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Retracting Anr's Onto Spheres, With Applications in the Theory of Fixed Points.

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ABSTRACT

This paper is concerned with necessary and sufficient conditions in terms of homology for retracting certain ANR's onto spheres. In fact however, the main theorems are proved for a larger class of spaces, the metrizable spaces which are dominated by locally finite polyhedra. Several corollaries in the theory of fixed points are obtained, using the fact that any space which retracts to a sphere does not have the fixed point property. The basic results concerning retractions are these:

(1) If $X$ is a metrizable space which is dominated by a finite $(m+1)$-dimensional polyhedron, and if $X$ contains an $m$-sphere $S$ such that the inclusion map $i:S \rightarrow X$ yields a split exact sequence

$$0 \rightarrow H_m(S) \rightarrow H_m(X) \rightarrow H_m(X)/H_m(S) \rightarrow 0$$

then $S$ is a retract of $X$.

(2) If $X$ is a metrizable space which is dominated by a locally finite polyhedron, and if $X$ contains a simple closed curve $S$ such that the inclusion map $i:S \rightarrow X$ yields a split exact sequence

$$0 \rightarrow H_1(S) \rightarrow H_1(X) \rightarrow H_1(X)/H_1(S) \rightarrow 0$$

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then $S$ is a retract of $X$.

Since a retraction $r:X \to S$ induces a homomorphism $r_*:H_n(X) \to H_n(S)$ such that $r_*i_*$ is the identity homomorphism of $H_n(X)$ for each $n$, the above conditions on the homology groups for retracting $X$ onto $S$ are also necessary.

The following results are obtained as corollaries to the retraction theorems.

(3) If $X$ is a metrizable space which is dominated by a finite $(m+1)$-dimensional polyhedron, and there is a mapping $f:S^m \to X$ such that the sequence

$$0 \to H_m(S^m) \overset{f_*}{\to} H_m(X) \to H_m(X)/f_*(H_m(S^m)) \to 0$$

is split exact, then $X$ has the homotopy type of a space $Y$ which retracts to an $m$-sphere. If $X$ is an ANR then there is such a $Y$ which is also an ANR. Furthermore, in the case that $S$ is a simple closed curve, the dimension and finiteness conditions on the dominating polyhedron are unnecessary.

It is not known if there exists a finite 2-dimensional polyhedron having Euler characteristic zero and also having the fixed point property. If $X$ is a finite 2-dimensional polyhedron with zero Euler characteristic, then the Euler-Poincaré formula assures that $H_1(X)$ is not a torsion group. Now it follows from (2) that
if $M$ is a compact manifold such that $H_1(M)$ is not a torsion group then $M$ retracts to a simple closed curve. It would be of interest to know if this is true for finite polyhedra, for in that case no finite polyhedron with zero Euler characteristic would have the fixed point property.
CHAPTER I

INTRODUCTION

This paper is concerned with necessary and sufficient conditions in terms of homology for retracting certain ANR's onto spheres. In fact however, the main theorems are proved for a larger class of spaces, the metrizable spaces which are dominated by locally finite polyhedra. Several corollaries in the theory of fixed points are obtained, using the fact that any space which retracts to a sphere does not have the fixed point property.

Items will be numbered consecutively throughout a chapter. For example, II.6 stands for item 6 of Chapter II, no matter if it is a theorem, lemma, definition, etc. Numerals in square brackets are bibliographical references.

I.1. If $X$ is a topological space and $A \subseteq X$, then $\overline{A}$ will denote the closure of $A$ in $X$. A mapping, or map, is a continuous function. If $A \subseteq X$, a map $r:X \to A$ is called a retraction if the restriction of $r$ to $A$, $r|A$, is the identity mapping of $A$. In the case that such a map exists, $A$ is said to be a retract of $X$, and $X$ is said to retract to $A$.

The standard $n$-cell will be denoted by $I^n$. In
particular, I will denote the unit interval. \( S^n \) will denote the n-sphere. If \( A \subseteq X \), \( A \) is said to be a deformation retract of \( X \) if there is a map \( H:X \times I \to X \) such that \( H(x,0) = x \) for all \( x \in X \), \( H(a,1) = a \) for all \( a \in A \), and \( H(X \times \{1\}) \subseteq A \). \( A \) is said to be a strong deformation retract of \( X \) if there is a map \( H:X \times I \to X \) such that \( H(x,0) = x \) for all \( x \in X \), \( H(a,t) = a \) for all \( a \in A \) and \( t \in I \), and \( H(X \times \{1\}) \subseteq A \). A space \( X \) is said to be contractible if there is a point \( x \in X \) which is a deformation retract of \( X \).

If \( f:X \to Y \) is a map, the mapping cylinder \( M(f) \) of \( f \) is the space formed by taking the disjoint union of \( X \times I \) and \( Y \) and identifying points \( (x,1) \in X \times I \) with corresponding points \( f(x) \in Y \). \( Y \) is a strong deformation retract of \( M(f) \), with deformation \( H:M(f) \times I \to M(f) \) being given by \( H(y,s) = y \) for \( y \in Y \subseteq M(f) \) and \( s \in I \), and \( H(x,t,s) = (x,t-st+s,s) \) for \( (x,t) \in M(f) \) and \( s \in I \). For a space \( X \), the cone over \( X \), \( C(X) \), is the mapping cylinder of a constant map. The point of \( C(X) \) which is the image of the constant map is called the coning point.

I.2. Let \( C \) be a class of topological spaces. An absolute retract (AR) for the class \( C \) is a space \( Y \) in \( C \) such that every homeomorphic image of \( Y \) as a closed subspace of a space \( Z \) in \( C \) is a retract of \( Z \). An ab-
solute neighborhood retract (ANR) for the class $C$ is a space $Y$ in $C$ such that every homeomorphic image of $Y$ as a closed subspace of a space $Z$ in $C$ is a neighborhood retract of $Z$. That is, if $h: Y \to Z$ is an imbedding and $h(Y)$ is closed in $Z$, then there is a neighborhood $V$ of $h(Y)$ in $Z$ such that $h(Y)$ is a retract of $V$.

In this paper the terms AR and ANR will mean, respectively, AR and ANR for the class of separable metric spaces.

It is proved in [6] that if $A$ is a deformation retract of a space $X$, and if $A$ and $X$ are ANR's, then $A$ is a strong deformation retract of $X$.

1.3. The word "complex" will always mean "countable simplicial complex." An $n$-complex is a complex of dimension $n$. If $K$ is a complex, then it is standard practice to let $|K|$ denote the topological space associated with $K$. Such a space is called a polyhedron. For any complex $K$, the space $|K|$ will have the weak topology with respect to the closed simplexes of $K$. Sometimes in this paper the words "complex" and "polyhedron" will be used interchangeably. The meaning will be clear from the context.

The Euler characteristic of a finite complex $K$, $\chi(K)$, is the number $\sum_{n=0}^{\infty} (-1)^n n(K)$, where $n(K)$ is the number of $n$-simplexes of $K$. The $n$-th Betti number of $K$
is the rank of the n-th homology group of K. The Euler-Poincaré formula states that the Euler characteristic of K is also equal to \( \sum_{n=0}^{\infty} (-1)^n \beta_n \), the alternating sum of the Betti numbers of K.

A complex K is said to be locally finite if each simplex of K intersects only finitely many other simplexes of K. If K is locally finite, then |K| is metrizable. A metric d is given as follows. For a point \( x \in |K| \) and a vertex v of K, let \( v(x) \) be the v-th barycentric coordinate of x. For x and y in |K|, let \( d(x,y) = \sqrt{\sum_{v \in K} (v(x) - v(y))^2} \). Note that the sum is finite, in the sense that almost all of the summands are zero. Thus if K is a locally finite complex (remember that complexes are countable), |K| is a separable metric space. It is shown in [6] that |K| is an ANR.

