Admissibility of Semigroup Structures on Continua.

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ABSTRACT

Let X be a continuum, i.e. a compact connected Hausdorff space. A semigroup structure on X is a continuous associative binary operation m on X. The space X is said to admit a semigroup structure if such an operation can be defined on X. In this dissertation we are concerned with various collections of topological properties which continua may have, and consider whether each collection P of properties determines an answer to the question "Does a continuum having the properties P admit the structure of a semigroup with identity?"

In Chapter II we consider retracts X of a two-cell. Using topological and cyclic element characterizations of X it is shown that X is a weakly ruled space (i.e. ruled in the sense of Eberhart) and thus X admits the structure of a semilattice (commutative idempotent semigroup) with identity if X has no free two-cell. We then show that any cyclic chain C(a,b) in X admits the structure of a semilattice with zero and identity. It is also shown that if t is a cut point of C(a,b) then there is a min thread from the zero of C(a,b) to t which is a homomorphic retract of C(a,b).

Assuming X has no free two-cell, an algorithm is
given combining the semilattice structures of a dense collection of cyclic chains so that the union of this collection admits the structure of a semilattice with identity. The operation on this dense subset of X is shown to be uniformly continuous and hence extendable to a semilattice structure with identity on X.

In Chapter III we define a class of continua called strongly ruled continua and prove that any such continuum admits the structure of a semigroup with zero and identity. Examples are given of spaces which are strongly ruled continua.

In Chapter IV we consider semilattices on generalized trees (arcwise connected hereditarily unicoherent continua X with distinguished element p so that \([p, x_\alpha] \rightarrow [p, x]\) whenever \(x_\alpha \rightarrow x\)). Two sufficient conditions are given in order that a generalized tree not admit the structure of a semilattice with identity. Examples are given of generalized trees satisfying each condition and hence not admitting the structure of a semilattice with identity.
CHAPTER I
PRELIMINARY REMARKS

Since we will be working with spaces which will not necessarily be metric, we employ nets. For a complete account, see Chapter 2 of Kelley [8]. Briefly, a set $D$ is directed by a reflexive, transitive relation $\leq$ if and only if for each $\alpha, \beta \in D$ there exists $\gamma \in D$ such that $\alpha, \beta \leq \gamma$. A net in a space $X$ is a function $f$ from a directed set $D$ to $X$. The net $f$ is frequently in a set $U$ if for each $\alpha \in D$ there exists $\beta \in D$ such that $\alpha \leq \beta$ and $f(\beta) \in U$; $f$ is eventually in $U$ if there exists $\alpha \in D$ such that $f(\beta) \in U$ for all $\beta \geq \alpha$. The net $f$ clusters at a point $x$ if and only if $f$ is frequently in each open set $U$ containing $x$; $f$ converges to $x$ if and only if $f$ is eventually in each open set containing $U$. For $\alpha \in D$ we write $f(\alpha) = x_\alpha$ and denote the net $f$ by $x_\alpha$ or sometimes $\{x_\alpha\}_{\alpha \in D}$. By $x_\alpha \to x$ we mean the net $x_\alpha$ converges to $x$. The main facts we use about nets are the following:

A function $g$ from a space $X$ to a space $Y$ is continuous if and only if $x_\alpha \to x$ in $X$ implies $g(x_\alpha) \to g(x)$ in $Y$. A subset $A \subset X$ is closed if and only if $x_\alpha \to x$ and $x_\alpha \in A$ implies $x \in A$. $X$ is compact if and only if each net in $X$ clusters at some point of $X$. 
Let \( 2^X \) denote the space of all closed subsets of \( X \). If \( \{X_\alpha \}_{\alpha \in D} \) is a net in \( 2^X \) then we denote by 
\[
\lim \inf X_\alpha
\]
the set of all points \( x \in X \) such that \( X_\alpha \) eventually intersects each open set \( U \) containing \( x \). We denote by 
\[
\lim \sup X_\alpha
\]
the set of all points \( x \in X \) such that \( X_\alpha \) frequently intersects each open set \( U \) containing \( x \). If \( \lim \sup X_\alpha = L = \lim \inf X_\alpha \), we say \( X_\alpha \) converges to \( L \).

Since \( \lim \inf X_\alpha \subseteq \lim \sup X_\alpha \) for any net \( X_\alpha \), in order to show \( X_\alpha \to L \) it suffices to show \( \lim \inf X_\alpha \subseteq L \subseteq \lim \sup X_\alpha \).

We denote the empty set by \( \Box \). \( A^* \) denotes the closure of \( A \) and \( X \setminus A \) denotes the complement of \( A \) in \( X \). For a subset \( A \) of a set \( X \), \( F(A) \) denotes the boundary of \( A \) and \( A^o = X \setminus (X \setminus A)^* \) denotes the interior of \( A \).

By an algebraic semigroup we mean a set \( S \) together with a binary operation \( m:S \times S \to S \) which is associative (i.e. \( m(x,m(y,z)) = m(m(x,y),z) \)). A semigroup is an algebraic semigroup such that the set \( S \) is a Hausdorff topological space and the function \( m:S \times S \to S \) is continuous. A Hausdorff space \( S \) is said to admit a semigroup structure if an operation can be defined on \( S \) so that \( S \) is a semigroup. Whenever possible this operation will be denoted by juxtaposition; i.e. \( m(x,y) = xy \). For two subsets \( A \) and \( B \) of \( S \), \( AB \) denotes \( \{ab \mid a \in A, b \in B\} \).

A nonvoid subset \( A \) of \( S \) is an ideal of \( S \) provided \( AS \cup SA \subseteq A \). It is easily shown that the intersection of
a collection of ideals of $S$ is again an ideal of $S$, provided it is non-void. The kernel of $S$, $K(S)$, is defined as the intersection of all ideals of $S$, provided this intersection is non-void. If $S$ is compact, then $K(S)$ exists.

A semilattice is a commutative idempotent semigroup; i.e. a semigroup $S$ such that $xy = yx$ and $x^2 = x$ for all $x, y \in S$. The kernel of a semilattice is a point if it exists, and thus compact connected semilattices have a zero.

Let $S$ be a semilattice. The relation $\leq$ defined on $S$ by $x \leq y$ iff $xy = x$ is reflexive, transitive, and antisymmetric. In addition, the graph of $\leq (\{(x,y) | x \leq y\})$ is a closed subset of $S \times S$. In general, if $X$ is a Hausdorff space on which a reflexive, transitive, and antisymmetric relation $\leq$ is defined such that the graph of $\leq$ is closed in $X \times X$ then we say $X$ is a partially ordered space. If in addition the operation $(x, y) \rightarrow (x \wedge y)$, where $x \wedge y$ denotes the greatest lower bound of $x$ and $y$, is continuous, then $X$ is a semilattice under this operation. We set $L(A) = \{ x \in X | x \leq a \text{ for some } a \in A \}$ and $M(A) = \{ x \in X | a \leq x \text{ for some } a \in A \}$. A subset $A$ of a partially ordered space $X$ is said to be convex if $x < y < z$ and $x, z \in A$ implies $y \in A$.

A continuum is a compact connected Hausdorff space. An arc is a continuum with exactly two non-cutpoints. Let $A$ be an arc with non-cutpoints $p$ and $q$. Define a
Multiplication on $A$ by $xy = x$ if $x \in [p,y]$ and $xy = y$ otherwise. Here $[p,y]$ denotes the subarc of $A$ with end points $p$ and $y$ if $p \neq y$ and $[p,y] = p$ if $p = y$. With this multiplication, $A$ is a compact connected semilattice with zero $p$ and identity $q$. The partial order induced by this operation is a total order (i.e. every pair of elements compare) and $xy = \min\{x,y\}$ with respect to this order. $A$ is then called a min thread. We reserve the symbol $I$ for the interval $[0,1]$ of real numbers under this min operation. In any compact, connected semilattice $A$, if $a \in A$ then there exists a min thread in $A$ from 0 to $a$.

In Chapter II we shall need the concept of, and results concerning, cyclic elements. For a complete account see Chapter IV of Whyburn [22]. Briefly, a simple link $E$ of a locally connected continuum $X$ is a non-degenerate connected subset of $X$ which has no cutpoint and is maximal with respect to the property of being a connected subset of $X$ having no cutpoint. If $p \in E$, then $E$ consists of $p$ together with all points which cannot be separated from $p$ in $X$. By a cyclic element of $X$ we mean a simple link, a cutpoint, or an end point (i.e. a point having a basis of neighborhoods with a one point boundary) of $X$. The cutpoints and end points of $X$ will be called degenerate cyclic elements and the simple links true cyclic elements. We list now some facts concerning cyclic elements and related topics.
Each point of \( X \) is contained in some cyclic element, and a non-cutpoint can belong to only one. If \( E \) is a true cyclic element of \( X \) and \( R \) is a component of \( X\setminus E \), then there exists \( p \in E \), which is also a cutpoint of \( X \), such that \( R \) is a component of \( M\setminus\{p\} \).

A subcontinuum \( A \) of a locally connected continuum \( X \) is called an A-set if it possesses any one of the following four equivalent properties:

a) If \( A \) contains more than one point of a cyclic element \( E \) of \( X \), then \( A \supset E \).

b) For any \( p \in A \) we have \( p = A \) or \( p \) belongs to some cyclic element \( E \subset A \).

c) The boundary in \( X \) of any component of \( X\setminus A \) is a single point.

d) For any arc \( [a,b] \) in \( X \), \( a, b \in A \) implies \( [a,b] \subset A \).

It follows that any cyclic element of \( X \) is an A-set and the intersection of a collection of A-sets is an A-set. Also any A-set \( A \) is locally connected and any cyclic element of \( A \) is a cyclic element of \( X \).

For any two distinct points \( a \) and \( b \) of a locally connected continuum \( X \), the intersection of all A-sets in \( X \) containing \( a \) and \( b \) is called the cyclic chain from \( a \) to \( b \) and is denoted by \( \mathcal{C}(a,b) \). Using the properties of A-sets,
\( \mathcal{C}(a,b) \) is an A-set. It should be noted that an equivalent characterization of \( \mathcal{C}(a,b) \) is the set \( L \), which consists of \( a \) and \( b \) together with all points of \( X \) which separate \( a \) and \( b \), and all cyclic elements \( E \) such that \( E \cap L \) consists of exactly two points.

A subset \( K \) of a locally connected continuum \( X \) is said to be **nodal** if \( F(K) \) is a point. The **nodes** of a locally connected continuum \( X \) are all cyclic elements (true or not) which are also nodal sets. It may be shown that a locally connected continuum which has a cutpoint has at least two nodes.
CHAPTER II
SEMLATTICES ON RETRACTS OF A TWO-CELL

In [15] and [16], Koch raised the question, "Does every absolute retract $X$ admit the structure of a semigroup with identity". The main theorem of this chapter gives an affirmative answer in the case $X$ is contained the plane, i.e. when $X$ is a retract of a two-cell.

2.1 Definition A subspace $A$ of a topological space $X$ is called a retract of $X$ if there exists a continuous map $r:X \rightarrow A$ such that $r(a) = a$ for every $a \in A$.

The following is a completely topological characterization of retracts of a two-cell.

2.2 Theorem (Borsuk [1]) A subset $A$ of a two-cell $X$ is a retract of $X$ if and only if $A$ is a locally connected continuum which does not separate the plane.

The cyclic element structure of a retract of a two-cell has been characterized by Whyburn as follows:

2.3 Theorem (Whyburn [22]) A locally connected continuum does not separate the plane if and only if every true cyclic element is a simple closed curve together with its interior, i.e. a two-cell.
The proof that a retract $X$ of a two-cell admits the structure of a semilattice with identity will be done in two cases. The first section will consider the case when $X$ has a free two-cell (i.e. a subset $A$ which is topologically a two-cell and which is such that $F(A)$ is a point), and the second section will consider the case when $X$ does not.

