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David Kekejian
Louisiana State University

Jerry P. Draayer
Louisiana State University

Tomáš Dytrych
Louisiana State University

Kristina D. Launey
Louisiana State University

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Overlaps of deformed and non-deformed harmonic oscillator basis states

David Kekejian [a,*], Jerry P. Draayer [a], Tomáš Dytrych [a,b], Kristina D. Launey [a]

[a] Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA
[b] Nuclear Physics Institute, Czech Academy of Sciences, 25068 Řež, Czech Republic

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ABSTRACT

A systematic approach for expanding non-deformed harmonic oscillator basis states in terms of deformed ones, and vice versa, is presented. The objective is to provide analytical results for calculating these overlaps (transformation brackets) between deformed and non-deformed basis states in spherical, cylindrical, and Cartesian coordinates. These overlaps can be used for reducing the complexity of different research problems that employ three-dimensional harmonic oscillator basis states, for example as used in coherent state theory and the nuclear shell-model, especially within the context of *ab initio* symmetry-adapted no-core shell model.

1. Introduction

The harmonic oscillator (HO) is perhaps the most frequently used concept in all of physics. Applications that employ HO concepts span from classical to quantum mechanics [1–6]. The fact that the Hamiltonian of the HO is very simple, incredibly intuitive, and analytically solvable makes it a desirable starting point for gaining initial insight into a broad range of physical phenomena.

The use of the three-dimensional (3D) HO for studying nuclear phenomena reaches back to the Nilsson single-particle model [7], and more generally to Elliott's many-particle SU(3) theory [8]. SU(3) enters because it is the symmetry of the 3D-HO, coupled with the fact that low-lying nuclear configurations can be considered in lowest order to be simple harmonic excitations around locally defined minima.

More recently, the use of an extended SU(3)-based theory has been shown to be advantageous in advanced *ab initio* nuclear structure studies because the SU(3) framework enables one to reduce the size of model spaces that are required to capture the dominant dynamics of complex nuclear systems. Applications using the so-called symmetry-adapted no-core shell-model (SA-NCSM) illustrate this well through successful studies up to and including medium-mass nuclei using SU(3) coupled basis states [9–13].

While all harmonic oscillator basis states can be considered to be equivalent, depending on the specific nature of the problem, some choices may be preferable to others; for example, spherical basis states ($|nlm\rangle$) have l as a good quantum number whereas cylindrical basis states ($|nn_zm\rangle$) do not; nevertheless, these are equivalent in the sense that both form complete sets. In particular, as first shown by the work of Nilsson cited above [7], it is advantageous to use a deformed basis to fold the dominant effects of deformation into smaller model spaces, an early result that anticipates and underpins the importance of providing easy-to-use transformations between these schemes – developed below – in anticipation of their use in more complex many-particle environments that these results can enable.

In this paper, we give analytic expressions for transformations between single-particle deformed and single-particle non-deformed spherical, cylindrical and Cartesian basis states; namely, for the overlaps $\langle\widetilde{nlm}|nlm\rangle$, $\langle\widetilde{nn_zm}|nn_zm\rangle$ and $\langle\tilde{n}_x\tilde{n}_y\tilde{n}_z|n_xn_yn_z\rangle$ respectively, where a tilde is used to denote deformation.

First we will calculate the transformation coefficients between non-deformed cylindrical and spherical basis states, namely $\langle nn_zm|nlm\rangle$ [14–16] which in turn will be used to calculate the $\langle\widetilde{nlm}|nlm\rangle$.

2. Non-deformed spherical versus cylindrical basis states

To calculate the $\langle nn_zm|nlm\rangle$, we begin by expanding a given $|nlm\rangle$ state in terms of all possible cylindrical states $|nn_zm\rangle$,

* Corresponding author.
E-mail address: dkekejian@lsu.edu (D. Kekejian).

