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Generalized Quotient Rings.

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by

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ABSTRACT

Let $D$ be a domain with quotient field $K$. In this thesis, we consider "generalized" quotient rings of $D$ obtained by replacing a multiplicative system of elements of $D$ by a multiplicative system $S$ of non-zero subsets of $D$. We call the ring $D_S = \{x \in K | \text{there exists } A \in S \text{ such that } xA \subseteq D\}$ a Generalized Quotient Ring (GQR) of $D$ and we call $S$ a Generalized Multiplicative System (GMS) of subsets of $D$. If $Q$ is a ring containing $D$ and there exists a GMS of subsets of $S$ of $D$ such that $Q = D_S$ and $A \in S$ implies $AQ = Q$, then $Q$ is said to be a Restricted GQR (RGQR) of $D$.

It is the purpose of this thesis to study GQR's of $D$. It is clear that ideal transforms introduced by Nagata in [N.2] are such rings. It is also true that ordinary quotient rings, intersections of localizations and flat overrings (as studied by Richman in [R] and Akiba in [A.1] and [A.2]) of $D$ are such rings.

If $R$ is a commutative ring with identity that is not necessarily a domain, then we may define a GQR of $R$ in much the same manner as we define an ordinary quotient ring of $R$. In the first chapter, we give this definition,
and study the properties of RGQR's of \( R \). We show that
the basic properties of ordinary quotient rings carry over
to RGQR's. We also show that the class of rings with
elements both flat over \( R \) and contained in homomorphic
images of \( R \) is the class of RGQR's of \( R \).

In Chapters II and III, we study properties of
GQR's of domains \( D \). We show each RGQR of \( D \) is an
intersection of localizations of \( D \) and that each inter­
section of localizations of \( D \) is a GQR of \( D \). Thus
if we let \( \mathcal{C}_1, \mathcal{C}_2, \text{ and } \mathcal{C}_3 \) be respectively the class
of RGQR's of \( D \), intersections of localizations of \( D \)
and GQR's of \( D \), then \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \). Much of this
thesis deals with the study of the containment relations
among \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \) over particular types of domains.
For example, if \( D \) is a Prüfer domain, RM-domain or
unique factorization domain, then \( \mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 \); and if
\( D \) is a noetherian or Krull domain, then \( \mathcal{C}_2 = \mathcal{C}_3 \) but
may properly contain \( \mathcal{C}_1 \). We also give an example of a
domain where \( \mathcal{C}_1 < \mathcal{C}_2 < \mathcal{C}_3 \), and give necessary and
sufficient conditions for a ring containing \( D \) to be an
element of \( \mathcal{C}_1, \mathcal{C}_2 \) or \( \mathcal{C}_3 \).

In Chapter IV, we study J-domains where a J-domain
is defined to be a one-dimensional domain in which proper
Ideals are contained in only finitely many maximal ideals. We show that $C_1 = C_2$ and may be properly contained in $C_3$ and also give a necessary and sufficient condition for every ring $Q$ such that $D \subseteq Q \subseteq K$ to be a J-domain.

Much of this chapter deals with generalizing results of RM-domains obtained by Grell in [HG]. For example, if $D$ is a J-domain with quotient field $K$, we give a necessary and sufficient condition for $D[\alpha_1, \ldots, \alpha_n]$ with $\alpha_1 \in K$ to be a RGQR of $D$.
CHAPTER I
Introduction

If $R$ is a commutative ring with identity, then a multiplicative system $M$ of $R$ is a subset of $R$ that is closed under multiplication and does not include zero. Recall that the quotient ring $R_M$ of $R$ is defined as follows [ZSI], page 221. Let $N = \{r \in R \mid$ there exists $m \in M$ such that $xm = 0\}$. Then $N$ is an ideal of $R$. Denote by $h$ the canonical homomorphism $R \rightarrow \mathbb{R}/N$ and observe that $h(M)$ is a subset of the regular elements of $h(R) = \mathbb{R}/N$. If we let $T(h(R))$ be the total quotient ring of $h(R)$, then we define $R_M = \{a \in T(h(R)) \mid$ there exists $h(m) \in h(M)$ such that $ah(m) \in h(R)\}$.

We define a "generalized" quotient ring of $R$ in a similar manner. The idea is to replace a multiplicative system of elements of $R$ by a multiplicative system of subsets of $R$. All rings considered here are commutative with an identity, and $R$ always denotes a ring. If $Q$ is a ring, $T(Q)$ denotes the total quotient ring of $Q$.

Definition 1.1. Let $R$ and $Q$ be rings.
(a) If $A$ and $B$ are subsets of $R$, then we define

$$ AB = \{\sum_{i=1}^n a_i b_i \mid a_i \in A \text{ and } b_i \in B\} \ . \text{ A Generalized Multi-}$$
plicative system (GMS) $S$ of subsets of $R$ is a non-empty collection of subsets of $R$ that does not include $\{0\}$ and that is closed under multiplication.

(b) We say $Q$ is a Generalized Quotient Ring (GQR) of $R$ provided there exists $S$, a GMS of subsets of $R$, such that $Q = R_S$ where $R_S$ is defined to be $R_S = \{a \in T(h(R)) | \text{there exists } A \in S \cdot ah(A) \subseteq h(R)\}$ and where $h$ is the canonical homomorphism $R \xrightarrow{h} R/N$, and $N = \{r \in R | \text{there exists } A \in S \cdot rA = 0\}$. (Notice that $h(R) \subseteq Q = R_S$; throughout this thesis, we shall use $\subseteq$ to denote containment and $<$ to denote proper containment).

If $S$ is a GMS of subsets of $R$, then, by the above, $R_S$ is a subset of the total quotient ring of a particular homomorphic image of $R$, and it is straightforward to show that it is a subring of this homomorphic image (see Theorem 1.1). For convenience, we shall always denote this homomorphism by $h$. Whence if $A \subseteq R$, then by $AR_S$, we mean $h(A)R_S$.

Definition 1.2. If $R$ and $Q$ are rings, then $Q$ is said to be a Restricted GQR (RGQR) of $R$ provided there exists a GMS of subsets $S$ of $R$ such that $Q = R_S$ and such that $A \in S$ implies $AR_S = R_S$. In this case,
S is said to be a **Restricted GMS (RGMS)** of R.

**Remark:** We observe that ordinary quotient rings of a ring R are RGQR's of R. Also note if S is a GMS of subsets of R and if \( J = \{ AR| A \in S \} \), then J is a GMS of subsets of R and \( R_S = R_J \). We therefore lose no generality in assuming that a GMS consists of ideals of R.

Also note that if S is a GMS such that each A \( \in \) S contains a regular element, then h is the identity map and \( R_S = \{ a \in T(R)| \text{there exists } A \in S \text{ such that } aA \subset R \} \). This holds in particular if R is a domain.

It is the primary purpose of this thesis to study GQR's of a ring R (especially when R is a domain).

In the first chapter, we study the properties of RGQR's of R. Although ordinary quotient rings of R are RGQR's of R, we show the converse is false. We also show that the basic properties of ordinary quotient rings carry over to RGQR's, but point out that many of these properties do not carry over to GQR's in general.

**Definition 1.3:** If M is an R-module, then M is R-flat provided \( [A:x] \cdot M = [AM:x] \) for every ideal A in R and every \( x \in R \) (see Bourbaki, Alg. Comm., Ch. I, §2, ex. 22). Recall that \( [A:x] = \{ r \in R| rx \in A \} \) and \( [AM:x] = \).
Akiba proves in [A.1] that if $R^*$ is a ring such that $R \subseteq R^* \subseteq T(R)$, then $R^*$ is $R$-flat if and only if $[R:r^*] \cdot R^* = R^*$ for every $r^* \in R^*$. In addition, Akiba establishes in [A.2] the following result. Let $R, R^*$ be rings and $f: R \rightarrow R^*$ a homomorphism such that $f(R) \subseteq R^* \subseteq T(f(R))$. Then $R^*$ is $R$-flat if and only if $R^*$ is $f(R)$-flat and $[o:xR] \cdot R^* = R^*$ for every element $x$ in the kernel of $f$, where $[o:xR] \cdot R^*$ means $f([o:xR]) \cdot R^*$. 

We show in this chapter that the class of rings that are both flat over $R$ and contained in homomorphic images of $R$ (as in [A.2]) is the class of RGQR's of $R$.

In Chapters II and III, we study properties of GQR's of domains $D$. If $P$ is a proper prime ideal of $D$, then $M = D - P$ is a multiplicative system of elements of $D$ and $D_M$ is denoted by $D_P$. We say that $D_P$ is the localization of $D$ at $P$. In Chapter II, we show each RGQR of $D$ is an intersection of localizations of $D$, and that each intersection of localizations of $D$ is a GQR of $D$. Thus if we let $C_1, C_2$, and $C_3$ be respectively the class of RGQR's of $D$, intersections of localizations of $D$ and GQR's of $D$, then $C_1 \subseteq C_2 \subseteq C_3$. 
Much of this thesis deals with the study of the containment relations among \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \) over particular types of domains. For example, if \( D \) is a noetherian or Krull domain, then \( \mathcal{C}_2 = \mathcal{C}_3 \) but may properly contain \( \mathcal{C}_1 \); and if \( D \) is a Prüfer domain, RM-domain (see definition preceding corollary 3.10) or unique factorization domain, then \( \mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 \). We also give an example of a domain where \( \mathcal{C}_1 < \mathcal{C}_2 < \mathcal{C}_3 \).

If \( A \neq \langle 0 \rangle \) is an ideal of \( D \) and \( n \) is a positive integer, then \( A^{-n} = \{ a \in K | aA^n \subseteq D \} \); and \( T_D(A) \), the transform of \( A \) over \( D \), is \( T_D(A) = \{ a \in K | a \in A^{-n} \text{ for some positive integer } n \} \). It is clear that \( T_D(A) \) is a GQR of \( D \). This notion is due to M. Nagata, and Nagata and several others have studied this ring (e.g. see [N.2] and [N.3]). We generalize several of their results in Chapters II and III.

By a proper ideal of a domain \( D \), we mean an ideal \( A \) such that \( 0 < A < D \). A J-domain is defined to be a domain in which proper ideals are contained in only finitely many maximal ideals and in which proper prime ideals are maximal. A property of J-domains is that each proper ideal can be given a finite, irredundant primary representation (see [ZSI], page 209). Since an RM-domain may be defined to be a noetherian domain in which proper prime ideals are
maximal, then an RM-domain is a J-domain.

In Chapter IV, we study J-domains. We show that \( C_1 \subseteq C_2 \) and may be properly contained in \( C_3 \) and also give a necessary and sufficient condition for every overring of a J-domain to be a J-domain. Most of this chapter, however, deals with generalizing results of RM-domains obtained by Grell in [HG]. For example, if \( D \) is a J-domain with quotient field \( K \), we give a necessary and sufficient condition for \( D[\alpha_1, \ldots, \alpha_n] \) with \( \alpha_i \in K \) to be a RGQR of \( D \).

Throughout this thesis \( R \) shall denote a commutative ring with identity with total quotient ring \( T(R) \) and \( D \) shall denote a domain with identity with quotient field \( K \).

Our notation and terminology is that of Zariski and Samuel, *Commutative Algebra*; in particular we use \( \subseteq \) to denote containment and \( < \) to denote proper containment; by an overring \( Q \) of \( R \), we mean a ring \( Q \) such that \( R < Q \subseteq T(R) \).

Theorem 1.1: If \( S \) be a GMS of subsets of \( R \), then \( R_S \) is a subring of \( T(h(R)) \).

Proof: If \( \alpha_1, \alpha_2 \in R_S \), there exists \( A_1, A_2 \in S \) such that \( \alpha_1 h(A_1) \subseteq h(R) \) and \( \alpha_2 h(A_2) \subseteq h(R) \). Let \( A = A_1 A_2 \in S \). Then \( \alpha_1 \alpha_2 h(A_1 A_2) \subseteq h(R) \) and \( (\alpha_1 + \alpha_2) h(A_1 A_2) \subseteq h(R) \) which implies
that \( R_S \) is a subring of \( T(h(R)) \).

**Example 1.1:** This is an example of a RGQR \( Q \) of \( D \) and a GMS of subsets \( S \) of \( D \) such that \( Q = D_S \) and there exists \( A \in S \) such that \( AQ < Q \). Let \( D = V \) be a valuation ring with a non-maximal proper prime \( P \) that is idempotent (i.e. \( P = P^2 \)). Then \( P^e = PD_P = PD \), \( T^e_P = T_D(P) \), and since \( (P^c)^2 = P^e \), we have

\[
T^e_P = D_P = V_P .
\]

Hence the transform of \( P \) over \( V \) is \( T_V(P) = V_P \), and if \( Q = V_P \), then \( Q \) is a RGQR of \( D = V \) (see Remark on page 3). But we have \( Q = D_S \) where \( S = \{P\} \) and \( PQ < Q \).

We shall presently show that the class of rings whose elements are both flat over \( R \) and contained in the total quotient ring of a homomorphic image of \( R \) is the class of RGQR's of \( R \). But first we need three lemmas.

**Lemma 1.2:** If \( Q \) is a ring such that \( R \subseteq Q \subseteq T(R) \), then \( Q \) is a RGQR of \( R \) if and only if \( [y:x]Q = Q \) for all \( \frac{x}{y} \in Q \) with \( x, y \in R \).

**Proof:** If \( Q \) is a RGQR of \( R \), then there exists a RGMS of subsets \( S \) of \( R \) such that \( R_S = Q \). Let \( \frac{x}{y} \in Q \) with \( x, y \in R \). Then there exists \( A \in S \) such that
$A^x_y \subseteq R$ which implies $A \subseteq [y:x]$. Since $AQ = Q$, then $[y:x]Q = Q$.

Conversely let $S$ be the GMS of $R$ generated by $\Sigma = \{ \frac{x}{y} \in Q \text{ and } x, y \in R \}$. That is, $S$ consists of the element of $\Sigma$ along with all finite products of elements in $\Sigma$. Now if $\frac{x}{y} \in Q$, then $\frac{x}{y}[y:x] \subseteq R$, so that $Q \subseteq R_S$. And since $[y:x]Q = Q$, then $[y:x]R_S = R_S$, so that $R_S$ is a RGQR of $R$. Now let $\alpha \in R_S$ and pick $A \in S$ such that $\alpha A \subseteq R$; note that $AQ = Q$. Then $\alpha \in \alpha Q = \alpha(AQ) = (\alpha A)Q \subseteq RQ = Q$ and therefore $R_S \subseteq Q$.

**Lemma 1.3:** If $Q$ is a ring such that $R \subseteq Q \subseteq T(R)$, then $Q$ is a RGQR of $R$ if and only if $Q$ is $R$-flat.

**Proof:** In view of lemma 1.2, this is obvious from [A.1], theorem 2, page 802 where Akiba proves that $Q$ is flat over $R$ if and only if $[y:x]Q = Q$ for all $\frac{x}{y} \in Q$ with $x, y \in R$. (Note that $[y:x] = [R:x/y]$ for all $x/y \in Q$).

**Lemma 1.4:** If $S$ is a GMS of $R$, then $R_S$ is a GQR of $h(R)$ and if $S$ is a RGMS of $R$, then $R_S$ is a RGQR of $h(R)$. 
Proof: Let \( N' = \{ r \in R \mid h(r)h(A) = o \} \) for some \( A \in S \). If \( N' = (o) \), then it is clear that \( h(S) = \{ h(A) \mid A \in S \} \) is a GMS of \( h(R) \) and that \( h(R)h(S) = R_S \).

Moreover, if \( h(A)R_S = R_S \) for all \( A \in S \), then \( N' = (o) \), and thus \( h(S) \) is a RGMS of \( h(R) \). Now assume \( N' \neq (o) \) and let \( \overline{S} \) be the GMS of \( h(R) \) generated by

\[
\{ [\theta; a] \mid a, \theta \in h(R), \theta \text{ is a regular element of } h(R) \}.
\]

We show \( h(R)\overline{S} = R_S \). Note that each generating element \( [\theta; a] \) of \( \overline{S} \) contains a regular element \( \theta \) of \( h(R) \) and therefore each element of \( \overline{S} \) contains a regular element of \( h(R) \). Hence \( \overline{N} = \{ \gamma \in h(R) \mid h(\overline{A}) = (o) \} \) for some \( \overline{A} \in \overline{S} = (o) \) and therefore \( R_{\overline{S}} \subseteq T(h(R)) \). Now let \( \delta = \frac{h(r)}{h(e)} \in h(R)\overline{S} \) and let \( \overline{A} = \prod_{1}^{k} [\theta; a_i] \) be an element of \( \overline{S} \) such that \( \delta \overline{A} \subseteq h(R) \).

For \( i = 1, \ldots, k \), let \( A_i \in S \) such that \( \frac{a_i}{\theta_i} \subseteq h(A_i) \subseteq h(R) \). Then \( A = \prod_{1}^{k} A_i \in S \) and each \( h(A_i) \subseteq [\theta; a_i] \), so that \( \delta h(A) \subseteq \delta h(\overline{A}) \subseteq h(R) \) and \( \delta \in R_S \). And if \( \lambda = \frac{h(r)}{h(s)} \in R_S \), then \( [h(s); h(r)] \in \overline{S} \) and since \( \lambda [h(s); h(r)] \subseteq h(R) \), we have that \( \lambda \in h(R)\overline{S} \).
Theorem 1.4: If $S$ is a RGMS of $R$, then $h(R) \subseteq R_S \subseteq T(h(R))$, where $h$ is the canonical homomorphism associated with $S$, and $R_S$ is $R$-flat. Conversely, if $Q$ is a ring and $f$ is a homomorphism $f: R \rightarrow Q$ such that $f(R) \subseteq Q \subseteq T(f(R))$, and $Q$ is flat over $R$, then $Q$ is a RGQR over $R$. In fact, if $S = \{ A \subseteq R | f(A)Q = Q \}$, then $S$ is a RGMS of $R$, $R_S = Q$ and $h = f$ (to within an isomorphism) where $h$ is the canonical homomorphism $h: R \rightarrow R_S$.

Proof: If $S$ is a RGMS of $R$, then $R_S$ is a RGQR of $h(R)$ by lemma 1.4; consequently $R_S$ is flat over $h(R)$ by lemma 1.3. Let $\mathfrak{h}$ be the kernel of $h$. Then by [A.2], theorem 1, page 40, $R_S$ is $R$-flat provided that $[o:a]_R Q = Q$ for all elements $a \in \mathfrak{h}$ (see remarks following definition 1.3). But this is clear, for if $a \in \mathfrak{h}$, then there exists $B \in S$ such that $aB = 0$. Consequently $B \subseteq [o:a]_R$, and since $BR_S = R_S$, then $[o:a]_R R_S = R_S$.

Now assume $Q$ is $R$-flat and let $\mathfrak{h}'$ be the kernel of $f$. Let $S = \{ A \subseteq R | f(A)Q = Q \}$. Note by the remarks following definition 1.3, that $S \supseteq \{ [o:a]_R | a \in \mathfrak{h}' \}$ and that $Q$ is flat over $f(R)$. We first show that $\mathfrak{h} = \mathfrak{h}'$ where $\mathfrak{h} = \{ r \in R | \text{there exists } A \in S \text{ such that } rA = 0 \}$.
is the kernel of $h$. Now if $r \in \mathfrak{m}$, then let $A \in S$ such that $rA = 0$. Then $0 = f(rA)Q = f(r)f(A)Q = f(r)Q$, so that $f(r) = 0$ and $r \in \mathfrak{m}'$. Let $r \in \mathfrak{m}'$. Since $[o:rR] \in S$ and $r[o:rR] = 0$, then $r \in \mathfrak{m}$. Whence $\mathfrak{m} = \mathfrak{m}'$ and $f = h$ (to within an isomorphism).

We conclude by showing that $R_S = Q$. If $a \in R_S$, then there exists $A \in S$ such that $af(A) \subseteq f(R)$. Hence $a \in aQ = af(A)Q \subseteq f(R)Q = Q$, and $R_S \subseteq Q$. If $a \in Q$, then $a = \frac{x}{y}$ where $x, y \in f(R)$ and $y$ is regular. By [A.2], Theorem 1, page 801, $Q$ is flat over $f(R)$ and thus by lemmas 1.2 and 1.3 we have $[y:x]Q = Q$. Since $f(R)$

$\frac{x}{y} [y:x] \subseteq f(R)$, then to conclude, it is sufficient to show $\frac{y}{f(R)} [y:x] = f(A)$ for some $A \in S$. But this is clear since $f(R)$

$f(f^{-1}[y:x]) = [y:x] = [y:x]$ (as $f:R \to f(R)$ is onto), and $f(R)$ $f(R)$ $f(R)$ $f(R)$

thus $f^{-1}[y:x] \in S$.

If $R_S$ is a GQR of $R$, and $A$ is an ideal of $R$, then $AR_S$, the extension of $A$ to $R_S$, is often denoted by $A^e = AR_S$; and if $\mathfrak{c}$ is an ideal of $R_S$, then $h^{-1}(\mathfrak{c} \cap h(R))$, the contraction of $\mathfrak{c}$ to $R$, is often denoted by $\mathfrak{c} \cap R = \mathfrak{c}^c$. Observe that the general properties of extended and contracted ideals as in [ZSI], page 218-219,
hold in our case (i.e. $R \xrightarrow{h} R_S$).

Throughout the rest of this chapter, $S$ shall be a RGMS of subsets of $R$. 

**Theorem 1.5:** Let $A$ and $B$ be ideals of $R$ and $\mathfrak{A}$ and $\mathfrak{B}$ be ideals of $R_S$. Then,

(a) $a \in AR_S$ if and only if there exists $C \in S$ such that $ah(C) \subseteq h(A)$.

(b) $(A \cap B)R_S = AR_S \cap BR_S$.

(c) $\mathfrak{A}^c = (\mathfrak{A} \cap R)R_S = \mathfrak{A}$. Hence each ideal of $R_S$ is the extension of an ideal of $R$.

