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# Analysis of Nonlinear Dispersive Model Equations

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# ANALYSIS OF NONLINEAR DISPERSIVE MODEL EQUATIONS

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# Abstract

The overall goal of this thesis is the following: to derive nonlinear dispersive model equations and report my results regarding the analysis of them. We begin with a brief survey of the classical fluid dynamics problem of water waves, and then proceed to derive well known nonlinear dispersive evolution equations via a Hamiltonian Variational approach. This method was first introduced in the seminal work of Walter Craig, et al. [CG94]. The distinguishing feature of this scheme is that the Dirichlet-Neumann operator of the fluid domain appears explicitly in the Hamiltonian.

In the second and third chapters, we employ the Hamiltonian perturbation theory introduced in [CG94] to derive the Benjamin-Bona-Mahony (BBM) and Benjamin-Bona-Mahony-Kadomtsev-Petviashvili (BBM-KP) equations. Finally, we briefly review the existence theory for their corresponding Cauchy problems and then proceed to mathematically study them.

In the fourth chapter, I motivate my first result and demonstrate how it ties in with the literature and previous chapters. In particular, we show that the solution of the Cauchy problem for the BBM-KP equation converges to the solution of the Cauchy problem for the BBM equation in a suitable function space whenever the initial data for both equations are close as the transverse variable $y \to \pm\infty$.

In the final chapter, we introduce a new modified Kadomstev-Petviashvili equation. This model was introduced in an effort to remedy the "odd" behavior of the mass of a given solution to the Kadomstev-Petviashvili model. This results in a model which does not impose certain restrictions, to be specified later, upon the initial data. After motivating and deriving the model we prove various linear estimates

for the operator equation arising from the Duhamel formulation of system. To this end I discuss the direction that my research is heading.

# Chapter 1
# Framework

Throughout this thesis, $n$ denotes the dimension of the Euclidean space $\mathbb{R}^n$. We also call $(\xi, \mu, \tau)$ the Fourier variable dual to $(x, y, z)$. Furthermore, all integrals will be with respect to the Lebesgue measure $dx$. We use $\langle x \rangle$ to denote the Japanese bracket $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$ of $x$. One should observe that it is an immediate consequence that $\langle x \rangle$ is comparable to $|x|$ for sufficiently large $x$ and 1 for small $x$. For $x \in \mathbb{R}^n$, we let $|x| := (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ be the magnitude of the vector $x$. As usual, let $(\cdot, \cdot) := |x|^2$ be the canonical inner product on $\mathbb{R}^n \times \mathbb{R}^n$. In addition, if $x$ and $y$ are two non-negative quantities, then $x \lesssim y$ means that there is an absolute constant $C > 0$, such that $x \leq Cy$. By $x \sim y$ it is meant that $x \lesssim y$ and $x \gtrsim y$. Extensive use will be made of the Lebesgue norms

$$\|f\|_{L^p_x(\mathbb{R}^n \to C)} := \Big( \int_{\mathbb{R}^n} |f|^p dx \Big)^{\frac{1}{p}}$$

where $1 \leq p \leq \infty$ for complex-valued measurable functions $f : \mathbb{R}^n \to \mathbb{C}$, with the convention that $\|f\|_{L^\infty_x(\mathbb{R}^n \to \mathbb{C})} := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f|$.

*Remark* 1.1. In this setting, $\mathbb{C}$ can be replaced with any Banach space $X$. For example, $\|f\|_{L^\infty_x(\mathbb{R}^n \to X)}$ denotes the space of all measurable functions $f : \mathbb{R}^n \to X$ endowed with the following norm.

$$\|f\|_{L^p_x(\mathbb{R}^n \to X)} := \Big( \int_{\mathbb{R}^n} \|f\|_X^p \, dx \Big)^{\frac{1}{p}}$$

A function $f : \mathbb{R}^n \to C$ is called rapidly decreasing if

$$\left\| \langle x \rangle^N f \right\|_{L^\infty_x(\mathbb{R}^n)} < \infty$$

for all non-negative $N$. A smooth function belongs to the Schwartz class $S(\mathbb{R}^n)$ provided that all of its derivatives are rapidly decreasing. It is well know that $S(\mathbb{R}^n)$ is a Frechet Space and, as a result, has a dual denoted $S'(\mathbb{R}^n)$, i.e. the space of tempered distributions cf. [FO14]. In many cases we will abbreviate $L^p_x(\mathbb{R}^n \to X)$ as $L^p_x(\mathbb{R}^n)$, or even $L^p$ provided that the context is clear. We will also adopt this abuse of notation for other species of function classes used throughout this work. For $f \in S'(\mathbb{R}^n)$ we denote by $\hat{f}$ or $\mathcal{F}(f)$ the space-time Fourier transform of $f$

$$\hat{f}(\xi, \mu, \tau) := \int_{\mathbb{R}^3} e^{-i(x\xi + y\mu + t\tau)} f(x, y, t) dx dy dt.$$

Moreover, we will use $\mathcal{F}_{x,y}$ and $\mathcal{F}_t$ to denote the Fourier transform in the spatial and temporal variables respectively. If $X$ and $Y$ are both Banach spaces, we denote a continuous embedding between them by $\hookrightarrow$, i.e. $X \hookrightarrow Y$ means that $\|u\|_Y \lesssim \|u\|_X$. When attempting to obtain a contraction mapping for the intergral formulation of a given Cauchy problem, one needs to first localize in time. Define the following function space $C^{t_0}_{t_1} X := C([t_0, t_1]; X)$ equipped with the norm

$$\|u\|_{C^a_b X} := \sup_{t \in [a,b]} \|u\|_X$$

It is well known that if $X$ is a Banach space, then so is $C^a_b X$ cf. [FO14]. For a detailed introduction to such spaces and their cousins we refer the reader to [Eva10].

*Remark* 1.2. The pseudo-differential operator $\partial_x^{-1} f$ is defined via the Fourier transform as

$$\widehat{\partial_x^{-1} f} = \frac{1}{i\xi} \hat{f}(\xi, y).$$

Due to the singularity of the symbol $\frac{1}{\xi}$ at $\xi = 0$, one requires that $\hat{f}(0, y) = 0$ (the Fourier transform in the variable $x$), which is clearly equivalent to

$$\int_{\mathbb{R}} f(x, y) dx = 0.$$

In what follows, $\partial_x^{-1} f \in L^2(\mathbb{R}^2)$ means there is an $L^2(\mathbb{R}^2)$ function $g$ such that $g_x = f$, at least in the distributional sense. When we write $\partial_x^{-k}\partial_y^m$ for $(k,m) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, we implicitly assume that the operator is well-defined. As pointed out in [MST07b], this imposes a constraint on the solution $u$. This condition implies that $u$ is an $x$ derivative of a suitable parent function. This can be achieved in the following two ways:

a.) If $u \in S'(\mathbb{R}^2)$ is such that $\xi^{-k}\mu^m \hat{u}(\xi,\mu,t) \in S'(\mathbb{R}^2)$.

b.) If $u(x,y,t) = \frac{\partial}{\partial_x} v(x,y,t)$ for $v \in C_x^1(\mathbb{R})$, i.e. the spaces of continuous functions possessing a continuous derivative with respect to $x$.

It is worth pointing out that the second possibility imposes a decay condition on $u$, i.e. for any fixed $y$ and $t \neq 0$ we must have $u \to 0$ as $x \pm \infty$. This results in

$$\int_{\mathbb{R}} u(x,y,t)dx = 0, y \in \mathbb{R}, \quad t \neq 0.$$

Formally, we define the follow class of functions and localize in time to arrive at

$$\mathcal{X}^s(\mathbb{R}^2) := \Big\{u : u \in H^s(\mathbb{R}^2) \cap H^{s-1}(\mathbb{R}^2)\Big\}.$$

Also, let $v(x,y,t) = \partial_x^{-1} u(x,y,t)$ and $w(x,y,t) = \int_{-\infty}^{x'} u(x',y,t)dx'$. It follows that $\hat{u}(t) = \hat{w}(t)$ in $S'(\mathbb{R}^2)$ for all $t \in [-T,T]$, thus $v = w$ due to that fact that the Fourier transform is an isomorphism on $S'(\mathbb{R}^2)$. As Mammeri points out in [Mam09], since $u \in C([-T,T]; \mathcal{X}^s(\mathbb{R}^2))$ and $s > 2$, it is a direct consequence that $u \in L^1_{x,y}$. Moreover, $\partial_x^{-1} u \in C([-T,T]; H^s(\mathbb{R}^2))$ and since $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^2)$, we must have that $v \to 0$ as $x \to \pm\infty$. So

$$\begin{aligned}\int_{\mathbb{R}} u(x', y, t)dx' &= \lim_{x\to\infty} \int_{-\infty}^{x'} u(x', y, t)dx' \\ &:= \lim_{x\to\infty} w(x, y, t) \\ &= \lim_{x\to\infty} v(x, y, t) \\ &= \lim_{x\to\infty} \partial_x^{-1} u(x, y, t) \\ &= 0.\end{aligned}$$

Due to this additional requirement of the initial data to a given Cauchy problem that contains the operator $\partial_x^{-k}\partial_y^m$. One is often forced to turn to, more regular, weighted anisotropic Sobolev spaces to look for a contraction to the corresponding Duhamel formulation of the IVP. Let $H^s(\mathbb{R}^2)$ denote the classical Sobolev space $H^s(\mathbb{R}^2)$ equipped with the norm

$$||\eta||_s = \Big(\int_{\mathbb{R}^2} (1 + \mu^2 + \xi^2)^s |\hat{\eta}(\xi, \mu)|^2 d\xi d\mu\Big)^{\frac{1}{2}}.$$

Analogously, $H_x^k(\mathbb{R})$ will denote the Sobolev space in just the spatial variable $x$ with the norm

$$||f||_{H_x^k(\mathbb{R})} = \Big(\int_{\mathbb{R}} (1 + \xi^2)^k |\hat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$

Define the space

$$H_{-1}^s(\mathbb{R}^2) = \{\eta \in S'(\mathbb{R}^2) : ||\eta||_{H_{-1}^s(\mathbb{R}^2)} < \infty\}$$

equipped with the norm

$$||\eta||_{H_{-1}^s(\mathbb{R}^2)} = \Big(\int_{\mathbb{R}^2} (1 + |\xi|^{-1})^2 (1 + \xi^2 + \mu^2)^s |\hat{\eta}(\xi, \mu)|^2 d\xi d\mu\Big)^{\frac{1}{2}}.$$

In Chapter 4, we will make use of the Dispersive Sobolev Spaces $X^{s,b}$.