If \( \sigma \) is an n-simplex of a complex K, then \( Bd(\sigma) \), the boundary of \( \sigma \), is the union of all of the faces of \( \sigma \) of dimension less than n. The interior of \( \sigma \), \( Int(\sigma) \), is equal to \( \sigma \setminus Bd(\sigma) \). If A is a subcomplex of a complex K, then \( (R(A), K) \), or simply R(A) when no confusion can arise, denotes the regular neighborhood of A in K. It is equal to \( \{ \tau \in K^{(2)} | \text{there is some } \sigma \in K^{(2)} \text{ such that } \tau \text{ is a face of } \sigma \text{ and } \sigma \cap A \neq \emptyset \} \), where \( K^{(2)} \) is the second barycentric subdivision of K.
Define $\text{Bd}(\text{R}(A))$ to be the set of simplexes of $\text{R}(A)$ which do not meet $A$. Notice that the definition of regular neighborhood given here differs slightly from that given by J.H.C. Whitehead.

**Lemma 1.4:** If $K$ is a locally finite complex and $A$ is a subcomplex of $K$, then there is a mapping $f: \text{Bd}(\text{R}(A)) \times I \rightarrow \text{R}(A)$ such that $f(x,0) = x$ for $x \in \text{Bd}(\text{R}(A))$, $f(\text{Bd}(\text{R}(A)) \times \{1\}) = f(\text{Bd}(\text{R}(A)) \times I) \cap A$, and $f|\text{Bd}(\text{R}(A)) \times [0,1)$ is a homeomorphism.

**Proof:** For each $n$-simplex $\sigma \in K$ which meets both $A$ and $K \setminus A$, $\sigma \cap \text{Bd}(\text{R}(A))$ is an $(n-1)$-cell $C_{\sigma}$. $C_{\sigma}$ separates $\sigma$ into two $n$-cells, one of which, $C'_{\sigma}$, is equal to $\text{R}(A) \cap \sigma$. Now $\sigma \cap A$ is a face of $\sigma$, and is thus a cell of dimension less than or equal to $n-1$ contained in the boundary of $C'_{\sigma}$. Thus a map $f_{\sigma}: C_{\sigma} \times I \rightarrow C'_{\sigma}$ may be defined which satisfies the conditions of the theorem. A map $f$ satisfying the conditions of the theorem may be constructed by inductively defining a collection of $f_{\sigma}$'s so that $f_{\sigma_1}$ and $f_{\sigma_2}$ agree on $\sigma_1 \cap \sigma_2$, for any $\sigma_1$ and $\sigma_2$. For further details, see [9].

**Lemma 1.5:** If $K$ is a locally finite complex and
A is a subcomplex of $K$, then $A$ is a strong deformation retract of $R(A)$.

**Proof:** Let $f: \text{Bd}(R(A)) \times I \rightarrow R(A)$ be a map as in Lemma 1.4. Then define a deformation $H: R(A) \times I \rightarrow R(A)$ by $H(x, t) = x$ for $x \in A$, and $H(x, t) = f(z, s - st + t)$, where $x = f(z, s)$.

I.6. The homology theory used will be singular theory with integer coefficients, although there will be occasion to use the fact that singular theory and simplicial theory are equivalent for locally finite polyhedra (see [2]). If $f: X \rightarrow Y$ is a map, then $f_*: H_n(X) \rightarrow H_n(Y)$ will be the corresponding induced homomorphism. In most cases the subscript on the induced homomorphism will be omitted.

If $G$ is an abelian group, let $T(G)$ denote the torsion subgroup of $G$. The sequence

$$0 \rightarrow T(G) \rightarrow G \rightarrow G/T(G) \rightarrow 0$$

is split exact for any abelian finitely generated group $G$. For a space $X$ and a positive integer $n$, the $n$-th weak homology group of $X$, denoted by $H^n_w(X)$, is the group $H_n(X)/T(H_n(X))$.

I.7. A CW-complex is a space $X$ together with a collection $\{f_j: I_j \rightarrow X\}_{j \in J}$ of mappings of cells into $X$, such that: i) $X = \bigcup_{j \in J} f_j(I_j)$. ii) For each $j \in J$, 

\( f_j \mid \text{Int}(I_{i_j}) \) is a homeomorphism, and if \( i \neq j \) then
\( f_i(\text{Int}(I_{i_i})) \cap f_j(\text{Int}(I_{i_j})) = \emptyset \). iii) For each \( j \in J \),
\( f_j(\text{Bd}(I_{i_j})) = \bigcup_{k=1}^{m} f_k(I_{i_k}) \), where the dimension of \( I_{i_k} \) is
less than the dimension of \( I_{i_j} \), for each \( k \). iv) \( X \) has
weak topology with respect of \( \{f_j(I_{i_j})\}_{j \in J} \).

The n-skeleton of a CW-complex \( X \) is the union of
all \( f_j(I_{i_j}) \), where the dimension of \( I_{i_j} \) is less than or
equal to \( n \). It is denoted by \( X^n \). A tree is an acyclic
1-complex. It is shown in [6] that every locally finite
tree is contractible. A bouquet is a CW-complex which
is formed by the wedge of a collection of n-spheres, for
some \( n \). If \( X \) is a CW-complex and \( x \in X \), then the star
of \( x \), \( \text{st}(x) \), is the union of all \( f_j(\text{Int}(I_{i_j})) \) such that
\( x \in f_j(I_{i_j}) \). Define \( \text{Bd}(\text{st}(x)) \) to be the union of all
\( f(I_{i_j}) \) in \( \overline{\text{st}(x)} \) which do not contain \( x \). If \( X \) is a
CW-complex and \( Y \) is a space, then a function \( g:X \to Y \) is
continuous if and only if the restriction to \( f_j(I_{i_j}) \) is
continuous, for each \( j \in J \). This follows from the fact
that \( X \) has the weak topology with respect to the \( f_j(I_{i_j}) \)'s.

Notice that every simplicial complex is also a
CW-complex.

I.8. If \( X \) is a space and \( \alpha \) is a cover of \( X \),
there is associated with a a simplicial complex called the nerve of a, N(a). The vertices of N(a) are the elements of a, and vertices $V_1, \ldots, V_n$ of N(a) span a simplex of N(a) if and only if $\cap_{i=1}^{n} V_i \neq \emptyset$. If X is a metric space with metric d, and a is a locally finite open cover of X, there is a map $f:S \to N(a)$, called the barycentric a-map, which is defined as follows. For $x \in X$, let $V_1, \ldots, V_m$ be the elements of a which contain $x$. Let $f(x)$ be the element of the simplex spanned by $V_1, \ldots, V_m$ having barycentric coordinates $b_1, \ldots, b_m$, where $b_j = d(x, X \cup \{j\}) / \sum_{i=1}^{m} d(x, X \cup \{i\})$.

I.9. A space Y is said to dominate a space X if there are maps $f:X \to Y$ and $g:Y \to X$ such that the composition $gf:X \to X$ is homotopic to the identity map of X. In this case, for each positive integer n the homomorphism $g_* f_* : H_n(X) \to H_n(Y)$ is the identity homomorphism, so $f_*$ is a monomorphism and the sequence

$$0 \to H_n(X) \to H_n(Y) \to H_n(Y) / f_* (H_n(X)) \to 0$$

is split exact. Note that if X is a retract of Y, then Y dominates X.
Theorem 1.10: If $X$ is an ANR then $X$ is dominated by a locally finite polyhedron.

The following is a sketch of a proof. Since $X$ is an ANR, there is an open cover $\alpha$ of $X$ such that if $f$ and $g$ are two maps from an arbitrary space $Y$ into $X$ having the property that for any $y \in Y$, $f(y)$ and $g(y)$ are in a common member of $\alpha$, then $f$ and $g$ are homotopic. If $\beta$ is a sufficiently fine locally finite open refinement of $\alpha$, and if $f: X \to N(\beta)$ is the barycentric $\beta$-map, then there is a map $g: N(\beta) \to X$ such that $gf(x)$ and $x$ are in a common member of $\alpha$, for each $x \in X$. Thus $gf: X \to X$ is homotopic to the identity map of $X$.

Notice that if $X$ is compact then a finite $\beta$ may be chosen. If $X$ is compact and if the dimension of $X$ is $n$, then a finite $\beta$ may be chosen so that the dimension of $N(\beta)$ is at most $n$. For further details of the proof, see Chapter IV of [6].