(d) $A^c = AR_S \cap R = \{ r \in R \mid \text{there exists } C \in S \text{ such that } h(r)h(C) \subseteq h(A) \}$.

(e) $AR_S = R_S$ if and only if there exists $C \in S$ such that $h(C) \subseteq h(A)$.

(f) $A = A^c$ if and only if $[h(A):h(C)] = h(A)$ for all $C \in S$.

(g) If $B$ is finitely generated, then $[AR_S:BR_S] = [AR_S:BR_S]$.

(h) $(\mathfrak{A}:\mathfrak{B})^c = (\mathfrak{A}^c:\mathfrak{B}^c)$.

(i) $\sqrt{A} = \sqrt{AR_S}$

(j) $AR_S = R_S$ if and only if $\sqrt{A} = R_S$. 

Proof:

(a) If $a \in AR_S$, then $a = \sum_{1}^{n} Eh(a_i)d_i$ with $a_i \in A$ and $d_i \in R_S$. Since $d_i \in R_S$, then there exists $C_i \in S$ such that $d_i h(C_i) \subseteq h(R)$. Let $C = \prod_{1}^{n} C_i$. Then $C \in S$ and for $i = 1, \ldots, n$, we have $d_i h(C) \subseteq h(R)$. Hence $ah(C) = \sum_{1}^{n} Eh(a_i)(d_i h(C)) \subseteq \sum_{1}^{n} Eh(a_i)h(R) = \sum_{1}^{n} Eh(a_i R) \subseteq h(A)$. Now if there exists $C \in S$ such that $ah(C) \subseteq h(A)$, then $a \in aR_S = a(h(C)R_S) = (ah(C)R_S) \subseteq h(A)R_S = AR_S$.

(b) See [A.1], lemma 2, page 801.

(c) See [A.2], corollary 2, page 41.

(d) This is clear from (a). That is, assume $r \in R$ and there exists $C \in S$ such that $h(r)h(C) \subseteq h(A)$. Then by (a), we have $h(r) \in AR_S = A^e$ and therefore $r \in A^{ec}$. Conversely, if $r \in A^{ec}$, then $h(r) \in A^e = AR_S$, and therefore by (a), we have that there exists $C \in S$ such that $h(r)h(C) \subseteq h(A)$.

(e) If $AR_S = R_S$, then $1 \in AR_S$ and thus by (d), there exists $C \in S$ such that $h(1)h(C) = h(C) \subseteq h(A)$. If there exists $C \in S$ such that $h(C) \subseteq h(A)$, then obviously $AR_S = R_S$.

(f) Assume $A = A^{ec}$. Now obviously $h(A) \subseteq [h(A):h(C)]$. If $h(r) \in [h(A):h(C)]$, then $h(R)$
h(r)h(C) ⊂ h(A) and thus by (d), we have \( r \in A^{ec} = A \).

Hence \( h(r) \in h(A) \) and \( h(A) = [h(A):h(C)] \). Conversely, \( h(R) \)

assume \( h(A) = [h(A):h(C)] \) and let \( r \in A^{ec} \). Then by (d),

there exists \( C \in S \) such that \( h(r)h(C) \subset h(A) \) and therefore

\( h(r) \in [h(A):h(C)] = h(A) \). Whence \( A^{ec} \subset A \), which means

that \( A = A^{ec} \).

(g) See \([A.2]\), lemma 2, page 40.

(h) Now \([\mathcal{S}:\mathcal{C}]^C \subset [\mathcal{S}^C:\mathcal{C}]^C \) is always true (see \([G_1]\),

page 33). Let \( a \in (\mathcal{S}:\mathcal{C})^C \). Then \( a\mathcal{C}^C \subset \mathcal{S}^C \) and therefore

\( h(\mathcal{S})^C \subset \mathcal{C}^C \). Whence by (c), we have \( h(\mathcal{S}) \subset \mathcal{S} \). Thus

\( h(a) \in [\mathcal{S}:\mathcal{C}] \) and therefore \( a \in [\mathcal{S}:\mathcal{C}]^C \).

(i) Let \( a \in \sqrt{A} R_S \). Then \( a = \sum_{i=1}^{k} h(a_i)r_i \) with \( a_i \in \sqrt{A} \)

and \( r_i \in R_S \). Since \( a_i \in \sqrt{A} \), there is a positive integer

\( n \) such that \( a_i^n \in A \). Hence \( h(a_i^n)r_i = (n(a_i))^nr_i^n = \)

\( = (h(a_i)r_i)^n \in AR_S \). Thus \( h(a_i)r_i \in \sqrt{AR_S} \) for each \( i = 1, \ldots, k \), and consequently \( \sum_{i=1}^{k} h(a_i)r_i = a \in \sqrt{AR_S} \). Now

let \( a \in \sqrt{AR_S} \). Then there exists a positive integer \( n \)

such that \( a^n \in AR_S \) and consequently by part (a), there is

an element \( C_1 \in S \) such that \( a^n h(C_1) \subset h(A) \). Then there

exists \( C \in S \) such that \( ah(C) \subset h(R) \) and \( (ah(C))^n \subset h(A) \).

Let \( c \) be an arbitrary element of \( C \) and pick \( r \in R \) such
that $\alpha h(c) = h(r)$. Then there exists $a \in A$ such that $(\alpha h(c))^n = h(r^n) = h(a)$. Hence $r^n - a \in N = \ker h$ and therefore there exists $B \in S$ such that $(r^n - a)B = 0$. This implies that $r^nB \subseteq aB \subseteq A$. Hence $(rB)^n = r^nB^n \subseteq r^nB = A$ and therefore $rB \subseteq \sqrt{A}$. Again by part (a), we have that $h(r) = \alpha h(c)$ is an element of $\sqrt{A}R_S$. Thus $\alpha h(c) \subseteq \sqrt{A}R_S$ and we have that $\alpha \in \alpha R_S = \alpha h(C)R_S \subseteq \sqrt{A}R_S$.

(j) Now obviously $AR_S = R_S$ implies $\sqrt{A}R_S = R_S$. If $\sqrt{A}R_S = R_S$, then $1 = \frac{1}{\sum_{i=1}^{k} h(a_i) r_i}$ with $a_i \in \sqrt{A}$ and $r_i \in R_S$. Since $a_1 \in \sqrt{A}$, then there is a positive integer $n$ such that $a_1^n \in A$ for all $i$. There exists an integer $M$ such that $1 = (\sum_{i=1}^{k} h(a_i) r_i)^M \in h(A)R_S = R_S$.

If $A$ is an ideal of $R$, then we say that $A$ misses $S$ provided $A$ contains no element of $S$.

Theorem 1.6: Let $Q$ be a primary ideal of $R$ such that $\sqrt{Q} = P$ misses $S$. Then $PR_S$ is prime, $QR_S$ is primary for $PR_S$, $P = PR_S \cap R$ and $Q = QR_S \cap R$.

Proof: By [A.2], corollary 2, page 41 and theorem 1.5, part (j), if we can show that $QR_S < R_S$, then our proof will be complete. Assume $QR_S = R_S$. Then $PR_S = R_S$ and therefore by theorem 1.5, part (e), there is an element $C$ of $S$ such that $h(C) \subseteq h(P)$. Since $C \notin P$, then there
is an element \( c \) of \( C - P \) such that \( h(c) = h(p) \) for some element \( p \in P \). Hence \( c - p \in N \) and therefore there is an element \( B \) of \( S \) such that \((c - p)B = (0)\). But this implies \( cB \subseteq pB \subseteq P \). Since \( P \) misses \( S \), then \( c \in P \) and this is a contradiction.

**Corollary 1.7:**

(a) The mapping \( P \rightarrow PR_S \) is a 1-1 mapping of the set of all prime ideals of \( R \) that miss \( S \) onto the set of all prime ideals of \( R_S \).

(b) If \( P \) is a prime of \( R \) that misses \( S \), then the mapping \( Q \rightarrow QR_S \) is a 1-1 mapping of the set of ideals of \( R \) that are primary for \( P \) onto the set of all ideals of \( R_S \) that are primary for \( PR_S \).

(c) Suppose \( A \) is an ideal of \( R \) and \( A = \bigcap_{1}^{n} Q_{1} \) is an irredundant primary representation of \( A \). Further assume that for \( 1 \leq i \leq r \), \( Q_{1} \) misses \( S \) and for \( r+1 \leq j \leq n, Q_{1} \) does not miss \( S \). Then \( AR_S = \bigcap_{1}^{r} Q_{1} R_S \) is an irredundant primary representation of \( AR_S \). Moreover, \( A^{ec} = AR_S \cap R = \bigcap_{1}^{r} (Q_{1} R_S \cap R) = \bigcap_{1}^{r} Q_{1} \); that is, \( AR_S \cap R \) is the intersection of those primary components of \( A \) whose radicals miss \( S \).

**Proof:** Clear.
Theorem 1.8:
(a) If $R$ is Noetherian, then $R_S$ is Noetherian.
(b) If $R$ is Noetherian and $A$ is an ideal of $R$, then $A$ is a contracted ideal of $R_S$ if and only if no prime ideal of $A$ contains an element of $S$.

Proof:
(a) Let $\mathfrak{O}$ be an ideal of $R_S$. Since $R$ is Noetherian, then $\mathfrak{O}^c$ is finitely generated. But $\mathfrak{O} = \mathfrak{O}^c e = h(\mathfrak{O}^c)R_S$, so that $\mathfrak{O}$ is finitely generated.

(b) This is clear by Corollary 1.7, part c.

Theorem 1.9: Let $Q = R_S$ and let

$$S' = \{A \subset R | AQ = Q\}$$

$$\Sigma = \{P_\alpha | P_\alpha \text{ is a prime of } R \text{ and } P_\alpha \text{ contains no elements of } S\}$$

$$S'' = \{A \subset R | A \nsubseteq P_\alpha \text{ for any } P_\alpha \in \Sigma\}.$$

Then $S', S''$ are RGMS's of subsets of $R$, $S \subset S' = S''$ and $Q = R_S = R_S' = R_S''$. Moreover, if $\mathfrak{O}$ is any RGMS of subsets of $R$ such that $R_\mathfrak{O} = Q$, then $\mathfrak{O} \subset S' = S''$.

Proof: Obviously $S'$ and $S''$ are GMS's of $R$ and $S \subset S''$. Since $Q$ is flat over $R$, then by Theorem 1.4, we know that $Q = R_S$. It is also clear that $S' \subset S''$, for if $A \in S'$ and $A \subset P$ where $P$ is a prime that contains no elements of $S$, then since $PR_S$ is a proper prime
of $R_g$ (Theorem 1.6), we have $AR_g < R_g$ which is a contradiction.

Let $A \in S''$. If $AR_g < R_g$, then $AR_g$ is contained in a maximal ideal $M$ of $R_g$. Since $M^c$ contains no elements of $S$, for otherwise $M = M^{c e}$ would not be a proper ideal of $R_g$ (Theorem 1.6), then $M^c \in \Sigma$ and $A \subseteq A^{e c} \subseteq M^c$. But this is a contradiction of the definition of $S''$. Hence $S' = S''$.

If $\mathcal{J}$ is a RGMS of $R$ such that $R_{\mathcal{J}} = Q$, then $A \in \mathcal{J}$ implies $AR_{\mathcal{J}} = AQ = Q$, so that $A \in S'$.

**Definition 1.4.** $S' = S''$ is defined to be the saturation of $S$, using the notation of Theorem 1.9.

**Remark:** We may characterize $S' = S''$ as the largest RGMS of $R$ such that $R_S = Q$. It, however, may not be the largest GMS of $R$ that generates $Q$. For example, let $Q = V_P$ as in example 1.1. Then $S' = S'' = \{A \in R | A \not\in P\}$. But if we let $\overline{S}$ be the GMS of $R$ generated by $P$ and $S'$, then $Q = R_{\overline{S}}$ but $S' < \overline{S}$.

**Definition 1.5.** $\mathcal{J}$ is a GMS of $R$ with the finite property provided $\mathcal{J}$ is a GMS of $R$ such that for each $E \in \mathcal{J}$ there exists $E' \in \mathcal{J}$ such that $E'$ is a finite set and $E' \subseteq E$. 
Theorem 1.10. If $S$ is a saturated RGMS of $R$, then $S$ has the finite property.

Proof: If $E \in S$, then $ER_S = R_S$ and therefore

$\prod_{i=1}^{k} \Sigma h(e_i)r_i$ with $e_i \in E$ and $r_i \in R_S$. If $E' = (e_1', \ldots, e_k')$, then $E' \subseteq E$, and $E' \in S$ since $S$ is a saturated RGMS.

Theorem 1.11. If $S$ is a RGMS of $R$ with the finite property, and $A$ is an ideal of $R$, then the following hold.

(a) $A^{ec} = AR_S \cap R = \{r \in R |$ there exists $C \in S$ such that $rC \subseteq A\}.$

(b) $AR_S = R_S$ if and only if there exists $C \in S$ such that $C \subseteq A$.

(c) $A = A^{ec}$ if and only if $[A:C] = A$ for all $C \in S$.

Proof: (a) If $r \in R$ and there exists $C \in S$ such that $rC \subseteq A$, then $h(r) \in h(r)R_S = h(r)(h(C)R_S) = h(C)R_S \subseteq h(A)R_S$ and therefore $r \in h^{-1}(h(r)) \subseteq A^{ec}$.

If $r \in A^{ec}$, then $h(r) \in AR_S$ and therefore by Theorem 1.5, there exists $B \in S$ such that $h(r)h(B) \subseteq h(A)$. Let $B'$ be a finite subset of $B$ that is in $S$. Then $h(r)h(B') \subseteq h(A)$ and therefore there exists $a_1, \ldots, a_k \in A$ such that $h(r)h(b_1) = h(a_1)$ for each $b_1 \in B'$. Hence
\[^{rb_1 - a_1} \in N \text{ (} = \text{ ker. } h \text{) and consequently there exists } B_1 \in S \text{ such that } (rb_1 - a_1)B_1 = 0. \text{ Thus } rb_1B_1 \subseteq a_1B_1 \subseteq A\text{ for all } i. \text{ Let } B'' = \prod_{1}^{k}B_i \in S. \text{ Then for each } i, \text{ we have } rb_iB'' \subseteq A. \text{ Thus if we let } C = B'B'', \text{ then } C \in S \text{ and } rC = r(B'B'') = r(b_1, \ldots, b_k)B'' \subseteq A.\]

(b) If \( AR_S = R_S \), then \( 1 \in A^{ec} \) and therefore by part (a) of this theorem, there exists \( C \in S \) such that \( 1.C = C \subseteq A \). The converse is obvious. (c) Assume \( A = A^{ec} \) and let \( x \in [A:C] \). Then \( xc \subseteq A \), so that by part (a) of this theorem, we have \( x \in A^{ec} = A \). The other containment if obvious. Now assume that \( A = [A:C] \) for all \( C \in S \).

If \( x \in A^{ec} \), then by part (a) of this theorem, there exists \( C \in S \) such that \( xc \subseteq A \). Hence \( x \in [A:C] = A \). It is obvious that \( A \subseteq A^{ec} \).

Assume \( \mathcal{J} \) is an arbitrary GMS of subsets of \( R \) and let \( h \) be the canonical homomorphism \( h:R \rightarrow R_\mathcal{J} \). Then,

(1) The kernel \( \mathcal{N} \) of \( h \) is \( \{ r \in R \mid \text{there exists } A \in \mathcal{J} \text{ such that } rA = 0 \} \)

(2) The elements of \( R_\mathcal{J} \) are the elements \( a \in T(h(R)) \)
such that there exists \( C \in \mathcal{J} \cdot a \cdot h(C) \subseteq h(R) \).

**Lemma 1.12.** Let \( Q \) be a ring and \( f \) a homomorphism \( f:R \rightarrow Q \) which satisfies (1) and (2) above when \( h \)
and $R'$ are replaced respectively by $f$ and $Q$. That is,

1. the kernel of $f$ is $\mathfrak{N}$, and
2. the elements of $Q$ are the elements $a \in T(f(R))$ such that there exists $A \in \mathfrak{A}$ such that $af(A) \subseteq f(R)$

are true statements. Then $R' \cong Q$.

Proof: Since the kernel of $f$ is the kernel of $h$, then there is a natural isomorphism $\phi = h^{-1}f$ from $h(R)$ onto $f(R)$ and $\phi$ can be extended to an isomorphism from $T(h(R))$ onto $T(f(R))$. Since $\phi(h(E)) = f(E)$ for $E = \mathfrak{A}$, then it follows that $\phi(R') = Q$.

Two important theorems dealing with quotient rings of $R$ are "the permutability of residue class ring and quotient ring formation" and "the transitive property of quotient ring formation" (see [ZSI], page 227). We now generalize these concepts for RGQR's of $R$.

**Theorem 1.13.** Let $S$ be a RGMS of $R$ with the finite property and let $A$ be an ideal of $R$ that contains no elements of $S$. Then,

$$R_S/AR_S \cong [R/A]_{S+A/A}$$

where $S+A/A = \{C+A/A | C \in S\}$.

Proof: If $S \in R_S$, let $\overline{S}$ denote the image of $S$ in $R_S/AR_S$; and if $B \subseteq R$, let $\overline{B}$ denote its image.
in $\mathbb{R}/\mathbb{A}$. Thus $\overline{\mathbb{R}} = \mathbb{R}/\mathbb{A}$ and $\overline{\mathbb{R}}_S = \mathbb{R}_S/\mathbb{A}S$. Also observe that $S+\mathbb{A}/\mathbb{A} = \overline{\mathbb{S}}$ is a GMS of $\overline{\mathbb{R}}$. Now since $\mathbb{A} \subseteq \mathbb{A}^{ec} = \mathbb{A}S \cap \mathbb{R}$, then there is a natural homomorphism $\overline{h}:\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_S$ defined by $\overline{h}(\overline{r}) = \overline{h(r)}$ for all $r \in \mathbb{R}$.

Let $g$ be the canonical homomorphism $g: \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_S$. We now prove the theorem by showing that the ring $\overline{\mathbb{R}}_S$ and the homomorphism $\overline{h}:\overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}_S$ satisfy conditions $(1')$ and $(2')$ of Lemma 1.11 which characterizes (to within an isomorphism) the ring $\overline{\mathbb{R}}_S$.

Let $\overline{r} = r + \mathbb{A}$ be an element of the kernel of $\overline{h}$. Then $\overline{h(r)} = \overline{\overline{h(r)}} = 0$ which means that $\overline{h(r)} \in \mathbb{A}S$. Hence $r \in \mathbb{A}^{ec} = \mathbb{A}S \cap \mathbb{R}$ and therefore by Theorem 1.11, there exists $C \in \mathbb{S}$ such that $rC \subseteq \mathbb{A}$. Hence $\overline{rC} = 0$ and therefore $\overline{r}$ is in the kernel of $g$. Now if $\overline{x} = x + \mathbb{A}$ is in the kernel of $g$, then there exists $C \in \mathbb{S}$ such that $xC \subseteq \mathbb{A}$. Hence $\overline{h(x)}h(C) = \overline{h(A)}$ which implies by Theorem 1.5, that $\overline{h(x)} \in \mathbb{A}S$. Hence $0 = \overline{\overline{h(x)}} = \overline{\overline{h(x)}}$ and $\overline{x}$ is in the kernel of $\overline{h}$.

To conclude, we must show that $\xi \in \overline{\mathbb{R}}_S$ if and only if there exists $C \in \mathbb{S}$ such that $\xi \overline{h(C)} \subseteq \overline{h}(\overline{r}) = \overline{h(r)}$.

Let $\overline{\alpha} = \alpha + \mathbb{A}S \in \overline{\mathbb{R}}_S$. Since $\alpha \in \mathbb{R}_S$, then there exists $C \in \mathbb{S}$ such that $\alpha h(C) \subseteq h(\mathbb{R})$. Hence $\overline{\overline{\alpha h(C)}} = \overline{\mathbb{R}}_S$.
and there is an element \( C \) of \( S \) such that \( \xi h(C) = \xi \widehat{h(C)} \in h(R) \). Now assume \( \xi \in T(\overline{h(R)}) = T(\overline{R_S}) \).

Theorem 1.14. Let \( S' \) be a RGMS of \( R \) with the finite property and let \( S \) be a RGMS of \( R \) such that \( S \subseteq S' \). If \( h \) is the canonical homomorphism \( h: R \rightarrow R_S \), then \( h(S') \) is a GMS of \( R_S \) and \( [R_S] h(S') = R_S \).

Proof: If \( E_1 \) and \( E_2 \) are in \( S' \) and \( h(E_1)h(E_2) = 0 \), then \( E_1E_2 \subseteq \text{ker} \ h \). Let \( E_1' \) and \( E_2' \) be finite subsets of \( E_1 \) and \( E_2 \) that are in \( S' \). Then \( E_1'E_2' \subseteq \text{ker} \ h \) and since \( E_1'E_2' \) is a finite subset of \( S' \), then there is an element \( B \) of \( S \) such that \( E_1'E_2'B = 0 \). But \( E_1'E_2'B \in S' \), so that this is a contradiction. Thus \( h(S') \) is a GMS of \( R_S \).

Now let \( h' \) and \( \phi \) be the canonical homomorphisms \( h': R \rightarrow R_\phi \) and \( \phi: R_S \rightarrow [R_S] h(S') \) and let \( \psi \) be the composite of \( h \) and \( \phi \). We show that \( \psi \) satisfies the conditions (1') and (2') of Lemma 1.12. Since \( \text{ker} \ h' = \{ r \in R \mid \text{there exists } A' \in S' \text{ such that } rA' = 0 \} \) and \( \text{ker} \ \psi = \{ r \in R \mid \text{there exists } A' \in S' \text{ such that } h(r)h(A') = 0 \} = \{ r \in R \mid \text{there exists } A' \in S' \text{ such that } rA' \subseteq \text{ker} \ h \} \),
then it is clear that \( \ker h' \subseteq \ker \psi \). Let \( r \in \ker \psi \). Since \( S' \) has the finite property, then there exists a finite subset \( E \) of \( S' \) such that \( rE \subseteq \ker h \). The finiteness of \( E \) implies that there exists \( B \in S \subseteq S' \) such that \( (rE)B = r(EB) = 0 \). Hence \( r \in \ker h' \) and \( \ker h' = \ker \psi \).