**Definition 1.3** (Dispersive Soblev Spaces)**.** Let $s, b$ be in $\mathbb{R}$, $(\xi, \tau) \in \mathbb{R}^d \times \mathbb{R}$ and $h \in C(\mathbb{R}^d)$. The space $X^{s,b}$ is defined to be the closure of the Schwartz functions $S_{x,t}(\mathbb{R}^d \times \mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}} := \|\langle \xi \rangle^s \langle \tau - h(\xi) \rangle^b \widehat{u}(\xi, \tau)\|_{L^2_{\xi_\tau}}.$$

These spaces are well adapted to the linear parts of the dispersive models in interest. One of the main ideas behind these spaces is to estimate the solution of a given Cauchy problem on levels sets of dispersion relation. We will give examples of this throughout this thesis. If one thinks about the KdV soliton then these spaces are the most natural Banach spaces to capture that "balance", between the nonlinear and dispersive terms. For a detailed study of these spaces, we refer the reader to [Tao06]. As Tao points out, if $b = 0$, then the dispersion doesn't matter and the space is the larger $L^2 H^s_x$. If $h = 0$, then there is no dispersion, is the smaller $H^b_t H^s_x$. For simplicity, we will abbreviate $X^{s,b}_{\tau=h(\xi)}$ as $X^{s,b}$. For our purposes, we will need to adapt these spaces to the specific equation or system of interest. To this end let $i = 1, 2.$ and define the following function spaces:

$$\|u\|_{X^{b,s}_i} = \left\| \langle \xi \rangle \langle \tau - \omega_i(\xi, \mu) \rangle^b \langle |\xi| + |\mu| \rangle^s \hat{u}(\xi, \mu, \tau) \right\|_{L^2_{\xi,\mu,\tau}}$$

$$\|u\|_{Y^s_\xi} = \| \langle \xi \rangle \langle |\xi| + |\mu| \rangle^s \hat{u}(\xi, \mu) \|_{L^2_{\xi,\mu}}$$

$$\|u\|_{H^{b,s}} = \left\| \langle \tau \rangle^b \langle |\xi| + |\mu| \rangle^s \hat{u}(\xi, \mu, \tau) \right\|_{L^2_{\xi,\mu,\tau}}$$

$$\|u\|_{H^b} = \left\| \langle \tau \rangle^b \hat{u}(\tau) \right\|_{L^2_\tau}.$$

*Remark* 1.4. The spaces listed above will be useful for our analysis of the model be presented in Chapter 5, for appropriate parameter values $b$ and $s$.

Listed below is a useful embedding theorem for the function class $X^{s,b}$.

**Theorem 1.5.** *$X^{s,b} \hookrightarrow C_t^0 H^s$ Let $u \in S_t(\mathbb{R}^n)$ be a Schwartz function in time, $h \in C(\mathbb{R}^n)$ be scalar valued. If $s \in \mathbb{R}$ and $b > \frac{1}{2}$ we have the embedding*

$$\|u\|_{C_0^t H_x^s(\mathbb{R}^d \times \mathbb{R})} \lesssim \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^d \times \mathbb{R})}$$

When attempting to obtain a contraction in a suitable space, the problem reduces to finding a suitable Banach space such that the map is a contraction on a ball of a specified radius. To accomplish this it is standard practice to first localize in time. This point begs the question of how the $X^{s,b}$ spaces behave with respect to time localization. Listed below is an important theorem concerning this matter. For a complete proof, we refer the reader to [Tao06].

**Theorem 1.6.** *$X^{s,b}$ is stable with respect to time localization. For all positive $\sigma$, $\eta \in S_t(\mathbb{R}^n)$, $u \in S_{t,x}(\mathbb{R} \times \mathbb{R}^n)$, and scalar valued $h \in C(\mathbb{R}^n)$. If we call $B$ to be the set of $b \in \mathbb{R}$ such that $\frac{-1}{2} < b' < b \leq \frac{1}{2}$, then for all $(s,b) \in \mathbb{R}^2$ we have that*

$$\|\eta(t)u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^d \times \mathbb{R})} \lesssim \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^d \times \mathbb{R})}.$$

*If we impose the additional restraints that $b \in B$ and time $T \in (0,1)$, then*

$$\left\|\eta(\tfrac{t}{T})u\right\|_{X^{s,b'}_{\tau=h(\xi)}(\mathbb{R}^d \times \mathbb{R})} \lesssim T^{b-b'} \|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}^d \times \mathbb{R})}.$$

*Remark* 1.7. The second estimate appearing in the above theorem is especially useful in large data theory, as it enables us to control certain $X^{s,b}$ norms of a solution after localizing in time.

*Remark* 1.8. These spaces enjoy all of the same Sobolev embeddings that free solutions of its corresponding linearization do. However, as remarked in [Tao06], these spaces are only well suited for analyzing nonlinear dispersive evolution equations, after one localizes in time. Fortunately, these spaces behave well with respect to time localization.

Now we briefly review some basics of Hamiltonian partial differential equations, as listed in [CG94]. Let $H : \mathcal{D} \subset P \to \mathbb{R}$ denote the Hamiltonian function defined on a dense subdomain $\mathcal{D}$ of $P$. The gradient is taken with respect to the inner product on $P$ and $J$ is a skew-adjoint operator called the structure map which defines the Poison bracket $\{\cdot,\cdot\} = (\text{grad}(\cdot), J\text{grad}(\cdot))$. It is well known that this arrangement fixes a Poison structure on $P$, cf. [FO14]. If the structure map $J$ is invertible, then we call the Poison structure symplectic. The water-wave problem falls into a class of Hamiltonian evolutionary systems in which $\mathcal{D} \subset P = (L^2(X))^n$ where $X \subset \mathbb{R}^n$. The structure map $J$ is independent of $v$, the $n$ components of the vector $v$ depend on a position variable $x \in X$ and the Hamiltonian $H$ has the form $\hat{H} = \int_X H$ for a Hamiltonian density function $H$. In practice, $H$ depends on spatial derivatives of $v$ and $\mathcal{D}$ is chosen so that $\hat{H}(v)$ is well-defined, e.g. $\mathcal{D} = (S(\mathbb{R}^n))^n$.

**Definition 1.9** (Hamiltonian Evolutionary System)**.** A Hamiltonian Evolutionary System is a system of partial differential equations of the form

$$v_t = J\text{grad}H(v)$$

where $v(t)$ describes a path in a Hilbert space $P$ equipped with inner product $(\cdot,\cdot)$.

The method employed to derive the nonlinear dispersive evolution equations analyzed in this thesis was introduced by Walter Craig et al. [CG94]. The advantage of this variational approach resides in its unifying fact that the well known water wave equations can be derived. Hamiltonian perturbation theory first appeared in the work of Benjamin [Ben84]. The main idea was to approximate a Hamiltonian evolutionary system by fixing its phase space and Poison structure and replace the Hamiltonian density function by an approximation $H_A(v)$. This results in the following approximation

$$v_t = J\text{grad}H_A(v)$$

of the original Hamiltonian system. Then if we consider a Hamiltonian evolutionary system in which $H = H(v, \epsilon)$ for some parameter $\epsilon$. If $H$ depends smoothly on $\epsilon$ one can expand it in the following power series

$$H(v, \epsilon) = H_0(v) + \epsilon H_1(v) + \epsilon^2 H_2(v) + n \cdots$$

Finally, we use this to construct a sequence of Hamiltonians which approximate the original system, i.e.

$$v_t = J\text{grad}\left(\sum_{i=0}^{N} \epsilon^i H_i(v)\right), \text{for } N = 1, 2, ...$$

# Chapter 2
# Introduction and Background

In the past few decades, research in nonlinear dispersive waves has been very active. The dispersive smoothing effect (see Figure 2.1 [1]) enjoyed by these equations helped to motivate the invention of the Bourgain spaces $X^{s,b}$, first utilized in [Bou93b, Bou93a, Bou93c], which provide a convenient way to control the "size" of the solution to the linear problem in terms of the initial data, or of the forcing term.

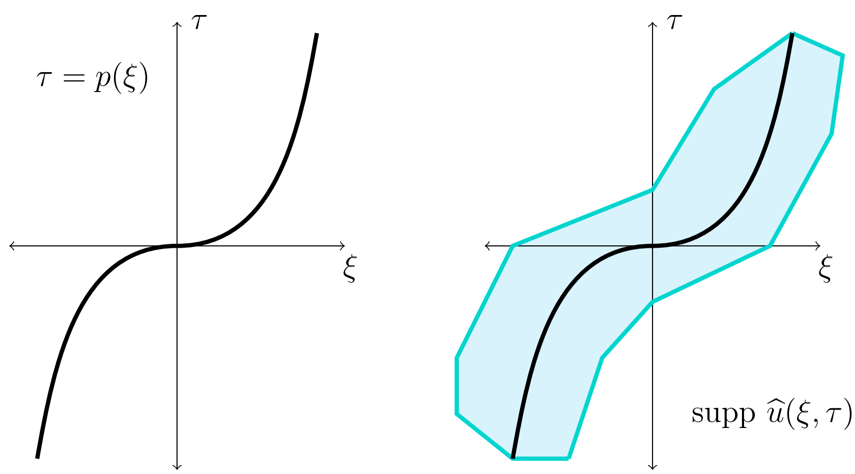


FIGURE 2.1. The dispersive smoothing effect

Also, due to the increasing demand of sea transport and off shore oil exploration, many applied mathematicians and engineers are concerned with evaluating forces applied by traveling waves on various off-shore structures [MA13], which usually reduces to solving a partial differential equation and using the obtained results to find out the pressure field.

[1] Let $u$ be a solution to a linear dispersive equation (e.g. the Airy equation). After taking the spacetime Fourier transform we obtain $i\tau\widehat{u}(\xi,\tau) = ip(\xi)\widehat{u}(\xi,\tau) \Rightarrow (\tau - p(\xi))\widehat{u}(\xi,\tau) = 0 \Rightarrow$ supp $\widehat{u}(\xi,\tau)$ is on the hypersurface $\{(\xi,\tau) : \tau = p(\xi)\}$ of the space-time frequency space. Solutions to nonlinear perturbations of that dispersive equation (e.g. the KdV), after localization in time, will have their space-time Fourier transform supported near that hypersurface. Thus, the nonlinearity did not significantly alter the space-time Fourier path of the solution for short time. The $X^{b,s}$ spaces provide an efficient way to capture this clustering.