Section I

2.4 Definition Suppose $X$ is a topological space and $E \subseteq X$, $0 \in X$. Let $\mathcal{A} = \{[0,e] : e \in E\}$ be a collection of arcs in $X$ satisfying

i) $X = \bigcup_{e \in E} [0,e]$

ii) $[0,e] \cap [0,f]$ is a proper subarc of each.

iii) For each $e \in I$, there is a unique arc $[0,e] \in \mathcal{A}$

iv) If $x_\alpha \to x$ then $[0,x_\alpha] \to [0,x]$ in the sense of

$\lim \sup \lim \inf$ convergence (Chapter I).

Then $\mathcal{A}$ is said to be a weak ruling of $X$.

Notation: $C(x,A)$ denotes the component of the point $x$ in the set $A$.

2.5 Theorem If $X$ is a retract of a two-cell then $X$ admits a weak ruling.

Proof By Theorems 2.2 and 2.3, $X$ must be a locally connected
continuum which does not separate the plane, and every true cyclic element of $X$ is a two-cell. Let $\{D_1, D_2, \ldots\}$ be the collection of true cyclic elements of $X$. Let $0 \in F(X) = X \cap (\pi \setminus X)^*$. Let $E = F(X) \setminus \{\text{cutpoints of } X\} \cup \{0\}$.

For each $D_i$ let $h_i$ be a homeomorphism of $I \times I$ ($I$ denotes the interval $[0,1]$ of real numbers) onto $D_i$ such that if $0 \in D_i$, then $h_i(0,0) = 0$ and if $0 \notin D_i$, then $h_i(0,0) = z_i$ where $z_i$ is that point of $D_i$ such that $C(0, X \setminus C_i) = C(0, X \setminus z_i)$. For each $e \in E$ construct $[0, e]$ as follows. Consider the cyclic chain $C(0, e)$. Let $\{D_k\}$ be the collection of true cyclic elements in $C(0, e)$. For each $D_k$ let $e_k$ be the boundary in $X$ of $C(e, X \setminus D_k)$ if this is non-empty, and $e$ otherwise. Let $A_k$ be the image under $h_k$ of the straight line in $I \times I$ from $h_k^{-1}(z_k)$ to $h_k^{-1}(e_k)$. Let $Q_1 = C(0, e) \setminus (D_1 \setminus A_1)$ and inductively $Q_{j+1} = Q_j \setminus (D_{j+1} \setminus A_{j+1})$. Then $[0, e] = \bigcap_k Q_k$. Since each $Q_k$ is a continuum and $Q_{j+1} \subset Q_j$, $[0, e]$ is a continuum.

We shall show that $0$ and $e$ are the only non-cutpoints of $[0, e]$. If $a \in [0, e] \setminus \{0, e\}$ is a degenerate cyclic element of $C(0, e)$, then $a$ is a cutpoint of $X$ separating $0$ and $e$ and hence cuts $[0, e]$. A similar argument handles the case when $a \in F(D_i)$ for some true cyclic element $D_i$ of $C(0, e)$. If $a \in D_i^0$ then $a$ cuts $D_i \cap [0, e]$ and hence cuts $[0, e]$. Thus any point of $[0, e] \setminus \{0, e\}$ cuts $[0, e]$. We show that $0$ does not cut $[0, e]$. First note that $0$ does not cut $C(0, e)$. If $0$
is an end point of \( C(0,e) \) then \( 0 \) cuts no subcontinuum of \( C(0,e) \) and hence does not cut \([0,e]\). If \( 0 \) is not an end point of \( C(0,e) \), then \( 0 \in D \) for some true cyclic element \( D \) of \( C(0,e) \). If \( 0 \) also cuts \([0,e]\), then \( 0 \) cuts \([0,e] \cap D \). But \([0,e] \cap D \) is an arc with end point \( 0 \). This is a contradiction, and so \( 0 \) does not cut \([0,e]\). A similar argument shows \( e \) does not cut \([0,e]\).

To show that \( \mathcal{A} = \{[0,e] : e \in E\} \) is a weak ruling of \( X \), we establish four claims.

**Claim 1** \( X = \bigcup_{e \in E} [0,e] \).

**Proof:** Let \( x \in X \). We show \( x \in \bigcup_{e \in E} [0,e] \). If \( x \in \bigcup_{e \in E} D_e \), then there exists \( b \in F(I \times I) \) so that \( h^{-1}_1(x) \) lies on the straight line from \((0,0)\) to \( b \), and hence \( x \) lies on the image of this straight line under \( h_1^{-1} \). Now \( h_1(b) \in F(X) \), so if \( h_1(b) \) is not a cutpoint then \( x \in [0,h_1(b)] \in \mathcal{A} \). If \( h_1(b) \) is a cutpoint then let \( K \) be a component of \( X \setminus \{h_1(b)\} \) not containing \( 0 \). \( F(K) = h_1(b) \), so \( K \) is a nodal set in \( X \) and hence contains at least one node of \( X \). This node must either be an element \( e \) of \( E \) or a two cell which in turn contains an element \( e \) of \( E \). Then \( h_1(b) \in [0,e] \in \mathcal{A} \) and hence \( x \in [0,e] \in \mathcal{A} \). Thus \( X \subset \bigcup_{e \in E} [0,e] \) so \( X = \bigcup_{e \in E} [0,e] \).

**Claim 2** For \( e,f \in E \), \( [0,e] \cap [0,f] \) is a proper subarc of
each.

Proof: By the definition of \([0,e]\) and \([0,f]\), if \(x \in [0,e] \cap [0,f]\) then any \(y\) which precedes \(x\) in either \([0,e]\) or \([0,f]\) is contained in \([0,e] \cap [0,f]\). Thus \([0,e] \cap [0,f]\) is a subarc of each.

Claim 3 For each \(e \in I\), there is a unique arc \([0,e] \in \mathcal{A}\).

Proof: Obvious from the definition of \([0,e]\).

Claim 4 If \(x_n \rightarrow x\) then \([0,x_n] \rightarrow [0,x]\).

Proof: Let \(x_n \rightarrow x\). We distinguish three cases.

Case I \(x \in X^0\). Then \(x \in D^{i^0}\) for some true cyclic element \(D_i\) and hence we may assume \(x_n \in D^{i^0}\) for all \(n\).

Then \([0,x] \cap (X \setminus D_i) = [0,x_n] \cap (X \setminus D_i) = [0,2_i]\) for all \(n\).

Let \([(0,0), h_i^{-1}(x_n)]\) denote the straight line in \(I \times I\) from \((0,0)\) to \(h_i^{-1}(x_n)\). Now \(h_i^{-1}(x_n) \rightarrow h_i^{-1}(x)\) so

\([(0,0), h_i^{-1}(x_n)] \rightarrow [(0,0), h_i^{-1}(x)]\), and

\(h_i[(0,0), h_i^{-1}(x_n)] = [0,x_n] \cap D_i \rightarrow h_i[(0,0), h_i^{-1}(x)] = [0,x] \cap D_i\)

since \(h_i\) is a homeomorphism. Thus \([0,x_n] \rightarrow [0,x]\).

Case II \(x \in F(X)\) and \(x\) is not an end point of \(C(0,x)\).

Then \(x \in D\) for some cyclic element \(D\) of \(C(0,x)\). We shall first show that \([0,x_n] \cap D\) is eventually non-empty.

If \(D = C(0,x)\) this is obviously true since \(0 \in D \cap [0,x_n]\).

Otherwise let \(d = D \cap (C(0,x) \setminus D)^*\). Let \(U\) be an open
connected set containing \( x \) so that \( d \notin U \). We may assume \( x_n \in U \) for all \( n \). Let \( A_n \) be an arc from \( x \) to \( x_n \) in \( U \). Then \( A_n \cup [0,x] \) contains an arc \( A \) from 0 to \( x_n \).

Since \( \mathcal{A}(0,x_n) \) is an A-set, \( A \subset \mathcal{A}(0,x_n) \). But \( d \in [0,x] \) so \( d \in A \) since \( d \notin A_n \subset U \). Thus \( d \) is a cutpoint of \( \mathcal{A}(0,x_n) \) and hence in \([0,x_n]\). So we may assume \([0,x_n] \cap D \neq \emptyset \) for all \( n \).

We now show that \( y_n = \sup([0,x_n] \cap D) \rightarrow x \), where \( \sup \) is in the order from 0 to \( x_n \) in \([0,x_n]\). It suffices to show that for any open connected set \( U \) containing \( x \), \( x_n \in U \) implies \( y_n \in U \). To this end let \( x_n \) be an element of an open connected set \( U \) containing \( x \). If \( x_n \in D \) then \( x_n = y_n \) and the implication is true. If \( x_n \notin D \), then \( y_n = F(\mathcal{C}(x_n, X \setminus D)) \) and hence \( y_n \) is an element of any arc from \( x \) to \( x_n \). But \( U \) is arcwise connected, so \( y_n \in U \).

Since \( \sup([0,x_n] \cap D) \rightarrow \sup([0,x] \cap D) \), an argument similar to Case I shows that \([0,x_n] \cap D \rightarrow [0,x] \cap D \). However, \([0,x_n] \cap (\mathcal{C}(0,x) \setminus D) = [0,x] \cap (\mathcal{C}(0,x) \setminus D) \), so \([0,x_n] \cap \mathcal{C}(0,x) \rightarrow [0,x] \cap \mathcal{C}(0,x) = [0,x] \). We thus have \([0,x] \subset \lim \inf [0,x_n] \).

We now show \( \lim \sup [0,x_n] \subset [0,x] \). Let \( y \in \lim \sup [0,x_n] \) and let \( x_n' \) be a subsequence of \( x_n \) such that \( y \in \lim \inf [0,x_n'] \). If \( y \in \mathcal{C}(0,x) \), then the above argument shows \( y \in [0,x] \). Suppose \( y \notin \mathcal{C}(0,x) \). Let \( K = \mathcal{C}(y,X \setminus \mathcal{C}(0,x)) \). Since \( K \) is open, we may assume \( x_n' \in K \)
for all $n$. Thus $\sup([0,x_n'] \cap C(0,x)) = F(K)$ for all $n$. However $\sup([0,x_n'] \cap C(0,x)) \to x$, so $x = F(K)$ and hence $x \in [0,y]$.

We shall show that $x$ and $y$ belong to the same cyclic element of $X$. It suffices to show that $x$ and $y$ cannot be separated in $X$. To this end suppose $X \setminus \{p\} = A \cup B$ where $A$ and $B$ are open disjoint sets such that $x \in A$, $y \in B$. We may assume that $x_n' \in A$ and $[0,x_n'] \cap B = \emptyset$ for all $n$. Let $q_n \in [0,x_n'] \cap B$, let $[x,q_n]$ be the subarc of $[0,x_n']$ from $x$ to $q_n$, and let $[q_n,x_n']$ be the subarc of $[0,x_n']$ from $q_n$ to $x_n'$. Since $x \in A$ and $q_n \in B$, $p \in [x,q_n]$; since $q_n \in B$ and $x_n \in A$, $p \in [q_n,x_n']$. This means $[0,x_n']$ is not an arc, a contradiction. Thus $x$ and $y$ cannot be separated in $X$ and must hence be contained in a cyclic element $D_1$.

