The Plastic Deformation of Clamped Rectangular Membranes Subjected to Impulsive Loads.

Franklin James Kay

Louisiana State University and Agricultural & Mechanical College

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Doctor of Philosophy

in

The Department of Mechanical, Aerospace and
Industrial Engineering

by

Franklin J. Kay
B.S., Oklahoma State University, 1964
M.S., Oklahoma State University, 1965
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**NOMENCLATURE**

\( x,y,z \)  Rectangular cartesian coordinates  
\( A,B \)  Plate half lengths  
\( V \)  Initial transverse velocity  
\( r,r_0,R,R_0 \)  Radius vectors (hinge-curve)  
\( t \)  Time  
\( n \)  Arbitrary number  
\( \alpha,\theta,\beta \)  Angles as defined in context  
\( H,h \)  Plate thickness  
\( dL \)  Elemental length  
\( q \)  Collapse pressure  
\( U \)  Radial displacement  
\( \dot{U} \)  Radial velocity  
\( \sigma_\theta \)  Circumferential normal stress  
\( \sigma_R \)  Radial normal stress  
\( \sigma_{R\theta} \)  Normal stress in sloping part  
\( \rho \)  mass/unit volume  
\( \nu \)  Poisson's ratio  
\( \sigma \)  Yield stress  
\( C \)  \((\sigma/\rho)^{1/2}\)  
\( a,b,c,w \)  Hardening parameters  
\( \Sigma \)  Summation  
\( \varepsilon \)  Young's modulus
\bar{\sigma}, \bar{\varepsilon} \quad \text{Effective stress, effective strain}

e^* \quad \text{Numerically largest strain}

\tau_{\text{oct}}, \gamma_{\text{oct}} \quad \text{Octahedral shearing stress, octahedral shearing strain}
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ABSTRACT

An analytical procedure has been developed for obtaining the solution to clamped rectangular membranes subjected to impulsive loading. The impulsive loading was due to an explosion causing the membrane to move with a uniform initial velocity. The inelastic deformation problem was formulated using the basic laws of mechanics for dynamic equilibrium, a hinge line concept, and the necessary relationships of plasticity.

Solutions were obtained for mild steel materials using both the incremental theory of plasticity and the deformation theory of plasticity. Included in this solution are cases for work hardening of the material and no work hardening of the material. The peripheral hinge was assumed to form as a hyper-ellipse and to propagate inward while retaining its original shape.

It was possible to obtain a closed form solution using the deformation theory and the no work hardening material. Solutions using the incremental theory are by nature obtained numerically. The topographical plots showing the propagation of the hinge toward the center of the membrane have shown the incremental theory to give the more
satisfactory solution for rectangular membranes.

Comparison of the results of this study and other work done in this area appear to validate the approach used in this study. The equations describing the rectangular membrane were reduced to those of the circular membrane and the deformation results obtained were compared with the solutions of Hudson and Frederick. The solution obtained for circular membranes approximated satisfactorily the experimental results shown in the work of Frederick.

Included in this study is a limited parametric study which illustrates the effects of varying the aspect ratio. Instead of the conical shape obtained for the shape of the deformed circular membrane, the deformed shape of the rectangular membrane can best be described as an elliptical shell.
CHAPTER I

INTRODUCTION

Dynamic deformations of structural members have long been an area of interest to scientists and engineers. The advent of the space age and its inherently severe requirements in the area of dynamic environments such as mechanical shock and vibration and the contrasting demand for lowest possible structural weight necessitate obtaining the maximum load carrying capacities for structural elements within allowable deformation limitations. The ability to predict deflections and lateral-load carrying capacities, including ultimate capacities, of plates is important where plates are an integral part of the structural design and are used as load-carrying elements or substructures. Since many of the materials used in engineering applications have considerable ductility beyond their elastic limit, the load-carrying capacity or the flexural capacity of a plate could be substantially greater than that at the first occurrence of yielding. Redistribution of the stresses
within the plate allows it to carry additional loads.

The elastic flexural behavior of plates is well known and documented, and the upper- and lower-bound ultimate capacities of plate structures of perfectly plastic material can be determined with the theorems of limit analysis. The inelastic behavior beyond the occurrence of first yielding of rectangular plates is comparatively unknown. Even less is known for plates of work-hardening materials, since the available methods of limit analysis do not apply.

Many practical and important problems in the area of load carrying capacity and deformation could be described, but in the interest of brevity only a few will be given. Structural integrity is of prime importance when it is related to the safety of people. Such a case would be the housing surrounding tests of explosive components or high pressure facilities where people working in the area could be injured from structural failure. The load carrying capacity and deformation is also of interest in various metal forming techniques, structural integrity of missiles against anti-missile attack, panel materials for protection of the people against nuclear attack, the response of submarine hulls to
large underwater explosions, and many other examples could be cited.

These considerations illustrate sufficient justification for the development of analytical methods of predicting the inelastic behavior and load carrying capacities of plates subjected to impulsive loading. Development in this area will aid the designer in the optimum usage of material and help delete some of the questions which necessitate large safety factors in design.

Definition of the Problem

The design of plates using mild steel, aluminum, and other materials is a problem frequently encountered in the analysis of structures. One problem of design is the response of steel plates to high intensity impulsive loadings, such as those associated with explosives. The exact solution of such a problem is very formidable and probably cannot be obtained at the present time because of the high degree of non-linearity of the describing equations. Therefore, the exact solution must be replaced with an approximate analysis that is more amenable to numerical computations.

The formation of this problem is applicable
to plates of arbitrary shape, but different assumptions would have to be made for the hinge line development. The primary emphasis of this solution is placed on the solution of isotropic rectangular plates for the cases of ideal plasticity and for work hardening.

The model describing the behavior of the rectangular membrane should be developed in such a manner that it will simplify the equations of the system, yet not adversely affect the values of the quantities that are sought. The quantities that can be obtained are thickness distribution, deflection, strain, strain-time relationships, geometry of the final shape, and the effects of work hardening.

The original rectangular plate will be replaced by a membrane of uniform thickness and of the same shape which by definition cannot sustain any bending moment or shear force. The loading will be an impulse in the lateral direction which imparts a uniform velocity, V, to the membrane material normal to its initial plane as shown in Figure 1.
Figure 1. Mechanism of Membrane Behavior
The static and dynamic elastic action and the plastic vibrations will be neglected. The material will be assumed to be an ideal incompressible plastic material, such that it will be isotropic in yielding. The loading causes a plastic bending wave which represents a discontinuity in the slope as shown in Figure 2.

![Figure 2. Cut away Sketch of the Membrane](image)

The discontinuity develops at the boundary and travels normal to the boundary with a uniform velocity if the membrane is of uniform thickness and is isotropic in yielding.

The hinge, or boundary of the discontinuity in slope, is assumed to be a curve geometrically...
similar to the boundary curve except at the corners where the added rigidity keeps the corner from deforming plastically. Thus the shape will resemble that of an ellipse with flat sides and ends which can be aptly described by the equation
\[
\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1
\]
where \( n \) is chosen to fit the configuration. As the hinge curve approaches the center of the membrane it is smaller than the boundary curve and is diminishing in size. The curve separates the membrane into two regions. The inner region is flat and moves with uniform transverse velocity \( V \), and the outer region is rigid and stationary as depicted in Figure 2. Plastic flow takes place only in the inner region.

Working from this model a mathematical formulation of the motion will be developed to satisfy the fundamental laws of mechanics. The equations are solved for intermediate and final displacements using the deformation theory for the non-work hardening case and incremental theory for both work- and non-work hardening.
CHAPTER II

PREVIOUS WORK

A complete history of previous work leading up to the problem illustrated in this study would be a voluminous report in itself. It would require covering the field of elastic materials and the many volumes of work done in this area. There will be no attempt to mention the many works done, but only the ones concerned with this undertaking. However, note should be made of the first mathematical account of the plastic straining of materials done in the latter part of the 19th century by Levy and Saint-Venant. Significant advances in the area of plasticity did not occur until after 1920. Plasticity is still a relatively young field, but it is finding increasing application, especially in the field of metal technology.