The rigorous mathematical analysis presented here is important since the equations considered model physical phenomena. As a result, the equations should have solutions which behave in an expected way, along with other desirable qualitative properties. After casting the equation of interest into a mathematical framework we proceed to investigate the notion of well-posedness for their corresponding Cauchy problems.

The notion of modeling an ideal fluid with a free surface, being acted on by gravity is a classical problem in fluid mechanics. The surface water wave problem is described by the Euler equations coupled with appropriate boundary conditions on the bottom surface; in combination with kinematic and dynamic boundary conditions on the free surface. We will see that the nonlinearities and time dependency are both a direct result of the boundary conditions on the free surface. In the presence of weak transverse effects the unknowns are: the surface elevation $\eta$, the horizontal and vertical fluid velocities, $u$ and $v$ respectively and the pressure $P$, which we will eliminate through means of the Bernoulli equation.

The basic conservation laws for the water wave problem are that mass, momentum and total energy (Hamiltonian) conservation. Additionally, for mathematical purposes we will treat a control volume element as a subspace of $\mathbb{R}^3$, i.e. the continuum hypothesis. To this end we consider a body of water of finite depth being acted on by gravity and bounded below by an undisturbed solid impermeable surface. If we ignore the effects of viscosity and assume the flow is incompressible and irrotational, then the fluid motion is modeled by the Euler equations coupled with appropriate boundary conditions on the rigid bottom and water-air interface. In the subsequent chapters, we will invoke simplifying assumptions that lead to various regularized model equations valid for small-amplitude long wavelength motion.

To this end, we let $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$ denote the velocity vector of the mass at point $(x, y, z)$ and time $t$. Also, we denote $\frac{DF}{Dt}$ to be the material or total derivative, i.e.

$$\begin{aligned} \frac{Df}{Dt} &= \lim_{\delta t \to 0} \frac{f(x + u\delta t, y + v\delta t, z + w\delta t) - f(x, y, z, t)}{\delta t} \\ &= \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v + \frac{\partial f}{\partial z} w + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f. \end{aligned}$$

*Remark* 2.1. The material derivative quantifies the rate of change of a function $f$ as it varies from the point $p(x, y, z)$ to $p(x + \delta t, y + \delta t, z + \delta t)$, for an infinitesimal increment of time $\delta t$. In our case, it is the derivative following the motion of the fluid particle. For this reason it is also called the convective derivative.

Consider a rectangular volume element $Q$, with volume $\delta x \delta y \delta z$. After calculating its rate of change over the time interval $[t, t + \delta t]$, we obtain that

$$\frac{1}{Q} \frac{DQ}{Dt} = \nabla \cdot \boldsymbol{u}.$$

Now for density $\rho = \rho(x, y, z, t)$, mass conservation means that

$$\frac{D(\rho Q)}{Dt} = 0. \qquad \text{Mass Conservation Law}$$

An application of the product rule and a substitution yield

$$\frac{D(\rho Q)}{Dt} = \frac{1}{Q} \frac{DQ}{Dt} + \frac{1}{\rho} \frac{D\rho}{Dt} = \rho \nabla \cdot \boldsymbol{u} + \frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla) \rho = 0.$$

It follows that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})$$

and if we make the assumption that the fluid is incompressible ($\rho = C$), we arrive at the following condition

$$\nabla \cdot \boldsymbol{u} = 0. \qquad \text{Incompressibility} \tag{2.1}$$

Newton's second law of motion state that the rate of change of momentum is equal to the net applied forces, i.e.

$$\frac{D(m\boldsymbol{u})}{Dt} = \boldsymbol{f}$$

where $m$ denotes the mass of the volume element $Q$ and $\boldsymbol{f}$ is the net force. If we assume the fluid to be inviscid, i.e. there are no internal frictional forces causing flow resistance. Since the mass is only acted on by pressure and gravity, we conclude that

$$m\frac{Du}{Dt} = (P(x,y,z) - P(x+\delta x,y,z))\delta y\delta z = -\frac{\partial P}{\partial x}Q$$

where $P = P(x,y,z,t)$ is the pressure. Since $m = \rho Q$, it follows that

$$\frac{du}{dt} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0.$$

Taking into account gravitational acceleration, a similar calculation yields

$$m\frac{dv}{dt} = -\frac{\partial P}{\partial y} - \rho Q g,$$

as a result

$$\frac{dv}{dt} + \frac{1}{\rho}\frac{\partial P}{\partial y} + g = 0.$$

Therefore, the conservation of momentum law becomes

$$\frac{D\boldsymbol{u}}{Dt} + \frac{1}{\rho}\nabla P + g\boldsymbol{e_2} = 0$$

or equivalently

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla P + g\boldsymbol{e_2} = 0, \tag{2.2}$$

where $\boldsymbol{e_2}$ is the unit vector in the direction opposite to gravitation. Since the fluid is irrotational, it follows that $\nabla \times \boldsymbol{u} = 0$. It follows that there exists a velocity potential $\phi(x,y,z,t)$ such that

$$\boldsymbol{u} = \nabla\phi. \tag{2.3}$$

This fact combined with (2.1) implies that $\phi$ solve Laplace's equation, i.e. $u$ is harmonic.

$$\Delta\phi = 0. \qquad \text{Laplace Equation} \tag{2.4}$$

Therefore that problem is reduced to solving Laplace's equation coupled with appropriate boundary conditions, then one can just calculate the velocity field $\boldsymbol{u}$ from (2.3).

The next step is to eliminate the pressure $P$ from equation (2.2). This is accomplished by making use of the identity $\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) = 2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + 2\boldsymbol{u}\times(\nabla\times\boldsymbol{u})$ and the Bernoulli equation. Accordingly, we can rewrite (2.2) as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) + \frac{1}{\rho}\nabla P + g\boldsymbol{e_2} = 0$$

Making use of the potential $\phi$ and the fact that $\nabla y = \boldsymbol{e_2}$ leads us to the following

$$\nabla\Big[\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{\rho}P + g\boldsymbol{e_2}\Big] = 0.$$

The above equation combined with the physically relevant assumption that the flow domain is simply connected, results in

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{\rho}P + g\boldsymbol{e_2} = C(t).$$

Let $P_0$ be the pressure in the air near the free surface of the fluid, we can include the pressure drop in the above equation to obtain that

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{1}{\rho}(P - P_0) + g\boldsymbol{e_2} = C(t).$$

*Remark* 2.2. In what follows we will take $P_0$ to be constant. The air-water interface has no mass. So, if surface tension is neglected then the pressure in the water and the air pressure must be equal on the interface. Although a disturbance in the surface causes some motion in the air, since the small density of air relative to the density of water, the air pressure is not changed significantly. As a result we approximate it by its undisturbed value.

Next, we define $\bar{\phi} = \phi - \int_0^t C(s)ds$ and rewrite the above equation in terms of $\bar{\phi}$, viz.

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{1}{2}\nabla\bar{\phi}\cdot\nabla\bar{\phi} + \frac{1}{\rho}(P - P_0) + g\boldsymbol{e_2} = 0.$$

Now we drop the bar and rewrite the previous equation as

$$\frac{1}{\rho}(P - P_0) = -\frac{\partial \phi}{\partial t} - \frac{1}{2}\nabla\phi\cdot\nabla\phi - g\boldsymbol{e_2}. \tag{2.5}$$

Suppose that the free surface of the liquid is described by and an equation of the form $\varphi(x, y, z, t) = 0$. Since the fluid doesn't cross the surface it follows that the velocity of the fluid at the surface equals the velocity of the surface. So, $\frac{D\varphi(x,y,z,t)}{Dt} = 0$, which leads to the following kinematic boundary condition

$$\varphi_t + u\varphi_x + v\varphi_y + w\varphi_z = 0.$$

If the free surface can be described by a single-valued function of $(x, z)$ for some time interval, say

$$\varphi(x, y, z, t) = \eta(x, z, t) - y$$

, then the previous equation can be written as

$$\eta_t + u\eta_x + w\eta_z = v.$$

After making use of the potential function we arrive at the following equation.

$$\eta_t + \phi_x\eta_x + \phi_z\eta_z = \phi_y \tag{2.6}$$

The second boundary condition on the free surface follows from the above remark and is $P = P_0$ at $y = \eta(x, z, t)$, where $P = P(x, \eta, z, t)$ is the pressure at the surface. Combining this with (2.6), we arrive at the following Bernoulli condition.

$$\eta_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad \text{for} \quad y = \eta \tag{2.7}$$

Since the lower boundary is impermeable, there is no flow through the bottom. As a result, the velocity normal to the bottom must be zero. If the bottom profile is $y = -h_0(x,z)$, then $\boldsymbol{u} \cdot \boldsymbol{n} = 0.$, where $\boldsymbol{n} = (h_{0x}, 1, h_{0z})$. This leads us to the following homogeneous Neumann boundary condition

$$\phi_x h_{0x} + \phi_z h_{0z} + \phi_y = 0 \quad \text{at} \quad y = -h_0(x,z).$$

In summary, assuming that the free surface and the bottom profile can be described as a single-valued function of $(x, z, t)$, the motion of a ideal fluid is described by the following system:

$$\begin{cases} \Delta\phi = 0 \qquad \text{in the flow domain} - h_0 < y < \eta \\ \eta_t + \phi_x \eta_x + \phi_z \eta_z = \phi_y \qquad \text{at the free surface} y = \eta \\ \phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \qquad \text{at the free surface} y = \eta \\ \phi_x h_{0x} + \phi_z h_{0z} + \phi_y = 0 \qquad \text{at the bottom} y = -h_0(x,z) \end{cases} \tag{2.8}$$

*Remark* 2.3. It is worth pointing out that all of the time dependence and nonlinearity in the problem is due to the top boundary conditions on the free surface $y = \eta$.