If $x_n' \in D_1$, then since $D_1$ is homeomorphic to $I \times I$, $[0,x_n'] \cap D_1 \to [0,x] \cap D_1 = x$, and hence $[0,x_n'] \to [0,x]$. If $x_n' \not\in D_1$ let $z_n = F(C(x_n',x \setminus D_1))$. Then for each open connected set $U$ containing $x$, $x_n' \in U$ implies $x_n \in U$. Also for each open connected set $V$ containing $y$, $[0,x_n'] \cap V \neq \emptyset$ implies $z_n \in V$. Thus $z_n \to x$ and $z_n \to y$, or $x = y$. This completes the proof of Case II.

**Case III** $x \in F(X)$ and $x$ is an end point of $C(0,x)$. Then there exists a sequence $\{t_j\}$ of cutpoints of $C(0,x)$,
and hence of \([0,x]\), such that \(\{t_j\} \to x\).

We shall show that \([0,x_n] \cap C(0,x) \to [0,x]\) by showing that \([0,x_n] \cap [0,x]\) eventually intersects any open set containing \(x\). To this end let \(U\) be an open set containing \(x\) and \(t_{j_0} \in U\). Let \(W\) be an open connected set so that \(t_{j_0} \notin W\). We may assume \(x_n \in W\) for all \(n\). If \(x_n \in C(0,x)\), then \(t_{j_0} \in [0,x_n]\) and hence \(t_{j_0} \in [0,x_n] \cap [0,x] \cap U\). If \(x_n \notin C(0,x)\), let \(y_n = \sup ([0,x_n] \cap C(0,x)) = F(C(x_n, X \setminus C(0,x))).\) Then \(y_n\) is an element of any arc from \(x\) to \(x_n\). Since \(U\) is arc wise connected, \(y_n \in U\). But then \(t_{j_0} \in [0,y_n] \subseteq [0,x_n]\), so \(t_{j_0} \in [0,x_n] \cap W\), and the statement is proven.

Since \([0,x_n] \cap C(0,x) \to [0,x]\), we have that \([0,x] \subseteq \lim \inf [0,x_n]\). An argument similar to Case II shows that \(\lim \sup [0,x_n] \subseteq [0,x]\). This completes the proof of Claim 4 and the proof of the theorem.

2.6 Corollary Let \(X\) be a retract of a two-cell such that \(X\) has a free two-cell. Then \(X\) admits the structure of a semilattice with identity.

Proof Let \(A\) be a free two-cell in \(X\) and let \(p = F(A)\). Then \((X \setminus A)^*\) is an \(A\)-set and hence a locally connected continuum every true cyclic element of which is a two-cell. So \((X \setminus A)^*\) is a retract of a two-cell. Since \(p \in F(A)\), by
Theorem 2.5 \((X \setminus A)^*\) admits a weak ruling \(a_1\) with \(p = 0\).
Since \(A\) is a two-cell we may consider it to be \(I \times I\) with \(p = (0,0)\). Then \(A\) may be weakly ruled by \(a_2\)

\[a_2 = \{\text{straight lines from } (0,0) \text{ to } (x,y): x \text{ or } y = 1\}.\]

Set \(a = a_1 \cup a_2\) and \(a\) is a weak ruling of \(X\) with \(p = 0\).
Moreover \(X\) may be embedded in a two-cell \(N\) so that

\[X \cap \{N \cap (\pi \setminus N)^*\} = [(0,0),(1,0)] \in a.\]

By a theorem of Eberhart [4], this implies that \(X\) admits the structure of a semilattice with identity.

Section II

We now consider the case when \(X\) has no free two-cell. Until otherwise stated, \(X\) is assumed to be a retract of a two-cell which has no free two-cell and which is not a point.

2.7 Lemma \(X\) contains at least two end points.

Proof First note that \(X\) has more than one cyclic element. It therefore contains a cutpoint and so \(X\) has at least two nodes, which in this case must be end points.

Let \(0\) and \(1\) be two of these end points and let \(K_0 = C(0,1)\).

2.8 Lemma There exist two distinct sequences \(\{p_i\}\) and \(\{q_i\}\) in \(X\) such that
1) \( p_1 = 0, q_1 = 1 \)

ii) Each \( p_1 \) and \( q_1 \) is an element of \( F(X) \) and such that the cyclic chains \( c(p_1, q_1) \) have the following properties

iii) For each \( i, p_1 = c(p_1, q_1) \cap \bigcup_{j=1}^{i-1} c(p_j, q_j) \)

iv) Setting \( H_n = \bigcup_{i=1}^{n} c(p_i, q_i) \) and \( H = \bigcup_{n=1}^{\infty} H_n \), every point of \( X \setminus H \) is an end point of \( X \) and thus \( H = X \).

v) The diameter of the components of \( S \setminus H_n \) tends to 0 uniformly with \( \frac{1}{n} \).

**Proof**  This lemma is proven in Whyburn ([1], p.73) without conclusions i) and ii). We shall show that these are also true. Whyburn's proof considers a sequence \( \{r_1\}_{i=1}^{\infty} \) such that \( (r_1^{\infty})^* = X \) and sets \( p_1 = r_1, q_1 = r_2 \). Clearly \( \{r_1\}_{i=1}^{\infty} \) may be chosen so that \( r_1 = 0 \) and \( r_1 = 1 \). Thus i) is satisfied. In each \( c(p_1, q_1) \), if \( q_1 \) is not in \( F(X) \), then \( q_1 \in D_j \) for some true cyclic element \( D_j \).

Since \( X \) has no free two-cells, \( F(D_j) \) consists of more than one point. One of these points is the boundary of \( C(p, X \setminus D_j) \). Let \( q_1' \) be any other point in \( F(D_j) \). Then \( c(p_1, q_1') = c(p_1, q_1) \). Hence we may assume each \( q_1 \in F(X) \).

In Whyburn's proof, \( p_1 \) was chosen to be in \( F(X) \). So ii) is satisfied.
2.9 Lemma  Let p and q be distinct elements of F(X).
Then \( C(p,q) \) admits the structure of a semilattice with zero p and identity q.

Proof  For each true cyclic element \( D_1 \) of \( C(p,q) \) let \( h_1 \) be a homeomorphism of \( IxI \) onto \( D_1 \) so that

\[
h_1(0,0) = \begin{cases} 
D_1 \cap \{ C(p,X\backslash D_1) \}^* & \text{if } p \notin D_1 \\
p & \text{if } p \in D_1 
\end{cases} \]

\[
h_1(1,1) = \begin{cases} 
D_1 \cap \{ C(q,X\backslash D_1) \}^* & \text{if } q \notin D_1 \\
q & \text{if } q \notin D_1 
\end{cases} \]

For each \( x \in C(p,q) \), let \( L(x) \) be the union of all cyclic elements \( D \) (true or degenerate) such that \( x \in C(1,X\backslash D) \), together with \( h_1(L(h_1^{-1}(x))) \) for each \( D_1 \) that contains \( x \), 
and \( x \). Then for \( x,y \in X \), if \( x \) and \( y \in D \) for some cyclic element \( D \) then \( x \land y \in D \). If some cyclic element \( D_x \) containing \( x \) separates \( y \) from 0, then \( x \land y = x \). It is clear that \( p \) is the zero and \( q \) the identity.

The preceding two lemmas have shown that there is a dense collection of cyclic chains in \( X \) each of which admits the structure of a semilattice with identity. The following two lemmas give the tools for putting these structures together.
2.10 Lemma Suppose $p_1, p_2, q_1,$ and $q_2$ are elements of $F(X)$ and suppose $C(p_1, q_1)$ and $C(p_2, q_2)$ are endowed with a semilattice structure by Lemma 2.9 so that $p_1$ is the zero and $q_1$ the identity of $C(p_1, q_1)$, and $p_2$ is the zero and $q_2$ the identity of $C(p_2, q_2)$. Let $t$ be a cutpoint of $C(p_1, q_1)$. Then there exists a min thread $T_2$ from $p_2$ to $q_2$ in $C(p_2, q_2)$ and a continuous onto homomorphism $h : C(p_1, q_1) \to T_2$ such that $h^{-1}(q_2) = [C(q_1, C(p_1, q_1) \setminus \{t\})]^*$. 

Proof A short proof of this lemma exists using the following unpublished result of Professor Jimmie Lawson: Let $X$ be a compact connected locally connected metric finite-dimensional semilattice. Then for $t \in X$ there exists a min thread $T$ from 0 to $t$ and a function $f : X \to T$ which is a homomorphism and a retraction.

If we let $T_1$ be such a min thread in $C(p_1, q_1)$ from $p_1$ to $t$ and $h$ the homomorphic retraction from $C(p_1, q_1)$ onto $T_1$, then $h$ followed by a homeomorphism between $T_1$ and $T_2$ yields the desired result.

For the sake of completeness, we include here a proof of Lawson's result in the case when $X$ is a cyclic chain $C(p, q)$ endowed with a semilattice structure by Lemma 2.9 so that $q$ is the identity and $p$ the zero, and $t$ is a cutpoint of $C(p, q)$.

For each true cyclic element $D_1$ of $C(p, t)$ let $h_1$ be
the homeomorphism of $I \times I$ onto $D_1$ defined in Lemma 2.9. Let $A_1$ be the image under $h_1$ of $\{(x,x) | x \in I\}$. Let $Q_1 = \mathcal{C}(p,t) \setminus (D_1 \setminus A_1)$ and inductively $Q_i = Q_{i-1} \setminus (D_1 \setminus A_i)$, and let $A = \cap Q_i$. By an argument similar to that used in showing that $[0,e]$ was an arc in Theorem 2.5, $A$ is an arc. Since the $h_i$'s used to define $A$ are the same as those used in Lemma 2.9 to define multiplication, $A$ is a min thread.

Define a homomorphism $h : \mathcal{C}(p,q) \to A$ by $h(x) = t$ if $x \in [\mathcal{C}(l,\mathcal{C}(p,q) \setminus \{t\})]^*$ and $h(x) = \sup \{a \in A | x \in M(a)\}$ if $x \in C(0,\mathcal{C}(p,q) \setminus \{t\})$. To show this is a homomorphism, let $x, y \in \mathcal{C}(p,q)$. If $x, y \in [\mathcal{C}(l,\mathcal{C}(p,q) \setminus \{t\})]^*$ then $xy \in [\mathcal{C}(l,\mathcal{C}(p,q) \setminus \{t\})]^*$ and so in this case $h(x)h(y) = tt = t = h(xy)$. If $x \in [\mathcal{C}(l,\mathcal{C}(p,q) \setminus \{t\})]^*$ and $y \in C(0,\mathcal{C}(p,q) \setminus \{t\})$ then $h(x) = t$, $h(y) \in C(0,\mathcal{C}(p,q) \setminus \{t\})$, and $xy = y$. Hence $h(x)h(y) = th(y) = h(y) = h(xy)$. A similar argument handles the case when the roles of $x$ and $y$ are interchanged. If $x, y \in C(0,\mathcal{C}(p,q) \setminus \{t\})$ and $x, y \in D_1$ for some $D_1$ in $\mathcal{C}(p,q)$, then $h(x)h(y) = h(xy)$ since $D_1$ is isomorphic to $I \times I$. If $x, y \notin D_1$ for any $D_1$ then there exists $t_o$ a cutpoint $t_o$ of $\mathcal{C}(p,q)$ so that $t_o$ separates $x$ and $y$. Using arguments similar to those above for $x$ and $y$ in various components of $\mathcal{C}(p,q) \setminus \{t_o\}$, we have $h(x)h(y) = h(x,y)$. Thus $h$ is a homomorphism.

