Inverse Semigroups on the 2-Cell.

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ABSTRACT

In this paper we investigate the structure of inverse semigroups on the 2-cell with an identity. Let $S$ denote such a semigroup. After the preliminaries in chapter I, it is shown that the minimal ideal, $K$, is a point and hence $S$ has a zero.

In chapter II, the case when $E$ has a cut point is considered. If $e \in E$ is a cut point, then $H(e)$ must be a circle group, $SeS$ is the union of $H(e)$ with the bounded component of $R^2 \setminus H(e)$, and $E \cap SeS$ is a min thread. Then a characterization of semigroups of this type is established by showing that if $E$ has a cut point, then $S$ must be a min cone over a circle group, and $E$ must be a min thread.

In chapter III, considering the case when $E$ has no cut point, it is shown that the inverse homeomorphism which maps $s$ onto $s^{-1}$, is the identity map and $s = s^{-1}$ for all $s \in S$. Hence it follows that $S$ is commutative and a union of groups. The maximal group $H(1)$ then must be either $Z_2 \times Z_2$, $Z_2$, or $\{1\}$. In the two cases when $H(1) \neq \{1\}$, $0 \in \overline{S \setminus E}$, and if $H(1) \approx Z_2 \times Z_2$, then $0 \in \text{int } S$. As a partial characterization of semigroups of this
type, it is shown that if $E$ has no cut point and $H(1) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$, then $S$ is the continuous monotone homomorphic image of the cartesian product of the min cone over $\mathbb{Z}_2$ with itself, if and only if $M(e)$ is connected for each $e \in E$. Using the min cone over $\mathbb{Z}_2$ and the min cone over the two element semilattice as building blocks, examples are constructed to illustrate the existence of semigroups for the various cases.

Chapter IV considers only algebraic inverse semigroups. It is shown that for each $a \in S$, the row and column idempotents of the powers of a generate a lattice denoted by $\wedge_a$. The following three conditions are considered for $a \neq b$:

1) $(a, b) \in \mu$, the maximum idempotent-separating congruence;
2) $\wedge_a = \wedge_b$;
3) $a^n a^{-n} = b^n b^{-n}$ where $a^n a^{-n}$ and $b^n b^{-n}$ are the row idempotents for $a^n$ and $b^n$, respectively.

It is shown that 1) implies 2) and 2) implies 3), and examples are given to show that the two implications are not reversible.
INTRODUCTION

One of the questions which arises in the study of topological semigroups is "given a topological space, can the semigroups defined on that space be classified?" The arc was considered first, and after semigroups on an arc were characterized, the next space to be considered was, naturally, the 2-cell. Various types of semigroups on the 2-cell have been studied and classified. Among them are the affine semigroups, the semigroups with group boundary, semilattices whose "upper sets" are connected, semigroups with trivial multiplication on the boundary, and semigroups with the boundary a union of threads.

In this paper we shall consider the problem of characterizing inverse semigroups with an identity on the 2-cell. For semigroups of this type, the set of idempotents $E$ is a retract, and we approach the problem by considering the following two cases: 1) when $E$ has a cut point, and 2) when $E$ does not have a cut point and is a 2-cell. In chapter II, we consider case one, with the result being that the min cone over the circle group is the only semigroup of this type.

The study of case two in chapter III was not as fruitful and a complete characterization was not established.
However, these important results were obtained: the semigroup must be commutative, a union of groups, and the maximal group containing 1 must be either $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2$, or \{1\}. The homeomorphisms, defined by mapping $x$ onto $x^{-1}$ and $x$ onto $ax$ for fixed $a \in H(1)$, were used in establishing these facts. Chapter III concludes the study of Topological inverse semigroups on the 2-cell with several examples.

In chapter IV, we consider general algebraic inverse semigroups and a lattice of idempotents is shown to exist for each element. An example is given to show that if two elements have the same lattice, they do not necessarily have to be related modulo the maximum idempotent-separating congruence.
CHAPTER I

Introduction. The purpose of this chapter is to establish notation, to introduce terminology and definitions, and to state previous results in the topology of the plane and in semigroups, which will be used and referred to in later chapters. For a more detailed and complete background, the suggested references are Kelley [11] and Whyburn [20] for topology, Clifford and Preston [5] and [6] for algebraic semigroups, and Hofmann and Mostert [8] for topological semigroups.

Topological Preliminaries. Throughout this paper, $R^2$ will denote the Cartesian plane with the usual topology. If $A$ and $B$ are subsets of $R^2$, then $A \setminus B$ is understood to mean the complement of $B$ in $A$. The closure of a subset $A$ is indicated by $\bar{A}$ and the empty set is denoted by $\Box$. The boundary of $A$, written $\text{Bd}A$, is $\bar{A} \cap R^2 \setminus A$; it will always mean the boundary with respect to $R^2$. If $A$ is a simple closed curve, then $\text{int}A$ represents the bounded component of $R^2 \setminus A$ and $\text{ext}A$ denotes the unbounded component of $R^2 \setminus A$.

The Alexander-Kolmogoroff-Spanier-Wallace cohomology theory is used with the integers as the coefficient group.
For $A \subseteq \mathbb{R}^2$, $A$ is said to be acyclic if $A$ has the cohomology groups of a point.

Because of their importance and frequent use in establishing results in this paper, the following well known theorems are presented without proofs. The statements are modified and restricted for clarity in the special cases considered here.

**Theorem 1.1** [20]. If $T$ is an arc from $a$ to $b$ in a 2-cell $S$ such that $(a,b) \subset \text{Bd}S$ and $T \cap \text{int} S \neq \emptyset$, then $T$ separates $S$.

**Theorem 1.2** [20]. If $x$ and $y$ are elements of $(\text{Bd}S) \setminus T$ and the two points $a$ and $b$ separate $x$ from $y$ in $\text{Bd}S$, then $T$ separates $x$ from $y$ in $S$.

**Theorem 1.3** [20]. If $T \cap \text{Bd}S = \{a,b\}$, then $T$ separates $S$, and $S \setminus T$ has exactly two components.

**Theorem 1.4** [20]. If $T$ is an arc spanning a 2-cell $S$ from $a$ to $b$, $R_1$ and $R_2$ are the two components of $S \setminus T$, and $E$ is a 2-cell contained in $S$ such that a subarc $T_1$ of $T$ is contained in $\text{Bd}E$, then if $t \in T_1$, there exists open set $U$, $U_1$, and $U_2$ and an arc $T_2$ such that $t \in U$, $U = U_1 \cup U_2 \cup T_2$, $U \cap R_1 = U_1$, $U \cap R_2 = U_2$, $U \cap T_1 = T_2$ and either $U_1 \subset E$ or
$U_2 \subseteq E$.

**Theorem 1.5** [20]. If $T$ is an arc spanning a 2-cell $S$ from $a$ to $b$ and $E$ is a 2-cell contained in $S$ such that $T \subseteq \text{Bd}E$, then $E$ is contained in the closure of one and only one of the two components of $S \setminus T$.

**Theorem 1.6** [3]. A subset $E$ of a 2-cell $S$ is a retract of $S$ if and only if $E$ is a locally connected continuum which does not separate the plane.

**Theorem 1.7** [21]. A retract $E$ of a 2-cell $S$ is a 2-cell if it is cyclicly connected, that is, if $E$ has no cut point.

**Semigroup Preliminaries.** A semigroup $S$ is a non-empty set together with an associative binary operation called multiplication. A topological semigroup $S$ is a semigroup which is a Hausdorff topological space such that the multiplication is continuous. An element $e$ of a semigroup $S$ is called idempotent if $e^2 = e$. Throughout the remainder of this chapter, $S$ will denote a topological semigroup and $E$ will denote the set of idempotents of $S$. If $1 \in S$, such that $1 \cdot a = a \cdot 1 = a$ for all $a \in S$, then $1$ is called an identity for $S$. If $0 \in S$, such that $a \cdot 0 = 0 \cdot a = 0$ for all $a \in S$, then $0$ is called a zero for $S$. 
By a left [right] ideal of S we mean a non-empty subset of S such that \( S \subseteq A [A \subseteq S] \). A is an ideal if A is both a right and left ideal. A compact semigroup \( S \) contains a unique minimal ideal \( K \), called the kernel of \( S \).

A continuous function \( f \) from \( S \) into a topological semigroup \( T \) is called a homomorphism if \( f(ab) = f(a)f(b) \). If, in addition, \( f \) is a homeomorphism, then \( f \) is called an isomorphism.

If \( a \in aSa \) for each \( a \in S \), then \( S \) is called a regular semigroup. For \( a, b \in S \), if \( aba = a \) and \( bab = b \), then \( b \) is called an inverse of \( a \). An inverse semigroup is a semigroup in which every element has a unique inverse. For \( a \in S \) denote the unique inverse of \( a \) by \( a^{-1} \). A semigroup is called Clifford if it is a union of groups.

Theorem 1.8 [5]. The following are equivalent:

i) \( S \) is regular, and any two idempotents of \( S \) commute;

ii) every principal right ideal and every principal left ideal of \( S \) has a unique idempotent generator;

iii) \( S \) is an inverse semigroup.

Theorem 1.9 [5]. A semigroup \( S \) is an inverse semigroup if and only if each \( L \)-class and each \( R \)-class of
S contains exactly one idempotent.

**Theorem 1.10** [5]. For any elements $a, b$ of an inverse semigroup $S$, we have

$$(a^{-1})^{-1} = a \quad \text{and} \quad (ab)^{-1} = b^{-1}a^{-1}.$$

Certain equivalence relations, called Green's relations, play an important role in the theory of semigroups. Four of these relations for a compact inverse semigroup are defined as follows:

- $\mathcal{L} = \{(a, b) : Sa = Sb\},$
- $\mathcal{R} = \{(a, b) : aS = bS\},$
- $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$
- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} = \{(a, b) : SaS = SbS\}.$

$H(a)$, $L(a)$, $R(a)$ and $D(a)$ will denote the corresponding equivalence classes containing $a$. If $e \in E$, then $H(e)$ is the maximal group containing $e$.

A semigroup is **stable** if $Sa \subseteq Sab$ implies $Sa = Sab$ and $xS \subseteq yxS$ implies $xS = yxS$ for each $a, b, x, y \in S$.

A **thread** is a semigroup on a closed real interval in which the endpoints act as zero and identity. A **min thread** is a thread whose multiplication is defined by $xy = \min\{x, y\}$ where the minimum is taken with respect to the usual ordering of the reals. I will denote the min
thread defined on the closed real interval from 0 to 1. Let $S$ and $D$ be compact semigroups with kernels $K_S$ and $K_D$ respectively. Define the congruence relation $C$ on $S \times D$ by:

If $y \in K_D$, then $(x, y)C(s, t)$ if, and only if $y = t$.

If $y \notin K_D$, then $(x, y)C(s, t)$ if, and only if $(x, y) = (s, t)$.

$C$ is a closed congruence and the semigroup $(S \times D)/C$ is called the D cone over $S$ [1]. If $I$ or any min thread is chosen as $D$, then $(S \times I)/C$ is called the min cone over $S$. Of particular interest in this paper is the min cone over the circle group, the min cone over $\mathbb{Z}_2$, and the min cone over the 2 element semilattice, all of which are inverse semigroups.

We now further restrict our attention by letting $S$ denote only an inverse semigroup on the 2-cell with an identity 1. There is a natural partial ordering on $S$ defined by $a \leq b$ if $a = ab^{-1}a$.

**Theorem 1.11** [13]. The following are equivalent:

1. $x \leq y$
2. $xx^{-1} = y^{-1}x$
3. $x^{-1}x = x^{-1}y$
4. $xx^{-1} = xy^{-1}$
5. $x = xx^{-1}y$
6. $yx^{-1}x = x$
7. $x^{-1} \leq y^{-1}$. 
Theorem 1.12 [13]. S is stable, and if \( x \leq y \) and \( x \prec y \), then \( x = y \).

Theorem 1.13 [8]. If \( D \) is a \( \mathcal{D} \)-class of \( S \), then \( \dim D \leq 1 \).

Theorem 1.14. The kernel \( K \) of \( S \) is the one point group, so \( S \) has a zero.

Proof: The existence of \( K \) in compact semigroups is well known as is the fact that \( K \) is connected and consists of one \( \mathcal{D} \)-class. If \( e \) and \( f \) are idempotents of \( K \), then \( ef \leq e \) and \( ef \leq f \), but \( ef \not\prec ef \), so \( e = ef = f \) by stability. Hence \( K \) contains one idempotent and \( K \) is a group. Now \( 0 = H^1(S) \approx H^1(K) \), so \( \dim K = 0 \), and since \( K \) is connected, \( K \) consists of a single point.

Remark: It will be assumed that \( S \) has a zero without reference to this theorem throughout the remainder of the paper.

Theorem 1.15 [2]. If \( e \in E \), then either \( \dim H(e) = 0 \) or \( H(e) \) is a circle group and \( \dim H(e) = 1 \).

Theorem 1.16 [15]. There exists a min thread \( T \in E \) from 0 to 1.

Theorem 1.17. If \( BdS = H(1) \), a circle group, then for each \( x \in S \), either \( xH(1) \) is homeomorphic to a
circle and \( N_x = \{ h \in H(1) : xh = x \} \) is finite and cyclic, or \( xH(1) = x \) and \( N_x = H(1) \).

**Proof:** This theorem follows from the well-known fact that the closed subgroups of the circle group are finite cyclic or the group itself.

**Theorem 1.18.** If \( e \in E \) and \( H(e) \) is a circle group, then \( S \subseteq H(e) \cup \text{int } H(e) \) and \( 0 \in \text{int } H(e) \).

**Proof:** Let \( T \) be a min thread from 0 to 1, and consider the map \( H \) from \( TX \times S \) onto \( S \) defined by \( H(t, ses_1) = tses_1 \). Clearly \( H \) is continuous and 
\[
H(1, ses_1) = ses_1 \quad \text{and} \quad H(0, ses_1) = 0 \quad \text{for all } ses_1 \in S.
\]
The Generalized Homotopy Theorem [18] implies 
\[
H^n(S) \cong H^n(0) \quad \text{for all } n, \quad \text{hence } S \text{ is acyclic. By [2], } 0 \in \text{int } H(e), \quad \text{and clearly } H(e) \subseteq S.
\]
Therefore, 
\[
H(e) \cup \text{int } H(e) \subseteq S.
\]
If \( a \in S \cap \text{ext } H(e) \), then \( aT \) is an arc from \( a \) to \( 0 \), so there exists \( t \in T \) such that \( at \in H(e) \) and \( a^{-1}at = e \). We may assume \( l \neq e \) since the theorem is obviously true if \( e = 1 \) and \( l \in \text{int } H(e) \) since \( H(l) \) is contained in the boundary of \( S \) [18]. Therefore, \( e \in T \) and \( e \) and \( t \) compare. If \( e \leq t \), then \( e = e \cdot e = a^{-1}at \cdot e = a^{-1}ae \) and \( e \leq a^{-1}a \). If \( t \leq e \), then \( t = et = a^{-1}at \cdot t = a^{-1}at = e \) and again \( e \leq a^{-1}a \).
Therefore,

\[ S_aS = SeS = Sa^{-1}aS = SaS, \]

hence \( SeS = Sa^{-1}aS \) and \( e \neq a^{-1}a \). By stability, \( e = a^{-1}a \) and similarly \( e = aa^{-1} \), hence \( a \in H(e) \), a contradiction. Therefore, \( SeS = H(e) \cup \text{int } H(e) \).

**Remark:** The theorems and results of this chapter will be used throughout the remainder of this paper without specific reference.

**A note of warning.** Many of the definitions and theorems stated in this chapter only apply to inverse semigroups, and in particular, some only apply to inverse semigroups on the 2-cell with an identity. Hence, for the more general theory, one should consult the references mentioned in the introduction.
CHAPTER II

Introduction. The purpose of this chapter is to characterize inverse semigroups with an identity on the 2-cell whose set of idempotents \( E \) has a cut point. Using Dimension Theory, it is shown that for a cut point \( e \) of \( E \), \( H(e) \), the maximal group associated with \( e \), is a circle group and further that if \( f \leq e \) then \( H(f) \) is also a circle group.

By considering the two cases \( \dim E = 1 \) and \( \dim E = 2 \), it is established that \( E \) must be a min thread and that the only inverse semigroup with an identity on the 2-cell whose set of idempotents \( E \) has a cut point is a min cone over a circle.