Lemma 1.11: If $K$ is a finite polyhedron such that $H_1(K)$ is not a torsion group, then there is a mapping $f: S^1 \to K$ such that the sequence

$$0 \to H_1(S^1) \xrightarrow{f_*} H_1(K) \to H_1(K)/f_*(H_1(S^1)) \to 0$$

is split exact.
Proof: There is a natural transformation $h$ from the functor $\pi_1$ to the functor $H_1$, called the Hurewicz homomorphism. For the space $K$, $h: \pi_1(K, x_0) \to H_1(K)$ is an epimorphism. Since $K$ is a finite polyhedron, $H_1(K)$ is a finitely generated abelian group. Choose a basis for $H_1(K)$, and let $b$ be a basis element of infinite order. Let $b'$ be an element of $h^{-1}(b)$. There is a map $f:S^1 \to K$ such that the homotopy class $[f]$ is $b'$. Let $a'$ generate the fundamental group of $S^1$, and let $a$ be the corresponding generator of $H_1(S^1)$, under $h$. Let $f_\#: \pi_1(S^1, z) \to \pi_1(K, x_0)$ be the homomorphism induced by $f$. Then, since $[f] = b'$, $f_\#(a') = b'$ or $f_\#(a') = b'^{-1}$.

Suppose, without loss of generality, that $f_\#(a') = b'$. Since $h$ is a natural transformation, $f_\#(a') = f_\#(h(a')) = h f_\#(a') = h(b') = b$. Thus the sequence

$$0 \to H_1(S^1) \to H_1(K) \to H_1(K)/f_\#(H_1(S^1)) \to 0$$

is split exact.

Note that the above lemma also holds when $K$ is a space which is dominated by a finite polyhedron.

The following theorems are quoted without proof.
Theorem I.1.2: (Borsuk's Homotopy Extension Theorem)

If $Y$ is an ANR, then every closed subspace $A$ of an arbitrary metrizable space $X$ has the homotopy extension property in $X$ with respect to $Y$. In other words, every partial homotopy $H:A \times I \to Y$ of an arbitrary map $f:X \to Y$ has an extension $F:X \times I \to Y$, such that $F(x,0) = f(x)$.

A proof of this theorem can be found in Chapter IV of [6].

Theorem I.1.3: Let $G$ be a finitely generated free abelian group of rank $n$, and let $H$ be a subgroup of $G$. Then $H$ is a finitely generated free abelian group of rank $m \leq n$, and there are bases $\{a_1, \ldots, a_n\}$ for $G$ and $\{b_1, \ldots, b_m\}$ for $H$, such that there exist integers $k_1, \ldots, k_m$ with $b_i = k_i a_i$, for $i = 1, \ldots, m$.

A proof of this theorem can be found in Chapter 5 of [4].

Theorem I.1.4: (Wecken) If $K$ is a connected finite polyhedron without local separating points and $X(K) = 0$, then $K$ admits a fixed point free map.

Corollary I.1.5: (Wecken) Let $K$ be a finite polyhedron with the property that no finite collection of points separates $K$. Then $K$ admits a fixed point free
map if \( \chi(K) = 0 \).

Theorem I.14 and Corollary I.15 are found in [8].
CHAPTER II

RETRACTING ANR'S ONTO SPHERES

This chapter is concerned with sufficient conditions for an m-sphere $S$ contained in an ANR $X$ to be a retract of $X$. It is shown that if $i:S \rightarrow X$ is the inclusion map, and the sequence

$$0 \rightarrow H_m(S) \xrightarrow{i_*} H_m(X) \rightarrow H_m(X)/i_*(H_m(S)) \rightarrow 0$$

is split exact, then under certain additional conditions $S$ is a retract of $X$. Paragraph I.7 insures that these conditions on the homology of $X$ are also necessary. The theorems are proved first for the case when $X$ is a locally finite polyhedron, and then they are generalized to the case when $X$ is a metrizable space which is dominated by such a polyhedron. Several preliminary lemmas are needed.

**Lemma II.1:** Let $B$ be a bouquet of m-spheres \( \{S_\alpha\}_{\alpha \in \Lambda} \), each with a given orientation. Let $S_\beta$ be one of the m-spheres of $B$, and let $i:S_\beta \rightarrow B$ be the inclusion map. Then for any homomorphism $h:H_m(B) \rightarrow H_m(S_\beta)$ such that $hi_*$ is the identity homomorphism of $H_m(S_\beta)$, there is a retraction $r:B \rightarrow S_\beta$ such that $r_* = h$. 

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Proof: For each $\alpha \in \Lambda$, let $[S_{\alpha}]$ be the fundamental homology class of $H_m(S_{\alpha})$. Then $h([S_{\alpha}]) = n_{\alpha}[S_{\beta}]$, where $n_{\alpha}$ is some integer. Let $\{z\} = \cap_{\alpha \in \Lambda} S_{\alpha}$ and for each $\alpha$ let $f_{\alpha}: S_{\alpha} \to S_{\beta}$ be a mapping of degree $n_{\alpha}$ such that $f_{\alpha}(z) = z$. In particular, let $f_{\beta}$ be the identity map of $S_{\beta}$. Define $r:B \to S_{\beta}$ so that $r|S_{\alpha} = f_{\alpha}$ for each $\alpha$. The homomorphisms $h$ and $r_*$ agree on $[[S_{\alpha}]]_{\alpha \in \Lambda}$, and this set is a basis for $H_m(B)$, so $h = r_*$. Since $f_{\beta}$ is the identity map of $S_{\beta}$, $r$ is a retraction.

Lemma II.2: Let $K$ be an $(m+1)$-dimensional CW-complex having an $m$-dimensional bouquet of spheres as $m$-skeleton. Let $S$ be one of the $m$-spheres of $K^m$. If the inclusion map $i:S \to K$ yields a split exact sequence

$$0 \to H_m(S) \xrightarrow{i_*} H_m(K) \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \to 0$$

then there is a retraction $r:K \to S$ such that $r_* = h$.

Proof: Let $j:K^m \to K$ and $t:S \to K^m$ be inclusion maps. Then the following diagram commutes and the rows are exact, where $I$ is the identity homomorphism of $H_m(S)$. 

```latex
\begin{align*}
0 & \xrightarrow{i_*} H_m(S) \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \to 0 \\
0 & \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \to 0 \\
0 & \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \xrightarrow{h} H_m(K) \xrightarrow{i_*} H_m(S) \to 0
\end{align*}
```
Let $\hat{h} = h j_*$. Then $\hat{h}$ splits the upper row of the diagram. Thus by Lemma II.1 there is a retraction $\hat{\iota}:K^m \rightarrow S$ such that $\hat{\iota}_* = \hat{h}$. Note that $\ker(j_*) \subseteq \ker(\hat{\iota}_*)$.

If $\alpha$ is an $(m+1)$-cell of $K$, then the homology class $[\text{Bd}(\alpha)]$ is an element of $\ker(j_*)$, and thus is an element of $\ker(\hat{\iota}_*)$. The groups $H_m(S)$ and $\pi_m(S)$ are naturally isomorphic under the Hurewicz homomorphism, so the homotopy class of $\text{Bd}(\alpha)$ is an element of the kernel of $\hat{\iota}_#:\pi_m(K^m) \rightarrow \pi_m(S)$. Thus $\hat{\iota}$ may be extended to a retraction $\hat{r}:K \rightarrow S$. Now $rj = \hat{r}$, so $r_*j_* = \hat{r}_* = \hat{h} = hj_*$. The homomorphism $j_*$ is an epimorphism, so $r_* = h$.

**Lemma II.3:** Let $K$ be a locally finite complex and let $L$ be a contractible subcomplex of $K$. Form the CW-complex $M$ by identifying $L$ to a point, and let $\eta:K \rightarrow M$ be the quotient map. Then $\eta$ is a homotopy
Proof: This proof follows the proof of the lemma for the case when K is finite, found in [9].

Let R(L) be the regular neighborhood of L in K. Adjoin C(R(L)), the cone over R(L), to K and call the resulting space P. By Lemma I.5, L is a strong deformation retract of R(L), so since L is contractible, R(L) is contractible. Thus for any point x ∈ R(L), [x] is a deformation retract of R(L). Let F:R(L) × I → R(L) be such a deformation, and define a map r:C(R(L)) → R(L) by r(x,t) = F(x,t). Then r is a retraction. Now define a map H:(C(R(L)) × [0,1)]∪(R(L) × I) → C(R(L)) by H(x,t) = x for x ∈ C(R(L)) and 0 ≤ t < 1, and H(x,1) = r(x). Since C(R(L)) is contractible, the homotopy groups of C(R(L)) are all trivial. The space C(R(L)) × I is a CW-complex, so an extension G:C(R(L)) × I → C(R(L)) of H may be constructed inductively by extending G to the 1-skeleton of C(R(L)) × I, then to the 2-skeleton, etc. Thus R(L) is a strong deformation retract of C(R(L)). From this it follows that K is a deformation retract of P. The inclusion map j:K → P is therefore a homotopy equivalence.