To conclude, we need to show that \( a \in [R_S]_h(S') \) if and only if \( a \in T[\psi(R)] \) and there exists \( E' \in S' \) such that \( a\psi(E') \subseteq \psi(R) \). It is easy to check that \( T[\phi(R_S)] = T[\psi(R)] \). Let \( a \in [R_S]_h(S') \). Then \( a\phi(h(E)) \subseteq \phi(R_S) \) for some \( E \in S' \) and we may assume that \( E \) is finite since \( S' \) has the finite property, say \( E = \{e_1, \ldots, e_n\} \).

Then \( a\phi(h(e_i)) = \phi(\theta_i) \) for some \( \theta_i \in R_S \) for \( i = 1, \ldots, n \). Since \( \theta_i \in R_S \), then there exists \( E_i \in S \) such that \( \theta_i h(E_i) \subseteq h(R) \) for \( i = 1, \ldots, n \). Let \( E' = \bigcup_{i=1}^n E_i \). Then \( a\phi(h(EE')) \subseteq \phi(h(R)) \). The other direction is clear since \( \phi(h(R)) \subseteq \phi(R_S) \).

**Corollary 1.15.** Let \( S \) be a RGMS of \( R \) and let \( P \) be a proper prime ideal of \( R \) that contains no elements of \( S \). Then \( [R_S]_{PR_S} \cong R_P \). (Note by Theorem 1.6 that \( PR_S \) is a proper prime ideal of \( R_S \).)

**Proof:** Let \( \overline{S} = \{ [x] | x \in R - P \} \) and \( S' = \{ A \subseteq R | A \notin P \} \). It is clear that \( R_{\overline{S}} = R_P \) and that \( \overline{S} \) is a RGMS
of \( R \). Also \( S' \) is the saturation of \( S \) since \( S' = \{ A \subseteq R | AR_S = R_S \} \), and therefore \( R_S = R_P \). Since \( S \subseteq S' \), then by theorems 1.10 and 1.14, we know that 
\[ R_P = R_S' \cong [R_S]_{h(S')} \] where \( h \) is the canonical homomorphism \( h : R \to R_S \). We now show that 
\[ [R_S]_{PR_S} \cong [R_S]_{h(S')} \].

Let \( S'' = \{ h(E)R_S | E \in S' \} \). Then \( [R_S]_{S''} = [R_S]_{h(S')} \), and since \( PR_S \cap R = P \) (theorem 1.6) and ideals of \( R_S \) are extensions of ideals of \( R \) (theorem 1.5, part (c)), then \( S'' \) consists of the ideals in \( R_S \) such that 
\[ \emptyset \in PR_S \]. Thus if \( S''' = \{ \{a\} | a \in R_S - PR_S \} \), then \( S'' \) is the saturation of \( S''' \) and 
\[ [R_S]_{h(S')} = R_S'' = R_S''' = [R_S]_{PR_S} \].

Remark: We point out now that most of the results in this chapter for RGQR's of \( R \) are not true in general for arbitrary GQR's. For example if \( D \) is a domain, then a necessary and sufficient condition for an overring \( Q \) of \( D \) to be flat over \( D \) is that \( (A \cap B)Q = AQ \cap BQ \) for all ideals \( A \) and \( B \) of \( D \) (see Theorem 2.1). Hence if \( Q \) is a GQR of \( D \) but is not flat over \( D \), then Theorem 1.5, part (b) is not true.

Also Eakin has given an example of a three dimensional Noetherian Krull ring \( D \) and an ideal \( A \) of \( D \) such that \( Q = T_D(A) > D \) is a three dimensional Krull ring with a minimal prime \( \emptyset \) such that no \( \emptyset \)-primary is
finitely generated (see [E], example 1). Thus Theorem 1.8 is not true, and since \( \mathfrak{p} \) is not finitely generated, then \( \mathfrak{p} \cap \mathfrak{q} < \mathfrak{q} \) which means Theorem 1.5, part (c) is not true.

Also since there is a one-to-one correspondence between the minimal primes of \( D \) that do not contain \( A \) and the minimal primes of \( Q \) (see Chapter III, Corollary 3.6) which can be realized by contractions of minimal primes of \( Q \), then \( \mathfrak{p} \) is a minimal prime of \( D \) that does not contain \( A \). Hence Theorem 1.6 is not true.

Example 2.1 in Chapter II illustrates that Theorem 1.5 parts (d), (e) and (f) need not be true.

If \( \mathcal{J} \) is a GMS of \( R \), then in view of the proof of Lemma 1.4, one may wonder when \( h(\mathcal{J}) \) is a GMS of \( h(R) \) (where \( h \) is the canonical homomorphism \( h:R \to R_\mathcal{J} \)). The final theorem of this chapter adds insight to this problem.

**Theorem 1.17.** If \( \mathcal{J} \) is a GMS of \( R \) and \( P \) is a prime ideal of \( R \) that misses \( \mathcal{J} \), then \( h(\mathcal{J}) \) is a GMS of \( h(R) \). Moreover, if \( h(\mathcal{J}) \) is a GMS of \( h(R) \) and \( \mathcal{J} \) has the finite property, then there is a prime ideal of \( R \) that misses \( \mathcal{J} \).

**Proof:** To show \( h(\mathcal{J}) \) is a GMS of \( h(R) \), we need to show \( h(E) \neq 0 \) for any \( E \in \mathcal{J} \). Since \( P \) misses \( \mathcal{J} \), then \( P \nsubseteq \ker h \); and therefore if there exists \( E \in \mathcal{J} \) such that \( h(E) = 0 \), then \( E \subseteq \ker h \subseteq P \) which is a contradiction. Now assume that \( h(E) \neq 0 \) for all \( E \in \mathcal{J} \).
and that $\mathfrak{J}$ has the finite property. Let $\Sigma = \{ A \subseteq R \mid A$ is an ideal of $R$ and $A$ misses $\mathfrak{J} \}$. Then $(0) \in \Sigma$, so that $\Sigma \neq \emptyset$. Also $\Sigma$ is partially ordered under $\subseteq$ and is inductive under this ordering since $\mathfrak{J}$ has the finite property. Zorn's lemma then implies that $\Sigma$ contains maximal elements (under this ordering). Let $P$ be such a maximal element and let $xy \in P$ where $x, y \in R$. If neither $x$ nor $y$ are elements of $P$, then $P + (x)$ and $P + (y)$ properly contain $P$, so that $P + (x)$ and $P + (y)$ contain elements of $\mathfrak{J}$. Hence $[P + (x)][P + (y)] \subseteq P$ contains an element of $\mathfrak{J}$ which is a contradiction. Thus $P$ is a prime ideal of $R$. 
CHAPTER II

Throughout this chapter, $D$ is a domain with identity with quotient field $K$. We primarily in this chapter study general properties of GQR's of $D$. We also give one theorem that deals with extension and contractions of ideals between a ring $R$ that is not necessarily a domain and a GQR of $R$ (Theorem 2.1?).

Recall that if $a = x/y \in K$ where $x, y \in D$, then $[D:a] = [y:x]$. We may characterize $[y:x] = [D:a]$ as the largest ideal $A$ of $D$ such that $A \subseteq D$.

Theorem 2.1. Let $Q$ be an overring of $D$. T.A.E.

(a) $Q$ is a RGQR of $D$
(b) $Q$ is flat over $D$
(c) $Q = D_S$ where $S = \{A \subseteq D | AQ = Q\}$
(d) $Q_P = D_P \cap D$ for all proper prime ideals $P$ of $Q$.
(e) $Q_M = D_M \cap D$ for all maximal ideals $M$ of $Q$.
(f) If $a = x/y \in Q$ with $x, y \in D$, then $[D:a]Q = [y:x]Q = Q$.
(g) $PQ = Q$ or $Q \subseteq D_P$ for all proper primes $P$ of $D$. 

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(h) \((A \cap B)Q = AQ \cap BQ\) for all ideals \(A\) and \(B\) of \(D\).

**Proof:** The equivalence of (a), (b), (c) and (f) follows from Lemma 1.2 and Theorem 1.4, and Richman in [R] shows the equivalence of (b), (d), (e) and (g). If \(Q\) is a RGQR of \(D\), then by Theorem 1.5, part (b), we know that (a) implies (h). Assume (h) is true and let \(\alpha\) be a non-zero element of \(Q\). Then \([D:\alpha] = \alpha^{-1}D \cap D\). If we let \(d \in D\) such that \(d\alpha^{-1} \in D\), then \(\alpha^{-1}D \cap D \subseteq 1/d(\alpha^{-1}D \cap D)\). We then have that \([D:\alpha]Q = (\alpha^{-1}D \cap D)Q \subseteq 1/d(\alpha^{-1}D \cap D)Q = 1/d[\alpha^{-1}DQ \cap DQ] = 1/d[DQ \cap Q] = 1/d[DQ] = Q\). Hence (h) implies (f) and proof is complete.

In chapter III, see remark following Theorem 3.2, we show that a RGQR of \(D\) need not be a regular quotient ring of \(D\). We now give a necessary and sufficient condition for a RGQR of \(D\) to be a quotient ring of \(D\).

**Theorem 2.2.** Let \(S\) be a RGMS of subsets of \(D\) with saturation \(S\) and let \(M = D - UP\), where \(\{P_\alpha\}\) consists of the primes of \(D\) that contain no elements of \(S\). Then,

(a) \(D_S = D_M\) if and only if \(A \in S\) implies \(A \notin UP_\alpha\).

(b) \(D_S\) is a quotient ring of \(D\) if and only if \(D_S = D_M\).
Proof: If $D_S = D_M$, then $A \in S$ implies $AD_M = D_M$. Hence $A$ contains elements of $M$ so that $A \not= UP_\alpha$.

Conversely, assume $A \in S$ implies $A \not= UP_\alpha$ and let $\beta \in D_S$. Then there exists $A \in S$ such that $\beta A = D$. Since there exists $m \in A - UP_\alpha$, then $m \in M$; and since $m \beta \in D$, then $\beta \in D_M$. Since no element of $M$ is in any $P_\alpha$, then by the definition of $\overline{S}$, we have $M \subseteq \overline{S}$ so that $D_M \subseteq D_S \subseteq D_S$ and (a) is proven.

Now assume $D_S$ is a quotient ring and let $M'$ be a multiplicative system of elements such that $D_M' = D_S$. Since $M'$ can be considered a RGMS of subsets of $D$, then $M' \subseteq \overline{S}$ and, by the definition of $\overline{S}$, we have that no element of $M'$ is in any $P_\alpha$. Hence $M' \subseteq M$ and $D_S = D_M' \subseteq D_M$. But in the proof of (a), we showed that $D_M$ is always contained in $D_S$.

Lemma 2.3. Let $S$ be a GMS of $D$ and $A$ be an ideal of $D$ containing an element $B$ of $S$. Let $S' = \{C \in S | C \subseteq A\}$. Then $S'$ is a GMS of $D$ and $D_{S'} = D_S$.

Proof: It is clear that $S'$ is a GMS of $D$ and since $S' \subseteq S$, then $D_{S'} \subseteq D_S$. If $\alpha \in D_S$, then there exists $C \in S$ such that $\alpha C \subseteq D$. Hence $\alpha CB \subseteq D$ and $CB \in S'$, so that $\alpha \in D_{S'}$. 

The proof of the next lemma is obvious and shall be omitted.

Lemma 2.4. If $P$ is a prime ideal of $D$ and $a \in K$, then $a \in D_P$ if and only if $[D:a] \notin P$.

Lemma 2.5. If $S$ is a GMS of $D$ and $S' = \{A \subseteq D | A \supseteq E$ for some $E \in S\}$, then $S'$ is a GMS of $D$ and $D_S = D_{S'}$.

Proof: It is clear that $S'$ is a GMS of $D$ and since $S \subseteq S'$, then $D_S \subseteq D_{S'}$. If $a \in D_{S'}$, then there exists $A \in S'$ such that $aA \subseteq D$. Let $E \in S$ such that $E \subseteq A$. Then $aE \subseteq D$, so that $a \in D_S$.

Lemma 2.6. An overring $Q$ of $D$ is a GQR of $D$ if and only if $Q = D_S$ where $S = \{\prod_{i=1}^{n} [D:a_i] \mid a_i \in Q \text{ and } S_1 \in D \}$ and $n$ is a positive integer.

Proof: Assume $Q$ is a GQR of $D$ and let $J$ be a GMS of $D$ such that $D_J = Q$. Let $a \in Q$ and let $A$ be an element of $J$ such that $aA \subseteq D$. Then $A \subseteq [D:a] \in S$ and therefore $[D:a] \in S_{1}$ and therefore $D_S \subseteq D_J$, so $D_S = D_J$. Since $a[D:a] \in D$, then it is clear that $Q \subseteq D_S$. 
Lemma 2.7. If $S$ is a GMS of $D$ and $P$ is a prime ideal of $D$ which misses $S$, then $D_S \subseteq D_P$; hence, if $\Sigma$ is the set of all prime ideals of $D$ which miss $S$, then $D_S \subseteq \bigcap_{P \in \Sigma} D_P$.

Proof: The proof is easy and we omit it.

Theorem 2.8. If $\Sigma$ is a collection of prime ideals of $D$, then $\cap_{P \in \Sigma} D_P$ is a GQR of $D$; in fact, $\cap_{P \in \Sigma} D_P = D_S$ where $S = \{A \subseteq D \mid A \notin P$ for any $P \in \Sigma \}$.

Proof: Lemma 2.7 shows that $D_S \subseteq \bigcap_{P \in \Sigma} D_P$. And if $a \in \bigcap_{P \in \Sigma} D_P$, then $[D; a] \notin P$ for any $P \in \Sigma$ by Lemma 2.4. Thus $[D; a] \in S$, and since $a[D; a] \subseteq D$, then $a \in D_S$.

Theorem 2.9. Let $S$ be a GMS of $D$, let $D'$ be a domain such that $D \subseteq D' \subseteq D_S$ and let $\Sigma, \Sigma'$ denote respectively the set of prime ideals of $D$ and $D'$ which miss $S$. Then the correspondence $P' \rightarrow P' \cap D$ is a one-to-one correspondence from the elements of $\Sigma'$ onto the elements of $\Sigma$. Moreover, if $P'$ corresponds to $P$, then $D'_{P'} = D_P$.

Proof: If $P \in \Sigma$, then by Lemma 2.7, we have that $D \subseteq D' \subseteq D_S \subseteq D_P$. Let $P' = PD_P \cap D'$. Then $P' \cap D = P$ and $P' \in \Sigma'$, so that our correspondence is onto. Moreover, since $P' \cap D = P$ and $D \subseteq D' \subseteq D_P$, then $D'_{P'} = D_P$. But this implies that our correspondence is one-to-one, so that our proof is complete.
Corollary 2.9. If $p$ is a non-zero element of $D$ such that $pD = P$ is a prime ideal of $D$ and $S$ is a GMS of $D$ such that $P$ contains no elements of $S$, then $PD_S$ is a prime ideal of $D$.

Proof: Since $P$ contains no element of $S$, then $D_S \subseteq D_P$ and $PD_P \cap D_S$ is a prime in $D_S$. Obviously $PD_S \subseteq PD_P \cap D_S$ and if $a \in PD_P \cap D_S$, then $a = ps/x$ with $s, x \in D$ and $x \not\in P$. Let $A \in S$ such that $aA \subseteq D$, and let $a$ be a non-zero element of $A$. Then $aa \in D$, so there exists an element $d \in D$ such that $psa = xd$. Since $xd \in P$, and $x \not\in P$, then $d = pr$ for some $r \in D$. Thus $sa = xr$ and $a(s/x) = r \in D$. Since $a$ is an arbitrary element of $A$, then $s/x A \subseteq D$, so that $s/x \in D_S$ and $a = p(s/x) \in PD_S$.

Corollary 2.10. If $S$ is a GMS of $D$, $D'$ is a domain such that $D \subseteq D' \subseteq D_S$, and $ED' = D'$ for all $E \in S$, then $D' = D_S = \bigcap_{P \in \Sigma} D_P$ where $\Sigma$ is the set of prime ideals of $D$ that miss $S$.

Proof: If we let $\Sigma'$ be the set of all proper prime ideals of $D'$, then (see [G], Theorem 3.10) $D' = \bigcap_{P' \in \Sigma'} D'_{P'}$. Since $ED' = D'$ for all $E \in S$, then the elements of $\Sigma'$ contain no elements of $S$, so that by Theorem 2.9, we have that $\bigcap_{P \in \Sigma} D_P = D' \subseteq D_S$. Since
\[ D_S \subseteq \bigcap_{P \in \mathcal{D}_P} \mathcal{D}_P \] by Lemma 2.7, then our proof is complete.

**Corollary 2.11.** If \( S \) is a RGMS of \( D \), then \( D_S \) is an intersection of localizations of \( D \).

**Proof:** This is immediate from Corollary 2.10.

If \( A \) is an ideal of a ring \( R \) and \( S \) is a GMS of \( R \), then (following [MG]) we define \( A_S \) to be \( \{ a \in T(h(R)) | ah(E) \subseteq h(A) \text{ for some } E \in S \} \) where \( h \) is the canonical homomorphism \( h: R \rightarrow R_S \). The next theorem, in part, is a generalization of Theorem 2.9 in the ring case.

**Theorem 2.12.** If \( R, S \) and \( h \) are as in the above paragraph, then the following are true.

(a) If \( A \) is an ideal of \( R \), then \( A_S \) is an ideal of \( R_S \) and \( A_S \supseteq AR_S = A^c \). Moreover, \( A_S \) may properly contain \( AR_S \), but if \( S \) is a RGMS of \( R \), then \( A_S = AR_S \).

(b) If \( P \) is a prime ideal of \( R \) that misses \( S \) and \( Q \) is a \( P \)-primary ideal of \( R \), then \( P_S \) is a prime ideal of \( R_S \), \( [Q_S]^c = Q = Q^c \) and \( Q_S \nsubseteq ER_S \) for \( E \in S \). Moreover, \( Q_S \) is primary for \( P_S \) if \( S \) has the finite property.

(c) If \( L \) is a primary ideal of \( R_S \) such that \( \sqrt{L} \nsubseteq ER_S \) for \( E \in S \), then \( L = [L^c]_S \) and \( L^c \) misses \( S \).
If $A$ is any ideal of $R_S$, then $A^c R_S \subseteq A \subseteq [A^c]_S$.

(d) The mapping $P \mapsto P_S$ is a one-to-one mapping from the prime ideals $P$ of $R$ that miss $S$ onto the prime ideals $P'$ of $R_S$ such that $P' \nsubseteq E R_S$ for $E \in S$.

**Proof:** If $a, b \in A_S$ and $\gamma \in R_S$, then there exist $E_1, E_2, E_3 \in S$ such that $a h(E_1) \subseteq h(A)$, $b h(E_2) \subseteq h(A)$ and $\gamma h(E_3) \subseteq h(R)$. Since $(a+b) h(E_1 E_2)$ and $a \gamma h(E_1 E_3)$ are both contained in $h(A)$, then $A_S$ is an ideal of $R_S$. By Theorem 1.5, part (a), it is clear that $A_S = A R_S$ provided $S$ is a RGMS of $R$, and by the proof of Theorem 1.5, part (a), it is clear that in general, $A R_S \subseteq A_S$. Now let $R = V$ in example 1.1 and let $S = \{M^n\}^\infty_1 = \{M\}$. Then $M_S = R_S > M R_S$. If $S' = \{\{x\}| x \in R - M\}$, then $R_S = R_{S'}$, $M R_S = M R_{S'}$, $M_S = M R_S < M_{S'}$. Hence, in general, $A_S$ depends upon the GMS $S$ while $A R_S$ depends on the ring $R_S$. Next, we consider (b).

Since $Q \subseteq [Q_S]^C$, then let $r \in [Q_S]^C$ and $B \in S$ such that $h(r) h(B) \subseteq h(Q)$. Since $P$ misses $S$, then there exists $b \in B - P$ and $q \in Q$ such that $h(r) h(b) = h(q)$. Hence $r b - q \in \ker h$ and therefore there exists $B' \in S$ such that $(r b - q) B' = 0$. But this implies
that \( rbB' \subseteq Q \); and since \( Q \) is primary for \( P \) and \( bB' \notin P \), then \( r \in Q \). Hence \( Q = [Q_S]^c \) and since \( QR_S \subseteq Q_S \), then \( Q^eS = Q \).

If \( P \) is a prime ideal of \( R \) and \( P \) misses \( S \), then \( P \ni \ker h \) and \( h(P) \) is a prime ideal of \( h(R) \). If \( \alpha \beta \in P_S \) with \( \alpha \notin P_S \), then there exist \( E_1, E_2 \in S \) such that \( \alpha h(E_1) \subseteq h(R) \), \( \beta h(E_2) \subseteq h(R) \) and \( \alpha h(E_1) \cdot \beta h(E_2) \subseteq h(P) \).

It follows that \( \beta h(E_2) \subseteq h(P) \), \( \beta \in P_S \) and \( P_S \) is prime.

Since \( [P_S]^c = P \), it is clear that \( Q_S \notin ER_S \) for \( E \in S \).

If \( P \) misses \( S \) and \( S \) has the finite property, it is easy to check that \( Q \ni \ker h \) for any \( P \)-primary ideal \( Q \) of \( R \), and therefore \( h(Q) \) is \( h(P) \)-primary. Let \( \alpha \beta \in Q_S \) with \( \alpha \notin Q_S \). There exist finite sets \( E_1, E_2 \in S \) such that \( \alpha h(E_1) \subseteq h(R) \), \( \beta h(E_2) \subseteq h(R) \) and \( \alpha h(E_1) \cdot \beta h(E_2) \subseteq h(Q) \).

Since \( \alpha h(E_1) \notin h(Q) \) and \( E_2 \) is finite, it follows that there exists a positive integer \( n \) such that \( \beta^n h(E_2^n) \subseteq h(Q) \) and consequently \( Q_S \) is primary.

