gamma(n/2)\,\lambda^{n+k-1}}\,, \qquad \text{as } \lambda \to \infty\,, \tag{6.4.2}$$

in the space $\mathcal{E}'_*(\mathbb{R}^n)$.

The projection $\Pi(f)$ is the distribution $\delta(\mathbb{S}) \in \mathcal{E}'(\mathbb{R}^n)$, with the obvious abuse of notation, and thus in the space $\mathcal{E}'(\mathbb{R}^n)$,

$$\delta\left(\mathbb{S}_{1/\lambda}\right) \sim \sum_{k=0}^{\infty} \frac{c_{k,n}\nabla^{2k}\delta(\mathbf{x})}{(2k)!\lambda^{n+2k-1}}\,, \tag{6.4.3}$$

where $c_{k,n} = 2\Gamma(k+1/2)\pi^{(n-1)/2}/\Gamma(k+n/2)$, *in agreement with [37, Example 94].*

Evaluation of (6.4.3) at a test function $\phi \in \mathcal{E}(\mathbb{R}^n)$ *yields the Pizzetti's formula [3, 64].*

Let us consider an example involving the expansion of radial distributions.

Example 19. $F(\mathbf{x}) = g(\mathbf{w})\delta(\mathbb{S})$.

Observe $\mu_k(\mathbf{w}) = g(\mathbf{w})$, *so*

$$F(\lambda\mathbf{x}) \sim \sum_{k=-\infty}^{\infty} \frac{g(\mathbf{w})\,2\pi^{n/2}\delta_*^{[k]}(\mathbf{x})}{\Gamma(n/2)\,\lambda^{n+k}}, \quad \text{as } \lambda \to \infty. \tag{6.4.4}$$

And by equation **??**,

$$\pi(F(\lambda\mathbf{x})) \sim \sum_{k=0}^{\infty}(-1)^k \sum_{|\mathbf{j}|=k} \frac{\langle g(\mathbf{w}), \mathbf{w}^{\mathbf{j}}\rangle}{\mathbf{j}!\lambda^{n+k}}\mathbf{D}^{\mathbf{j}}\delta(\mathbf{x}) \quad \text{as } \lambda \to \infty. \tag{6.4.5}$$

In particular if $g(\mathbf{w}) = 1$, then $F(\mathbf{x}) = \delta(\mathbb{S})$, and $F(\lambda\mathbf{x}) = \frac{1}{\lambda}\delta(\lambda\mathbb{S})$. The moment expansion of $\pi(F(\lambda\mathbf{x}))$ should agree with the Example 1. Indeed, let us compute

$$\langle 1, \mathbf{w}^{\mathbf{j}}\rangle = \int_{\mathbb{S}^{n-1}} w_1^{j_1}...w_n^{j_n}\,d\sigma(\mathbf{w}),$$

If any of the j_i, $i = 1,..,n$ is an odd number, it's easy to see that the integral is 0. So equation 6.4.5 can be simplified as

$$\pi\left(\frac{1}{\lambda}\delta\left(\mathbb{S}_{1/\lambda}\right)\right) \sim \sum_{k=0}^{\infty}\sum_{|\mathbf{j}|=k} \frac{\langle 1, \mathbf{w}^{2\mathbf{j}}\rangle}{(2j_1)!...(2j_n)!\lambda^{n+2k}}\mathbf{D}^{2\mathbf{j}}\delta(\mathbf{x}) \quad \text{as } \lambda \to \infty.$$

Meanwhile, the only surviving terms, when all $j_i's$ are even, yields:

$$\langle 1, \mathbf{w}^{2\mathbf{j}}\rangle = \int_{\mathbb{S}^{n-1}} w_1^{2j_1}...w_n^{2j_n}\,d\sigma(\mathbf{w}) = \frac{2\Gamma\left(j_1+\frac{1}{2}\right)...\Gamma\left(j_n+\frac{1}{2}\right)}{\Gamma\left(j_1+...+j_n+\frac{n}{2}\right)}. \tag{6.4.6}$$

Notice

$$\frac{\Gamma\left(j+\frac{1}{2}\right)}{(2j)!} = \frac{\pi^{\frac{1}{2}}}{2^{2j}j!},$$

and

$$\Gamma\left(k+\frac{1}{2}\right)=\frac{(2k-1)!!}{2^k}\Gamma\left(\frac{1}{2}\right),$$

hence

$$\frac{\left\langle 1,\mathbf{w}^{2\mathbf{j}}\right\rangle}{(2j_1)!...(2j_n)!}=\frac{k!2\pi^{\frac{n-1}{2}}\Gamma\left(k+\frac{1}{2}\right)}{j_1!...j_n!\Gamma\left(k+\frac{n}{2}\right)(2k)!}.$$

Also note

$$\sum_{|\mathbf{j}|=k}\frac{1}{j_1!...j_n!}\mathbf{D}^{2\mathbf{j}}\delta(\mathbf{x})=\frac{1}{k!}\nabla^{2k}\delta(\boldsymbol{x}),$$

so

$$\pi\left(\frac{1}{\lambda}\delta\left(\mathbb{S}_{1/\lambda}\right)\right)\sim\sum_{k=0}^{\infty}\frac{2\Gamma\left(k+\frac{1}{2}\right)\pi^{\frac{1}{2}(n-1)}}{\Gamma\left(k+\frac{n}{2}\right)(2k)!\lambda^{n+2k}}\nabla^{2k}\delta(\mathbf{x})\quad\text{as }\lambda\to\infty,$$

which is exactly the same with the result in Example [**?**].

6.5 Laplace Formula

Let us consider the asymptotic evaluation of multidimensional integrals of the type

$$I(\lambda)=\int_R e^{-\lambda h(\mathbf{x})}\phi(\mathbf{x})\,d\mathbf{x},$$

where $h(\mathbf{x})$ is a real function and where R is a region of $\mathbb{R}^n$. We assume that $h(\mathbf{x})$ is smooth in a region containing the closure of the region R. It was shown by Focke that the main contribution to $I(\lambda)$ for $\lambda\gg 1$ comes from the vicinity of the minima of $h(\mathbf{x})$ in $\overline{R}$. The isolated critical points of h where minima occur are classified into three types according to their location: the interior critical points; the critical points of the smooth parts of the boundary; the critical points situated on the nonsmooth parts of the boundary. The asymptotic expansion takes a different form for each type of critical points. In this section we want to discuss the asymptotic expansion of $I(\lambda)$ when R contains a first type critical point.$\mathbf{x}_0$.

We also assume that $\phi(\mathbf{x})$ is smooth except for at the point $\mathbf{x}_0$. In fact, we assume that $\phi(\mathbf{x})\in\mathcal{K}_{*,\mathbf{x}_0}(\mathbb{R}^n)$.That is, it is tempered when $|\mathbf{x}|\to\infty$ and, it admits an asymptotic expansion on $r=|\mathbf{x}|$ when $\mathbf{x}\to\mathbf{x}_0$.

By assumption, x is an interior minimum of h, non-degenerate. Namely, $\left.\frac{\partial h}{\partial x_i}\right|_{\mathbf{x}_0} = 0$ and its Hessian matrix

$$A = \left.\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)\right|_{\mathbf{x}_0}$$

is positive definite. Thus we can find $\mathbf{y} = \Psi(\mathbf{x}) = (\psi_i(\mathbf{x}))$, with $\Psi(\mathbf{x}_0) = \mathbf{0}$, with positive Jacobian near $\mathbf{x}_0$, and such that,

$$h(\mathbf{x}) = h(\mathbf{x}_0) + |\Psi(\mathbf{x})|^2 .$$

Thus

$$I(\lambda) = e^{-\lambda h_0(\mathbf{x})} \int_R e^{-\lambda |\Psi(x)|^2} \phi(\mathbf{x})\, d\mathbf{x}.$$

By the computation we did in example **??**,

$$e^{-\lambda|\Psi(\mathbf{x})|^2} \sim \sum_{m=-\infty}^{-1} C_{n-1} \left(\frac{\sqrt{\pi}(-2)^{m+1} \delta_*^{[2m-n+1]}(\Psi(\mathbf{x}))}{2(2m+1)!!\lambda^{m+1/2}} - \frac{\gamma \delta_*^{[2m-n+2]}(\Psi(\mathbf{x}))}{k!\lambda^{m+1}} \right)$$
$$+ \sum_{m=0}^{\infty} C_{n-1} \left(\frac{\sqrt{\pi}(2m)! \delta_*^{[2m-n+1]}(\Psi(\mathbf{x}))}{(m!2^{2m+1})\lambda^{m+1/2}} + \frac{2m!! \delta_*^{[2m-n+2]}(\Psi(\mathbf{x}))}{2^{m+1}\lambda^{m+1}} \right).$$

Note here m is the index, n is the dimension of the space.$\mathbb{R}^n$.

Now let us consider the leading term of this expansion. Let us pick up a test function $\phi(\mathbf{x}) \in \mathcal{K}_{*,\mathbf{x}_0}(\mathbb{R}^n)$. We know that it admits an asymptotic expansion

$$\phi(\mathbf{x}) \sim \sum_{k=M}^{\infty} a_k(\mathbf{w}) |\mathbf{x}|^k$$

with some integer M. In the following context let us use b_{k-n+1} to denote the constant coefficients of the expansion of $e^{-\lambda|\Psi(\mathbf{x})|^2}$ for simplicity. i.e.

$$e^{-\lambda|\Psi(\mathbf{x})|^2} \sim \sum_{k=-\infty}^{\infty} b_{k+n-1} \frac{\delta_*^{[k]}(\Psi(\mathbf{x}))}{\lambda^{(k+n)/2}}.$$

Before discussing the expansion, let us present the following fact:

Proposition 24. *Let $g(\mathbf{w}) \delta_*^{[k]}(\mathbf{x}) \in \mathcal{K}'_*(\mathbb{R}^n)$, then*

$$g(\mathbf{w}) \delta_*^{[k]}(\mathbf{x} - \mathbf{x}_0) = g(\mathbf{w}) \delta_{*,\mathbf{x}_0}^{[k]}(\mathbf{x}) \in \mathcal{K}'_{*,\mathbf{x}_0}(\mathbb{R}^n).$$

Recall, if $\phi(\mathbf{x}) \in \mathcal{K}_{*,\mathbf{x}_0}(\mathbb{R}^n)$*, it admits an asymptotic expansion when* $|\mathbf{x}-\mathbf{x}_0| \to 0$ *as* $\sum_{j=M}^{\infty} a_j(\mathbf{w})|\mathbf{x}-\mathbf{x}_0|^j$*, and* $g(\mathbf{w})\,\delta_{*,\mathbf{x}_0}^{[k]}(\mathbf{x})$ *acts on* $\phi(\mathbf{x})$ *as*

$$\left\langle g(\mathbf{w})\,\delta_{*,\mathbf{x}_0}^{[k]}(\mathbf{x}), \phi(\mathbf{x})\right\rangle_{\mathcal{K}'_{*,\mathbf{x}_0}(\mathbb{R}^n)\times\mathcal{K}_{*,\mathbf{x}_0}(\mathbb{R}^n)} = \langle g(\mathbf{w}), a_k(\mathbf{w})\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})}.$$

Proof.

$$\left\langle g(\mathbf{w})\,\delta_*^{[k]}(\mathbf{x}-\mathbf{x}_0), \phi(\mathbf{x})\right\rangle = \left\langle g(\mathbf{w})\,\delta_*^{[k]}(\mathbf{x}), \phi(\mathbf{x}+\mathbf{x}_0)\right\rangle$$

the statement follows if one observes that $\mathbf{x}+\mathbf{x}_0$ is to move the coordinate system towards $\mathbf{x}_0$. □