Let f:Bd(R(L)) × I → R(L) be a mapping as in Lemma I.4. Define a map H:C(Bd(R(L))) × I∪R(L) × [0,1] → C(R(L)) as follows. Let H(x,t) = x for x ∈ C(Bd(R(L))) and
t \in I$, let $H(x,0) = x$ for $x \in R(L)$, let $H(x,1) = c$, the coning point, for $x \in L$, and let $H(f(x,t),1) = (x,t)$ for $f(x,t) \in R(L)$. Since $C(R(L))$ is contractible, $H$ has an extension $G: C(R(L)) \times I \to C(R(L))$. Thus $C(\text{Bd}(R(L)))$ is a strong deformation retract of $C(R(L))$, and it follows that there is a deformation $F: P \times I \to P$ of $P$ into $Q = (K \setminus R(L)) \cup C(\text{Bd}(R(L)))$ which is an extension of $G$. Let $F_1: P \to Q$ be defined by $F_1(p) = F(p,1)$. Then $F_1$ is a homotopy equivalence.

Note that $F_1j: K \to Q$ is one-to-one on $K \setminus L$ and $F_1j(x) = c$ for $x \in L$. The spaces $Q$ and $M$ both have the weak topology with respect to their closed cells, so $Q$ and $M$ are homeomorphic and $F_1j = \eta$. Since $j$ and $F_1$ are both homotopy equivalences, so is $\eta$.

The following theorem is one of the main results of this chapter.

**Theorem 11.4:** Let $K$ be a finite $(m+1)$-complex containing an $m$-sphere $S$ in its $m$-skeleton. If the inclusion map $i: S \to K$ yields a split exact sequence

\[
0 \to H_m(S) \xrightarrow{i_*} H_m(K) \to H_m(K)/i_*(H_m(S)) \to 0
\]

then $S$ is a retract of $K$.

**Proof:** Let $a$ be an $m$-cell of $S$. Let $L$ be the
CW-complex formed by identifying $\Box\alpha$ to a point, and let $\mu: K \to L$ be the quotient map. By Lemma II.3, $\mu$ is a homotopy equivalence, so there is a split exact sequence.

$$0 \to H_m(\mu \circ \iota) \to H_m(L) \to H_m(L)/H_m(\mu(S)) \to 0.$$  

Since sequence (2) splits, the sequence

$$0 \to \omega H_m(\mu(S)) \to \omega H_m(L) \to H_m(L)/\omega H_m(\mu(S)) \to 0$$

is also split exact.

Form the CW-complex $M$ from $L$ by identifying the $(m-1)$-skeleton of $L$ to a point, and let $\eta: L \to M$ be the quotient map. Suppose that the weak homology class $\omega(\eta \circ \iota) \in H_m(M)$ is divisible by the integer $k$. Then there exists a chain $b' \in C_m(M)$ such that $\omega[k_1 b'] = \omega(\eta \circ \iota)$. Therefore, $k_1 b' - \eta \circ \iota$ is a weak boundary. In other words, there exists a $c' \in C_{m+1}(M)$ and an integer $k_2$ such that $k_2 (k_1 b' - \eta \circ \iota) = \partial (c')$. The map $\eta$ induces isomorphisms of $m$-chains and $m$-boundaries, so there are chains $b$ and $c$ in $C_m(L)$ and $C_{m+1}(L)$, respectively, such that $k_2 (k_1 b - \eta \circ \iota) = \partial (c)$. Thus $k_1 b - \eta \circ \iota$ is a weak boundary, so $\omega[k_1 b - \eta \circ \iota] \in \omega H_m(L)$ is divisible by $k_1$. Since sequence (3) splits, $k_1 = \pm 1$. Thus $\omega[\eta \circ \iota]$ is divisible only by $\pm 1$. 
From the above conclusion and Theorem I.13, it can be seen that the sequence

\[ 0 \rightarrow \text{w}_m(\eta \mu(S)) \rightarrow \text{w}_m(M) \rightarrow \text{w}_m(M)/\text{w}_m(\eta \mu(S)) \rightarrow 0 \]

is split exact. The following diagram commutes, and \( I \) is the identity homomorphism.

Since the bottom row and the middle column are split exact, the top row is also split exact. Thus by Lemma II.2 there is a retraction \( \hat{\eta}:M \rightarrow \eta \mu(S) \). Now the map \( \eta \mu|S \) is a homotopy equivalence, so there is a homeomorphism \( \theta: \eta \mu(S) \rightarrow S \) such that \( \theta \eta \mu|S \) is of degree 1. Thus \( \hat{\eta} \eta \mu|S \) is homotopic to the identity mapping of \( S \), so by Theorem I.12 \( \hat{\eta} \eta \mu:K \rightarrow S \) is homotopic to a retraction \( r:K \rightarrow S \).

II.5. The dimension restriction on \( K \) in Theorem II.4 is necessary. The complex projective plane, \( \mathbb{C}P^2 \),
contains a 2-sphere $S$ such that the inclusion map $i : S \to \mathbb{CP}^2$ induces an isomorphism $i_* : H_2(S) \to H_2(\mathbb{CP}^2)$.

However, $S$ cannot be a retract of $\mathbb{CP}^2$, because $\mathbb{CP}^2$ has the fixed point property and $S$ does not.

The following theorem is a generalization of Theorem II.4 in the case when $S$ is a simple closed curve.

**Theorem II.5:** Let $K$ be a complex and let $S$ be a simple closed curve contained in $K^1$. If the inclusion map $i : S \to K$ yields a split exact sequence

$$(1) \quad 0 \to H_1(S) \xrightarrow{i_*} H_1(K) \to H_1(K)/i_*(H_1(S)) \to 0$$

then $S$ is a retract of $K$.

**Proof:** Since sequence (1) is split exact, and the inclusion $K^2 \to K$ induces an isomorphism between $H_1(K^2)$ and $H_1(K)$, the sequence

$$(2) \quad 0 \to H_1(S) \xrightarrow{i_*} H_1(K^2) \to H_1(K^2)/i_*(H_1(S)) \to 0$$

is also split exact. Let $\alpha$ be a 1-cell of $S$, and let $T$ be a maximal tree in $K^1$ containing $S \setminus \alpha$. Form the CW-complex $M$ from $K^2$ by identifying $T$ to a point, and let $\mu : K^2 \to M$ be the quotient map. By Lemma II.3 the map $\mu$ is a homotopy equivalence, so the sequence

$$(3) \quad 0 \to H_1(\mu(S)) \to H_1(M) \to H_1(M)/H_1(\mu(S)) \to 0$$
is split exact. Thus by Lemma II.2 there is a retraction \( \hat{\mu}: M \to \mu(S) \). The map \( \hat{\mu}|_S \) is a homotopy equivalence, so there is a homeomorphism \( \theta: \mu(S) \to S \) such that \( \theta|_S: S \to S \) is a degree 1. This map is therefore homotopic to the identity map of \( S \). It follows from Theorem I.12 that \( \theta\mu: K^2 \to S \) is homotopic to a retraction \( r_0: K^2 \to S \). Since \( \pi_1(S) = 0 \) for \( i > 1 \), \( r_0 \) may be extended to \( K^3 \), then to \( K^4, \ldots \), and inductively, to \( K \).

II.7. Let \( K \), \( S \), and \( \mu \) be as in Theorem II.6. The following diagram then commutes, the rows are exact, and the vertical homomorphisms are isomorphisms.

\[
\begin{array}{ccc}
0 & \to & H_1(S) \\
\downarrow & & \downarrow i_* \\
0 & \to & H_1(K) \\
\downarrow & & \downarrow \mu_* \\
0 & \to & H_1(M) \\
\downarrow & & \downarrow \hat{\mu} \\
0 & \to & H_1(\mu(S)) \\
\end{array}
\]

Suppose that \( h: H_1(K) \to H_1(S) \) splits the top row of the diagram. Define \( \hat{h}: H_1(M) \to H_1(\mu(S)) \) so that

\[
\begin{array}{ccc}
H_1(S) & \xleftarrow{h} & H_1(K) \\
\downarrow & & \downarrow h \\
H_1(\mu(S)) & \xleftarrow{\hat{h}} & H_1(M) \\
\end{array}
\]
commutes. The bottom row is then split by $\hat{\pi}$. By Lemma II.2 a retraction $\hat{\pi}:M \rightarrow \mu(S)$ may be chosen so that $\hat{\pi}_* = \hat{\pi}$. Therefore a retraction $r:K \rightarrow S$ may be chosen so that $r_* = h$.