In many applications it is appropriate to consider the case where the fluid motion is two-dimensional, i.e. motions that are independent of $z$. If we assume that the bottom is flat, so that $h_0$ is constant, then system (2.8) reduces to

$$\begin{cases} \phi_{xx} + \phi_{yy} = 0 \qquad \text{in the flow domain} - h_0 < y < \eta \\ \eta_t + \phi_x \eta_x = \phi_y \qquad \text{at the free surface} y = \eta \\ \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta = 0 \qquad \text{at the free surface} y = \eta \\ \phi_y = 0 \qquad \text{at the bottom} y = -h_0 \end{cases} \tag{2.9}$$

In what follows we define the fluid domain to be $S(\eta) = \{(x,y) : -h_0 < y < \eta(x,t)\}$. It is well known that these equations can be written as a Hamiltonian system. It is a straightforward choice that the Hamiltonian function should be total energy of the system. The kinetic energy is the Dirichlet integral over the fluid domain $S(\eta)$;

$$K = \int_{\mathbb{R}} \int_{-h_0}^{\eta} \frac{1}{2} (\nabla \phi)^2 \; dydx$$

and the potential energy is

$$V = \int_{\mathbb{R}} \int_{0}^{\eta} y \; dydx.$$

Therefore the total energy is

$$\begin{aligned} H &= K + V \\ &= \int_{\mathbb{R}} \int_{-h_0}^{\eta} \frac{1}{2} (\nabla \phi)^2 \; dydx + \int_{\mathbb{R}} \int_{0}^{\eta} y \; dydx \\ &= I + \int_{\mathbb{R}} \frac{1}{2} \eta^2 \; dx. \end{aligned}$$

However

$$\begin{aligned} I &= \int_{\mathbb{R}} \int_{-h_0}^{\eta} \frac{1}{2} (\nabla \phi)^2 \; dydx \\ &= -\int_{\mathbb{R}} \int_{-h_0}^{\eta} \frac{1}{2} \phi \Delta \phi \; dydx + \int_{y=\eta} \phi \nabla \phi \cdot \boldsymbol{n} \; dS + \int_{y=-h_0} \phi \nabla \phi \cdot \boldsymbol{n} \; dS. \end{aligned}$$

Where integration by parts was utilized. The first term vanishes since $\phi$ is harmonic in $S(\eta)$. Due to the bottom boundary conditions we have that $\nabla \phi \cdot \boldsymbol{n} = \phi_x h_{0x} + \phi_y + \phi_z h_{0z} = 0$. As a result, the Hamiltonian is

$$H(\eta, \Phi) = \frac{1}{2} \int_{\mathbb{R}} \left[ \eta^2 + \Phi G(\eta) \Phi \right] \; dx.$$

This is the Hamiltonian formulation of the water wave problem as in found in [CG94]. The operator $G(\eta)$ appearing in the Hamiltonian is the Dirichlet-Neumann operator for the fluid domain $S(\eta)$. This linear operator takes the initial data $\Phi$ and produces the normal derivative of the solution $\phi$ of the boundary value problem (2.10), where

$\Phi(x,t) = \phi(x,\eta(x,t),t)$ is the trace of the potential at the free surface.

$$\begin{cases} \Delta\phi = 0 \quad \text{for} \quad (x,y) \in S(\eta) \\ \nabla\phi \cdot \boldsymbol{n} = 0 \quad \text{for} \quad \{(x,y) : y = -h_0\} \end{cases} \tag{2.10}$$

More explicitly, if $\phi$ solves (2.10) with Dirichlet data $\Phi(x,t) = \phi(x,\eta(x,t),t)$, then

$$G(\eta)\Phi(x)\ dx = (\nabla\phi \cdot \boldsymbol{n})(x)\ dS(x). \tag{2.11}$$

Most of the variety in water wave models is a consequence of how this positive, symmetric, bounded, analytic operator is approximated. The analyticity of the operator $G(\eta)$ means that we may represent it in terms of its Taylor series expansion.

$$G(\eta) = \sum_{j\geq 0} G_j(\eta), \tag{2.12}$$

where each term $G_j(\eta)$ is homogeneous in $\eta$ of degree $j$. The scheme for deriving the water wave models is to substitute (2.12) into the Hamiltonian and truncate terms up to a specified order obtaining an approximate Hamiltonian.

*Remark* 2.4. A derivation for the recursive formula of the operator $G(\eta)$ is included in the Appendix for the sake of completeness.

For our purposes, it will be convenient to formulate the Hamiltonian in terms of the dependent variable $u = \Phi_x$. In this spirit we define the operator $\mathcal{K}(\eta)$ by

$$G(\eta) = D\mathcal{K}(\eta)D \quad \text{for} \quad D = -i\partial_x.$$

Since $\mathcal{K}$ inherits the analyticity of $G$, it has a Taylor expansion around zero, i.e.

$$\mathcal{K}(\eta)\zeta = \sum_{j\geq 0} \mathcal{K}_j(\eta)\zeta, \quad \text{for} \quad \mathcal{K}(\eta) = D^{-1}G_j(\eta)D^{-1}. \tag{2.13}$$

The operator $\mathcal{K}$ enables us to write the Hamiltonian in terms of $u$, viz.

$$H(\eta,\Phi) = \frac{1}{2}\int_{\mathbb{R}} \left[\eta^2 + u\mathcal{K}(\eta)u\right]\ dx.$$

The water wave problem can then be written as a Hamiltonian system using variational derivatives of $H$ and posing the Hamiltonian equations

$$\begin{cases} \eta_t = -\frac{\delta H}{\delta u} \\ u_t = -\frac{\delta H}{\delta \eta} \end{cases} \tag{2.14}$$

*Remark* 2.5. Observe that the structure map associated with (2.14) is symmetric.

This concludes the Hamiltonian formulation of the water wave problem. The advantage of this approach lies in the fact that one can obtain most of the well known nonlinear dispersive evolution equations by making appropriate choice of operators. This will be explained in detail as the thesis progresses.

# Chapter 3
# The BBM Equation

The pure initial value problem for the Benjamin-Bona-Mahony (BBM) equation

$$\begin{cases} u_t + uu_x - u_{xxt} = 0, \quad \text{for } x \in \mathbb{R} \text{ and } \quad t > 0, \\ u(x,0) = u_0. \end{cases} \tag{3.1}$$

has been studied by various authors. The BBM equation (3.13) was introduced in [BBM72] as the regularized counterpart of the Korteweg-de-Vries (KdV) equation (see e.g. [BPS81, HS74, ZG71]). This equation was introduced as a model for one-dimensional, unidirectional, small-amplitude long-waves in nonlinear dispersive media. For shallow-water waves, $u = u(x,t)$ represents the displacement of the water surface or velocity at time $t$ and location $x$. One can obtain (3.13) from the KdV equation by observing that under suitable conditions $u_x \approx -u_t$, which implies $u_{xxx} \approx -u_{xxt}$. This derivative approximation will be explained in detail later. It is worth mentioning that, as in the case of the KdV, the BBM admits stable solitary wave solutions cf. [Zen03].

It is worth mentioning that the BBM equation covers not only surface waves of long wavelength in liquids, but also hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals and acoustic gravity waves in compressible fluids.

### 3.1 Derivation of the model equation

To derive the BBM equation, we begin by implementing a Boussinesq scaling regime. In this parameter regime one specializes to the case of long waves which have a small amplitude compared to the depth of the water. More precisely, one chooses $\alpha = \beta = \mu^2$ to be the small parameter. To this end we define the stretched independent variables

$$x_1 = \mu x, \qquad t_1 = \mu t,$$

and scaled dependent variables

$$\eta_1 = \frac{1}{\mu^2}\eta, \qquad \Phi_1 = \frac{1}{\mu}\Phi.$$

Define the operator $D = -i\frac{\partial}{\partial_x}$. We introduce $\mu$ dependency on the Hamiltonian through the induced transformation $D = \mu D_1$ and the small amplitude scaling presented above. In this small-amplitude, long wave regime we will construct a sequence of approximate Hamiltonians by expanding the Dirichlet-Neumann operator in the integrand of (2) and retaining terms up to $O(\mu^{\mu n})$, $n = 2, 3, \ldots$. This process is assisted by the fact that $G_j$ is homogeneous of degree $j$ in $\eta$ and of degree 2 in $\Phi$, so that in the new variables, $\Phi_1 G_j(\eta_1)\Phi_1$ is at least of order $\mu^{2+2j}$. In the first approximation to the Hamiltonian, terms up to $O(\mu^4)$ are retained. As a result, we neglect all but the first term $G_0(\eta)$ in the Taylor series for $G(\eta)$. This leads us to the following Hamiltonian

$$H = \int_{\mathbb{R}} \left(\frac{1}{2} g\mu^4\eta_1^2 + \frac{1}{2}\mu\Phi_1 G_0 \mu\Phi_1\right)\, dx + O(\mu^6).$$

From the Appendix we know that $G_0 = \mu D_1 \tanh(\mu h D_1)$. The next step involves expanding $G_0$ and retaining terms up to $O(\mu^4)$, i.e.

$$\begin{aligned}
H &= \int_{\mathbb{R}} \left(\frac{1}{2} g\mu^4\eta_1^2 + \frac{1}{2}\mu\Phi_1 G_0 \mu\Phi_1\right)\, dx + O(\mu^6) \\
&= \int_{\mathbb{R}} \left[\frac{1}{2} g\mu^4\eta_1^2 + \frac{1}{2}\mu\Phi_1 \frac{\mu}{i}\frac{\partial}{\partial_{x_1}}\Big(\frac{\mu h}{i}\frac{\partial}{\partial_{x_1}}\Big)\mu\Phi_1\right]\, dx + O(\mu^6) \\
&= H_{\mathbf{W}} + O(\mu^6)
\end{aligned}$$

where $H_{\mathbf{W}} = \int_{\mathbb{R}} \left( \frac{1}{2} g\mu^4 \eta_1^2 + \frac{1}{2} h\mu^4 \Phi_{1_{x_1 x_1}}^2 \right) dx$. The approximate equations of motion are, in unscaled variables,

$$\begin{cases} \eta_t = \frac{\delta H_{\mathbf{W}}}{\delta \Phi} = -h\Phi_{xx} \\ \Phi_t = -\frac{\delta H_{\mathbf{W}}}{\delta \eta} = -g\eta. \end{cases} \tag{3.2}$$

Now, we let $u = \Phi_x$ to arrive at the factored wave equation

$$\begin{cases} \eta_t + hu_x = 0, \\ u_t + g\eta_x = 0. \end{cases} \tag{3.3}$$

*Remark* 3.1. Since this thesis is primarily focused on unidirectional models, the above system merits a remark. Suppose that we couple system (3.3) with initial data, say, $\eta(x,0) = f(x)$ and $u(x,0) = g(x)$ and form the pure initial value problem. The solution is

$$\begin{aligned} \eta(x,t) &= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}[g(x+t) - g(x-t)] \\ u(x,t) &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}[f(x+t) - f(x-t)]. \end{aligned}$$

If we focus on waves propagating just to the right, then $f = g$. As a result, $\eta(x,t) = f(x-t) = u(x,t)$. Thus, at the lowest order we have that $u = \eta + O(\mu^6)$ and $\eta_t = -\eta_x + O(\mu^6)$ as $\mu^2 \to 0$ (since the amplitude scaling parameter $\mu$ is assumed to be small). It follows that we can use $\eta_t$ and $-\eta_x$ interchangeably without affecting the overall level of approximation.