Now, in order to discuss the leading term of the expansion, let us consider the following

$$\begin{aligned}&\left\langle \delta_*^{[k]}(\Psi(\mathbf{x})), \phi(\mathbf{x})\right\rangle \\ &= \left\langle \frac{\delta_*^{[k]}(\mathbf{y})}{\det(\mathbf{D}\Psi(\mathbf{x}_0))}, \phi\left(\Psi^{-1}(\mathbf{y})\right)\right\rangle = \left\langle \frac{\delta_*^{[k]}(\mathbf{y})}{\sqrt{\det(\mathbf{D}^2 h(\mathbf{x}_0))}}, \phi\left(\Psi^{-1}(\mathbf{y})\right)\right\rangle,\end{aligned}$$

where $\mathbf{D}^2 h(\mathbf{x}_0) = A$ denotes the Hessian matrix at $\mathbf{x}_0$. Note $\Psi^{-1}(\mathbf{0}) = \mathbf{x}_0$, if we write $\Psi^{-1}(\mathbf{y}) = \mathbf{x}_0 + \widetilde{\Psi}^{-1}(\mathbf{y})$, then $\widetilde{\Psi}^{-1}(\mathbf{0}) = \mathbf{0}$. We know that $\phi(x)$ admits an asymptotic expansion as $\sum_{j=M}^{\infty} a_j(\mathbf{w})|\mathbf{x}-\mathbf{x}_0|^j$ when $|\mathbf{x}-\mathbf{x}_0| \to 0$. One can see that

$$\phi\left(\Psi^{-1}(\mathbf{y})\right) = \sum_{j=M}^{\infty} a_j\left(\frac{\widetilde{\Psi}^{-1}(\mathbf{y})}{\left|\widetilde{\Psi}^{-1}(\mathbf{y})\right|}\right)\left|\widetilde{\Psi}^{-1}(\mathbf{y})\right|^j$$

Since $\widetilde{\Psi}^{-1}(\mathbf{0}) = \mathbf{0}$, there is an n by n matrix B such that

$$\widetilde{\Psi}^{-1}(\mathbf{y}) = B\mathbf{y} + O\left(\left|\mathbf{y}^2\right|\right).$$

In fact, $B = \mathbf{D}\left(\Psi^{-1}(\mathbf{x}_0)\right)$

Hence

$$\phi\left(\Psi^{-1}(\mathbf{y})\right) \sim a_M\left(\frac{B\mathbf{y}}{|B\mathbf{y}|}\right)|B\mathbf{y}|^M \qquad \text{as } \mathbf{y} \to \mathbf{0}$$

Namely, the leading term of the expansion can be viewed as a linear change of variable.

Observe, if we denote $\mathbf{y} = r\mathbf{w}$,

$$a_M\left(\frac{B\mathbf{y}}{|B\mathbf{y}|}\right)|B\mathbf{y}|^M = a_M\left(\frac{B\mathbf{w}}{|B\mathbf{w}|}\right)|B\mathbf{w}|^M r^M.$$

Thus in conclusion,

$$e^{-\lambda|\Psi(\mathbf{x})|^2} \sim b_{M+n-1}\frac{g_M\delta_{*,\mathbf{x}_0}^{[M]}(\mathbf{x})}{\sqrt{\det\left(\mathbf{D}^2 h\left(\mathbf{x}_0\right)\right)}\lambda^{(M+n)/2}},$$

where $g_M \in \mathcal{D}'\left(\mathbb{S}^{n-1}\right)$ and

$$\langle g_M(\mathbf{w}), a(\mathbf{w})\rangle = \left\langle 1, a\left(\frac{B\mathbf{w}}{|B\mathbf{w}|}\right)|B\mathbf{w}|^M\right\rangle.$$

Chapter 7
Regularization using different surfaces

7.1 Introduction

It is well known that each locally integrable function has associated a unique regular distribution. However, it seems to be not so well understood that other, more singular functions may or may not have associated distributions, and, more importantly, if such distributions exist, they are not unique, but rather depend on the *regularization* method employed [25]. We have mentioned in Chapter one that Hnidzo [**?**] studied such regularization issues for the second order derivatives of potential functions, a situation already pointed out by Farassat [**?**]. More specifically, Hnizdo [50] studied such regularization issues for the kernel $K(\mathbf{x}) = \left(3x_i x_j - r^2\delta_{ij}\right)/r^5$, and proved that (7.1.1) holds for the *spherical* regularization, $\lim_{\varepsilon\to 0} H(r-\varepsilon) K(\mathbf{x})$, but not for other regularizations

Indeed, Frahm [39] gave the formulas for the second order distributional derivatives of $1/r$ in $\mathbb{R}^3$, $(r = |\mathbf{x}|)$, and they are usually written as

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \frac{4\pi}{3}\delta_{ij}\delta(\mathbf{x})\,. \tag{7.1.1}$$

Actually formulas for the distributional derivatives of any order of r^α in $\mathbb{R}^n$ are available [30, 32], [37, Section 2.7], [58, Section 5.10]. However, the right side of formula (7.1.1) is not so clearly defined as a distribution, since the kernel $K(\mathbf{x}) = \left(3x_i x_j - r^2\delta_{ij}\right)/r^5$ is not locally integrable near the origin of $\mathbb{R}^3$. As Hnidzo explains, the correct interpretation of (7.1.1) is obtained if one understands the first term of the right side as the *spherical* regularization, $\lim_{\varepsilon\to 0} H(r-\varepsilon) K(\mathbf{x})$, where we denote the Heaviside function by H. Moreover, as he proves, other regularizations, such as the spheroidal one, $\lim_{\varepsilon\to 0} H\left(\gamma^2 x_1^2 + x_2^2 + x_3^2 - \varepsilon^2\right) K(\mathbf{x})$, also exist,

but (7.1.1) is no longer valid but needs to be replaced by a new formula [50]. The fact that spherical regularizations are used is explicit in [30, 32], and it is also the default setting when spherical coordinates are used [49]; as Frahm explains [39], formula (7.1.1) is "to be understood in the sense that, if necessary, angular integrations are to done before the radial integrations," which in fact means that a spherical regularization is employed.

Note that the concern here related to (7.1.1) is different from the one in [40], which we have discussed in the previous chapters.

Our aim in this chapter is to study the surface dependence of the regularization of homogeneous functions in $\mathbb{R}^n$. We concentrate on the case of homogeneous functions of degree $-n$, but also give results for homogenous functions of other orders.

7.2 Description of the problem

Let Σ be a closed surface in $\mathbb{R}^n$ that encloses the origin. We describe Σ by an equation of the form $g(\mathbf{x}) = 1$, where $g(\mathbf{x})$ is continuous in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and homogeneous of degree 1.

Let $K(\mathbf{x})$ be a real-valued C^∞ function on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, homogeneous of degree $-d$, that is, $K(t\mathbf{x}) = t^{-d}K(\mathbf{x})$ for any real number $t > 0$. The function K gives a distribution in $\mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$, but in general in order to have a distribution defined even at the origin, that is, a distribution of $\mathcal{D}'(\mathbb{R}^n)$, we need to employ *regularizations* of K. Let us now consider the ensuing regularization of $K(\mathbf{x})$ as a distribution in $\mathcal{D}'(\mathbb{R}^n)$,

$$\begin{aligned} \langle \mathcal{R}_\Sigma (K(\mathbf{x})), \phi(\mathbf{x}) \rangle &= \lim_{\varepsilon \to 0} \langle K(\mathbf{x}) H(g(\mathbf{x}) - \varepsilon), \phi(\mathbf{x}) \rangle \\ &= \lim_{\varepsilon \to 0} \int_{g(\mathbf{x}) \geqslant \varepsilon} K(\mathbf{x}) \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,, \end{aligned} \tag{7.2.1}$$

where H is the Heaviside function, and where $\phi \in \mathcal{D}(\mathbb{R}^n)$ is a test function. The question we would like to consider is the following: Does this regularization depend on the surface?

Naturally, the limit in (7.2.1) may or may not exist, and when it does not one needs to consider the Hadamard finite part of the limit,

$$\begin{aligned} \langle \mathrm{Pf}_{\Sigma}(K(\mathbf{x})), \phi(\mathbf{x}) \rangle &= \mathrm{F.p.}\lim_{\varepsilon \to 0} \langle K(\mathbf{x}) H(g(\mathbf{x}) - \varepsilon), \phi(\mathbf{x}) \rangle \\ &= \mathrm{F.p.}\lim_{\varepsilon \to 0} \int_{g(\mathbf{x}) \geqslant \varepsilon} K(\mathbf{x}) \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,, \end{aligned} \tag{7.2.2}$$

but even then the question still is whether the regularization $\mathrm{Pf}_{\Sigma}(K(\mathbf{x}))$ depends on Σ or not. Let us recall at this point that the finite part of the limit of $F(\varepsilon)$ as $\varepsilon \to 0^+$ exists and equals A if we can write $F(\varepsilon) = F_{\mathrm{fin}}(\varepsilon) + F_{\mathrm{infin}}(\varepsilon)$, where the *infinite* part $F_{\mathrm{infin}}(\varepsilon)$ is a linear combination of functions of the type $\varepsilon^{-p} \ln^q \varepsilon$, where $p > 0$ or $p = 0$ and $q > 0$, and where the *finite* part $F_{\mathrm{fin}}(\varepsilon)$ is a function whose limit as $\varepsilon \to 0^+$ is A [37, Section 2.3].

Observe that if $d < n$, then $K(\mathbf{x})$ is integrable at $\mathbf{x} = 0$, and thus the limit exists and does not depend on the surface. The question of the dependence of the regularization on the surface then arises only when $d \geqslant n$.

Usually, Σ is chosen to be the unit sphere, $\Sigma = \mathbb{S}$, that is, $g(\mathbf{x}) = r = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. In this case the standard spherical regularization, namely, $\mathcal{R}_{\mathbb{S}}(K(\mathbf{x})) = \lim_{\varepsilon \to 0} K(\mathbf{x}) H(r - \varepsilon)$, is obtained. However, we can also choose many other surfaces, such as a spheroid, or a cylinder, or a cube. Each type of surface may have some advantages in some problems; for instance, spheroidal regularization may be suitable in applications involving effects of special relatively [50].

7.3 A special case: When $K(\mathbf{x})$ is a derivative

In order to discuss the dependence of $\mathcal{R}_\Sigma(K(\mathbf{x}))$ on Σ, we first consider a special case. Suppose that

$$K = \frac{\partial L}{\partial x_i}\,, \tag{7.3.1}$$

where $L(\mathbf{x})$ is homogeneous of degree $-(n-1)$. Observe that this equation holds in $\mathbb{R}^n\backslash\{\mathbf{0}\}$, but it does not exist at $\mathbf{x}=\mathbf{0}$, since K and L are not defined at the origin.

Since L is locally integrable in $\mathbb{R}^n$, then $L \in \mathcal{D}'(\mathbb{R}^n)$ is a regular distribution. Therefore $\mathcal{R}_\Sigma(L(\mathbf{x})) = L(\mathbf{x})$ is independent of the surface Σ. Thus if $\Sigma_1 : g_1(\mathbf{x}) = 1$ and $\Sigma_2 : g_2(\mathbf{x}) = 1$ are two different closed surfaces, we have that

$$L(\mathbf{x}) = \lim_{\varepsilon\to 0} L(\mathbf{x})H(g_1(\mathbf{x})-\varepsilon) = \lim_{\varepsilon\to 0} L(\mathbf{x})H(g_2(\mathbf{x})-\varepsilon)\,, \tag{7.3.2}$$

in the distributional sense.