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$$|nlm\rangle = \sum_{n_z=0}^{n} \langle nn_z m|nlm\rangle \, |nn_z m\rangle \,, \quad (1)$$

where $n \geq 0$ is the major oscillator shell quantum number, l is the angular momentum quantum number that can take any even number $0, 2, ..., n$ if n is even or any odd number $1, 3, ..., n$ if n is odd, and m is the projection of l on to the z-axis which can take on any value from $-l$ to l. The spherical symmetry dictates that the m on the left must be equal to the m on the right, and therefore the sum is only over n_z which runs from 0 to n. To actually determine the $< nn_z m|nlm >$ we write these states in terms of their respective coordinate representations,

$$\Psi_{n_r lm}(r,\theta,\phi) = \omega^{(l+1)/2}\omega^{1/4}\sqrt{\frac{2n_r!}{\Gamma(n_r+l+3/2)}}\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}$$
$$r^l e^{-\omega r^2/2} L_{n_r}^{l+1/2}(\omega r^2) P_l^m(cos(\theta)) e^{im\phi}, \quad (2)$$

$$\Psi_{n_\rho n_z m}(\rho,z,\phi) = \omega^{(|m|+1)/2}\omega^{1/4}\sqrt{\frac{n_\rho!}{2^{n_z} n_z! \Gamma(n_\rho+|m|+1)}}$$
$$\rho^{|m|} e^{-\omega\rho^2/2} e^{-\omega z^2/2} L_{n_\rho}^{|m|}(\omega\rho^2) H_{n_z}(\sqrt{\omega}z) e^{im\phi}/\pi^{3/4}, \quad (3)$$

where in eq. (2) $n_r = (n-l)/2$ and in eq. (3) $n_\rho = (n - n_z - |m|)/2$ are the radial quantum numbers of their respective geometries. In what follows we will use radial quantum numbers in our derivations because they simplify the resulting expressions. Also for simplicity we put $\hbar = M = 1$. If one desires to recover the SI units, one can simply replace $\omega \to M\omega/\hbar$. Because of axial symmetry, we can also, without loss of generality, drop the absolute value of m and consider only $m \geq 0$. The overlaps of the spherical and cylindrical basis states are therefore given by

$$\langle nn_z m|nlm\rangle = \int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi} r^2 sin(\theta)$$
$$\Psi^*_{n_\rho n_z m}(rsin(\theta), rcos(\theta), \phi)\Psi_{n_r lm}(r,\theta,\phi) dr d\theta d\phi. \quad (4)$$

The integral over ϕ in eq. (4) is simply 2π. For the other integrals over θ and r, we expand the special functions – two associated Laguerre functions plus a Hermite polynomial and a Legendre polynomial – in their respective polynomial forms where we substitute $\rho = rsin(\theta)$, $z = rcos(\theta)$ and integrate the resulting expressions; that is,

$$L_{n_\rho}^m(\omega r^2 sin^2(\theta)) = \sum_{k_\rho=0}^{n_\rho} (-1)^{k_\rho} \frac{(n_\rho+m)!}{(n_\rho-k_\rho)!(k_\rho+m)!k_\rho!} (\sqrt{\omega} rsin(\theta))^{2k_\rho}, \quad (5)$$

$$L_{n_r}^{l+1/2}(\omega r^2) = \sum_{k_r=0}^{n_r} (-1)^{k_r} \frac{(n_r+l+1/2)!}{(n_r-k_r)!(k_r+l+1/2)!k_r!} (\sqrt{\omega} r)^{2k_r}, \quad (6)$$

$$H_{n_z}(\sqrt{\omega} rcos(\theta)) = \sum_{k_z=0}^{[n_z/2]} (-1)^{k_z} \frac{n_z!}{(n_z-2k_z)!k_z!} (2\sqrt{\omega} rcos(\theta))^{n_z-2k_z}, \quad (7)$$

$$P_l^m(cos(\theta))$$
$$= (-1)^m sin^m(\theta) \sum_{k_l=0}^{[\frac{l-m}{2}]} (-1)^{k_l} \frac{(2l-2k_l)!}{2^l (l-k_l)!(l-2k_l-m)!k_l!} (cos(\theta))^{l-2k_l-m}. \quad (8)$$

The $[n_z/2]$ and $[l-m/2]$ in the upper limits of the sums above denote the integer part of those quantities. If we collect the θ dependent terms first and exploit the equivalence

$$\int_0^{\pi} sin(\theta) sin^{2m+2k_\rho}(\theta) cos^{n_z-2k_z+l-2k_l-m}(\theta) d\theta =$$
$$-\int_0^{\pi} (1-cos^2(\theta))^{m+k_\rho} cos^{n_z-2k_z+l-2k_l-m}(\theta) dcos(\theta), \quad (9)$$

and substitute $u = cos(\theta)$, the θ integration simply reduces to

$$\int_{-1}^{+1} (1-u^2)^{m+k_\rho} u^{n_z-2k_z+l-2k_l-m} du = \left(1+(-1)^{n_z-2k_z+l-2k_l-m}\right)$$
$$(m+k_\rho)! \frac{\Gamma(\frac{n_z-2k_z+l-2k_l-m+1}{2})}{2\Gamma(\frac{m+2k_\rho+n_z+l-2k_z-2k_l+3}{2})}, \quad (10)$$