Theorem II.4 may be generalized as follows.

Theorem II.8: Let $X$ be a metrizable space containing an $m$-sphere $S$. If the inclusion map $i:S \rightarrow X$ yields a split exact sequence

$$0 \rightarrow \mathcal{H}_m(S) \xrightarrow{i_*} \mathcal{H}_m(X) \xrightarrow{i_*} \mathcal{H}_m(S) \mathcal{H}_m(X) \rightarrow 0$$

and if $X$ is dominated by a finite $(m+1)$-dimensional polyhedron, then $S$ is a retract of $X$.

Proof: Let $K$ be an $(m+1)$-dimensional polyhedron which dominates $X$, and let $f:X \rightarrow K$ and $g:K \rightarrow X$ be maps such that $gf:X \rightarrow X$ is homotopic to the identity map of $X$. Let $h:S \rightarrow K$ be a piecewise linear approximation of $f|S$, and let $M(h)$ be the mapping cylinder of $h$. Note that there is an inclusion $S \rightarrow M(h)$, which is homotopic to the composition $jh:S \rightarrow M(h)$, where $j:X \rightarrow M(h)$ is the inclusion. Sequence (1) is split exact, and the sequence

$$0 \rightarrow \mathcal{H}_m(X) \xrightarrow{f_*} \mathcal{H}_m(K) \xrightarrow{f_*} \mathcal{H}_m(K) / \mathcal{H}_m(X) \rightarrow 0$$

is also split exact. Thus the sequence...
The diagram

\[
\begin{array}{c}
0 \rightarrow H_m(S) \rightarrow H_m(K) \rightarrow H_m(K)/H_m(S) \rightarrow 0 \\
\uparrow h_* \uparrow \downarrow h_* \downarrow \uparrow \downarrow \\
0 \rightarrow H_m(S) \rightarrow H_m(M(h)) \rightarrow H_m(M(h))/H_m(S) \rightarrow 0
\end{array}
\]

commutes, and the vertical homomorphisms are isomorphisms. Thus the bottom row is split exact. The space \( M(h) \) is a finite \((m+1)\)-dimensional polyhedron containing \( S \) in the \( m \)-skeleton of some triangulation, so by Theorem II.4 there is a retraction \( \hat{r}:M(h) \rightarrow S \). The map \( \hat{r}i:S \rightarrow S \) is of degree 1, and is therefore homotopic to the identity map of \( S \). Thus by Theorem I.12 there is a retraction \( r:X \rightarrow S \).

**Corollary II.9:** Let \( X \) be a compact \((m+1)\)-dimensional ANR containing an \( m \)-sphere \( S \). If the inclusion map \( i:S \rightarrow X \) yields a split exact sequence

\[
0 \rightarrow H_m(S) \overset{i_*}{\rightarrow} H_m(X) \rightarrow H_m(X)/H_m(S) \rightarrow 0
\]

then \( S \) is a retract of \( X \).

**Proof:** Theorem I.10 assures that \( X \) satisfies the hypotheses of Theorem II.8.

Theorem II.6 has a similar generalization.
Theorem II.10: Let $X$ be a metrizable space and let $S$ be a simple closed curve contained in $X$. If the inclusion map $i:S \to X$ yields a split exact sequence

$$0 \to H_1(S) \xrightarrow{i_*} H_1(X) \to H_1(X)/H_1(S) \to 0$$

and if $X$ is dominated by a locally finite polyhedron, then $S$ is a retract of $X$.

Proof: Choose a dominating polyhedron for $X$ and proceed as in the proof of Theorem II.8, but apply Theorem II.6 rather than II.4.

Corollary II.11: Let $X$ be an ANR, and let $S$ be a simple closed curve contained in $X$. If the inclusion map $i:S \to X$ yields a split exact sequence

$$0 \to H_1(S) \xrightarrow{i_*} H_1(X) \to H_1(X)/H_1(S) \to 0$$

then $S$ is a retract of $X$.

For a final generalization of Theorem II.6 we need the following lemma.

Lemma II.12: Let $T$ be a product of circles $S_1, \ldots, S_n$, and let $r:T \to S_1$ be a mapping such that $r_*(H_1(S_i)) = 0$ for $i = 2, \ldots, n$. Then $r$ is homotopic to the projection map $p_1:T \to S_1$. 
Proof: Consider $T$ as a CW-complex having a bouquet as its 1-skeleton. The mapping $r|T^1$ is homotopic to retraction $s:T^1 \to S_1$ which is a constant map on each of the circles $S_2, \ldots, S_n$. Theorem I.12 says that $s$ may be extended to a retraction $s:T \to S_1$. Define a map $H:(T \times [0,1]) \cup (T \times I) \to S_1$ by $H(x,0) = \hat{r}(x)$, for $x \in T$, $H(x,t) = \hat{r}(x)$ for $x \in T^1$ and $t \in I$, and $H(x,1) = \pi_1(x)$ for $x \in T$. Then $H$ may be extended inductively to a homotopy $F:T \times I \to S_1$, since $\pi_1(S_i) = 0$ for all $i > 1$.

Theorem II.13: Let $X$ be a metrizable space containing a subspace $T$ which is a product of circles $S_1, \ldots, S_n$. If the inclusion map $i:T \to X$ yields a split exact sequence

$$(1) \quad 0 \to H_1(T) \overset{i_*}{\longrightarrow} H_1(X) \to H_1(X)/i_*(H_1(T)) \to 0$$

and if $X$ is dominated by a locally finite polyhedron, then $T$ is a retract of $X$.

Proof: Choose a base point $z$ for $T$. Then there is a natural inclusion $S_i \to T$ for each $i$, and the sequence

$$(2) \quad 0 \to H_1(S_i) \to H_1(T) \to H_1(T)/H_1(S_i) \to 0$$
is split exact for each $i$. Since sequence (1) is split exact, the sequence

$$(3) \quad 0 \rightarrow H_1(S_i) \rightarrow H_1(X) \rightarrow H_1(X)/H_1(S_i) \rightarrow 0$$

is also split exact, for each $i$. By Theorem II.8 there is for each $i$ a retraction $\hat{r}_i : X \rightarrow S_i$. It follows from paragraph II.7 that the $\hat{r}_i$'s may be chosen so that $\hat{r}_i^*(H_1(S_j)) = 0$ for $i \neq j$. Thus by Lemma II.12, $\hat{r}_i | T$ is homotopic to the projection map $p_i : T \rightarrow S_i$. By Theorem I.12 there is a retraction $r_i : X \rightarrow S_i$ such that $r_i | T = p_i$, for each $i$. Let $r : X \rightarrow T$ be the product of the $r_i$'s. Then $r$ is a retraction.
CHAPTER III

APPLICATIONS IN THE THEORY OF FIXED POINTS

The following was for many years an outstanding conjecture in the fixed point theory of polyhedra: If $X$ and $Y$ are finite polyhedra with the fixed point property, then $X \times Y$ also has the fixed point property.

A counterexample to this conjecture has recently been given by William Lopez [7]. Lopez proceeds as follows. He first constructs a polyhedron $X$ with even Euler characteristic having the fixed point property. In fact, $\chi(X) = 8$. He then lets $Z = X \vee Y$, the wedge of $X$ and $Y$, where $Y$ is the suspension of complex projective 8-space. Now $Y$ has the fixed point property, and the wedge of two spaces having the fixed point property also has the fixed point property. Therefore $Z$ has the fixed point property. Since $\chi(Y) = -7$, it follows that $\chi(Z) = 0$. The Betti numbers of $Z \times I$ are the same as those of $Z$, so $\chi(Z \times I) = 0$. The space $Z \times I$ has no local separating points, so it follows from Theorem I.14 that $Z \times I$ does not have the fixed point property. The dimension of $X$ is 8, and the dimension of $Y$ is 17, so the dimension of $Z$ is also 17.
This construction has led R.H. Bing [1] to ask, "Is there a two dimensional polyhedron with the fixed point property which has even Euler characteristic?" and more specifically, is there such a polyhedron with Euler characteristic zero? If $K$ were such a polyhedron, then Theorem I.14 assures that $K \times I$ would admit a fixed point free map.