Let us now consider the *distributional* derivative of L. Indeed, L is a well defined distribution, hence so is its distributional derivative, $\overline{\partial}L/\partial x_i$. In view of (7.3.1), the restriction of $\overline{\partial}L/\partial x_i$ to $\mathbb{R}^n\backslash\{\mathbf{0}\}$ is equal to the distribution $K \in \mathcal{D}'(\mathbb{R}^n\backslash\{\mathbf{0}\})$; hence $\overline{\partial}L/\partial x_i$ is another regularization of K. We have,

$$\frac{\overline{\partial}L}{\partial x_i} = \frac{\overline{\partial}}{\partial x_i}(\lim_{\varepsilon\to 0} L(\mathbf{x})H(g(\mathbf{x})-\varepsilon)) = \lim_{\varepsilon\to 0}\frac{\overline{\partial}}{\partial x_i}(L(\mathbf{x})H(g(\mathbf{x})-\varepsilon))\,. \tag{7.3.3}$$

Let us now use the formula for the derivative of a function with a jump across a hypersurface in $\mathbb{R}^n$ [58, pg.113]:

$$\frac{\overline{\partial}}{\partial x_i}(\lim_{\varepsilon\to 0} L(\mathbf{x})H(g(\mathbf{x})-\varepsilon)) = \lim_{\varepsilon\to 0}\left(K(\mathbf{x})H(g(\mathbf{x})-\varepsilon) + n_i L(\mathbf{x})\delta(\Sigma_{(\varepsilon)})\right)\,, \tag{7.3.4}$$

where $K = \partial L/\partial x_i$, the ordinary derivative of L, $\Sigma_{(\varepsilon)}$ is the surface $g(\mathbf{x}) = \varepsilon$, and $\mathbf{n} = (n_i)$ is the unit normal vector to the surface $\Sigma_{(\varepsilon)}$. Observe now that if $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$\left\langle n_i L(\mathbf{x})\delta(\Sigma_{(\varepsilon)}), \phi\right\rangle = \int_{\Sigma_{(\varepsilon)}} n_i L(\mathbf{v})\phi(\mathbf{v})\, \mathrm{d}\sigma(\mathbf{v})\,, \tag{7.3.5}$$

where $\mathrm{d}\sigma$ is the surface measure on $\Sigma_{(\varepsilon)}$, and thus if we use the change $\mathbf{v} = \varepsilon\mathbf{w}$ and notice that the normal vector remains the same while $L(\varepsilon\mathbf{w}) = \varepsilon^{1-n} L(\mathbf{w})$, we obtain

$$\left\langle n_i L(\mathbf{x})\delta(\Sigma_{(\varepsilon)}), \phi \right\rangle = \int_{\Sigma_{(1)}} n_i L(\mathbf{w})\phi(\varepsilon\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{7.3.6}$$

Thus

$$\begin{aligned} \lim_{\varepsilon\to 0} \left\langle n_i L(\mathbf{x})\delta(\Sigma_{(\varepsilon)}), \phi \right\rangle &= \phi(\mathbf{0}) \int_{\Sigma} n_i L(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \\ &= \left\langle c_i(\Sigma)\delta(\mathbf{x}), \phi(\mathbf{x}) \right\rangle \ , \end{aligned} \tag{7.3.7}$$

where $\Sigma = \Sigma_{(1)} : g(\mathbf{x}) = 1$ is the "unit surface," and where the constant $c_i(\Sigma) = c_i(\Sigma; L)$ is given by

$$c_i(\Sigma) = \int_{\Sigma} n_i L(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{7.3.8}$$

Therefore from equations (7.3.3) and (7.3.4), we obtain the following result.

Theorem 13. Let $L \in \mathcal{D}'(\mathbb{R}^n)$, homogeneous of degree $-(n-1)$, and let $K = \partial L/\partial x_i$, the ordinary derivative of L, which is homogeneous of degree $-n$ and a distribution of $\mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$. Then the distributional derivative $\overline{\partial} L/\partial x_i$ is a regularization of K, and for each closed surface Σ the limit (7.2.1) exists and gives another regularization, $\mathcal{R}_\Sigma(K)$, of K. These regularizations are related as

$$\frac{\overline{\partial} L}{\partial x_i} = \mathcal{R}_\Sigma(K) + c_i(\Sigma)\delta(\mathbf{x}) \, , \tag{7.3.9}$$

where the constant $c_i(\Sigma)$ is given by (7.3.8).

Relation (7.3.9) shows how the regularization $\mathcal{R}_\Sigma(K)$ depends on the surface Σ. We can also express this in a different way. Indeed, let $\Sigma_1 : g_1(\mathbf{x}) = 0$ and $\Sigma_2 : g_2(\mathbf{x}) = 0$ be two surfaces. Then

$$\mathcal{R}_{\Sigma_1}(K) - \mathcal{R}_{\Sigma_2}(K) = (c_i(\Sigma_2) - c_i(\Sigma_1))\delta(\mathbf{x}) \, . \tag{7.3.10}$$

It is interesting to observe that the constants $c_i(\Sigma)$ depend only on the shape of the set $\Sigma \cup \{\mathbf{0}\}$ but not on its size.

Proposition 25. *For any $t > 0$ and any L,*

$$c_i(t\Sigma, L) = c_i(\Sigma, L)\,, \tag{7.3.11}$$

and

$$\mathcal{R}_{t\Sigma}(K) = \mathcal{R}_{\Sigma}(K)\,.$$

Proof. Indeed, if we use (7.3.8), $c_i(t\Sigma, L) = \int_{t\Sigma} n_i L\left(\mathbf{w}\right)\,\mathrm{d}\sigma\left(\mathbf{w}\right)$, and a change of variables transforms it into $\int_{\Sigma} n_i L\left(t\mathbf{v}\right)\,\mathrm{d}\sigma\left(t\mathbf{v}\right)$, which is nothing but $\int_{\Sigma} n_i L\left(\mathbf{v}\right)\,\mathrm{d}\sigma\left(\mathbf{v}\right) = c_i(\Sigma, L)$. $\square$

We can also consider the situation when L is homogeneous of degree $-d$, where $n-1 < d < n$. Again, in this case L is locally integrable in $\mathbb{R}^n$, but the limit of $\langle K(\mathbf{x})H(g(\mathbf{x})-\varepsilon), \phi(\mathbf{x})\rangle$ may not exist. Thus one needs to consider the finite part regularization $\mathrm{Pf}_{\Sigma}(K(\mathbf{x}))$. Equations (7.3.3) and (7.3.4) remain valid, but now

$$\left\langle n_i L(\mathbf{x})\delta(\Sigma_{(\varepsilon)}), \phi\right\rangle = c_i(\Sigma)\varepsilon^{n-d-1}\phi\left(\mathbf{0}\right) + o\left(1\right)\,, \tag{7.3.12}$$

as $\varepsilon \to 0$. Therefore the *infinite* part of $K(\mathbf{x})H(g(\mathbf{x})-\varepsilon)$ as $\varepsilon \to 0$ is $c_i(\Sigma)\varepsilon^{n-d-1}\delta\left(\mathbf{x}\right)$, which depends on Σ, while the finite part of the limit is $\overline{\partial}L/\partial x_i$. We thus obtain that $\mathrm{Pf}_{\Sigma}(K(\mathbf{x}))$ is independent of Σ in this case.

Theorem 14. Let $L \in \mathcal{D}'(\mathbb{R}^n)$, homogeneous of degree $-d$, where $n-1 < d < n$. Let $K = \partial L/\partial x_i$, the ordinary derivative of L, which is homogeneous of degree $-d-1$ and a distribution of $\mathcal{D}'(\mathbb{R}^n \backslash \{\mathbf{0}\})$. Then the distributional derivative $\overline{\partial}L/\partial x_i$ is a regularization of K, and for each closed surface Σ the finite part regularization $\mathrm{Pf}_{\Sigma}(K(\mathbf{x}))$ gives the same regularization of K,

$$\frac{\overline{\partial}L}{\partial x_i} = \mathrm{Pf}_{\Sigma}(K(\mathbf{x}))\,. \tag{7.3.13}$$

7.4 Four examples of the regularization of $\frac{\partial^2}{\partial x_i \partial x_j}(\frac{1}{r})$ in $\mathbb{R}^3$

We shall illustrate the ideas of the previous section by taking the homogeneous function of degree -3 in $\mathbb{R}^3$

$$K(\mathbf{x}) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5}\,. \tag{7.4.1}$$

Clearly K is the ordinary second order derivative $\partial^2/\partial x_i \partial x_j (1/r)$, so that

$$K = \frac{\partial L}{\partial x_i} = \frac{\partial}{\partial x_i}\left(-\frac{x_j}{r^3}\right)\,. \tag{7.4.2}$$

Observe that $1/r$ is of degree -1, and hence for any surface $\Sigma : g(\mathbf{x}) = 1$ we have that

$$\frac{\overline{\partial}}{\partial x_j}\left(\frac{1}{r}\right) = \mathcal{R}_\Sigma\left(-\frac{x_j}{r^3}\right) = \lim_{\varepsilon\to 0}\left(-\frac{x_j}{r^3}\right)H(g(\mathbf{x})-\varepsilon) = -\frac{x_j}{r^3}. \tag{7.4.3}$$

Because $-x_j/r^3$ is independent of the surface. In fact, $L = -x_j/r^3$ is of degree -2, and therefore locally integrable.

Using equation (7.3.9) we obtain the formula

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \mathcal{R}_\Sigma\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) + \eta_{ij}(\Sigma)\,\delta(\mathbf{x})\,, \tag{7.4.4}$$

where

$$\eta_{ij}(\Sigma) = c_i\left(\Sigma; -\frac{x_j}{r^3}\right) = -\int_\Sigma \frac{n_i w_j}{|\mathbf{w}|^3}\,\mathrm{d}\sigma(\mathbf{w})\,. \tag{7.4.5}$$

The constants $\eta_{ij}(\Sigma)$ will depend on the surface. Some of their properties are the following.

Proposition 26. *The constants $\eta_{ij}(\Sigma)$ are symmetric,*

$$\eta_{ij}(\Sigma) = \eta_{ji}(\Sigma)\,, \tag{7.4.6}$$

and satisfy

$$\eta_{11}(\Sigma) + \eta_{22}(\Sigma) + \eta_{33}(\Sigma) = -4\pi\,. \tag{7.4.7}$$

Proof. Formula (7.4.6) is obtained from the symmetry of the other terms of equation (7.4.4), while (7.4.7) follows from the well known formula $\nabla(1/r) = -4\pi\delta(\mathbf{x})$ [58]. □

In many cases we have that $\eta_{ij}(\Sigma) = 0$ if $i \neq j$. Indeed, the ensuing property is clear from (7.4.5).

Proposition 27. *If Σ is invariant under the change $x_j \to -x_j$, namely, if it is symmetric with respect to the plane $x_j = 0$ for a fixed j, then $\eta_{ij}(\Sigma) = 0$ if $i \neq j$.*

7.4.1 Spherical regularization

The spherical regularization is the usual one. Here $\mathbb{S} : (x_1^2 + x_2^2 + x_3^2)^{1/2} = 1$, the unit sphere. Equation (7.4.5) gives

$$\eta_{ij}(\mathbb{S}) = -\int_{\mathbb{S}} n_i w_j \, \mathrm{d}\sigma(\mathbf{w}) = -\int_{\mathbb{S}} n_i n_j \, \mathrm{d}\sigma(\mathbf{w}) = -\frac{4\pi}{3}\delta_{ij}\,,$$

so we obtain the usual formula [58],

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{\overline{\partial}}{\partial x_i}\left(-\frac{x_j}{r^3}\right) = \mathcal{R}_{\mathbb{S}}\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) - \frac{4\pi}{3}\delta_{ij}\delta(\mathbf{x})\,. \tag{7.4.8}$$

7.4.2 Spheroidal regularization

Let us consider the spheroid $\Upsilon : g(\mathbf{x}) = (\gamma^2 x_1^2 + x_2^2 + x_3^2)^{1/2} = 1$, where $\gamma = 1/\sqrt{1-v^2}$, $0 < |v| < 1$. We can describe it in spherical coordinates as

$$x_1 = \frac{\cos\theta}{\gamma}\,, \qquad x_2 = \sin\theta\cos\phi\,, \qquad x_3 = \sin\theta\sin\phi\,, \tag{7.4.9}$$

so that in terms of θ and ϕ,

$$\mathbf{n} = \left(\frac{\gamma\cos\theta}{\sqrt{\gamma^2\cos^2\theta + \sin^2\theta}}, \frac{\sin\theta\cos\phi}{\sqrt{\gamma^2\cos^2\theta + \sin^2\theta}}, \frac{\sin\theta\sin\phi}{\sqrt{\gamma^2\cos^2\theta + \sin^2\theta}}\right)\,, \tag{7.4.10}$$

and

$$\mathrm{d}\sigma(\mathbf{w}) = \left|\frac{\partial\mathbf{F}}{\partial\theta} \times \frac{\partial\mathbf{F}}{\partial\phi}\right| \mathrm{d}\theta\,\mathrm{d}\phi = \sin\theta\sqrt{1 - v^2\sin^2\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \tag{7.4.11}$$

where $\mathbf{F}$ is the transformation described in equation (7.4.9).