For the second approximation we retain the first two terms of $G(\eta)$ and expand $\mu D_1 \tanh(\mu h D_1)$. Upon retaining terms up to $O(\mu^6)$, it transpires that

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\frac{1}{2} g\mu^4 \eta_1^2 + \frac{1}{2}\mu\Phi_1 (G_0 + G_1)\mu\Phi_1\right] dx + O(\mu^8) \\
=&\int_{\mathbb{R}} \left[\frac{1}{2} g\mu^4 \eta_1^2 + \frac{1}{2}\mu\Phi_1 (\mu D_1 \tanh(\mu h D_1) + \mu D_1 \eta_1 \mu D_1 - \mu D_1 \tanh(\mu h D_1)\eta_1 \right.\\
&\left. \times \mu D_1 \tanh(\mu h D_1))\mu\Phi_1\right] dx + O(\mu^8) \\
=&\int_{\mathbb{R}} \frac{1}{2} g\mu^4 \eta_1^2 + \frac{1}{2}\mu\Phi_1 \left[\frac{\mu}{i}\frac{\partial}{\partial_{x_1}}\left(\frac{\mu h}{i}\frac{\partial}{\partial_{x_1}} - \frac{1}{3}\left(\frac{\mu h}{i}\frac{\partial}{\partial_{x_1}}\right)^3\right)\right. \\
&\left. + \frac{\mu}{i}\frac{\partial}{\partial_{x_1}}\left(\frac{\mu^2\eta_1\mu}{i}\frac{\partial}{\partial_{x_1}}\right)\right]\mu\Phi_1 \; dx + O(\mu^8) \\
=&H_{\mathbf{B}} + O(\mu^6),
\end{aligned}$$

where

$$H_{\mathbf{B}} = \int_{\mathbb{R}} \left(\frac{1}{2} g\mu^4 \eta_1^2 + \frac{1}{2}\mu^4 (h + \mu^2\eta_1)\Phi_{1_{x_1}}^2 - \frac{1}{6} h^3 \mu^6 \Phi_{1_{x_1 x_1}}^2\right) \; dx.$$

The approximate equations of motion are, in unscaled variables,

$$\begin{cases} \eta_t = \frac{\delta H_{\mathbf{B}}}{\delta \Phi} = -((\eta + h)\Phi_x)_x - \frac{1}{3} h^3 \Phi_{xxxx} \\ \Phi_t = -\frac{\delta H_{\mathbf{B}}}{\delta \eta} = -\frac{1}{2}\Phi_x^2 - g\eta. \end{cases} \tag{3.4}$$

Now, we let $u = \Phi_x$ to arrive at the Hamiltonian version of the Boussinesq equations

$$\begin{cases} \eta_t + ((\eta + h)\Phi_x)_x + \frac{1}{3} h^3 u_{xxx} = 0, \\ u_t + u u_x + g\eta_x = 0. \end{cases} \tag{3.5}$$

*Remark* 3.2. As pointed out in [CG94], this version differs from the usual Boussinesq system cf. [Whi11], but it is the natural system that arises from the Hamiltonian perturbation theory. In the mid seventies it was shown to be completely integrable, cf. [Kau75].

To derive the BBM equation, we first derive the KdV, then make use of the previous remark regarding the substitution of $\eta_x$ with $-\eta_t$. The starting point in deriving the KdV is system (3.5) and the Hamiltonian formulation corresponding to the change of variables $G(\eta) = D\mathcal{K}(\eta)D$. For the sake of simplicity we introduce the non-dimensionalisation.

*Remark* 3.3. According to the transformation theory in [CG94], if the independent variables $r$ and $s$ are used in place of $u$ and $\eta$, then the structure map changes to

$$J_{r,s} = \begin{pmatrix} -\frac{1}{2}\frac{\partial}{\partial_x}K & 0 \\ 0 & \frac{1}{2}\frac{\partial}{\partial_x}K \end{pmatrix}$$

Thus, one can see that the change of variables results in the loss of symmetry.

$$(x, y, z) = \frac{1}{h}(x', y', z'), \quad t = t'\left(\frac{g}{h}\right)^{\frac{1}{2}}, \quad u = \frac{u'}{\sqrt{gh}},$$

which is equivalent to setting $g = h = 1$.

The first step in deriving the KdV from system (3.5) is to transform to a frame of reference which is stationary with respect to the right-moving wave front associated with the longest waves. In nondimensional variables this wave front has unit speed. This is accomplished by subtracting the momentum $I_{\mathbf{m}} = \int_{\mathbb{R}} u\eta \, dx$ from the Hamiltonian, this process is equivalent to the introduction of the two independent variables

$$x_2 = x_1 - t_1, \quad t_2 = \mu^2 t_1.$$

In the new reference frame the Boussinesq system is

$$\begin{cases} \eta_t = -\frac{\partial}{\partial_x}\left(\frac{\delta(H_{\mathbf{B}} - I_{\mathbf{m}})}{\delta\Phi}\right), \\ \Phi_t = - - \frac{\partial}{\partial_x}\left(\frac{\delta(H_{\mathbf{B}} - I_{\mathbf{m}})}{\delta\eta}\right), \end{cases} \tag{3.6}$$

where

$$H_{\mathbf{B}} - I_{\mathbf{m}} = \int_{\mathbb{R}} \left(\frac{1}{2}\mu^4(u_1 - \eta_1)^2 + \frac{1}{2}\mu^6\eta_1 u_1^2 - \frac{1}{6}\mu^6 u_{1_{x_2}}^2\right) \, dx + O(\mu^8).$$

In order to derive the KdV equation for uni-directional wave propagation, it is important to understand how the solutions can be restricted to either left or right-going waves. As it turns out, if $\eta$ and $u$ are such that $\eta = Ku$, then this pair of functions represents a solution which is propagating to the right. To see this, we analyze the linearized system emanating from the Hamiltonian. Making use of the first approximation and the change of variables $G(\eta) = D\mathcal{K}(\eta)D$ we arrive at the following system.

$$\begin{cases} \eta_t = -\mathcal{K}_0 u_x, \\ u_t = - - \eta_x. \end{cases} \tag{3.7}$$

If we consider a solution of the above system in the form

$$\eta(x,t) = \delta e^{i\xi x - i\omega t}, \quad u(x,t) = \epsilon e^{i\xi x - i\omega t}$$

then we obtain the following matrix equation

$$\begin{pmatrix} -\omega & \frac{\tanh\xi}{\xi}\xi \\ \xi & -\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the existence of a nontrivial solution is to be guaranteed, we must have that

$$\omega^2 - \frac{\tanh\xi}{\xi}\xi^2 = 0.$$

As $c = \frac{\omega}{\xi} = \pm\sqrt{\frac{\tanh\xi}{\xi}}$, the choice $c > 0$ corresponds to right-going solutions of the system. Information from the above system, i.e. the relationship between $\eta$ and $u$ suggests the following combination variables

$$r = \frac{1}{2}(\eta + Ku), \quad s = \frac{1}{2}(\eta - Ku)$$

where $K = \sqrt{\mathcal{K}_0} = \sqrt{\frac{\tanh D}{D}}$.

*Remark* 3.4. The expression for the mass of a solution to the water wave problem is

$$I_{\mathbf{M}} = \int_{\mathbb{R}} \eta \, dx.$$

If one restricts to the case of waves propagating to the right, say, $\eta = Ku$ for an appropriate operator $K$, then

$$I_{\mathbf{M}} = \int_{\mathbb{R}} Ku \; dx.$$

Consequently, in the case of the KdV and BBM-KP ($K = 1$), one can interpret the velocity $u$ and displacement relative to the undisturbed free surface $\eta$ interchangeably.

In accordance with the above discussion we focus on right moving evolution by defining the variables

$$r = \frac{1}{2}(\eta + u), \quad s = \frac{1}{2}(\eta - u),$$

and the isotropic scaling

$$r_1 = \frac{1}{\mu^2} r, \qquad s_1 = \frac{1}{\mu^2} s.$$

Accordingly, system (3.6) becomes

$$\begin{cases} r_t = -\frac{1}{2}\frac{\partial}{\partial_x}\left(\frac{\delta(H_{\mathbf{B}} - I_{\mathbf{m}})}{\delta r}\right), \\ s_t = \frac{1}{2}\frac{\partial}{\partial_x}\left(\frac{\delta(H_{\mathbf{B}} - I_{\mathbf{m}})}{\delta s}\right), \end{cases} \tag{3.8}$$

where

$$H_{\mathbf{B}} - I_{\mathbf{m}} = \int_{\mathbb{R}} \Big[\frac{1}{2}\mu^4 s_1^2 + \frac{1}{2}\mu^6 r_1^3 - \frac{1}{6}\mu^6 r_{1_{x_2}}^2 - \frac{1}{2}\mu^6 r_1 s_1^2 - \frac{1}{2}\mu^6 s_1 r_1^2 + \frac{1}{2}\mu^6 s_1^3 \\ - \frac{1}{6}\mu^6 s_{1_{x_2}}^2 + \frac{1}{3}\mu^6 r_{1_{x_2}} s_{1_{x_2}}\Big] \, dx + O(\mu^8).$$

The resulting system is a Hamiltonian evolutionary system of the form (**??**) in which $v$ is the dependent variable $(r, s)^T$ and $J$ is the skew-symmetric structure map

$$J_{\mathbf{KdV}} = \frac{1}{2}\begin{pmatrix} -\frac{\partial}{\partial_x} & 0 \\ 0 & \frac{\partial}{\partial_x} \end{pmatrix},$$

By taking variational derivatives, we can write the system (3.8) as

$$\begin{cases} r_t + \frac{3}{2} r r_x + \frac{1}{6} r_{xxx} - \frac{1}{2}(rs)_x - \frac{1}{2} s s_x - \frac{1}{6} s_{xxx} = 0, \\ s_t - \frac{1}{2}(s)_x - \frac{3}{2} s s_x - \frac{1}{6} s_{xxx} + \frac{1}{2}(rs)_x + \frac{1}{2} r r_x + \frac{1}{6} s_{xxx} = 0. \end{cases} \tag{3.9}$$