If $K$ is a 2-complex and $\chi(K) \leq 0$, then the first Betti number of $K$ is not zero. Thus $w_{1}(K) \neq 0$. If $w_{1}(K) \neq 0$, then from Lemma I.11 we see that there is a map $f:S^{1} \rightarrow K$ such that the sequence

$$0 \rightarrow H_{1}(S^{1}) \xrightarrow{f_{*}} H_{1}(K) \rightarrow H_{1}(K)/f_{*}(H_{1}(S^{1})) \rightarrow 0$$

is split exact. If there is such a map $f$ which is also an imbedding, then by Theorem II.8 $f(S^{1})$ is a retract of $K$, so $K$ does not have the fixed point property. This is indeed the case when $K$ is a compact manifold.

Theorem III.1: If $M$ is a compact connected manifold with boundary such that $w_{1}(M) \neq 0$, then $M$ retracts to a simple closed curve.

Proof: If the dimension of $M$ is 1, the theorem is trivial.

Suppose that $M$ is of dimension 2. If $M$ has
empty boundary, then \(M\) is a 2-sphere with handles and cross-caps. Since \(w_1(M) \neq 0\), \(M\) has a handle. Thus there is a connected 2-manifold \(M' \subseteq M\) such that \(M = M' \cup (S^1 \times I)\), and \(M' \cap (S^1 \times I) = S^1 \times (0,1)\). Choose a point \(z \in S^1\), and let \(A\) be an arc in \(M'\) from \((z,0)\) to \((z,1)\). By Tietze's extension theorem there is a retraction \(r_1 : M' \to A\) such that \(r_1(S^1 \times \{0\}) = (z,C)\) and \(r_1(S^1 \times \{1\}) = (z,1)\). There is also a retraction \(r_2 : S^1 \times I \to \{z\} \times I\) which is the map which takes a point \((x,t)\) to the point \((z,t)\). Define \(r : M \to A \cup \{(z) \times I\}\) so that \(r|_{M'} = r_1\) and \(r|_{S^1 \times I} = r_2\). Then \(r\) is a retraction of \(M\) onto the simple closed curve \(A \cup \{(z) \times I\}\).

Suppose now that the boundary of \(M\) is not empty. The boundary of \(M\), \(\text{Bd}(M)\), is then a compact 1-dimensional manifold with empty boundary. Thus \(\text{Bd}(M)\) is the union of disjoint simple closed curves \(C_1, \ldots, C_n\). Form the manifold \(M'\) from \(M\) by attaching a collection of 2-cells \(I_1, \ldots, I_n\) to the boundary of \(M\), so that \(\text{Bd}(I_i) = C_i\) for each \(i\). If \(w_1(M') \neq 0\) then there is an imbedding \(f : S^1 \to M'\) such that \(f(S^1)\) is a retract of \(M\). Now \(R(I_i)\) is a 2-cell, and \(I_i \subseteq \text{Int}(R(I_i))\), for each \(i\). Choose for each \(i\)
a point \( x_1 \in \text{Int}(I_1) \) such that \( x_1 \not\in f(S^1) \). Then there is a 2-cell \( V_1 \subset I_1 \) such that \( x_1 \in V_1 \) and \( V_1 \cap f(S^1) = \emptyset \). There is an isotopy \( H_1 : R(I_1) \times I \rightarrow R(I_1) \) which takes \( V_1 \) onto \( I_1 \) but leaves \( x_1 \) and \( \text{Bd}(R(I_1)) \) fixed. Let \( H : M' \times I \rightarrow M' \) be defined so that \( H(x,t) = x \) for \( x \not\in \bigcup_{i=1}^{n} R(I_1) \), and \( H(x,t) = H_1(x,t) \) for \( x \in R(I_1) \). Then the map \( h : S^1 \rightarrow M' \) defined by \( h(z) = H(f(z),1) \) is an imbedding homotopic to \( f \). Since \( f(S^1) \) is a retract of \( M' \), the sequence

\[
0 \rightarrow H_1(S^1) \rightarrow H_1(M') \rightarrow H_1(M')/f_*(H_1(S^1)) \rightarrow 0
\]

is split exact. Therefore the sequence

\[
0 \rightarrow H_1(S^1) \rightarrow H_1(M') \rightarrow H_1(M')/h_*(H_1(S^1)) \rightarrow 0
\]

is split exact, and thus \( h(S^1) \) is a retract of \( M' \). But \( h(S^1) \subset M \), so \( h(S^1) \) is a retract of \( M \). If \( H_1(M') = 0 \) then \( H_1(M') \) is a torsion group. In that case \( M' \) is either the projective plane or the 2-sphere. Thus \( M'' = M \cup I_1 \cup \ldots \cup I_{n-1} \) is either a Mobius band or a 2-cell. In the first case \( M'' \) retracts to a simple closed curve, so \( M \) also retracts to a simple closed curve. In the second case, \( M \neq M'' \) because \( H_1(M) \neq 0 \). Thus \( M \) is a
2-cell with holes poked in it, so $M$ retracts to a simple closed curve.

Now let us consider the case when the dimension of $M$ is greater than or equal to 3. Let $d$ be a metric for $M$. $M$ is a compact ANR (see [6]), so there exists a number $\epsilon > 0$ such that if $f: X \to M$ and $g: X \to M$ are maps of an arbitrary space $X$ into $M$ and $d(f(x), g(x)) < \epsilon$ for every $x \in X$, then $f$ and $g$ are homotopic. Cover $M$ with finitely many Euclidean neighborhoods $E_1, \ldots, E_n$, each having diameter less than $\epsilon$. Let $f: I \to M$ be a map such that $f(0) = f(1)$. Then $\{f^{-1}(E_i)\}_{i=1}^n$ is a cover of $I$. Let $\delta$ be a Lebesgue number for this cover, and let

$\{0 = x_0, x_1, \ldots, x_m = 1\}$ be a partition of $I$ of mesh less than $\delta$. Now $f([x_0, x_1])$ is contained in some $E_{i_1}$. Let $A_1$ be an arc in $E_{i_1}$, with endpoints $z_0$ and $z_1$, such that $z_0 = f(x_0)$, and $z_1 \in E_j$ if and only if $f(x_1) \in E_j$, for $j = 1, \ldots, n$. There is some $i_2$ such that $f([x_1, x_2]) \subset E_{i_2}$. Let $A_2$ be an arc in $E_{i_2}$ with endpoints $z_1$ and $z_2$, such that $A_1 \cap A_2 = \{z_1\}$ and $z_2 \in E_j$ if and only if $f(x_2) \in E_j$, for $j = 1, \ldots, n$. Inductively choose such a collection $(A_i)_{i=1}^{m-1}$ so that $A_j \cap (\cup_{i=1}^{j-1} A_i) = \{z_{j-1}\}$, and finally choose $A_m$ with endpoints $z_{m-1}$ and $z_0$ such that $A_m \cap (\cup_{i=1}^{m-1} A_i) = \{z_0, z_{m-1}\}$. Let
$g: I \rightarrow \bigcup_{i=1}^{m} A_i$ be a mapping such that for each $i = 1, \ldots, m$, $g|_{[x_{i-1}, x_i]}$ is a homeomorphism onto $A_i$ with $g(x_i) = z_i$.

Then for $x \in I$, $d(g(x), f(x)) < \epsilon$. Now $f$ and $g$ are closed paths, and they may therefore be considered as mappings of $S^1$ into $M$. Considered as mappings of $S^1$, $f$ and $g$ are homotopic. The map $g$ is an imbedding of $S^1$.