To show that $h$ is continuous we let $x_n \to x$. If
$x \in C(1, C(p, q) \setminus \{t\})$ then eventually $h(x_n) = t = h(x)$. If $x \in C(0, C(p, q) \setminus \{t\})$ we consider two cases. First suppose $x$ is not a cutpoint of $C(p, q)$. Then $x \in D_1$ for some cyclic element $D_1$ of $C(p, q)$ and $x_n$ is eventually in $D_1$. Since $D_1$ is isomorphic to $I_x I$, $h(x_n) \to h(x)$. Hence we may assume that $x$ is a cutpoint of $C(p, q)$. Let $\{x'_n\} = \{x_n : x_n \in C(1, C(p, q) \setminus \{x\})\}$ and $\{x''_n\} = \{x_n : x_n \in C(0, C(p, q) \setminus \{x\})\}$. If $x$ is an endpoint of $C(0, C(p, q) \setminus \{x\})$ then using the fact that for any $y \in C(0, C(p, q) \setminus \{t\}) y$ and $h(y)$ lie in the same cyclic element, we see that $h(x''_n) \to h(x)$. If $x$ is not an endpoint of $C(0, C(p, q) \setminus \{x\})$ then $x \in D_1$ for some $D_1$ in $[C(0, C(p, q) \setminus \{x\})]^*$. Again since $D_1$ is isomorphic to $I_x I$, $h(x'_n) \to h(x)$. A similar argument shows $h(x''_n) \to h(x)$. Thus $h$ is continuous and the proof of the lemma is complete.

**Notation** If $x, y \in X$, then by $X/x \sim y$ we mean the quotient space of $X$ in which $x$ and $y$ are identified.

**2.11 Lemma** Suppose $S$ and $Q$ are abelian semigroups such that $0$ is the zero and $1$ the identity of $S$, $e$ the zero and $f$ the identity of $Q$, satisfying

i) $S = RU P$ with $rd = r$ for $r \in R$, $d \in P$.

ii) $R \cap P = \{y\}$.

iii) There exists $a = a^2 \in R$ such that $aP = a$.

iv) There exists a thread $[e, f]$ in $Q$ and a continuous
onto homomorphism \( h: P \rightarrow [e, f] \).

Then \( B = SUQ/e \) admits the structure of a commutative semigroup with zero \( 0 \) and identity \( 1 \) which agrees with each of the original structures on \( S \) and \( Q \) and such that any \( d \in H^{-1}(f) \) acts as an identity for \( RUQ \).

**Proof** Define \(*\) on \( SUQ \) by

\[
y*x = x*y = \begin{cases} 
x y & \text{if } x, y \in S \text{ or } x, y \in Q \\
x a & \text{if } x \in R, y \in Q \\
h(x)y & \text{if } x \in P, y \in Q \\
\end{cases}
\]

Here juxtaposition means product in \( S \) if both elements are in \( S \) or product in \( Q \) if both elements are in \( Q \). Note that this multiplication is commutative.

**Associativity:** Let \( r \in R, p \in P, q \in Q \).

1) Show that the product of \( r, p, \) and \( q \) in any order and grouping is \( ra \).

\[
(r*p)*q = q*(r*p) = (p*r)*q = q*(p*r) = q*(pr) = a(pr) = ar \text{ since } pr = r.
\]

\[
(r*q)*p = (q*r)*p = p*(q*r) = p*(r*q) = p*(ra) = pra = ra \text{ since } pr = r.
\]

\[
(p*q)*r = (q*p)*r = r*(q*p) = r*(p*q) = r*(h(p)q) = ra \text{ since } h(p)q \in Q.
\]

ii) Show that if \( r_1, r_2 \in R \) and \( q \in Q \) then the product of \( r_1, r_2, \) and \( q \) in any order is \( ar_1r_2 \).

\[
(r_1*r_2)*q = q*(r_1*r_2) = q*(r_1r_2) = q*(r_1r_2) = ar_1r_2.
\]

\[
(r_1*q)*r_2 = (q*r_1)*r_2 = r_2*(q*r_1) = r_2*(r_1*q) = r_2*(r_1a) = r_2r_1a = ar_1r_2.
\]
iii) Show the product of $p_1, p_2$ and $q$ in any order and grouping is $h(p_1)h(p_2)q$.

$$(p_1*p_2)*q = q*(p_1*p_2) = qh(p_1)h(p_2).$$

$$(p_1*q)*p_2 = (q*p_1)*p_2 = p_2*(q*p_1) = p_2*(p_1*q) = p_2*(h(p_1)q) = h(p_2)h(p_1)q.$$ 

iv) Show that the product of $r, q_1,$ and $q_2$ in any order and grouping is $ac$. $r*(q_1*q_2) = (q_1*q_2)*r = ar$ since $q_1q_2 \in Q$.

$$(q_1*r)*q_2 = (r*q_1)*q_2 = q_2*(r*q_1) = q_2*(q_1*r) = q_2*(ar) = a(ar) = ar.$$ 

v) Show that the product of $p, q_1,$ and $q_2$ in any order and grouping is $q_1q_2h(p)$

$$p*(q_1*q_2) = (q_1*q_2)*p = q_1q_2h(p) = (q_1*p)*q_2 = q_2*(q_1*p) = q_2*(p*q_1) = (p*q_1)*q_2 = (h(p)q_1)*q_2 = h(p)q_1q_2.$$ 

Thus $SUQ$ is a commutative semigroup. Define an equivalence relation on $SUQ$ by $a \sim e$. This is a closed congruence since $ax = ex$ for all $x \in SUQ$. Hence $B = SUQ/e \sim$, the quotient space of this congruence, admits the structure of a commutative semigroup with zero $0$ and identity $1$.

If $x \in h^{-1}(f)$ then for any $r \in R$, $xr = r$ since $x \in P$, and for any $q \in Q$, $xq = h(x)q = fq = q$. So any element of $h^{-1}(f)$ acts as an identity for $RUQ$. This completes the proof of the lemma.

Recall from Lemma 2.8 that $H = \bigcup_{i=1}^{\infty} c(p_i, q_i)$.

We now give
2.12 Algorithm for defining multiplication on $\mathbb{H}$.

Since 1 is an end point, there exists $(t_n)_{n=1}^\infty$ a sequence of cutpoints of $\mathcal{C}(0,1)$ such that $t_{n+1}$ separates $t_n$ from 1 for each $n$, and $\{t_n\} \to 1$. Recall that $K_0 = \mathcal{C}(0,1) = \mathcal{C}(p_1,q_1)$ in the collection of cyclic chains constructed in Lemma 2.8. Let $A_1 = \mathcal{C}(p_2,q_2)$. Then $A_1 \cap K_0 = p_2$. Let $n_1$ be the smallest integer such that $t_{n_1}$ separates $p_2$ from 1. Make the following definitions:

i) $Q = A_1 = \mathcal{C}(p_2,q_2)$ with the semilattice structure of Lemma 2.9 so that $p_2$ is the zero, $q_2$ the identity.

ii) $P = [\mathcal{C}(1,K_0 \setminus \{t_{n_1}\})]^*$, $R = [\mathcal{C}(0,K_0 \setminus \{t_{n_1}\})]^*$, and $R \cap D = y = t_{n_1}$.

iii) $a = p_2$.

iv) $h:P \to T_1$ is the continuous onto homomorphism of Lemma 2.10 where $T_1$ is a thread from $p_2$ to $q_2$ and $t = t_{n_2}$ when $n_2$ is the smallest integer $> n_1$ such that there exists a $\mathcal{C}(p_1,q_1)$ so that $\mathcal{C}(p_1,q_1) \cap [K_0 \cup A_1] = p_1 \in \mathcal{C}(0,(K_0 \cup A) \setminus \{t_{n_2}\})$. If no such $n_2$ exists, then $H = K_0 \cup A$ and so let $n_2 = n_1 + 1$.

Then the conditions of Lemma 2.11 are satisfied and hence $K_1 = K_0 \cup A_1$ admits the structure of a semilattice with
zero 0 and identity 1. Note by Lemmas 2.9 and 2.11 any element of $[C(1,K_1 \setminus \{t_{n_2}\})]^*$ acts as an identity for any element of $[C(0,K_1 \setminus \{t_{n_2}\})]^*$.

Suppose that a semilattice structure with zero 0 and identity 1 has been defined on $K_1$ so that any element $[C(1,K_1 \setminus \{t_{n_{i+1}}\})]^*$ acts as an identity for any element of $[C(0,K_1 \setminus \{t_{n_{i+1}}\})]^*$.

Let $A_{i+1}$ be that $C(p_j,q_j)$ of least index such that $C(p_j,q_j) \cap K_1 = p_j \in C(0,K_1 \setminus \{t_{n_{i+1}}\})$. (If no such $C(p_j,q_j)$ exists then $K_1 = H$ since at the $i$ stage $t_{n_{i+1}}$ was chosen so that such a $C(p_j,q_j)$ would exist were the collection of Lemma 2.9 not exhausted, and so we would be done). Let $n_{i+2}$ be the smallest subscript larger than $n_{i+1}$ so that there exists a $C(p_k,q_k)$ such that $C(p_k,q_k) \cap (K_1 \cup A_{i+1}) = p_k \in C(0,K_1 \cup A_{i+1} \setminus \{t_{n_{i+2}}\})$. If no such $n_{i+2}$ exists then $H = K_1 \cup A_{i+1}$ so let $n_{i+2} = n_{i+1} + 1$. Satisfy the conditions of Lemma 2.11 as follows:

1) $R = [C(0,K_1 \setminus \{t_{n_{i+1}}\})]^*, P = [C(1,K_1 \setminus \{t_{n_{i+1}}\})]^*$, and $y = t_{i+1} = R \cap P$.

ii) $Q = A_{i+1}$ with the semilattice structure of Lemma 2.8 so that the zero is $p_j$ and the identity is $q_j$.

iii) $a = p_j$. 
iv) Let \( h: P \to T_j \) be the continuous onto homomorphism of Lemma 2.10 where \( T_j \) is a thread from \( p_j \) to \( q_j \) in \( C(p_j, q_j) \) and \( t_{n+2} \) is the \( t \) of Lemma 2.10.

Set \( K_{i+1} = K_i \cup A_{i+1} \). Then by Lemma 2.11 \( K_{i+1} \) admits the structure of a semilattice with zero 0 and identity 1 so that any element of \( [C(1, K_{i+1} \setminus \{t_{n+2}\})]^* \) acts as an identity for \( [C(0, K_{i+1} \setminus \{t_{n+2}\})]^* \).

By induction we have that each \( K_n \) and also \( \bigcup_{n=1}^{\infty} K_n \) admits the structure of a semilattice with zero 0 and identity 1. We now show that \( H = \bigcup_{i=1}^{\infty} K_i \).

For any \( C(p_j, q_j) \) in \( H \), \( C(p_j, q_j) \) can be joined to \( C(0, 1) = K_0 \) by a finite chain \( C(p_{j_1}, q_{j_1}), \ldots, C(p_{j_m}, q_{j_m}) \).

This chain intersects \( K_0 \) in a point \( x \), and \( x \) can be separated from 1 by some \( t_{n_0} \) since \( \{t_n\} \to 1 \) an end point. Let \( k = n_0 + j_1 + \ldots + j_m \). Then \( C(p_j, q_j) \subset K_k \) and the claim is proven.

We now show that this multiplication is uniformly continuous on \( H \) and hence extendable to \( X \).

**2.13 Definition** For each \( n \in \omega \), define \( n: X \to K_n \) by

\[
n(x) = \begin{cases} 
  x & \text{if } x \in K_n \\
  F(x, X \setminus K_n) & \text{if } x \notin K_n 
\end{cases}
\]
2.14 Lemma. For each $n$, $n$ is a retraction, and the restriction of $n$ to $\bigcup_{i=1}^{\infty} K_i$ is a homomorphism.