When $i \neq j$, $\eta_{ij}(\Upsilon) = 0$ on account of the integration with respect to the azimuthal angle ϕ. When $i = j = 1$,

$$\begin{aligned} \eta_{11}(\Upsilon) &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \frac{\cos^2\theta \sin\theta \sqrt{1-v^2}}{(1-v^2\cos^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta \\ &= -2\pi \left(\frac{2}{v^2} - \left(\frac{2}{\gamma v^3}\right) \arcsin v \right). \end{aligned} \tag{7.4.12}$$

When $i = j = 2$,

$$\begin{aligned} \eta_{22}(\Upsilon) &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \frac{\cos^2\phi \sin^3\theta \sqrt{1-v^2}}{(1-v^2\cos^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta \\ &= -2\pi(1 - \frac{1}{v^2} + \left(\frac{1}{\gamma v^3}\right) \arcsin v). \end{aligned} \tag{7.4.13}$$

The case $i = j = 3$ yields the same as $i = j = 2$ because of the symmetry. These results agree with [50].

7.4.3 Cylindrical regularization

Consider now a cylinder, $\mathfrak{C} : x_1 = kz, z \in [-1, 1]$, k is a constant, $x_2 = \cos\theta$, $x_3 = \sin\theta$.

At the top, $\mathbf{n} = (1, 0, 0)$; at the bottom, $\mathbf{n} = (-1, 0, 0)$, while on the side, $\mathbf{n} = (0, \cos\theta, \sin\theta)$.

We have that $\eta_{ij}(\mathfrak{C}) = 0$ if $i \neq j$ by symmetry. When $i = j = 1$, because $n_1 = 0$ on the side, only the integrals on the top and on the bottom do not vanish, and there $\frac{x_1}{r^3} = \frac{kz}{(\sqrt{k^2z^2+s^2})^3}$, and $z = 1$ at the top, $z = -1$ at the bottom. So

$$\eta_{11}(\mathfrak{C}) = -2 \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \frac{kds}{(\sqrt{k^2+s^2})^3}\,\mathrm{d}s = 2\pi \left(\frac{kz}{\sqrt{1+k^2z^2}} - 1 \right). \tag{7.4.14}$$

When $i = j = 2$, only the side integral is not zero, and there $\frac{x_2}{r^3} = \frac{\cos\theta}{(\sqrt{(kz)^2+1})^3}$, so

$$\eta_{22}(\mathfrak{C}) = \int_0^{2\pi} \mathrm{d}\theta \int_{-1}^1 -\frac{\cos^2\theta (k\,\mathrm{d}z)}{(\sqrt{(kz)^2+1})^3} = -2\pi \left(\frac{k}{\sqrt{k^2+1}} \right). \tag{7.4.15}$$

Finally, $\eta_{33}(\mathfrak{C}) = \eta_{22}(\mathfrak{C})$ because of the symmetry. The cylindrical regularization yields the same result as that of [50]

7.4.4 Cubic regularization

We may also consider regularizations with respect to non smooth surfaces, such as the cube $\Psi : \max_{1\leq j\leq 3} |x_j| = 1$. In this case the symmetry of the cube with respect to interchanging x_i and x_j, and its symmetry with respect to the coordinate planes yields $\eta_{ij}(\Psi) = -4\pi\delta_{ij}/3$.

7.5 The case when $K(\mathbf{x})$ is homogeneous of degree $-n$

Let now K be an arbitrary homogeneous function of degree $-n$ in $\mathbb{R}^n$. Then, if $\phi \in \mathcal{D}(\mathbb{R}^n)$, the limit $\lim\limits_{\varepsilon\to 0} \langle K(\mathbf{x})H(g(\mathbf{x})-\varepsilon), \phi(\mathbf{x})\rangle$ will not exist, in general. However, we may always consider the Hadamard finite part of the limit, and this will define a regularization of K which we shall denote as $\mathrm{Pf}_\Sigma(K(\mathbf{x}))$,

$$\langle \mathrm{Pf}_\Sigma(K(\mathbf{x})), \phi(\mathbf{x})\rangle = \mathrm{F.p.}\lim_{\varepsilon\to 0} \langle K(\mathbf{x})H(g(\mathbf{x})-\varepsilon), \phi(\mathbf{x})\rangle \; . \tag{7.5.1}$$

When the limit exists, of course, we have that $\mathrm{Pf}_\Sigma(K(\mathbf{x})) = \mathcal{R}_\Sigma(K)$.

Theorem 15. The regularization of $K(\mathbf{x})\, H(g(\mathbf{x}))$ in the sense of Hadamard finite part is

$$\begin{aligned}\langle \mathrm{Pf}_\Sigma(K(\mathbf{x})), \phi(\mathbf{x})\rangle = \int_{g\geq 1} & K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \\ & \int_{1\geq g\geq 0} K(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{0}))\,\mathrm{d}\mathbf{x}\,.\end{aligned} \tag{7.5.2}$$

Proof. We have that as $\varepsilon \to 0$,

$$\begin{aligned}
\langle K(\mathbf{x})H(g(\mathbf{x})-\varepsilon),\phi(\mathbf{x})\rangle &= \int_{g(\mathbf{x})\geqslant\varepsilon} K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= \int_{g\geq 1} K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{1\geq g\geq\varepsilon} K(\mathbf{x})(\phi(\mathbf{x})-\phi(\mathbf{0}))\,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{1\geq g\geq\varepsilon} K(\mathbf{x})\phi(\mathbf{0})\,\mathrm{d}\mathbf{x} \\
&= \int_{g\geq 1} K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{1\geq g\geq 0} K(\mathbf{x})(\phi(\mathbf{x})-\phi(\mathbf{0}))\,\mathrm{d}\mathbf{x} \\
&\qquad + \phi(\mathbf{0})\int_{1\geq g\geq\varepsilon} K(\mathbf{x})\,\mathrm{d}\mathbf{x} + o(1)\ ,
\end{aligned}$$

since $K(\mathbf{x})(\phi(\mathbf{x})-\phi(\mathbf{0}))$ is locally integrable at $\mathbf{x}=\mathbf{0}$.

Now we claim that

$$F(\varepsilon) = \int_{1\geq g\geq\varepsilon} K(\mathbf{x})\,\mathrm{d}\mathbf{x} = D\ln\varepsilon\,, \tag{7.5.3}$$

where D is a constant independent of ε. In fact, the change of variables $\mathbf{x}=t\mathbf{y}$ in the integral in equation (7.5.3) yields $F(\varepsilon)=F(\varepsilon t)-F(t)$ for any $t>0$, and the only continuous solution of this equation is $F(\varepsilon)=D\ln\varepsilon$ for some constant D. Therefore the infinite part of $\langle K(\mathbf{x})H(g(\mathbf{x})-\varepsilon),\phi(\mathbf{x})\rangle$ is $D\ln\varepsilon$ and the finite part limit is given by (7.5.2). □

Does the regularization $\mathrm{Pf}_\Sigma(K(\mathbf{x}))$ depend on Σ, or, what is the same, does it depend on $g(\mathbf{x})$?. We have the following results.

Proposition 28. *Let $\Sigma_1 : g_1(\mathbf{x})=0$ and $\Sigma_2 : g_2(\mathbf{x})=0$ be two surfaces, then*

$$\begin{aligned}
&\mathrm{Pf}_{\Sigma_1}(K(\mathbf{x})) - \mathrm{Pf}_{\Sigma_2}(K(\mathbf{x})) \\
&\qquad = \left(\int_{g_1\geq 1, g_2\leq 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{g_2\geq 1, g_1\leq 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x}\right)\delta(\mathbf{x})\,.
\end{aligned} \tag{7.5.4}$$

Proof. Indeed, if $\phi \in \mathcal{D}(\mathbb{R}^n)$ is a test function,

$$\begin{aligned}
&\langle R_{\Sigma_1} - R_{\Sigma_2}, \phi(\mathbf{x})\rangle \\
&= \int_{g_1\le 1} K(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{0}))\,\mathrm{d}\mathbf{x} + \int_{g_1\ge 1} K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&\quad - \int_{g_2\le 1} K(\mathbf{x})(\phi(\mathbf{x}) - \phi(\mathbf{0}))\,\mathrm{d}\mathbf{x} - \int_{g_2\ge 1} K(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} \\
&= \int_{g_1\ge 1, g_2\le 1} K(\mathbf{x})\phi(\mathbf{0})\,\mathrm{d}\mathbf{x} - \int_{g_2\ge 1, g_1\le 1} K(\mathbf{x})\phi(\mathbf{0})\,\mathrm{d}\mathbf{x} \\
&= \left(\int_{g_1\ge 1, g_2\le 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{g_2\ge 1, g_1\le 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x}\right) \langle \delta(\mathbf{x}), \phi(\mathbf{x})\rangle \,,
\end{aligned}$$

as required. □

Interestingly, while the finite part depends on the surface, the infinite part does not.

Proposition 29. *The constant D in (7.5.3) is independent of the surface Σ.*

Proof. Let $\Sigma_1 : g_1(\mathbf{x}) = 0$ and $\Sigma_2 : g_2(\mathbf{x}) = 0$ be two surfaces. We have that

$$K(\mathbf{x})H(g_j(\mathbf{x}) - \varepsilon) = \mathrm{Pf}_{\Sigma_1}(K(\mathbf{x})) + D_j \ln \varepsilon + o(1) \,, \tag{7.5.5}$$

as $\varepsilon \to 0$, where $D_j = \int_{1\ge g_j\ge\varepsilon} K(\mathbf{x})\,\mathrm{d}\mathbf{x}/\ln\varepsilon$.

Observe now that if f is a distribution of rapid decay at infinity, in particular if is of compact support, it satisfies the moment asymptotic expansion [37, Chp. 4]

$$f(\lambda\mathbf{x}) \sim \sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}\mu_{\mathbf{k}} D^{\mathbf{k}}\delta(\mathbf{x})}{\mathbf{k}!\lambda^{|\mathbf{k}|+n}} \,, \quad \text{as } \lambda \to \infty \,, \tag{7.5.6}$$

where $\mu_{\mathbf{k}} = \left\langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\right\rangle$, $\mathbf{k} \in \mathbb{N}^n$, are the moments of f. In particular, $\lambda^n f(\lambda\mathbf{x}) = O(1)$ as $\lambda \to \infty$. If we now take $f(\mathbf{x}) = K(\mathbf{x})H(g_1(\mathbf{x}) - 1) - K(\mathbf{x})H(g_2(\mathbf{x}) - 1)$ and use the facts that K is homogeneous of degree $-n$ while g_j are homogeneous of degree 1, we obtain

$$\lambda^n f(\lambda\mathbf{x}) = K(\mathbf{x})H(g_1(\mathbf{x}) - \lambda^{-1}) - K(\mathbf{x})H(g_2(\mathbf{x}) - \lambda^{-1}) \,, \tag{7.5.7}$$

and setting $\varepsilon = 1/\lambda$ we obtain that $K(\mathbf{x})H(g_1(\mathbf{x})-\varepsilon) - K(\mathbf{x})H(g_2(\mathbf{x})-\varepsilon) = O(1)$ as $\varepsilon \to 0$, and it follows from (7.5.5) that $D_1 = D_2$. Actually the first term of the expansion (7.5.6), $\lambda^n f(\lambda\mathbf{x}) = \mu_{\mathbf{0}}\delta(\mathbf{x}) + O(1/\lambda)$, allows us to recover (7.5.4), since

$$\mu_{\mathbf{0}} = \int_{g_1\geq 1, g_2 \leq 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{g_2\geq 1, g_1\leq 1} K(\mathbf{x})\,\mathrm{d}\mathbf{x}\,. \tag{7.5.8}$$