Preliminaries. Because of their importance in establishing this characterization, the following four theorems from Dimension Theory [8] will be stated without proofs. The statements will be modified and restricted to the special cases of this chapter.

Theorem 2.1 [8, pp. 48]. The 2-cell cannot be disconnected by a subset of dimension \( \leq 0 \).

Theorem 2.2 [8, pp. 34]. If \( B \) is either
0-dimensional or one-dimensional and \( A \) is compact then

\[
\dim (A \times B) = \dim A + \dim B.
\]

**Theorem 2.3** [8, pp. 91] If \( f \) is a mapping of a compact space \( X \) onto a space \( Y \) and

\[
\dim X - \dim Y = 1,
\]

then there is a point of \( Y \) whose inverse image has dimension \( \geq 1 \).

**Theorem 2.4** [8, pp. 46] Let \( U \) be an open set in \( \mathbb{R}^2 \) which is neither empty nor dense in the space and let \( B \) be the boundary of \( U \); then \( \dim B = 1 \).

**Results.**

**Theorem 2.5.** Let \( S \) be an inverse semigroup with an identity on the 2-cell. If \( e \) is a cut point of \( E \), then \( H(e) \) is a circle group.

**Proof:** Let \( f \) be the retraction of \( S \) onto \( E \) defined by \( r(x) = ss^{-1} \) [11]. Let \( E_1 \cup E_2 = E \setminus e \) be a separation of \( E \) by \( e \). Clearly \( r^{-1}(E_1) \cap r^{-1}(E_2) = \emptyset \), so suppose \( x \in r^{-1}(E_1) \cap r^{-1}(E_2) \). Then there is a sequence \( \{x_n\} \) contained in \( r^{-1}(E_1) \) converging to \( x \). Inversion is continuous [8], so \( \{x_n^{-1}\} \) converges to \( x^{-1} \) and it
follows that \( \{x_nx_n^{-1}\} \) converges to \( xx^{-1} \). But \( x_nx_n^{-1} \in E_1 \) for all \( n \) and \( xx^{-1} \in E_2 \), a contradiction since 
\( E_1 \cap E_2 = \emptyset \). Thus \( r^{-1}(e) \) separates \( S \) and by Theorem 2.1, 
\( \dim r^{-1}(e) \geq 1 \). Clearly \( R(e) \subseteq r^{-1}(e) \) and for \( x \in r^{-1}(e) \),
\( xx^{-1} = e \) and \( x = xx^{-1}x = ex \). It follows that \( x \in R(e) \),
so \( R(e) = r^{-1}(e) \) and \( \dim R(e) \geq 1 \).

By Theorem 1., \( \dim D(e) \leq 1 \) and since \( R(e) \subseteq D(e) \),
\( \dim R(e) \leq 1 \). Thus \( \dim R(e) = 1 \) and hence \( \dim D(e) = 1 \).
Inversion restricted to \( R(e) \) is a homeomorphism onto \( L(e) \)
and dimension is topologically invariant, so \( \dim L(e) = 1 \).
\( L(e) \) is compact, so by Theorem 2.2

\[ \dim (L(e) \times R(e)) = \dim L(e) + \dim R(e) = 2. \]

Let \( m \) be the multiplication map from \( L(e) \times R(e) \) onto
\( D(e) \) defined by \( m(\ell, r) = \ell r \). By Theorem 2.3, there is an
element \( \ell r \) of \( D(e) \) such that \( m^{-1}(\ell r) \) has dimension
\( \geq 1 \) and \( m(\ell, r) = \ell r \). Now for \( \ell r = \ell_1 r_1 \),
\( \ell S = \ell eS = \ell rS = \ell_1 r_1 S = \ell_1 eS = \ell_1 S \) and
\( Sr = seS = S\ell r = S\ell_1 r_1 = S\ell_1 eS = Sr_1 \), so \( \ell \ell_1 \), \( r\ell_1 \)
and \( m^{-1}(\ell r) = H(\ell) \times H(r) \), hence \( \dim (H(\ell) \times H(r)) \geq 1 \).

By Green's translational lemmas, \( H(e) \) is
homeomorphic to \( H(\ell) \) and \( H(r) \), so \( H(\ell) \times H(r) \) is
homeomorphic to \( H(e) \times H(e) \). Hence \( \dim (H(e) \times H(e)) \geq 1 \).
If \( \dim H(e) = 0 \), then by Theorem 2.2
\[
\dim (H(e) \times H(e)) = \dim H(e) + \dim H(e) = 0,
\]
so by Theorem 1.15, \( \dim H(e) = 1 \) and \( H(e) \) is a circle group.

Note that by this theorem 0 is not a cut point of \( E \).

**Corollary 2.6.** If \( e \) is a cut point of \( E \) then \( E \cap S \) is a min thread.

**Proof:** There exists a min thread \( T \subseteq E \) from 1 to 0, [6], which clearly contains \( e \) and if \( f \in T \) such that \( f < e \), then \( f \in S \). Clearly \( T_1 = T \cap S \subseteq E \cap S \). \( H(e) \) is a circle group, so for \( x \in H(e) \backslash \{ e \} \), let \( U \) be an open connected set about \( e \) and \( V \) and open connected set about \( x \), such that \( U \cap V = \emptyset \) and \( U \times V \subseteq V \). Let \( f \in T_1 \cap U \) such that \( f < e \). Then \( fx \in V \), \( fx \neq f \) and \( fH(e) \) is a connected set containing at least two distinct points \( f \) and \( fx \), so \( \dim fH(e) \geq 1 \). For all \( x \in H(e) \), \( fxS = fS \subseteq S \), so \( fH(e) \subseteq R(f) \) and \( \dim R(f) \geq 1 \). By the same argument as in the proof of the theorem \( H(f) \) is a circle group.

Let \( f = \inf A \) where \( A = \{ t \in T_1 : H(t) \) is a circle group and if \( t < t_1 \) for \( t_1 \in T_1 \) then \( H(t_1) \) is a circle group\}. By the above argument \( H(f) \) is not a circle group and \( \dim H(f) = 0 \). Let \( B = \cap StS \). Clearly \( B \) is compact, connected and contains \( f \). Let \( x \in BdB \). Then there is a
sequence \{x_n\} converging to x such that \( x_n \in H(f_n) \), \( f_{n+1} < f_n \) for all \( n \), and \( \{f_n\} \) converges to \( f \).

Inversion is continuous so \( \{x_n^{-1}\} \) converges to \( x^{-1} \). Now \( \{x_n x_n^{-1}\} \) converges to \( xx^{-1} \) and \( \{x_n^{-1} x_n\} \) converges to \( x^{-1}x \); but \( x_n x_n^{-1} = f_n = x_n^{-1} x_n \), so \( xx^{-1} = f = x^{-1}x \) and \( x \in H(f) \). Hence \( BdB \in H(f) \), \( f \in BdB \) and \( \dim BdB = 0 \).

If \( f \neq 0 \) then \( 0 \notin BdB \), so \( \text{int} B \neq \emptyset \) and by Theorem 2.4 \( \dim BdB = 1 \), a contradiction.

Therefore \( f = 0 = BdB = B = \bigcap_{t \in T} S_t \) and clearly \( T_1 = E \cap S \).

**Theorem 2.7.** If \( \dim E = 1 \), then \( E \) is a min thread and \( S \) is a min cone over a circle.

**Proof:** \( E \) is a tree, so the unique arc from 1 to 0 is a min thread \( T \) where every point of \( T \setminus \{0,1\} \) is a cut point of \( E \) and hence whose \( \mathcal{N} \)-class is a circle group.

Suppose \( e \in E \setminus T \). Then \( e \) lies on an arc \( A \) of idempotents for \( e \) to 0. Since \( T \) is closed, there exists \( f \in A \) such that \( f \notin T \) and \( f \) is a cut point of \( E \). Hence \( H(f) \) is a circle group, containing 0 in its interior, so \( H(f) \cap T \neq \emptyset \), a contradiction. Therefore \( E = T \).

Let \( e \in T \setminus \{0,1\} \). Then \( S_e \subseteq S \cdot f \cdot S = H(f) \cup \text{int} H(f) \) for \( e < f \). But \( H(e) \cap H(f) = \emptyset \), so \( H(e) \subseteq \text{int} H(f) \subseteq S \),
the 2-cell. Thus \( \text{int } H(f) \subseteq \text{int } BdS \) and it follows that \( H(e) \cap BdS = \emptyset \).

Clearly \( H(1) \subseteq BdS \), so let \( x \in BdS \). Then \( SxS = SfS \) for some \( f \in E \). Now \( f \not\in T \setminus \{0,1\} \) since \( SSeS = H(e) \cup \text{int } H(e) \subseteq \text{int } BdS \) for all \( e \in T \setminus \{0,1\} \).

Clearly \( f \neq 0 \), so \( f = 1 \) and \( H(1) = BdS \) is a circle group. Hence \( S = \bigcup_{e \in T} H(e) \), where each \( H(e) \) is a circle group if \( e \neq 0 \), and \( BdS = H(1) \). Let \( t \in xH(1) \); then \( t = xh \) and \( tS = xhS = xS \) since \( RS = hS = S \), so \( t \not\in x \) and it follows that \( xH(1) \subseteq H(x) \). Now for each \( x \in S \), either \( xH(1) = x \) or \( xH(1) = H(x) \). Suppose \( 0 \neq f = f^2 \) and \( H(f) = fH(1) \). Such an \( f \) exists since \( H(1) = 1 \cdot H(1) \).

\( S \) is Hausdorff and multiplication is continuous so there exists an open connected set \( U \) about \( f \) and an open connected set \( V \) about \( x \) for some \( x \in H(f) \setminus \{f\} \), such that \( U \cap V = \emptyset \) and \( Ux \subseteq V \). Let \( T = \{ t \in T : t \leq f \} \) and consider \( e \in T \setminus \{f\} \cap U \). Then \( ex \in V \), but \( x = fh \) for some \( h \in H(1) \) and \( ef = e \), so \( ex = efh = eh \). Now \( eh \in eH(1) \) and \( eh \neq e \), so \( eH(1) = H(e) \).

Let \( e = \inf \{ t \in T \setminus \{0\} : H(t) = tH(1) \} \) and suppose \( e \neq 0 \). Then \( eH(1) = e \neq 1 \) by the argument above. For \( x \in H(e) \), \( x \neq e \), let \( U \) be an open connected set about \( e \) and \( V \) an open connected set about \( x \) such that \( U \cap V = \emptyset \). Now there exists a sequence \( \{ en \} \) converging to \( x \) such that the sequence \( \{ e_n \} \) converges to \( e \).
Let \( e_n > e_{n+1} \), \( e_n < e \) and \( h_n \in H(l) \) for all \( n \). Let \( h \in H(l) \) such that a subsequence \( \{h_{n_j}\} \) of \( \{h_n\} \) converges to it. Now \( \{e_{n_j}\} \) converges to \( e \), so \( \{e_{n_j}h_{n_j}\} \) converges to \( eh = e \); but \( \{e_{n_j}h_{n_j}\} \) is a subsequence of \( \{e_nh\} \) which converges to \( x \), a contradiction. Therefore \( e = 0 \), \( S = T \cdot H(l) \) and \( N_x = \{h \in H(l) : xh = x\} \) is finite for all \( x \neq 0 \) by \([8]\). Suppose \( fh = f \) for \( f \in T \setminus \{0\} \) and \( h \in H(l) \setminus \{1\} \); that is, \( N_f \neq \emptyset \). For \( e < f \) then \( ef = e \) and \( eh = efh = ef = e \), so \( f \) is assumed to be the maximal element in \( T \) such that \( fh = f \). Let \( T_2 = T \setminus T_1 U \{f\} \) where \( T_1 = \{t \in T : t \leq f\} \). Now \( Th = T_1 UT_2h \) is an arc from \( 0 \) to \( h \) where \( T_2h \cap H(l) = h \). Consider the connected set \( T_2 UT_2h \) which separates \( S \) and in particular it separates an arc \( A \) of \( H(l) \) from \( 0 \). Let \( g \in A \setminus \{1, h\} \) and consider the arc \( Tg \) from \( g \) to \( 0 \). Now either \( Tg \cap T_2 \neq \emptyset \) or \( Tg \cap T_2h \neq \emptyset \).
If $e \in Tg \cap T_2$, then $e = \lambda g \neq 0$ for some $\lambda \in T$. Therefore $e = \lambda h \in H(\lambda)$, $e = \lambda$ and $e = eg$. Hence $fg = feg = fe = f$. If $0 \neq eg = \lambda h \in Tg \cap T_2$, then $e = \lambda hg^{-1}$, and $e = e^2 = \lambda hg^{-1}gh^{-1}g = \lambda^2 = \lambda$, so $e = ehg^{-1}$. Now $e = \lambda \in T_2$, so $fe = e$. So $fg^{-1} = fhg^{-1} = fehg^{-1} = fe = f$.

Therefore $A \not\subset N_f$, a contradiction, since $N_f$ is finite.

Hence $N_f = \{1\}$ for all $f \in T \setminus \{0\}$ and the maps from $H(l)$ onto $fH(l) = H(f)$ defined by mapping $h$ onto $fh$ are homeomorphisms.

Now $fh^{-1} \in fH(l) - H(f)$, $(fh^{-1})^{-1} = hf \in fH(l)$, so $hf \cdot fh^{-1} = hfh^{-1} = H(f)$, but $hfh^{-1} \in T = E$, so $hfh^{-1} = f$ or $hf = fh$. That is to say $T$ is contained in the center of $S$.

Consider the multiplication map $m$ from $T \times H(l)$ onto $T \cdot H(l) = S$. Now $m(e,g)m(f,h) = egfh = efgh = m(ef, gh)$ so $m$ is a homomorphism. Note that $m^{-1}(0) = O \times H(l)$.

Clearly, $\frac{T \times H(l)}{O \times H(l)}$ is homomorphic to $S$ and if $fh = eg$, then as above $f = e$, so $fh = fg$ or $fhg^{-1} = f$.

Therefore $hg^{-1} \in N_f$ or $hg^{-1} = 1$ and $h = g$. Hence the canonical map is one to one, thus a homeomorphism and it follows that $S$ is a min cone over a circle.
We now consider the possibility of $E$ having a cut point and $\dim E = 2$. It will be shown that this case does not exist.

Throughout the remainder of this chapter $e$ will denote $\sup\{t \in T : h(t) \text{ is a circle group}\}$ where $T$ still denotes a min thread from 1 to 0. Since $E$ is assumed to have a cut point, $e$ does exist.

**Lemma 2.8.** $H(e)$ is a circle group.

**Proof:** Let $A = UH(t)$ for all $t \in T$ such that $t < e$. Each $H(t)$ is a circle group, so $A$ is open since $H(t) \subset \text{int} H(g)$ for $t < g < e$. Clearly $A$ is not dense in the plane, so by Theorem 2.4, $\dim \text{Bd}A = 1$. For $x \in \text{Bd}A$, there exists a sequence $\{x_n\}$ of $A$ converging $x$ such that $x_n^{-1}x_n = x_n^{-1} = e_n$, $e_n < e_{n+1}$ for all $n$, and the sequence $\{e_n\}$ converges to $e$. By the continuity of inversion, the sequence $\{x_n^{-1}\}$ converges to $x^{-1}$. Now both $\{x_n^{-1}\}$ and $\{x_n^{-1}x_n\}$ converge to $e$, so $xx^{-1} = e = x^{-1}x$, $x = H(e)$ and $\text{Bd}A \subset H(e)$. Hence $\dim H(e) = 1$ and $H(e)$ is a circle group [8].

**Remark:** Since $e$ is a cut point of $E$ and $e$ separates $E \cap S \setminus SeS = E \cap (\text{ext}H(e))$ from 0, then for any $t \in S \setminus SeS$, $e < tt^{-1}$ and $e < t^{-1}t$. 
**Lemma 2.8.** The idempotents of $SeS$ are contained in the center of $S$.

**Proof:** Note that $E \cap SeS = \{ f \in T : f \leq e \}$ is a min thread from $e$ to $0$ by corollary 2.6: For $f \in E \cap SeS$ and $t \in S \setminus SeS$, $ftS = ftt^{-1}S = fS$, since $f \leq e < tt^{-1}$, so $ft \leq f$. Similarly $Stf = St^{-1}tf = Sf$ and $tf \leq f$. $H(f)$ is a circle group, so $H(f) = D(f) = R(f) = L(f)$ and it follows that $(tf)^{-1} = ft^{-1} \in H(f)$. Now $tt^{-1} \in E$ and $tt^{-1} = tf \cdot ft^{-1} \in H(f)$. Therefore $tt^{-1} = f$ or $tf = ft$.