Since the formulation of a general theory is so difficult, the particular problems of circular membranes have attracted most analytical investigations. The first solution of plastic deformation of circular
diaphragms subjected to impulsive loading is that of Hudson (1) in 1951. Hudson used a method which employed a plastic hinge and developed the governing differential equations using the conservation of energy. Solutions were obtained using the deformation theory of plasticity for the cases of work hardening and no work hardening. Frederick (2) in 1959 used the incremental theory of plasticity and formulated his governing differential equations using work-energy and impulse-momentum principles. Frederick's solution followed the same form as Hudson's in that he used a circular hinge being propagated inward from the perimeter. Included in Frederick's solution is a comparison with experimental data obtained from explosive tests. In 1955 Wang (3) applied a limit analysis approach for circular plates which considered only bending stresses. Witmer, Salmer, Leech, and Pian (4) published their work concerning large dynamic deformation of plates and shells during 1963. Their theory was most general since it included bending and membrane stresses, elastic-plastic deformation responses, and the incremental theory of plasticity. The most serious deficiency of their theory is the extensive computer time required for each problem, making it unsuitable
for parametric investigations. Neglecting changes in the geometry of the continuum, Martin (5) published impulsive loading theorems for rigid-plastic, strain-rate insensitive continua. The response time given by his theory for certain beam and circular plate problems was comparable with other theories, but the magnitude of central deflection was too much in error. In 1966 Boyd (6) analyzed the circular membrane problem using deformation theory of plasticity in conjunction with work-energy principles. He, also, neglected the radial inertia and the work done by forces acting in the radial direction. This solution provided a very efficient means of performing parametric investigations for circular membranes. In 1968 Thomson (7) published his work on the plastic behavior of circular plates subjected to transverse impulsive loadings of Gaussian distribution. This is the most recent work found in the literature for dynamic loading.

The literature survey shows that solutions to the problem of the dynamic deformation response of rectangular plates in the inelastic region are practically nonexistant. A dissertation by Lindbergh (8) in 1967 is the only published solution to this problem found in the literature. Lindbergh solved
the rectangular membrane problem using finite difference techniques applied to the Flügge and Geyling formulation of the deformation of membrane shells. He considered the membrane as a pseudo-shell and used the incremental theory to describe the total response as the sum of numerous increments of small displacements. The material was assumed to be rigid-plastic strain-hardening in response to the applied load. One of the latest contributions in the area of dynamic response of rectangular membranes is a thesis by Srivastava (9). Some ideas are presented for the formulation of the problem using a hinge line approach, but no solution is obtained, and no definite formulation of the total problem is achieved. This writing does present a rather extensive list of references and points out many important ideas to keep in mind during the solution of a plastic deformation problem.

Other analytical and numerical investigations of rectangular membranes found in the literature were related to static load conditions. Lately much interest has been shown in plate and membrane problems using the finite element approach. During 1966, Pope (10) published a paper on the application of finite element analysis to the rectangular membrane problem.
using an elastic-plastic material, but the loading was not in the transverse direction. Also during 1966, Oden and Sato (11) developed a finite element formulation for the finite strain and displacement of elastic-membranes of general shape. In 1968, Ang and Lopez (12) developed a discrete model analysis of elastic-plastic plates and applied it for square plates with fixed and simply supported edge conditions.

One other work of special interest for this study was the work done by Rzhanitsyn (13) concerning the shape at collapse of elastic-plastic plates. This work is primarily concerned with the shape of the hinge lines formed when plates of various shapes and various boundary conditions are loaded to a limiting state. Hinge lines are shown for uniformly loaded plates and for point loaded plates.
CHAPTER III

FORMULATION OF THE PROBLEM

This chapter includes the development of the mathematical equations for the model described in the introduction. The complete formulation of the problem using hinge line theory and plastic deformation as depicted in Figure 2 is characterized and subdivided into three distinct subproblems:

A. Determination of the hinge line shape and its propagation toward the center of the membrane,

B. Behavior of the material in the neighborhood of the bending wave, and

C. Development of the governing equations of motion in the flat central region.

The theory developed in this work is based on the results of previous theoretical and experimental investigations performed using circular membranes. Results obtained from explosively loaded circular membranes and the theoretical work for circular membranes (2) have been shown to be in good agreement.
In the formulation of this problem an attempt will be made to justify all simplifying assumptions and their contribution to deviations from the actual or true solution.

Hinge Line Geometry

When an impulsive velocity is imparted normal to the plane of a clamped membrane, the membrane moves as if an initial uniform velocity had been impressed upon it as shown in Figure 1. This is analogous to a hinge being formed at the boundary and being propagated toward the center of the membrane. Levy (14) demonstrated experimentally using rectangular plates subjected to hydrostatic pressures that points midway along the edges become plastic under high pressures. The rigidity of a rectangular clamped plate near the corners is increased, thus it seems reasonable to assume that the corners do not deform plastically. This phenomenon has been shown experimentally by Levy.

Previous work on the shape of clamped plates at collapse (13), loaded by a uniformly distributed pressure have shown the peripheral hinge to take the shape shown in Figure 3.
The hinge line has been found to consist of two parts, one part being segments of the fixed edges and the second part being the corners. The fixed edges consist of straight line segments and the corners are approximated by curved segments. In the work of Rzhanitsyn (13) for a rectangular plate a hinge line was obtained that consisted of the boundary of the plate and a curved segment of the form

$$r(\theta) = \frac{b \sqrt{12 M}}{\sqrt{q} \cosh b (\theta + \beta)}$$

Where

\(\beta\) and \(b\) are arbitrary constants of the solution,

\(M\) is the limiting moment, and

\(q\) is the collapse pressure.
In the interest of simplifying the solution to the problem, but in congruence with the desire of keeping the results as accurate as possible, a single continuous equation defining the hinge curve is desirable. Since the corners of the membrane do not deform plastically and plastic flow starts along the fixed sides, then a reasonable assumption for the hinge curve would be of the form

\[ \left( \frac{x}{A} \right)^n + \left( \frac{y}{B} \right)^n = 1 \]

Where

A is the half length of the plate in the x-direction,  
B is the half length of the plate in the y-direction, and  
n is an arbitrary integer.

It was suggested by Srivastava (9) that a curve of this type could be applied. Since the hinge curve attempts to remain geometrically similar to the boundary curve this assumption would appear to be quite accurate. The hinge line may be forced to approximate the boundary of a rectangular membrane as closely as desired, if discrimination is used when selecting n. For a value of n = 2 the curve represents an ellipse. It will be shown later that, when n = 2 and A is equal to B, the equation will
satisfy the solution for the circular membrane. As 
n increases the curve will move toward the corners, 
as shown in Figures 4, 5, and 6, resulting in a much 
closer approximation. A value of \( n = 6 \) was 
selected for this work. A comparison of the reduction 
in area using \( n = 6 \) has shown that even if the hinge 
line was the boundary \( (n = \infty) \) the error involved 
would be less than 5 per cent.

The curves shown in Figures 4, 5, and 6 are 
shown to illustrate the proximity of the curve to the 
actual corner of the membrane for various aspect 
ratios and different values of \( n \).

The shape of the hinge curve as it progresses 
inward toward the center of the membrane is assumed 
to remain similar to the initial peripheral hinge. 
In the absence of experimental data for impulsively 
loaded, clamped, rectangular membranes certification 
of this assumption is based on available comparative 
results. In this case the hinge curve may be compared 
with known work for statically loaded membranes and 
the one known solution of dynamically loaded membranes 
given by Lindbergh (8). The true measure of the 
acceptability of the proposed hinge line will be 
shown in the final results of the solution.