where $(1+(-1)^{n_z-2k_z+l-2k_l-m}) = 2$ because it follows from the definition of n_ρ and l that $n_z + l - m$ should always be even. It is also the only acceptable value for the integral above to be nonzero. As for the radial integrals we get

$$\int_0^{+\infty} r^{m+l+2+2k_\rho+2k_r+n_z-2k_z} e^{-\omega r^2} dr =$$
$$\omega^{-(m+l+3+2k_\rho+2k_r+n_z-2k_z)/2}\Gamma(\frac{m+l+2+2k_\rho+2k_r+n_z-2k_z+1}{2})/2. \quad (11)$$

Inserting these three factors into eq. (4) yields the following analytical expression for the overlaps, which is a four-fold alternating sum,

$$\langle nn_z m|nlm\rangle = \frac{2\pi}{\pi^{3/4}}\sqrt{\frac{2n_r!}{\Gamma(n_r+l+3/2)}}\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\sqrt{\frac{n_\rho!}{2^{n_z}n_z!\Gamma(n_\rho+m+1)}}$$
$$\sum_{k_\rho=0}^{n_\rho}\sum_{k_r=0}^{n_r}\sum_{k_z=0}^{[n_z/2]}\sum_{k_l=0}^{[\frac{l-m}{2}]} (-1)^{m+k_\rho+k_r+k_z+k_l} 2^{n_z-2k_z}$$
$$\frac{(n_\rho+m)!}{(n_\rho-k_\rho)!k_\rho!}\frac{(n_r+l+1/2)!}{(n_r-k_r)!(k_r+l+1/2)!k_r!}\frac{n_z!}{(n_z-2k_z)!k_z!}\frac{(2l-2k_l)!}{2^l(l-k_l)!(l-2k_l-m)!k_l!}$$
$$\frac{\Gamma(\frac{n_z-2k_z+l-2k_l-m+1}{2})}{2\Gamma(\frac{m+2k_\rho+n_z+l-2k_z-2k_l+3}{2})}\Gamma(\frac{m+l+3+2k_\rho+2k_r+n_z-2k_z}{2}). \quad (12)$$

Eq. (12) was tested and benchmarked against the results of [16] given in eqs. (33)-(36). In this reference, the author presents interbasis expansions between cylindrical and spherical coordinates up to $n = 3$. The formula in eq. (12) can be considered to be an extension to the work of [16] since it provides the values of these types of interbasis expansions for any set of quantum numbers and is not limited to the size of the model space. In Table 1, we present a comparison for some of the overlaps in [16] to our results using the formula in eq. (12). Note that the overlaps in [16] are given in the $\langle n_\rho n_z m|nlm\rangle$ notation where $n_\rho = (n - n_z - |m|)/2$ as mentioned above.

Table 1
Long table caption.

$\langle n_\rho n_z m\|nlm\rangle$	$\langle nn_z m\|nlm\rangle$	Results in [16] eq. (33)-(36)	Results using our eq. (12)
$\langle 000\|000\rangle$	$\langle 000\|000\rangle$	1	1
$\langle 001\|111\rangle$	$\langle 101\|111\rangle$	−1	−1
$\langle 020\|220\rangle$	$\langle 220\|220\rangle$	$\sqrt{2/3}$	0.816497
$\langle 020\|200\rangle$	$\langle 220\|200\rangle$	$-\sqrt{1/3}$	−0.577350
$\langle 110\|330\rangle$	$\langle 310\|330\rangle$	$\sqrt{3/5}$	0.774597
$\langle 110\|310\rangle$	$\langle 310\|310\rangle$	$\sqrt{2/5}$	0.632456
$\langle 021\|331\rangle$	$\langle 321\|331\rangle$	$-\sqrt{4/5}$	−0.894427
$\langle 021\|311\rangle$	$\langle 321\|311\rangle$	$\sqrt{1/5}$	0.447214

3. Deformed basis states expanded in terms of spherical basis states

Next we turn our attention to an expansion of non-deformed cylindrical states ($|nn_zm\rangle$) of eq. (1) in terms of their deformed ($|\widetilde{n}\widetilde{n}_zm\rangle$) counterparts,

$$|nn_zm\rangle = \sum_{\widetilde{n}=0}^{\infty}\sum_{\widetilde{n}_z=0}^{\widetilde{n}} \langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle \, |\widetilde{n}\widetilde{n}_zm\rangle \,. \tag{13}$$