For $x \in SeS$, either (i) $f \leq xx^{-1} = x^{-1}x$ or (ii) $x^{-1}x = xx^{-1} < f$. In case (i), $fxS = fxx^{-1}S = fS$ and $xfx^{-1} = f$. Hence $xf = fx$.

In case (ii), $fxS = fxx^{-1}S = xx^{-1}S = x^{-1}xS$, so $fxx^{-1} = xx^{-1}$ and $x^{-1}xf = x^{-1}x$. Multiplying by $x$ on the appropriate side implies $fx = x$ and $xf = x$. Hence $fx = x = xf$. Therefore $f$ is an element of the center of $S$.

**Lemma 2.9.** If $f \in E$ such that $e < f$ then $H(f) \cdot e$ is a closed subgroup of $H(e)$.

**Proof:** Clearly $H(f)e$ is closed and is contained in $H(e)$. By the previous lemma, $e$ commutes with all elements of $H(f)$, so for $xe$, $ye \in H(f)e$ where $x, y \in H(f)$, $(ye)^{-1} = ey^{-1} = y^{-1}e \in H(f)e$ and
\[ xe(ye)^{-1} = xe \cdot ey^{-1} = xey^{-1} = xy^{-1}e \in H(f)e. \] Therefore \( H(f)e \) is a closed subgroup of \( H(e) \).

**Remark:** \( H(f) \) is not a circle group and \( \dim H(f) = 0 \). Therefore \( \dim H(f)e = 0 \) and \( H(f)e \) is a finite cyclic subgroup [8]. Also note that the map from \( H(f) \) into \( H(e) \), mapping \( x \) onto \( xe \) is a homomorphism.

**Lemma 2.10.** For \( t \in S \setminus S_e \), and \( x \in H(e) \),
\[ tx = xt. \]

**Proof:** Note \( H(e) \) is abelian, \( te \in H(e) \) by [14], and \( te = et \) by lemma 2.8. So \( xe = x = ex \) and \( tx = t \cdot ex = te \cdot x = x \cdot te = x \cdot et = xe \cdot t = xt. \)

**Lemma 2.11.** For \( t \in S \setminus S_e \) and \( x, y \in H(e) \),
then \( tx = ty \) implies \( x = y. \)

**Proof:** \( tx = t \cdot ex = t \cdot ey = ty \) and multiplying by \( (te)^{-1} \) implies \( x = ex = (te)^{-1}te \cdot x = (te)^{-1}tex = (te)^{-1}tey = ey = y. \)

**Lemma 2.12.** If \( t \in S \setminus S_e \) and \( tx = x \) for some \( x \in H(e) \), then \( th = h \) for all \( h \in H(e) \).

**Proof:** Multiplying \( tx = x \) by \( x^{-1} \) implies \( txx^{-1} = xx^{-1} \) or \( te = e. \) Therefore \( th = teh = eh = h \) for all \( h \in H(e). \)
Lemma 2.13. Let \( C = \{ t \in S \setminus \text{int } H(e) : te = e \} \). Then \( C \) is a closed connected inverse subsemigroup such that \( C \cap H(e) = \{ e \} \).

**Proof:** Let \( \{ t_n \} \) be a sequence contained in \( C \) converging to \( t \). \( \{ t_n e \} \) converges to \( te \), but \( t_n e = e \) for all \( n \), so \( te = e \) and \( C \) is closed. Let \( T_1 = \{ t \in T : t \geq e \} \). For \( t \in C \) and \( x = ft \in T_1 t \), \( xe = fte = fe = e \) and \( T_1 t \subseteq C \). Therefore \( C \) is connected. For \( t \in C \), \( te = e \) and \( (te)^{-1} = et^{-1} = e \), but \( e \) is in center of \( S \), so \( t^{-1} e = e \) and \( t^{-1} \in C \). For \( x, t \in C \) then \( txe = te = e \) and hence \( C^2 \subseteq C \). Therefore \( C \) is a compact connected inverse semigroup such that \( C \cap H(e) = \{ e \} \).

**Remark:** If \( f \) lies on a min thread from 1 to 0 then \( M(f) = \{ fg \in E : gf = f \} \) is connected.

Lemma 2.14. \( 1 \not\in \text{Bd} \cap \overline{S \setminus C} \).

**Proof:** Suppose there is a sequence \( \{ t_n \} \) converging to 1 such that \( t_n \not\in C \) for all \( n \). \( \dim E = 2 \) implies there exists \( f \in \text{int } E \) [10, pp. 44], and since \( E \cap S \subseteq S \) is an arc, \( f \in S \setminus S \subseteq S \) and \( fe = e \).

Now the sequence \( \{ t_n f \} \) converges to \( f \), so there exists an integer \( m \) such that \( t_m f \in E \) and \( t_n f \in S \setminus S \subseteq S \). Therefore \( t_m^{-1} e = t_m^{-1} f e = t_m f \cdot e = e \), a contradiction since
Lemma 2.15. If $aT \cap bT \neq \emptyset$ then there exists $f \in T$ such that (i) $a_t = b_t$ for all $t \leq f$ and (ii) if $T_1 = \{t \in T : t \geq f\}$ then $aT_1 \cap bT_1 = \{af\}$.

Proof: Let $f = \sup \{t \in T : a_t = b_t\}$ and clearly $af = bf$. For $t \leq f$, multiplying $af = bf$ on the right by $t$ yields $a_t = b_t$.

Suppose $x \in aT_1 \cap bT_1$. Then there exists $t \in T_1$, $t_1 \in T_1$ such that $x = a_t = b_{t_1}$. $t$ and $t_1$ compare, so without loss of generality suppose $t \leq t_1$. Then $a_t = a_{t'} = b_{t_1}t = bt$ and $t \leq f$. But $t \in T_1$, so $t = f$ and $x = af$.

Remark: Note $aT \cap bT$ is connected.

Lemma 2.16. $C$ is acyclic.

Proof: Let $T_1 = \{t \in T : t \geq e\}$ and consider the map $H$ from $T_1 \times C$ onto $C$ defined by $H(t, c) = tc$.

Clearly $H$ is continuous and $H(1, c) = c$ and $H(e, c) = e$ for all $c \in C$. The Generalized Homotopy Theorem [18] implies

$t_m \not\in C$. Therefore $1 \not\in bdC \cap \mathbb{S} \setminus C$.
$H^n(C) = H^n(e)$ for all $n$, hence $C$ is acyclic.

**Remark:** $\text{BdS} \not\subseteq C$ since $\text{int } H(e) \cap C = \emptyset$ and $C$ does not cut $R^2$.

**Lemma 2.18.** $\text{BdS} \cap C$ is connected.

**Proof:** Let $a \in \text{BdS} \cap C$ such that $a \neq 1$. It will suffice to show that one of the arcs of $\text{BdS}$, from $a$ to $1$ is contained in $C$. Let $A$ and $B$ be the arcs from this separation. Suppose neither arc, $A$ or $B$, is contained in $C$ and consider the set $T \cup aT$. Let $x \in A \setminus C$, $y \in B \setminus C$, $f = \sup \{t \in T : t = at\}$ and $T_1 = \{t \in T : t \geq f\}$. Clearly $f \geq e$. Lemma 2.15 implies $T_1 \cap aT_1 = \{f\}$, so $T_1 \cup aT_1$ is an arc from $1$ to $a$, not contained in $\text{BdS}$. Hence $T_1 \cup aT_1$ separates $S$ and in particular it separates $x$ from $y$. $H(e) \setminus \{e\} \cap (T_1 \cup aT_1) = \emptyset$ and $H(e) \setminus \{e\}$ is connected, so either $x$ or $y$ is separated from $H(e) \setminus \{e\}$. $x \in xT \cap H(e) \setminus \{e\}$ and $y \in yT \cap H(e) \setminus \{e\}$, a contradiction, since $xT \cup yT \cap H(e) \setminus \{e\}$ is a connected set containing $x$ and $y$ missing $C$ and in particular $T_1 \cup aT_1$. Hence $\text{BdS} \cap C$ is connected.

Throughout the remainder of this chapter let $A$ denote the arc $\text{BdS} \cap C$ from $a$ to $b$ and let $B$ denote the complement arc $\overline{\text{BdS} \setminus C}$.
Lemma 2.19. $b = a^{-1}$ and $aT_1 \cup a^{-1}T_1 = \overline{S \setminus C} \cap C$

where $T_1 = \{t \in T : t \geq e\}$.

Proof: There exists a unique $x \in H(e) \setminus \{e\}$ such that $x^2 = e$ or $x = x^{-1}$. For $s \in S \setminus SeS$, if $se = x$ then $es^{-1} = s^{-1}e = x$, so by lemma 2.11, $s = s^{-1}$.

Now if $\{t_n\}$ is a sequence converging to $a$, $t_n \in S \setminus C$ for all $n$, then $\{t_n e\}$ converges to $e$, so we choose a sequence $\{t_n\} \subset B$ converging to $a$ such that $t_n \neq t_n^{-1}$ for all $n$, $t_n e \neq t_m e$ for $n \neq m$, and $t_n$ and $t_n^{-1}$ are separated by $e$ and $x$ in $H(e)$, $\{t_n\} = t_n X \cap BdS$, $\{t_n^{-1}\} = t_n^{-1}X \cap BdS$, and $\{t_n e\}$ is contained in one of the two arcs from $e$ to $x$. Inversion is a homeomorphism, so $\{t_n^{-1}\}$ converges to $a^{-1}$ and both the sequence $\{t_n^{-1}\}$ and $a^{-1}$ are contained in $BdS$. $C$ is an inverse semigroup by lemma 2.13, so either $a^{-1} = a$ or $a^{-1} = b$ since $\{a,b\} = A \cap B$.

Let $H'$ be the arc of $H(e)$ from $e$ to $t_1 e$ which does not contain $x$ and consider the arc $aT_1 \cup t_1 T_1 \cup H'$. Note it is an arc since $aT_1 \cap t_1 T_1 = \emptyset$, $aT_1 \cap H' = \{e\}$ and $t_1 T_1 \cap H' = \{t_1 e\} \neq \{e\}$. Now the arc $aT_1 \cup t_1 T_1 \cup H'$ separates $t_n$, for $n > 1$, from $H(e) \setminus H'$ and in general separates any point of $B$ between $a$ and $t_1$ from $H(e) \setminus H'$. 
Consider $t_n^{-1}t_1$ for $n > 1$ and by the choice of the sequence $\{t_n\}$, $t_n^{-1}t_1 \cap t_nT_1 = \emptyset$, $t_n^{-1}T_1 \cap aT_1 = \emptyset$, and $t_n^{-1}e \not\in H'$. Now $t_n^{-1}T_1$ is a connected set containing $t_n^{-1}$ and meeting $H(e) \setminus H'$, so $t_n^{-1}$ does not lie between $a$ and $t_1$ in $B$.

Therefore the sequence $\{t_n^{-1}\}$ converges to $b$ and $a^{-1} = b$.

**Diagram**

![Diagram](image-url)
For \( t \in T_1 \), the sequence \( \{t_n t\} \) converges to \( a t \) and \( t_n t \in S \setminus C \) for all \( n \), so \( aT_1 \subseteq S \setminus \overline{C} \) and similarly \( a^{-1}T_1 \subseteq S \setminus \overline{C} \).

Let \( H_n \) be the arc of \( H(e) \) containing \( l \) from \( t_n e \) to \( t_n^{-1}e \) for each \( n \) and consider the arcs \( C_n = t_n T \cup t_n^{-1} T \cup H_n \). Let \( B_n \) be the arc of \( BdS \) from \( t_n \) to \( t_n^{-1} \) contained in \( B \) and \( A_n = \overline{BdS \setminus B} \). Note \( A = A_n \) and \( A_n \cap B_n = \{t_n, t_n^{-1}\} \). Now clearly \( C_n \cup A_n \cup B_n \) is a \( \theta \)-curve for each \( n \). Denote the bounded regions of the separation of \( \mathbb{R}^2 \) by each \( \theta \)-curve by \( P_n \) and \( Q_n \) where \( P_n \) has edges \( C_n \) and \( B_n \) and \( Q_n \) has edges \( C_n \) and \( A_n \). From the definitions of the \( \theta \)=curves it is clear that \( C \subseteq \overline{Q_n} \) for each \( n \) and \( \overline{Q_{n+1}} \subseteq \overline{Q_n} \) for all \( n \). Let \( Q = \bigcap_{n=1}^{\infty} \overline{Q_n} \) and clearly \( Q \cap H(e) = \{e\} \). By lemma 2.15, for \( s \in Q \), \( sT_1 \cap H(e) = \{e\} \) and \( Q = C \) and it follows that \( A \cup aT_1 \cup a^{-1}T_1 = BdC \) and hence \( aT_1 \cup a^{-1}T_1 = S \setminus \overline{C} \).

**Theorem 2.20.** There does not exist an inverse semigroup on the 2-cell with an identity whose set of idempotents \( E \) has a cut point and \( \dim E = 2 \).

**Proof:** If such a semigroup \( S \) exists, then since \( \{t_n^{-1}\} \) converges to \( a^{-1} \), \( \{at_n^{-1}\} \) converges to \( aa^{-1} \).
Now \( \{at_n^{-1}\} \subseteq S \setminus C \), so \( aa^{-1} \in S \setminus C \cap C = aT_1 \cup a^{-1}T_1 \).

Therefore either \( aa^{-1} \leq a \) and \( a^{-1} \leq a^{-1}a \) or \( aa^{-1} \leq a^{-1} \) and \( a \leq a^{-1}a \). Hence in either case \( a = a^{-1} \in E \) and \( C = T_1 \), a contradiction since \( \dim C = \dim E = 2 \).

Combining Theorem 2.7 and Theorem 2.20, we have established the following main result.

**Theorem 2.21.** If \( S \) is an inverse semigroup with an identity on the 2-cell whose set of idempotents \( E \) has a cut point then \( S \) is a min cone over a circle and \( E \) is a min thread.

**Concluding Remarks:** The following result is a consequence of Theorems 2.20, 2.7 and corollary 2.6.

**Theorem 2.22.** If \( S \) is an inverse semigroup with an identity on an annulus whose kernel \( K \) is one of the boundary circles, then \( S \) is isomorphic to \( K \times E \) where the set of idempotents \( E \) is a min thread.
CHAPTER III

Introduction. In this chapter we investigate inverse semigroups on the 2-cell with an identity 1 and whose set of idempotents E has no cut point. The inversion homeomorphism is shown to be the identity map, and it then follows that S must be a commutative Clifford semigroup. The maximal group H(l) containing 1 then must be a subgroup, not necessarily proper, of the four group, \(Z_2 \times Z_2\). Examples are given for all possible cases of H(l).

In the special case when \(M(e) = \{f \in E : e \preceq f\}\) is connected for each \(e \in E\) and \(H(1) \cong Z_2 \times Z_2\), then the semigroup S must be the continuous monotone homomorphic image of the \((\text{min cone over } Z_2) \times (\text{min cone over } Z_2)\).

Preliminaries and Results. Throughout this chapter, S will denote an inverse semigroup on the 2-cell with an identity 1 and whose set of idempotents E has no cut point.

By [13], E is a retract and by Theorem 1.7, E is a 2-cell. T will denote a min thread from 1 to 0 and let \(C = \{s \in S : s = s^{-1}\}\). It is clear that \(T \subseteq E \subseteq C\).

Lemma 3.1. C is closed.
Proof: Let \( x \in S \) and \( \{x_n\} \) a sequence of elements of \( C \) converging to \( x \). Now \( x_n = x_n^{-1} \), so \( \{x_n^{-1}\} \) is a sequence converging to \( x \) also, but since inversion is continuous, \( \{x_n^{-1}\} \) converges to \( x^{-1} \). Hence \( x = x^{-1} \) and \( x \in C \). Now it follows that \( C \) is closed.