Conversion of the hinge curve to polar
Figure 4. Peripheral Hinge for a Membrane with $A/B = 1$ and Varying $n$. 
Figure 5. Peripheral Hinge for a Membrane with $A/B = 2/3$ and Varying $n$. 
Figure 6. Peripheral Hinge for $A/B = \frac{1}{2}$ and Varying $n$. 
coordinates changes the equation to a more applicable form. With the appropriate substitutions the equation is converted to the following form

\[ R = \frac{A}{\left((\cos \theta)^n + \left(\frac{A}{B} \sin \theta\right)^n\right)^{1/n}} \]  \hspace{1cm} (1)

Behavior in the Neighborhood of the Bending Wave

The problem of a circular membrane subjected to impulsive loading has been treated by Hudson (1) and by Frederick (2) using a mechanism similar to the one used in this study. Application of the basic laws of mechanics and the geometrical configuration of the membrane as it deforms will determine the governing equations for this problem.

At a time, \( t \), after the start of the motion, when the hinge line has propagated inward and passed a given reference point, it has tilted the portion behind it into the form of a sixth order elliptical shell as shown in Figure 7. Let \( dL \) be the width of an incremental ring of thickness \( H \) just ahead of the bending wave, when the bending wave is a distance \( R \) from the center of the membrane as shown in Figure 8. The elemental ring of width \( dL \) will be swept over and rotated through an angle \( \alpha \) in the time interval \( dt \). This ring has rotated without elongation onto the
Figure 7. Plate in the State of Deformation
rigid outer portion of the deformed membrane. The wave has traveled a radial distance of $-\dot{R}dt$ in time interval $dt$, $R$ is decreasing which causes $\dot{R}$ to be negative and accounts for the negative sign. Point $P$ has moved to point $P'$ and has a radial displacement of $U$. This may be written as $\dot{U}dt$ which is the equivalent radial displacement in terms of radial velocity. Then using the geometry of Figure 8

$$dL = -\dot{R}dt + \dot{U}dt.$$ (2)

Thus the rate at which the material is swept over by the wave is shown as

$$\frac{dL}{dt} = \dot{U} - \dot{R}. $$ (3)

The inner edge of the element has displaced an amount $Vdt$ normal to the original plane of the membrane while the outer edge has remained stationary.

Observation of the geometrical illustration will show the wave has traveled a distance $dL$ along the generator of the deformed profile while it has traveled inward a distance of $-\dot{R}dt$. Then observation of the right triangle with $dL$ as the hypotenuse, and $\dot{R}dt$ and $Vdt$ as the two sides has provided the relationships

$$\frac{dL}{dt} = (\dot{R}^2 + V^2)^{\frac{1}{2}}$$ (4)

$$\cos \alpha = \frac{-\dot{R}dt}{dL}$$

and

$$\sin \alpha = \frac{Vdt}{dL}.$$
Figure B. Membrane Mechanism at Two Successive Instants, t and t + dt.
Figure 9. A Small Element at the Bending Wave
Thus a second relationship has been obtained for the rate of travel along the profile of the deformed membrane.

Application of the basic laws of mechanics will provide equations in addition to the ones obtained using the geometrical relations. Infinitesimals of higher order, such as changes in the thickness and changes in the stresses, will be neglected. The basic laws of mechanics will be applied to an element as shown in Figure 9. Just behind the hinge the stress shown acting on the sloping portion of the element is $\sigma_{R\theta}$. The other stresses are shown as $\sigma_{\theta}$, the circumferential stress, and $\sigma_{R}$, the radial stress.

Calculation of the momentum component has shown $\rho HRd\theta dLV$ to be the vertical component in the time interval $dt$. The impulse delivered in the vertical direction is found to be $[\sigma_{R\theta} \sin \alpha HRd\theta]dt$. Equating the impulse and the momentum will result in

$$\frac{\rho dl}{dt} = \sigma_{R\theta} \sin \alpha$$  \hspace{1cm} (5)$$

where $\rho$ is defined as the mass per unit volume of the material.

The momentum component in the radial direction is evaluated as $\rho HRd\theta dLU$ in the time interval $dt$. Utilizing a summation of forces in the radial direction the impulse delivered in the time $dt$ is obtained as
\((\sigma_R - \sigma_R \theta \cos \alpha)HRd\theta dt\). Then application of the impulse-momentum equation will result in

\[
\frac{rdL\dot{U}}{dt} = \sigma_R - \sigma_R \theta \cos \alpha.
\] (6)

The next relationship will be obtained from application of the work-energy principle with the assumption that no plastic work is done by the internal stresses as the element is rotated onto the rigid portion. Since the stress \(\sigma_R \theta\) is in the rigid stationary material it will not contribute to the work. As the bending wave progresses inward over the element the stress \(\sigma_R\) will contribute work of the amount \(-\sigma_R HRd\theta \ddot{U} dt\). The increase in kinetic energy is equal to the work done in time \(dt\), so that

\[-\sigma_R HRd\theta \ddot{U} dt = -\frac{1}{2}pdLHRd\theta(\dot{U}^2 + V^2)\]

which may be rewritten as

\[\sigma_R \ddot{U} = \frac{1}{2}pdL(\dot{U}^2 + V^2).\] (7)

From the elimination of \(dL/dt\) in equations (3) and (4) the following equation is obtained

\[-\dot{R} + \ddot{U} = (\dot{R}^2 + V^2)^{\frac{1}{2}}.\]

Then squaring both sides

\[\dot{R}^2 - 2\dot{R}\dot{U} + \ddot{U}^2 = \dot{R}^2 + V^2\]

or

\[\ddot{U}^2 - 2R\dot{U} - V^2 = 0\] (8)

and adding \(\dot{U}^2\) to both sides results in
\[ 2\dot{U}^2 - 2\dot{R}\dot{U} = \dot{V}^2 + \dot{U}^2 \]
\[ 2\dot{U}(\dot{U} - \dot{R}) = \dot{V}^2 + \dot{U}^2. \]  
(9)

Equation (9) may be substituted in equation (7) to obtain

\[ \sigma_R \dot{U} = \rho \ddot{U} \dot{U}(\dot{U} - \dot{R}) \]

or

\[ \sigma_R = \rho \ddot{U}(\dot{U} - \dot{R}). \]  
(10)

When combined with equation (6), equation (10) is found to yield

\[ -\rho \ddot{R} + \sigma_R = \sigma_R \theta \cos \alpha = \sigma_R \]

or

\[ \rho \ddot{R} = -\sigma_R \theta \cos \alpha. \]  
(11)

Now making the substitution \( \dot{R} = -\frac{dL}{dt} \cos \alpha \) from the geometrical relationship into equation (11) yields

\[ \rho \ddot{L} \left( -\frac{dL}{dt} \cos \alpha \right) = -\sigma_R \theta \cos \alpha \]

\[ \frac{dL}{dt} = \left( \frac{\sigma_R \theta}{\rho} \right)^{\frac{1}{2}}. \]  
(12)

The substitution of equation (3) into equation (10) may be used to obtain

\[ \sigma_R = \rho \frac{dL}{dt} \left( \frac{dL}{dt} \right) = \sigma_R \theta. \]  
(13)

Hence, the stresses are shown to be the same on both sides of the bending wave. \( \sigma_R \) may be expressed in terms of \( \dot{R} \) and \( \theta \) using equations (4) and (13) as
\[ \sigma_R = \rho(\dot{r}^2 + v^2) \quad (14) \]

\( \sigma_R \) can be considered a known quantity because it is determined from the yield condition. Therefore, the above equations have defined the unknowns \( dL, \sigma_R \), \( \alpha \), \( \dot{R} \), and \( \dot{U} \) in terms of the known quantities \( \sigma_R \), \( V \), and \( \rho \). The velocity \( V \) is known since it is determined from the well-known relationships as explained by Cole (16), Lindbergh (8), and Srivastava (9). These equations will be included as Appendix A in this thesis.