*Remark* 3.5. Formally, we are restricting our analysis to the space

$$\mathcal{R} = \{(\eta, u) \in \mathcal{D} : \|\eta - u\|_{H^1(\mathcal{S}_0)} < \mu^2\} = \{(r, s) \in \mathcal{D} : \|s\|_{H^1(\mathcal{S}_0)} < \frac{1}{2}\mu^2\}$$

where

$$H_{\mathbf{KdV}} = \int_{\mathbb{R}} \left[ \frac{1}{2} \mu^6 r_1^3 - \frac{1}{6} \mu^6 r_{1_{x_2}}^2 \right] dx, \tag{3.10}$$

$$\begin{cases} r_t = -\dfrac{1}{2} \dfrac{\partial}{\partial_x} \left( \dfrac{\delta(H_{\mathbf{KdV}})}{\delta r} \right) = -\dfrac{1}{2} \dfrac{\partial}{\partial_x} \left( \dfrac{3}{2} r^2 + \dfrac{1}{3} r_{xx} \right), \\ s_t = \dfrac{1}{2} \dfrac{\partial}{\partial_x} \left( \dfrac{\delta(H_{\mathbf{KdV}})}{\delta s} \right) = 0. \end{cases} \tag{3.11}$$

Disregarding the constants and making a slight abuse of notion, viz. $r = u$ we obtain the KdV equation

$$u_t + u u_x + u_{xxx} = 0. \tag{3.12}$$

Taking into account (3.4), we obtain the BBM equation

$$u_t + u u_x - u_{xxt} = 0. \tag{3.13}$$

*Remark* 3.6.

$$\omega(\xi) = \frac{\xi}{1 + \xi^2}. \qquad \text{Dispersion relation (BBM)} \tag{3.14}$$

The dispersion relation relates the time evolution of a system to its spatial structure. This function uniquely characterizes the linear part of a system and encodes information about how traveling wave solutions propagate. If one compares the dispersion relation of the BBM to that of the KdV, it is immediate that (3.14) is preferable. Indeed (3.14) is bounded, however the dispersion relation for the KdV is not. This means that wave solutions to the KdV can propagate with infinite speed, which is physically impossible. The dispersion relation for the BBM does not have such bad limiting behavior.

### 3.2 Existence theory

The question of global well-posedness of (3.13) with data given by (2.2) was recently answered in [BT09]. Bona and Tzvetkov proved that the Cauchy problem associated with (2.1) is globally well-posed in the $L^2$-based Sobolev class $H^s(\mathbb{R})$, for $s \geq 0$. More precisely, they proved the following result.

**Theorem 3.7.** *[BT09] Fix $s \geq 0$. For any $u_0 \in H^s(\mathbb{R})$, there exists a $T = T(||u_0||_{H^s}) > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}))$ of the IVP (3.13). Moreover, for $R > 0$, let $B_R$ connote a ball of radius $R$ centered at the origin in $H^s(\mathbb{R})$ and let $T = T(R) > 0$ denote a uniform existence time for the IVP (3.13) with $u_0 \in B_R$. Then the correspondence $u_0 \mapsto u$ which associates to $u_0$ the solution $u$ of the IVP (3.13) with initial data $u_0$ is a real analytic mapping of $B_R$ to $C([T, -T]; H^s(\mathbb{R}))$.*

The above theorem improved the earlier known results proven by Benjamin et. al. [BBM72] where the IVP (3.13) was shown to be globally well-posed for data in $H^k$, $k \in \mathbb{Z}$ such that $k \geq 1$. It should also be noted that the authors in [BT09] proved that the IVP (3.13) is ill-posed for given data in $H^s(\mathbb{R})$ for $s < 0$. Indeed, the flow map $u_0 \mapsto u(t)$ is not even $C^2$. The precise result proved in [BT09] reads as follows.

**Theorem 3.8.** *[BT09] For any $s < 0$, $T > 0$ the flow-map $u_0 \mapsto u(t)$ established in (3.7) is not of class $C^2$ from $H^s$ to $C([0,T]; H^s(\mathbb{R}))$.*

# Chapter 4
# The BBM-KP Equation

Listed below is the pure initial value problem for the Benjamin-Bona-Mahony-Kadomtsev-Petviashvili (BBM-KP) equation

$$\begin{cases} (\eta_t + \eta_x + \eta\eta_x - \eta_{xxt})_x + \gamma\eta_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, t > 0, \\ \eta(x, y, 0) = \psi(x, y) \end{cases} \tag{4.1}$$

where $\gamma = \pm 1$.

A restriction in the application of the BBM equation as a practical model for water waves is that the BBM is strictly one-dimensional, (one spatial dimension plus time), whereas the surface of a water wave is two-dimensional.

The BBM-KP (4.1) is the regularized version of the usual KP equation which arises in various contexts where nonlinear dispersive waves propagate principally along the $x$-axis with weak dispersive effects being felt in the direction parallel to the $y$-axis perpendicular to the main direction of propagation. As in the case of the BBM, the BBM-KP has nontrivial solitary wave solutions cf. [dBS97].

To obtain the BBM-KP from the BBM we first orient the horizontal coordinates so that the $x$-direction is the principal direction of wave propagation. In addition we assume that wave amplitudes are small, the water is shallow typical to horizontal wavelengths, and the waves are nearly one dimensional.

The resulting equation is the BBM formulated in the KP sense, and is naturally referred to as the BBM-KP equation. This equation models small amplitude long waves in shallow water moving mainly in the $x$ direction, in 2+1 space. Similar to the KP equations, (4.1) is called the BBM-KP I if $\gamma = -1$ and BBM-KP II if $\gamma = 1$.

In a physical context, the sign of $\gamma$ corresponds to whether the surface tension is neglected or not. Also, one should observe that the linearized dispersion relation for the KP equation, i.e. $-\eta_{xxt}$ replaced with $\eta_{xxx}$, is

$$\omega_1(\xi,\mu) = \xi(1-\xi^2+\gamma\frac{\mu^2}{\xi^2}), \quad \gamma = \pm 1.$$

While that of (4.1) is

$$\omega_2(\xi,\mu) = \frac{\xi^2+\gamma\mu^2}{\xi(1+\xi^2)}, \quad \gamma = \pm 1.$$

As pointed out in [BLT02], $\omega_2$ is a good approximation for $\omega_1$ and does not possess the unwanted limiting behavior as $m \to +\infty$.

## 4.1 Derivation of the model equation

Craig and Groves [CG94] explain how the Hamiltonian structure can be preserved under changes of dependent or independent variables. Following their method, we separate out the right and left-going waves by defining

$$r = \frac{1}{2}(L_1^{-1}\eta + u), \quad s = \frac{1}{2}(L_1^{-1}\eta - u). \tag{4.2}$$

The right and left-moving interpretation of $r$ and $s$ can be extended away from the long wave limit by a better choice for the constant coefficient self-adjoint operator $L_1$ than its long wave asymptote of unity. The exact Hamiltonian equations for the evolution of $r, s$ and $w$ are

$$\begin{bmatrix} \frac{\partial r}{\partial T} \\ \frac{\partial s}{\partial T} \\ \frac{\partial w}{\partial T} \end{bmatrix} = \frac{1}{2}L_1^{-1} \begin{bmatrix} -\frac{\partial}{\partial X} & 0 & -\frac{\partial}{\partial z} \\ 0 & -\frac{\partial}{\partial X} & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta H}{\delta r} \\ \frac{\delta H}{\delta s} \\ \frac{\delta H}{\delta w} \end{bmatrix} \tag{4.3}$$

$$H = \frac{1}{2}\iint \Big[ r(L_1^2+K-2UL_1)r + wKw + 2r(L_1^2+K)s + s(L_1^2+K-2UL_1)s \Big]\, dXdz.$$

The Kadomstev-Petviashvili equations are obtained by replacing the dependent variable $w(X,z,T)$ in terms of $r(X,z,T)$ i.e. to neglect $s(X,z,T)$ in the exact relationship $w_x = (r-s)_z$. For exceeding long waves the operator $rKr$ is asymptotically $r + r^3$. However, any constant coefficient self-adjoint linear operator $L_2$ with a long wave asymptote of unity can be used in a change of dependent variables $r = L_2 R$, $w = L_2 W$ to replace the cubic contributions $r^3$ by $R^3$. This leads to approximate $rKr$ by $RK_1 L_2^2 R + R^3$, where the constant coefficient self-adjoint operator $K_1$ will be chosen as some long wave approximation to the exact linear operator $K(0)$. The resulting Hamiltonian only involves the single dependent variable $R$. However, it is convenient to write it in terms of both $R$ and $W$ with the four linear operators $K_1, K_2, L_1$ and $L_2$.

$$H = \frac{1}{2} \iint \left[ RL_2(L_1^2 + K_1 - 2UL_1)L_2 R + R^3 + wK_2 L_2^2 W \right] dXdz \tag{4.4}$$

The corresponding generalization of the KP equations are

$$\begin{cases} L_1 L_2^2 (R_T - UR_X) + (\frac{3}{4}R^2 + \frac{L_1^2 + K_1}{2} L_2^2 R)_X + \frac{1}{2} K_2 L^2 L_2 W, \\ W_X - R_z = 0. \end{cases} \tag{4.5}$$

One obtains the usual KP equations with

$$K_1 = 1 - \partial_X^2, \quad K_2 = L_1 = L_2 = 1. \tag{4.6}$$

Another neat selection that is related to the BBM model is

$$K_1 = \frac{1 + \frac{1}{6}\partial_X^2}{1 - \frac{1}{6}\partial_X^2}, \quad K_2 = L_1 = 1, \quad L_2 = \sqrt{1 - \frac{1}{6}\partial_X^2}. \tag{4.7}$$

### 4.2 Existence theory

Bona et al. [BLT02] has shown that the Cauchy problem 4.1 can be solved by Picard Iteration yielding to local and global well-posedness results. In particular, it is shown that the pure initial-value problem (4.1), regardless of the sign of $\gamma$, is globally well-posed in

$$W_1(\mathbb{R}^2) = \{\psi \in L^2(\mathbb{R}^2) : ||\psi||_{L^2} + ||\psi_x||_{L^2} + ||\psi_{xx}||_{L^2} + ||\partial_x^{-1}\partial_y\psi||_{L^2} + ||\psi_y||_{L^2} < \infty\}.$$

Saut and Tzvetkov [ST04] improved this global well-posedness to the space

$$Y = \{\psi \in L^2(\mathbb{R}^2) : \psi_x \in L^2(\mathbb{R}^2)\}.$$

**Theorem 4.1.** *Let $\psi \in H^s_{-1}(\mathbb{R}^2)$ with $s > \frac{3}{2}$. Then there exist a $T_0$ such that the initial-value problem (2.2) has a unique solution $\eta \in C([0,T]; H^s_{-1}(\mathbb{R}^2))$, $\partial_x^{-1}\eta_y \in C([0,T]; H^{s-1}_{-1}(\mathbb{R}^2))$, with $\eta_t \in C([0,T]; H^{s-2}(\mathbb{R}^2))$. Moreover, the map $\psi \to \eta$ is continuous from $H^s_{-1}(\mathbb{R}^2)$ to $C([0,T_0]; H^s_{-1}(\mathbb{R}^2))$.*

# Chapter 5
# An Analysis of the Above Models

## 5.1 Statement of results

The previous chapter sheds light on the fact that the derivation of the BBM-KP hinges on the same physical assumptions utilized for the BBM, with the additional condition that the wave motion simultaneously experiences weak variation along the transverse direction. To make the above discussion precise, we utilize the mathematical framework presented in [MST07b]. To this end we start with a one dimensional long-wave dispersive equation of the BBM type, i.e.

$$u_t + u_x + uu_x - Lu_t = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \quad t \geq 0. \tag{5.1}$$

In (5.1) $L$ is a Fourier multiplier defined by

$$\widehat{L\varphi(\xi)} = m(\xi)\hat{\varphi(\xi)} \tag{5.2}$$

for a real function $m$. For example, the case $m(\xi) = \pm\xi^2$ ($L = \mp\partial_x^2$) corresponds to the classical KdV equation. In the context of water waves $m(\xi)$ is the phase velocity and its sign depends on the surface tension parameter, as in the case for the KP equations.