Proof. Let $\{x_i\} \rightarrow x$. If $x \notin K_n$ then $x$ is an element of some component $C$ of $X \setminus K_n$. Since $C$ is open, $x_i \in C$ for sufficiently large $i$ and hence $n(x_i) = n(x)$ for sufficiently large $i$. So in this case, $n(x_i) \rightarrow n(x)$. If $x \in K_n$, break $\{x_i\}$ into two sequences $\{x_i'\} \subset K_n$ and $\{x_i''\} \subset X \setminus K_n$. Then $n(x_i') = x_i' \rightarrow x = n(x)$. As for the sequence $\{x_i''\}$, we let $U$ be an open set containing $x$ and show $\{x_i''\}$ is eventually in $U$. There exists an open connected set $V$ such that $x \in \forall U$. For each $x_i'' \in V$, there exists an arc $[x, x_i''] \subset V$. Since $x_i'' \notin K_n$, $C(x_i'', X \setminus K_n)$ has $n(x_i'')$ as its boundary. Thus $n(x_i'') \in [x, x_i''] \subset V$, and so $x_i'' \in V$ implies $n(x_i'') \in V$ and $n(x_i'')$ is eventually in $U$. Hence $n$ is a retraction.

We now show that $n \big| \bigcup_{i=1}^{\infty} K_i$ (the restriction of $n$ to $\bigcup_{i=1}^{\infty} K_i$) is a homomorphism. First note that if $m \geq n$ then $m \circ n = n$. We show that $n \big| K_{n+1}$ is a homomorphism. Let $P_{n+1}, R_{n+1}$, and $Q_{n+1}$ be the sets to which Lemma 2.11 was applied at the $(n+1)$st stage. If $p \in P_{n+1}$, $r \in R_{n+1}$, and $q \in Q_{n+1}$ then $n(pq) = n(h(p)q) = a = n(p)a = n(p)n(q)$ and $n(cq) = n(ca) = ca = n(c)n(q)$. So $n \big| K_{n+1}$ is a homomorphism. Now suppose that $n \big| K_{m+1}$ is a homomorphism for some $m > n$. Since $m \big| K_{m+1}$ is a homomorphism and $n \big| K_{m+1} = n \big| K_m \circ m \big| K_{m+1}$ we have that $n \big| K_{m+1}$ is a
homomorphism. Thus by induction \( n|_{i=1}^{\infty} \mathbb{U}_i K_i \) is a homomorphism.

**Notation:** If \( \mathcal{O}(p_j, q_j) = A_m \), set \( p_j = p_j^m \).

We now prove the lemma which allows the extension of the multiplication on \( \bigcup_{i=1}^{\infty} K_i \) to \( X \).

**2.15 Lemma** Let \( x, y \in X \) and let \( \{x_n\} \to x \) and \( \{y_n\} \to y \) where \( x_n \) and \( y_n \) are elements of \( \bigcup_{i=1}^{\infty} K_i \) for each \( n \). Then there exists \( z \in X \) such that \( \{x_n y_n\} \to z \) and \( z \) is independent of the choice of \( \{x_n\} \) and \( \{y_n\} \).

**Proof** We distinguish four cases.

**Case I.** \( x = y = 1 \). We shall show \( z = 1 \). Let
\[
V_i = \{ s \in X | 0(s) \in C(1, K_0 \setminus \{t_1\}) \}
\]
where \( 0(s) \) is the function defined for the natural number \( 0 \) in Definition 2.13. Then \( V_i = C(1, X \setminus \{t_1\}) \) and hence \( \{V_i\}_{i=1}^{\infty} \) is an open neighborhood basis at \( 1 \). We shall show the following:
If \( z, w \in V_i \cap K_m \) then \( zw \in V_i \cap K_m \).

If \( z, w \in V_i \cap K_0 \) this is clear. Suppose it is true whenever \( z, w \in V_i \cap K_{m-1} \) for some \( m \geq 1 \), and suppose \( z, w \in V_i \cap K_m \). Recall that \( K_m = K_{m-1} \cup A_m \). There are two possibilities. If \( z \in A_m \) and \( w \in K_{m-1} \) then by the extension algorithm either \( zw \in A_m \subset V_i \cap K_m \), which is the desired result, or \( zw = p_j^m w \). In this second case \( 0(p_j^m) = 0(z) \) so \( p_j^m \in V_i \cap K_{m-1} \), and thus by the induction
hypothesis \( zw = p_j^m w \in (V_i \cap K_{m-1})^* \subseteq (V_i \cap K_m)^* \). The second possibility is that \( z \) and \( w \) are both elements of \( A_m \cap V_i \). But then \( zw \in A_m \cap V_i \subseteq (V_i \cap K_m)^* \) and we are done.

Thus \( x_ny_n \in V_i^* \) for sufficiently large \( n \), and hence \( \{x_ny_n\} \to 1 \). This is easily seen to be independent of the choice of the sequences \( \{x_n\} \) and \( \{y_n\} \).

**Case II** \( x \nparallel y \parallel 1 \).

Let \( N \) be an integer so large that \( 0(x), 0(y) \in C(0, K_0 \setminus \{t_N\}) \) and the diameter of any component of \( X \setminus K_N^* < \frac{d(x,y)}{2} \). We may assume that \( d(x_n, y_n) > \frac{d(x,y)}{2} \) and \( 0(x_n), 0(y_n) \in C(0, K_0 \setminus \{t_N\}) \) for each \( n \). We now wish to show that \( x_ny_n = N(x_n)N(y_n) \); this will be done by induction on \( \lim_{i=1}^\infty K_i \). The statement is obviously true if \( x_n, y_n \in K_N \). Suppose it is true whenever \( x_n, y_n \in K_{N+m} \) for some \( m \geq 0 \), and let \( x_n, y_n \in K_{N+m+1} = K_{N+m} + A_{N+m+1} \). If \( x_n \in A_{N+m+1} \) and \( y_n \in K_{N+m} \) then by definition of multiplication, \( x_ny_n = p_j^{N+m+1}y_n = N(p_j^{N+m+1})N(y_n) = N(x_n)N(y_n) \). By symmetry the statement is true if \( y_n \in A_{N+m+1} \) and \( x_n \in K_{N+m} \). The case \( x_n, y_n \in A_{N+m+1} \) is impossible for this would mean that \( d(x_n, y_n) < \frac{d(x,y)}{2} < d(x_n, y_n) \).

We know that \( K_N^* \) is a semigroup and hence \( x_ny_n = N(x_n)N(y_n) \to N(x)N(y) = z \). None of the above depends upon the sequences \( \{x_n\} \) and \( \{y_n\} \).
**Case III** \( x = y \not\in 1. \)

a) \( x = y \not\in \bigcup_{i=1}^{\infty} K_i \). Then \( x = y \) is an end point of \( X \) and
the neighborhoods of \( x \) with a one point boundary form a
neighborhood basis at \( x \). Moreover we need only consider
those neighborhoods \( U \) such that \( U^* \cap K_0 = \emptyset \). We shall
show that if \( U \) is such a neighborhood of \( x \) and if
\( x_n, y_n \in U \cap (\bigcup_{i=1}^{\infty} K_i) \) then \( x_n y_n \in U^* \). The proof will be by
induction on the \( K_m \) containing \( x_n \) and \( y_n \). Let \( L \) be
the smallest subscript such that \( K_L \cap U \not\in \emptyset \). Then for
\( x_n, y_n \in K_L \cap U, x_n, y_n \in A_L \cap U \) so \( x_n y_n \in (A_L \cap U)^* \) by
definition of the multiplication in the cyclic chain \( A_L \).

Suppose the statement has been shown true whenever
\( x_n, y_n \in U \cap K_{L+m} \) for some \( m \geq 0 \) and let \( x_n, y_n \in U \cap K_{L+m+1} \).

If \( x_n \in A_{L+m+1} \) and \( y_n \in K_{L+m} \) then \( x_n y_n = p_j^{L+m+1} y_n \in (K_{L+m} \cap U)^* \)
by the induction hypothesis. By symmetry the statement is
true if \( y_n \in A_{L+m+1} \) and \( x_n \in K_{L+m} \). If \( x_n, y_n \in A_{L+m+1} \cap U \)
then \( x_n y_n \in (A_{L+m+1} \cap U)^* \) by the definition of multiplication
on the cyclic chain \( A_{L+m+1} \).

Using this result we have that \( \{x_n y_n\} \) is eventually
in any open set containing \( x = y \), and hence \( \{x_n y_n\} \to x = y \).

b) \( x = y \in K_N \), some \( N \). Let \( \epsilon > 0 \). There exists \( L > N \) so
that the diameter of any component of \( x \setminus K_L < \epsilon/2 \). We may
assume that \( d(x_n, x) < \epsilon/2 \) and \( d(y_n, y) < \epsilon/2 \) for each \( n \).
If \( x_n y_n \in K_L \) then
\[ x_n y_n = L(x_n) L(y_n) = L(x_n y_n) = L(x) L(y) = x y = x = y. \]

If \( x_n y_n \notin K_L \) then \( x_n y_n \in C(x_n, X \setminus K_L) \) or \( x_n y_n \in C(y_n, X \setminus K_L) \).

Thus \( d(x, x_n y_n) \leq d(x, x_n) + d(x_n, x_n y_n) < \varepsilon \) or

or \( d(y, x_n y_n) \leq d(y, y_n) + d(y_n, x_n y_n) < \varepsilon \), so that in either case \( d(x, x_n y_n) = d(y, x_n y_n) < \varepsilon \). This completes Case III.

**Case IV** \( y \ni x = 1 \). We first establish two facts:

(A) If \( a, b \in \bigcup_{i=1}^{\infty} K_i \) so that \( 0(a) \in C(1, K_0 \setminus \{t_m\}) \) and \( 0(b) \in C(0, K_0 \setminus \{t_m\}) \) then \( ab = 0(a)b \).

The proof is by induction on \( K_i \). For \( a = 0(a) \), it is clear. Suppose the statement is true for \( a \in K_m, m \geq 0 \). Let \( a \in A_{m+1} = C(p_i^{m+1}, q_i^{m+1}) \). Then \( b \notin A_{m+1} \) and so

\[ ab = p_i^{m+1} b = 0(p_i^{m+1}) b = 0(a)b \] by the way multiplication was defined on \( K_{m+1} \) and by the induction hypothesis. Thus (A) is established.

(B) If \( a, b \in \bigcup_{i=1}^{\infty} K_i \) so that \( a \in C(1, K_0 \setminus \{t_m\}) \) and \( 0(b) \in C(0, K_0 \setminus \{t_m\}) \) then \( ab = aM(b) \) or \( ab \in C(b, M(b)) \).

Again the proof is by induction on \( K_i \). If \( b \in K_M \) then \( b \in P_M \) and \( a \in P_M \) so \( ab = b \in C(b, M(b)) \). Suppose the statement is true for \( 'b \in K_M+k \) and let \( b \in A_{M+k} \). If \( a \in C(1, K_0 \setminus \{t_M+k\}) \) then \( ab \in C(p_j^{M+k}, y_n) \subseteq C(b, M(b)) \).

If \( a \in [C(0, K_0 \setminus \{t_M+k\})]* \) then \( ab = aM(p_i^{M+k}) = aM(p_i^{M+k}) = aM(b) \) by the definition of multiplication on \( K_{M+k} \) and the
induction hypothesis. Thus (B) is established. We now distinguish two subcases of Case IV.