Equations of Motion in the Flat Central Region

The velocity component of any particle in the flat central region has been assumed to have the constant value \( V \) in the direction normal to the membrane until the bending wave has passed over it. This normal motion is independent of the radial motion. The importance of the radial motion is enhanced because the distribution of thinning is partly dependent upon it.

Observation of Figure 8 has shown the thickness at time \( t \) is \( H \) throughout the undeformed region and the radius is \( R - dL \). At a time \( t + dt \), the radius has become \( R - dL - \dot{U}dt \) and the thickness has become \( H + Hdt \). Application of the condition of
conservation of volume in plastic flow, has led to
\[ \left[ \int_0^{2\pi} \int_0^R r dr \right. \right. 
= \left[ \int_0^{2\pi} \int_0^R \right. \right. 
\left. \left. r dr \right] H + \dot{H} dt \]

or
\[ (H + \dot{H} dt) \int_0^{2\pi} \right. \right. 
\left. \left. (2R \dot{U} dt - 2U dL dt + \dot{U}^2 (dt)^2 \right) 
+ \dot{H} dt \int_0^{2\pi} \right. \right. 
\left. \left. (R^2 - 2R dL + (dL)^2) d\theta = 0 \right. \right. 
\]

Then dropping product terms of differential quantities
\[ H \int_0^{2\pi} \left. \left. (2R \dot{U} dt) d\theta + H \int_0^{2\pi} \right. \right. 
\left. \left. (R dt) d\theta = 0 \right. \right. 
\]

This equation may be rewritten to produce
\[ \frac{\dot{H}}{H} = \frac{-2 \int_0^{2\pi} \right. \right. 
\left. \left. R \dot{U} d\theta }{ \int_0^{2\pi} \right. \right. 
\left. \left. R^2 d\theta } \quad (15) \]

Another relationship may be obtained from the constant volume relationship and observation of Figure 10.

\[ \text{Figure 10. One Quadrant of the Plate} \]

at \( r_0 \), thickness = \( h \)

at \( r \), thickness = \( H \)
The general formulation of the problem has been completed. For a solution it is necessary to apply the theory in Appendix A to obtain the velocity $V$ for the specific loading situation. The plasticity theory included in Appendix B is also necessary to define the strains, stresses and work-hardening criteria to be used with this formulation.
CHAPTER IV

SOLUTION OF THE PROBLEM

It is the purpose of this chapter to apply the theory as delineated in Chapter III to the problem of the rectangular membrane. Since the availability of tests to verify the rectangular solution is limited to hydrostatic loading, a comparison will be made for this theory as applied to circular membranes.

The solution is completely analytical and can be separated into two types of solutions. The first, or "elementary approximation" is the solution based on the assumption of no work hardening, which in this application is the perfectly plastic case. With this assumption the solution may be solved in finite form and the total time of deformation, the final deformed membrane profile, and the thickness distribution after deformation may be obtained. The second type of solution is based on work hardening in the form of Ludwik's Power Law. For the second type of solution the problem becomes more
complex and requires a numerical solution.

For convenience, the relevant equations have been grouped and rewritten

\[ \ddot{U}^2 - 2R \dot{U} - V^2 = 0 \] (8)

\[ \rho (\dot{R}^2 + \psi^2) = \sigma_R \] (14)

\[
\begin{align*}
\dot{H} &= -2 \int_0^{2\pi} R \psi d\theta \\
\dot{H} &= \frac{1}{2\pi} \int_0^{2\pi} R^2 d\theta \\
\end{align*}
\]

\[
\begin{align*}
\dot{h} &= \int_0^{2\pi} \frac{R^2}{R^2} d\theta \\
\dot{H} &= \int_0^{2\pi} R \psi d\theta \\
\end{align*}
\]

\[ R = \frac{A}{(\cos \theta)^n + (\frac{A}{B} \sin \theta)^n}^{1/n} \] (1)

The initial conditions can be written as

\[ R(0) = R_0 \]

\[ H(0) = h \]

\[ U(0) = \psi \left( \frac{A\sigma}{\rho} \cdot \frac{\sigma(1 - \nu)}{E} \right)^{1/2} \]

where

\[ R(0) \] is the initial radius of the bending wave

\[ H(0) \] is the initial thickness of the membrane
\[ U(0) \] is the initial radial velocity of the bending wave.
\[ \sigma \] is the yield stress.

The equation for the initial radial velocity has been developed by Hudson (1) and has been demonstrated experimentally by Rinehart and Pearson (17). The initial value of the radial velocity would be obtained from the elastic conditions for the general case. But for the ideal plastic material the stress would jump immediately from zero to the yield value. Hudson has shown the value of the time for total deformation for steel, as used in this study, is only affected by approximately 3% from neglecting elastic effects. An increase in accuracy of this amount does not compensate for the increase in complexity.

The final shape of the membrane profile can be defined by the equation

\[ Z = Vt \tag{18} \]

where \( Z(t) \) is defined as the distance of the flat region from the initial plane of the membrane. \( R \) can be found as a function of \( t \), then \( t \) can be eliminated from \( Z \) and the equation of the profile is then defined as \( Z(R) \).
Circular Membrane

For a circular membrane the equation of the hinge line has been observed to take the shape of the boundary. With this knowledge the hinge line equation (1) can be observed to degenerate to the form

\[ R = \frac{A}{\sqrt{(\cos \theta)^2 + \left(\frac{A}{B} \sin \theta\right)^2}} \]

Since for a circle \( A = B \), this can be rewritten for \( t = 0 \) as

\[ R(0) = A \]

Using equation (14)

\[ p(\dot{R}^2 + V^2) = \sigma \], where \( \sigma \) is the yield stress

\[ \dot{R} = (\frac{\sigma}{p} - V^2)^{\frac{1}{2}} \]

and recalling that \( \dot{R} \) is negative because \( R \) is decreasing

\[ \dot{R} = -(\frac{\sigma}{p} - V^2)^{\frac{1}{2}} \]

Upon integration and substitution of the initial condition

\[ R = -(\frac{\sigma}{p} - V^2)^{\frac{1}{2}} t + A \] \hspace{1cm} (19)

\( \ddot{U} \) will be obtained from the manipulation of equation (8).

\[ \ddot{U}^2 - 2\dot{R}\dot{U} - V^2 = 0 \]
\[
\frac{U^2}{v^2} - \frac{2\ddot{U}R}{v^2} - 1 = 0
\]

or

\[
\frac{U^2}{v^2} + \frac{2\ddot{U}}{v^2} \left( \frac{\sigma}{\rho v^2} - 1 \right) V - 1 = 0.
\]

If \( \frac{\sigma}{\rho v^2} \gg 1 \), as will be shown numerically in Chapter V, the above equation will become

\[
\frac{2\ddot{U}}{v} \left( \frac{\sigma}{\rho v^2} \right)^{\frac{1}{2}} = 1
\]

and solving for \( \ddot{U} \)

\[
\ddot{U} = \frac{v^2}{2} \left( \frac{\rho}{\sigma} \right)^{\frac{1}{2}}.
\]

If \( \frac{\sigma}{\rho} \) is defined as equal to \( C^2 \), this equation may be written as

\[
\ddot{U} = \frac{v^2}{2C}.
\] (20)

With \( C \) as defined and \( \frac{\sigma}{\rho v^2} \gg 1 \) equation (19) can be simplified to obtain

\[
R = A - Ct.
\] (21)