The correction to (5.1) due to weak transverse effects does not depend on the dispersion in $x$ and is only related to the finite propagation speed properties of the linear transport operator $M = \partial_t + \partial_x$. We recall that $M$ gives rise to right moving unidirectional waves with unit speed i.e. an initial wave profile $\varphi(x)$ evolves under the flow of $M$ as $\varphi(x-t)$. In accordance with the terminology in [MST07b] we define a weak transverse perturbation of $\varphi(x)$ to be a two dimensional function

$\psi(x,y)$ close to $\varphi(x)$, localized in the frequency region $\left|\frac{\mu}{\xi}\right| \ll 1$, where $\xi$ and $\mu$ are the Fourier modes dual to $x$ and $y$ respectively.

The idea is to look for a two dimensional perturbation $\tilde{M} = \partial_t + \partial_x + \omega(D_x, D_y)$ of $M$ such that the wave profile of $\psi(x,y)$ does not change much when evolving under the the flow of $\tilde{M}$. Where $\omega(D_x, D_y)$ stands for the Fourier multiplier with real symbol $\omega(\xi,\mu)$. A natural generalization of the flow of $M$ in $\mathbb{R}^2$ is the flow of the wave operator $M = \partial_t + \sqrt{-\Delta}$ (which also enjoy the finite propagation speed property. Now, since $\xi + \frac{1}{2}\xi^{-1}\mu^2 \approx \pm\sqrt{\xi^2+\mu^2}$, if $\left|\frac{\mu}{\xi}\right| \ll 1$. As a result we deduce that

$$\partial_t + \partial_x + \frac{1}{2}\partial_x^{-1}\partial_y^2 \sim \partial_t + \sqrt{-\Delta}$$

which leads to the correction $\omega(D_x, D_y) = \frac{1}{2}\partial_x^{-1}\partial_y^2$ in (2.2). Therefore we arrive at the following two dimensional model

$$u_t + u_x + uu_x - Lu_t + \partial_x^{-1}\partial_y^2 = 0. \tag{5.3}$$

This is consistent with the fact that the KP formulation of the BBM models long weakly dispersive waves which essentially propagate in one direction with weak transverse effects. Thus when considering such a formulation, it is assumed that the average wave length is the $x$ direction is much larger than the average wave length in the $y$ direction.

From this viewpoint, it is natural to think of the BBM-KP as a weak transverse perturbation of the BBM. This observation leads us to the following question: Provided the initial datum of the Cauchy problem corresponding to the BBM-KP is a weak transverse perturbation of the initial data of the Cauchy problem pertaining to the BBM, then is the solution of the first a weak transverse perturbation of the second? The answer to this question is a direct consequence of my first result. More formally we arrived at the following theorem [GT].

**Theorem 5.1.** *Let $u^{\pm}$ and $\eta$ solve the Cauchy problems (3.13) and (4.1), for initial data $\phi^{\pm} \in H^k(\mathbb{R})$ and $\psi \in H^s_{-1}(\mathbb{R}^2)$, respectively. Also, let $k \geq 1$ and $s \geq k+1$. If*

$$\lim_{y \to \pm\infty} ||\psi(\cdot, y) - \phi^{\pm}(\cdot)||_{H^k_x(\mathbb{R})} = 0,$$

*it follows that*

$$\lim_{y \to \pm\infty} ||\eta - u^{\pm}||_{H^k_x(\mathbb{R})} = 0.$$

Loosely speaking, theorem 5.1 serves as a measure of stability between the two systems. From a physical standpoint, our result states that the initial wave profile evolving under the flow of the BBM is a weak transverse perturbation of the wave profile evolving under the flow of the BBM-KP, provided their respective initial data are.

*Remark* 5.2. The relationship between $s$ and $k$ is primarily due to the $\partial_x^{k-1}\eta_{yy}$ term. The fact that $s \geq k+1$ insures enough regularity so that $\partial_x^{k-1}\eta_{yy} \to 0$ as $y \to \pm\infty$. For expositional purposes we summarize this in the following table.

TABLE 5.1: Relationship between $k$ and $s$

| $k$ | $\partial_x^{k-1}\eta_{yy}$ | $s$ |
|---|---|---|
| 0 | $\partial_x^{-1}\eta_{yy}$ | $s \geq 1$ |
| 1 | $\eta_{yy}$ | $s \geq 1$ |
| 2 | $\partial_x\eta_{yy}$ | $s \geq 3$ |
| 3 | $\partial_x^2\eta_{yy}$ | $s \geq 4$ |
| $k$ | $\partial_x^{k-1}\eta_{yy}$ | $s \geq k-1+2 = k+1$ |

To mathematically formalize the problem at hand, we assume that $\psi \in H^s_{-1}(\mathbb{R}^2)$ with $s > \frac{3}{2}$, and let $u^+$ be the solution to the initial-value problem (4.1) corresponding to the initial data $\phi^+$ where

$$\phi^+(\cdot) = \lim_{y \to +\infty} \psi(\cdot, y),$$

and $u^-$ be the solution corresponding to the initial data $\phi^-$ where

$$\phi^-(\cdot) = \lim_{y \to -\infty} \psi(\cdot, y).$$

For example, we could let $\psi(x, y) = \phi(x) + \operatorname{sech} y$, where $\phi \in H^s(\mathbb{R})$. Let $\eta$ be the solution to the IVP 4.1 corresponding to the initial data $\psi$.

Define the function

$$w(x, y, t) = \eta - \frac{1}{2}[u^+ + u^-] - \frac{1}{2}[u^+ - u^-] \tanh y. \tag{5.4}$$

Observe that

$$w(x, y, 0) = \psi(x, y) - \frac{1}{2}[\phi^+(x) + \phi^-(x)] - \frac{1}{2}[\phi^+(x) - \phi^-(x)] \tanh y \tag{5.5}$$

and hence

$$\lim_{y \to \pm\infty} w(x, y, 0) = 0.$$

A straightforward calculation shows that $w$ satisfies the following initial value problem

$$\begin{cases} w_t + w_x - w_{xxt} - \partial_x^{-1} \eta_{yy} + w w_x + \frac{1}{2}(1 + \tanh y)(u^+ w)_x + \frac{1}{2}(1 - \tanh y)(u^- w)_x \\ -\frac{1}{4}(1 - \tanh^2 y)(u^+ u_x^+ + u^- u_x^- - u^+ u_x^- - u^- u_x^+) = 0, \\ w(x, y, 0) = \psi(x, y) - \frac{1}{2}[\phi^+ + \phi^-] - \frac{1}{2}[\phi^+ - \phi^-] \tanh y. \end{cases} \tag{5.6}$$

We now venture into the task of estimating $||w||_{H^1_x(\mathbb{R})}$ for any $y \in \mathbb{R}$.

*Proof.* To this end, we multiply equation (5.6) by $w$ and integrate over $\mathbb{R}$, in the spatial variable $x$, to obtain the following integral

$$\int_{\mathbb{R}}[ww_t+ww_x-ww_{xxt}-w\partial_x^{-1}\eta_{yy}+w^2w_x+w\frac{1}{2}(1+\tanh y)(u^+w)_x+w\frac{1}{2}(1-\tanh y)(u^-w)_x$$

$$-w\frac{1}{4}(1-\tanh^2 y)(u^+u_x^+ + u^-u_x^- - u^+u_x^- - u^-u_x^+)]\,dx = 0.$$

After a few integration by parts, we arrive at the following estimate

$$\frac{1}{2}\frac{d}{dt}\Big[\int_{\mathbb{R}} w^2dx + \int_{\mathbb{R}} w_x^2dx\Big] \leq \Big|\int_{\mathbb{R}} w\partial_x^{-1}\eta_{yy}dx\Big| + \frac{1}{2}\Big|\int_{\mathbb{R}}(1+\tanh y)u^+w_xwdx\Big| +$$

$$\frac{1}{2}\Big|\int_{\mathbb{R}}(1-\tanh y)u^-w_xwdx\Big| + \frac{1}{4}\Big|\int_{\mathbb{R}}(1-\tanh^2 y)wu^+u_x^+\,dx\Big| + \frac{1}{4}\Big|\int_{\mathbb{R}}(1-\tanh^2 y)wu^-u_x^-dx\Big|$$

$$+\frac{1}{4}\Big|\int_{\mathbb{R}}(1-\tanh^2 y)wu^+u_x^-dx\Big| + \frac{1}{4}\Big|\int_{\mathbb{R}}(1-\tanh^2 y)wu^-u_x^+dx\Big|.$$

Making use of Hölders inequality, it follows that

$$\frac{1}{2}\frac{d}{dt}||w||^2_{H_x^1(\mathbb{R})} \leq ||w||_{L^2(\mathbb{R})}||\partial_x^{-1}\eta_{yy}||_{L^2(\mathbb{R})} + \frac{(1+\tanh y)}{2}|u^+|_\infty||w||_{L^2(\mathbb{R})}||w_x||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh y)}{2}|u^-|_\infty||w||_{L^2(\mathbb{R})}||w_x||_{L^2(\mathbb{R})} + \frac{(1-\tanh^2 y)}{4}\;|u^+|_\infty||w||_{L^2(\mathbb{R})}||u_x^+||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\;|u^-|_\infty||w||_{L^2(\mathbb{R})}||u_x^-||_{L^2(\mathbb{R})} + \frac{(1-\tanh^2 y)}{4}\;|u^+|_\infty||w||_{L^2(\mathbb{R})}||u_x^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\;|u^-|_\infty||w||_{L^2(\mathbb{R})}||u_x^+||_{L^2(\mathbb{R})}.$$