In this case the inner sum runs over $\widetilde{n}_z$ from 0 to $\widetilde{n}$, while the outer sum runs over $\widetilde{n}$ from 0 to infinity which in practice is taken to be some $\widetilde{N}_{\max}$ cutoff, which is the maximum number of HO excitations (quanta) considered within a given model space. The transformation coefficients, $\langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle$ in eq. (13), are the key elements in this expansion. Once the $\langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle$ are known, the corresponding transformation between the non-deformed spherical states and their deformed counterparts follows directly through a double application of eq. (12); specifically,

$$\langle \widetilde{n}\widetilde{l}m|nlm\rangle = \sum_{\widetilde{n}_z}^{\widetilde{n}}\sum_{n_z}^{n} \langle \widetilde{n}\widetilde{l}m|\widetilde{n}\widetilde{n}_zm\rangle\langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle\langle nn_zm|nlm\rangle, \tag{14}$$

where the first $\langle \widetilde{n}\widetilde{l}m|\widetilde{n}\widetilde{n}_zm\rangle$ and third $\langle nn_zm|nlm\rangle$ terms in this double sum follow from applications of eq. (12). Note that the overlap in eq. (12) is independent of ω, which is a result that follows simply from the completeness of the basis states as long as the transformation is carried out in the same space, deformed or non-deformed.

The missing ingredient in this whole picture is the transformation coefficient, $\langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle$, between deformed cylindrical states and their non-deformed counterparts, to which we now turn our attention. A coordinate representation for the non-deformed ket, $|nn_zm\rangle$, was introduced in section 2, eq. (3). The corresponding coordinate representation for the deformed bra, $\langle \widetilde{n}\widetilde{n}_zm|$, captures the effect of the deformation; that is, $\omega_x = \omega_y$ (cylindrical symmetry) not equal to ω_z and where $\omega_x\omega_y\omega_z = \omega^3$ to ensure overall volume conservation.

$$\Psi_{\widetilde{n}_\rho\widetilde{n}_zm}(\rho,z,\phi) = \omega_x^{(m+1)/2}\omega_z^{1/4}\sqrt{\frac{\widetilde{n}_\rho!}{2^{\widetilde{n}_z}\widetilde{n}_z!\Gamma(\widetilde{n}_\rho+m+1)}} \rho^m e^{-\omega_x\rho^2/2}e^{-\omega_z z^2/2}L^m_{\widetilde{n}_\rho}(\omega_x\rho^2)H_{\widetilde{n}_z}(\sqrt{\omega_z}z)e^{im\phi}/\pi^{3/4}. \tag{15}$$

It follows from this that

$$\langle \widetilde{n}\widetilde{n}_zm|nn_zm\rangle = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_0^{2\pi} \Psi^*_{\widetilde{n}_\rho\widetilde{n}_zm}(\rho,z,\phi)\Psi_{n_\rho n_zm}(\rho,z,\phi)\rho d\rho dz d\phi. \tag{16}$$

The ϕ part again gives 2π. However, unlike the previous case considered in section 2 above, these integrals are separable. First, for the radial integral we have

$$\int_0^{+\infty} \rho^{2m}L^m_{\widetilde{n}_\rho}(\omega_x\rho^2)L^m_{n_\rho}(\omega\rho^2)e^{-(\omega+\omega_x)\rho^2/2}\rho d\rho = \int_0^{+\infty} u^m L^m_{\widetilde{n}_\rho}(\omega_x u)L^m_{n_\rho}(\omega u)e^{-(\omega+\omega_x)u/2}du/2, \tag{17}$$

where we made the $\rho^2 = u$ and $2\rho d\rho = du$ substitution. Now, we write the associated Laguerre polynomials in their explicit forms as we did in eq. (5) and eq. (6) to get

$$\sum_{k_\rho=0}^{n_\rho}\sum_{\widetilde{k}_\rho=0}^{\widetilde{n}_\rho}(-1)^{k_\rho+\widetilde{k}_\rho}\frac{(n_\rho+m)!}{(n_\rho-k_\rho)!(k_\rho+m)!k_\rho!}\frac{(\widetilde{n}_\rho+m)!}{(\widetilde{n}_\rho-\widetilde{k}_\rho)!(\widetilde{k}_\rho+m)!\widetilde{k}_\rho!}\omega^{k_\rho}\omega_x^{\widetilde{k}_\rho} \int_0^{+\infty} u^{k_\rho+\widetilde{k}_\rho+m}e^{-(\omega+\omega_x)u/2}du/2, \tag{18}$$

where the integral gives us

$$\int_0^{+\infty} u^{k_\rho+\widetilde{k}_\rho+m}e^{-(\omega+\omega_x)u}du/2 = 2^{k_\rho+\widetilde{k}_\rho+m}(k_\rho+\widetilde{k}_\rho+m)! (\omega+\omega_x)^{-(k_\rho+\widetilde{k}_\rho+m+1)}. \tag{19}$$