The solution for the thickness variation will be obtained using equations (15), (20), and (21).

\[
\frac{H}{\dot{\theta}} = \frac{2\int_{\frac{2\pi}{3}}^{\frac{2\pi}{3}} RU \, d\theta}{\int_{0}^{2\pi} R^2 \, d\theta}
\]
which gives after integration with respect to theta

\[ \frac{dH}{H} = -\frac{2u}{R} \]

\[ \frac{dH}{H} = \frac{-2V^2}{2C(A - Ct)} \ dt \]

Finally, after integration of this equation and substitution of the initial condition for thickness, the result is

\[ \frac{H}{h} = \left(\frac{A - Ct}{A}\right)^2 \]

which may be written as

\[ \frac{H}{h} = \left(\frac{R}{A}\right)^2 \]  \hspace{1cm} (22)

Solving equation (21) for \( t \) has given

\[ t = \frac{A - R}{C} \]

Then the substitution of \( t \) into equation (18) has defined the deformation profile as

\[ Z = \frac{V}{C}(A - R) \]  \hspace{1cm} (23)

The circular membrane without work hardening has been completed with the attainment of equations for deformation profile, the shape of the hinge curve, the radial velocity, and the thickness distribution. Other values such as strain and strain time can be
obtained using these equations in conjunction with the plasticity theory.

This particular solution is not claimed as a contribution of this study, but only as check on the validity of the equations as compared with other solutions. The numerical comparison will be shown in Chapter V.

Rectangular Membrane Without Work Hardening

This solution will be the elementary solution for the rectangular membrane. The equation of the hinge line is defined in equation (1). For a rectangular membrane the initial condition for $R$ is defined as

$$R(0) = \frac{A}{(\cos \theta)^n + (\frac{A}{B} \sin \theta)^n}^{1/n}$$

and is observed to be a function of theta and the actual membrane boundary. The solution is begun by finding $\dot{R}$ from equation (14).

$$\dot{R} = -\left(\frac{\sigma}{p} - V^2\right)^{\frac{1}{2}}$$

Solving for $R$ has resulted in

$$R = -\left(\frac{\sigma}{p} - V^2\right)^{\frac{1}{2}} t + R(0)$$

or
Equation (8), with the same manipulations and assumptions used for circular membranes, is found to yield for the radial velocity

$$\dot{U} = \frac{v^2}{2c}$$  \hspace{1cm} (25)

The solution for the thickness distribution is found to be much more difficult and more lengthy than the solution for circular membranes. Starting with equation (15) and substituting for $R$ and $\dot{U}$ with equations (24) and (25) has resulted in the following equation:

$$\frac{dH}{H} = \left[ \frac{\int_{0}^{2\pi} \left( -\left( \frac{\sigma}{p} - V^2 \right)^{1/2} t + \left[ (\cos \theta)^n + (\frac{A}{B} \sin \theta)^n \right]^{1/n} \right] d\theta}{\int_{0}^{2\pi} \left( -\left( \frac{\sigma}{p} - V^2 \right)^{1/2} t + \left[ (\cos \theta)^n + (\frac{A}{B} \sin \theta)^n \right]^{1/n} \right)^2 d\theta} \right] dt$$

First, evaluation of the numerator inside the square brackets has resulted in

$$\text{Numerator} = A_1 t - 2\dot{U}A_2$$

where

$$A_1 = 4\pi \dot{U} \left( \frac{\sigma}{p} - V^2 \right)^{1/2}$$
\[ A_2 = \int_0^{2\pi} \left[ (\cos \theta)^2 + \left( \frac{A}{B \sin \theta} \right)^2 \right]^{1/n} d\theta. \]

Evaluation of the denominator has obtained:

\[ \text{Denominator} = A_3 t^2 - A_4 t + A_2 A_5 \]

where

\[ A_3 = 2\pi \left( \frac{\sigma}{p} - v^2 \right) \]
\[ A_4 = 2AA_2 \left( \frac{\sigma}{p} - v^2 \right)^{3/2} \]
\[ A_5 = \int_0^{2\pi} \left[ (\cos \theta)^2 + \left( \frac{A}{B \sin \theta} \right)^2 \right]^{2/n} d\theta. \]

Observation of \( A_2 \) and \( A_5 \) will show their dependence on the \( A/B \) ratio. These constants must be evaluated separately for individual aspect ratios and this can be done quite handily using the computer simulation language MIMIC with a predetermined value for \( n \). The thickness distribution equation can be shown in the form:

\[ \frac{dH}{H} = \frac{A_1 t - 2UAA_2}{A_3 t^2 - A_4 t + A_2 A_5} \, dt \]

or

\[ \frac{dH}{H} = \frac{A_1 t - A_6}{A_3 t^2 - A_4 t + A_7} \, dt \]

where
\[ A_6 = 2UA_2 \]

\[ A_7 = A^2A_5 \]

Finally the solution is obtained, and with the initial condition \( H(0) = h \) has resulted in

\[ \frac{H}{h} = \left( A_3t^2 - A_4t + A_7 \right) C_1 + \left( \frac{2A_3t + C_2}{2A_3t + C_3} \right) C_4 \\
-\left( \frac{2A_3t + C_2}{2A_3t + C_3} \right) C_5 - (A_7) C_1 - \left( \frac{C_2}{C_3} \right) + \left( \frac{C_2}{C_3} \right) \]

(26)

where

\[ C_1 = \frac{A_1}{2A_3} \]

\[ C_2 = -A_4 - \sqrt{A_4^2 + 4A_3A_7} \]

\[ C_3 = -A_4 + \sqrt{A_4^2 + 4A_3A_7} \]

\[ C_4 = \frac{A_4A_1}{2A_3 \sqrt{A_4^2 - 4A_3A_7}} \]

\[ C_5 = \frac{A_6A_1}{\sqrt{A_4^2 - 4A_3A_7}} \]

The solution of equation (24) for time has given

\[ t = \frac{A}{(\frac{\sigma}{p} - \frac{V^2}{2}) \int (\cos \theta)^n + (\frac{A}{B} \sin \theta)^{n-1} - \frac{R}{(\frac{\sigma}{p} - \frac{V^2}{2})}} \]
Using the assumption $\frac{\sigma_{xx}}{\rho V^2} \gg 1$, this equation may be rewritten as

$$t = \frac{A}{C[(\cos \theta)^n + (\frac{A}{B} \sin \theta)^n]^{1/n} - \frac{R}{C}}.$$  \hspace{1cm} (27)

And the solution for the deformation profile may be shown as

$$z = \frac{V}{C[(\cos \theta)^n + (\frac{A}{B} \sin \theta)^n]^{1/n} - R}.$$  \hspace{1cm} (28)

The elementary solution for the rectangular membrane has been completed. Equations (24), (25), (26), and (28) describe respectively the shape of the hinge curve, the radial velocity, the thickness distribution, and the deformation profile. This solution could be defined as the elementary solution using the "deformation theory of plasticity." Observation of the topographical plots will show the hinge curve to differ quite radically from the expected results. This has encouraged the author to employ the "incremental theory of plasticity" and obtain another solution.

The incremental theory has not changed the equations derived above. It has required only a change in the method of solution of these equations. The hinge line will be propagated inward in time, but
will remain as derived in equation (24). Equation (14) has provided
\[ \frac{dR}{dt} = -\left(\frac{\sigma}{\rho} - v^2\right)^{\frac{1}{2}} \]
and the incremental equation for time may be shown as
\[ dt = \frac{\frac{dR}{\rho}}{-\left(\frac{\sigma}{\rho} - v^2\right)^{\frac{1}{2}}} \quad \text{(29)} \]
The solution for the total time for deformation is obtained by a summation of the time increments that are required to propagate the wave inward to the final deformation profile.