Lemma 3.2. For \( c \in C \cap \text{Bd}S \), let \( U \) be an open connected disc about \( c \) such that \( \overline{U} \) is a 2-cell, \( \overline{U} = \overline{U}^{-1} \), and \( \overline{U} \cap \text{Bd}S \) is an arc from \( a \) to \( b \) containing \( c \). Denote \( \overline{U} \cap \text{Bd}S \) by \( A \) and let \( A_1 \) be the arc of \( A \) from \( a \) to \( c \) and \( A_2 \) the arc of \( A \) from \( c \) to \( b \). If \( \{t_n\} \) is a sequence in \( A_1 \setminus \{a, c\} \) converging to \( c \) such that \( t_n \not\in C \) for all \( n \), then there exists an integer \( N \geq 1 \) such that \( t_n^{-1} \in A_2 \setminus \{b, c\} \) for all \( n \geq N \).

Proof: Suppose there is a subsequence \( \{t_{n_j}\} \subseteq \{t_n\} \) such that \( \{t_{n_j}^{-1}\} \subseteq A_1 \setminus \{a, c\} \). Clearly both \( \{t_n\} \) and \( \{t_{n_j}^{-1}\} \) converge to \( c \). Consider the subarcs of \( A_1 \setminus \{a, c\} \) from \( t_{n_j} \) to \( t_{n_j}^{-1} \), denoted by \( A(j) \).

The inversion map \( i \) is a homeomorphism, mapping \( \text{Bd}S \) onto \( \text{Bd}S \), \( c \) onto \( c \) and, in particular, \( A \) onto \( A \). Therefore, the connected set \( i(A(j)) \) is contained in \( A \setminus \{1\} \) since \( i(1) = i^{-1}(1) = 1 \not\in A(j) \) for all \( j \), and \( \{t_{n_j}^{-1}, t_{n_j}^{-1}\} \subseteq A(j) \subseteq i(A(j)) \subseteq A \setminus \{a, c\} \). Now homeomorphisms
preserve cut points, so $i(A(j)) = A(j)$ for all $j$.

The restriction of $i$ to $A(j)$ has a fixed point, so there exists $x_j \in A(j) \setminus \{t^n_j, t^{-1}_n\}$ such that $i(x_j) = x_j$ or $x_j = x^{-1}_j$. Clearly $x_j$ and $c$ separate $t^n_j$ from $t^{-1}_n$ in $BdS$. The convergence of $\{t^n_j\}$ to $c$ and $\{t^{-1}_n\}$ to $c$ implies the existence of $x_h$ and $x_k$, elements of $A \setminus \{a, c\}$ such that $x_h = x^{-1}_h$, $x_k = x^{-1}_k$, and $x_h$ and $x_k$ separate $t^n_h$ from $t^{-1}_n$ in $BdS$. That is, the arc from $x_h$ to $x_k$ contained in $A \setminus \{a, c\}$ contains either $t^n_h$, or $t^{-1}_n$ but not both.

Now the homeomorphism $i$ maps this arc from $x_h$ to $x_k$ onto itself, a contradiction since the arc contains one but not both of $t^n_h$ and $t^{-1}_n$. Therefore, $\{t^{-1}_n\}$ is residually in $A_2 \setminus \{c, b\}$ and the proof is complete.

**Corollary 3.3.** Either $BdS \subset C$ or $BdS \cap C$ contains no arcs.

**Proof:** Suppose $BdS \not\subset C$ and suppose there exists $c \in BdS \cap C$ such that $c$ is an end point of an arc in $BdS \cap C$ and $c \in BdS \setminus C$. 
Now let \( U \) be an open connected disc about \( c \) as described in the lemma, such that \( A_2 \subset C \) and \( A_1 \not\subset C \). Let \( \{t_n\} \) be a sequence converging to \( c \) such that \( \{t_n\} \subset A_1 \setminus C \).

Therefore, \( t_n \neq t_n^{-1} \) for all \( n \) and by the lemma \( \{t_n^{-1}\} \) is residually in \( A_2 \), a contradiction, for if \( t_n^{-1} \in A_2 \) then \( t_n^{-1} = t_n \). Therefore, \( \text{Bd}S \cap C \) contains no arcs, or, it is totally disconnected if \( \text{Bd}S \not\subset C \). We proceed now to show that the inversion homeomorphism is the identity map. That is \( C = S \). The first step is to establish that \( \text{Bd}S \not\subset C \).

This will be done by considering the following two cases:

1. If \( T \setminus \{1\} \cap \text{Bd}S \neq \emptyset \) and
2. If \( T \setminus \{1\} \cap \text{Bd}S = \emptyset \).

**Case 1:** For \( t \in T \setminus \{1\} \cap \text{Bd}S \), let \( T_1 = \{t_1 \in T : t_1 \geq t\} \) and let \( A \) and \( B \) be the two arcs of \( \text{Bd}S \) from \( 1 \) to \( t \) such that \( A \cup B = \text{Bd}S \) and \( A \cap B = \{1,t\} \).

Note that if \( \text{Bd}S \not\subset C \), then \( A \not\subset T_1 \) and \( B \not\subset T_1 \). That is, \( T_1 \cap \text{int} S \neq \emptyset \).

**Lemma 3.4.** Either \( aT \cap T_1 \neq \emptyset \) for all \( a \in A \) or \( bT \cap T_1 \neq \emptyset \) for all \( b \in B \).

**Proof:** Suppose there exists \( a \in A \setminus \{1,t\} \) and \( b \in B \setminus \{1,t\} \) such that \( aT \cap T_1 = \emptyset = bT \cap T_1 \), or, that \( (aT \cup bT) \cap T_1 = \emptyset \). Since \( T_1 \), \( A \), and \( B \) are all arcs
from 1 to t such that \( a \notin T_1 \) and \( b \notin T_1 \), it follows

\[ T_1 \cap B = A \cup B \]

or that \( T_1 \cap \text{int } S \neq \emptyset \). Therefore, by

Theorem 1.1, \( T_1 \) cuts \( S \) and separates \( a \) from \( b \) by

Theorem 1.2. Now \( 0 \in aT \cap bT \), so \( aT \cup bT \) is a

connected set containing both \( a \) and \( b \). But

\[ (aT \cup bT) \cap T_1 = \emptyset \]

is a contradiction.

Remark: Without loss of generality, we may assume

\( aT \cap T_1 \neq \emptyset \) for all \( a \in A \). Note also that the arcs \( aT \)

and \( bT \) in the corollary may be replaced with the arcs \( Ta \)

and \( Tb \).

Corollary 3.5. If \( af \in aT \cap T_1 \) for \( f \in T \), then

\[ f > t \]

Proof: If \( f < t \), then \( af \cdot t = a \cdot ft \), and since

\( af \in T_1 \), \( af > t \) and \( af \cdot t = t \). So \( af = t \), and since

\( af \in E \), \( af < f \) or \( t < f \), a contradiction. Therefore,

\[ f > t \] and \( aT \cap T_1 = aT_1 \cap T_1 \).

Lemma 3.6. For case 1, \( BdS \subseteq C \).

Proof: Suppose \( BdS \not\subseteq C \) and by corollary 3.3 \( BdS \not\subseteq C \)

contains no arcs. If follows that \( T_1 \) separates \( S \). Let

\( \{t_n\} \) be a sequence converging to 1 contained in \( A \),

\( t_n \notin C \) for all \( n \), and such that the sequence \( \{t_n^{-1}\} \)
is contained in $B$. Such a sequence exists since $BdS \cap C$
is totally disconnected and by lemma 3.2 the inverses are
in $B$. Clearly $\{t_n^{-1}\}$ converges to 1. Let

$$f_n = \sup \{ t_1 \in T_1 : t_n t_1 \in T_1 \}. \quad \text{The existence of } f_n \text{ is}
$$
clear from the assumption that $aT_1 \cap T_1 \neq \emptyset$ for all $a \in A$.

Now it follows that $t_n f_n \in T_1$, and since

$$t_n f_n \in E, \quad t_n f_n = (t_n f_n)^{-1} = f_n t_n^{-1} \quad \text{and } T_1 t_n^{-1} \cap T_1 = \emptyset.$$  

Clearly $f_n = \sup \{ t_1 \in T_1 : t_n t_1^{-1} \in T_1 \}$ and $f_n \neq 1$ for all $n$. Let $T_n = \{ t_1 \in T_1 : t_1 \geq f_n \}$ and by the definition of

$$f_n, \quad t_n T_n \cap T_1 = \{ t_n f_n = f_n t_n^{-1} \} = T_n t_n^{-1} \cap T_1.$$  

Let $C(t_n^{-1})$ be
the components of $S T_1$ containing $t_n$ and $t_n^{-1}$ repectively. By lemma 1.2, $C(t_n) \cap C(t_n^{-1}) = \emptyset$ for all $n$.

Since $T_n \setminus f_n$ is a connected set and

$$t_n (T_n \setminus f_n) \cap T_1 = \emptyset, \quad t_n (T_n \setminus f_n) \subset C(t_n) \quad \text{for all } n \quad \text{and}
$$
similarly $(T_n \setminus f_n) t_n^{-1} \subset C(t_n^{-1})$ for all $n$. For $h \in T_n$,

if $t_n h \in t_n T_n \cap E$, then $t_n h = (t_n h)^{-1} = h t_n^{-1}$ and

$$t_n h \in t_n T_n \cap T_n t_n^{-1}. \quad \text{But } t_n T_n \cap T_n t_n^{-1} = \{ t_n f_n \}, \quad \text{so}
$$

$$t_n T_n \cap E = \{ t_n f_n = f_n t_n^{-1} \} = T_n t_n^{-1} \cap E.$$  

Now $K_n = t_n T_n \cup T_n t_n^{-1}$
is an arc from $t_n$ to $t_n^{-1}$ such that $K_n \cap E = \{ t_n f_n \}$. 
We consider now the two cases:

(i) there exists an $n$ such that $f_n t_n \neq t$ or

(ii) $t_n f_n = t$ for all $n$.

**Case (i).** Since $1 \notin K_n$ and $t \notin K_n$, $K_n \not\subseteq B_dS$ if $t_n f_n \neq t$, so $K_n$ separates $S$ and in particular $K_n$ separates $1$ from $t$. It follows that $K_n$ separates $E$. Therefore, $\{t_n f_n\} = K_n \cap E$ is a cut point of $E$, a contradiction.

**Case (ii).** Suppose $t_n f_n = t$ for all $n$. If $t_1 \in (T_1 \cap B_dS) \setminus \{1, t\}$, then either $t_1 \in A \setminus \{1, t\}$ or $t_1 \in B \setminus \{1, t\}$. Letting $t_1 \in B$ without loss of generality, the arc $T(t_1)$ from $1$ to $t_1$ of $T_1$ separates $S$ and would also separate $t_n^{-1}$ from $t$ for some $n$. But $T_n t_n^{-1}$ is a connected set from $t_n^{-1}$ to $t$ such that $T_n t_n^{-1} \cap T_1 = \{f_n t_n^{-1} = t\}$, so $T(t_1) \subseteq T_1 \setminus t$, $T(t_1) \cap T_n t_n^{-1} = \emptyset$ and $t_n^{-1}$ and $t$ lie in the same component of $S \setminus T(t_1)$, a contradiction. Therefore, $T_1 \cap B_dS = \{1, t\}$ and $T_1$ separates $S$ into exactly two components, $C_A$ containing the arc $A \setminus \{1, t\}$ and $C_B$ containing the arc $B \setminus \{1, t\}$. 

If \( 1 = \sup\{f_n \in T_1\} \), then since \( f_n \neq 1 \) for all \( n \), there is a subsequence \( \{f_{n_j}\} \) converging to \( 1 \). Now \( \{t_{n_j}\} \) converges to \( 1 \), so \( \{t_{n_j}f_{n_j}\} \) converges to \( 1 \), a contradiction since \( t_{n_j}f_{n_j} = t \) for all \( j \) and \( t \neq 1 \).

Therefore, \( e < 1 \) where \( e = \sup\{f_n \in T_1\} \). Let \( f \) be any element of \( T_1 \setminus \{1\} \) larger than \( e \) and by the definition of the \( f_n \)'s, \( t_n f \) and \( f t_n^{-1} \) are not idempotents. Note \( t_n f \in T_1 \setminus C_A \) and \( f t_n^{-1} \in T_1 t_n^{-1} \setminus C_B \).

If \( f \in \text{int } E \), then there is an open set \( U \) about \( f \) such that \( U \subseteq E \). Since \( \{t_n f\} \) converges to \( f \), \( \{t_n f\} \) is residually in \( U \), a contradiction, since \( t_n f \not\in E \) for all \( n \).

Therefore, the arc of \( T_1 \) from \( 1 \) to \( e \) is contained in \( \text{Bd}E \), a simple closed curve, and in particular \( f \in \text{Bd}E \).

There exists an open set \( U \) about \( f \) such that either \( U \cap C_A \subseteq E \) or \( U \cap C_B \subseteq E \). The sequence \( \{t_n f\} \) is residually in \( U \cap C_A \) and the sequence \( \{f t_n^{-1}\} \) is residually in \( U \cap C_B \), a contradiction since \( \{t_n f\} \cap E = \emptyset = \{f t_n^{-1}\} \cap E \). Therefore \( \text{Bd}S \subseteq C \) under the conditions of case 1.
Case 2. Suppose $T \setminus \{1\} \cap \text{Bd}S = \emptyset$ and let $U$ be an open connected disc about 1, such that $\overline{U}$ is a 2-cell, $\overline{U}^{-1} = \overline{U}$, $0 \not\in \overline{U}$, and $\overline{U} \cap \text{Bd}S$ is an arc $A$ from $a$ to $b$ where $1 \in A \setminus \{a,b\}$. Let $A_1$ be the arc of $A$ from $a$ to $1$ and let $A_2$ be the arc of $A$ from $1$ to $b$.

Lemma 3.7. In case 2, $\text{Bd}S \subseteq C$.

**Proof:** Suppose $\text{Bd}S \not\subseteq C$ and then by corollary 3.3 $\text{Bd}S \cap C$ contains no arcs. Hence there is a sequence $\{t_n\} \subseteq A_1 \setminus \{a,1\}$, $t_n \not\in C$ for all $n$, converging to 1 and by lemma 3.2, the sequence $\{t_n^{-1}\}$ is residually in $A_2 \setminus \{1,b\}$, also converging to 1. Let $t = \sup\{t_1 \in T : t_1 \in \overline{U} \setminus U\}$ and $T_1 = \{t_1 \in T : t_1 \geq t\}$. Clearly, $t \neq 1$ and $T_1$ is a subarc of $T$ spanning $\overline{U}$ from 1 to $t$, separating $\overline{U}$ into exactly two components $R_1$ containing $A_1 \setminus \{a,1\}$ and $R_2$ containing $A_2 \setminus \{1,b\}$. Let $f_n = \sup\{t_1 \in T : t_1 \in \overline{U} \setminus U\}$. Then $t_n f_n \in \overline{U} \setminus U$, and it follows that $(t_n f_n)^{-1} = f_n t_n^{-1} \overline{U} \setminus U$ and $f_n = \sup\{t_1 \in T : t_1 t_n^{-1} \in \overline{U} \setminus U\}$ since inversion is a homeomorphism.

For all $n$, $f_n \neq 1$ since $t_n \not\in \overline{U} \setminus U$. Let $T_n = \{t_1 \in T : t_1 \geq f_n\}$ and clearly $t_n \cdot T_n \setminus \{f_n\}$ and $T_n \setminus f_n \cdot t_n^{-1}$ are contained in $U$. If there exists an $n$
such that \( t_n T_n \cap E = \emptyset \), let \( f = \sup \{ t \in T_n : t_n t \in E \} \).

Then \( t_n f = (t_n f)^{-1} = ft_n^{-1} \), \( f \neq 1 \), and if \( T_n = \{ t \in T_n : t \leq f \} \), then \( t_n (T_n \setminus f) \cap E = \emptyset = (T_n \setminus f) t_n^{-1} \cap E \), \( t_n (T_n \setminus f) \subset R_1 \), and \( (T_n \setminus f) t_n^{-1} \subset R_2 \). Therefore \( t_n f = ft_n^{-1} \in T_1 \).

If \( t_n f = 1 \), then \( t_n^{-1} t_n f = t_n^{-1} e E \), and hence \( t_n^{-1} = t_n \), a contradiction. Therefore \( t_n f \neq 1 \). The arc \( K_n = t_n T_n \cup T_n t_n^{-1} \) from \( t_n \) to \( t_n^{-1} \) is contained in \( \overline{U} \) such that \( t_n T_n \cap t_n^{-1} = \{ t_n f = ft_n^{-1} \} \) and

\[
t_n T_n \cap E = \{ t_n f \} = T_n t_n^{-1} \cap E.
\]

Clearly \( K_n \not\subset A \) since \( l \not\in K_n \), so \( K_n \) separates \( \overline{U} \) and \( S \) and in particular it separates \( l \) from the complement of \( \overline{U} \) which contains \( 0 \). Therefore, it separates \( E \) and \( t_n f \) is a cut point of \( E \), a contradiction. Therefore for all \( n \), \( t_n T_n \cap E = \emptyset \).