Subcase 1 \( y \in K_M \), some \( M \). Let \( \epsilon > 0 \). Choose \( M \) so large that \( 0(y) \in C(0,K_0 \setminus \{ t_M \}) \) and the diameter of any component of \( X \setminus K_M \) is \( < \epsilon/2 \). We may assume that
\[
0(y_n) \in C(0,K_0 \setminus \{ t_M \}) \quad \text{and} \quad 0(x_n) \in C(1,K_0 \setminus \{ t_M \})
\]
for each \( n \). Then by (A), \( x_n y_n = 0(x_n) y_n \) and by (B) \( 0(x_n) y_n = 0(x_n) M(y_n) \) or \( 0(x_n) y_n \in C(y_n,M(y_n)) \). If \( 0(x_n) y_n = 0(x_n) M(y_n) \) then \( x_n y_n = 0(x_n) M(y_n) \to 1 \cdot M(y) = y \) by the continuity of multiplication on \( K_M \). If \( 0(x_n) y_n \in C(y_n,M(y_n)) \) then \( 0(x_n) y_n \in C(y_n,X \setminus K_M) \) and hence \( d(0(x_n) y_n, y_n) < \epsilon/2 \). We may assume \( d(y_n, y) < \epsilon/2 \), and so \( d(y_n,0(x_n) y_n) < \epsilon \). Thus we conclude that \( x_n y_n \to y \). A similar argument works for any other pair of sequences converging to \( x \) and \( y \).

Subcase 2 \( y \notin K_1 \), all i. Then \( y \) is an end point of \( X \). Let \( U \) be an open connected neighborhood of \( y \) with boundary \( p_i^m \). We will show the following: If
\[
x_n \in C(l,K_0 \setminus t_m) \quad \text{and} \quad y_n \in \bigcup_{i=1}^\infty K_i \cap U \quad \text{then} \quad x_n y_n \in U.
\]
Now if \( y_n \in A_m = C(p_1^m,q_i^m) \) then \( x_n y_n \in A_m \) from the manner in which multiplication was defined. Suppose \( y_n \in C(p_1^j,q_i^j) \) where \( p_1^j \in C(p_1^k,q_i^k) \). Then \( t_j \in C(l,K_0 \setminus t_k) \) and so
\[
x_n y_n \in C(p_1^j,q_i^j) \subseteq U \quad \text{or} \quad x_n y_n = x_n p_1^j \in C(p_1^k,q_i^k) \]
by the previous argument and \( C(p_1^k,q_i^k) \subseteq U^* \). Notice however that for each \( y_n \in \bigcup_{i=1}^\infty K_i \cap U \), there is a finite sequence
\( p^1, \ldots, p^r \) such that \( y \in \mathcal{C}(p^1, q^1), p^1 \in \mathcal{C}(p^2, q^2) \ldots p^r \in \mathcal{C}(p^k, q^k) \) and so the statement follows by induction.

Now \( 0(x_n) \in \mathcal{O}(1, K \setminus t_k) \) for sufficiently large \( n \) and \( y_n \in U \) for sufficiently large \( n \). Combining the above result with (A) we see that \( x_n y_n = 0((x_n)y_n \in U \) for sufficiently large \( n \). We conclude that \( x_n y_n \to y \). Again this is independent of the choice of the sequences \( \{x_n\} \) and \( \{y_n\} \). This completes the proof of 2.15.

2.16 Theorem A retract \( X \) of a two-cell \( N \) admits the structure of a semilattice with identity.

Proof If \( X \) has a free two-cell then by Corollary 2.5 \( X \) admits the structure of a semilattice with identity. If not define multiplication on \( X \) as follows. For \( x, y \in X \) let \( \{x_n\} \to x, \{y_n\} \to y \) where \( x_n, y_n \in \bigcup_{i=1}^{\infty} K_i \) for each \( n \). Define \( x \cdot y = \lim_{n} x_n y_n \) where \( x_n y_n \) is the product defined in 2.12. By Lemma 2.15 this limit exists and is independent of the choice of \( \{x_n\} \) and \( \{y_n\} \). Hence \( \cdot \) is well-defined. It follows that \( \cdot \) is continuous, associative, commutative, and idempotent on \( X \) with identity \( 1 \).

2.17 Remark In the preceding proof, the semilattice was constructed so that \( 0 \in F(X) \). It is conjectured that \( 0 \) may be chosen to be any point in \( X \).
2.18 Definition A space $X$ is **homogeneous** if for each $x,y \in X$ there is a homeomorphism of $X$ onto itself carrying $x$ to $y$.

2.19 Corollary Any homogeneous retract of a two-cell is a point.

**Proof** Let $X$ be a homogeneous retract of a two-cell. By 2.16 $X$ admits the structure of a semilattice with identity. However any homogeneous compact connected semigroup with identity is a group [5]. If $X$ has more than one point then the group $X$ has more than one idempotent, a contradiction. Thus $X$ must be a point.

2.20 Remark This result for semilattices with identity is not extendable to higher dimensions. Professor Jimmie Lawson has shown (unpublished) that a semilattice which does not contain a two-cell must be of dimension less than two. This means Borsuk's example of a two-dimensional retract of $I^3$ which contains no disks cannot be a semilattice. The question of whether a retract admits the structure of a semigroup with zero and identity remains open.
CHAPTER III
SEMIGROUPS ON STRONGLY RULED SPACES

In this chapter we present a class of metric continua called strongly ruled continua and show that any member of this class admits the structure of a semigroup with zero and identity.

3.1 Definition A metric $d$ on a partially ordered space $X$ is said to be radially convex if $x < y$ implies there exists $z \in X$ such that $x < z < y$ and $d(x, z) = d(z, y)$.

3.2 Definition Let $S$ be a metric weakly ruled continuum (see Chapter II). Partially order $S$ by $x \leq y$ if and only if $x \in [0, y]$. Then $S$ admits a radially convex metric $d[2]$. It follows that if $x \not\leq y$ are elements of $[0, e]$ where $e \in E$, then $d(0, x) \not\leq d(0, y)$. Also there exists $u \in E$ so that $d(0, u)$ is maximal among $\{d(0, e) : e \in E\}$. We assume that $d(0, u) = 1$. Let $E_x = \{e : x \in [0, e]\}$.

Suppose further that $E$ may be totally ordered with maximal element $u$ so that the following are satisfied:

(A) For each $x \in S$, $E_x$ is convex

(B) Whenever $x_n \to x$ and $y_n \to y \not\leq [0, u]$, so that $x \not\leq [0, y]$ and $y \not\leq [0, x]$ (i.e. $E_x \cap E_y = \emptyset$), and whenever
there exist \( e_x \in E_x, e_y \in E_y \) so that \( e_x < e_y \), then there exist subsequences \( x_{n_k} \) and \( y_{n_k} \), and \( e_{x_{n_k}} \in E_{x_{n_k}} \) and \( e_{y_{n_k}} \in E_{y_{n_k}} \), so that \( e_{x_{n_k}} < e_{y_{n_k}} \).

(C) Let \( K_1 = \{ y \in [0,u] : \text{there exists } x \notin [0,u] \text{ and } x_n \to x, y_n \to y \text{ such that for each } n \text{ there exists } e_{x_n} \in E_{x_n} \text{ and } e_{y_n} \in E_{y_n}, \text{ such that } e_{y_n} \leq e_{y_n} \} \). Let \( k = \sup(K_1 \cup \{0\}) \) and \( K = [0,k] \). If \( K \notin \{0\} \) then \( S \) can be remetrized with a radially convex metric \( d \) so that \( d(0,u) = 1 \), \( d(0,k) = \frac{1}{2} \), and if \( e \in I \) and \([0,p] = [0,e] \cap [0,u] \) then \( d(e,p) \leq \frac{1}{2} \).

A strongly ruled continuum is a metric weakly ruled space whose set \( E \) may be totally ordered with maximal element \( u \) so that (A), (B) and (C) are satisfied. The set \( K \) in (C) may be thought of as the points of discontinuity induced by the ordering on \( E \).

We shall now show that any strongly ruled continuum admits the structure of a semigroup with zero and identity. This will be done by defining an algebraic semigroup \( T_S \) with zero and identity which induces an algebraic semigroup structure on \( S \), and then proving this operation on \( S \) is continuous. It remains an open question whether any metric weakly ruled continuum \( S \) admits the structure of a semigroup with zero and identity.
3.3 Lemma Any strongly ruled continuum $S$ admits the structure of an algebraic semigroup.

Proof Let $T_S = \{(a,e) | e \in E, 0 \leq a \leq d(0,e)\}$. For $(a,e)$ and $(b,f) \in T_S$, define $(a,e)(b,f) = (a \circ b, \min\{e,f\})$ where $a \circ b = \max\{a+b-1,0\}$ and where $\min$ is in the ordering on $E$. Note that $a \circ b \leq \min\{a,b\}$. This product is an element of $T_S$, and the operation is easily checked to be associative and commutative. Thus $T_S$ is a commutative algebraic semigroup with identity $(1,u)$.

Define $F:T_S \to S$ by $F((a,e))$ equals that point $x \in S$ such that $x \in [0,e]$ and $d(0,x) = a$. From the definition of $T_S$, $F$ is well-defined. $F$ is easily seen to be onto, but not necessarily one-to-one. Define a relation $\rho$ on $T_S$ by $(a,e)\rho(b,f)$ if and only if $a = b$ and $a \leq d(0,p)$, where $[0,p] = [0,e] \cap [0,f]$. Thus $(a,e)\rho(b,f)$ if and only if $F((a,e)) = F((b,f))$. We show that $\rho$ is a congruence relation on $T_S$.

The relation $\rho$ is easily seen to be reflexive and symmetric. To show transitivity suppose $(a,e)\rho(b,f)$ and $(b,f)\rho(c,g)$. Then $a = b = c$, so $a = c$. Also $b \leq d(0,p)$ where $[0,p] = [0,e] \cap [0,f]$, and $b \leq d(0,q)$ where $[0,q] = [0,f] \cap [0,q]$. Then $a = b \leq d(0,r)$ where $[0,r] = [0,p] \cap [0,q] = [0,e] \cap [0,f] \cap [0,g] \subset [0,e] \cap [0,g]$ since $[0,p]$ and $[0,q]$ are contained in $[0,f]$. Thus $a \leq d(0,s)$ where $[0,s] = [0,e] \cap [0,g]$, so $(a,e)\rho(c,g)$. 

Thus \( \rho \) is transitive.

We now show \( \rho \) is a congruence. Toward this end let \((a,e) \rho (c,g)\) and let \((b,f) \in T_S\). We shall show that 

\[(c,g)(b,f) \rho (a,e)(b,f),\text{ i.e. } (a \circ b, \min \{g,f\}) \rho (a \circ b, \min \{e,f\}).\]

Since \((c,g) \rho (a,e)\), \(a = c \leq d(0,p)\) where \([0,p] = [0,e] \cap [0,g]\).

Thus \(a \circ b = b \circ c\) and we must show \(a \circ b \leq d(0,q)\) where 
\([0,q] = [0,\min \{f,g\}] \cap [0,\min \{e,f\}]\). We distinguish four cases.

**Case I** \(f \leq \) both \(e\) and \(g\). Then \(f = \min \{e,f\} = \min \{f,g\}\) and so 
\([0,q] = [0,f] \cap [0,f] = [0,f].\)

**Case II** \(e \leq f \leq g\). Recall that \(a = c \leq d(0,p)\) where 
\([0,p] = [0,e] \cap [0,q]\). So \(F(a,e) = F(c,g) \in [0,e] \cap [0,g]\). 
By the convexity of \(E\), \(F(a,e) = F(c,g) \in [0,f]\). Thus 
\((a \circ b) \leq a \leq d(0,r)\) where \([0,r] = [0,e] \cap [0,f]\).

**Case III** \(g \leq f \leq e\). Similar to Case II.

**Case IV** Both \(e\) and \(g \leq f\). Then \([0,q] = [0,e] \cap [0,g] = [0,p]\) and so \(a \circ b \leq a \leq d(0,p) = d(0,q)\). This completes the
proof that \(\rho\) is a congruence.