The final equation for the deformation profile is resolved to be
\[ \sum_{i=1}^{n} Z = \sqrt{R \left(\frac{dR}{dt}\right)} \quad \text{(30)} \]
The resulting equations for incremental theory have provided the necessary solutions for the rectangular membrane, but have required extensive use of the computer to obtain the results.

**Rectangular Membrane with Work Hardening**

The solution for the work hardening case is more complicated than the previous case and will be solved only using the incremental theory. The loading function, which is denoted as \( \sigma_R \) for this
study, must be defined in a manner that will account for work hardening. Many work hardening relationships are known to exist. Suggestions of different empirical relationships may be obtained from any text on plasticity. It has been decided in this study to use Ludwik's Power Law which is of the form

$$\sigma = a + b \varepsilon^c$$

where $a$, $b$, and $c$ are arbitrary constants and $\varepsilon$ is the strain. $\varepsilon$ can be defined in many ways; it can be defined in terms of octahedral shearing strain, effective strain, maximum strain, or in other ways. In this study the power law for work hardening has been simplified to

$$\sigma = a + b \varepsilon$$

which has been suggested by Hill (19). At very large strains many metals, such as high carbon steels and steel alloys, have shown this equation to be a close representation (19) and (21).

The particular work hardening relation used in this study has been chosen as

$$\sigma_R = \sigma + \omega \ln \frac{H}{h}$$

where

$\sigma = \text{yield stress}$

$\omega = \text{work hardening stress per unit natural strain in a tensile test (lb/in.}^2)$.
Equation (14) can be written as
\[ p(\dot{R}^2 + \dot{V}^2) = \sigma + \omega \ln \frac{H}{h} \]

or
\[ \dot{R} = \left(\frac{\sigma}{p} + \frac{\omega}{p} \ln \frac{H}{h} - \dot{V}^2\right)^{\frac{1}{2}}. \quad (32) \]

and \( \dot{R} \) is negative because \( R \) is decreasing as noted previously. The equation for time in incremental form can be written from equation (32) as
\[ dt = \frac{dR}{-(\frac{\sigma}{p} + \frac{\omega}{p} \ln \frac{H}{h} - \dot{V}^2)^{\frac{1}{2}}}. \quad (33) \]

With the substitution of equation (16) into equation (33) this equation will become
\[ dt = \frac{dR}{-(\frac{\sigma}{p} + \frac{\omega}{p} \ln \left(\int_0^{2\pi} R^2 d\theta\right) - \dot{V}^2)^{\frac{1}{2}}} \quad (34) \]

where
\[ R_0 = \text{the radius of the previous increment} \]
\[ R = \text{the radius of the new location} \]

Values are chosen for \( dR \) and a solution is then obtained for the incremental time. A summation of all the time increments will give the value for the total time required to deform the plate.

The hinge line shape has been determined to be identical with the elementary solution using incremental methods. This is not an unexpected result
since work hardening has not changed the propagation of the wave, but only affects the profile of the deformed membrane. The incremental method is applied by solving for the condition at the desired point, then time is set equal to zero and the final results of the previous increment become the initial conditions for the new increment.

The deformation of the membrane is determined from the equation

\[ Z = V \sum_{i=1}^{n} (dt)_i \]  \tag{35}  

and from the solution for \( dt \) it is readily observed that the deformation profile is a function of \( R \).

The solutions to the proposed problems have now been completed. Solutions of rectangular membranes have been obtained for the hinge shape, the deformation profile, the total time of deformation, and the thickness distribution after deformation. The actual numerical values will be carried out in Chapter V.
CHAPTER V

NUMERICAL RESULTS

The essential goal of this chapter is to present numerical solutions to the problems of the previous chapter. Major areas of presentation include:

(a) The circular membrane with clamped edges,
(b) The clamped rectangular membrane without work-hardening. (Results are presented using both the deformation theory of plasticity and the incremental theory of plasticity.)
(c) The clamped rectangular membrane with work-hardening using incremental theory of plasticity, and
(d) A variation of the aspect ratio in conjunction with parts (b) and (c).

Material parameters used for the rectangular membranes and circular membranes with one exception are:
\[ \sigma = 60000 \text{ lb/in.}^2 \]

\[ E = 30 \times 10^5 \text{ lb/in.}^2 \]

\[ v = 0.3 \]

\[ p = 0.75 \times 10^{-3} \text{ lb sec}^2/\text{in.}^4 \]

\[ w = 4.0 \times 10^5 \text{ lb/in.}^2 \]

The basic reason for the selection of these values affords one the opportunity to compare results for a circular membrane with Hudson (1). A secondary reason was that these are rather typical values for mild steel. The exception previously noted will be the test case comparison with the results shown by Frederick (2).

**Circular Membrane**

The equations derived for the circular membrane in Chapter IV will be repeated here for convenience.

\[ R = A - ct \]

\[ U' = \frac{v^2}{2c} \]

\[ H = \left( \frac{v}{c} \right)^2 \]

\[ H = \left( \frac{R}{A} \right) \]

\[ Z = \frac{V}{C} (A - R) \]
Although the formulation was somewhat different from Hudson's formulation, the final equations are precisely the same. Using the same values for thickness, radius, and initial velocity have led to the results shown in Figure 11. Figure 11 has been plotted for a circle with a 5 inch radius and initial velocities of 1000.0 and 2000.0 inches per second.

Frederick (2) has shown experimental results done by the Underwater Explosions Research Division, Norfolk Naval Shipyard in his paper. The deflection results are reproduced here as Figure 12 with the results obtained from this study superimposed. The figure used for this comparison was taken directly from Frederick's paper. Mild steel with a yield stress of 49200 psi was used. A ½ inch thick membrane with a radius of 10 inches was loaded by 10 pound pentolite charges placed seven feet from the center of the membrane to provide a uniform wave front.

The essential purpose for the presentation of the material on circular membranes has been to give an indication as to the validity of the formulation. Since experimental data were not available for rectangular membranes some measure of
Figure 11. Circular Membrane Without Work-Hardening
Figure 12. Deflection Profiles for a Circular Membrane Taken From Reference 2.
accuracy of the formulation was deemed desirable.
With this idea in mind the simplification to circular
theory has been represented.

Rectangular Membrane Without Work-Hardening

As explained in Chapter IV the clamped
rectangular membrane has been solved by the two
theories of plasticity. Solutions have been obtained
for the three different aspect ratios listed in
Table I and for two initial velocities. Initial
velocities of 1000.0 in./sec. and 2000.0 in/sec.
were selected for this study. These velocities
have been determined using the impulsive loading
criteria for explosives as listed in the appendix.

| TABLE I |
| ASPECT RATIO DATA |

<table>
<thead>
<tr>
<th>Case</th>
<th>X-length</th>
<th>Y-length</th>
<th>Z (thickness)</th>
<th>Aspect Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.00</td>
<td>24.00</td>
<td>0.05</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>15.98</td>
<td>33.96</td>
<td>0.05</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>19.60</td>
<td>29.40</td>
<td>0.05</td>
<td>2/3</td>
</tr>
</tbody>
</table>

Figures 13 and 14 show topographical plots
for the solutions using the deformation theory with
n = 6. Figures 15, 16, 17, 18, and 19 show solutions using other values for n. Figures 20, 21, and 22 show the topographical plots and profiles of the deflection using the incremental theory of plasticity. The topographical plots for the work hardening and no work hardening conditions have been shown on the same figures. This is possible because the hinge line propagates inward with the same shape in both cases. Displacement profiles for the no work hardening case are shown in Figures 20, 21, and 22.

It should be noted that the values in parenthesis in the topographical plots are the displacements corresponding to an initial velocity of 1000 in/sec. and the other values correspond to an initial value of 2000 in/sec.