Let \( e = \sup \{ f_n \in T \} \). If \( e = 1 \), then since \( f_n \neq 1 \) for all \( n \), there is a sequence \( \{ f_n \} \) converging to \( 1 \).

But the sequence \( \{ t_n f_n \} \) converges to \( 1 \) and also it is contained in \( \overline{U} \setminus U \), a contradiction since \( U \) was chosen open about \( 1 \). Therefore \( e \neq 1 \).

For \( f \in T_1 \), between \( 1 \) and \( e \), \( t_n f \in R_1 \), \( ft_n^{-1} \in R_2 \), and \( \{ t_n f, ft_n^{-1} \} \cap E = \emptyset \) for all \( n \).
Consider the two cases when either the arc of $T_1$ from $1$ to $e$ is contained in $\text{BdE}$, or there is an $f$ between $1$ and $e$ contained in the interior of $E$.

By the same arguments as presented at the end of case 1 in lemma 3.6, a contradiction is reached in both cases. Therefore, $\text{BdS} \subseteq C$ for case 2. By combining the two cases, the following theorem has been established.

**Theorem 3.8.** $\text{BdS} \subseteq C$.

We proceed now to one of the main results of this chapter.

**Theorem 3.9.** $S = C$ or the inverse homeomorphism is the identity map.

**Proof:** Define a map $m$ from $C \times T$ onto $C$ by the following: $m(c,t) = tct$. Clearly $m$ is continuous and $(tct)^{-1} = t^{-1}c^{-1}t^{-1} = tct$ for all $c \in C$. Note that $m$ is an action since

$$m(c,th) = thcht = thcht = m(hch,t) = m(m(h,c),t).$$

For all $c \in C$, $m(c,1) = c$ and $m(c,0) = 0$. Hence by the generalized homotopy theorem, $C$ is acyclic [18]. Therefore, $C$ does not cut $\mathbb{R}^2$, and since $\text{BdS} \subseteq C$, $S = C$. 
Corollary 3.10. S is commutative.

Proof: For any two elements \( a \) and \( b \) of \( S \),
\[ a = a^{-1}, \quad b = b^{-1}, \quad \text{and} \quad ab = (ab)^{-1} \]
so
\[ ab = (ab)^{-1} = b^{-1}a^{-1} = ba. \]

Corollary 3.11. S is a Clifford semigroup. That is, S is the union of groups.

Proof: For all \( a \in S \), \( aa^{-1} = a^2 = a^{-1}a \), so the right and left units of each element of \( S \) are equal.

The corollary now follows from [5, pp.41]. We now turn our attention to the maximal group \( H(1) \). By the previous theorem and corollaries, \( H(1) \) is commutative and every element has order 2.

Lemma 3.12. The order of \( H(1) \) is less than or equal to 4.

Proof: Using the notation that \( O(G) \) denotes the order of a group \( G \), suppose \( O(H(1)) > 4 \). Hence there are elements \( a, b, \) and \( c \) of \( H(1) \) different from 1, such that \( ab \neq c \). Let \( G \) be the subgroup of \( H(1) \) generated by these three elements. It follows that \( O(G) = 8 \) and \( G \subseteq \text{BdS} \). Let \( x \) be one of the two elements of \( G \) such that \( x \) and 1 separate \( \text{BdS} \) into an arc \( A \) having only one element \( y \) of \( G \) between 1 and \( x \), and
the complementary arc \( B \) containing 5 elements of \( G \) between 1 and \( x \). Multiplication by \( x \) is a homeomorphism, implying either \( xA = A \) or \( xA = B \). Remembering that \( xG = G \), \( xA \neq B \), since \( xA \) contains 3 elements of \( G \) and \( B \) contains 7 elements of \( G \). But if \( xA = A \), then \( xy = y \), a contradiction. Therefore \( O(H(1)) \leq 4 \).

**Corollary 3.13.** Either (i) \( H(1) = \{1\} \)

(ii) \( H(1) \approx \mathbb{Z}_2 \)

or (iii) \( H(1) \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \).

**Proof:** The corollary follows immediately from the theory of groups, the preceding lemma, and the fact that all elements are of order 2.

We now give examples showing the existence of an inverse semigroup of the type investigated in this chapter in each of the three cases for \( H(1) \).

First we define two inverse semigroups on an interval \([13]\). Let \( I_1 \) be the inverse semigroup defined on the closed interval \([-1,1]\) by

\[
ab = \begin{cases} 
\min \{|a|, |b|\} & \text{if } ab \geq 0 \\
-\min \{|a|, |b|\} & \text{if } ab < 0.
\end{cases}
\]

\( I_1 \) is a min cone over \( \mathbb{Z}_2 \) or \( I_1 \approx \frac{\text{IX}[{-1,1}]}{0xx[-1,1]} \) where \( I \) is a min thread and \([-1,1]\) is the 2 element group.
Let $I_2$ be the inverse semigroup defined on the closed interval $[-1,1]$ by

$$ab = \begin{cases} \min\{a,b\} & \text{if } ab \geq 0 \\ \min\{|a|,|b|\} & \text{if } ab < 0. \end{cases}$$

$I_2$ is a min cone over a 2 element semilattice or $I_2 \cong \mathbb{I} \times \{0,1\}$ where $\{0,1\}$ has the usual semilattice multiplication. Note that $I_2$ is a semilattice.

**Example 1.** $S_1 = I_1 \times I_1$

Properties: (1) $H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, (2) $0 \in \text{int } S$, (3) $E = \{(a,b): a \geq 0 \text{ and } b \geq 0\}$, (4) $H(a,b) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ for all $(a,b)$ such that $a \neq 0$, $b \neq 0$, and (5) $S_1 = \cup (i,j)E$ where $(i,j) \in H(1)$ and $i$ and $j$ take on values of 1 and -1.

**Example 2.** $S_2 = I_1 \times I_1$ or equivalently the inverse subsemigroup of $S_1$, $S_2' = \{(a,b) \in S_1: b \geq 0\}$.

Properties: (1) $H(1) \cong \mathbb{Z}_2$, (2) $E = \{(a,b): a \geq 0\}$, (3) $S_2 = E \cup (-1,1)E$, (4) $E \cap (-1,1)E = I$, (5) $0 \in \text{Bd } S_2$ and (6) $H(a,b) \cong \mathbb{Z}_2$ for all elements $(a,b)$ such that $a \neq 0$.

**Example 3.** $S_3 = I_1 \times I_2$

Properties: (1) $H(1) \cong \mathbb{Z}_2$, (2) $0 \in \text{int } S_3$, (3)
$E = \{ (a,b) : a \geq 0 \}$  
$S_3 = E \cup (-1,1)E$,  
$E \cap (-1,1)E \approx I_2$  
and  
$H(a,b) \approx \mathbb{Z}_2$  
for all elements $(a,b)$ such that $a \neq 0$.

**Example 4.**  
$S_4 = \{ (a,b) \in S_1 : b \geq -1/2 \}$.

Properties:  
(1) $H(1) \approx \mathbb{Z}_2$,  
(2) $0 \in \text{int } S_4$,  
(3) $E = \{ (a,b) : a \geq 0 \}$ and  
(4) $S_4$ contains groups of orders 4 and 2.

**Example 5:**  
$S_5 = \{ (a,b) \in S_1 : -3/2 \leq b-a \leq 3/2 \}$.

Properties:  
(1) $H(1) \approx \mathbb{Z}_2$,  
(2) $0 \in \text{int } S_5$,  
(3) $E = \{ (a,b) : a \geq 0 \}$ and  
(4) $S_5$ contains groups of order 4 and 2.

**Example 6:**  
$S_6 = \{ (a,b) \in S_1 : a \geq -1/2 \text{ and } b \geq 0 \}$.

Properties:  
(1) $H(1) = \{1\}$,  
(2) $E = \{ (a,b) : a \geq 0 \}$,  
(3) $S_6 = E \cup (-1/2,1)E$,  
(4) $E \cap (-1/2,1)E \approx I_2$,  
(5) $0 \in \text{Bd}S_6$ and  
(6) $H(a,b) \approx \mathbb{Z}_2$  
for all elements $(a,b)$ such that $a \neq 0$ and $a \leq 1/2$.

**Example 7:**  
$S_7 = \{ (a,b) \in S_3 : a \geq -1/2 \}$.

Properties:  
(1) $H(1) = \{1\}$,  
(2) $0 \in \text{int } S_7$,  
(3) $E = \{ (a,b) : a \geq 0 \}$,  
(4) $S_7 = E \cup (-1/2,1)E$,  
(5) $E \cap (-1/2,1)E \approx I_2$ and  
(6) $H(a,b) \approx \mathbb{Z}_2$  
for all $(a,b)$ such that $a \neq 0$ and $a \leq 1/2$. 
We now prove two lemmas which will lead to the establishment of the fact that $0 \in S \setminus E$ if $H(1) \cong Z_2$ or $H(1) \cong Z_2 \times Z_2$.

**Lemma 3.14.** If $a \in H(1) \setminus \{1\}$, then $a \in T$, and if $at \in a \cap E$ for $t \in T$, then $at = t$.

**Proof:** Let $f \in a \cap E$. Then $f = at$ for some $t \in T$ and $at \in E$. So, $f = at = (at)^2 = a^2t^2 = t$ and $f = at = t \in T$.

**Lemma 3.15.** For $f \in T \setminus \{0\}$, let $C_f = \{ s \in S : sf = f \}$. Then $C_f$ is a compact, connected inverse subsemigroup such that $C_f \cap \text{Bd}S$ is connected.

**Proof:** Clearly $C_f$ is a compact inverse subsemigroup. Let $T_1 = \{ t \in T : t \geq f \}$. For $s \in C_f$, $st_1$ is connected and contains both $s$ and $f$. If $st \in st_1$, the $tf = f$ and $st \cdot f = sf = f$, so $st_1 \subseteq C_f$ and hence $C_f$ is connected.

Define the map $m$ from $C_f \times T_1$ onto $C_f$ by $m(c, t) = ct$. Now $m(ckt) = cth = m(m(c, t), h)$, so $T_1$ acts on $C_f$. $m(c, 1) = c$ and $m(c, f) = f$ for all $c \in C_f$, hence it follows that $C_f$ is acyclic. Since $C_f \neq S$, $BdS \not\subseteq C_f$. 
Suppose \( \text{BdS} \cap C_f \) is not connected, and let \( p \) and \( q \) be elements of two different components of \( \text{BdS} \cap C_f \). Let \( A \) and \( B \) be the two arcs of \( \text{BdS} \) from \( p \) to \( q \). It follows that there exists \( a \in A \) and \( b \in B \) such that \( \{a, b\} \cap C_f = \emptyset \). Let \( g = \sup \{t \in T : at = bt\} \) and let \( T_0 = \{t \in T : t \geq g\} \), and hence \( aT_0 \cap bT_0 = \{ag = bg\} \) and by a straightforward argument, \( (aT_0 \cup bT_0) \cap C_f = \emptyset \).

Since \( \{p, q\} \cap (aT_0 \cup bT_0) = \emptyset \), \( aT_0 \cup bT_0 \not\subseteq \text{BdS} \), and hence \( aT_0 \cup bT_0 \) is an arc from \( a \) to \( b \) separating \( S \) and in particular separating \( p \) from \( q \), a contradiction, since \( C_f \) is connected containing both \( p \) and \( q \) and \( (aT_0 \cup bT_0) \cap C_f = \emptyset \). Therefore, \( C_f \cap \text{BdS} \) is connected.

**Theorem 3.16.** If \( H(1) \neq \{1\} \) then \( 0 \in \overline{S \setminus E} \).

**Proof:** It is necessary to consider only the case when \( aT \cap E = aT \cap T \neq \{0\} \), for all \( a \in H(1) \setminus \{1\} \). Since \( O(H(1)) = 4 \) or \( 2 \), there exists \( a \in H(1) \setminus \{1\} \), and a decomposition of \( \text{BdS} \) into two arcs \( A_1 \) and \( A_2 \) from \( 1 \) to \( a \). Let \( f = \sup \{t \in T : at = t\} \) and it follows that \( af = f \neq 0 \).

Now multiplication by \( a \) is a homeomorphism, so either \( aA_1 = A_1 \) or \( aA_1 = A_2 \).

**Case 1.** Suppose \( aA_1 = A_1 \). The arc \( A_1 \) has the
fixed point property, so there exists $x \in A_1 \setminus \{1,a\}$ such that $ax = x$, and clearly $axt = xt$ for all $xt \in xT$. Similarly there exists $y \in A_2 \setminus \{1,a\}$ such that $ay = y$ and $ayt = yt$ for all $yt \in yT$. It is clear that

$(yT \cup xT) \cap B_dS = \{x,y\}$, since each arc $A_1$ and $A_2$ can have only one fixed point under the multiplication map by $a$. Let the arcs $X$ and $Y$ from $x$ to $y$ be the decomposition of $B_dS$ such that $1 \in X$ and $a \in Y$. Let $e = \sup \{t \in T : st = yt\}$ and let $T_e = \{t \in T : t \geq e\}$.

Consider the arc $A = xT_e \cup yT_e$ from $x$ to $y$ which certainly separates $S$ into exactly two regions, $R_1$ with boundary $X \cup A$, and $R_2$ with boundary $Y \cup A$. Note that $A \cup X \cup Y$ is a $\theta$-curve.

Multiplication by $a$ maps $X$ onto $Y$, $Y$ onto $X$, and $R_1$ onto $R_2$. Therefore, $A = \{s \in S : as = s\} = xT \cup yT$, $e = 0$ and $xT \cap yT = \{0\}$.

Since $ax^2 = x^2$ and $ay^2 = y^2$, either $x^2 \in xT$ and $x^2 \leq x$ or $y^2 \in yT$ and $y^2 \leq y$, or, $x = x^2$ or $y = y^2$ since $S$ is stable. Without loss of generality, assume $x$ is idempotent. So $xT \subseteq E$. Note that if either $x$ or $y$ is zero then the other one must be idempotent, so assume $x \neq 0$.

Let $t \in xT \setminus \{x,0\}$ and suppose $t \in \text{int} E$. Let $U$
be an open set about $t$ contained in $E$ such that $aU = U$. Let $e \in U \cap R_1$. Now $ae \in U \cap R_2 \subseteq E$, so $ae = (ae)^2 = a^2 e^2 = e$, a contradiction, since $R_1 \cap R_2 = \emptyset$. Therefore, $xT \subseteq \overline{S \cap E}$ and in particular $0 \in \overline{S \cap E}$.

**Case 2.** Suppose $aA_1 = A_2$. Under this multiplication by $a$, there is not a point $x$ in $BdS$ such that $ax = x$. Since $0 = a \cdot 0$, $0 \in \text{int } S$.

Consider $C_f$ which contains $a$ and $1$ but not $BdS$. Since $C_f \cap BdS$ is connected, lemma 3.15, either $A_1 \subseteq C_f$ or $A_2 \subseteq C_f$. But $C_f$ is a subsemigroup, so either $A_1 \cup aA_1 \subseteq C_f$ or $A_2 \cup aA_2 \subseteq C_f$, a contradiction, since $aA_1 = A_2$ and $A_1 \cup aA_1 = A_2 \cup aA_2 = BdS$. Therefore, $aT \cap E = aT \cap T = \{0\}$ and the theorem is established.

**Remark:** If $H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then there exists $c \in H(1) \setminus \{1\}$ such that $C_1 \setminus (c, 1) \cap H(1) \neq \emptyset \neq C_2 \setminus (c, 1) \cap H(1)$ where $C_1$ and $C_2$ are the two arcs of $BdS$ from $1$ to $c$. It follows that $cC_1 = C_2$, and from the proof of case 2, $0 \in \text{int } S$ and $cT \cap E = cT \cap T = \{0\}$. 
Section I

Throughout this section, $H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $a, b, c$ will denote the elements of $H(1) \setminus \{1\}$ such that:

(i) The decomposition of $BdS$ into arcs $A_1$ and $A_2$ from 1 to $a$, has the properties that $aA_1 = A_1$, $aA_2 = A_2$ and $(c, b) \subseteq A_2 \setminus \{1, a\}$,

(ii) The decomposition of $BdS$ into arcs $B_1$ and $B_2$ from 1 to $b$, has the properties that $bB_1 = B_1$, $bB_2 = B_2$ and $(a, c) \subseteq B_2 \setminus \{1, b\}$, and

(iii) The decomposition of $BdS$ into arcs $C_1$ and $C_2$ from 1 to $c$, has the properties that $cC_1 = C_1$, $cC_2 = C_2$, $a \in C_1 \setminus \{1, c\}$ and $b \in C_2 \setminus \{1, c\}$.