And finally, for eq. (17) we obtain the following expression

$$I_\rho = \sum_{k_\rho=0}^{n_\rho}\sum_{\widetilde{k}_\rho=0}^{\widetilde{n}_\rho}(-1)^{k_\rho+\widetilde{k}_\rho}2^{k_\rho+\widetilde{k}_\rho+m}\frac{(n_\rho+m)!}{(n_\rho-k_\rho)!(k_\rho+m)!k_\rho!} \frac{(\widetilde{n}_\rho+m)!}{(\widetilde{n}_\rho-\widetilde{k}_\rho)!(\widetilde{k}_\rho+m)!\widetilde{k}_\rho!}(k_\rho+\widetilde{k}_\rho+m)!\frac{\omega^{k_\rho}\omega_x^{\widetilde{k}_\rho}}{(\omega+\omega_x)^{k_\rho+\widetilde{k}_\rho+m+1}}. \tag{20}$$

Now consider the integral over z,

$$\int_{-\infty}^{+\infty} e^{-(\omega+\omega_z)z^2/2}H_{n_z}(\sqrt{\omega}z)H_{\widetilde{n}_z}(\sqrt{\omega_z}z)dz = \sqrt{\tfrac{2}{\omega+\omega_z}}\int_{-\infty}^{+\infty} e^{-u^2}H_{n_z}(\sqrt{\tfrac{2\omega}{\omega+\omega_z}}u)H_{\widetilde{n}_z}(\sqrt{\tfrac{2\omega_z}{\omega+\omega_z}}u)du, \tag{21}$$

where we made the substitution $z = \sqrt{\frac{2}{\omega+\omega_z}}u$. We can simplify this result further by utilizing a multiplication theorem [17] for Hermite polynomials,

$$H_n(\gamma z) = \sum_{k=0}^{[n/2]}\gamma^{n-2k}(\gamma^2-1)^k\frac{n!}{(n-2k)!k!}H_{n-2k}(z). \tag{22}$$

Inserting eq. (22) into eq. (21), we get the following intermediate result

$$\sqrt{\tfrac{2}{\omega+\omega_z}}\sum_{k=0}^{[n_z/2]}\sum_{\widetilde{k}=0}^{[\widetilde{n}_z/2]}(-1)^{\widetilde{k}}(\sqrt{\tfrac{2\omega}{\omega+\omega_z}})^{n_z-2k}(\sqrt{\tfrac{2\omega_z}{\omega+\omega_z}})^{\widetilde{n}_z-2\widetilde{k}}(\tfrac{\omega-\omega_z}{\omega+\omega_z})^{k+\widetilde{k}} \tfrac{\widetilde{n}_z!}{(\widetilde{n}_z-2\widetilde{k})!\widetilde{k}!}\tfrac{n_z!}{(n_z-2k)!k!}\int_{-\infty}^{+\infty} e^{-u^2}H_{\widetilde{n}_z-2\widetilde{k}}(u)H_{n_z-2k}(u)du, \tag{23}$$

where $\int_{-\infty}^{+\infty} e^{-u^2}H_{\widetilde{n}-2\widetilde{k}}(u)H_{n_z-2k}(u)du = \sqrt{\pi}2^{n_z-2k}(n_z-2k)!\,\delta_{n_z-2k,\widetilde{n}_z-2\widetilde{k}}$. And finally, we obtain the following for eq. (21)

$$\sqrt{\tfrac{2\pi}{\omega+\omega_z}}\sum_{k=0}^{[n_z/2]}\sum_{\widetilde{k}=0}^{[\widetilde{n}_z/2]}(-1)^{\widetilde{k}}2^{n_z-2k}(\sqrt{\frac{2\omega}{\omega+\omega_z}})^{n_z-2k}(\sqrt{\frac{2\omega_z}{\omega+\omega_z}})^{\widetilde{n}_z-2\widetilde{k}} \left(\frac{\omega-\omega_z}{\omega+\omega_z}\right)^{k+\widetilde{k}}\frac{\widetilde{n}_z!n_z!}{(n_z-2k)!k!\widetilde{k}!}\delta_{n_z-2k,\widetilde{n}_z-2\widetilde{k}}. \tag{24}$$