$\square$

Let us now find an alternative expression for D. Let us observe first that (7.5.3) yields,

$$D = \frac{\int_{1\geq g \geq \varepsilon} K(\mathbf{x})\,\mathrm{d}\mathbf{x}}{\ln\varepsilon}\,, \tag{7.5.9}$$

independent of ε. Let us now make a change of variable: Let $\mathbf{x} = t\mathbf{v}(u_1, u_2, ..., u_{n-1})$. where $\varepsilon \leq t \leq 1$, and $g(\mathbf{v}) = 1$. We have that $\mathrm{d}\mathbf{x} = |J|\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\mathrm{d}t$, where J is the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial t} & \cdots & \frac{\partial x_n}{\partial t} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial x_1}{\partial u_{n-1}} & \cdots & \frac{\partial x_n}{\partial u_{n-1}} \end{bmatrix} = \frac{\partial \mathbf{x}}{\partial t} \cdot \left(\frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} \times ... \times \frac{\partial \mathbf{x}}{\partial u_{n-1}} \right).$$

Here $\frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} \times ... \times \frac{\partial \mathbf{x}}{\partial u_{n-1}}$ denotes the vector having the i^{th} component as the $(1,i)$ cofactor of the Jacobian matrix J. Clearly, the direction of $\frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} \times ... \times \frac{\partial \mathbf{x}}{\partial u_{n-1}}$ is normal to the surface $g(\mathbf{v}) = 1$, because $\frac{\partial \mathbf{x}}{\partial u_j} \cdot (\frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} \times ... \times \frac{\partial \mathbf{x}}{\partial u_{n-1}}) = 0$ for any $j = 1, 2, ..., n-1$. And

$$\begin{aligned} &\left\| \frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} \times ... \times \frac{\partial \mathbf{x}}{\partial u_{n-1}} \right\| \mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_{n-1} \\ &\qquad = t^{n-1} \left\| \frac{\partial \mathbf{v}}{\partial u_1} \times \frac{\partial \mathbf{v}}{\partial u_2} \times ... \times \frac{\partial \mathbf{v}}{\partial u_{n-1}} \right\| \mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_{n-1} \\ &\qquad = t^{n-1}\mathrm{d}\sigma_{n-1}\,, \end{aligned}$$

where $\mathrm{d}\sigma_{n-1}$ denotes the volume element of $\Sigma : g(\mathbf{v}) = 1$ determined by the Riemannian metric it acquires as a submanifold of $\mathbb{R}^n$ [**?**, Prob. 13, pg. 351]. On the other hand, $\partial\mathbf{x}/\partial t = \mathbf{v}\left(u_1, u_2, ..., u_{n-1}\right)$. Hence in conclusion,

$$\int_{1\geq g\geq\varepsilon} K(\mathbf{x})\,\mathrm{d}\mathbf{x} =$$

$$\int_{1\geq g(t\mathbf{v})\geq\varepsilon} K(t\mathbf{v})\left|\frac{\partial\mathbf{x}}{\partial t}\cdot\left(\frac{\partial\mathbf{x}}{\partial u_1}\times\frac{\partial\mathbf{x}}{\partial u_2}\times ...\times\frac{\partial\mathbf{x}}{\partial u_{n-1}}\right)\right|\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\mathrm{d}t$$

$$= \int_{1\geq t\geq\varepsilon}\int_{\Sigma} K(t\mathbf{v})|\mathbf{v}\cdot\mathbf{n}|t^{n-1}\mathrm{d}\sigma_{n-1}\mathrm{d}t\,,$$

But K is of degree $-n$, $K(t\mathbf{v}) = t^{-n}K(\mathbf{v})$, so the above expression becomes

$$\int_{1\geq t\geq\varepsilon}\frac{\mathrm{d}t}{t}\int_{\Sigma} K(\mathbf{v})|\mathbf{v}\cdot\mathbf{n}|\,\mathrm{d}\sigma_{n-1} = -\ln\varepsilon\int_{\Sigma} K(\mathbf{v})|\mathbf{v}\cdot\mathbf{n}|\,\mathrm{d}\sigma_{n-1}\,,$$

and (7.5.9) gives $D = \int_{\Sigma}(\mathbf{v}K(\mathbf{v}))\cdot\mathbf{n}\,\mathrm{d}\sigma_{n-1}$. Let us summarize.

Proposition 30. *The constant D is given by*

$$D = \int_{\Sigma}(\mathbf{v}K(\mathbf{v}))\cdot\mathbf{n}\,\mathrm{d}\sigma_{n-1}\,. \tag{7.5.10}$$

In particular,

$$D = \int_{\mathbb{S}} K(\mathbf{w})\,\mathrm{d}\sigma\left(\mathbf{w}\right)\,, \tag{7.5.11}$$

where $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$.

Formula (7.5.10) allow us to verify that D is indeed independent of the surface, since if we consider two different surfaces Σ_1 and Σ_2, we have

$$D_{\Sigma_1} - D_{\Sigma_2} = \int_{\Sigma_1}(\mathbf{v}K(\mathbf{v}))\cdot\mathbf{n}\,\mathrm{d}\sigma_{n-1} - \int_{\Sigma_2}(\mathbf{v}K(\mathbf{v}))\cdot\mathbf{n}\,\mathrm{d}\sigma_{n-1}$$

$$= \int_X \operatorname{div}(\mathbf{x}K(\mathbf{x}))\,\mathrm{d}\mathbf{x}\,,$$

where X is the set bounded by the surfaces. Notice now that away from the origin,

$$\operatorname{div}(\mathbf{x}K(\mathbf{x})) = \sum_{i=1}^{n}\frac{\partial x_i}{\partial x_i}K(\mathbf{x}) + x_i\frac{\partial K(\mathbf{x})}{\partial x_i} = 0\,,$$

by Euler's equation. Thus $D_{\Sigma_1} = D_{\Sigma_2}$.

Actually we can write the divergence theorem as

$$D = \int_{\Sigma} (\mathbf{v}K(\mathbf{v})) \cdot \mathbf{n}\, \mathrm{d}\sigma_{n-1} = \int_{g\leq 1} \overline{\mathrm{div}}(\mathbf{x}K(\mathbf{x}))\, \mathrm{d}\mathbf{x}\,, \tag{7.5.12}$$

where we use $\overline{\mathrm{div}}$ to denote the divergence in the sense of distributions. Indeed, notice that $x_iK(\mathbf{x})$ is of degree $-n+1$, and so by equation (7.3.9),

$$\frac{\overline{\partial}}{\partial x_i}(x_iK(\mathbf{x})) = \mathcal{R}_{\mathbb{S}}\left(\frac{\partial}{\partial x_i}(x_iK)\right) + c_i(\mathbb{S}, x_iK)\delta(\mathbf{x})\,, \tag{7.5.13}$$

and because $\mathrm{div}(\mathbf{x}K(\mathbf{x})) = 0$ for $\mathbf{x} \neq \mathbf{0}$,

$$\overline{\mathrm{div}}(\mathbf{x}K(\mathbf{x})) = \sum_{i=1}^{n} c_i(\mathbb{S}, x_iK)\delta(\mathbf{x})\,. \tag{7.5.14}$$

But $\sum_{i=1}^{n} c_i(\mathbb{S}, x_iK) = \sum_{i=1}^{n} \int_{\mathbb{S}} n_i w_i K(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) = \int_{\mathbb{S}} K(\mathbf{w})\, \mathrm{d}\sigma\,(\mathbf{w})$, thus

$$\overline{\mathrm{div}}(\mathbf{x}K(\mathbf{x})) = \left(\int_{\mathbb{S}} K(\mathbf{w})\, \mathrm{d}\sigma\,(\mathbf{w})\right)\delta(\mathbf{x})\,. \tag{7.5.15}$$

Therefore,

$$\begin{aligned}\int_{g\leq 1} \overline{\mathrm{div}}(\mathbf{x}K(\mathbf{x}))\, \mathrm{d}\mathbf{x} &= \int_{g\leq 1}\left(\int_{\mathbb{S}} K(\mathbf{w})\, \mathrm{d}\sigma\,(\mathbf{w})\right)\delta(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{S}} K(\mathbf{w})\, \mathrm{d}\sigma\,(\mathbf{w}) \\ &= D\,.\end{aligned} \tag{7.5.16}$$

7.5.1 An example: $1/r^3$ in $\mathbb{R}^3$

Employing Theorem 15 the formula

$$\begin{aligned}\left\langle \mathrm{Pf}_{\Sigma}\left(\frac{1}{r^3}\right), \phi(\mathbf{x})\right\rangle = &\int_{g\geq 1} \frac{\phi(\mathbf{x})\, \mathrm{d}\mathbf{x}}{(x_1^2+x_2^2+x_3^2)^{3/2}} \\ &+ \int_{1\geq g\geq 0} \frac{(\phi(\mathbf{x})-\phi(\mathbf{0}))\, \mathrm{d}\mathbf{x}}{(x_1^2+x_2^2+x_3^2)^{3/2}}\,,\end{aligned} \tag{7.5.17}$$

is obtained. In this case,

$$D = \int_{\Sigma} \frac{(\mathbf{r}\cdot\mathbf{n})}{r^3}\, \mathrm{d}\sigma = \int_{\Sigma}\left(\frac{\mathbf{r}}{r^3}\right)\cdot d\mathbf{S} = 4\pi\,,$$

by Gauss' Law.

We thus derive the asymptotic estimate,

$$\frac{H(g(\mathbf{x})-\varepsilon)}{r^3} = \mathrm{Pf}_{\Sigma}\left(\frac{1}{r^3}\right) + 4\pi \ln\varepsilon\,\delta(\mathbf{x}) + o\left(1\right)\,, \tag{7.5.18}$$

that is, if ϕ is a test function of the space $\mathcal{D}(\mathbb{R}^n)$,

$$\int_{g\geq\varepsilon} \frac{\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \left\langle \mathrm{Pf}_{\Sigma}\left(\frac{1}{r^3}\right), \phi(\mathbf{x})\right\rangle + 4\pi\ln\varepsilon\,\phi(\mathbf{0}) + o\left(1\right)\,. \tag{7.5.19}$$

We also obtain that if $\Sigma_1 : g_1\left(\mathbf{x}\right) = 1$ and $\Sigma_2 : g_2\left(\mathbf{x}\right) = 1$ are two surfaces, then

$$\mathrm{Pf}_{\Sigma_1}\left(\frac{1}{r^3}\right) - \mathrm{Pf}_{\Sigma_2}\left(\frac{1}{r^3}\right) = \left(\int_{g_1\geq 1, g_2\leq 1} \frac{\mathrm{d}\mathbf{x}}{r^3} - \int_{g_2\geq 1, g_1\leq 1} \frac{\mathrm{d}\mathbf{x}}{r^3}\right)\delta(\mathbf{x})\,. \tag{7.5.20}$$

Chapter 8

Distributional Asymptotic Expansion in Heat Kernel Invariants, Eigenvalue Distributions and Quantum Vacuum Energy

8.1 Introduction

This chapter is based on the paper ”Some subtleties in the relationships among heat kernel invariants, eigenvalue distributions, and quantum vacuum energy” [J. Phys. A: Math. Theor. **48** (2015) 045402] by Professor Stephen Fulling and myself. My main contribution is to use mathematical tools to get our main results, comparing it with the observations of Kolomeisky et al; while the interpretation of our results of its importance in physics and Riesz means is given by Professor Fulling. The relation between cylinder kernel expansion and heat kernel expansion came from a discussion between me and Professor Ricardo Estrada.

A common tool in quantum field theory (and many other areas) is the asymptotic (high-frequency) expansion of eigenvalue densities, employed as either input or output of calculations of the asymptotic behavior of various Green functions. Here we clarify some fine points and potentially confusing aspects of the subject. In particular, we show how recent observations of Kolomeisky et al. [Phys. Rev. A **87** (2013) 042519] fit into the established framework of the distributional asymptotics of spectral functions.