Using the fact that \( \sqrt{h(Q)} = h(P) \) and that \( S \) has the finite property, it is easy to show that \( \sqrt{Q_S} = P \).

If \( \alpha \in [L^c]^S \), then there exists \( E \in S \) such that \( \alpha h(E) \subseteq h(L^c) \subseteq L \). Since \( h(E) \notin \sqrt{L} \), then \( \alpha \in L \) and \( [L^c]^S \subseteq L \). Let \( \beta \in L \) and let \( E \in S \) such that
\( \beta h(E) \subseteq h(R) \). Let \( e \in E \) and let \( r \in R \) such that 
\( \beta h(e) = h(r) \). Then \( h(r) \in L \cap h(R) \), so that \( r \in L^C \) and 
\( h(r) = \beta h(e) \in h(L^C) \). But \( e \) is an arbitrary element of 
\( E \), so that \( \beta h(E) \subseteq h(L^C) \) and \( \beta \in [L^C]_S \). It is clear 
that \( L^C \) misses \( S \) and the proof of (d) follows 
immediately from parts (b) and (c).

**Theorem 2.13.** If \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \) denote respect­
ively the class of all RGQR's of \( D \), the class of all 
intersections of localizations of \( D \), and the class of 
all GQR's of \( D \), then \( \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \).

**Proof:** The proof follows directly from Theorem 
2.8 and Corollary 2.11.

**Theorem 2.14.** Let \( \Sigma \) be a collection of prime 
ideals of \( D \), let \( N = D - \bigcup_{P \in \Sigma} P \), and let \( S = \)
\[ \{ E \subset D \mid E \not\subset P \ \text{for any } P \in \Sigma \} \]. Then, \( D_N \cap D_S = \bigcap_{P \in \Sigma} D_P \).

If \( \Sigma \) is finite, then \( D_{N=D_S} = \bigcap_{P \in \Sigma} D_P \) and \( S \) is a RGMS of \( D \).

**Proof:** By Theorem 2.8, \( D_S = \bigcap_{P \in \Sigma} D_P \), and since \( N \subset S \), then \( D_N \subset D_S \). If \( \Sigma \) is a finite set, then 
since an ideal contained in the union of a finite number 
of prime ideals implies that it must be contained in one of 
them (see [ZSI], page 215), we have that \( S = \)
\[ \{ A \subset D \mid A \not\subset \bigcup_{P \in \Sigma} P \} \]. Since \( N \) is obviously a RGMS of \( D \),
then $S$ is a RGMS and it is clear that $D_S \subseteq D_N$.

**Definition 2.1.** Let $\Sigma$ be a collection of prime ideals of $D$ and let $S$ be a GMS of $D$ such that $P$ misses $S$ for all $P \in \Sigma$. If $D_S = \bigcap_{P \in \Sigma} D_P$, then $S$ is said to be **complete with respect to** $\Sigma$.

**Definition 2.2.** If $\Sigma$ is a collection of prime ideals of $D$ and $S$ is a GMS of $D$, then $S$ is said to be **maximal with respect to** $\Sigma$ provided $S = \{E \subset D | E \not\subset P \text{ for any } P \in \Sigma\}$.

**Remark:** It follows from Theorem 2.8 that if $S$ is maximal with respect to $\Sigma$, then $S$ is complete with respect to $\Sigma$. The converse is false (e.g. see Lemma 2.3 and example 1.1). Theorem 2.16 gives a sufficient condition for $S$ to be complete with respect to $\Sigma$—in particular, if $S$ is a RGMS and $\Sigma$ is the set of prime ideals missing $S$, then $S$ is complete with respect to $\Sigma$. Also, in Theorem 2.14, if $\Sigma$ is a finite set of prime ideals, $N = D - \bigcup_{P \in \Sigma} P$ a multiplicative system of elements, $S^* = \{\{x\} | x \in N\}$, $S = \{E \subset D | E \not\subset P \text{ for all } P \in \Sigma\}$, and $S'$ is a GMS such that $S^* \subset S' \subset S$, then $S'$ is complete with respect to $\Sigma$.

**Theorem 2.15.** Let $D_1$ and $Q$ be subrings of $K$ such that $D \subset D_1 \subset Q$ and let $S$ and $S_1$ be GMS's of
D and $D_1$ respectively. Then, the following are true.

(a) If $Q$ is a GQR of $D$, then $Q$ need not be a GQR of $D_1$.

(b) If $Q$ is a RGQR of $D$, then $Q$ is a RGQR of $D_1$.

(c) If $Q = D_S$ and $S$ has the finite property, then $Q = [D_1]_S$.

(d) If $Q$ is an intersection of localizations of $D$, then $Q$ is an intersection of localizations of $D_1$. In fact, if $Q = \bigcap_{p \in \mathcal{P}} D_{p}$ and $S$ is complete with respect to $\mathcal{P}$, then $Q = [D_1]_S$.

(e) If $Q$ is a RGQR of $D$, then $D_1$ need not be a GQR of $D$.

(f) If $Q$ is a RGQR of $D_1$, and $D_1$ is a RGQR of $D$, then $Q$ is a RGQR of $D$.

(g) If $Q$ is a RGQR of $D_1$ (or $Q = [D_1]_{S_1}$ where $S_1$ has the finite property), and $D_1$ is a GQR of $D$, then $Q$ is a GQR of $D$.

(h) If $Q = [D_1]_{S_1}$ and $S$ is a RGMS of $D$ such that $D_S = D_1$, then $[D_1](S,S_1) = Q$ where by $(S,S_1)$ we mean the GMS of $D_1$ generated by the elements of $S$ and $S_1$. 
Proof: See [H], example 2.9 to see that (a) is true. If $S$ is a RGMS of subsets of $D$ and $D_S = Q$, then $Q = D_S \subseteq [D_1]_S$. Let $\alpha \in [D_1]_S$ and let $A \in S$ such that $\alpha A \subseteq D_1$. Then $\alpha \in \alpha Q = \alpha (AQ) = (\alpha A)Q \subseteq D_1Q = Q$. Hence $[D_1]_S = Q$ and (b) is true.

Assume that $S$ has the finite property and that $D_S = Q$. Obviously $[D_1]_S \supseteq Q$. Let $\alpha \in [D_1]_S$ and let $A = (a_1, \ldots, a_n)$ be a finitely generated element of $S$ such that $\alpha A \subseteq D_1$. Since $\alpha a_i \in D_1$ for $i = 1, \ldots, n$, then there exists $B_i \in S$ such that $\alpha a_i B_i \subseteq D$. If $B = \prod_{i=1}^n B_i$, then $B \in S$, and $\alpha a_i B \subseteq D$ for each $i$. Hence $\alpha (AB) \subseteq D$ and since $AB \in S$, then $\alpha \in D_S$ and (c) is true.

If $Q = \bigcap D_{P_{\alpha}}$, then $[D_1]_{P_{\alpha}}' = D_{P_{\alpha}}$ where $P_{\alpha}' = P_{\alpha}D_{P_{\alpha}} \cap D_1$, so that $Q = \bigcap [D_1]_{P_{\alpha}}'$. And if $S$ is complete with respect to $\Gamma$, then $S$ is complete with respect to $[P_{\alpha}]'$, so that $[D_1]_S = Q$ and (d) follows.

To see that (e) is true we take example 2.1, and let $k'$ be a field such that $k_0 < k' < k$ and let $D = k_0 + M$, $D_1 = k' + M$ and $Q = V_P$ where $P$ is a prime of $V$ such that $P < M$ (see the remark following example 2.1). By Theorem A, in [G], page 560, we know that $Q = D_P$ so that
Q is a RGQR of D. But since M is the unique maximal ideal of D and \( T_D(M) = k + M = V \), then all other GQR's of D contain V, so that \( D_V \) is not a GQR of D.

We now consider (f) and (g). If \( Q \) is a RGQR of \( D_V \), then by Theorem 2.1, we may assume that \( Q = [D_V]_{S_1} \) where \( S_1 \) consists of all ideals \( \mathcal{O} \) of \( D_V \) such that \( \mathcal{O}Q = Q \). By Definition 1.4, \( S_1 \) is a saturated RGMS of \( D_V \) and therefore by Theorem 1.10, \( S_1 \) has the finite property.

Thus assume \( D_V = D_S \). If \( \mathcal{O} \) is a finitely generated element of \( S_1 \), then there exists \( A \in S \) such that \( \mathcal{O}A \subseteq D \). Let \( S' = \{ \mathcal{O}A | \mathcal{O} \) is a finitely generated element of \( S_1 \), \( A \in S \) and \( \mathcal{O}A \subseteq D \} \). Then \( S' \) is a GMS of \( D \) and we show \( D_S = Q \).

If \( a = Q \), let \( \mathcal{O} \) be a finitely generated element of \( S_1 \) such that \( \mathcal{O} \subseteq D \). Since \( \mathcal{O} \) is finitely generated over \( D \), then there exists \( A \in S \) such that \( (\mathcal{O})A \subseteq D \). Pick \( B \in S \) such that \( \mathcal{O}B \subseteq D \). Then \( \mathcal{O}AB \in S' \) and \( a(\mathcal{O}AB) \subseteq D \), so that \( a \in D_S \). If \( y \in D_S \), then there exists \( \mathcal{O}A \in S' \) such that \( y(\mathcal{O}A) \subseteq D \). Since \( yaA \subseteq D \) for all \( a \in \mathcal{O} \), then \( y \subseteq D_S = D_1 \) and therefore
\gamma \in [D_1]_{S_1} = \emptyset$. Hence (g) is proven and if $S$ is a RGMS of $D$, then $S'$ is a RGMS of $D$, so that (f) is true.

Under the hypothesis of (h), it is clear that $Q \subseteq D(S, S_1)$. And if $a \in D(S, S_1)$, then there exists $A \in S$ and $\emptyset \in S_1$ such that $aA \emptyset \subseteq D$. (Without loss of generality, we may assume $D \in S$ and $D_1 \in S_1$.) Hence $a\emptyset \subseteq a\emptyset D_1 = a(AD_1) = (aA\emptyset)D_1 \subseteq DD_1 = D_1$ and therefore $a \in [D_1]_{S_1} = \emptyset$.

Remark: If $Q$ is a GQR of $D_1$ and $D_1$ is a GQR of $D$, then we do not know whether $Q$ must be a GQR of $D$.

Theorem 2.16. If $S$ is a GMS of $D$ with the finite property and $\Sigma$ is the collection of prime ideals of $D$ that miss $S$, then $S$ is complete with respect to $\Sigma$.

Proof: By Lemma 2.7, we have that $D \subseteq D_S \subseteq \bigcap_{P \in \Sigma} D_P$. Let $a \in \bigcap_{P \in \Sigma} D_P$. Then by Theorem 2.8, there exists an ideal $A$ of $D$ such that $A \notin P$ for any $P \in \Sigma$ and such that $aA \subseteq D$. Now if $A$ contains an element of $S$, then $a \in D_S$ and $D_S = \bigcap_{P \in \Sigma} D_P$. Thus assume $A$ misses $S$ and
let \( \Sigma = \{ B \subseteq D \mid B \text{ is an ideal of } D \text{ containing } A \text{ and } B \text{ misses } S \} \). Then \( \tau \neq \emptyset \) and since \( S \) has the finite property, then by the proof of Theorem 1.7, we know that \( \Sigma \) contains a maximal element \( P \) that is a prime ideal of \( D \). But this implies \( P \in \Sigma \) which is a contradiction.

If \( S \) is a GMS of \( D \) and \( D_s \) is an intersection of localizations of \( D \), then must \( D_s = \bigcap_{P \in \Sigma} D_P \) where \( \Sigma \) consists of the prime ideals of \( D \) that miss \( S \)? Example 1.1 illustrates that we may have \( D_s \subseteq D_p \) where \( P \notin \Sigma \), but in this case, \( D_s \) is still \( \bigcap_{P \in \Sigma} D_P \). Also if \( S \) is a RGM3 of \( D \) or if \( S \) has the finite property, then we have seen that the amswer to the above is yes. The next theorem adds further light to this question, but, in general, we have been unable to answer it.

**Theorem 2.17.** Let \( Q \) be a GQR of \( D \), let \( S = \{ \prod_{D: a_1} \mid a_1 \in Q \text{ and } n \text{ is a positive integer} \}, \) and let \( \Sigma \) be the collection of prime ideals that miss \( S \). Then \( Q \) is an intersection of localizations of \( D \) if and only if \( Q = \bigcap_{P \in \Sigma} D_P \) (i.e. \( S \) is complete with respect to \( \Sigma \)) if and only if \( Q \) is an intersection of localizations of \( D \).

**Proof:** If \( \Gamma = \{ q_\beta \} \) is a subset of the prime
ideals of $D$ such that $Q = \bigcap_{\beta} q_{\beta}$, then by Lemma 2.7, \[ \bigcap_{\beta} q_{\beta} = \bigcap_{P \in \Sigma} D_P. \] By Lemma 2.4, no element of $\Gamma$ contains any element of the form $[D:a]$ with $a \in Q$, so that no element of $\Gamma$ contains any element of $S$. Consequently $\Gamma \subseteq \mathfrak{p}$ which means that $\bigcap_{P \in \Sigma} D_P \subseteq \bigcap_{\beta} q_{\beta}$.

**Lemma 2.18.** If $\Sigma$ is a collection of prime ideals of $D$, then $\bigcap_{P \in \Sigma} D_P = \{a \in K|[D:a] \not\in P \text{ for any } P \in \Sigma\}.$

**Proof:** This follows immediately from Lemma 2.4.

**Theorem 2.19.** Let $S$ be a GMS of $D$ and let $\Sigma$ be a set of prime ideals of $D$. Then,

(a) $D_S = \bigcap_{P \in \Sigma} D_P$ if and only if $D_S = \{a \in K|[D:a] \not\in P \text{ for any } P \in \Sigma\}$ for any $P \in \Sigma$ and

(b) $D_S = \bigcap_{P \in \Sigma} D_P$ if and only if $D_S \subseteq D_P$ for all $P \in \Sigma$ and $\bigcap [D_S]_{P'} = D_S$ where $P' = PD_P \cap D_S$ for all $P \in \Sigma$.

**Proof:** The proof of (a) is obvious from Lemma 2.18, and (b) is clear since $D_S \subseteq D_P$ implies that $[D_S]_{P'} = D_P$.

**Theorem 2.20.** Let $Q$ be an overring of $D$ and let $S = \{\prod_{i=1}^{n} [D:a_i] | a_i \in Q \text{ and } n \text{ is a positive integer}\}$
\[ S_1 = \{ E \subseteq D | EQ = Q, \ E \text{ is an ideal of } D \} \]
\[ S_2 = \{ E \subseteq D | E \supseteq C \text{ for some } C \in S \} \]
\[ \Sigma = \{ P \subseteq D | P \text{ is a prime ideal that misses } S \} \]
\[ S_3 = \{ A \subseteq D | A \notin P \text{ for any } P \in \Sigma \} \].

Then \( S, S_1, S_2, S_3 \) are GMS's of \( D \) such that
\[ S_1 \subseteq S_2 \subseteq S_3, \ S \subseteq S_2 \text{ and } D_{S_1} \subseteq Q \subseteq D_S = D_{S_2} \subseteq D_{S_3} = \]
\[ \bigcap P \subseteq D \]. Furthermore, \( Q \) is a GQR of \( D \) if and only if \( Q = D_S \). Moreover, if \( Q \) is a GQR of \( D \), then

(a) \( Q \) is a RGQR of \( D \) if and only if \( D_{S_1} = D_{S_2} = D_{S_3} \).

(b) \( Q \) is an intersection of localizations of \( D \) but is not a RGQR of \( D \) if and only if \( D_{S_1} < D_{S_2} = D_{S_3} \).

(c) \( Q \) is not an intersection of localizations of \( D \) if and only if \( D_{S_1} < D_{S_2} < D_{S_3} \).

**Proof:** It is clear that \( S, S_1, S_2 \) and \( S_3 \) are GMS's of \( D \) and that \( S \subseteq S_2 \subseteq S_3 \). If \( E \in S_1 \), then
\[ l = \sum_{i=1}^{k} e_i a_i \text{ with } e_i \in E \text{ and } a_i \in Q. \text{ Hence } \prod_{i=1}^{k} [D:a_i] = \]
\[ = \sum_{i=1}^{k} (a_i \prod_{i=1}^{k} [D:a_i]) \subseteq \sum_{i=1}^{k} D \subseteq E, \text{ and therefore } E \in S_2. \text{ Thus } S_1 \subseteq S_2 \text{ and by applying Lemma 2.5 and Theorem 2.8, we have} \]
\[ D_{S_1} \subset D_S = D_{S_2} \subset D_{S_3} = \cap_{P \in \Sigma} D_P. \] It is also clear that \( Q \subset D_S \) and if \( \alpha \in Q \), then there exists \( E \in S_1 \) such that \( \alpha E \subset D \). Hence \( \alpha \in \alpha Q = \alpha EQ \subset DQ = Q \) and \( D_{S_1} \subset Q \). By Lemma 2.6, \( Q \) is a GQR of \( D \) if and only if \( Q = D_S \). Now assume \( Q \) is a GQR of \( D \).

If \( Q \) is a RGQR of \( D \), then by Theorem 2.1, if \( A \in S \), then \( AQ = Q \). Hence \( S \) is a RGMS of \( D \), so that \( S_1 = S_3 \) is the saturation of \( S \) and \( D_{S_1} = D_{S_2} = D_{S_3} \). The converse of (a) follows from Theorem 2.1.

If \( Q \) is not a RGQR of \( D \), then by (a), we must have \( D_{S_1} < D_{S_2} = Q \); and if \( Q \) is an intersection of localizations of \( D \), then Theorem 2.8 shows that \( Q = D_{S_3} \). Conversely, if \( D_{S_1} < D_{S_2} = D_{S_3} \), then \( Q = \cap_{P \in \Sigma} D_P \) and by (a), \( Q \) cannot be a RGQR of \( D \).

If \( Q \) is not an intersection of localizations of \( D \), then by Theorem 2.8, we have that \( Q = D_{S_2} < D_{S_3} \). Moreover \( D_{S_1} < Q \) for otherwise \( Q \) would be a RGQR of \( D \) and hence an intersection of localizations of \( D \). The converse of (c) is clear by Theorem 2.8.
Corollary 2.21. Let $S$ and $Q$ be as in Theorem 2.20. If there exists $A \subseteq D$ such that $AQ = Q$ and $\alpha A \subseteq D$ for some $\alpha \in Q$, then there are elements $B_1$ and $B_2$ of $S$ such that $B_1 \subseteq A \subseteq B_2$.

Proof: Since $\alpha A \subseteq D$, then $A \subseteq [D;\alpha]$, and since $AQ = Q$, then by Theorem 2.20, there exists $B_1 \in S$ such that $B_1 \subseteq A$.

We point out that Corollary 2.21 may not be true if we omit the hypothesis that $AQ = Q$ - even if $Q$ is a GQR of $D$. For example, in Example 1.1, we have that $T_V(P) = V_P$, so that $\alpha P \subseteq V$ for all $\alpha \in V_P$. But $V_P$ is a RGQR of $V$ and therefore by Theorem 2.1, we have that $AV_P = V_P$ for all $A \in S$. And since $PV_P < V_P$, then $P$ contains no element of $S$. (In fact, since $V$ is a valuation ring, then $P$ is properly contained in each element of $S$.)

Proposition 2.22. If $\{S_\alpha\}_{\alpha \in \Gamma}$ is a collection of GMS's of $D$, then $\bigcap_{\alpha \in \Gamma} S_\alpha$ is a GMS of $D$ provided $\bigcap_{\alpha \in \Gamma} S_\alpha \neq \emptyset$. (And we may assume that $\bigcap_{\alpha \in \Gamma} S_\alpha$ is not empty by assuming that $D$ is in $S_\alpha$ for all $\alpha \in \Gamma$. This will not change $D_{S_\alpha}$).
Also \( D_\cap S_\alpha \subset \cap DS_\alpha \). Moreover if for each \( \alpha \in \Gamma \), \( \cap \in a = \{ P_\alpha, \beta \subset D | P_\alpha, \beta \ \text{is a prime ideal of} \ \ D \ \text{and} \ P_\alpha, \beta \)

contains no element of \( S_\alpha \} \) and if each \( S_\alpha \) is maximal with respect to \( \Sigma_\alpha \), then \( D_\cap S_\alpha \subset \cap DS_\alpha \).

**Proof:** It is clear that \( \cap S_\alpha \) is a GMS of \( D \) and that \( D_\cap S_\alpha \subset \cap DS_\alpha \). If each \( S_\alpha \) is maximal with respect to \( \Sigma_\alpha \), then \( D_\alpha = \cap DP_\alpha, \beta \), so that \( \cap DS_\alpha = \cap D_\alpha \).

It is clear that \( \cap S_\alpha = \{ A \subset D | A \notin P_\alpha, \beta \} \) for any \( \alpha, \beta \} \) and therefore \( D_\cap S_\alpha = \cap D_\alpha \).

We point out that in the above theorem, \( \cap S_\alpha \) may indeed be empty - even if there are only finitely many \( S_\alpha \) and the \( D_\alpha \) are equal. For example, in Example 1.1, \( S_1 = \{ P \} \) and \( S_2 = \{ A \subset V | A \notin P \} \) are two GMS's of \( V \) such that \( S_1 \cap S_2 = \emptyset \) and \( V_{S_1} = V_{S_2} \). Also since intersections of localizations over a domain \( D \) may not be a RGQR of \( D \) (for example, see Example 2.1), then each \( D_\cap S_\alpha \) may be a RGQR of \( D \) without \( D_\cap S_\alpha \) being a RGQR of \( D \).
Proposition 2.23. Let $\alpha_1, \ldots, \alpha_n \in K$, $Q = \prod_{i=1}^{n} D[\alpha_i]$ and $A = \prod_{i=1}^{n} D[\alpha_i]$. Then $Q$ is a GQR of $D$ if and only if $Q = T_D(A)$, and $Q$ is a RGQR of $D$ if and only if $AQ = Q$.