Rectangular Membrane With Work Hardening

The solution for the clamped rectangular membrane is presented in this section for the three different aspect ratios, and material dimensions noted in Table I. Results are shown in Figures 20, 21, and 22 for the hinge line at different stages in its progress toward the center of the plate. Values are plotted for the previously mentioned initial velocities. It may be observed that the hinge line
propagation shown in the topographical plots has the same form for the different velocities. However, at the same point in time the displacement magnitude is greater for the higher initial velocity.

The work hardening relationship used is

\[ \sigma_R = \sigma + \omega \ln \frac{W}{h} \],

where these values are defined for a mild steel at the beginning of this chapter. From equations (34) and (35) it may be ascertained that the displacement relationship for work hardening is not linear. This may also be noted from the observation of the plots for varying aspect ratios in Figures 20, 21, and 22. In all of these plots the profile view has been shown for the XZ-plane. In Figure 23 the observation of the displacement profile in the YZ-plane shows a flat top on the rectangular plates. This will not be the case for the square plate as its final profile will come to a point and will appear somewhat like a pyramid with rounded corners. Figure 24 provides some insight to the appearance of the final deformed shape of the clamped rectangular membrane as viewed from above the plate. Figure 25 shows the deformation-time response for a square membrane in which the center of the membrane reached its maximum displacement to a 1000 in/sec. velocity in 1.4 msec.
Figure 13. Topographical View of the Deformation Solution for a Membrane with $A/B = 1$ and $n = 6$. 
Figure 14. Topographical View of the Deformation Solution for a Membrane with $A/B = 2/3$ and $n = 6$. 
Figure 15. Topographical View of the Deformation Solution for a Membrane with $A/B = 1$ and $n = 3$. 
Figure 16. Topographical View of the Deformation Solution for a Membrane with \( A/B = 1 \) and \( n = 4 \).
Figure 17. Topographical View of the Deformation Solution for a Membrane with $A/B = 2/3$ and $n = 4$. 
Figure 13. Topographical View of the Deformation Solution for a Membrane with $A/3 = 2/3$ and $n = 3$. 
Figure 19. Topographical View of the Deformation Solution for a Membrane with $A/3 = 2/3$ and $n = 2$. 
Figure 20. Profile and Topographical View For an Incremental Solution
**Figure 21. Profile and Topographical View for an Incremental Solution**

- **Profile (X-Axis):**
  - Work Hardening: ---
  - No Work Hardening: ----
  - A/B = 2/3

- **Velocity:**
  - V = 2000.0 in/sec
  - V = 1000.0 in/sec

**Transverse Displ.:**

**Note:** Dimensions in inches.
Figure 22. Profile and Topographical View for an Incremental Solution

- Profile (X-Axis)
- Work Hardening: ---
- No Work Hardening: ---
- \( A/B = 1/2 \)

- \( V = 2000.0 \) in/sec
- \( V = 1000.0 \)
Profile
No Work-Hardening
Size: 19.6" X 29.40"
Thickness: 0.05"

Figure 23. Typical Transverse Displacement
Viewed From the ZY-Plane
Figure 24. Top View of the Final Deformed Shape of a Clamped Rectangular Membrane
Figure 25. Variation of Center Displacement with Time of the Square Membrane \((A/B = 1)\)

Square Plate
Size: 24" X 24"
Thickness: 0.05"

V = 1000 in/sec
CHAPTER VI

SUMMARY AND DISCUSSION OF THE RESULTS

This study has culminated with the attainment of a solution for the inelastic deformation of a clamped, rectangular membrane subjected to impulsive loading. The dynamic equilibrium of a representative element has been mathematically formulated using the principles of mechanics and the geometry of the system. The governing differential equations have been solved in a closed form for the deformation theory and numerically for the incremental theory. Use of the digital computer was necessary for the numerical solution.

In conjunction with the solution for the rectangular membrane, a limited parametric investigation of aspect ratios was performed to illustrate the development.

Discussion of the Results

A complete evaluation of the results is not possible because of the lack of experimental results.
for rectangular membranes. The formulation of the problem appears to be quite acceptable in view of the comparison with circular membranes in Figures 11 and 12.

Figures 13 thru 19 showing the results using deformation theory have indicated the necessity for using incremental theory. This was not completely unexpected due to the limited applicability suggested for the deformation theory. Application of various values of \( n \) have shown that a more realistic solution using deformation theory could be obtained using high values of \( n \) near the boundary and decreasing \( n \) as the center is approached. For \( n = 6 \) the results have shown an area resembling a four-leaved rose lemniscate with the leaves remaining elastic. If the final value of \( n \) was chosen as two and the value of \( A \) set equal to \( B \), then the hinge line would close on the center as a circle and all areas would be plastic. This is more realistic, but the final shape should resemble the initial peripheral hinge. Deformation theory is acceptable for circular membranes because of their inherent symmetry which allows the principal stress ratios to remain constant. Lack of this same quality in rectangular membranes necessitates the usage of
incremental theory.

An interesting sidelight obtained from this study that will not elicit exclamations of surprise from the supporters of the incremental theory is shown when comparing the results of the two theories. These results lend support to the incremental theory as the method for solving plasticity problems with varying principal stress ratios.

The results obtained using the incremental theory in Figures 20, 21, and 22 are more generally acceptable as a solution. These results compared favorably with the shapes shown by Lindbergh (8) in his solution for rectangular plates. Direct comparison with Lindbergh's results was not possible due to his definition of the material used as half-hard aluminum. Exact comparison with his topographical plot using incremental theory could be obtained by slightly varying \( n \) as the hinge line approaches the center of the membrane. Lindbergh's solution has shown results verifying the formation of a peripheral wave which traverses inward retaining somewhat the original boundary shape.

Application of the work hardening criteria to rectangular membranes has been shown in Figures 20, 21, and 22 and results in a rounding off effect in
the slope of the displacement profiles at the apex. This is in agreement with the effect of work hardening on circular membranes. The work hardening relationship used in the solution appears to be acceptable. The effect of substituting effective strain for maximum strain will only tend to move the work hardening and the no work hardening curves into closer proximity at the apex. In fact, the results of this solution indicate for expediency in design work hardening could be neglected and would result in only small errors.

Observance of Figure 24 indicates that the shape of a completely deformed membrane as viewed from above has the appearance of an envelope. Although the corners would not be as prominent as shown in the illustration because they are actually rounded. This result is in good agreement with the final collapse shape as indicated by Gvozdev (18) for clamped rectangular plates subjected to pressure loading.

The results of this study appear to validate the approach used as a means of analytically investigating the response of clamped rectangular membranes to impulsive loads. Complete confirmation of the accuracy involved in this solution is not
possible at this time. Verification of the analytical work can only be done when experimental results are available.

Recommendations for Future Analysis

In addition to the conclusions just mentioned, this study has suggested pertinent areas of development which are recommended for further analysis and scrutiny.

Specifically, the following areas are proposed for extended study.

1. An experimental investigation of this problem is needed to complete the evaluation of this study.

2. A solution to this problem using other than a uniform initial velocity is desirable. This solution is attainable using the same approach to the problem, but describing the initial velocity with a double sine or cosine series.

3. For the results to be beneficial to a designer, then a parametric study of different materials, and membrane parameters would be very desirable.
4. A solution including the initial elastic effects as a comparison with this solution is another problem that could be studied.
BIBLIOGRAPHY


APPENDIX A

DETERMINATION OF THE INITIAL VELOCITY

The initial velocity $V$ of the membrane may be evaluated from the well known relations (15), (16). For the present, only final expressions are given here.