Theorem 3.17. If $H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ then:

(1) There exists $e \in A_1$ and $f \in B_1$ such that $ae = e$, $bf = f$, $ce \in C_2 \cap B_2$, and $cf \in C_1 \cap A_2$.

(2) $eT \cup ceT = \{s \in S : as = s\}$, $(eT \cup ceT) \cap BdS = \{e, ce\}$ and $eT \cap ceT = \{0\}$.

(3) $fT \cup cfT = \{s \in S : bs = s\}$, $(fT \cup cfT) \cap BdS = \{f, cf\}$ and $fT \cap cfT = \{0\}$.

(4) $eT \cap fT = \{0\}$

(5) $eT \cap cfT = \{0\}$ and dually $fT \cap ceT = \{0\}$.

(6) If $Q$ denotes the 2-cell bounded by the arc of $BdS$ from $e$ to $f$ containing $1$ and $eT \cup fT$, then

$S = Q \cup aQ \cup bQ \cup cQ$, $Q \cap aQ = eT$, $Q \cap bQ = ft$,
\( aQ \cap cQ = cfT , \ bQ \cap cQ = ceT , \) and \( Q \cap cQ = \{0\} = aQ \cap bQ. \)

(7) \( \{ s \in S : cs = s \} = \{0\}. \)

(8) \( e \in E \) and \( f \in E \)

(9) \( eT \cup fT = BdE \cap S \setminus E \) and

(10) \( E \subset Q. \)

Diagram

\[
\begin{array}{cccc}
  & a & e & l \\
  aQ &   &   &   \\
  cf &   &   &   \\
  cQ &   &   & bQ \\
  c &   & ce & b \\
\end{array}
\]

Note that \( A_1 \subset B_2, \ B_1 \subset A_2, \ A_1 \subset C_1, \ B_1 \subset C_2, \ C_2 \subset A_2, \)

and \( C_1 \subset B_2. \)

**Proof of 1, 2, and 3:** From case 1, Theorem 3.16, there is a unique element \( e \) of \( A_1 \) such that \( ae = e. \)

Since \( c = ab \) and \( ac = b \), \( ce = abe = be \), \( ce \in cA_1 \subset C_1 = C_2 \), and \( ce = be \in bA_1 \subset bB_2 = B_2. \)

Therefore, \( ce \in C_2 \cap B_2 \) and \( a \cdot ce = be = ce. \) Hence \( ce \) is
is the unique element of $A_2$ such that $ace = ce$. The proof of case 1 of Theorem 3.16 implies $eT \cup ceT = \{s \in S : as = s\}$ and $eT \cap ceT = \{0\}$. Clearly

$$(eT \cup ceT) \cap B_dS = \{e, ce\}. $$

Similarly there exists a unique element $f$ of $B_1$ such that $bf = f$, $bcf = cf$, $cf \in C_1 \cap A_2$, $fT \cup cfT = \{s \in S : bs = s\}$, $(fT \cap cfT) \cap B_dS = \{f, cf\}$, and $fT \cap cfT = \{0\}$.

**Proof of 4:** Suppose $x \in eT \cap fT$. Then there exists $t \in T$ such that $x = et = ft$. Multiplying by $a$ and $b$ implies $aft = aet = et = ft$ and $bft = ft$; so $cft = abft = aft = ft = x$, and hence $x \in fT \cap cfT$. By part (3), $x = 0$ and $eT \cap fT = \{0\}$.

**Proof of 5:** For $x \in eT \cap cfT$, there exists $t \in T$ such that $x = et = cft$. Hence

$$x = et = aet = acft = bft = ft,$$

and $x \in eT \cap fT$ and by part (4) $x = 0$. Similarly, $fT \cap ceT = \{0\}$.

**Proof of 6:** It is clear now that the arcs, $eT$, $ceT$, $fT$, and $cfT$ separate $S$ into 4 regions, each region containing one and only one element of $H(1)$. Also, multiplication by elements of $H(1)$ maps these regions onto themselves in the natural way. Therefore (6) is obvious.
Proof of 7: Let \( x \in \{ s \in S : cs = s \} \), and since \( cs = abs = x \), \( bx = as \).

If \( x \in Q \), then \( ax \in aQ \cap bQ \), and hence \( ax = 0 \) and \( x = 0 \).

If \( x \in aQ \), then \( bx \in baQ = cQ \) and \( ax \in a \cdot aQ = Q \) so \( ax \in Q \cap cQ \), and hence \( ax = 0 \) and \( x = 0 \).

If \( x \in bQ \), then \( ax \in abQ = cQ \) and \( bx \in b \cdot bQ = Q \) so \( ax \in cQ \cap Q \), and hence \( ax = 0 \) and \( x = 0 \).

If \( x \in cQ \), then \( ax \in acQ = bQ \) and \( bx \in bcQ = aQ \) so \( ac \in bQ \cap aQ \), and hence \( ax = 0 \) and \( x = 0 \).

Proof of 8, 9, and 10: By the proof of case one of Theorem 3.16, either \( e \in E \) or \( ce \in E \). If both are idempotent, then \( e \in e^2 \) and \( ce = (ce)^2 = c^2e^2 = e \), a contradiction, since \( ce \neq e \). Therefore, either \( e \in E \) or \( ce \in E \) but not both, and similarly, either \( f \in E \) or \( cf \in E \), but not both.

If \( e \in E \), then \( ceT \cap E = \{0\} \); for if \( cet \in E \), then

\[
 cet = (cet)^2 = c^2e^2t^2 = et ,
\]

and \( cet \in ceT \cap eT = \{0\} \). Similarly if \( ce \in E \), then \( eT \cap E = \{0\} \). Note that the same statements hold if \( e \) is replaced by \( f \).
Suppose \( ce \in E \) and \( cf \in E \); then \((e \cup f) \cap E = \{0\}\) and the arc \( e \cup f \) separates 1 from \( cf \) and \( ce \), or, 0 is a cut point of \( E \), a contradiction. If \( ce \in E \) and \( f \in E \), then by the proof of Theorem 3.16, \( f \cup ce \subset S \setminus E \) and, therefore, either \( E = bQ \) or \( E \subset aQ \cup aQ \cup cQ \). \( E \not\subset bQ \) since 1 \( \not\in bQ \) and \( e \) separates 1 from \( ce \) in \( aQ \cup aQ \cup cQ \), so \( E \subset aQ \cup aQ \cup cQ \), a contradiction.

By replacing \( e \) with \( f \) and \( f \) with \( e \), it becomes clear that both \( cf \) and \( e \) cannot be idempotent.

Now, after exhausting all other possible cases, \( e \) and \( f \) must be idempotent. Clearly \( e \cup f = E \cap S \setminus E \), and either \( E = cQ \) or \( E = aQ \cup bQ \cup cQ \). Since \( c \) separates \( e \) from \( f \) in \( aQ \cup bQ \cup cQ \) and \( c \cap E = \{0\} \), \( E \not\subset aQ \cup bQ \cup cQ \). Therefore, \( E = cQ \) and the proof of the theorem is complete.

**Corollary 3.18.** \( ef = 0 \).

**Proof:**

\[ cef = ab\epsilon = aebf = ef, \]

so by part (7), \( ef = 0 \).

**Corollary 3.19.** If \( T \) and \( T_1 \) are min threads from 1 to 0, then \( fT = fT_1 \) and \( eT = eT_1 \).
Proof: From the theorem it is clear that

\[ fT = \{ s \in S : bs = s \} \cap E = fT_1. \]

**Corollary 3.20.** If \( x = aQ \) and \( T \) is any min thread from 1 to 0, then \( xT \subseteq aQ \).

**Proof:** Suppose \( xT \not\subseteq aQ \); then there exists \( t \in T_1 \) such that \( xt_1 \in ET \cup cfT \). Assume \( t = \sup \{ t_1 \in T : xt_1 \in ET \cup cfT \} \) and let \( h \in T \) such \( h < t \). If \( xt \in ET \), then \( ext = xt \) and \( exh = exth = eth = xh \). For \( xt \in ET \), \( x^2t = xt \) and it follows that \( xh \in ET \). Hence \( xT \subseteq aQ \). If \( xt \in cfT \), then \( bxt = xt \) and \( bxh = bxth = xth = xh \), so \( xh \in cfT \cup fT \). Now \( xt = cft \) for some \( t_1 \in T \), so \( xh = xth = cft \) \( h = cf(t_1 h) \in cfT \) and the corollary is established.

Note that \( aQ \) may be replaced by \( Q \), \( bQ \), or \( cQ \) and the corollary still holds.

**Remark:** It is clear from corollary 3.20 that if \( xt \in \text{int}(aQ) \) for \( t \in T \), then \( xT \subseteq aQ \).

**Lemma 3.2.** If \( B \) is the arc of \( BdS \) from \( e \) to \( f \) containing \( l \) and \( T \) is any min thread from 1 to 0, then \( BT = Q \).

**Proof:** \( BT \subseteq Q \) by corollary 3.20 and clearly the boundary of \( Q \), \( ET \cup fT \cup B \), is contained in \( BT \). Now
T acts on BT by \( m(bt, t_1) = bt t_1 \), and \( m(bt, 1) = bt \) and \( m(bt, 0) = 0 \) for all \( bt \in BT \), hence BT is acyclic and it follows that \( BT = \mathbb{Q} \).

We continue our investigation in this section by considering a class of semigroups where the semilattice structure of \( E \) has the following additional hypothesis: For \( e \in E \) define \( M(e) = \{f \in E : e \leq f \} \) and assume \( M(e) \) is connected for all \( e \in E \).

Semilattices on a 2-cell with this property have been studied in [4] with one of the results being:

**Theorem 3.22.** [4] Suppose \( E \) has a \( 1 \).

These are equivalent:

(i) For each \( e \), \( M(e) \) is a connected set;

(ii) The boundary of \( E \) is the union of two min threads from \( 1 \) to \( 0 \); and

(iii) \( E \) is the continuous homomorphic image of \( I \times I \).

Throughout the remainder of this section, assume \( M(e) \) is connected for each \( e \in E \).

By Theorems 3.22 and 3.17, the arc of \( \text{Bd}E \) from \( 1 \) to \( e \) is a min thread \( J \), and the arc of \( \text{Bd}E \) from \( 1 \) to \( f \) is a min thread \( K \).

**Lemma 3.23.** \( JK = E \)
Proof: It is clear that \( eK = eT \) and \( fJ = fT_1 \), so \( BdE = J \cup K \cup eK \cup fJ \subseteq JK \). It follows that \( (BdE)^2 \subseteq JK \) and by \([4]\), \( (BdE)^2 = E \). Hence \( JK = E \).

**Theorem 3.24.** E is a continuous homomorphic image of IXI such that \((0,1)\) and \((1,0)\) are mapped onto \(e\) and \(f\) respectively.

**Remark:** The homomorphism of Theorem 3.22 \([4]\) does not have the properties required by this Theorem.

**Proof:** Denote \( IX[1] \) by \( V \) and \([1]\times I \) by \( W \) and let \( \alpha_1 \) and \( \alpha_2 \) be isomorphisms from \( V \) and \( W \) onto \( J \) and \( K \) respectively.

Define the map \( \alpha \) from \( IXI \) onto \( E \) by

\[
\alpha(a,b) = \alpha_1(a,1)\alpha_2(1,b)
\]

For \((a,b)\) and \((c,d)\) in \( IXI \), \( \alpha((a,b)(c,d)) = \alpha(\min\{a,c\}, \min\{b,d\}) = \alpha_1(\min\{a,c\}, 1)\alpha_2(1, \min\{b,d\}) \)

and \( \alpha(a,b)\alpha(c,d) = \alpha_1(a,1)\alpha_2(1,b)\alpha_1(c,1)\alpha_2(1,d) = \min\{\alpha_1(a,1), \alpha_1(c,1)\}, \min\{\alpha_2(1,b), \alpha_2(1,d)\} \). But since \( \alpha_1 \) and \( \alpha_2 \) are isomorphisms, \( \alpha_1(\min\{a,c\}, 1) = \min\{\alpha_1(a,1), \alpha_1(c,1)\} \) and \( \alpha_2(1, \min\{b,d\}) = \min\{\alpha_2(1,b), \alpha_2(1,d)\} \), so it follows that \( \alpha \) is a homomorphism.
Consider the following commuting diagram:

\[
\begin{array}{ccc}
V \times W & \xrightarrow{\alpha_1 \times \alpha_2} & J \times K \\
| \downarrow m_1 \quad \alpha \quad \downarrow m_2 | & & | \downarrow & E \\
I \times I & & \\
\end{array}
\]

The maps \( m_1 \) and \( m_2 \) are the corresponding multiplication functions. Now \( m_1 \) maps the compact set \( V \times W \) onto \( I \times I \), and \( \alpha m_1 = m_2(\alpha_1 \times \alpha_2) \) is continuous, hence \( \alpha \) is continuous. By lemma 3.23, \( m_2 \) is onto, hence \( \alpha \) is onto and the proof is complete.

**Remark:** By [4] \( \alpha \) is monotone and by [20] a simple closed curve (or arc) is mapped by \( \alpha \) onto a simple closed curve (or arc), possibly degenerate. It follows from [7] that the monotone continuous homomorphism \( \alpha \) maps a min thread onto a min thread or a single point.

**Lemma 3.25.** Let \( \Delta = \{(a,b) \in I \times I : a = b\} \) and let \( T = \alpha(A) \); then \( T \) is a min thread from 1 to 0 such that \( T \setminus \{0,1\} \cap B \setminus E = \emptyset \).

**Proof:** Clearly \( T \) is a min thread from 1 to 0 [7]. Consider the simple closed curves \( C_1 = (0,1) \cup U \cup J \wedge \Delta \) and \( C_2 = (1,0) \cup U \cup W \cup \Delta \). Since 1 and 0 are elements of both
\( \alpha(C_1) \) and \( \alpha(C_2) \) are mapped onto simple closed curves.

Now \( \alpha(V \cup (0,1)W) = E \cup eK \) and \( \alpha(W \cup (1,0)V) = K \cup fJ \), so \( \alpha(C_1) = cE \cup E B_1 \) and \( \alpha(C_1) = fE \cup E B_2 \) where \( B_1 \) and \( B_2 \) are arcs from 1 to 0 such that \( B_1 \setminus [0,1] \cap (E \cup E J) = \emptyset \) and \( B_2 \setminus [0,1] \cap (fE \cup E K) = \emptyset \). The min thread \( T \) is an arc from 1 to 0 contained in \( E \cup E J \cup B_1 \) and also contained in \( fE \cup E K \cup B_2 \). The simple closed curves \( \alpha(C_1) \) and \( \alpha(C_2) \) contain only two arcs each from 1 to 0, namely \( eE \cup E J \) and \( B_1 \) for \( \alpha(C_1) \) and \( fE \cup E K \) and \( B_2 \) for \( \alpha(C_2) \). Since \( \alpha(C_1) = \alpha(V \cup (0,1)W \cup \Delta) = E \cup E J \cup T \) and \( \alpha(C_2) = fE \cup E K \cup T \), it follows that \( T = B_1 = B_2 \) and \( T \setminus [0,1] \cap \text{BdE} = \emptyset \).

**Remark:** By corollary 3.19, \( eE = eT \) and \( fJ = fT \).

Let \( x \in J \cup K \setminus \{e,f\} \) and \( (a,b) \in V \cup W \) such that \( \alpha(a,b) = x \). Note \( (a,b) \not\in (1,0)V \cup (0,1)W \). Consider the min thread \( (a,b)\Delta \) from \( (a,b) \) to \( (0,0) \). By the multiplication of \( \text{IXI} \), \( (a,b)\Delta \) meets the boundary of \( \text{IXI} \) only at \( (a,b) \) and \( (0,0) \).