Proof: Since $\alpha_i D[\alpha_i] \subseteq D$, then $Q \subseteq T_D(A)$. If $Q$ is a GQR of $D$, then $T_D(A) \subseteq Q$ by Lemma 2.6. If $Q$ is a RGQR of $D$, then $AQ = Q$ by Theorem 2.1, and if $AQ = Q$, then $A^n Q = Q$ and $\alpha A^n \subseteq D$ implies that $\alpha \in \alpha Q = \alpha A^n Q \subseteq D Q = Q$ and $T(A) \subseteq Q$.

Proposition 2.24. Let $Q$ be an overring of $D$.

(a) If $D[\alpha_1, \ldots, \alpha_n]$ is a GQR of $D$ for all finite subsets $\{\alpha_1, \ldots, \alpha_n\}$ of $Q - D$, then $Q$ is a GQR of $D$.

(b) If $D[\alpha]$ is a RGQR of $D$ for all elements $\alpha$ of $Q - D$, then $Q$ is a RGQR of $D$.

Proof: Let $S = \{\prod_{i=1}^{n} D[\beta_i] | \beta_i \in Q$ and $n$ is a positive integer $\}$ and let $\alpha \in D_S$. Then there exists an element $A = \prod_{i=1}^{k} D[\beta_i]$ of $S$ where $\beta_i \in Q$ such that $\alpha A \subseteq D$. But by Proposition 2.23, $\alpha \in T_D(A) = D[\beta_1, \ldots, \beta_k] \subseteq Q$. Since $Q$ is obviously contained in
Now if $D[a]$ is a RGQR of $D$ for all $a \in Q - D$, then by Theorem 2.1, we have that $1 \in [D:a]D[a] \subseteq [D:a]Q$. Hence again by Theorem 2.1, $Q$ is a RGQR of $D$.

**Proposition 2.25.** If $\{Q_a\}_{a \in \Gamma}$ is a chain of GQR's of $D$, then $Q = \bigcup_{a \in \Gamma} Q_a$ is a GQR of $D$. Moreover, if $Q_a$ is a RGQR of $D$ for all $a \in \Gamma$, then $Q$ is a RGQR of $D$.

**Proof:** If for each $a \in \Gamma$, we let $S_a$ be the GMS of $D$ generated by $\{[D:y]| y \in D_a\}$, then $D_a = D_{S_a}$ (Lemma 2.6), and the $S_a$ are linearly ordered under inclusion. Let $S$ be the GMS of $D$ generated by the elements of the $S_a$. Then it is clear that $Q \subseteq D_S$. And if $y \in D_S$, then let $A$ be an element of $S$ such that $yA \subseteq D$. Since $A = A_1 \cdots A_k$ with $A_i \in S_a$ and the $S_a$ are linearly ordered, then there is a $\beta \in \Gamma$ such that each $A_i = S_{a_i}$. Hence $A \in S_{a_1} \subseteq D_S = Q_\beta \subseteq Q$.

Now if $Q_a$ is a RGQR of $D$ for all $a \in \Gamma$, then
$S_\alpha$ is a RGMS of $D$ for all $\alpha \in \Gamma$ by Theorem 2.1. Thus define $S$ as in the preceding paragraph and let $A \in S$. Then $A \in S_\beta$ for some $\beta \in \Gamma$, and therefore $1 \in AD_{S_\beta} = AQ_\beta \subseteq AQ$, so that $Q$ is a RGQR of $D$.

**Proposition 2.26.** If $S$ is a GMS of $D$ and $D_1, \ldots, D_k$ are overrings of $D$ such that $D = \bigcap_{l=1}^{k} D_l$, then $D_S = \bigcap_{l=1}^{k} [D_l]_S$.

**Proof:** Since $D \subseteq D_l$ for $i = 1, \ldots, k$, then it is clear that $D_S \subseteq \bigcap_{l=1}^{k} [D_l]_S$. Let $\alpha \in \bigcap_{l=1}^{k} [D_l]_S$ and let $A_i \in S$ for $i = 1, \ldots, k$ such that $\alpha A_i \subseteq D_i$. Then $A = \bigcap_{l=1}^{k} \alpha A_i \in S$ and $\alpha A \subseteq D_i$ for each $i$. Hence $\alpha A \subseteq D$ and $\alpha \in D_S$.

An ideal $A$ of $D$ is said to be invertible provided $AA^{-1} = D$ where $A^{-1} = \{ \alpha \in K | \alpha A \subseteq D \}$. If $Q$ is an overring of $D$, then the conductor $C$ of $D$ in $Q$ is $\{ d \in D | dQ \subseteq D \}$. $C$ may be characterized as the largest ideal that is common to both $D$ and $Q$. We now give several basic properties concerning the transform of an ideal.

**Theorem 2.27.** Let $A$ and $B$ be ideals of $D$ and let $P$ be a prime of $D$. Then,
(a) If \( A \not\in P \), then \( T_D(A) \subseteq D_P \). Moreover, if \( AT_D(A) = T_D(A) \), then \( A \not\in P \) if and only if \( T_D(A) \subsetneq D_P \) and \( FT_D(A) = T_D(A) \) if and only if \( A \subseteq P \).

(b) If \( A \) is invertible, then \( AT_D(A) = T_D(A) \).

(c) If there exists a positive integer \( n \) such that \( A^n \subseteq B \), then \( T_D(B) \subseteq T_D(A) \).

(d) If \( A \) and \( B \) are finitely generated and \( \sqrt{A} = \sqrt{B} \), then \( T_D(A) = T_D(B) \).

(e) The conductor of \( D \) in \( T_D(A) \) contains \( \sum_{1}^{n} A^n \).

(f) If \( a \not\in \mathfrak{o} \) is an element of \( A \), then \( A^{-1} = \{ x/a | x \in [a:A] \} \). Consequently, if \( \mathfrak{o} \) is a GMS of \( D \), then an arbitrary element \( a \) of \( D_S \) is of the form \( a = d/c \) where \( c \not\in \mathfrak{o} \) is in some element \( C \) of \( S \) and \( d \in [c:C] \).

(g) If \( D \) is a noetherian domain and \( n \) is a positive integer, then \( A^{-n} \) is a finite \( D \)-module. In fact, if \( a \not\in \mathfrak{o} \) is an element of \( A^n \), then \( A^{-n} = \Sigma_{1}^{n} (x_i/1)D \) where \( [a:A^n] = (x_1, \ldots, x_n)D \).

(h) \( T_D(A+B) = T_D(A) \cap T_D(B) \), and if \( S \) is the GMS of \( D \) generated by \( A \) and \( B \), then \( D_S = T_D(AB) \).

(i) If \( x \) is a non-zero element of \( D \) and \( N \) is the
multiplicative system of elements \( \{x^n\}_{n=1}^\infty \), then \( T_D(x) = D[1/x] = D_N \).

**Proof:** The proof of (a) follows from Theorem 2.1 and Lemma 2.7. If \( A \) is invertible, then \( 1 \in {AA^{-1}} \subseteq T_D(A) \), so that (b) is true. The proofs of (c), (e) and (f) are trivial and (g) follows immediately from (f). If \( A = \langle a_1, \ldots, a_k \rangle_D \) and \( \\bar{A} = \bar{B} \), then there is a positive integer \( N \) such that \( a_i^N \in B \) for all \( i \) such that \( 1 \leq i \leq k \). Hence if \( \ell = kN \), then \( A^\ell = \sum a_1^{e_1} \cdots a_k^{e_k} D \subseteq B \).

Therefore (d) follows from (c). If \( a \in A^{-1} \) and \( a \neq 0 \) is an element of \( A \), then \( aa \in D \), so that \( a = x/a \) with \( x \in D \). But \( AA = x/a A \subseteq D \) implies that \( xA \subseteq aD \), so that \( x \in [a:A] \) and (f) is true.

It is clear that \( T_D(A+B) \subseteq T_D(A) \cap T_D(B) \). If \( a \in T_D(A) \cap T_D(B) \), then there is a positive integer \( n \) such that both \( aA^n \subseteq D \) and \( aB^n \subseteq D \). Now \( (A+B)^{2n} = \sum a_i^i B^j \), so that \( i \geq n \) or \( j \geq n \). Hence \( a(A+B)^{2n} = \sum a_i^i B^j \subseteq D \) and \( a \in T_D(A+B) \). Obviously, \( T_D(AB) \subseteq D_S \) and if \( a \in D_S \), then \( aA^i B^j \subseteq D \) where \( i \geq 0 \), \( j \geq 0 \) and either \( i \) or \( j \) is greater than zero. Let
k = \max \{i,j\}. Then \(a(AB)^k \subseteq aA^iB^j\), so that \(D_S = T_D(AB)\) and \((h)\) is true.

**Remark:** In Example 1.1, we have that \(T_D(P) = D_P\) and therefore the converse of part (a) of Theorem 2.27 is not true in general. In Chapter IV (see Corollary 4.10 and Corollary 4.23) we show that if \(A\) is an ideal of an RM-domain \(D\), then \(AT_D(A) = T_D(A)\). Since there exist non-invertible proper ideals of RM-domains (see [M], Example 1.3), then the converse of part (b) of Theorem 2.9 is also not always true.

If we define \(C_1, C_2\) and \(C_3\) over \(D\) as in Theorem 2.13, then \(C_1 \subseteq C_2 \subseteq C_3\). In Chapter III, we show that if \(D\) is a Prufer domain, then \(C_1 = C_2 = C_3\) (see Corollary 3.3). We now give three examples illustrating the other containment possibilities. The first example is an example of a domain \(D\) such that \(C_1 \subset C_2 = C_3\) and is due to Robert Gilmer.

**Example 2.1.** Let \(k\) be a field with a subfield \(k_0 \neq k\) and let \(V\) be a valuation ring of the form \(V = k + M\) where \(M = M^2\) is the maximal ideal of \(V\) and \(M = \bigcup_{\alpha} P_{\alpha}\) where \(\{P_{\alpha}\}\) consists of the primes properly contained in \(M\). (For such an example, see [G.H.], page...
Let $D = k_0 + M$.

Now $\bigcup_{a \in P_a}$ is a prime ideal of $\bigcap_{a \in P_a}$, so that $\bigcap_{a \in P_a} = V$. Since $M \notin P_a$ for any $a$, then $T_V(M) \subset \bigcap_{a \in P_a}$ (see part (a) of Theorem 2.27) which means that $T_V(M) = V$.

But since $M$ is a common ideal of $D$ and $V$, then $T_V(M) = T_D(M) = V$. By Theorem A in [G], page 560, we know that the primes of $D$ are exactly the primes of $V$ and that if $P < M$ is a proper prime of $D$, then $D_P = V_P$.

Hence $T_D(M) = V = \bigcap_{a \in P_a}$. And since any proper ideal $A$ of $D$ has the property that $A V < V$, then there can be no RGMS of subsets $S$ of $D$ such that $D_S = V$. Thus $V$ is a GQR of $D$ that is not restricted and we have $c_1 < c_2 \subseteq c_3$.

Now let $S$ be any GMS of ideals of $D$ such that $D < D_S$. Since $M$ is the unique maximal ideal of $V$ and of $D$ then $T_D(M) = V \subset T_D(A)$ for all proper ideals $A \in S$ and therefore $V \subset D_S$. But $V$ is a Prufer domain, so that there is a subcollection of the $\{P_a\}$, say $\{Q_\beta\}$, such that $D_S = \bigcap_{\beta} Q_\beta = \bigcap_{\beta} D_{Q_\beta}$ (see Corollary 3.3). Hence each GQR of $D$ is an intersection of localizations of $D$ and $c_1 < c_2 = c_3$.

Remark: If we let $k$ be algebraic over $k_0$, then
V is integral over D and since V is a valuation ring, then V is the integral closure of D. Also it is easy to show in this case that there is a one-to-one correspondence between the fields between k₀ and k and the domains between D and V. Hence if there are no fields between k₀ and k and k is algebraic over k₀, then there are no rings properly between D and V and all overrings of D are rings containing V. Whence each overring of D is an intersection of localizations of D.

We now give an example of a domain D such that C₁ = C₂ ⊂ C₃.

Example 2.2. If k is a field and x and y are indeterminates over k, then the function v defined from k[x,y] into the extended reals by v(o) = ∞ and 
v(ΣΣ_{1j}x^iy^j) = \min(i+j\sqrt{2} | a_{1j} ≠ 0) can be extended to the quotient field k(x,y) to give a non-discrete exponential valuation on k(x,y). If V = {t ∈ k(x,y) | v(t) ≥ 0}, then V is a rank 1 non-discrete valuation ring of the form V = k+M where M is the maximal ideal of V. Since M = {t ∈ k(x,y) | v(t) > 0}, then there is no element of minimal v-value in M = M² and therefore T_V(M) = V. Hence if we let D = k₀+M where k₀ is a proper subfield of k, then since M is an ideal common to D and V, we have that T_D(M) = T_V(M) = V. Since M is the only proper prime of D and D_M = D, then V is a GQR of D that is not
an intersection of localizations of \( D \). Since localizations of \( D \) are RGQR's of \( D \), then \( \mathcal{C}_1 < \mathcal{C}_2 < \mathcal{C}_3 \).

We point out that if we chose \( k ( \neq k_0 ) \) algebraic over \( k_0 \) such that there are no fields properly between \( k \) and \( k_0 \), then each overring of \( D \) is a ring containing \( V \) (see remark following Example 2.1), so that the only overrings of \( D \) are \( V \) and \( k(x,y) \). Hence each overring of \( D \) is a GQR of \( D \).

Our last example is an example where \( \mathcal{C}_1 < \mathcal{C}_2 < \mathcal{C}_3 \).

**Example 2.3.** Let \( K = k(x_1, \ldots, x_n, \ldots) \) and \( \overline{K} = k(x_2, \ldots, x_n, \ldots) \) where \( k \) is a field and the \( x_i \) are indeterminates over \( k \). Let \( x \) and \( y \) be indeterminates over \( k \) and let \( V \) be the rank 1 non-discrete valuation ring of \( K(x,y) \) as defined in Example 2.2. Then \( V \) is of the form \( V = K + M \) where \( M \) is the maximal ideal of \( V \). Let \( W \) be the valuation ring of \( \overline{K} \) induced by defining the function \( w \) from \( k[x_2, x_3, \ldots] \) into the countable weak direct sum of the integers, ordered lexicographically, by \( w(o) = \infty \) and \( w(ax_2^r \ldots x_n^r) = (r_2, r_3, \ldots, r_n, o, \ldots) \) for \( a \neq o \). Then \( W \) is of the form \( W = k + \overline{M} \) where \( \overline{M} = \overline{M}^2 \) is the maximal ideal of \( W \) and \( \overline{M} = \bigcup_{\alpha \in \Sigma} P_\alpha \) where \( \Sigma \) consists of the primes of \( W \) properly contained in \( \overline{M} \). (For a complete discussion of \( W \), see [G.H.], page 145.)
Let \( k_0 \) be a proper subfield of \( k \) and let \( D = k_0 + \overline{M} \). By Example 2.1, the primes of \( D \) are exactly the primes of \( W \) and \( W = \bigcap_{P_a \in \Sigma} P_a \).

Let \( D_1 = W + M \) and \( D_2 = D + M \). Since \( M \) is a non-zero ideal of \( D_2 \), \( D_1 \) and \( V \), then \( D_2 \), \( D_1 \) and \( V \) have quotient field \( k(x,y) \). We have the following:

\[
\begin{align*}
&k(x,y) \\
&\downarrow \\
&V = K + M \\
&\downarrow \\
&D_1 = W + M \quad \text{where} \quad W = k + \overline{M} \\
&\downarrow \\
&D_2 = D + M \quad \text{where} \quad D = k_0 + \overline{M}.
\end{align*}
\]

Since \( M \) is a common ideal of \( D_2 \) and \( V \), then by Example 2.2, we have \( T_{D_2}(M) = T_V(M) = V \). By Theorem A, in [G], page 560, we have that \( M \) is the unique minimal proper prime of \( D_2 \). Hence if \( P \) is a proper prime of \( D_2 \), then \( [D_2]_P \subseteq [D_2]_M \). Thus if we show that \( [D_2]_M \subset V \), then \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \) with respect to \( D_2 \). And \( [D_2]_M \subset V \), for \( D_2 = D + M \subseteq K + M < K + M = V \), so that \( [D_2]_M \subseteq [K + M]_M = K + M \subset V \).

Now \( W = \bigcap_{P_a \in \Sigma} P_a \), so that \( D_1 = W + M = \bigcap_{P_a \in \Sigma} P_a + M \).
But \( \cap D_{p_\alpha} + M = \cap (D_{p_\alpha} + M) \), for \( \alpha \), let \( z = d_\alpha + m_\alpha \in (D_{p_\alpha} + M) \), where \( d_\alpha \in D_{p_\alpha} \) and \( m_\alpha \in M \). Since the \( p_\alpha \) are linearly ordered under inclusion, then if \( d_\alpha \notin D_{p_\beta} \) for some \( p_\beta \in \Sigma \), then \( D_{p_\alpha} \subseteq D_{p_\beta} \). But \( z = d_\beta + m_\beta \) with \( d_\beta \in D_{p_\beta} \) and \( m_\beta \in M \) which means that \( d_\beta + m_\beta \) is a representative of \( z \) in \( D_{p_\alpha} + M \). Since \( d_\alpha + m_\alpha \) is the only representative of \( z \) in \( D_{p_\alpha} + M \) in the form \( x + m \) with \( x \in D_{p_\alpha} \) and \( m \in M \), then \( d_\alpha \in D_{p_\beta} \) for all \( p_\beta \in \tau \) and therefore \( z \in \cap D_{p_\alpha} + M \). The other inclusion is obvious.

Now by Theorem A in [G], \( P_\alpha + M \) is a prime ideal of \( D_2 \) for all \( P_\alpha \in \Sigma \). Thus if we can show that \( D_{p_\alpha} + M = [D_2](P_\alpha + M) \) for all \( P_\alpha \in \Sigma \), then \( D_1 = \cap [D_2 + M] = \cap [D_2](P_\alpha + M) \) is an intersection of localizations of \( D_2 \).

Let \( d_1/d_2 + m \in D_{p_\alpha} + M \) with \( d_1, d_2 \in D \) and \( d_2 \in D - P_\alpha \). Since the element of \( D_2 = D + M \) have a unique representation of the form \( d + m \) with \( d \in D \) and \( m \in M \), then \( d_2 \in D_2 - (P_\alpha + M) \). Hence \( d_1/d_2 + m = (d_1 + d_2 m)/d_2 \in [D_2](P_\alpha + M) \).

Let \((d_1 + m_1)/(d_2 + m_2) \in [D_2](P_\alpha + M) \) with \( d_1, d_2 \in D \),
\( m_1, m_2 \in M \) and \( d_2 + m_2 \in D_2 - (P_\alpha + M) \). Then \( d_2 \in D - P_\alpha \) and \( d_2 + m_2 \) is a unit of \( V \) which means that

\[ d_1/d_2 + m' \in D_{P_\alpha} + M \] where \( m' = (m_1 d_2 - d_1 m_2) / d_2 (d_2 + m_2) \).

But it is straightforward to show that

\[ (d_1 + m_1)/(d_2 + m_2) = d_1/d_2 + m' \] and therefore \( D_{P_\alpha} + M = [D_2](P_\alpha + M) \).

To conclude, we need to show that \( D_1 \) is not a RGQR of \( D_2 \). By Theorem A, in \([G]\), \( \overline{M} + M \) is the unique maximal ideal of \( D_2 \). Since \( \overline{M} + M \) is also a proper ideal of \( D_1 \), then for all proper ideals \( A \) of \( D_2 \), we have that \( AD_1 < D_1 \). Thus \( D_1 \) cannot be a RGQR of \( D \).
CHAPTER III

Much of the remainder of this thesis is concerned with the study of the containment relations between $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ over certain types of domains (where $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ are defined as in Theorem 2.13). Recall that (Theorem 2.13) $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3$ over any domain. We now list some of our results in this direction and give references to their proofs.

(1) $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$ if $D$ is a Prüfer, RM or unique factorization domain (see Corollary 3.3; Corollary 4.10; Corollary 3.13).

(2) $\mathcal{C}_2 = \mathcal{C}_3$ and may properly contain $\mathcal{C}_1$ if $D$ is a noetherian or Krull domain (see Corollary 3.5; Theorem 3.10).

(3) $\mathcal{C}_1 = \mathcal{C}_2$ and may be properly contained in $\mathcal{C}_3$ if $D$ has a finite number of prime ideals or if $D$ is a J-domain (see Theorem 2.14 and Example 2.2; Theorem 4.9).

Throughout this chapter, $D$ is a domain with identity with quotient field $K$. 
D is said to be a Prüfer domain provided each localization of D is a valuation ring.

Lemma 3.1. D is a Prüfer domain if and only if each integrally closed overring of D is a RGQR of D.

Proof: Assume D is Prüfer, Q is an overring of D and $\mathfrak{q}$ is a proper prime of Q. Since $Q_{\mathfrak{q}} \supseteq D \cap D_{\mathfrak{q}}$ and $D \cap D_{\mathfrak{q}}$ is a valuation ring, then $Q_{\mathfrak{q}} = D \cap D_{\mathfrak{q}}$ and by part (d) of Theorem 2.1, we have that Q is a RGQR of D. Conversely, assume that all integrally closed overrings of D are RGQR's of D and let Q be a ring containing D (Q may be equal to D here). Then, by part (c) of Theorem 2.1, the integral closure $\bar{Q}$ of Q is $D_S$ where $S = \{ A \subseteq D | A\bar{Q} = \bar{Q} \}$. But by the lying over theorem (for example, see [ZSI], page 257), we know that $A\bar{Q} = \bar{Q}$ if and only if $AQ = Q$. Hence $\bar{Q} = Q$, for if $a \in K$ and $aA \subseteq D$ for some $A \in S$, then $a \in aQ = a(AQ) = (aA)Q \subseteq DQ = Q$. We therefore have that all overrings of D are integrally closed and therefore by Theorem 22.2, page 309 in [G], we have that D is a Prüfer domain.