Since \(\rho\) is a congruence, \(\frac{T_S}{\rho}\) is a commutative algebraic semigroup with zero \((0,e)\) (where \((0,e)\) denotes the congruence class of \((0,e)\)) and identity \((1,u)\).

Let \(\eta: T_S \rightarrow \frac{T_S}{\rho}\) be the canonical map. Define
Since \( \rho \) is a congruence, \( G \) is well-defined. Moreover \( G \) is onto since \( F \) is, and \( G \) is one-to-one by the definition of \( \rho \). Thus \( S \) admits the structure of an algebraic semigroup with zero and identity. The following three lemmas show this induced operation on \( S \) is continuous.

3.4 Lemma Suppose \( x_n \to x \) with \((a_n, e_n) \in F^{-1}(x_n)\) and \((a, e) \in F^{-1}(x)\). Let \( \{t_n\} \subseteq [0,1] \subseteq \mathbb{R} \) with \( t_n \to t \) and \( t_n \leq a_n \) for each \( n \). Then \( F(t_n, e_n) \to f(t, e) \).

Proof We first must show that each \((t_n, e_n)\) and \((t, e) \in T_S\). Since \( t_n \leq a_n \) for each \( n \), \((t_n, e_n) \in T_S\) for each \( n \). Also \( t_n \leq a_n \) for each \( n \), so \( t \leq a \), and hence \((t, e) \in T_S\).

To show that \( F(t_n, e_n) \to f(t, e) \), we show that any subsequence of \( F(t_n, e_n) \) clusters at \( f(t, e) \). Toward this end, consider a subsequence of \( F(t_n, e_n) \) which we shall again denote by \( F(t_n, e_n) \). Then \( F(t_n, e_n) \in [0, x_n] \to [0, x] \), so there exists a subsequence \( F(t_{n_k}, t_{n_k}) \) converging to some
element of \([0, x] \subseteq [0, e]\); i.e. \(F(t_{n_k}, e_{n_k}) \to F(t_o, e)\) for some \(t_o\). But \(t_{n_k} \to t_o\) and \(t_n \to t\), so \(t_o = t\). Thus \(F(t_{n_k}, e_{n_k}) \to F(t, e)\) and we are done.

In the following two lemmas, let \(x_n \to x\), \(y_n \to y\). Without loss of generality we may assume that \(e_x \leq e_y\).

3.5 Lemma If \(x \in [0, y]\) or \(y \in [0, x]\), then \(x_n y_n \to xy\).

Proof We distinguish two cases.

Case I \(x \neq y\). We first suppose that \(x \in [0, y]\). By choosing subsequences we may assume that either i) there exists \(e_{x_n} < e_{y_n}\) for each \(n\), or ii) there exists \(e_{y_n} < e_{x_n}\) for each \(n\). If i) then \(x_n y_n = F(a_{x_n} o a_{y_n}, e_{x_n}) \to F(a_{x_n} o a_{y}, e_{x}) = xy\) by Lemma 3.4. If ii) then \(x_n y_n = F(a_{x_n} o a_{y_n}, e_{y_n}) \to F(a_{x} o a_{y}, e_{y})\) by Lemma 3.4. Moreover \(F(a_{x_n} o a_{y}, e_{y_n}) = F(a_{x_n} o a_{y}, e_{y})\), for \((a_{x_n} o a_{y}, e_{y_n}) \rho (a_{x_n} o a_{y}, e_{y})\) since \(a_{x_n} o a_{y} \leq a_{x}\) and \(x \in [0, y]\). So \(x_n y_n \to xy\).

If \(y \in [0, x]\), a similar argument works.

Case II \(x = y\). Again we may choose subsequences such that either i) there exist \(e_{x_n} \leq e_{y_n}\) for each \(n\) or ii) there exist \(e_{y_n} \leq e_{x_n}\) for each \(n\). If i) then \(x_n y_n = F(a_{x_n} o e_{y_n}, e_{x_n}) \to F(a_{x} o a_{y}, e_{x}) = xy\) by Lemma 3.4.
If ii) then $x_n y_n = F(a_{x_n} o a_{y_n}, e_{y_n})$ by Lemma 3.4 and $F(a_{x_n} o a_{y_n}, e_{y}) = F(a_{x} o a_{y}, e_{x})$ since $F(a_{x} o a_{y}, e_{y}) \in [0, y] = [0, x] \subset [0, e_{x}] \cap [0, e_{y}]$. Thus $x_n y_n \rightarrow xy$. This completes the proof of Lemma 3.5.

3.6 Lemma If $x \notin [0, y]$ and $y \notin [0, x]$ then $x_n y_n \rightarrow xy$.

Proof We distinguish two cases.

**Case I** $y \notin [0, u]$. Then by (B) we may choose subsequences so that $e_{x_n} \leq e_{y_n}$ for each $n$. Then $x_n y_n = F(a_{x_n} o a_{y_n}, e_{x_n}) \rightarrow F(a_{x} o a_{y}, e_{x}) = xy$ by Lemma 3.4.

**Case II** $y \in [0, u]$. We may choose subsequences so that either i) $e_{x_n} \leq e_{y_n}$ for each $n$ or ii) $e_{y_n} \leq e_{x_n}$ for each $n$. If i) then $x_n y_n = F(a_{x_n} o a_{y_n}, e_{x_n}) \rightarrow F(a_{x} o a_{y}, e_{x}) = xy$ by Lemma 3.4. If ii) then $y \in K$ as defined by (C), where $a = x$. Also $y \notin 0$. Let $k = \sup K$. By the remetrization, $d(0, k) = \frac{1}{2}$. Then $d(0, x) \leq d(0, e_{x}) \leq d(0, p_{p_{x}}) + d(p_{x}, e_{x})$, where $[0, p_{x}] = [0, e_{x}] \cap [0, u]$, for each $e_{x}$. By (C) this means $a_{x} \leq a_{p_{x}} + \frac{1}{2}$. Since $y \in K$, $a_{y} \leq \frac{1}{2}$. Thus $a_{x} o a_{y} \leq (a_{p_{x}} + \frac{1}{2}) o (\frac{1}{2}) = a_{p_{x}}$ by the definition of $o$. So $x_n a_{x} o a_{y} \leq (a_{p_{x}} + \frac{1}{2}) o (\frac{1}{2}) = a_{p_{x}}$ by the definition of $o$. So $F(x_n y_n) = F(a_{x_n} o a_{y_n}, e_{y_n}) \rightarrow F(a_{x} o a_{y}, e_{x})$ by Lemma 3.4, and $F(a_{x} o a_{y}, e_{y}) = F(a_{x} o a_{y}, u)$ since $y \in [0, u]$ and $a_{x} o a_{y} \leq a_{y}$.

But $F(a_{x} o a_{y}, u) = F(a_{x} o a_{y}, e_{x})$ since $a_{x} o a_{y} \leq a_{p_{x}}$, and
hence \( x_ny_n \rightarrow F(a_x \circ a_y, e_x) = xy \).

We have therefore proven

3.7 Theorem [14] Let \( S \) be a strongly ruled space. Then \( S \) admits the structure of a commutative semigroup with zero and identity.

3.8 Remark One should notice that if \( S \) is a strongly ruled continuum such that \( K = \{0\} \), then any operation on \([0,1]\) in which 0 is a zero and 1 the identity may be used for \( \circ \) in the multiplication on \( S \). In particular \( \text{min} \) may be used, and so if \( S \) is a continuum strongly ruled so that \( K = \{0\} \), then \( S \) admits the structure of a semilattice with zero and identity.

3.9 Examples Because of the imposing nature of the conditions upon strongly ruled continua, we now give some examples of strongly ruled continua. The first example is the one which inspired the concept of strongly ruled continua, namely the Cantorian swastika.

I. Let \( C \) be a Cantor set in the interval \([0,1]\). Then the \textit{Cantorian swastika}, denoted \( CS \), is \( C \times I \) together with the rotations of \( C \times I \) through 90, 180, and 270 degrees about the point \((0,0)\). (See figure 1).
CS is easily seen to be a weakly ruled space. Let $u = (1,1)$ and order $E$ clockwise as indicated. This ordering is convex, so (A) is satisfied. Condition (B) is satisfied because of the continuity of the ordering away from $[0,u]$. The set $K$ described in (C) is the intersection of the non-negative $x$-axis with CS, and $k = (1,0)$. It is clear that condition (C) is satisfied. Thus $S$ admits the structure of a commutative semigroup with identity.

II. Cones tangent along a line segment.

Let $S$ be the cone over a decreasing tower of circles having a common point of tangency, together with a free arc at the point of tangency (See Figure 2). The ordering is as
indicated, from the outer circles toward the inner circles, with \( u \), the end point of the free arc, being maximal. This example illustrates what might be called discontinuity of higher order, and also the technique of separating the identity from the discontinuities. It is an open question whether this space without the free arc is a strongly ruled space.

Figure 2.
4.1 Definition A continuum $X$ is said to be hereditarily unicoherent if each pair of points $x$ and $y$ is contained in a unique minimal continuum $[x,y]$.

4.2 Definition A generalized tree is an arcwise connected hereditarily unicoherent continuum $X$ containing a point $p$ so that $[p,x_\alpha] \to [p,x]$ whenever $x_\alpha \to x$.

The concept of a generalized tree is due to L. E. Ward [20]. He defined a generalized tree as a hereditarily unicoherent continuum which admits an order-dense (i.e. $x < y$ implies there exists $z$ such that $x < z < y$) partial order with unique minimal element. Koch and Krule [12] showed that "order-dense" could be replaced by "monotone" ($L(x)$ connected for each $x \in X$) and also gave the purely topological characterization which we use. Note that the class of all generalized trees properly contains the class of all trees (locally connected continua in which each pair of points can be separated by a third).

Our interest in generalized trees stems from their connection with semigroups and semilattices. In particular, R. P. Hunter has shown [6] that a one-dimensional compact
connected semigroup with zero and identity must be a
generalized tree. It is also easily shown that a semilattice
on a hereditarily unicoherent continuum must necessarily be
a generalized tree.

In [16], Koch also raised the question, "Under
what conditions does a generalized tree admit the structure
of a semigroup with identity"? Only a few results have been
obtained on this question. Koch and McAuley [13] have shown
that any tree admits the structure of a semilattice with
identity, and also that any strongly ruled continuum admits
the structure of a semigroup with zero and identity [14]
(see Chapter III). Eberhart [4] has given an example of a
non-metric generalized tree which does not admit the
structure of a semigroup with zero and identity. We give
here some necessary conditions for a generalized tree to
admit the structure of a semilattice with identity, and also
some examples of generalized trees which do not admit the
structure of a semilattice with identity.

Suppose $S$ is a generalized tree. For $x, y \in S$, let
$[x,y]$ denote the unique arc from $x$ to $y$ in $S$. Let $0$
de note the unique minimal element of $S$.

4.3 Definition The quadrant of $x$, $Q(x)$, is $C(x,S\setminus\{0\})$.

Note that $Q(x) = \{y \in S | [0,x] \cap [0,y] \text{ properly contains } zero\}$.
4.4 Lemma ([6]) If $S$ is a semigroup on a generalized tree so that $S^2 = S$, then for $p, q \in S$, $p[0,q] = [0,pq] = [0,p]q$ and $[0,p][0,q] = [0,pq]$.

4.5 Theorem Let $S$ be a semilattice with identity on a generalized tree. Suppose there exists a net $\{e_\alpha\} \to 1$ so that $[0,e_\alpha] \cap [0,1] \to \{0\}$. Then for each $x \in S \setminus [0,1]$, $[0,e_\alpha x] \cap [0,1] \to \{0\}$.