Since $w_1(M) \neq 0$, there is a mapping $f: S^1 \rightarrow M$ such that the sequence

$$0 \rightarrow H_1(S^1) \xrightarrow{f_*} H_1(M) \rightarrow H_1(M) / f_*(H_1(S^1)) \rightarrow 0$$

is split exact. Let $g: S^1 \rightarrow M$ be an imbedding which is homotopic to $f$. Then the sequence

$$0 \rightarrow H_1(S^1) \xrightarrow{g_*} H_1(M) \rightarrow H_1(M) / g_*(H_1(S^1)) \rightarrow 0$$

is also split exact. Thus by Corollary II.9, $g(S^1)$ is a retract of $M$.

**Conjecture III.2:** Theorem III.1 holds when $M$ is an arbitrary finite polyhedron.

**Conjecture III.3:** If $K$ is a finite 2-complex such that $\chi(K) = 0$, then there is an imbedding $i: S^1 \rightarrow K$ such that the sequence
$$0 \rightarrow H_1(S^1) \xrightarrow{i_*} H_1(X) \rightarrow H_1(X)/i_*(H_1(S^1)) \rightarrow 0$$

is split exact.

**Conjecture III.4:** If $K$ is a finite 2-complex such that $\chi(K) < 0$, then there is an imbedding $i : S^1 \rightarrow X$ such that the sequence

$$0 \rightarrow H_1(S^1) \rightarrow H_1(X) \rightarrow H_1(X)/i_*(H_1(S^1)) \rightarrow 0$$

is split exact.

**Conjecture III.5:** If $K$ is a finite 2-complex such that $\chi(K) = 0$, then $K$ does not have the fixed point property.

The following three results are related to these conjectures.

**Lemma III.6:** Let $K$ be a complex containing a point $x$ which is a local cut point of $K$ but which is not a cut point of $K$. Then $K$ retracts to a simple closed curve.

**Proof:** Since $x$ is a local cut point of $K$, $\text{st}(x) \setminus \{x\}$ is not connected. Let $C_1$ and $C_2$ be sets which form a separation of $\text{st}(x) \setminus \{x\}$, and let $A_1$ be an arc in $\text{st}(x) \cup \{y_1, y_2\}$ from $y_1$ to $y_2$, where
$y_1 \in C_1 \setminus \text{st}(x)$ and $y_2 \in C_2 \setminus \text{st}(x)$. The point $x$ is not a cut point of $K$, so $K \setminus \text{st}(x)$ is connected. Let $A_2$ be an arc in $K \setminus \text{st}(x)$ from $y_2$ to $y_1$, and let $S = A_1 \cup A_2$.

Since $\text{st}(x) \setminus \{x\}$ is not connected, $\text{Bd} \left( \text{st}(x) \right)$ is not connected. There is thus a retraction $r_0 : \text{Bd} \left( \text{st}(x) \right) \rightarrow \{y_1, y_2\}$. This map may be extended to retractions $r_2 : K \setminus \text{st}(x) \rightarrow A_2$ and $r_1 : \text{st}(x) \rightarrow A_2$. Let $r : K \rightarrow S$ be defined so that $r \mid \text{st}(x) = r_1$ and $r \mid (K \setminus \text{st}(x)) = r_2$.

**Lemma III.7:** If $K$ is a finite complex, $V$ is the set of vertices of $K$, and $C$ is the closure of a component of $K \setminus V$, then $C$ is a retract of $K$.

**Proof:** Let $T$ be a maximal tree in $C^1$. Since $T$ is an absolute retract, there is a retraction $r_0 : (K \setminus C) \cup T \rightarrow T$. The map $r_0$ may be extended to a retraction $r : K \rightarrow C$.

**Lemma III.8:** Suppose that $K$ is a finite complex with the fixed point property having Euler characteristic zero, and let $V$ be the set of vertices of $K$. Let $C_1$, $\ldots$, $C_n$ be the closures of the components of $K \setminus V$. Then each $C_i$ has the fixed point property, and some $C_i$ has negative Euler characteristic.

**Proof:** By Lemma III.7 each $C_i$ is a retract of $K$,
so each $C_i$ has the fixed point property. From Lemma III.6 it follows that every local cut point of $K$ is also a cut point of $K$. Thus by induction on the number of $C_i$'s we see that the nerve $N((C_i)_{i=1}^n)$ is a finite tree. There is therefore some $C_j$ such that $C_j \cap (K \setminus C_j)$ is a single point. It follows by induction on the number of $C_i$'s that $\chi(K) = \sum_{i=1}^n \chi(C_i) - n + 1$. Thus $\sum_{i=1}^n \chi(C_i) = n - 1$. By Lemma III.6 no $C_i$ contains a local cut point, since each $C_i$ has the fixed point property. From Theorem I.14 it follows that $\chi(C_i) \neq 0$ for each $i$. Thus for some $i$, $\chi(C_i) < 0$.

III.9 The following implications hold.

\[
\begin{array}{ccc}
III.2 & 1 & III.3 \\
III.5 & 2 & \end{array}
\]

III.2 \perp III.3

Numbers 1, 2, 3, and 4 are obvious. Lemma III.8 shows that a counterexample to III.5 gives a counterexample to III.4, and this establishes implication 5.

The following argument establishes implication 6. Suppose that $K$ is a finite complex and $\mathcal{H}_1(K) \neq 0$. Then
$\omega H_1(K^2) \neq 0$. The proof proceeds by induction on the number of 2-simplexes of $K^2$. If $K^2$ has no 2-simplexes then $K^2$ retracts to a simple closed curve. Since $K^2$ has no 2-simplexes, $K^2 = K$ so this is a retraction of $K$ onto $S$. Suppose that the theorem holds for all 2-complexes having at most $(n-1) \geq 0$ 2-simplexes, and suppose that $K^2$ has $n$ 2-simplexes. If $\omega H_2(K^2) = 0$, then $\chi(K^2) \leq 0$, so by III.3 and III.4 $K^2$ retracts to a simple closed curve.

If $\omega H_2(K^2) \neq 0$, let $\alpha$ be a 2-cycle in $K^2$, and let $\sigma$ be a 2-simplex which has non-zero coefficient in $\alpha$. Let $L = K^2\sigma$. Then $\omega H_1(L) \neq 0$, so by the induction hypothesis there is a retraction $r_0$ of $L$ onto a simple closed curve $S$. Now $\alpha = \sum_{i=1}^{m} n_i \sigma_i + n \sigma$, so $n \delta \sigma = -\delta (\sum_{i=1}^{m} n_i \sigma_i)$. Thus $\text{Bd}(\sigma)$ represents a torsion element of $H_1(S)$. Therefore $r_0^*(\text{Bd}(\sigma))$ is a torsion element of $H_1(S)$, so $r_0^*(\text{Bd}(\sigma)) = 0$. Thus $r_0$ may be extended to $K^2$. But then $r_0$ may be extended to all of $K$.

**Theorem III.10:** If $X$ is a metrizable space which is dominated by a finite polyhedron, and if $\omega H_1(X) \neq 0$, then there is a space $Y$ of the homotopy type of $X$ which retracts to a simple closed curve. In fact, $X \times I^2$ retracts to a simple closed curve.
Proof: Let \( f: S^1 \to X \times I^2 \) be a mapping. Then \( f \) is homotopic to a map \( g: S^1 \to X \times \{(c,0)\} \subseteq X \times I^2 \). Consider \( g \) as a mapping of the unit interval \( I \) such that \( g(0) = g(1) \). Define \( h: I \to X \times I^2 \) by \( h(t) = (g(2t), 0, 2t) \) for \( 0 \leq t \leq 1/2 \), and \( h(t) = (g(1), -4t^2 + 6t - 2, 2 - 2t) \) for \( 1/2 \leq t \leq 1 \). Then \( h \), considered as a mapping of \( S^1 \) into \( X \times I^2 \), is homotopic to \( g \). Notice that \( h \) is an imbedding of \( S^1 \).

Since \( \pi_1(X) \neq 0 \) and \( X \) is dominated by a finite polyhedron, \( \pi_1(X \times I^2) \neq 0 \) and \( X \times I^2 \) is also dominated by a finite polyhedron. Thus there is a map \( f: S^1 \to X \times I^2 \) such that the sequence

\[
0 \to H_1(S^1) \xrightarrow{f_*} H_1(X \times I^2) \to H_1(X \times I^2)/f_* (H_1(S^1)) \to 0
\]

is split exact. The map \( f \) is homotopic to an imbedding \( h: S^1 \to X \times I^2 \), and the sequence

\[
0 \to H_1(S^1) \xrightarrow{h_*} H_1(X \times I^2) \to H_1(X \times I^2)/h_* (H_1(S^1)) \to 0
\]

is also split exact. By Theorem II.10, \( h(S^1) \) is a retract of \( X \times I^2 \).