By the proof of the above lemma, we obtain the following theorem.

Theorem 3.2. D is a Prüfer domain if and only if each overring of D is a RGQR of D.
Remark: Since an overring of a Prüfer domain $D$ need not be a quotient ring of $D$ (for example, see [G.O.], page 102), then by Theorem 3.2, a RGQR of $D$ need not be a quotient ring of $D$. (Theorem 2.2 gives a necessary and sufficient condition for a RGQR of $D$ to be a quotient ring of $D$.)

**Corollary 3.3.** If $D$ is a Prüfer domain, then $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{c}_3$.

**Proof:** This follows immediately from Theorem 3.2 and Corollary 2.11.

We say $D$ is a *Krull domain* (see [N.L], page 115) provided $D = \bigcap_{P \in \Sigma} D_P$ where $\Sigma$ is the set of minimal prime ideals of $D$, each $D_P$ is a rank one discrete valuation ring, and each non-zero element of $D$ is contained in only finitely many elements of $\Sigma$. We now study GQR's of Krull domains.

**Theorem 3.4.** If $D$ is a Krull domain, $S$ is a GMS of $D$, and $\Sigma$ is the set of minimal prime ideals of $D$ that miss $S$, then $D_S = \bigcap_{P \in \Sigma} D_P$.

**Proof:** Lemma 2.7 shows that $D_S \subseteq \bigcap_{P \in \Sigma} D_P$. Let $y = x/y \in \bigcap_{P \in \Sigma} D_P$ with $x$ and $y$ in $D$. Now $y$ is
contained in only finitely many prime ideals $q_1, \ldots, q_s$ of $D$. Let $\Gamma = \{P_\beta\}$ be the set of minimal primes of $D$ outside of $q_1, \ldots, q_s$. Then $\gamma \in \bigcap_{\beta \in \Gamma} P_\beta$, and if $\gamma \in \bigcap_{l=1}^s D_{q_l}$, then $\gamma \in \left( \bigcap_{\beta \in \Gamma} P_\beta \right) \cap \left( \bigcap_{l=1}^s D_{q_l} \right) = D \subseteq D_S$. Thus assume there is a $q_l$, $1 \leq l \leq s$, such that $\gamma \not\in D_{q_l}$.

Since $q_l$ cannot be any element of $\Sigma$, then $q_l$ contains an element $A_1$ of $S$. But $D_{q_l}$ is a rank 1 discrete valuation ring and therefore there is a $z \in D_{q_l}$ such that $q_1D_{q_l} = zD_{q_l}$. Hence there are positive integers $n$ and $m$ such that $A_1D_{q_l} = z^nD_{q_l}$ and $yD_{q_l} = z^mD_{q_l}$. We then have

$$A_1^mD_{q_l} = (z^n)^mD_{q_l} = (z^m)^nD_{q_l} = y^nD_{q_l}$$

which implies that $\gamma A_1^mD_{q_l} \subseteq D_{q_l}$. Now for each $j$ between 1 and $s$ such that $\gamma \not\in D_{q_l}$, let $A_j$ be an element of $S$ such that $A_j \subseteq q_l$. Then $A = \bigcup_{j \in S} A_j$ and $\gamma A \subseteq \left( \bigcap_{l=1}^s D_{q_l} \right) \cap \left( \bigcap_{\beta \in \Gamma} P_\beta \right) = D$ which means $\gamma \in D_S$.

**Corollary 3.5.** If $D$ is a Krull domain, then $C_2 = C_3$ and may properly contain $C_1$. 
Proof: By Theorem 3.4, we have \( C_2 = C_3 \). And Eakin's example (see remark following Corollary 1.15) shows that we may have \( C_1 < C_2 \) over a Krull domain. (That is, \( Q = T_D(A) \) is not a RGQR of the Krull domain \( D \) in Eakin's example by Theorem 1.8, since \( Q \) is not a noetherian domain.)

Corollary 3.6. If \( D \) is a Krull domain and \( S \) is a GMS of \( D \), then \( D_S \) is a Krull domain and there is a one-to-one correspondence between the minimal prime ideals of \( D \) that miss \( S \) and the minimal prime ideals of \( D_S \).

Proof: By Theorem 3.4, \( D_S = \cap_{\alpha} D_{\alpha} \) where \( \{\alpha\} \) consists of the minimal prime ideals of \( D \) that miss \( S \). Since \( D \subseteq D_S \subseteq D_{\alpha} \), then \( [D_S]_{P_{\alpha}} = D_{P_{\alpha}} \) where \( P_{\alpha}' = P_{\alpha} D_{P_{\alpha}} \cap D_S \) and \( P_{\alpha}' \) is a minimal prime ideal of \( D_S \). Thus \( D_S \) is a Krull domain, and the \( P_{\alpha}' = P_{\alpha} D_{P_{\alpha}} \cap D_S \) consist of all the minimal prime ideals of \( D_S \) by [ZSII], pages 82-83.

Corollary 3.7. If \( D \) is a Krull domain and \( A \) and \( B \) are proper ideals of \( D \), then (a) \( T_D(A) = T_D(\cap P_I) \) where the \( P_I \) are the minimal prime ideals of \( D \) that
contain $A$, (b) $T_D(A) = T_D(A \cap B)$ provided $B$ is contained in no minimal prime ideal of $D$.

**Proof:** The proof of (a) is immediate from Theorem 3.4 and (b) follows from (a) since $A$ and $A \cap B$ are contained in exactly the same minimal prime ideals of $D$.

**Proposition 3.8.** If $\Sigma$ is a subset of the prime ideals of $D$ and $S$ is a complete GMS of $D$ with respect to $\Sigma$, then $A \in S$ implies $T_{DS}(AD_S) = D_S$.

(Hence if $D$ is a Krull domain, then $AD_S$ is contained in no minimal prime ideal of $D_S$.)

**Proof:** Let $Q = D_S = \bigcap_{P \in \Sigma} D_P$ and let $\Sigma'$ be the set of prime ideals of $Q$ that miss $S$. By the one-to-one correspondence $P' \rightarrow P$ between the prime ideals of $\Sigma'$ and $\Sigma$ (see Theorem 2.9) and by Lemma 2.7, we have $Q_S \subseteq \bigcap_{P' \in \Sigma'} Q_{P'} = \bigcap_{P \in \Sigma} D_P = D_S$. Hence $D_S = Q_S$, and if $A \in S$, then $T_{DS}(AD_S) \subseteq Q_S = D_S$, so that $T_{DS}(AD_S) = D_S$.

The remainder of Proposition 3.8 follows from Corollary 3.7.

**Proposition 3.9.** Let $A$ and $S$ be as in Proposition 3.8 and let $d$ be a non-zero element of $A^n$ where $n$ is a positive integer. Then, $dD_S = [dD_S : A^nD_S]$. 
Proof: If \( a \in \left[ d_{DS} : A_{DS}^n \right] \), then \( aA_{DS}^n \subseteq d_{DS} \), so that \( a/d A_{DS}^n \subseteq d_{DS} \). Hence \( a/d \in T_{DS}(AD_{S}) = D_{S} \) and \( a \in d_{DS} \). The other containment is obvious.

**Theorem 3.10.** If \( D \) is a noetherian domain, then \( \mathcal{C}_2 = \mathcal{C}_3 \) and may properly contain \( \mathcal{C}_1 \).

**Proof:** \( \mathcal{C}_2 = \mathcal{C}_3 \) follows from Theorem 2.16 and by Theorem 1.8. Eakin's example (see the remark following Corollary 1.15) is an example of a noetherian domain \( D \) such that \( \mathcal{C}_1 < \mathcal{C}_2 \).

We may wonder if a noetherian (Krull) overring of a noetherian (Krull) domain \( D \) must be a GQR of \( D \)? Again by using Eakin's example, we show the answer is no. Let \( R \), \( A' \), \( A \) and \( t \) be as in Eakin's example (see [E], Example 1). That is, \( R \) is a 3-dimensional, noetherian, integrally closed (and hence Krull) domain, \( t \) is an indeterminate over \( R \), and \( A' \) and \( A \) are overrings of \( R[t^{-1}] \) properly contained in \( R[t^{-1}, t] \) such that \( A' \) is noetherian and \( A \) is Krull. Let \( D = R[t^{-1}] \). Then \( D \) is both noetherian and Krull, and we show that neither \( A' \) nor \( A \) are GQR's of \( D \). Assume the contrary and let \( S \) be a GMS of \( D \) such that \( D_S \) is \( A \) or \( A' \). By (i) of Theorem 2.27 and Theorem 3.4, we know that \( T_D(t^{-1}) = R[t, t^{-1}] = \bigcap_{P \in \Gamma} D_P \) where \( \Gamma \).
\( \mathcal{T} \) is the set of all minimal prime ideals of \( D \) excluding \( t^{-1}D \). Now \( t \not\in D_S \) for otherwise \( D_S \) would be \( R[t, t^{-1}] \). Hence \( t^{-1}D = P \) is a minimal prime of \( D \) that contains no elements of \( S \). Therefore \( D_S \subset D_P \) and \( D_S \subset D_P \cap (\cap_{P \in \mathcal{T}} D_P) = D \) which is a contradiction as \( D < D_S \).

If \( D \) is a noetherian domain and \( A \) is a proper ideal of \( D \), then the (uniquely determined) associated prime ideals of the primary ideals occurring in an irredundant primary representation of \( A \) are called the prime ideals of \( A \) (see \([ZSI]\), pages 209-211).

**Theorem 3.11.** If \( S \) is a GMG of ideals of \( D \), then the following statements hold.

(a) If \( D \) is a noetherian domain and each prime ideal of \( D \) that is a prime ideal of an element of \( S \) contains an invertible primary ideal of \( D \), then \( S \) is a RGMC of \( D \).

(b) If \( D \) is a Krull domain and each minimal prime ideal of \( D \) that contains an element of \( S \) contains an invertible primary ideal of \( D \), then \( S \) is a RGMG of \( D \).

**Proof:** Let \( A \) be an element of \( S \). By part (d) of Theorem 2.27 (Corollary 3.7), we know that \( T_D(A) = T_D(\sqrt{A}) = T_D(\cap P_i) \) where \( \{P_i\}_1^k \) is the set of prime ideals of \( A \) (minimal prime ideals that contain \( A \)). For \( i = 1, \ldots, k \), let \( B_i \) be an invertible ideal of \( D \) such that \( \sqrt{B_i} = P_i \) and let \( B = \prod_{1}^{k} B_i \). Then \( B \) is an invertible and hence finitely generated ideal of \( D \) (for example, see Lemmas 3 and 4 in \([ZSI]\), page 272), so
that by the proof of part (d) of Theorem 2.27, we can find a positive integer \( l \) such that \( B^l \subset A \). We have

\[
T_D(A) \subset T_D(B^l) = T_D(\bigcap_{1 \leq i \leq k} B^l_i) = T_D(\bigcap_{1 \leq i \leq k} P_i) \quad \text{and therefore}
\]

\[
T_D(A) = T_D(B^l). \quad \text{Since } B^l \text{ is invertible, then}
\]

\[
1 \in B^l T_D(B^l) \subset A T_D(A) \subset A D_S.
\]

**Theorem 3.12.** An intersection of localizations over a unique factorization domain \( D \) is a regular (i.e. classical, as in \([Z.3.I]\)) quotient ring of \( D \).

**Proof:** Let \( \Gamma \) be a set of prime ideals of \( D \) and let \( M \) be the multiplicative system of elements

\[
M = D - \bigcup_{P \in \Gamma} P. \quad \text{Theorem 2.14 shows that } D_M = \bigcap_{P \in \Gamma} D_P = Q.
\]

Let \( y = x/y \in Q \) where \( x, y \in D \) and \( x \) and \( y \) have no common "prime element" factor and let \( y = p_1 \cdots p_k \) be the prime factorization of \( y \) in \( D \). Assume there exists \( i \) such that \( 1 \leq i \leq k \) and \( p_i \) is in an element \( P \) of \( \Gamma \). Since \( y \in D_P \), then there exists \( z \in D - P \) such that \( zy \in D \). But \( x \) and \( y \) are relatively prime, so that \( p_i \) must be a prime factor of \( z \). But this implies \( z \in P_a \) which is a contradiction. Thus the prime factors of \( y \) are in \( M \) which means \( y \in M \) and \( y \in D_M \).
Corollary 3.13. If $D$ is a unique factorization domain, then $c_1 = c_2 = c_3$.

Proof: $c_1 = c_2$ by Theorem 3.12 and the remark on page 3. Since a unique factorization domain is a Krull domain (for example, see page 82 in [Z.S.II], then $c_2 = c_3$ by Corollary 3.5.

The next few theorems are concerned with integral dependence of generalized quotient rings. If $S$ is a GMS of $D$ and $D'$ is integral over $D$, then we were not able to answer whether $D'_S$ must be integral over $D_S$. We can answer several related questions, however.

Theorem 3.14. If $S$ is a GMS of $D$ and $D$ is integrally closed, then $D'_S$ is integrally closed.

Proof: If $a$ is integral over $D_S$, then there exists $d_{n-1}, \ldots, d_0 \in D_S$ such that $a^n + d_{n-1}a^{n-1} + \ldots + d_0 = 0$. Let $a \in S$ such that $d_iA \subseteq D$ for each $i = 0, 1, \ldots, n-1$. Then if $a$ is a non-zero element of $A$, we have $0 = a^n(a^n + d_{n-1}a^{n-1} + \ldots + d_0) = (a^na^n + a_dn-1(a^na)^{n-1} + \ldots + a^nd)$, so that $a^na$ is integral over $D$. Since $D$ is integrally closed, then $aA \subseteq D$ and $a \in D_S$.

Corollary 3.15. If $Q$ is an intersection of
localizations of \( D \), \( D' \) is the integral closure of \( D \), and \( D' \subset Q \), then \( Q \) is integrally closed.

**Proof:** By Theorem 2.8 and part (d) of Theorem 2.15, we know that \( Q \) is a GQR of \( D' \). Thus \( Q \) is integrally closed by Theorem 3.14.

**Corollary 3.16.** Let \( \Sigma \) be a subset of the prime ideals of \( D \) and let \( S \) be a complete GMS of \( D \) with respect to \( \Sigma \). Then, \( D_S \) is integrally closed if and only if \( D_S = D'_S \) where \( D' \) is the integral closure of \( D \).

**Proof:** If \( D_S \) is integrally closed, then \( D \subset D' \subset D_S = \bigcap_{P \in \Sigma} D_P \), so that by part (d) of Theorem 2.15, we have that \( D_S = D'_S \). The converse if clear by Theorem 3.14.

**Theorem 3.17.** Let \( S \) be a RGMS of \( D \) and \( D' \) be an overring of \( D \). If \( D' \) is integral over \( D \), then \( D'_S \) is integral over \( D_S \) and if \( D' \) is the integral closure of \( D \), then \( D'_S \) is the integral closure of \( D_S \).

**Proof:** If \( \alpha \in D'_S \), then there exists \( A \in S \) such that \( \alpha A \subset D' \). Let \( Q \) be the integral closure of \( D_S \). Since \( AD_S = D_S \), then by the lying over theorem (see for example, [Z.S.I], page 257), we have \( AQ = Q \). Moreover,
since \( Q \) is integrally closed and \( Q \supset D_S \supset D \), then
\( Q \supset D' \) and therefore \( \alpha \in \alpha Q = \alpha AQ \subset D' Q = Q \). Thus \( \alpha \) is
integral over \( D_S \). The rest of the theorem follows from
Theorem 3.14.

Theorem 3.18. Let \( D' \) be the integral closure of \( D, C \) be the conductor of \( D \) in \( D' \), and \( S \) be a GMS
of \( D \). Then the following statements hold.

(a) If \( C \) contains an element of \( S \), then \( D' \subseteq D_S \)
and if \( D_S \) is an intersection of localizations of \( D \), then
\( D_S \) is integrally closed.

(b) \( CD_S \) is a subset of the conductor of \( D_S \) in \( D' \).
Moreover, if \( S \) is a RGMS of \( D \) and \( D' \) is a finite
\( D \)-module, then \( CD_S \) is the conductor of \( D_S \) in \( D' \).

(c) If \( S \) is a RGMS of \( D \) and \( D' \) is a finite
\( D \)-module, then \( D_S \) is integrally closed if and only if \( C \)
contains an element of \( S \).

Proof: If \( C \supset A \) where \( A \in S \), then \( AD' \subset D \)
implies \( D' \subset T_D(A) \subseteq D_S \). The rest of (a) follows from
Corollary 3.15.

If \( d' \in D_S \), then there exists \( A \in S \) such that
\( d' A \subset D' \). But \( CD' \subset D \), so that \( dcA \subset D \) and
\( d'c \in D_S \) for all \( c \in C \). Hence \( (CD_S)D_S' = CD_S' \subset D_S \) which
means $CD_S$ is a subset of the conductor of $D_S$ in $D'_S$.

Now assume $D'$ is a finite $D$-module and $S$ is a RGMS of $D$ and let $x$ be in the conductor of $D_S$ in $D'_S$. Since $xD'_S \subseteq D'_S$, then $xD' \subseteq D_S$. But $xD'$ is finitely generated over $D$, so that we can find an element $A$ of $S$ such that $(xD')A = (xA)D' \subseteq D$. Thus $xA \subseteq C$ and we have $x \in xD_S = x(AD_S) = (xA)D_S \subseteq CD_S$.

To prove (c), we need only show that if $D_S$ is integrally closed, then $C$ contains an element of $S$.

By Corollary 2.11 and Corollary 3.16, $D_S = D'_S$ and since $CD_S$ is the conductor of $D_S$ in $D'_S$, then $CD_S = D_S$.

Therefore by Theorem 1.5, $C$ contains an element of $S$.

$D$ is defined to be quasi-local provided it has only one maximal ideal, and $D$ is said to be an almost Dedekind domain provided each localization of $D$ is a rank 1 discrete valuation ring.

**Theorem 3.19.** Assume $D$ is not quasi-local. Then,

(a) $D$ is integrally closed if and only if each GQR overring of $D$ is integrally closed.

(b) $D$ is a Prüfer domain if and only if each GQR overring of $D$ is a Prüfer domain.

(c) $D$ is an almost Dedekind domain if and only if each GQR overring of $D$ is an almost Dedekind domain.
(d) $D$ is a Dedekind domain if and only if each GQR overring of $D$ is a Dedekind domain.

**Proof:** If each GQR overring of $D$ is integrally closed, then $D_{M_{\alpha}}$ is integrally closed for all maximal ideals $M_{\alpha}$ of $D$. But $D = \cap D_{M_{\alpha}}$, so that $D$ is integrally closed. The converse is clear by Theorem 3.14.

Now overrings of Prüfer, almost Dedekind and Dedekind domains are again Prüfer, almost Dedekind and Dedekind (for example, see [G], Theorems 22.1, 29.3 and [C], page 31). And if each GQR overring of $D$ is Prüfer (almost Dedekind) and $P$ is a prime of $D$, then $D_P$ is a Prüfer (almost Dedekind) domain, so that $D_P = [D_P]$ is a valuation (rank 1 discrete) ring. Hence (b) and (c) are true. Part (d) follows from Corollary 2.8 in [B].

**Remark:** It is clear that if $D$ is integrally closed, Prüfer, almost Dedekind, or Dedekind, then each GQR of $D$ is the same, even if $D$ is quasi-local. But the converse need not be true. For example, in Example 2.2, if we let $k$ be algebraic over $k_0$ and have no fields between $k$ and $k_0$, then $D = k_0 + M$ is neither integrally closed nor Prüfer, but all GQR overrings are both integrally closed and Prüfer. And if $D$ is a rank 1 non-discrete valuation ring; then $D$ is neither almost Dedekind nor
Dedekind, but it has the property that its only overring \( K \)
is both almost Dedekind and Dedekind.

We now study GQR's of \( D \) that are valuation rings.

Proposition 3.20. Let \( V \) be a valuation overring of \( D \) with maximal ideal \( M \). If \( V \) is a GQR of \( D \) and \( T_D(M \cap D) \nsubseteq V \), then \( V \) is a RGQR of \( D \).

Proof: Let \( S \) be a GMS of \( D \) such that \( D_S = V \)
and let \( A \in S \). If \( AV \subsetneq V \), then \( AV \subset M \), so that
\( A \subset M \cap D \). Hence \( T_D(M \cap D) \subset T_D(A) \subset D_S \) which is a
contradiction.

We point out that if \( V \) is a GQR of \( D \) and \( T_D(M \cap D) \subseteq V \), then \( V \) may or may not be a RGQR of \( D \)
as Examples 1.1 and 2.2 illustrate. We also note that the
condition \( T_D(M \cap D) \nsubseteq V \) is not sufficient to insure that
\( V \) is a GQR of \( D \). For if we let \( k \) be a field with
subfield \( k_o \) and let \( V \) be a rank 1 discrete valuation
ring of the form \( V = k + M \) where \( M \) is the maximal
ideal of \( V \) \((k[x](x)) \) is such an example, where \( x \) is an
indeterminate over \( k \) \) and set \( D = k_o + M \), then
\( M = M \cap D \) and \( T_D(M) = T_V(M) = K \). \([\text{For } M \text{ is a common
ideal of } D \text{ and } V, \text{ } M \text{ is a principal ideal of } V \text{ and
the only overring of } V \text{ is } K \text{ (See Example 2.2)}. \text{ Since} \)
M is the unique maximal ideal of \( D \), then each GQR overring of \( D \) must be \( K \) so that \( V \) is not a GQR of \( D \).

**Proposition 3.21.** If \( V \) is a valuation overring of \( D \) with maximal ideal \( M \), then \( V \) is a RGQR of \( D \) if and only if \( V = D_M \cap D \). Thus if \( T_D(M \cap D) \not\subseteq V \) and \( V \) is a GQR of \( D \), then \( V = D_M \cap D \).

**Proof:** If \( V \) is a RGQR of \( D \), then by Theorem 2.1, we know that \( V = D_S \) where \( S = \{ A \subseteq D \mid AV = V \} \). But \( AV = V \) if and only if \( A \not\subseteq M \cap D \) and therefore by Corollary 2.10, \( V = D_M \cap D \).