The density of eigenvalues of a differential operator (such as a Hamiltonian), with the closely related subjects of semiclassical approximations and the asymptotic dependence of various Green functions on various coordinates and parameters, is a central tool in nuclear, atomic, and molecular physics [6], condensed-matter physics [**?**], general-relativistic quantum field theory [18], and quantum vacuum

energy (Casimir physics) [43]. (These four references are merely illustrative.) A key point is that a variety of differential equations (relativistic and nonrelativistic, classical and quantum) are all associated with the same elliptic, spatial differential operator; a variety of Green functions (heat, Schrödinger, wave, ...) are related to the same eigenvalue distribution and hence to each other, and the study of one may yield valuable insight into another. Nevertheless, the subject contains some complications and pitfalls that are likely occasionally to rise up and may somehow confuse even some experts in it.

Here is a synopsis of this chapter in mathematical language: Let $-H$ be the Laplacian on scalar functions in a compact region in $\mathbf{R}^3$ with smooth Dirichlet boundary; let $K(t) = \mathrm{Tr}\, e^{-tH}$ be the heat kernel trace, $T(t) = \mathrm{Tr}\, e^{-t\sqrt{H}}$ the trace of the cylinder (Poisson) kernel, and $N(\omega^2)$ the eigenvalue counting function. Loosely speaking, the small-t asymptotics of K and T are in close correspondence with the averaged large-ω asymptotics of N, but there are some subtleties that can be confusing. (1) Nonnegative integer powers of t in the expansion of K do not correspond to (negative) integer powers of ω^2 in the expansion of $dN/d(\omega^2)$ (even after the latter has been well defined by averaging). Instead, they give rise to terms $\delta^{(n)}(\omega^2)$ in the *moment asymptotic expansion* of a distribution (which is actually an expansion in a parameter, not in ω). (2) The expansion of T contains additional, nonlocal spectral invariants, which show up in $dN/d\omega = 2\omega\, dN/d(\omega^2)$, filling in the missing odd negative integer powers. The first of these, $O(t^0)$, or its electromagnetic analogue, gives the Casimir energy in quantum field theory with idealized boundary conditions. (3) Negative powers in T physically represent "divergences" that must be explained or argued away. The term of order t^{-1} is particularly subtle because it corresponds to the first "moment" term in N (also to the topological (index) or Kac term, $O(t^0)$, in K). Thus there is no $O(\omega^{-1})$ term in $dN/d\omega$, so this term in T

has correctly been said to come from low-frequency oscillations of the eigenvalue density rather than high-frequency asymptotics; however, that does not mean that it is one of the nonlocal cylinder-kernel terms. Also, although a divergent local energy density near the boundary does exist, because of an algebraic accident (unrelated to the "moment" issue!) the coefficient of this term in $T(t)$ actually turns out to be zero. However, recent work in physics indicates that the T expansion does not give a trustworthy model of the energy in a realistic system, and in a better model a nonzero $O(t^{-1})$ contribution reappears (rendered finite by regularization). This chapter works through these observations, roughly in order.

8.2 Mathematical setting and notation

Although the setting could be greatly generalized, here we assume that Ω is a compact region in $\mathbb{R}^d$ with smooth boundary, and H is an associated positive self-adjoint operator with pure point spectrum. In the situation of most interest for Casimir physics, $-H = \nabla^2$ is the Laplacian on scalar functions in Ω with the Dirichlet boundary condition. We use the two notations

$$\lambda_n = \omega_n^2 \tag{8.2.1}$$

for the nth eigenvalue of H.

The heat kernel, $K(t, \mathbf{x}, \mathbf{y})$, solves the initial-value problem for $-\frac{\partial u}{\partial t} = Hu$:

$$u(t, \mathbf{x}) = e^{-tH} u(0, \mathbf{x}) = \int_\Omega K(t, \mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y}. \tag{8.2.2}$$

We have the famous heat kernel expansion,

$$K(t, \mathbf{x}, \mathbf{y}) \sim (4\pi t)^{-d/2} e^{|\mathbf{x}-\mathbf{y}|^2/4t} \sum_{s=0}^{\infty} a_s(\mathbf{x}, \mathbf{y}) t^{s/2}, \tag{8.2.3}$$

$$\mathrm{Tr} K = \int_\Omega K(t, \mathbf{x}, \mathbf{x}) \, d\mathbf{x} \sim (4\pi t)^{-d/2} \sum_{s=0}^{\infty} a_s[\Omega] t^{s/2}. \tag{8.2.4}$$

Less well known, but more pertinent to vacuum energy, is the cylinder (or Poisson) kernel, $T(t, \mathbf{x}, \mathbf{y})$. It solves the initial-value problem for

$$-\frac{\partial^2 u}{\partial t^2} = Hu, \qquad \lim_{t \to +\infty} u(t, \mathbf{x}) = 0 : \tag{8.2.5}$$

$$u(t, \mathbf{x}) = e^{-t\sqrt{H}} u(0, \mathbf{x}) = \int_\Omega T(t, \mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y}. \tag{8.2.6}$$

T is the t-derivative of another kernel, $\bar{t}$, which solves the same problem as 8.2.5 except that $\frac{\partial u(0,\mathbf{x})}{\partial t}$ is the initial data. (Note that t in 8.2.2 and 8.2.6 can be thought of as related by a Wick rotation to the physical time in nonrelativistic (Schrödinger) and relativistic (wave) equations, respectively.) The cylinder kernel has a trace expansion 8.6.1 similar to, and related to, 8.2.4, to which we turn in 8.6.

Let $N(\lambda)$ be the number of eigenvalues less than or equal to λ. Then the density of eigenvalues — the derivative of N — is a distribution, having a Dirac delta function at each eigenvalue. Also, it depends on the variable of integration: $\frac{dN}{d\omega} \neq \frac{dN}{d\lambda}$. Instead,

$$\frac{dN}{d\omega} = 2\omega \frac{dN}{d(\omega^2)} = 2\sqrt{\lambda} \frac{dN}{d\lambda} . \tag{8.2.7}$$

Famously, the counting function N obeys *Weyl's law*: As $\omega \to \infty$,

$$N(\omega^2) \propto \omega^d. \tag{8.2.8}$$

(This holds for a second-order operator in dimension d acting in a *compact* region Ω.)

8.3 Properties and problems of the Weyl series

An obvious question is whether 8.2.8 is the start of an asymptotic expansion; that is, whether one can write something like

$$N(\omega^2) \sim \sum_{s=0}^{N} g_s \omega^{d-s} .$$

Because TrK is the Laplace transform of $\frac{dN}{d\lambda}$, one can show, by calculations like those in [41], that the coefficients in this series, if it existed, would be determined by the coefficients $a_s[\Omega]$ in the rigorous asymptotic series 8.2.4. (The Laplace transform of λ^{p-1} is proportional to t^{-p}, at least for $p > 0$.) However, it turns out that only the leading term of the Weyl series is genuinely asymptotic:

$$N(\omega^2) = \sum_{s=0}^{M} g_s \omega^{d-s} + E_M(\omega) \tag{8.3.1}$$

where $E_M(\omega)$ is usually *not* of order $O(\omega^{d-M-1})$. Instead, when $M > 0$, in general E_M is of the same order as the previous terms in the series (but is oscillatory). This problem has been understood for ages; the oscillations are related to periodic orbits of the classical system with Hamiltonian H.

A related but less well known problem is that the proposed series and its formal derivatives become quite problematical when the exponents of ω cease to be positive. Consider, for example, the term $g_d\,\omega^0$. Its derivative vanishes, so the eigenvalue density $\frac{dN}{d\lambda}$ cannot contain a term proportional to λ^{-1}. In fact, if a term with that asymptotic behavior did exist, its Laplace transform would contain $\log t$, and the same is true of any negative integral power of λ; but one knows that such terms do not appear in the heat kernels of second-order differential operators. Where, then, did the heat-kernel coefficient $a_0[\Omega]$ come from? It, and the $a_s[\Omega]$ with $d - s$ both negative and even, come entirely from the contributions of small values of λ to the Laplace transform integral, not from the asymptotic behavior of N at large λ.

Two approaches have been followed to bring clarity and precision into this seemingly rather muddled situation:

- *Riesz–Cesàro means:* The heat-kernel coefficients can be related by the Laplace transformation to the coefficients of the asymptotic expansions of sufficiently high-order iterated antiderivatives of the counting function [44, 43].

- *Generalized Weyl series and distributional moments:* The "missing" coefficients (which "should" accompany negative integral powers of λ) can be identified as those multiplying terms $\delta^{(j)}(\lambda)$ in the asymptotic analysis of $N(\lambda)$ [21, 60]. These are known as "moment terms" in the theory of distributions [33, 27, 24, 37].

One of our principal goals is to elaborate on the second of these.

8.4 Distributions and moment expansions

8.4.1 Review of general theory

In order to make this chapter easy to read, let me give a very brief review of the theory of distributions. Recall that distributions are defined as linear functionals on spaces of very well-behaved functions called *test functions*. A convenient test-function space is

$$\mathcal{D}(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) \colon \phi \text{ compactly supported.}\} \tag{8.4.1}$$

The corresponding space of distributions is the *dual space* $\mathcal{D}'(\mathbb{R}^n)$, comprising the linear functionals on $\mathcal{D}(\mathbb{R}^n)$ (that are continuous in the weak topology). The action of $f \in \mathcal{D}'$ on $\phi \in \mathcal{D}$ is written $\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle$ and generalizes the inner-product integral

$$\int_{\mathbb{R}^n} f(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}, \tag{8.4.2}$$

to which it reduces when f is a nonsingular ordinary function. These concepts extend to, for example, functions and distributions defined on Ω, with technicalities we shall not discuss here. Also, we sometimes need to refer to distributions taking values in some other vector space (rather than $\mathbb{R}$ or $\mathbb{C}$), as in the case of $D'(\mathbb{R}, L(\mathcal{X}, \mathcal{H}))$ in Definition 25.

Moments are defined as the results of applying distributions to power functions. Because the powers do not have compact support, it is necessary to work with a

larger test-function space and hence a more restricted distribution space. Loosely speaking, we need the integral 8.4.2 to converge when $\phi(x) = x^q$. Let

$$\mathcal{K}(\mathbb{R}^n) = \left\{\phi \in C^\infty(\mathbb{R}^n) \colon \exists q \in \mathbb{N} \colon D^{\mathbf{k}}\phi(\mathbf{x}) = O\big(|\mathbf{x}|^{q-|\mathbf{k}|}\big) \quad \text{as } |\mathbf{x}| \to \infty\right\}. \quad (8.4.3)$$

(More technically, one defines the space $\mathcal{K}^{[q]}$ of functions satisfying the condition with a fixed q, which is a Frechét space under certain seminorms, and then defines $\mathcal{K}$ as the union of those spaces with the inductive limit topology.)

The dual space, $\mathcal{K}'(\mathbb{R}^n)$, is the distribution space where moment asymptotic expansions work. Crudely speaking, $f \in \mathcal{K}'(\mathbb{R}^1)$ says that f falls off sufficiently fast at infinity that all the moments $\langle f(x), x^q\rangle$ exist [33].

Theorem 16 (moment asymptotic expansion theorem)**.** [37, Sec. 3.3] Function f is in $\mathcal{K}'(\mathbb{R}^n)$ if and only if $f(\lambda\mathbf{x})$ admits the *moment asymptotic expansion*:

$$f(\lambda\mathbf{x}) \sim \sum_{|\mathbf{k}|=0}^{\infty} \frac{(-1)^{|\mathbf{k}|}\,\mu_{\mathbf{k}} D^{\mathbf{k}}\delta(\mathbf{x})}{\lambda^{|\mathbf{k}|+n}\mathbf{k}!} \quad \text{as } \lambda \to \infty. \quad (8.4.4)$$

Here

$$\mu_{\mathbf{k}} = \left\langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\right\rangle = \left\langle f(\mathbf{x}), x_1^{k_1}...x_n^{k_n}\right\rangle, \qquad \mathbf{k} \in \mathbb{N}^n. \quad (8.4.5)$$

Formula 8.4.4 means that if $\phi \in \mathcal{K}(\mathbb{R}^n)$, then for each M,

$$\langle f(\lambda\mathbf{x}), \phi(\mathbf{x})\rangle = \sum_{|\mathbf{k}|=0}^{M} \left\langle \frac{(-1)^{|\mathbf{k}|}\,\mu_{\mathbf{k}} D^{\mathbf{k}}\delta(\mathbf{x})}{\lambda^{|\mathbf{k}|+n}\mathbf{k}!}, \phi(\mathbf{x})\right\rangle + o\left(\frac{1}{\lambda^{M+n}}\right)$$

as $\lambda \to \infty$. Notice that it is an expansion in the parameter λ, not in the variable x (which will be ω or ω^2 in our spectral applications). It is this change in point of view that makes it logically possible to incorporate part of the content of f at small x into the series formally describing the behavior of f at large x [21, 60]. In one dimension, the theorem says that $f \in \mathcal{K}'(\mathbb{R})$ is equivalent to the applicability of the moment expansion

$$f(\lambda\mathbf{x}) \sim \sum_{k=0}^{\infty} \frac{(-1)^k \mu_k \delta^{(k)}(x)}{\lambda^{k+1}k!} \quad \text{as } \lambda \to \infty, \quad (8.4.6)$$

where $\mu_k = \langle f(x), x^m \rangle$.