**Lemma 3.26.** \( \alpha((a,b)\Delta) = xT \) is a min thread from \( x \) to 0 such that \( xT \cap \text{BdE} = \{x,0\} \).

**Proof:** Clearly \( xT \) is a min thread from \( x \) to 0,
and by using the same arguments of the preceding lemma, it follows that \( xT\{x,0\}\cap BdE = \Box \).

**Lemma 3.27.** If \( x \in E\setminus eT \cup fT \), then 
\[ xT\{x,0\}\cap BdE = \Box. \]

**Proof:** Let \((a,b) \in I \times I\) such that \( a \neq 0 \), \( b \neq 0 \) and \( \varphi(a,b) = x \). If \( a < b \), consider \((a,1)\Delta\); and if \( b < a \), consider \((1,b)\Delta\). Without loss of generality, assume \( a < b \). Now \((a,b)\Delta = (a,1)\Delta\) and hence \( xT\cap(a,1)T \). But by Lemma 3.26, \( \alpha(a,1)T\{\alpha(a,1),0\}\cap E = \Box \), so clearly 
\[ xT\{x,0\}\cap BdE = \Box. \]

We now turn our attention to the structure of \( Q \), remembering that \( B \) denotes the arc of \( BdQ \) from \( e \) to \( f \).

**Lemma 3.28.** If \( x \in B \) and \( x^2 \in eT \cup fT \), then \( x = x^2 \) and \( x \in E \).

**Proof:** For some \( t \in T \), either \( x^2 = et \) or \( x^2 = ft \). Hence \( x = xet \) or \( x = xft \) and \( ax = axet = xet = x \) or \( bx = bxft = xft = x \). Now \( e \) is the only element of \( B \) left fixed by multiplication by \( a \), and \( f \) is the only element of \( B \) left fixed by multiplication by \( b \). Therefore, \( x = e \) or \( x = f \), and in either case \( x^2 \leq x \) and \( x = x^2 \).

**Theorem 3.29.** \( E = Q \).
Proof: It will suffice to show $B \subseteq E$ since $BT = Q$ by lemma 3.21.

For $x \in B$ consider $xT$ where $T = \alpha(\Delta)$. The arc $cTUxT$ cuts $S$ and separates $e$ from $f$ where $cT \cap E = \{0\}$ by the proof of Theorem 3.16. Therefore, for $h = \sup \{t \in T: xt \in E\}$, $xh \neq 0$. Clearly $xh \in BdE$, so $xh = (xh)^2 = x^2h$ and $x^2h \in x^2T \cap BdE$.

If $x^2 \in eTUfT$, then $x \in E$ by lemma 3.28. If $x^2 \not\in eTUfT$, then $x^2T \setminus \{x^2, 0\} \cap BdE = \emptyset$ by lemma 3.27. Therefore, $x^2h = x^2$ since $x^2h = xh \neq 0$ and multiplying by $x$ yields $x = xh = x^2h = x^2$ and $x \in E$. Therefore $B \subseteq E$ and $E = Q$.

Theorem 3.30. $S$ is the continuous monotone homomorph ic image of $S_1 = I_1 x I_1$, the semigroup defined in Example 1.

Proof: Express $S_1$ as $IxI \cup (-1,1)(IxI) \cup (1,-1)$ \([IxI \cup (-1,1)(IxI)]\) and remember that $S_2$, the semigroup of Example 2 is $IxI \cup (-1,1)(IxI)$. That is, $S_1 = S_2 \cup (1,-1)S_2$. We now extend $\alpha$, the map defined in Theorem 3.24, to $S_2$ in the obvious way. Consider the following commutative diagram
The maps $f$ and $\rho_a$ are the homeomorphism of multiplication by $(-1,1)$ and $a$, respectively, and $i$ is the injection map.

$\beta_1$ is defined by $\beta_1(s,t) = \beta_1((-1,1)(-s,t)) = a\alpha(-s,t)$.

Clearly $\beta_1$ is continuous and onto, and

$\beta_1|\{0\}xI = \alpha|\{0\}xI$ where $\{0\}xI = IXI \cap (-1,1)(IXI)$.

Define the map $\beta$ from $S_1$ onto $EUaE$ by

$$\beta(s,t) = \begin{cases} 
\alpha(s,t) & \text{if } (s,t) \in IXI \\
\beta_1(s,t) & \text{if } (s,t) \in (-1,1)(IXI).
\end{cases}$$

It follows that $\beta$ is continuous, and if $(s,t)$ and $(x,y)$ are elements of $IXI$, then $\beta((s,t)(x,y)) = \beta((s,t)\beta(x,y))$.

If $(s,t)$ and $(x,y)$ are elements of $(-1,1)(IXI)$, then $(s,t) = (-1,1)(-s,t)$ and $(x,y) = (-1,1)(-x,y)$, and $(-s,t)$ and $(-x,y)$ are elements of $IXI$.

$$(s,t)(x,y) = (-s,t)(-x,y)$$

$$\beta((s,t)(x,y)) = \beta((s,t)(-x,y)) = \beta(-s,t)\beta(-s,y)$$

$$= a\beta(-s,t)a\beta(-s,y) = \beta(s,t)\beta(x,y).$$
If \((s,t)\in\mathbb{I}\times\mathbb{I}\) and \((x,y)\in(-1,1)\mathbb{I}\times\mathbb{I}\), then

\[(x,y) = (-1,1)(-x,y)\]

where \((-x,y)\in\mathbb{I}\times\mathbb{I}\) and

\[\beta((s,t)(x,y)) = \beta((-1,1)(s,t)(x,y)) = \alpha((s,t)(-x,y)) = \alpha(s,t)\alpha(-s,y) = \beta(s,t)\cdot\alpha(-x,y) = \beta(s,t)\beta(x,y).\]

Therefore, \(\beta\) is a homomorphism and it follows from the diagram that \(\beta_1\) is monotone. Hence \(\beta\) is clearly monotone.

Next, consider the following commutative diagram.

The maps \(f_1\) and \(\rho_b\) are the homeomorphisms of multiplication by \((1,-1)\) and \(b\), respectively, and again \(i\) is the injection map. Define \(\gamma\) by

\[\gamma(s,t) = \gamma((1,-1)(s,-t)) = b\beta(s,-t).\]

By the same arguments as before, \(\gamma\) is an onto continuous monotone homomorphism.

Define the map \(\hat{\beta}\) from \(S_1\) onto \(S\) by

\[\hat{\beta}(s,t) = \begin{cases} 
\beta(s,t) & \text{if } (s,t) \in S_2 \\
\gamma(s,t) & \text{if } (s,t) \in (1,-1)S_2
\end{cases}\]

and it follows that \(\hat{\beta}\) is the desired continuous monotone
homomorphism.

We conclude this section with several questions.

**Question 1.** Without any additional hypothesis on $E$, is $Q$ an inverse subsemigroup of $S$?

**Question 2.** What is a necessary and sufficient condition for $E = Q$?

**Section II**

Throughout this section, $H(1) \cong Z_2$, and $a$ will denote the element of $H(1) \setminus \{1\}$ where the arcs $A_1$ and $A_2$ from 1 to $a$ are the decomposition of $BdS$. By Theorem 3.16, this section is clearly divided into the following two cases:

1. when $aA_1 = A_2$ and $aA_2 = A_1$
2. when $aA_1 = A_1$ and $aA_2 = A_2$.

**Case 1.** Example 5 illustrates the existence of a semigroup of this type and indicates a procedure of how other examples can be derived. A characterization of this type is unknown at this time so we conclude case 1 with the following questions:
Question 3. Does there exist a semigroup of this type such that $E \cap BdS = \{1\}$?

Question 4. What is a necessary and sufficient condition for a semigroup of this type to be the continuous monotone homomorphic image of $S_5$?

Case 2. By the proof of Theorem 3.16, we can assume without loss of generality that the unique element $e$ of $A_1$, such that $ae = e$, is idempotent. Let $y$ denote the unique element of $A_2$ such that $ay = y$.

There are three natural classes of semigroups to consider in this case;

(i) when $y = 0$

(ii) when $y \in E \setminus \{0\}$ and

(iii) when $y \not\in E$.

Remark: Examples 2, 3, and 4 illustrate the existence of semigroups of each of the three classes.

Theorem 3.31. If $y = 0$, then

1) $eT \cap BdS = \{x, 0\}$

2) $eT - BdE \cap S \setminus E$

3) If $Q$ is the 2-cell bounded by $eT$ and the arc $B$ of $BdS$ from $e$ to 0 containing 1, then $S = Q \cup aQ$

where $Q \cap aQ = eT$. 
4) $E \subset Q$.

**Proof:** Since $eT = \{s \in S : a_s = s\}$, the theorem follows from Theorems 3.16, 3.17 and lemma 1.5.

Assume $M(f)$ is connected for each $f \in E$, and let $J$ be the min thread of $BdE$ from 1 to $e$, and let $K$ be the min thread from 1 to 0 which does not contain $e$. The following lemmas and Theorem are stated without proof since their proofs are almost identical to the corresponding proofs of Lemmas 3.23, 3.25, 3.26, 3.27, 3.28 and Theorem 3.24 in Section I.

**Lemma 3.32.** $JK = E$.

**Theorem 3.33.** $E$ is a continuous monotone homomorphic image of $IXI$ such that $(0, 1)$ is mapped onto $e$.

**Lemma 3.34.** Let $\Delta = \{(s, t) \in IXI : s = t\}$ and let $T = a(\Delta)$; then $T$ is a min thread from 1 to 0 such that $T \setminus \{0, 1\} \cap E = \emptyset$.

**Lemma 3.35.** If $x \in J \cup K \setminus \{0, e\}$ and $(s, t) \in IXI$ such that $a(s, t) = x$, then $a((s, t)\Delta) = xT$ is a min thread from $x$ to 0 such that $xT \cap BdE = \{x, 0\}$.

**Lemma 3.36.** If $x \in E \setminus eT$, then $xT \setminus \{0, x\} \cap BdE = \emptyset$. 
Lemma 3.37. If \( B \) denotes the arc of \( Q \) from \( e \) to \( 0 \) containing \( 1 \), \( x \in B \), and \( x^2 \in eT \), then \( x = x^2 \) and \( x \in E \).

The parallel between this development and that of Section I ends, since \( Q \) does not necessarily equal \( E \).

Example 8. Consider \( S_4 \), the semigroup of Example 4, and let \( R = \{(0,b) \in S_4 : -1/2 \leq b \leq 1/2\} \), a closed ideal of \( S_4 \). Let \( S_8 \) be the Rees quotient modulo \( R \) [8]. This semigroup has the properties of case one class (i), \( M(f) \) is connected for each \( f \in E \), and \( E \) is a proper subset of \( Q \).

Theorem 3.38. If \( E = Q \), then \( S \) is the continuous monotone homomorphic image of \( S_1 = I_1 \times I_1 \), the semigroup defined in Example 2.

Proof: Extend the homomorphism of Theorem 3.33 in the obvious way, as in the proof of Theorem 3.30.

For classes (ii) and (iii) there are the obvious theorems similar to Theorem 3.31, which will be stated without proof.

Theorem 3.39. If \( y \in E \setminus \{0\} \), then

1) \((eT \cup yT) \cap B_dS = \{e, y\}\),

2) \(eT \cup yT = B_dE \cap \overline{S \setminus E}\),

3) If \( Q \) is the 2-cell bounded by \( eT \cup yT \) and the arc \( B \) of \( B_dS \) from \( e \) to \( y \) containing \( 1 \), then \( S = Q \cup aQ \).
where \( Q \cap aQ = eT \cup yT \)

4) \( E \subset Q \).

**Theorem 3.40.** If \( y \not\in E \), then

1) \((eT \cup yT) \cap \text{Bd}_S = \{e, y\}\)
2) \(eT \subset \text{Bd}_E \cap S \setminus E\)
3) \(eT \cup yT = \{s \in S : as = s\}\)
4) If \( Q \) is the 2-cell bounded by \( eT \cup yT \) and the arc \( B \) of \( \text{Bd}_S \) from \( e \) to \( y \) containing 1, then \( S = QUaQ \) where \( Q \cap aQ = eT \cup yT \).
5) \( E \subset Q \).

We conclude this section with the following example.

**Example 9.** Consider the subsemigroup of \( S_3 = I_1 \times I_2 \)
defined by \( S = \{(s, t) \in S_9 : t \geq -1/2\} \) and let the closed ideal \( R \) of \( S_9 \) be \( \{(s, t) \in S_9 : s = 0 \text{ and } -1/2 \leq t \leq 1/2\} \). Let \( S_9 \) be the Rees quotient modulo \( R [8] \). Properties of this semigroup are: 1) It is of case one class \( i \) type, 2) \( M(f) \) is not connected for all \( f \in E \) and 3) \( E = Q \). If \( e \) and \( f \) correspond to the points \((0, 1)\) and \((1, -1/2)\) then \( ef = 0 \).

**Conclusion.** By combining Theorem 3.30 and lemma 1 of [4], we have the following result.

**Theorem 3.41.** If \( S \) is an inverse semigroup on a
2-cell with an identity 1 whose set of idempotents \( E \) has no cut point and \( H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( S \) is the continuous monotone homomorphic image of \( S_1 \) if and only if \( M(e) \) is connected for each \( e \in E \).

**Example 11.** Consider \( S_9 \) embedded in the plane on the rectangle \([-1,1] \times [0,1]\) where \((1,1), (0,0), (-1,1), (0,1)\) and \((1,0)\) correspond to \(1, 0, a, e\) and \(f\) and \(a \in H(1) \setminus \{1\} \). Let \( S_{10} = S_9 \cup \{(s,t) \in \mathbb{R}^2 : (s-t) \in S_9 \} \) or \( S \cup (1,-1)S_9 \), where multiplication of \( S_{10} \) is defined by

\[
(s,t)(x,y) = \begin{cases} (s,t)(x,y) & \text{if } (s,t),(x,y) \in S_9 \\ (s_1-t)(x,-y) & \text{if } (s,t),(x,y) \notin S_9 \\ (1,-1)[(s,t)(x,-y)] & \text{if } (s,t) \in S_9 \text{ and } (s,y) \notin S_9. \end{cases}
\]

Properties of \( S_{10} \) are: (1) \( H(1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \)
(2) \( E = \{(s,t) : s \geq 0 \text{ and } t \geq 0 \} \), (3) \( 0 \in \text{ int } S_{10} \), (4) \( E = Q \) where \( Q \) is defined in Theorem 3.17 and (5) \( M(e) \) is not connected for all \( e \in E \).

It is clear from Examples 9 and 10 that if \( E \) is a semilattice on a 2-cell with an identity 1 and with min threads from \( e \) to 0 and \( f \) to 0 such that \( ef = 0 \), then examples of the type of Section I and Section II class (i) can be constructed.
Question 5. For Theorem 3.17, 3.31, 3.39 and 3.40, is \( Q \) necessarily a subsemigroup?

Question 6. If \( M(e) \) is connected for each \( e \in E \), must \( \mathcal{S} \) be a continuous monotone homomorphic image of a subsemigroup of \( \mathcal{S}_1 \)?
CHAPTER IV

Introduction. In the field of algebraic inverse semigroups, idempotent-separating congruences and their relation to the semilattice of idempotents have been studied in [5], [6], [9], and [17]. The existence and characterization of a maximal idempotent-separating congruence \( U \) is presented in [9]. In [17], a condition on the semilattice of idempotents is given which is sufficient to imply that the \( v \) equivalence is a congruence and hence \( v = u \).

The purpose of this chapter is to show the existence of a lattice of idempotents for each element of the semigroup. These lattices are generated by the row and column idempotents of the powers of the elements. The original goal was to characterize \( u \) in terms of these lattices. That is, if \( a v b \) and \( a \) and \( b \) generate the same lattice, then is \( (a,b) \in v \)? The answer to this question is no and an example is given. However, the condition that two \( v \)-equivalent elements have the same lattice is shown to be stronger than the condition that the row idempotents of their powers are equal.

Preliminaries and Results. Throughout this chapter,
S will denote an algebraic inverse semigroup and E will denote its semilattice of idempotents.

**Definition 4.1.** The row (column) idempotent of an element \( a \) is the unique idempotent \( e \) such that \( a e (a e) \).

For \( a \in S \), the row idempotent is \( aa^{-1} \) and the column idempotent is \( a^{-1}a \) [5].