**Theorem 3.22.** Let \( S \) be a GMS of \( D \) and \( V \) be a valuation overring of \( D \) such that \( V = D_S \). If \( \Sigma = \{ P_\alpha \} \) is the set of prime ideals that contain no elements of \( S \), and \( S' \) is a complete GMS of \( D \) with respect to \( \Sigma \), then \( V' = D_S \) is a valuation ring since \( V' \supset V \). Let \( M' \) and \( M \) be the maximal ideals of \( V' \) and \( V \) respectively, and let \( Q \) and \( Q' \) be their respective contractions to \( D \). Then the following statements hold.

(a) The elements of \( \Sigma \) are ordered under inclusion, and \( Q' = \bigcup_{\alpha \in \Sigma} P_\alpha \).
(b) If \( Q' \in \Sigma \), then \( V' = D_{Q'} \).

(c) If \( P \) is a prime of \( D \) such that \( P \not\in \Sigma \) and \( D_S \subseteq D_P \), then \( D_P \subseteq V' \) and \( D_{Q'} = V' \).

(d) If the elements of \( S \) are finitely generated, then \( Q' \in \Sigma \) and 
\[ \Sigma = \{ P \leq D \mid P \text{ is a prime ideal of } D \text{ and } P \subseteq Q' \} \].

**Proof:** We recall (see Theorem 14.6 in [G]) that if \( V_1 \) and \( V_2 \) are valuation overrings of \( D \) with maximal ideals \( M_1 \) and \( M_2 \), then \( V_1 \subseteq V_2 \) if and only if \( M_2 \subseteq M_1 \). We have the following diagram.

\[
\begin{array}{c}
M \\
\downarrow \quad \downarrow \quad \downarrow \\
V' = D_{S'} = \bigcap_{P \in \Sigma} D_{P_{\alpha}} \\
\downarrow \\
V = D_S \\
\downarrow \\
D \\
\end{array}
\]

Now since overrings of a valuation ring are valuation rings and are ordered under inclusion (for example, see Theorem 14.6 in [G]), then the \( D_{P_{\alpha}} \) are ordered under inclusion for all \( P_{\alpha} \in \Sigma \). Thus the \( P_{\alpha} = P_{\alpha} D_{P_{\alpha}} \cap D \) are ordered under inclusion and consequently \( \bigcup P_{\alpha} \) is a prime ideal of \( D \). Now \( P_{\alpha} D_{P_{\alpha}} \subseteq M' \) for all \( P_{\alpha} \in \Sigma \).
\( P_\alpha \in \Sigma \), so that \( \bigcup P_\alpha \subset M' \); and if \( \gamma \in M' \), then

\[
1/\gamma \not\in V' = \cap D_{P_\beta},
\]

so that \( 1/\gamma \not\in D_{P_\beta} \) for some \( P_\beta \in \Sigma \).

But this implies \( \gamma \in P_\beta D_{P_\beta} \) as \( D_{P_\beta} \) is a valuation ring.

Since this is a contradiction, we have that \( M' = \bigcup P_\alpha D_{P_\alpha} \).

Therefore, \( Q' = M' \cap D = (\bigcup P_\alpha D_{P_\alpha}) \cap D = \bigcup (P_\alpha D_{P_\alpha} \cap D) = \bigcup P_\alpha \) and (a) is true.

If \( Q' \in \Sigma \), then \( V' \subset D_{Q'} \), so that \( V' = D_{Q'} \),

and (b) is true.

If \( P \not\in \Sigma \) and \( D_S = V \subset D_P \), then \( D_P \) is a valuation ring so that \( D_P \subset V' \) or \( V' < D_P \). If \( V' < D_P \), then \( PD_P < M' \) and therefore \( P < Q' = \bigcup P_\alpha \).

Since \( D_P \) is a valuation ring, then the above argument shows that there exists \( P_\beta \in \Sigma \) such that \( P \subset P_\beta \).

Hence \( P \in \tau \) which is a contradiction and therefore \( D_P \subset V' \). But this means that \( M' \subset PD_P \), so that \( Q' \subset P \) and \( D_P \subset D_{Q'} \). Since \( D_{Q'} \) is a valuation ring and \( Q' \) is the center of \( V' \) on \( D \), then \( D_{Q'} = V' \).

If \( A \) is finitely generated ideal of \( D \) and
A \subset Q' = \bigcup_{P_\alpha \in \Sigma} P_\alpha$, then since the $P_\alpha$ are linearly ordered, we have that $A \subset P_\beta$ for some $P_\beta \in \Sigma$. Hence if $A \in S$, then $A \notin Q'$ and therefore $Q' \in \Sigma$ if the elements of $S$ are finitely generated. The rest of (d) is clear.

In Theorem 3.2, we give a necessary and sufficient condition for a domain $D$ to be Prüfer. We can strengthen this somewhat if $D$ is integrally closed.

**Theorem 3.23.** If $D$ is integrally closed and each overring of $D$ is a GQR of $D$, then $D$ is a Prüfer domain.

**Proof:** By Theorem 3.14, each overring of $D$ is integrally closed and therefore by Theorem 22.2 on page 309 in [G], we have that $D$ is Prüfer.

Heinzer in [H] conjectures that if each overring of $D$ is a GQR of $D$, then the integral closure $\overline{D}$ of $D$ is Prüfer and he proves this when $\overline{D}$ is a finite $D$-module. Now in the case that each overring of $D$ is an intersection of localizations of $D$, then by part (d) of Theorem 2.15, we have that each overring of $\overline{D}$ is a GQR of $\overline{D}$, so that $\overline{D}$ is Prüfer by Theorem 3.23. In fact, all we need is that each overring of $D$ (excluding $\overline{D}$) is an intersection of localizations to insure that $\overline{D}$ is Prüfer. We are not able to answer Heinzer's conjecture.
If we modify Example 2.2 so that \( V = k + M \) is a rank 1 discrete valuation ring instead of a rank 1 non-discrete valuation ring (see the discussion following Proposition 3.20), then we have that \( \overline{D} \) is Prüfer and \( \overline{D} \) is not a GQR of \( D \). Thus the converse of Heinzer's conjecture is false. In fact, even if \( \overline{D} \) is both Prüfer and a GQR of \( D \) is not enough to insure that all overrings of \( D \) are GQR's of \( D \). For example, if we take Example 2.1 and let \( D_1 = k' + M \) where \( k_0 < k' < k \) and \( k \) is algebraic over \( k_0 \), then \( \overline{D} = V \) is a GQR of \( D \) but \( D_1 \) is not a GQR of \( D \). (See Remark following Example 2.1).

But if \( \overline{D} \) is Prufer and is a GQR of \( D \) and is the unique minimal overring of \( D \), then every overring of \( D \) is a GQR of \( D \). For let \( Q \) be an overring of \( D \) such that \( Q \neq \overline{D} \). Then \( D < Q \), so that by Theorem 3.2, \( Q \) is a RGQR of \( \overline{D} \). But this implies by part (g) of Theorem 2.15 that \( Q \) is a GQR of \( D \).
Recall that a J-domain is a one-dimensional domain whose non-zero ideals are contained in only finitely many maximal ideals. Two proper ideals $A$ and $B$ of a domain $D$ are said to be relatively prime provided $A + B = D$. By [ZSI], page 177, $A$ and $B$ are relatively prime if and only if their radicals are; and if $A$ and $B$ are relatively prime, then $A \cap B = AB$.

**Definition 4.1.** If $A$ is a proper ideal of a domain $D$, then $A$ is said to have a regular representation if and only if it can be represented as a finite intersection of pairwise relatively prime, primary ideals (and thus also as a finite product of relatively prime, primary ideals).

Proofs of the next five theorems can be found in [GD].

**Theorem 4.1.** A domain $D$ is a J-domain if and only if $D$ is one-dimensional and all proper ideals have regular representations.

**Theorem 4.2.** Regular representations are unique in a J-domain.
Theorem 4.3. Let $A$ and $B$ be proper ideals of a $J$-domain with regular representations $A = Q_1 \cdots Q_m$ and $B = Q_1' \cdots Q_n'$. Then $A \subseteq B$ if and only if $n \leq m$ and by renumbering $Q_i' \supseteq Q_i$ for $i = 1, \ldots, n$.

Theorem 4.4. Let $A = Q_1 \cdots Q_n$ be a regular representation of a proper ideal $A$ in a $J$-domain. Then for $1 \leq i \leq n$, $Q_i$ is the intersection of all ideals in $J$ that contain $A$ and that are primary for $P_i = \sqrt{Q_i}$.

Theorem 4.5. If $P$ is a prime in a $J$-domain, then the intersection of all its $P$-primarys is the zero ideal.

We point out that if $P$ is a prime ideal in a $J$-domain, then $\cap_{i=1}^{\infty} P^n$ may not be the zero ideal as the idempotent maximal ideal in any rank one non-discrete valuation ring illustrates.

If we take Example 2.2, and let $k$ be the field $F(x)$ where $F$ is a field and $x$ is an indeterminate over $F$ and let $k_0 = F$, then $D_1 = F[x] + M$ is a domain such that $D < D_1 < V$. By Theorem A, page 560, in [G], we know that $D$ is an integrally closed $J$-domain and that the dimension of $D_1$ is two. Hence overrings of $J$-domains are not necessarily $J$-domains and an integrally closed $J$-domain need not be Prüfer. We now give a necessary and
sufficient condition for an overring of a J-domain to be a J-domain.

**Theorem 4.6.** Let $D$ be a J-domain with integral closure $\overline{D}$. Then every overring of $D$ is a J-domain if and only if $\overline{D}$ is a J-domain that is Prüfer.

**Proof:** If every overring of $D$ is a J-domain, then $\overline{D}$ must be one-dimensional, and if $\overline{D}$ is not Prüfer, then there exists a maximal ideal $P$ of $\overline{D}$ such that $\overline{D}_P$ is not a valuation ring. But this implies (for example, see [G], Theorem 16.10, page 218) that there is a valuation overring $V$ of $\overline{D}$ having primes $P_1 < P_2$ such that $P_1 \cap \overline{D} = P_2 \cap \overline{D} = P$. Hence $V$ is not a J-domain which is a contradiction. Conversely, assume $\overline{D}$ is a Prüfer J-domain and let $Q$ be an overring of $D$. Since the integral closure $\overline{Q}$ of $Q$ contains $\overline{D}$, then by Theorem 3.2, $\overline{Q}$ is a RGQR of $\overline{D}$. Hence by Theorem 1.5 (c), $\overline{Q}$ is one-dimensional, and therefore by the lying over theorem (see [ZSI], page 257), $Q$ is one-dimensional. Let $A$ be a proper ideal of $Q$ and assume $A$ is contained in infinitely many maximal ideals of $Q$. Then again by the lying over theorem, $A\overline{Q}$ is contained in infinitely many maximal ideals of $\overline{Q}$ and therefore by Theorem 1.5 (c), $A\overline{Q} \cap \overline{D}$ is contained in infinitely many maximal ideals of $\overline{D}$ which is a contradiction.
Theorem 4.7. Let $D$ be a domain such that
$D = \bigcap_{\alpha \in \Gamma} D_{\alpha}$ is an intersection of quasi-local domains (all of
which are contained in the quotient field $K$ of $D$) and
such that
(1) Each $D_{\alpha}$ has a single proper prime $M_{\alpha}$,
(2) An element of $D$ is in only finitely many $M_{\alpha}$,
(3) $P_{\alpha} = D \cap M_{\alpha}$ is maximal, and all the $P_{\alpha}$ are
distinct.
Then $D$ is a J-domain with maximal ideals $\{P_{\alpha}\}_{\alpha \in \Gamma}$ and
$D_{\alpha} = D_{P_{\alpha}}$.

Proof: The proof is identical to the first part of
that in [C], Lemma 2, page 39 and we omit it.

Lemma 4.8. If $Q$ is a $P$-primary ideal of a
J-domain $D$, then $Q$ contains an invertible $P$-primary.

Proof: Let $a \neq 0$ be an element of $Q$ and let
$Q_1 \cdots Q_k$ be the regular representation of $aD$. Since a
product of ideals is invertible if and only if each factor
is invertible (for example, see [ZSI], page 272), and $aD$
is invertible, then each $Q_1$ is invertible. But $Q \supset aD$
and therefore by Theorem 4.3, $Q$ must contain $Q_1$ for
some $i$. 
Theorem 4.9. \( C_1 = C_2 \) over a J-domain \( D \) and it is possible that \( C_1 = C_2 < C_3 \).

Proof: Let \( Q = \cap D_{M_\alpha} \) be an intersection of localizations over \( D \) where \( \Gamma = \{M_\alpha\} \) is a subset of the primes of \( D \) and let \( \Sigma = \{P_\beta\} \) be the set of maximal ideals outside of \( \Gamma \). (If \( \Sigma = \emptyset \), then \( Q = D \) which is a RGQR of \( D \)). Let \( Q_\beta \) be an invertible \( P_\beta \)-primary for each \( P_\beta \in \Sigma \) (Lemma 4.8) and let \( S \) be the GMS of \( D \) generated by the \( Q_\beta \). Since each \( Q_\beta \) is invertible, then by Theorem 2.27, part (b), we know that \( S \) is a RGMS of \( D \) and therefore by Corollary 2.10, we have that \( D_S = \cap_{M_\alpha \in \Gamma} D_{M_\alpha} = Q \). Example 2.2 shows that we may have \( C_1 = C_2 < C_3 \).

Remark: Note that we could have let \( S \) be the GMS of \( D \) generated by all (or any subcollection) of the invertible primarysts of each \( P_\beta \) in \( \Sigma \) and we would still have had that \( S \) is a RGMS of \( D \) and that \( D_S = Q \).

We say \( D \) is an **RM-domain** provided \( D \) is a one-dimensional, noetherian domain \([C]\). It is clear that RM-domains are J-domains and therefore by Theorem 3.10, we have the following corollary.
**Corollary 4.10.** If $D$ is an RM-domain, then $c_1 = c_2 = c_3$.

**Definition.** A domain is said to be a GRM-domain provided its integral closure is a Dedekind domain.

Since Dedekind domains are one-dimensional and have the property that proper ideals are contained in only finitely many maximal ideals, then by the lying over theorem, we know that GRM-domains are $J$-domains. These domains have been studied by Butts and Smith in [B.S.] and they give an example of a GRM-domain that is not noetherian. In fact, if we let $k$ be any infinite algebraic extension field of $k_o$ and let $V$ be a rank one discrete valuation ring of the form $V = k + M$ (see the discussion following Proposition 3.20) where $M$ is the maximal ideal of $V$ and let $D = k_o + M$, then by Theorem A in [G], page 560, we have that $V$ is not noetherian, but that $V$ is the integral closure of $D$.

Our next few theorems are concerned with GRM-domains.

**Theorem 4.11.** Let $D$ be a GRM-domain with quotient field $K$ and let $Q$ be a domain integrally dependent on $D$ with quotient field $K'$ finite over $K$. Then $Q$ is also a GRM-domain.
Proof: Let $\overline{Q}$ and $\overline{D}$ be the integral closures of $Q$ and $D$. By transitivity of integral dependence (for example, see [ZSI], page 256), we have $\overline{Q}$ is integral over $D$ and therefore over $\overline{D}$. Since $\overline{D}$ is an RM-domain, then by Theorem 3 in [HG], we know that $\overline{Q}$ is an RM-domain. Since $\overline{Q}$ is integrally closed, then it must be a Dedekind domain (for example, see [ZSI], page 275) and therefore $Q$ is a GRM-domain.

Theorem 4.12. Overrings of GRM-domains are GRM-domains.

Proof: Since overrings of Dedekind domains are Dedekind domains (for example, see [C], page 31), then the proof is clear.

Theorem 4.13. A domain $D$ is a GRM-domain if and only if $D_M$ is a GRM-domain for each proper prime $M$ of $D$ and the non-zero elements of $D$ are contained in only finitely many maximal ideals of $D$.

Proof: Assume $D_M$ is a GRM-domain for all proper primes $M$ of $D$ and let $\overline{D}$ be the integral closure of $D$. If $\overline{M}$ is a maximal ideal of $\overline{D}$ that lies over $M$, then $\overline{D}_M$ is integrally closed by Theorem 3.14 and since $\overline{D}_M \supset D_M \cap D$, then $\overline{D}_M$ contains the integral closure of $D_M \cap D$. Thus $\overline{D}_M$ is an overring of a Dedekind domain.
and is therefore a Dedekind domain. Since $\overline{D_M}$ has only one maximal ideal, then $\overline{D_M}$ is a rank one discrete valuation ring; therefore $\overline{D}$ is an almost Dedekind domain.

Now assume that each non-zero element of $D$ is contained in only finitely many maximal ideals of $D$. By Theorem 9.11 in [G], page 108, the integral closure of $D_M$ is $\bigcap_{\alpha} \overline{D_{M_{\alpha}}} = Q$ where the $M_{\alpha}$ consist of the prime ideals in $\overline{D}$ that lie over $M$. Since $Q$ is a Dedekind domain and $MQ \subseteq M_{\alpha} \overline{D_{M_{\alpha}}} \cap Q$ for all $\alpha$, then there must be only finitely many $M_{\alpha}$ that lie over $M$. Thus the non-zero elements of $D$ are contained in only finitely many maximal ideals of $\overline{D}$. Let $\gamma = x/y$ with $x, y \in D$ be a non-zero element of $\overline{D}$. If $\gamma$ is contained in infinitely many maximal ideals of $\overline{D}$, then so is $x$ which is a contradiction. Thus by Theorem 30.2 in [G], then $\overline{D}$ is a Dedekind domain. Since the converse of this theorem is clear, then our proof is complete.

**Theorem 4.14.** Let $D$ be a GRM-domain with integral closure $\overline{D}$ and assume that the conductor $C$ of $D$ in $\overline{D}$ is non-zero. Let $A$ be an ideal of $D$ such that $A + C = D$. Then,

(a) $A = AD \cap D$

(b) $A$ can be expressed uniquely as a product of prime
ideals.

(c) $A$ is invertible, and hence also finitely generated.

**Proof:** Note that no proper prime ideal contains both $A$ and $C$ since $A + C = D$. Let $M_1, \ldots, M_k$ be the primes of $D$ that contain $A$. By Theorem 3.10, in [G], we know that $A = \bigcap_{l=1}^{k} (A D_{M_l} \cap D)$. Furthermore, since $C \not\subseteq M_i$ for $1 \leq i \leq k$, then $\overline{D} \subseteq T_D(C) \subseteq D_{M_l}$ for each $i$ and therefore $A \subseteq A \overline{D} \cap D \subseteq \bigcap_{l=1}^{k} A D_{M_l} \cap D = A$, so that $A = A \overline{D} \cap D$. Since $\overline{D}$ is a Dedekind domain, then each $D_{M_l}$ is a rank one discrete valuation ring and therefore $A D_{M_l} = M_i^{e_i} D_{M_l}$ is a power of $M_i D_{M_l}$ for each $i$. By Theorem 1.6, it then follows that $A = \bigcap_{l=1}^{k} M_i^{e_i} = \prod_{l=1}^{k} M_i^{e_i}$ and the uniqueness follows by Theorem 4.2.

To show $A$ is invertible, it is sufficient to show each $M_i$ is invertible as the product of invertible ideals is invertible. Let $L_1, \ldots, L_n$ be the prime ideals containing $C$ and let $x \in M_1 - \bigcup_{l=1}^{n} L_l$. (We can find such an $x$, for otherwise $M_1$ would be contained in $\bigcup_{l=1}^{n} L_l$ and would therefore be one of the $L_1$ (see [G], page 40).
Since \( x \not\in \bigcup_{i=1}^{k} L_i \), then \( xD + C = D \) and therefore \( xD \) is a product of prime ideals. Since \( x \in M_1 \), then \( M_1 \) must be one of its prime factors and since \( xD \) is invertible, then so is \( M_1 \).

**Theorem 4.15.** If \( D \) is a domain, then

(a) \( D \) is a J-domain (GRM-domain) if and only if each quotient overring of \( D \) is a J-domain (GRM-domain).

(b) \( D \) is a J-domain (GRM-domain) if and only if each RGQR overring of \( D \) is a J-domain (GRM-domain).

(c) \( D \) is a GRM-domain if and only if each GQR overring of \( D \) is a GRM-domain.

**Proof:** The proof of (a) (in the case of J-domains) can be found in \([D]\), Theorem 10, page 59. If each quotient overring of \( D \) is a GRM-domain, then each localization of \( D \) is a GRM-domain, so that by Theorem 4.13, we need only to show that non-zero elements of \( D \) are contained in only finitely many maximal ideals of \( D \). But this is clear since GRM-domains are J-domains and therefore \( D \) is a J-domain. Since overrings of GRM-domains are GRM-domains, then the proof of (a) is complete.

If \( D \) is a J-domain and \( Q \) is a RGQR of \( D \), then by part (c) of Theorem 1.5, \( Q \) is one-dimensional and proper ideals of \( Q \) are contained in only finitely many maximal ideals of \( Q \). (For otherwise, their contractions
in $D$ would be contained in infinitely many maximal ideals of $D$. Thus $Q$ is a $J$-domain. Since quotient rings of $D$ are RGQR's of $D$, then the converse of (b) is clear.

The proof of (c) is obvious.

**Theorem 4.16.** If $A$ and $B$ are proper ideals of a GRM-domain $D$, then $A \nsubseteq AB$.

**Proof:** If $A = AB$, then $AD = (AD)(BD)$ where $\overline{D}$ is the integral closure of $D$. Since $\overline{D}$ is a Dedekind domain, then $A\overline{D}$ is invertible, so that $B\overline{D} = \overline{D}$. But by the lying over theorem, this implies that $B = D$ which is a contradiction.

We point out that Theorem 4.16 may not be true over a $J$-domain. For if $V$ is a rank one non-discrete valuation ring with maximal ideal $M$, then $M = M^2$.