8.4.2 The spectral density

Let $\mathcal{H}$ be a Hilbert space and let H be a self-adjoint operator with domain $\mathcal{X} \subset \mathcal{H}$. Then H admits a spectral decomposition $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$, such that (in a weak sense)

$$H = \int_{-\infty}^{\infty} \lambda \, dE_\lambda \, .$$

In most cases of interest in this paper, this equation is a discrete sum

$$H = \sum_{n=1}^{\infty} \lambda_n P_n \, ,$$

where $P_n = E_{\lambda_n} - E_{\lambda_{n-1}}$ is the orthogonal projection onto the eigenvectors with eigenvalue λ_n.

Definition 25. *The* spectral density $e_\lambda = dE_\lambda/d\lambda$ *is the distribution in* $D'(\mathbb{R}, L(\mathcal{X}, \mathcal{H}))$ *such that*

$$\langle e_\lambda, \phi(\lambda) \rangle_\lambda = \int_{-\infty}^{\infty} \phi(\lambda) \, dE_\lambda \, .$$

One sometimes denotes e_λ by $\delta(\lambda - H)$.

For example, the identity operator is $I = \langle e_\lambda, 1 \rangle = \int_{-\infty}^{\infty} dE_\lambda$, and H itself is $H = \langle e_\lambda, \lambda \rangle = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$, meaning that

$$(Hx|y) = (\langle e_\lambda, \lambda \rangle_\lambda \, x|y) = \int_{-\infty}^{\infty} \lambda \, d\left(E_\lambda x|y\right) \text{ for all } x \in \mathcal{X} \text{ and } y \in \mathcal{H}.$$

(Here we write the inner product as $(\cdot|\cdot)$ to avoid confusion with the evaluation of a distribution, $\langle \cdot, \cdot \rangle$.)

Let $\mathcal{X}_n$ denote the domain of H^n and let $\mathcal{X}_\infty = \cap_{n=1}^{\infty} \mathcal{X}_n$. Then $\langle e_\lambda, \lambda^n \rangle = H^n \in L\left(\mathcal{X}_\infty, \mathcal{H}\right)$ exists. Hence $e_\lambda \in \mathcal{K}'(\mathbb{R}, L(\mathcal{X}_\infty, \mathcal{H}))$.

Theorem 17. [37, (6.328)] The moment asymptotic expansion of $e_{\lambda\sigma}$ is

$$e_{\lambda\sigma} \sim \sum_{k=0}^{\infty} \frac{(-1)^k H^k \delta^{(k)}(\lambda)}{\sigma^{k+1} k!} \quad \text{as } \sigma \to \infty. \tag{8.4.7}$$

We are now ready to address the Weyl expansion specifically. The trace of P_n is the number of eigenvectors with eigenvalue λ_n , so the trace of E_λ is $N(\lambda)$;

$$N(x) = \sum_{\lambda_n \le x} 1 = \sum_n \theta(x - \lambda_n). \tag{8.4.8}$$

Observe that

$$N'(x) = \sum_{\lambda_n \le x} \delta(x - \lambda_n), \tag{8.4.9}$$

the trace of e_λ. However, if we integrate 8.4.7 term by term, we get a wrong answer, inconsistent with 8.2.8. The problem is that when we take the trace of E_λ to get $N(\lambda)$, the result is no longer in $\mathcal{K}'$, so the moment expansion theorem does not apply without modification. The (Cesàro-averaged) asymptotic expansion (see [33] and [37, Chap. 6]) of $N(\lambda)$ contains not only the moment terms but also powers of λ. The exponents may be positive or negative, but negative integers do not occur (being replaced by the moments). Terms of negative half-integer order become literally asymptotic only after repeated indefinite integration to form the Riesz means (see **??** and [44, 43, 2]).

In the Riesz means of sufficiently high order, the difference between moments and power terms is washed out. Correspondingly, in TrK there is no deep distinction between integral and half-integral powers of t. Taking the direct product with a one-dimensional system (the interval or the one-torus, with $K \propto t^{-\frac{1}{2}}$) interchanges odd and even powers of $\sqrt{t}$.

8.4.3 Green kernels associated with spectral decompositions

The same ideas apply to the Green functions, or integral kernels, associated with operators. An operator H on $\mathcal{D}(\Omega)$ can be realized by a distributional kernel $h \in \mathcal{D}'(\Omega \times \Omega)$:

$$Hf\,(x) = \langle h(x,y),\, f(y)\rangle_y \ .$$

For example,

$$\langle \delta(x-y),\, f(y) \rangle = If\,(x)\,,$$

$$\langle H\delta\,(x-y)\,,\, f\,(y) \rangle = \langle \delta(x-y),\, Hf\,(y) \rangle = Hf\,(x)\,.$$

Similarly, e_λ has an associated kernel $e(x,y;\lambda) \in \mathcal{K}'(\mathbb{R}, \mathcal{D}'(\Omega \times \Omega))$, such that

$$\Big\langle \langle e(x,y;\lambda),\, f(y) \rangle_y\,,\, \phi(\lambda) \Big\rangle_\lambda = \langle e_\lambda\,,\, \phi(\lambda) \rangle_\lambda\, f\,(x) = \phi(H) f(x). \tag{8.4.10}$$

Observe that $\langle e(x,y;\lambda), \lambda^n \rangle_\lambda = H^n \delta\,(x-y)\,.$ So we have

Corollary 4. *[24, (61)]*

$$e(x,y;\sigma\lambda) \sim \sum_{k=0}^{\infty} \frac{(-1)^k\, H^k \delta\,(x-y)\, \delta^{(k)}(\lambda)}{\sigma^{k+1} k!} \quad \text{as } \sigma \to \infty. \tag{8.4.11}$$

Let $K(t,x,y) = \left\langle e(x,y;\lambda),\, e^{-\lambda t} \right\rangle$ be the heat kernel, as introduced in 8.2.2. Then

$$\begin{aligned} \left\langle e(x,y;\lambda),\, e^{-\lambda t} \right\rangle &= \frac{1}{t} \left\langle e(x,y;\lambda/t),\, e^{-\lambda} \right\rangle \\ &\sim \sum_{k=0}^{\infty} \left\langle \frac{(-1)^k\, H^k \delta\,(x-y)\, \delta^{(k)}(\lambda) t^k}{k!},\, e^{-\lambda} \right\rangle \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, H^k \delta\,(x-y)\, t^k}{k!} \quad \text{as } t \to 0. \end{aligned} \tag{8.4.12}$$

This is a representation of K as a distribution on $\Omega \times \Omega$, so it does not directly give the asymptotic expansion "on diagonal", 8.2.4.

8.5 Example: The Dirac comb

Kolomeisky et al. [60] work out some examples where the moment terms can be calculated by the Euler–Maclaurin formula, because the eigenvalues are equally spaced (possibly after a change of variable). Here we study the simplest such case and reproduce the conclusions of [60] by another route.

Theorem 18. [27, Lemma 2.11] If $g \in \mathcal{K}$ and if $\int_0^\infty g(x)dx$ is defined, then

$$\sum_{n=1}^{\infty} g(n\varepsilon) = \left\langle \sum_{n=1}^{\infty} \delta(x-n),\, g(\varepsilon x) \right\rangle$$
$$= \frac{1}{\varepsilon}\int_0^\infty g(x)\,dx + \sum_{n=0}^{\infty} \frac{\zeta\,(-n)\,g^{(n)}(0)}{n!}\varepsilon^n + o(\varepsilon^\infty), \tag{8.5.1}$$

where $\zeta(x)$ is the zeta function. Thus $\zeta(-n)$ are the moments in this case. That is,

$$\sum_{n=1}^{\infty} \delta\left(\frac{x}{\varepsilon} - n\right) \sim \theta(x) + \sum_{n=0}^{\infty} \frac{(-1)^n\,\zeta(-n)\delta^{(n)}(x)}{n!}\varepsilon^{n+1} \quad \text{as } \varepsilon \downarrow 0. \tag{8.5.2}$$

The *generalized Weyl expansion* found in [60] is

$$\sum_{n=1}^{\infty} g\left(\frac{\pi n}{a}\right) = \frac{\pi}{a}\int_0^\infty g(q)\,dq - \frac{g(0)}{2} - \frac{\pi g'(0)}{12a} + \frac{\pi^3 g^{(3)}(0)}{720a^3} - \cdots. \tag{8.5.3}$$

That is, if one defines $G(q) = \sum \delta(q-n)$, then

$$G\left(\frac{aq}{\pi}\right) \sim \frac{a}{\pi} - \frac{\delta(q)}{2} + \frac{\pi\delta'(q)}{12a} - \frac{\pi^3\delta^{(3)}(q)}{720a^3} + \cdots \quad \text{as } a \to \infty. \tag{8.5.4}$$

Here a denotes the radius of the one-dimensional interval, half the distance between the (Dirichlet) boundaries.

By Theorem 18, if $g \in \mathcal{K}$ and if $\int_0^\infty g(x)\,dx$ is defined, then

$$\sum_{n=1}^{\infty} g(n\varepsilon) \sim \frac{1}{\varepsilon}\int_0^\infty g(x)\,dx + \sum_{n=0}^{\infty} \frac{\zeta(-n)g^{(n)}(0)}{n!}\varepsilon^n \quad \text{as } \varepsilon \downarrow 0. \tag{8.5.5}$$

If we let $\varepsilon = \frac{\pi}{a}$ and compute the zeta-function terms, we obtain the same equation as 8.5.3. Note that the first term is a Weyl (high-frequency asymptotic) term and the rest are moment terms, related to the low-frequency content of N'.

The most elementary application of this expansion is when the eigenvalues are proportional to n^2. Then

$$N(x) = \sum_{n\geq 0} \theta(x-n^2), \qquad N'(x) = \sum_{n\geq 0} \delta(x-n^2).$$

If we set $g(x^2) = f(x)$, we can then calculate

$$\left\langle \sum_{n=1}^{\infty} \delta(x-n^2),\ g(\varepsilon x) \right\rangle = \sum_{n=1}^{\infty} f(\varepsilon^{1/2} n)$$
$$\sim \frac{1}{2\varepsilon^{1/2}} \int_0^{\infty} \frac{g(x)}{\sqrt{x}} dx \quad \text{as } \varepsilon \downarrow 0 \tag{8.5.6}$$

for the first term. But, actually, the first term is all: For the moment terms we can calculate the moments of N' to be

$$\mu_k = \left\langle \sum_{n \geq 0} \delta(x-n^2), x^k \right\rangle = \zeta(-2k) = 0.$$

That is,

$$\sum_{n=1}^{\infty} \delta\left(\frac{x}{\varepsilon} - n^2\right) = \frac{\theta(x)}{2\varepsilon^{1/2}\sqrt{x}} + o(\varepsilon^{\infty}) \quad \text{as } \varepsilon \downarrow 0. \tag{8.5.7}$$

As an example of the example, let $Hy = y''$ be considered on the domain $\mathcal{X} = \{y \in C^2[0,\pi] : y(0) = y(\pi) = 0\}$ inside $L^2[0,\pi]$. The eigenvalues are $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$, with normalized eigenfunctions $\phi_n(x) = \sqrt{2/\pi} \sin nx$. Therefore,

$$e(x, y; \lambda) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin nx \sin ny\, \delta(\lambda - n^2),$$

and

$$e(x, x, \lambda) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin^2 nx\, \delta(\lambda - n^2)$$
$$= \frac{1}{\pi} \sum_{n=1}^{\infty} (1 - \cos 2nx)\, \delta(\lambda - n^2).$$

Hence

$$e(x, x, \lambda/t) \sim \frac{1}{\pi} \frac{\theta(\lambda) t^{1/2}}{2\sqrt{\lambda}} \quad \text{as } t \downarrow 0 \tag{8.5.8}$$

for $0 < x < \pi$. From this we recover the asymptotics of the heat kernel:

$$\left\langle e(x, x; \lambda), e^{-\lambda t} \right\rangle \sim \frac{1}{2\pi t^{1/2}} \int_0^{\infty} \frac{e^{-\lambda}}{\sqrt{\lambda}} d\lambda$$
$$= \frac{1}{(4\pi t)^{1/2}} \quad \text{as } t \downarrow 0. \tag{8.5.9}$$

This result is nonuniform in x, so it does not give the correct trace expansion 8.2.4, which contains an additional term representing the effect of the Dirichlet (or alternative) boundaries.