The only term that survives in eq. (24) is when $\widetilde{k} = (\widetilde{n}_z - n_z + 2k)/2$ which reduces it to

$$I_z = \sqrt{\frac{2\pi}{\omega+\omega_z}} \sum_{k=0}^{[n_z/2]} (-1)^{(\widetilde{n}_z - n_z + 2k)/2} 4^{n_z-2k} \left(\frac{\sqrt{\omega\omega_z}}{\omega+\omega_z}\right)^{n_z-2k} \left(\frac{\omega-\omega_z}{\omega+\omega_z}\right)^{(\widetilde{n}_z-n_z+4k)/2} \frac{\widetilde{n}_z! n_z!}{(n_z-2k)!((\widetilde{n}_z-n_z+2k)/2)!k!}. \quad (25)$$

It follows from all of the above that the desired overlap is given by

$$\langle \widetilde{n}\widetilde{n}_z m | n n_z m\rangle = \frac{2\pi}{\pi^{3/2}} (\omega\omega_x)^{(m+1)/2} (\omega\omega_z)^{1/4} \times \sqrt{\frac{\widetilde{n}_\rho!}{2^{\widetilde{n}_z}\widetilde{n}_z!\Gamma(\widetilde{n}_\rho+m+1)}} \sqrt{\frac{n_\rho!}{2^{n_z}n_z!\Gamma(n_\rho+m+1)}} \times I_\rho \times I_z. \quad (26)$$

Knowing $\langle \widetilde{n}\widetilde{n}_z m | n n_z m\rangle$ allows us to obtain an analytic expression for the $\langle \widetilde{n}\widetilde{l}m | nlm\rangle$ through eq. (14). In particular, note that these expressions only depend on ω and ω_z, since $\omega_x = \omega_y$ follows from cylindrical symmetry and $\omega_x\omega_y\omega_z = \omega^3$ from volume conservation.

Finally, it is easy to see from eq. (25) that the overlaps between deformed and non-deformed Cartesian basis states would be given by

$$\langle \widetilde{n}_x\widetilde{n}_y\widetilde{n}_z | n_x n_y n_z\rangle = \frac{\omega^{3/4}(\omega_x\omega_y\omega_z)^{1/4} I_x \times I_y \times I_z}{\pi^{3/2}\sqrt{2^{\widetilde{n}_x+\widetilde{n}_y+\widetilde{n}_z+n_x+n_y+n_z}\widetilde{n}_x!\widetilde{n}_y!\widetilde{n}_z!n_x!n_y!n_z!}}. \quad (27)$$

In eq. (27), I_x and I_y are given by eq. (25) where the index z is replaced by x and y respectively. Unlike $\langle \widetilde{n}\widetilde{n}_z m | n n_z m\rangle$ and $\langle \widetilde{n}\widetilde{l}m | nlm\rangle$, these overlaps hold even for the $\omega_x \neq \omega_y$ case but don't have m as a good quantum number.

4. Conclusion

In this paper we revisited various features of the three-dimensional harmonic oscillator (3D-HO). While the isotropy of space is commonly invoked which implies equal oscillator lengths in the three (x, y and z) directions, for many applications this is not an optimal choice since deformation often dominates the dynamics, for example, in nuclear physics deformation dominates in nearly all cases therefore it is best to incorporate deformation into the picture from the onset. In the above, we introduced analytic expressions for transformation brackets between deformed and non-deformed basis states of the 3D-HO, under a constant volume constraint, which means $\omega_x\omega_y\omega_z = \omega^3$ where ω is the oscillator strength of the equivalent (non-deformed) isotropic 3D-HO. The above constraint has applications in deformed HO models that study equipotential surfaces [3] and is a direct implication of the incompressibility of nuclear matter [7].

Further, we have also provided analytic results for transformation coefficients between spherical, cylindrical and Cartesian basis states of the 3D-HO. So while our interest in these expressions is driven by our need for them in nuclear physics studies, they can be invoked whenever and wherever the 3D-HO comes into play. Such applications include studies of coupled oscillators [1], the interaction of coherent and squeezed states with different frequencies [2], the construction of a deformed effective field theory [4], as well as for gaining a better understanding of how damped oscillators interact [5].

Declaration of competing interest

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