For the most part, the remaining theorems in this chapter are generalizations of theorems in [HG]. Throughout the rest of this chapter, we shall use $D$ to denote a $J$-domain with quotient field $K$. Note that if $A$ and $B$ are ideals of $D$ and $B$ is invertible, then $[A:B] = AB^{-1}$. By a fractional ideal $G$ of $D$, we mean a $D$-module with the property that there exists a non-zero element $d$ of $D$ such that $dG \subseteq D$. A fractional ideal $G$ of $D$ is said to be proper if $G \nsubseteq D$ and is said to be invertible if
\[ G^{-1} = D \] where \( G^{-1} = \{ a \in K | ag \subset D \} \).

**Definition 4.2.** If \( P \) is a proper prime of \( D \), then a fractional ideal \( G \) of \( D \) is said to be a **generalized \( P \)-primary** of \( D \) provided there exist ideals \( Q \) and \( Q_1 \) of \( D \) such that \( Q_1 \) is invertible and \( G = QQ_1^{-1} \), where \( Q \) is either \( P \)-primary or \( Q = D \), and \( Q_1 \) is either \( P \)-primary or \( Q_1 = D \), and \( Q \) or \( Q_1 \) is \( P \)-primary.

**Theorem 4.17.** If \( F \) is a proper fractional ideal of \( D \), then \( F = G_1 \cdots G_n \) where each \( G_i \) is a generalized \( P_i \)-primary and \( P_i \neq P_j \) for \( i \neq j \). Moreover, this representation is unique.

**Proof:** Let \( d \) be a non-zero element of \( D \) such that \( dF \subset D \). Then we can let \( dF = Q_1 \cdots Q_n \) and \( dD = \overline{Q}_1 \cdots \overline{Q}_n \) be primary representations of \( dF \) and \( dD \) where for each \( i \), either \( Q_i \) or \( \overline{Q}_i \) is primary for \( P_i \), and if \( Q_i \) (or \( \overline{Q}_i \)) is not primary for \( P_i \), then \( Q_i \) (or \( \overline{Q}_i \)) is \( D \). Since \( dD \) is invertible, then each \( \overline{Q}_i \) is invertible, and since \( dF = \overline{Q}_1 \cdots \overline{Q}_n F = Q_1 \cdots Q_n \), then \( F = Q_1\overline{Q}_1^{-1} \cdots Q_n\overline{Q}_n^{-1} \). Hence by letting \( G_i = Q_iQ_i^{-1} \) for each \( i \) such that \( 1 \leq i \leq n \), then \( F = G_1 \cdots G_n \) is a generalized primary representation of \( F \). Now suppose \( F = G'_1 \cdots G'_K \) is any other generalized primary represen-
tation of $F$. Since any given proper prime $P$ of $D$ contains an invertible $P$-primary $Q$, and $QQ^{-1} = D$ is a generalized $P$-primary, then by renumbering, we have that $F = G_1 \cdots G_k = G_1' \cdots G_k'$ are generalized primary representations of $F$ where for each $i$, $G_i$ and $G_i'$ are generalized $P_i$-primarys. Now for each $i$, let $G_i' = M_i \overline{M}_i^{-1}$ where $\overline{M}_i^{-1}$ is invertible. Then, $Q_1 \cdots Q_k M_1 \cdots M_k F = Q_1 \overline{M}_1 \cdots Q_k \overline{M}_k = M_1 Q_1 \cdots M_k Q_k$. In the manner in which we choose $Q_i, Q_i', M_i$ and $\overline{M}_i$, then we know that $Q_1 \overline{M}_1$ and $M_1 Q_1'$ are primary for $P_i$ for each $i$. Hence by Theorem 4.2, we have that $Q_1 \overline{M}_1 = M_1 Q_1'$ and therefore $Q_1 \overline{Q}_1 = M_1 \overline{M}_1^{-1}$ for each $i$ and our proof is complete.

The proofs of the next two theorems are identical to Theorems II.13 and II.15 in [M], so we omit them.

**Theorem 4.18.** Let $P$ be a proper prime of $D$ and let $G$ be a $D$-module such that $P^n \subseteq G \subseteq P^m$ for two integers (not necessarily positive) $n$ and $m$. Then $G$ is a generalized $P$-primary of $D$.

The converse of Theorem 4.18 is not true as any rank one non-discrete valuation ring illustrates. That is, primarys (and hence generalized primarys) of a $J$-domain need not contain a power of their radical.

**Theorem 4.19.** If $F$ is an invertible fractional ideal of $D$ and if $F_i$ is a non-zero fractional ideal of
D such that $F_1 \subseteq F$, then there exists $u \in K$ such that

$$F = F_1 + uD.$$

**Corollary 4.20.** An invertible fractional ideal of

D has a basis of two or fewer elements over D.

**Proof:** Let $u_1$ be a non-zero element of an invertible fractional ideal $F$. Then $u_1D \subseteq F$, so that there exists an element $u \in F$ such that $F = u_1D + uD$.

If $F$ is a fractional ideal of D and if $\Gamma$ denotes the collection of all proper primes in D, then

$$F = M(P_1) \cdots M(P_n)$$

where $M(P_i)$ is a generalized $P_i$-primary ideal and $P_i \neq P_j$ if $i \neq j$. It is clear that $F$ may be considered as a formal product over all $P \in \Gamma$ - that is, $F = \prod M(P)$ where $M(P) = D$ for all but finitely many factors. The following definition is given in [HG].

**Definition 4.3.** Let $N$ be a collection of prime ideals of D. Then $D_N = \{0\} \cup \{u \in K| \text{ in the representation } uD = \prod M(P) \text{ of generalized primarys, if } M(P) \notin D, \text{ then } P \in N \}$.

By the prime ideals of a proper subset $A$ of D, we shall mean those prime ideals that contain A.
Theorem 4.21. If $N$ is a collection of prime ideals of $D$ and $S = \{A \in D |$ the prime ideals of $A$ are in $N\}$, then $S$ is a GMS of $D$ and $D_S = D_N$ (where $D_N$ is as defined in Definition 4.3).

Proof: It is clear that $S$ is a GMS of $D$ by Theorem 4.2. Let $u \in K$ and let $uD = \prod_{P} M(P) = M(P_1) \cdots M(P_n) \cdots M(P_k)$ be the generalized primary representation of $uD$ where $n \leq k$ and for $1 \leq i \leq n$, $M(P_i) \notin D$. If for each $i$, we let $M(P_i) = Q_i^{-1}$, then $Q_1 \cdots Q_n u = Q_1 \cdots Q_n M(P_{n+1}) \cdots M(P_k) \in D$. Now if $u \in D_N$, then $Q_1, \ldots, Q_n$ are elements of $S$, so that $u \in D_S$ and $D_N \subseteq D_S$. Thus assume $u \in D_S$. We must show $P_1, \ldots, P_n$ are elements of $N$. Let $A \in S$ such that $uA \subseteq D$. Then $\mu A = M(P_1) \cdots M(P_n) \cdots M(P_k) A \subseteq D$ and therefore $Q_1 \cdots Q_n u A = Q_1 \cdots Q_n M(P_{n+1}) \cdots M(P_k) A = \overline{Q}_1 \cdots \overline{Q}_n$.

By Theorem 4.3, each $Q_i$ for $1 \leq i \leq n$ contains a primary of $Q_1 \cdots Q_n M(P_{n+1}) \cdots M(P_k) A$. Since $\overline{Q}_i \notin Q_i$ (for if $\overline{Q}_i \supseteq Q_i$, then $D \supseteq Q_i \overline{Q}_i^{-1}$) and since $\overline{Q}_i \notin M(P_j)$ for $j > n$ (for their belonging primes are different), then $Q_1$ must contain a primary of the form $Q_1 Q_i'$ where $Q_i'$ is a primary ideal of $A$. But by hypothesis, the
primarys of A are in N, so that \( P_1 \in N \). Thus \( D_S \subseteq D_N \) and our proof is complete.

A GQR of D as defined in Definition 4.3 shall be referred to as a "Grell GQR" of D. Our next theorem shows that the RGQR's of D are exactly the "Grell GQR's" of D.

**Theorem 4.22.** If Q is an overring of D, then Q is a "Grell GQR" of D if and only if D is a RGQR of D. In fact, if \( D_N \) is a "Grell GQR" of D associated with the collection N of prime ideals and \( \overline{N} \) denotes the complement of N in the set of all prime ideals of D, then \( D_N = \cap D_P = D_S \) where \( S = \{ A \in D \mid A \notin P \) for any \( P \in \overline{N} \} \).

**Proof:** Assume \( Q = D_N \) is a "Grell GQR" of D. By Theorem 2.8, \( D_S = \cap D_P \) which is RGQR of D by Theorem 4.9. Moreover, S is a RGMS of D for let \( A \in S \) and let \( AD = Q_1 \cdots Q_r \) be the regular representation of AD where \( \sqrt{Q_i} = P_i \) for each \( i = 1, \ldots, r \). By Lemma 4.8, each \( Q_i \) contains an invertible \( P_i \)-primary ideal \( Q_i' \), and since each \( P_i \supseteq A \), then no \( P_i \in \overline{N} \). Thus by the remark following Theorem 4.9, we know that \( D_S = Q_1' \cdots Q_r' D_S \subset (AD)D_S = \).
Hence $AD_S = D_S$ and $S$ is a RGMS of $D$. Since it is clear that $S = \{A \subseteq D | \text{the prime ideals of } A \text{ are in } N \}$, then by Theorem 4.21, $D_S = D_N = Q$ and $Q$ is a RGQR of $D$.

Conversely, if $Q$ is a RGQR of $D$, then by Corollary 2.10 and Theorem 1.9, we know that $Q = D_S = \bigcap_{P \in \pi} D_P$ where $\pi$ is a subset of the prime ideals of $D$ and

$S' = \{A \subseteq D | A \notin P \text{ for any } P \in \pi \} = \{A \subseteq D | AQ = Q \}$. Hence if $N$ is the set of prime ideals of $D$ that lie outside of $\pi$, then $S' = \{A \subseteq D | \text{the prime ideals of } A \text{ are in } N \}$ and therefore by Theorem 4.21, we have that $D_{S'} = D_N = Q$.

By the proof of Theorem 4.22, we obtain the following corollary.

**Corollary 4.23.** If $\Sigma$ is a collection of prime ideals of $D$ and $S$ is a GMS of $D$ that is complete with respect to $\Sigma$, then $S$ is a RGMS of $D$.

**Theorem 4.24.** If $I$ is a fractional ideal with respect to $D$ such that $I^0 = t \subseteq I$, then $Q = \bigcup_{n=0}^{\infty} I^n$ is a ring such that $D \subseteq Q \subseteq K$. If $I$ is invertible, then $Q = T_D(I^{-1}) = \bigcap_{P \in \Sigma} D_P$ where $\Sigma$ consists of the primes of $D$ which do not occur in the generalized primary decom-
position of \( t \). Hence \( Q \) is the "Grell GQR" \( t_N \) where \( N \) consists of the associated prime ideals of \( t \) in its generalized primary decomposition.

**Proof:** Since \( D \subseteq t \), then \( t^n \subseteq t^{n+1} \) for all \( n \geq 0 \) and it follows easily that \( Q \) is a ring such that \( D \subseteq Q \subseteq K \). Since \( D \subseteq t \), then \( t^{-1} \subseteq t \). If \( t \) is invertible, it is easy to check that \( (t^{-1})^n = (t^n)^{-1} \) and hence \( Q = T_D(t^{-1}) \). Since \( t^{-1} \) is a finitely generated ideal of \( D \), then by Theorem 2.16 and 4.22, we have that \( Q = \cap_{P \in \mathfrak{N}} P = D_N \).

**Remark:** If \( \alpha_1, \ldots, \alpha_n \in K-D \) and \( Q = D[\alpha_1, \ldots, \alpha_n] \), then there is a fractional ideal \( t \supset D \) such that
\[ Q = \bigcup_{n=0}^{\infty} t^n \] (where \( t^0 = D \)). For let \( t = (D, D\alpha_1, \ldots, D\alpha_n) = (D, M) \) where \( M = (D\alpha_1, \ldots, D\alpha_n) \). Then \( t^n = (D, M, M^2, \ldots, M^n) \) for all positive integers \( n \) and it is clear that \( Q = \bigcup_{n=0}^{\infty} t^n \). Notice by the previous theorem, if \( t \) is invertible, then \( Q = D[\alpha_1, \ldots, \alpha_n] = T_D(t^{-1}) = D_N \) where \( N \) consists of the prime ideals of \( t^{-1} \).

**Theorem 4.25.** Let \( \alpha_1, \ldots, \alpha_n \in K-D \),
\[ Q = D[\alpha_1, \ldots, \alpha_n], \quad A = \prod_{i=1}^{n} D[\alpha_i], \quad \text{and} \quad \Sigma = \{ P \subseteq D \mid P \text{ is a prime ideal of } D \text{ and } P \notin A \}. \] Then the following state-
ments hold.

(a) \( Q \) is a GQR of \( D \) if and only if \( Q = T_D(A) \).

(b) \( Q \) is a RGQR of \( D \) if and only if \( Q = T_D(A) \) and \( AQ = Q \).

(c) \( Q \) is a RGQR of \( D \) if and only if there exists an invertible ideal \( B \subset D \) such that \( \sqrt{B} = \sqrt{A} \) and \( Q = T_D(B) = \bigcap_{P \in \Sigma} D_P \).

(d) If \( Q \) is a RGQR of \( D \), then \( Q \) is a simple extension of \( D \), i.e. \( Q = D[\lambda] \) for some \( \lambda \in K \).

**Proof:** Parts (a) and (b) follows from Proposition 2.23. We consider (c). If \( Q \) is a RGQR of \( D \), then by (b), we have that \( Q = T_D(A) \) and \( AQ = Q \), so that by Corollary 2.10, \( T_D(A) = Q = \bigcap_{P \in \Sigma} D_P \). Let \( A = Q_1 \cdots Q_m \) be the regular representation of \( A \) where for each \( i \), \( Q_i \) is primary for \( P_i \). By Lemma 4.8, each \( Q_i \) contains an invertible \( P_i \)-primary \( Q'_i \). Thus \( B = Q'_1 \cdots Q'_m \) is an invertible ideal of \( D \) and \( \sqrt{B} = \sqrt{A} \). Since \( B \) is finitely generated, then by Theorem 2.16, we have that \( T_D(B) = \bigcap_{P \in \Sigma} D_P = Q \). The converse of (c) is clear by part (b) of Theorem 2.27. In (d), we have from (c) that \( B^{-1} \) is an invertible fractional ideal such that \( B^{-1} \supset D \). Hence by Theorem 4.19, there exists \( u \in K \) such that \( B^{-1} = D + uD \), and therefore \( T_D(B) = \bigcup_{n} B^{-n} = D[u] \) (see the remark following
Theorem 4.24). 

Remark: If $S_1$ and $S_2$ are RGMS's of $D$ and $D_{S_1} = D_{S_2}$, then the set of prime ideals of $D$ which miss $S_1$ is the same as the set of prime ideals which miss $S_2$, both sets consisting of the prime ideals of $D$ which extend and contract to themselves.

By a finite ring extension of $D$, we mean a ring of the form $D[\alpha_1, \ldots, \alpha_n]$ where each $\alpha_i \in K$ and where at least one of the $\alpha_i$ is not an element of $D$.

**Corollary 4.26.** Let $S$ be a RGMS of $D$ and let $\Sigma$ be the set of prime ideals of $D$ that miss $S$. The following are equivalent.

(a) $D_S$ is a finite ring extension of $D$.

(b) $D_S$ is a simple ring extension of $D$.

(c) $\Sigma$ consists of all but a finite number of prime ideals of $D$.

**Proof:** Now (a) $\Rightarrow$ (b) $\Rightarrow$ (c) by Theorem 4.25. Suppose (c) holds and let $P_1, \ldots, P_n$ be the prime ideals of $D$ not in $\Sigma$. By Corollary 2.10, $D_S = \bigcap_{P \in \Sigma} D_P$. Let $Q_i$ be an invertible $P_i$-primary ideal for $i = 1, \ldots, n$ (Lemma 4.8) and set $A = \prod_{i=1}^n Q_i$. Since $A$ is invertible, then
$A^{-1}$ is finitely generated over $D$, say $A^{-1} = (a_1, \ldots, a_n)D$.

Since $A$ is finitely generated, then by Theorem 2.16, we have that $D_S = \bigcap_{P \in \Sigma} D_P = T_D(A) = U A^{-n} = D[a_1, \ldots, a_n]$.

We now give an example of a simple extension of $D$ that is a GQR of $D$ but that is not a RGQR of $D$ and an example of a simple extension of $D$ that is not a GQR of $D$. Let $V = k + M$ as in Example 2.2 where $k = R[\sqrt{2}]$ and $R$ is the set of rationals, and let $D = k_0 + M$ where $k_0 = R$. Then $V$ is a GQR of $D$ but is not a RGQR of $D$, and $V = D[\sqrt{2}]$, so that $V$ is a simple ring extension over $D$. And if we let $V$ be a rank 1 discrete valuation ring of the form $V = R[\sqrt{2}] + M$ (see discussion following Proposition 3.20) and let $D = R + M$, then $V = D[\sqrt{2}]$ but $V$ is not a GQR of $D$.

**Proposition 4.27.** If $D$ is a J-domain such that every primary ideal contains a power of its radical, then $c_1 = c_2 = c_3$. If in addition, each prime ideal is the radical of a principal ideal, then every GQR of $D$ is a quotient ring of $D$.

**Proof:** Let $S$ be a GMS of $D$, $A \in S$ and $A = Q_1 \cdots Q_k$ be the regular representation of $A$ where $\sqrt{Q_i} = P_i$ for each $i$. For $i = 1, \ldots, k$, let $Q'_i$ be an
invertible $P_i$-primary ideal contained in $Q_i$ and set $B = \prod_{i=1}^{k} Q_i$. Then $B \subseteq A$ and since $\sqrt{B} = \sqrt{A}$, then there is a positive integer $n$ such that $A^n \subseteq B$. Thus by part (c) of Theorem 2.27, $T_D(A) = T_D(B)$; and since $B$ is invertible, then by part (b) of Theorem 2.27, $T_D(B)$ is a RGQR of $D$. We therefore have that $1 \in B \cdot T_D(B) \subseteq A \cdot T_D(B) = A T_D(A) \subseteq A D S$, so that $S$ is a RGMS of $D$ and $C_2 = C_3$. Assume now that each prime ideal of $D$ is the radical of a principal ideal and let $\Sigma$ be any collection of prime ideals of $D$. We show $\bigcap_{P \in \Sigma} D_P$ is a quotient ring of $D$. Let $\Gamma$ be the prime ideals of $D$ that are outside of $\Sigma$ and let $\Psi$ be the elements $d \in D$ such that $\sqrt{d D}$ is an element of $\Gamma$. If we let $M$ be the multiplicative system of elements generated by the elements of $\Psi$, then $\Sigma$ consists exactly of the prime ideals of $D$ containing no elements of $M$. Since $M$ is a RGMS of subsets of $D$, then by Corollary 2.10, $D_M = \bigcap_{P \in \Sigma} D_P$.

**Theorem 4.28.** If $A$ is a proper ideal of $D$ that is contained in an invertible proper prime ideal $P$ of $D$, then $A = P^k B$ where $k$ is a positive integer and $B$ is an ideal of $D$ such that $B \not\subseteq P$. Thus if the prime ideals of $A$ are invertible, then $A$ is a product of prime ideals
and is therefore invertible and finitely generated.

**Proof:** By Theorem 6.6 in [G], we know that if \( P \) is an invertible proper prime ideal of a ring \( Q \) with identity, then \( \{P^k\}_{k=1}^{\infty} \) is the set of \( P \)-primary ideals of \( Q \). The proof then easily follows.

**Theorem 4.29.** A ring \( R \) is noetherian if and only if each prime ideal of \( R \) has a finite basis (Cohen's theorem).

**Proof:** See [N.1], Theorem 3.4, page 8.

**Theorem 4.30.** If \( A \) is a proper ideal of \( D \) and there exists a non-zero element \( \gamma \) of \( A \) such that the prime ideals of \( \gamma D \) are finitely generated, then \( A \) is finitely generated.

**Proof:** If \( P \) is finitely generated, then by Cohen's theorem, we have the \( D_P \) is a RM-domain. Hence \( AD_P \) is finitely generated for all prime ideals \( P \) that contain \( \gamma \) and therefore by Theorem 30.3 in [G], we know that \( A \) is finitely generated.

**Theorem 4.31.** If \( Q \) is an overring of \( D \) and if the conductor \( C \) of \( D \) in \( Q \) is non-zero, then the prime ideals of \( C \) in \( D \) are non-invertible in \( D \).

**Proof:** Assume \( P \) is an invertible prime ideal of
D that contains C. Then by Theorem 4.28, we have \( C = P^kA \) where \( k \) is a positive integer and \( A \) is an ideal of \( D \) such that \( P \not\supset A \). Hence \( A = CP^{-k} \) and \( AQ = (P^{-k}C)Q = P^{-k}(CQ) = P^{-k}C = A \), so that \( A \) is an ideal in \( Q \); and consequently \( A \subset C \) since \( C \) is maximal with this property. However \( C \subset A \) and therefore \( C = A \) which is a contradiction since \( P \supset C \) and \( P \not\supset A \).

**Theorem 4.32.** If \( S \) is a RGMS of \( D \) and \( A \) is a proper ideal of \( D \) such that the prime ideals of \( A \) miss \( S \), then \( D_S/AD_S \cong D/A \). Moreover, if \( Q \) is a domain such that \( D \subseteq Q \subseteq D_S \), then \( A = AQ \cap D \) and \( D/A \cong Q/AQ \cong D_S/AD_S \).

**Proof:** The proof that \( D_S/AD_S \cong D/A \) is identical to that in [M], Theorem III.6, page 35. Since all of the prime ideals of \( A \) miss \( S \), then by part (c) of Corollary 1.7, \( A = AD_S \cap D \), so that \( A = AQ \cap D \) and \( D/A \cong Q/AQ \cong D_S/AD_S \).
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AUTobiography

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