Impulse $I_0$ can be obtained from the expression

$$I_0 = K \left( \frac{w^3}{D_0} \right)^P, \text{ from (16)} \tag{1}$$

or

$$I_0 = 8w^3 \left( \frac{1}{D_0} \right)^P, \text{ from (15)} \tag{2}$$

where

$I_0 =$ Peak impulse-intensity $\left( \frac{\text{lb-sec}}{\text{in}^2} \right)$

$K, p, B = \text{Explosive material constants}$

$w = \text{Weight of explosive (lbs)}$

$D_0 = \text{Normal distance from charge to membrane.}$

The normal component of the total impulse can be obtained from the relation

$$dI_v = B(w)^{\frac{1+p}{3}} \left[ \frac{1}{D} \right]^p dA$$

where $D$ is shown in Figure 26.

$$dI_v = B(w)^{\frac{p+1}{3}} \left( \frac{1}{D_0^2 + x^2 + y^2} \right)^{\frac{p}{2}} dx \, dy$$

which gives, for a rectangular membrane,

$$I_v = B(w)^{\frac{p+1}{3}} \int_a^b \int_{-a-b}^{b} \left( D_0^2 + x^2 + y^2 \right)^{-\frac{p}{2}} dx \, dy$$

The impulse momentum equation is

$$I_v = \int_a^b \int_{-a-b}^{b} m(x,y) V(x,y) \, dx \, dy$$

where

$$m(x,y) = \text{mass of the membrane at the position (x,y)}.$$  

$$V(x,y) = \text{initial velocity of the membrane at (x,y).}$$

If $V_0$ is not uniform a general expression for $V_0$ can be assumed or an expression for $V(x,y)$
Figure 26. Development of Impulse From a Charge
can be obtained corresponding to the shape and distance of the charge from the membrane (16). In the present problem, $V_0$ has been taken as uniform.

For a general case, let

$$V_0 = V f(x,y)$$  \hspace{1cm} (6)

then

$$I_v = V \rho h \int_{a-b}^{a+b} f(x,y) \, dx \, dy$$  \hspace{1cm} (7)

where

- $\rho$ = density of the membrane material
- $V$ = initial velocity at the origin
- $h$ = thickness of the membrane.

If $V_0 = V = \text{constant}$, then

$$I_v = V \rho h (2a \times 2b)$$

$$= 4 \rho h V ab$$  \hspace{1cm} (8)

If $V_0 = V \cos \frac{\pi x}{2a} \cdot \cos \frac{\pi y}{2b}$, then

$$I_v = \frac{16 \rho h V ab}{\pi^2}$$  \hspace{1cm} (9)

Hence, $V$ can be obtained from equations (5) and (7) in terms of $I_v$. 
Several theories of plasticity are available to be used in a problem solution. The two most commonly used theories for plastic deformation are the incremental theory and the deformation theory. Batdorf and Budiansky have published a theory that is based on slip. It has not attained the stature of the two previously mentioned theories, possibly because it requires lengthy tedious computations.

The deformation theory has the advantage of being the least complex mathematically, but it is limited in its applicability as pointed out by (19), (20), (21). Application of the deformation theory is limited to the cases where the principal directions remain fixed during deformation and the principal stress ratios remain constant during deformation. It has been determined that the deformation theory and the incremental theory give the same results for these two cases. The incremental theory is not limited to these conditions which makes it applicable for a wider range of problems. One
disadvantage of the incremental theory is that it requires time consuming numerical analysis. Since the advent of the digital computer this disadvantage has been greatly reduced.

The major difference between a volume element that is elastically deformed and one that is inelastically deformed is in the stress-strain relation for the material. The usual assumption is that the stress-strain relation for the material is given either by the average diagram obtained from tension and compression specimens or by the diagram obtained from hollow torsion specimens. For most metals at room temperature the stress-strain diagram can be accurately approximated by two straight lines as indicated in Figure 27.

In order to solve a plasticity problem the following three relationships are needed:

(a) A universal stress-strain relation,
(b) A flow rule, and
(c) The constant volume condition.

Three relationships are necessary, in addition to Hooke's law and the equilibrium of force and compatibility conditions, because of the three additional quantities, the plastic components of the three principal strains.
Figure 27. Stress-Strain Curve Approximated by Two Lines
For a particular plasticity problem the following three relationships provide the necessary equations:

(a) A universal stress-strain relationship,
(b) The Von Mises condition, and
(c) The constant volume condition.

Since the elastic strains are negligible in an ideally plastic problem, the equations can be written for only the plastic portion of the strain. When working with large deformations this is a good approximation as the plastic strains account for practically the entire deformation.

For the universal stress-strain relationship any one of the following sets of functional relations could be used:

(1) Effective stress and effective strain,
(2) Maximum shear stress and maximum shear strain,
(3) Maximum shear and numerically largest strain, or
(4) Octahedral shear stress and the octahedral shear strain.

For problems where a solution in equation form is desired and where the differences between principal stresses change signs, it is usually most convenient
to use the effective stress and the effective strain (20). With this choice the sign is inconsequential because all of the stress or strain differences are squared. The simpler functions are easier to work with, particularly, for solutions using numerical methods.

The effective stress and effective strain, in conjunction with the Von Mises yield condition, can be defined as follows:

\[ \bar{\sigma} = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{\frac{1}{2}} \]

\[ \bar{\varepsilon} = \frac{2}{3} \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right]^{\frac{1}{2}} \]  

where

\[ \sigma_k (k = 1, 2, 3) \] are the true principal stresses, and

\[ \varepsilon_k (k = 1, 2, 3) \] are logarithmic principal strains. A loading function valid for all states of stress can now be written as

\[ \bar{\sigma} = F(\bar{\varepsilon}). \]  

If it is desired to define the strain-hardening behavior of the material using the octahedral shearing stress and octahedral shearing strain, the following functional relation is used
\[ \tau_{\text{oct}} = F(\gamma_{\text{oct}}) \] (3)

where

\[ \tau_{\text{oct}} = \frac{1}{3} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{\frac{1}{2}} \]

and

\[ \gamma_{\text{oct}} = \frac{2}{3} \left[ (\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_2)^2 \right]^{\frac{1}{2}}. \]

Choosing the combination of the maximum shearing stress and the numerically largest strain will provide the universal stress-strain relationship as

\[ \sigma_1 - \sigma_3 = F(\epsilon^*) \] (4)

where \( \epsilon^* \) is the numerically largest strain.

If the maximum shearing stress is combined with the maximum shearing strain the functional relationship will appear as

\[ \sigma_1 - \sigma_3 = F \left[ \frac{2}{3} (\epsilon_1 - \epsilon_3) \right]. \] (5)

In all of these expressions \( F \) represents the stress-strain curve for strain-hardening materials. The most common function of \( F \) for strain-hardening materials is the Power Law proposed by Ludwik and written as

\[ \sigma = b \epsilon^c \] (6)

where \( b \) and \( c \) are two characteristic constants of the
material.

The Von Mises condition may be expressed as

\[
\frac{\sigma_1 - \sigma_2}{d\varepsilon_1 - d\varepsilon_2} = \frac{\sigma_2 - \sigma_3}{d\varepsilon_2 - d\varepsilon_3} = \frac{\sigma_1 - \sigma_3}{d\varepsilon_1 - d\varepsilon_3}
\]  

(7)

This law states that the intermediate principal strain increment lies between the extremes in the same relative position that the intermediate principal stress lies between the extremes. If the principal stress ratios remain constant, the Von Mises condition can be applied to the total plastic strains. Then the incremental values in the denominator can be replaced with the corresponding total value of strain.

The other condition necessary for a complete formulation of the plasticity problem is the conservation of volume for incompressible materials. This condition can be stated as

\[
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0
\]

(8)

or

\[
d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 = 0
\]

(9)
VITA

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Thesis: THE PLASTIC DEFORMATION OF CLAMPED RECTANGULAR MEMBRANES SUBJECTED TO IMPULSIVE LOADS

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