Proof We assume that there exists $r \in (0,1]$ so that $[0,r]$ is frequently contained in $[0,e_\alpha x] \cap [0,1]$ and show this leads to a contradiction. Since $e_\alpha x \to x$, $[0,e_\alpha x] \to [0,x]$ and hence $[0,r] \subset \limsup [0,e_\alpha x] = [0,x]$. Since $r \in (0,1]$, $e_\alpha r \in e_\alpha [0,1] = [0,e_\alpha]$. Were $e_\alpha r$ frequently in $[0,1]$ then $e_\alpha r$ would frequently be in $[0,1] \cap [0,e_\alpha] \to \{0\}$ and hence $e_\alpha r$ would converge to both $r$ and $0$, a contradiction. So we may assume $e_\alpha r \notin [0,1]$ for each $\alpha$.

Since $[0,r] \subset [0,x]$, we have $[0,e_\alpha r] \subset [0,e_\alpha x]$. Combining this with the fact that $e_\alpha r \notin [0,1]$ we have that $[0,e_\alpha r] \cap [0,1] = [0,e_\alpha x] \cap [0,1]$. Thus $[0,r] \subset [0,e_\alpha r] \cap [0,1] \subset [0,e_\alpha] \cap [0,1]$. This is a contradiction since $[0,e_\alpha] \cap [0,1] \to \{0\}$, and the proof is complete.

4.6 Corollary Let $S$ be a semilattice with identity on a generalized tree. Suppose there exists a net $\{e_\alpha\} \to 1$ so that $[0,e_\alpha] \cap [0,1] \to \{0\}$. Then for each $x \in Q(1)$, $S$ is
not locally connected at $x$.

**Proof** The statement is obviously true for $x \in (0,1]$. For $x \in Q(1) \setminus (0,1]$, let $U$ be an open set containing $x$ such that $U^* \cap [0,1] = \emptyset$. We show that for any open set $V$ such that $x \in V \subseteq U$, $V$ is not locally connected. Let $[0,p] = [0,x] \cap [0,1]$. Let $\alpha_0$ be such that $[0,1] \cap [0,e_{\alpha_0} x] \subseteq [0,p]$ and such that $e_{\alpha_0} x \in V$. Then $[0,e_{\alpha_0} x] \cap [0,1] = [0,1]$, for otherwise $p \in [0,e_{\alpha_0}] \cap [0,1]$, a contradiction to the choice of $\alpha_0$. But $[x,e_{\alpha_0} x] \subseteq [0,x] \cup [0,e_{\alpha_0} x]$ and $[x,e_{\alpha_0} x] \cap ([0,x] \cap [0,e_{\alpha_0} x]) = \emptyset$. So it suffices to apply Theorem 4.5.

The preceding results may be stated for any point in $(0,1]$.  

**4.7 Corollary** Let $S$ be a semilattice with identity on a generalized tree and let $a \in (0,1]$. Suppose there exists a net $\{e_{\alpha}\} \rightarrow a$ so that $[0,e_{\alpha}] \cap [0,a] \rightarrow \{0\}$. Then for each $x \in Sa$, $[0,e_{\alpha} x] \cap [0,1] \rightarrow \{0\}$.

**Proof** Note that $Sa$ is a semilattice with identity $a$. Also $[e_{\alpha} a] \rightarrow a$ and $[0,e_{\alpha} a] \cap [0,1] = [0,e_{\alpha}] \cap [0,1] \rightarrow \{0\}$. So it suffices to apply Theorem 4.5.
4.8 Corollary Under the same hypotheses as Corollary 4.7, for \( x \in S_a \cap Q(a) \), \( S \) is not locally connected at \( x \).

**Proof** Applying Corollary 4.6 to the semilattice \( S_a \) with identity \( a \), we have that \( S_a \) is not locally connected at \( x \). Thus given an open set \( U \) containing \( x \) there exists a point \( y \in S_a \) so that \( [x,y] \cap (S_a \setminus U^*) \neq \emptyset \). Since \( S \) is uniquely arcwise connected, \( [x,y] \cap (S \setminus U^*) \neq \emptyset \). So \( S \) is not locally connected at \( x \).

4.9 Example Let \( a_n = (2 - \frac{1}{n}, 0) \), \( b_n = (2 + \frac{1}{n}, \frac{1}{n}) \), \( c_n = (2 + \frac{1}{n}, -\frac{1}{n}) \), and \( e_n = (0, -\frac{1}{n}) \) for \( n = 1, 2, \ldots \). Let \( A_n = [a_n, b_n] \) (the straight line from \( a_n \) to \( b_n \) in the plane), \( B_n = [b_n, c_n] \), and let \( C_n = [c_n, e_n] \). Let \( D = [(0,0), (2,0)] \) and \( e_0 = (0,0) \). Finally let \( G = D \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup C_n) \) (see Figure 3). We shall show that \( G \) does not admit the structure of a semilattice with identity.

![Figure 3](image-url)
Note that the point \((2,0)\) is the only possibility for the zero \(0\) of \(G\). As for the identity \(1\) of \(G\), by Corollary 2, p. 613, of [9], \(1\) must be one of the non weak-cutpoints \(e_0, e_1, e_2, \ldots\). If \(1 = e_0\), then by Corollary 4.6 \(S\) is not locally connected at any point of \(Q(1) = S \setminus \{0\}\), a contradiction. If \(1 = e_{n_o}\) for some \(n_o\), let \([0,a] = [0,e_{n_o}] \cap [0,e_{n_o}]\). Since \(\{e_k\}_{k=1}^{\infty} \to e_0\), we have \(\{e_k a\}_{k=1}^{\infty} \to e_{n_o} a = a\). Moreover, \(e_k a \in e_k [0,e_{n_o}] = [0,e_k a] \subset [0,e_k]\) since \(e_{n_o} = 1\). Hence

\([0,e_k a] \cap [0,a] \subset [0,e_k] \cap [0,e_{n_o}] \to \{0\}\). By Corollary 4.8, \(S a \setminus \{0\}\) is not locally connected at any point. However \(e_k a \in S a\) and \(e_k a\) is eventually not in \([0,a]\), so \(e_k a\) is a point at which \(S a\) is locally connected, a contradiction. Thus \(G\) cannot be a semilattice with identity.

We now prove some lemmas concerning multiplication between quadrants. In each, \(S\) denotes a semilattice with identity on a generalized tree with zero \(0\).

4.10 Lemma If \(x \in [0,1]\), \(y \in S\), and \(xy = 0\), then \(xQ(y) = 0\).

Proof Let \(p \in Q(y)\), and let \([0,r] = [0,p] \cap [0,y]\). Then \(x[0,r] \subset x[0,y] = [0,0] = 0\). Now \(x[0,p] \leq [0,1][0,p] = [0,p]\). If \(xp \not\in 0\), then there exists \(p_1 \in [0,r]\) so that \(xp_1 \in [0,r]\) and \(xp_1 \not\in 0\). But \(xp_1 = x(xp_1) = 0\) since \(xp_1 \in [0,r]\). So we must have \(xp = 0\), and the lemma is proved.
4.11 Lemma If $x \in Q(1)$ and $y \in S$ so that $Q(y)^* \cap Q(1) \nsubseteq \Box$, then $xy \in Q(y)$.

Proof Let $[0,z] = [0,1] \cap [0,x]$. We shall first show that $zy \nsubseteq 0$ by supposing not and arriving at a contradiction. If $zy = 0$, then by Lemma 4.10 $zQ(y) = 0$ and so $[0,z]Q(y) = 0$. Let $q \in Q(y)^* \cap Q(1)$ and let $[0,r] = [0,z] \cap [0,q]$. Then $r \in Q(y)^* \cap Q(1)$ since $r \in [0,q]$, and also $rQ(y) = 0$ since $r \in [0,z]$. By continuity of multiplication, $rQ(y)^* = 0$ which means $rr = 0$, a contradiction. Thus $zy \nsubseteq 0$. However, $zy \in ([0,x] \cap [0,1])y = [0,xy] \cap [0,y]$, and so $xy \in Q(y)$.

4.12 Theorem Let $S$ be a generalized tree with zero 0 and let $1 \in S$. Suppose there exists a quadrant $Q(x) \nsubseteq Q(1)$ such that

a) $Q(1)$ is the only quadrant whose closure meets $Q(x)$

b) There exists a sequence $(Q_i)_{i=1}^\infty$ of quadrants such that $Q_1 = Q(x)$, $Q_\infty = Q(1)$, and $Q_i^* \cap Q_{i+1} \nsubseteq \Box$

for each $i$.

Then $S$ does not admit the structure of a semilattice with zero 0 and identity 1.

Proof Suppose that $S$ does admit the structure of a semilattice with a zero 0 and identity 1. We shall first
show there exists \( a \in (0,1] \cap Q_{n-1}^* \) such that \( aq(x) = 0 \). Let \( q \in Q(1)^* \setminus Q(x) \) and \( a \in (0,1] \cap Q_{n-1}^* \). There exists a net \( \{y_\alpha\} \subseteq Q_{n-1} \) so that \( y_\alpha \to a \) and a net \( \{q_\beta\} \subseteq Q(1) \) so that \( q_\beta \to q \). By Lemma 4.11 \( y_\alpha q_\beta \in Q_{n-1}^* \), and hence \( aq = \lim y_\alpha q_\beta \in Q_{n-1}^* \). However \( aq \in [0,1]q = [0,q] \) also. By assumption, \( Q_{n-1}^* \cap Q(x) = \emptyset \), so \( aq \notin (0,q] \). This implies \( aq = 0 \). By Lemma 4.10, \( aq(x) = aq(q) = 0 \), and the statement is proven.

Since \( aq(x) = aq_1 = 0 \), \( aq_1^* = 0 \) by continuity. If \( z \in Q_2 \cap Q_1^* \), then \( az = 0 \). By Lemma 4.10, \( aq_2 = 0 \). Continuing this process we have \( aq_n = 0 \). However \( a \in Q_n = Q(1) \), so \( aa = 0 \). This is a contradiction and the theorem is proven.

4.13 Example Recall the Cantorian swastika CS from Chapter III. It was shown there that CS admits the structure of a semigroup with zero and identity. CS also admits the structure of a semilattice. Let \( T \) be the subsemilattice of \( I \times I \) consisting of \( (I \times 0) \cup (0 \times I) \). Then CS is a subsemilattice of \( T \times T \).

However CS does not admit the structure of a semilattice with identity. Since CS is a generalized tree, it must be locally connected at zero [12]. Thus its zero must be \((0,0)\). As with the previous example, the identity \( I \) of CS must be a non weak-cutpoint. Thus it satisfies the
hypotheses of Theorem 4.12, and hence cannot be a semilattice with identity.

4.14 Remark The hypotheses of Theorem 4.12 apply to other spaces than CS, but these spaces are in some sense similar to CS. Consider the example shown in Figure 4. If this space is to be a semilattice with identity then the zero must be the "center" point. So Theorem 4.12 applies no matter where 1 is chosen (again 1 must be a non weak cutpoint), and the space does not admit the structure of a semilattice with identity.

Figure 4.
BIBLIOGRAPHY


BIOGRAPHY

William Wiley Williams was born May 16, 1942, in Lake Charles, Louisiana. He attended public schools in Houston, Texas, and was graduated from Lamar High School with highest honors in 1960. He entered Rice University, Houston, Texas, in the fall of 1960 and in June, 1964, received his B. A. degree in Mathematics. In September, 1964, he entered Louisiana State University. He received his M.S. in August, 1966, and is presently a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.
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Major Field: Mathematics

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EXAMINING COMMITTEE:


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