**Corollary III.11:** If \( K \) is a finite polyhedron and \( \pi_1(K) \neq 0 \), then there is an imbedding \( h: S^1 \to K \times I \) such that the sequence
\[ 0 \rightarrow H_1(S^1) \xrightarrow{h_*} H_1(K \times I) \xrightarrow{\partial} H_1(K \times I)/h_*(H_1(S^1)) \rightarrow 0 \]

is split exact. Thus \( K \times I \) retracts to a simple closed curve.

**Proof:** Since \( H_1(K) \neq 0 \), there is a mapping \( f : S^1 \rightarrow K \) such that the sequence

\[ 0 \rightarrow H_1(S^1) \xrightarrow{f_*} H_1(K) \rightarrow H_1(K)/f_*(H_1(S^1)) \rightarrow 0 \]

is split exact. It may be supposed that \( f \) is simplicial, so that \( f(S^1) \subseteq K^1 \).

Suppose that \( f(S^1) \) does not meet a 2-simplex of \( K \). The sequence above is exact, so \( f(S^1) \) is not a tree. Let \( C \) be a simple closed curve in \( f(S^1) \), and let \( \tau \) be a 1-simplex of \( C \). Then \( C \setminus \tau \) is an arc, and thus \( C \setminus \tau \) is a retract of \( K \setminus \tau \). Since \( \tau \) is the face of no 2-simplex of \( K \), \( C \) is a retract of \( K \).

Now suppose that \( f(S^1) \) does meet a 2-simplex \( \sigma \) of \( K \). Let \( v \) be a point of \( \sigma \cap f(S^1) \). Consider \( f \) as a mapping of \( I \) into \( K \), with \( f(0) = f(1) \), and without loss of generality suppose that \( f(0) = v \). Let \( p : I \rightarrow \sigma \) be an imbedding such that \( p(I) \cap \text{Bd}(\sigma) = \{p(0)\} = \{v\} \). Define a map \( h : I \rightarrow K \times I \) by \( h(t) = (f(2t), 2t) \) for
$0 \leq t \leq 1/2$, and $h(t) = (p(-4t^2 + 6t - 2), 2 - 2t)$ for $1/2 \leq t \leq 1$. Then $h$, considered as a mapping of $S^1$ into $K \times I$, is an imbedding, and $h$ is homotopic to $f' : S^1 \rightarrow K \times I$, where $f'(x) = (f(x), 0)$. Thus the sequence

$$0 \rightarrow H_1(S^1) \xrightarrow{h_*} H_1(K \times I) \rightarrow H_1(K \times I) / h_* (H_1(S^1)) \rightarrow 0,$$

is split exact.

**Corollary III.12:** If $X$ is a metrizable space which is dominated by a finite polyhedron, $\omega H_1(X) \neq 0$, and $X$ has the fixed point property, then either: i) $X$ has the fixed point property and $X \times I$ does not, or ii) $X \times I$ has the fixed point property and $(X \times I) \times I$ does not.

**Example III.13:** The following example shows that it is necessary in Corollary III.12 and in Theorem II.10 that $X$ be dominated by a finite polyhedron.

For each positive integer $n$, let $S_n$ be the circle in the plane having center $(1/n, 0)$ and radius $1/n$. Let $Z = \bigcup_{n=1}^{\infty} c_n$, let $c = (0, 0, 1) \in \mathbb{E}^3$, and let $X$ be the union of all possible line segments from $c$ to $Z$. The following argument shows that $Z$ has the fixed point
property. For each positive integer \( k \), let

\[ Z_k = \bigcup_{n=1}^{k} S_n \]

and let \( X_k \subseteq X \) be the cone over \( Z_k \). For each \( k \) there is a retraction \( r_k: X \rightarrow X_k \) such that

\[ d(x, r(x)) \leq 2/k + 1. \]

Such a map may be defined by letting

\[ r_k(t_1, t_2, t_3) = (0, 0, t_3) \quad \text{for} \quad (t_1, t_2, t_3) \in X \setminus X_k, \]

and

\[ r_k(t_1, t_2, t_3) = (t_1, t_2, t_3) \quad \text{otherwise}. \]

Suppose that there is a map \( f: X \rightarrow X \) which is fixed point free. The space \( X \) is compact, so there is a number \( \epsilon > 0 \) such that

\[ d(x, f(x)) > \epsilon \quad \text{for} \quad x \in X. \]

Let \( k \) be a positive integer such that \( 2/(k+1) < \epsilon \). Consider the map \( r_k f |_{X_k} \).

For \( x \in X \), \( d(x, f(x)) > \epsilon \), and \( d(f(x)) \),

\[ r_k f(x) \leq 2/(k+1) < \epsilon. \]

Thus \( r_k f(x) \neq x \), so

\[ r_k f |_{X_k}: X_k \rightarrow X_k \]

is fixed point free. But \( X_k \) is a
contractible finite complex, so $X_k$ has the fixed point property. This contradiction shows that $X$ has the fixed point property.

Now for each positive integer $n$, let $S_n'$ be the circle in the plane having center at $(-1/n, 0)$ and radius $1/n$. Let $Z' = \bigcup_{n=1}^{\infty} S_n'$, let $c' = (0, 0, -1)$, and let $X'$ be the union of all possible line segments from $c'$ to $Z'$. Let $Y = X \cup X'$. The group $H_1(Y)$ is not a torsion group (see [3]). However, since $X$ and $X'$ each has the fixed point property and $X \cap X' = \{(0,0,0)\}$, then $Y$ has the fixed point property. It can be shown that for any positive integer $n$, $Y \times I^n$ also has the fixed point property, using an argument similar to the one given for $X$. In fact, the product of $Y$ with the Hilbert cube also has the fixed point property. It is shown in [3] that $H_1(Y)$ has the cardinality of the continuum. Since $H_1(K)$ is countable for any countable complex $K$, the space $Y$ is not dominated by such a complex. Therefore $Y$ does not provide a counterexample to any of the theorems of this paper.

Finally, as a corollary to Theorem II.8, we have the following theorem.

**Theorem III.14**: If $X$ is a metrizable space which is dominated by a finite $(m+1)$-dimensional polyhedron, and if there is a map $f:S^m \to X$ such that the sequence
is split exact, then there is a compact metrizable space
$Y$ of the homotopy type of $X$ which retracts to an $m$-sphere.
Thus $X$ has the homotopy type of a space which does not have the fixed point property.

**Proof:** Let $f:S^m \to X$ be a mapping which yields a split exact sequence as above. Let $Y = M(f)$, the mapping cylinder of $f$. The inclusion $i:S^m \to Y$ is homotopic to the map $jf:S^m \to Y$, where $j:X \to Y$ is the inclusion. Since $j$ is a homotopy equivalence, the sequence

$$0 \to H_m(S^m) \xrightarrow{j_*f_*} H_m(Y) \to H_m(Y)/j_*f_*(H_m(S^m)) \to 0$$

is split exact. Therefore the sequence

$$0 \to H_m(S^m) \xrightarrow{i_*} H_m(Y) \to H_m(Y)/i_*(H_m(S^m)) \to 0$$

is also split exact. It follows from Theorem II.8 that $i(S^m)$ is a retract of $Y$.

**III.15** In the above theorem, if $X$ is an ANR, then so is $Y$. If $X$ is a locally finite polyhedron and $g:S^m \to X$ is piecewise linear, then $M(g)$ is also a locally finite polyhedron. Thus $Y$ can be taken to be a locally finite polyhedron. If $S^m$ is the circle, then the finite-
ness and dimension restrictions on the dominating polyhedron of $X$ are unnecessary.
BIBLIOGRAPHY


The author was born on May 14, 1942, near Popejoy, Iowa. He graduated from Iowa Falls High School in 1960, and in 1964 he received the degree of Bachelor of Science from Tulane University in New Orleans, Louisiana. He is currently working toward the degree of Doctor of Philosophy at Louisiana State University. The author is married to the former May Gwin of Starkville, Mississippi. He is a member of The American Mathematical Society and Phi Kappa Phi.
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