Furthermore, we can study the relation to the expansion of the cylinder kernel. We know from 8.5.7 that

$$\sum_{n=1}^{\infty} \delta\left(\frac{x}{\varepsilon} - n^2\right) = \frac{\theta(x)}{2\varepsilon^{1/2}\sqrt{x}} + o(\varepsilon^{\infty}) \quad \text{as } \varepsilon \downarrow 0.$$

If we set $x = \omega^2$, we might expect

$$\sum_{n=1}^{\infty} \delta\left(\frac{\omega^2}{\varepsilon} - n^2\right) = \frac{\theta(\omega)}{2\varepsilon^{1/2}\omega} + o(\varepsilon^{\infty}) \quad \text{as } \varepsilon \downarrow 0.$$

But in fact, it's not so trivial:

$$\sum_{n=1}^{\infty} \delta(\omega^2 - n^2) = \sum_{n=1}^{\infty} \frac{1}{2n}\left[\delta(\omega - n) + \delta(\omega + n)\right], \tag{8.5.10}$$

so by 8.5.2 we have

$$\sum_{n=1}^{\infty} \delta\left(\frac{\omega^2}{\varepsilon} - n^2\right) \sim \frac{\theta(\omega)}{2\varepsilon^{1/2}\omega} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,\zeta(-n)\delta^{(n+1)}(\omega)}{2\,(n+1)!}\varepsilon^{\frac{n+1}{2}} + o(\varepsilon^{\infty}) \quad \text{as } \varepsilon \downarrow 0. \tag{8.5.11}$$

The significance of the extra terms in 8.5.11 will become clear in the next two sections.

8.6 The cylinder kernel

Because the material of this section and the next has been extensively covered before [44, 43, 2], we shall be relatively brief. But to atone for a certain vagueness at certain points in earlier sections, we give complete and precise formulas. Please consult those papers for references to the fundamental work of Hardy and Hörmander on which the theorems are based.

The cylinder kernel of 8.2.6 has the trace expansion

$$\sum_{n=1}^{\infty} e^{-t\omega_n} = \mathrm{Tr}T = \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s t^{-d+s} \log t. \tag{8.6.1}$$

It is convenient to redefine the expansion coefficients in the heat trace 8.2.4 by

$$\sum_{n-1}^{\infty} e^{-t{\omega_n}^2} = \mathrm{Tr}K = \sum_{s=0}^{\infty} b_s t^{(-d+s)/2} \tag{8.6.2}$$

when treating its relations with the cylinder trace and with Riesz means of N.

Theorem 19. The coefficients in the cylinder and heat expansions are related by

$$e_s = \pi^{-1/2} 2^{d-s} \Gamma\left(\frac{d-s+1}{2}\right) b_s \quad \text{if } d-s \text{ is even or positive,}$$

whereas if $d-s$ is odd and negative,

$$f_s = \frac{(-1)^{(s-d+1)/2} 2^{d-s+1}}{\sqrt{\pi}\Gamma\big((s-d+1)/2\big)}\, b_s\,, \quad \text{but } e_s \text{ is undetermined by the } b_r\,.$$

The new coefficients e_s (with $d-s$ odd and negative) are *new* spectral invariants. They are *nonlocal* in their dependence on the geometry of Ω. The first one, e_{d+1}, has the interpretation of renormalized Casimir vacuum energy in quantum field theory.

Riesz means are a generic tool of long standing, but we consider only those of the counting function, N. The "old" Riesz means (with respect to λ) are defined by the α-fold iterated (simplex) indefinite integration:

$$R^\alpha_\lambda N\,(\lambda) = \frac{1}{\alpha!}\,\lambda^{-\alpha} \int^\lambda \overset{\alpha}{\cdots} \int N(\tilde{\lambda})\, d\tilde{\lambda}.$$

But we may also have Riesz means with respect to $\omega = \sqrt{\lambda}$:

$$R^\alpha_\omega N\,(\omega) = \frac{1}{\alpha!}\,\omega^{-\alpha} \int^\omega \overset{\alpha}{\cdots} \int N(\tilde{\omega}^2)\, d\tilde{\omega}.$$

Theorem 20. There exist asymptotic formulas of the forms

$$R^\alpha_\lambda N = \int_{s=0}^{\alpha} a_{\alpha s}\lambda^{(d-s)/2} + O\left(\lambda^{(d-\alpha-1)/2}\right),$$

$$R^\alpha_\omega N = \int_{s=0}^{\alpha} c_{\alpha s}\omega^{d-s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\alpha} d_{\alpha s}\omega^{d-s}\log\omega + O\left(\omega^{d-\alpha-1}\log\omega\right).$$

Contrast 8.3.1. In words, the oscillations that prevent the Weyl series from being asymptotic beyond the first term are averaged out by the integrations, so that the corresponding series for a Riesz mean is a valid asymptotic approximation to a certain higher order.

Only the coefficients a_{ss} are truly important; the $a_{\alpha s}$ with $s < \alpha$ contain redundant information.

Theorem 21. The heat-kernel coefficients are proportional to the old Riesz means:

$$b_s = \frac{\Gamma\left((d+s)/2+1\right)}{\Gamma(s+1)}\, a_{ss}\,.$$

The cylinder-kernel coefficients are related to the new Riesz means by

$$e_s = \frac{\Gamma(d+1)}{\Gamma(s+1)}\, c_{ss} \quad \text{if } d-s \text{ is even or positive,}$$

$$f_s = -\frac{\Gamma(d+1)}{\Gamma(s+1)}\, d_{ss}\,, \quad e_s = \frac{\Gamma(d+1)}{\Gamma(s+1)}\left[e_{ss} + \psi(d+1)d_{ss}\right]$$

if $d-s$ is odd and negative.

It is therefore no surprise that the asymptotic coefficients of the old and new Riesz means are related by formulas [44] very similar to those relating the heat and cylinder coefficients. In particular, when $d-s$ is odd and negative, c_{ss} is *undetermined* by the a_{rr}. There are integral operations leading from old Riesz means to new Riesz means and vice versa:

- When going old $\to$ new, the new c_{ss} arise from the lower limit of integration, bringing in new information about $N(\omega^2)$ at low frequencies.

- When going new $\to$ old, the c_{ss} are multiplied by numerical coefficients that turn out to equal 0 when $d - s$ is odd and negative, so their information is lost in those cases.

8.7 One very special term

The term of order t^0 in a heat kernel is geometrically dimensionless and has topological significance. It counts eigenfunctions with eigenvalue zero.

More precisely, when $H_1 = A^*A$ and $H_2 = AA^*$,

$$\mathrm{Tr}K[H_1] - \mathrm{Tr}K[H_2]$$

is independent of t (only the $O(t^0)$ terms fail to cancel) and equals the *index* of the operator A. Kolomeisky et al. [60] call it the *Kac term*. We propose that *index term* is more descriptive.

The index term in K corresponds to a constant term in (the averaged) N. More precisely, since $N(\lambda) = 0$ for $\lambda < 0$, the term is a multiple of the Heaviside function, $\theta(\lambda)$. Therefore, it gives rise to a multiple of the Dirac delta distribution in the eigenvalue density $dN/d\lambda$ or $dN/d\omega$. Thus, whether this term is a moment or a high-frequency asymptotic term depends on which function one is looking at.

Because there is no $\log\omega$ term in N, there is no $O(\omega^{-1})$ term in $dN/d\omega$. Therefore, it is correctly said that the index term comes from the low-frequency, not high-frequency, behavior of the spectral density. Note, however, that it is a true heat-kernel term, locally determined by the geometry, not one of the new cylinder-kernel terms.

The index term corresponds to a term $O(t^0)$ in the cylinder trace, T. There is no $O(\log t)$ term. Hence there is no $O(t^{-1})$ term in dT/dt. dT/dt has the physical interpretation of (-2 times) the vacuum energy of a scalar field, subjected to *timelike point-splitting regularization* (see references in [**?**, 67]). Negative powers of

t here represent buildups of energy against the idealized boundary, which must be absorbed into the description of the boundary itself (*renormalization*).

We see that no renormalization is needed in order t^{-1}; nevertheless, calculations of local energy density show that large boundary energy is indeed there! (Energy density proportional to z^{-2} (z = distance to boundary) formally implies energy proportional to t^{-1}.)

Candelas [68] correctly stresses the physical necessity of the $O(z^{-2})$ energy density and argues (apparently incorrectly) that an $O(t^{-1})$ term in the total energy arises from $dN/d\omega$. Other authors [10, 60] correctly point out that no such term exists. Here we attempt to dispose of this controversy.

First, in timelike point-splitting regularization the large energy density near the boundary is compensated by a larger opposite-sign density concentrated even closer to the boundary. Therefore, there is no mathematical contradiction between the local and global statements.

Second, there are now physical reasons to believe that for certain purposes, spacelike point-splitting gives a more trustworthy model of the energy in a realistic system [**?**, 67]. In that framework the contribution of the index term does not vanish after all.

Third, quite apart from the technical criticisms of [68] in [60], a close examination of [68] shows that its argument (as concerns the total energy, not the boundary energy density) does not note the possibility that the overall coefficient of the term in question might turn out to be zero. In fact, as just remarked, it *is* zero in timelike but not in spacelike point-splitting.

Thus a 30-year-old controversy has been revived, resolved, and rendered irrelevant in roughly the same year (2012–13).

In summary, the index term has a number of special properties, which are not particularly closely related to each other:

- It does not contribute to the total energy in timelike regularization, because its contribution to TrT is killed by the differentiation.

- Because it sits on the boundary between moments and high-frequency asymptotics, there is no corresponding $O(\lambda^{-1})$ term in the eigenvalue density.

- Some moments (low-frequency contributions) in TrT are new spectral invariants that do not appear in TrK. *However, this term is not one of them.* They start immediately afterwards.

There is a tendency for these three issues to become muddled together in our thinking.

8.8 Summary of the subtleties

- Certain powers in the heat-kernel expansion match moment (delta function) terms (not powers) in the "generalized Weyl expansion". A generalized Weyl expansion that is introduced in the paper [60] can be realized as the moment asymptotic expansion of the Dirac comb $\sum_{n=1}^{\infty} \delta\left(x/\varepsilon - n\right)$ acting on a test function.

- The missing powers match terms in the cylinder-kernel expansion. These new, nonlocal spectral invariants show up in the Riesz-Cesàro asymptotics of N with respect to ω (but not ω^2).

- Some confusion and controversy in the physics literature is related to this fact: The term in the t-derivative of the cylinder kernel trace corresponding to the "index" term in the heat kernel trace vanishes because of an algebraic

accident, but nevertheless quantum field theory predicts a divergent boundary energy density proportional to that spectral invariant. That this term is a moment, not a high-frequency part of the eigenvalue density, is beside the point. Recent physics suggests that this method of regularizing the energy is not reliable anyway.

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Vita

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