**Definition 4.2.** A congruence \( \rho \) on \( S \) is called idempotent-separating if each \( \rho \) equivalence class contains at most one idempotent. The maximal idempotent-separating congruence \( \mu \) is contained in \( \mathcal{N} \) and characterized by [17];

\[(a, b) \in \mu \text{ if and only if } aea^{-1} = beb^{-1} \text{ for all } e \in E.\]

**Theorem 4.3.** The \( \mathcal{N} \)-equivalence is a congruence if and only if for any two idempotents \( e \) and \( f \), \( xe = ex \) for \( x \in H(f) \).

**Proof:** If \( \mathcal{N} \) is a congruence, then \( \mathcal{N} = \mu \) and for \( x \in H(f) \), \( xx^{-1} \) and \( x2xx^{-1} \). Hence \( x2ex^{-2} = x^{-1}ex \), and by multiplying by \( x \), \( x2ex^{-2}x = x^{-1}ex^2 \). But \( x^{-1}x = xx^{-1} = f \), \( x = fx = xf \) and \( x^{-1} = x^{-1}f = fx^{-1} \) since \( x \in H(f) \) so,

\[x^{-1}ex^2 = x^2ex^{-2}x = x^2ex^{-1}x^{-1}x = x^2ex^{-1}f = x^2ex^{-1}.\]
also, \( xux^{-1} \) implies \( xex^{-1} = x^{-1}ex \) so,
\[
xe = xfe = xef = xex^{-1}x = x^{-1}ex\cdot x = x^{-1}ex^2
\]
\[
= x^2ex^{-1} = x\cdot xex^{-1} = xx^{-1}ex = fex = ef_0 = ex.
\]
Conversely, for \( a^\#b \), \( aa^{-1} = bb^{-1} \), \( a^{-1}a = b^{-1}b \), \( ab^{-1} \)
and \( ba^{-1} \) are elements of \( H(aa^{-1}) \), and \( a^{-1}b \) and \( b^{-1}a \)
are elements of \( H(a^{-1}a) \). Therefore, \( ab^{-1} \), \( ba^{-1} \), \( a^{-1}b \),
and \( b^{-1}a \) commute with all idempotents. For any \( e \in E \)
\[
aea^{-1} = ae\cdot e\cdot a^{-1}aa^{-1} = ae\cdot a^{-1}a\cdot ea^{-1} = aeb^{-1}b\cdot ea^{-1}aa^{-1}
= aeb^{-1}beb^{-1}ba^{-1} = aeb^{-1}beb^{-1}ba^{-1} = aeb^{-1}ba^{-1}beb^{-1}
= ae.b^{-1}b.a^{-1}b.eb^{-1} = ae.a^{-1}bb^{-1}b.eb^{-1} = ae.a^{-1}b.eb^{-1}
= a.e.a^{-1}b.eb^{-1} = aa^{-1}b.e.eb^{-1} = bb^{-1}beb^{-1} = beb^{-1}.
\]
Therefore, \( (a,b) \in U \) and \( U = N \).

Remark: After this theorem had been established, it was found in a slightly different form as an exercise in [6].

Lemma 4.4. For \( a \in S \), the row idempotent of \( a^n \) is \( a^n a^{-n} \) and the column idempotent is \( a^{-n}a^n \).

Proof: \( aa^{-1}a = a \) and if \( a^ka^{-k}a^k = a^k \), then \( a^{-k}a^k \in E \) and
Therefore, $a^n = a^n a - n$ for all $n$, and similarly
$a^{-n} = a^{-n} a - n$. Hence $a^n a - n a - n a - n = a^n S$ and
$a^n a - n a - n a - n = a^n S$ and $a^n S = a^n a - n a - n a - n$.

Lemma 4.5. For $a \in S$, let $e_n = a^n a - n$ and
$f_n = a^{-n} a - n$; then $e_n e_{n+r} = e_{n+r} = e_{n+r}$ for $r \geq 1$ and
dually $f_n f_{n+r} = f_{n+r}$ for $r \geq 1$.

Proof:

$$a^n a - n a + r a - (n+r) = a^n a - n a a - r a - (n+r) = a^n a - n a r a - (n+r) = a^n a r a - (n+r)$$

Hence $e_{n+r} \leq e_n$ and $e_{n+r} e_n = e_{n+r}$. Similarly

$$f_{n+r} f_{n+r} = f_{n+r}$$

Lemma 4.6. If $e_n f_m \leq e_s f_t$, then

$$e_n f_m = e_{\max(n,x)} f_{\max(m,s)}$$

Proof:

$$e_n f_m = e_n f_m \cdot e_s f_t = e_n e_s f_m f_t = e_{\max(n,s)} f_{\max(m,t)}$$

Lemma 4.7. If $e_n f_m = e_{n+1} f_m$, then
\[ e_{n^r} = e_{n+r}^f \] for all \( r \geq 1 \).

**Proof:** Multiply \( e_{n^r}^f = e_{n+1}^f \) on the left by \( a \) and on the right by \( a^{-m}a^{m-1} \).

\[
a \cdot e_{n^r}^f a^{-m}a^{m-1} = ae_{n+1}^f a^{-m}a^{m-1}
\]

\[
a \cdot a^n a^{-n} a^{-m}a^{m}a^{-m}a^{m-1} = a a ^{n+1} a^{-n} a^{-m}a^{m}a^{-m}a^{m-1}
\]

\[
a^{n+1} a^{-n} a^{-m}a^{m}a^{m-1} = a^{n+2} a^{-n} a^{-m}a^{m}a^{m-1}
\]

\[
a^{n+1} a^{-n} a^{-m}a^{m}a^{m-1} = a^{n+2} a^{-n} a^{-m}a^{m}a^{m-1}
\]

Hence \( e_{n+1}^f a^{m-1} = e_{n+2}^f a^{m-1} \) and multiplying by \( f_m \) implies
\[ e_{n+1}^f a^{m-1} = e_{n+2}^f a^{m-1} \] and the lemma follows by induction.

**Remark:** Clearly if \( e_{n^r}^f = e_{n^r}^f m+1 \), then
\[ e_{n^r}^f = e_{n^r}^f m+r \] for \( r \geq 1 \), by a dual argument.

**Theorem 4.8.** For \( a \in S \)

\[ \wedge_a = \{ e_{n^r}^f : n \geq 1, m \geq 0 \text{ where } e_{n^r}^f o = e_n \text{ for all } n \} \]

is a lattice.

**Proof:** By lemma 4.5, \( \wedge_a \) is closed under multiplication, and it is well known that a commutative idempotent semigroup is a semilattice where multiplication is the semilattice operation. Hence it will suffice to show the existence of a "join" or "cup" operation.
For elements $e_{n^m}$ and $e_{s^t}$ of $\wedge_a$ define

$$e_{n^m} \vee e_{s^t} = \begin{cases} 
  e_{n^m} & \text{if } e_{s^t} \leq e_{n^m} \\
  e_{s^t} & \text{if } e_{n^m} \leq e_{s^t} \\
  e_{p^q} & \text{otherwise, where } p = \min \{m, s\} \\
  & \text{and } q = \min \{n, t\}.
\end{cases}$$

If the two elements compare, it is clear that $e_{n^m} \vee e_{s^t}$ is the least upper bound. Hence assume that they do not compare and suppose $e_{n^m} \leq e_{i^f} \leq e_{i^f}$ and $e_{s^t} \leq e_{j^f} \leq e_{j^f}$. We now consider all possible cases for $i$ and $j$ with respect to $p$ and $q$.

**Case 1.** If $i < p$ and $j < q$, then $e_{p} \leq e_{i}$ and $f_{q} \leq f_{j}$ by lemma 4.5, and it follows that $e_{p} f_{q} \leq e_{i} f_{j}$.

**Case 2.** If $i > p$ and $j < q$, then

$$e_{n^m} = e_{n^m} \cdot e_{i^f} = e_{\max(n, i)} \cdot f_{m}$$

$$e_{s^t} = e_{s^t} \cdot e_{i^f} = e_{\max(s, i)} f_{t}.$$ 

If $i = \max(n, i) = \max(s, i)$, then $e_{n^m} = e_{i^f}$ and $e_{s^t} = e_{i^f}$, and it follows that $e_{n^m}$ and $e_{s^t}$ compare.

Since $i > p = \min \{n, s\}$, we may assume $i > n$ without loss of generality, so the only other case to
consider is when $s = \max(s, i)$ and $s > i$. Therefore, $e_{n \cdot m} = e_{i \cdot m}$ and by lemmas 4.5 and 4.7 $e_{n \cdot m} = e_{s \cdot m}$. Hence $e_{n \cdot m}$ and $e_{s \cdot t}$ compare, and the conditions of case 2 can not exist if $e_{n \cdot m}$ and $e_{s \cdot t}$ do not compare.

**Case 3.** If $i \leq p$ and $j > q$, then by a dual argument of case 2, $e_{n \cdot m}$ and $e_{s \cdot t}$ must compare, so case 3 does not exist if $e_{n \cdot m}$ and $e_{s \cdot t}$ do not compare.

**Case 4.** If $i > p$ and $j > q$, then

$$e_{n \cdot m} = e_{n \cdot m} e_{i \cdot j} = e_{\max(i, n)} \cdot \max(j, m)$$

and

$$e_{s \cdot t} = e_{s \cdot t} e_{i \cdot j} = e_{\max(i, s)} \cdot \max(j, t).$$

If $i = \max(i, n) = \max(i, s)$ or $j = \max(j, m) = \max(j, t)$, then by the argument of case 2, $e_{n \cdot m}$ and $e_{s \cdot t}$ compare. Hence we can assume that $s > i > n$ and consider the subcases when (a) $m > j > t$ or (b) $t > j > m$. If $m > j > t$, then $e_{n \cdot m} = e_{i \cdot m}$, and by lemmas 4.5 and 4.7,

$$e_{n \cdot m} = e_{i \cdot m} = e_{s \cdot m},$$

and hence $e_{n \cdot m}$ and $e_{s \cdot t}$ compare. If $t > j > m$, then $e_{n \cdot m} = e_{i \cdot j}$, so again $e_{n \cdot m}$ and $e_{s \cdot t}$ compare.

Therefore, if $e_{n \cdot m}$ and $e_{s \cdot t}$ do not compare
and \( e_i f_j \) is larger than or equal to both, then \( i \leq p \)
and \( j \leq q \) and \( e_p f_q \leq e_i f_j \).

Hence \( e_p f_q \) is the less upper bound of the set of
elements larger than or equal to \( e_n f_m \) and \( e_s f_t \), and
\( \wedge_a \) is a lattice.

**Lemma 4.9.** If \((a,b) \in \mu\), then \( \wedge_a = \wedge_b \) and
\( \wedge_a^{-1} = \wedge_b^{-1} \).

**Proof:** It will suffice to show that \( a^n a^{-n} = b^n b^{-n} \)
and that \( a^{-n} a^n = b^{-n} b^n \) for all \( n \geq 1 \).

For \((a,b) \in \mu\), it follows that \( a \neq b \), \( a^{-1} a = b^{-1} b \)
and \( a^{-1} a = b^{-1} b \), so

\[
a^2 a^{-2} = a(aa^{-1})a^{-1} = b(aa^{-1})b^{-1} = b(bb^{-1})b^{-1} = b^2 b^{-2}
\]

and by induction \( a^n a^{-n} = b^n b^{-n} \) for all \( n \geq 1 \). \( \mu \) is a
congruence on \( S \), so if \((a,b) \in \mu\), then \((a^{-1}, b^{-1}) \in \mu\)
[17]. Therefore

\[
a^{-2} a^2 = a^{-1} (a^{-1} a) a = b^{-1} (a^{-1} a) b = b^{-1} (b^{-1} b) b = b^{-2} b^2
\]

and again by induction \( a^{-n} a^n = b^{-n} b^n \) for all \( n \geq 1 \).

The next lemma is obvious and is stated without a
proof.

**Lemma 4.10.** If \( \wedge_a = \wedge_b \), then the row idempotent
of \( a^n \) is equal to the row idempotent of \( b^n \) for all \( n \geq 1 \).
Examples.

Definition 4.11 [5]. By a one-to-one partial transformation of a set X we mean a one-to-one mapping \( \alpha \) of a subset \( Y \) of X onto a subset \( Y_1 = YA \) of X. By the inverse \( \alpha^{-1} \) of \( \alpha \) we mean the mapping of \( YA \) onto \( Y \) which is inverse to \( \alpha \) in the usual sense of mappings. Let \( J_X \) denote the set of all one-to-one partial transformations of X, including that of the empty set \( \square \) of X onto itself; this "empty transformation" will be denoted by 0. The product of two elements \( \alpha \) and \( \beta \) of \( J_X \) is defined as follows. Let \( Y \) be the domain of \( \alpha \) and \( Z \) that of \( \beta \). If \( YA \cap Z = \square \), define \( \alpha \beta = 0 \). Otherwise let \( W = (YA \cap Z)\alpha^{-1} \) and define \( \alpha \beta \) to be the iterate of \( \alpha|_W \) and \( \beta|_{WA} \) in the usual sense. Clearly, \( \alpha \beta \) is a one-to-one transformation of \( W \) onto \( WA \beta \), and so belongs to \( J_X \). Associativity is easily verified. Hence \( J_X \) is a semigroup, which is called the symmetric inverse semigroup on the set X.

If the set X is finite, then denote \( \alpha \in J_X \) by

\[
(x_1, \ldots, x_n, y_1, \ldots, y_n)
\]

where \( \{x_1, \ldots, x_n\} \) is the domain of \( \alpha \), \( \{y_1, \ldots, y_n\} \) is the range of \( \alpha \), and \( \alpha(x_i) = y_i \).
Example 1. Let $X = \{a,b,c,d,e,f\}$,

$\alpha = (\begin{array}{cccc} a & b & c & d \\ a & e & b & f \end{array})$ and $\beta = (\begin{array}{cccc} a & b & c & d \\ a & f & b & e \end{array})$.

It follows that $\alpha^2 = (\begin{array}{cc} a & c \\ a & e \end{array})$, $\alpha^n = (\begin{array}{c} a \end{array})$ for $n \geq 3$, $\beta^2 = (\begin{array}{cc} a & c \\ a & f \end{array})$, and $\beta^n = (\begin{array}{c} a \end{array})$ for $n \geq 3$. Hence

$$\wedge_\alpha = \wedge_\beta$$

But $\wedge_{\alpha^{-1}} \neq \wedge_{\beta^{-1}}$, since $\alpha^{-2} \alpha^2 = (\begin{array}{cc} a & e \\ a & e \end{array})$ and $\beta^{-2} \beta^2 = (\begin{array}{cc} a & f \\ a & f \end{array})$; therefore $(\alpha, \beta) \not\in u$.

Example 2. Let $X = \{a,b,c,d,e,f,g,h\}$,

$\alpha = (\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ a & c & e & f & g & h \end{array})$ and $\beta = (\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ a & c & f & e & g & h \end{array})$.

Now $\alpha^2 = (\begin{array}{cccc} a & b & c & d \\ a & e & g \end{array})$, $\alpha^3 = (\begin{array}{ccc} a & b \\ a & g \end{array})$, $\alpha^n = (\begin{array}{c} a \end{array})$ for $n \geq 4$, $\beta^2 = (\begin{array}{cccc} a & b & c & d \\ a & f & h & g \end{array})$, $\beta^3 = (\begin{array}{ccc} a & b \\ a & h \end{array})$, and $\beta^n = (\begin{array}{c} a \end{array})$ for $n \geq 4$.

It follows that $\alpha^n \alpha^{-n} = \beta^n \beta^{-n}$ for all $n$, $\alpha^{-1} \alpha^{-2} \alpha^2 = (\begin{array}{cc} a & e \\ a & e \end{array})$, and $\beta^{-1} \beta^{-2} \beta^2 = (\begin{array}{cc} a & f \\ a & f \end{array})$. Therefore, the row
idempotents of the powers of $\alpha$ and $\beta$ are equal but $\wedge_\alpha \neq \wedge_\beta$. 


VITA

James Edward L'heureux was born August 5, 1934, in Edgefield, South Carolina. He attended the public schools in Greenville and Georgetown, South Carolina, graduating from high school in 1952. In the fall of that year he entered Louisiana State University, where he received his B.S. degree in Physics in the Spring of 1956. For the next three years he served as an officer in the United States Air Force. He re-entered L.S.U. in September, 1959 and received his M.S. degree in Mathematics at the Summer Commencement, 1961.

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