An application of Young's inequality yields

$$\frac{1}{2}\frac{d}{dt}||w||^2_{H_x^1(\mathbb{R})} \leq ||w||_{H_x^1(\mathbb{R})}||\partial_x^{-1}\eta_{yy}||_{L^2(\mathbb{R})} + \frac{(1+\tanh y)}{2}|u^+|_\infty||w||^2_{H_x^1(\mathbb{R})}$$

$$+\frac{(1-\tanh y)}{2}|u^-|_\infty||w||^2_{H_x^1(\mathbb{R})} + \frac{(1-\tanh^2 y)}{4}\;|u^+|_\infty||w||_{H_x^1(\mathbb{R})}||u_x^+||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\;|u^-|_\infty||w||_{H_x^1(\mathbb{R})}||u_x^-||_{L^2(\mathbb{R})} + \frac{(1-\tanh^2 y)}{4}\;|u^+|_\infty||w||_{H_x^1(\mathbb{R})}||u_x^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\;|u^-|_\infty||w||_{H_x^1(\mathbb{R})}||u_x^+||_{L^2(\mathbb{R})}.$$

After utilizing an elementary embedding theorem, the last inequality can be written in the form

$$\frac{1}{2}\frac{d}{dt}||w||^2_{H^1_x(\mathbb{R})} \leq \Big[||\partial_x^{-1}\eta_{yy}||_{L^2(\mathbb{R})} + (1-\tanh^2 y)a\Big(||u^+||_{H^1(\mathbb{R})}, ||u^-||_{H^1(\mathbb{R})}\Big)\Big]||w||_{H^1_x(\mathbb{R})}$$

$$+\frac{1}{2}\Big[(1+\tanh y)||u^+||_{H^1(\mathbb{R})} + (1-\tanh y)||u^-||_{H^1(\mathbb{R})}\Big]||w||^2_{H^1_x(\mathbb{R})},$$

or

$$\frac{1}{2}\frac{d}{dt}||w||^2_{H^1_x(\mathbb{R})} \leq (C_\eta + C_1)||w||_{H^1_x(\mathbb{R})} + (C_+ + C_-)||w||^2_{H^1_x(\mathbb{R})}.$$

From this the following inequality is derived

$$\frac{d}{dt}||w||_{H^1_x(\mathbb{R})} \leq (C_\eta + C_1) + (C_+ + C_-)||w||_{H^1_x(\mathbb{R})}.$$

By a variant of Gronwall's lemma, it follows that

$$||w||_{H^1_x(\mathbb{R})} \leq ||w(x,y,0)||_{H^1_x(\mathbb{R})}e^{(C_+ + C_-)t} + \frac{C_\eta + C_1}{C_+ + C_-}\Big(e^{(C_+ + C_-)t} - 1\Big). \tag{5.7}$$

By letting $y \to +\infty$, we observe that $C_\eta = ||\partial_x^{-1}\eta_{yy}||_{L^2(\mathbb{R})} \to 0$ since, for $s \geq 2$, $\partial_x^{-1}\eta_{yy} \to 0$ as $y \to \pm\infty$. In addition, it follows that

$$C_1 = (1-\tanh^2 y)a\Big(||u^+||_{H^1(\mathbb{R})}, ||u^-||_{H^1(\mathbb{R})}\Big) \to 0,$$

since $(1-\tanh^2 y) \to 0$ as $y \to \pm\infty$ and $a\Big(||u^+||_{H^1(\mathbb{R})}, ||u^-||_{H^1(\mathbb{R})}\Big) < \infty$ by the global well-posedness of the BBM. Clearly $C_- = \frac{1}{2}(1-\tanh y)||u^-||_{H^1(\mathbb{R})} \to 0$ as $y \to +\infty$, but $C_+$ does not. Let $y \to +\infty$ on both sides of inequality (2.4), to conclude

$$\lim_{y\to+\infty} ||w||_{H^1_x(\mathbb{R})} \leq 0 \cdot e^{C_+ t} + 0 \cdot \Big(e^{C_+ t} - 1\Big) = 0.$$

We have established that

$$\lim_{y\to+\infty} ||w||_{H^1_x(\mathbb{R})} = \lim_{y\to+\infty} ||\eta(x,y,t) - u^+(x,t)||_{H^1_x(\mathbb{R})} = 0.$$

A similar reasoning follows in the case when $y \to -\infty$, since $C_\eta \to 0$ and $C_1 \to 0$. However, in this case $C_+ = \frac{1}{2}(1 + \tanh y)||u^+||_{H^1(\mathbb{R})} \to 0$, but $C_-$ does not. Hence,

$$\lim_{y\to-\infty} ||\eta(x, y, t) - u^-(x, t)||_{H^1_x(\mathbb{R})} = \lim_{y\to-\infty} ||w||_{H^1_x(\mathbb{R})} \leq 0 \cdot e^{C_- t} + 0 \cdot \left(e^{C_- t} - 1\right) = 0.$$

Therefore,

$$\lim_{y\to\pm\infty} ||\eta(x, y, t) - u^\pm(x, t)||_{H^1_x(\mathbb{R})} = 0.$$

### 5.2 The general case $k \geq 1$

As both of the models in consideration are globally well-posed in $H^k_x(\mathbb{R})$, for appropriate $k$. One should expect that for regular enough initial data, the solutions behave as they do in Theorem 5.1. To investigate this issue we let $k \geq 1$ and apply the operator $\partial^k_x$ to both sides of the differential equation (5.6), multiply the result by $\partial^k_x w$, and integrate the result over $\mathbb{R}$ in the spatial variable $x$. After a few integration by parts we arrive at the following integral equation

$$\frac{1}{2}\frac{d}{dt}\Big[\int_{\mathbb{R}} (\partial^k_x w)^2\, dx + \int_{\mathbb{R}} (\partial^{k+1}_x w)^2\, dx\Big] = \int_{\mathbb{R}} \partial^k_x w \partial^{k-1}_x \eta_{yy}\, dx$$

$$+\frac{(-1)^k}{2}(1+\tanh y)\int_{\mathbb{R}} \left(\partial^{2k+1}_x w\right)(u^+ w)\, dx + \frac{(-1)^k}{2}(1-\tanh y)\int_{\mathbb{R}} \left(\partial^{2k+1}_x w\right)(u^- w)\, dx$$

$$+\frac{(-1)^k}{4}(1-\tanh^2 y)\int_{\mathbb{R}} \left(\partial^{2k}_x w\right)(u^+ u^+_x)\, dx + \frac{(-1)^k}{4}(1-\tanh^2 y)\int_{\mathbb{R}} \left(\partial^{2k}_x w\right)(u^- u^-_x)\, dx$$

$$+\frac{(-1)^{k+1}}{4}(1-\tanh^2 y)\int_{\mathbb{R}} \left(\partial^{2k}_x w\right)(u^+ u^-_x)\, dx + \frac{(-1)^{k+1}}{4}(1-\tanh^2 y)\int_{\mathbb{R}} \left(\partial^{2k}_x w\right)(u^- u^+_x)\, dx.$$

Similar to the case for $H^1$, the above equation delivers the bound

$$\frac{1}{2}\frac{d}{dt}\Big[||\partial^k_x w||^2_{L^2(\mathbb{R})} + ||\partial^{k+1}_x w||^2_{L^2(\mathbb{R})}\Big] \leq \int_{\mathbb{R}} |\partial^k_x w \partial^{k-1}_x \eta_{yy}|\, dx$$

$$+\frac{(1+\tanh y)}{2}|u^+|_\infty \int_{\mathbb{R}} \left|\partial^k_x w \partial^{k+1}_x w\right| dx + \frac{(1-\tanh y)}{2}|u^-|_\infty \int_{\mathbb{R}} \left|\partial^k_x w \partial^{k+1}_x w\right| dx$$

$$+\frac{(1-\tanh^2 y)}{4}|u^+|_\infty \int_{\mathbb{R}} \left|\partial^{k+1}_x w \partial^k_x u^+\right| dx + \frac{(1-\tanh^2 y)}{4}|u^-|_\infty \int_{\mathbb{R}} \left|\partial^{k+1}_x w \partial^k_x u^-\right| dx$$

$$+\frac{(1-\tanh^2 y)}{4}|u^+|_\infty \int_{\mathbb{R}} \left|\partial_x^{k+1} w \partial_x^k u^-\right| dx + \frac{(1-\tanh^2 y)}{4}|u^-|_\infty \int_{\mathbb{R}} \left|\partial_x^{k+1} w \partial_x^k u^+\right| dx.$$

An appeal to Hölders inequality results in

$$\frac{1}{2}\frac{d}{dt}\left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right] \leq ||\partial_x^k w||_{L^2(\mathbb{R})}||\partial_x^{k-1}\eta_{yy}||_{L^2(\mathbb{R})}+$$

$$\frac{(1+\tanh y)}{2}|u^+|_\infty ||\partial_x^k w||_{L^2(\mathbb{R})}||\partial_x^{k+1} w||_{L^2(\mathbb{R})} + \frac{(1-\tanh y)}{2}|u^-|_\infty ||\partial_x^k w||_{L^2(\mathbb{R})}||\partial_x^{k+1} w||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^+|_\infty ||\partial_x^{k+1} w||_{L^2(\mathbb{R})}||\partial_x^k u^+||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^-|_\infty ||\partial_x^{k+1} w||_{L^2(\mathbb{R})}||\partial_x^k u^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^+|_\infty ||\partial_x^{k+1} w||_{L^2(\mathbb{R})}||\partial_x^k u^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^-|_\infty ||\partial_x^{k+1} w||_{L^2(\mathbb{R})}||\partial_x^k u^+||_{L^2(\mathbb{R})}.$$

After invoking Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right] \leq \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}} ||\partial_x^{k-1}\eta_{yy}||_{L^2(\mathbb{R})}$$

$$+\frac{(1+\tanh y)}{2}|u^+|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]$$

$$+\frac{(1-\tanh y)}{2}|u^-|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^+|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}} ||\partial_x^k u^+||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^-|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}} ||\partial_x^k u^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^+|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}} ||\partial_x^k u^-||_{L^2(\mathbb{R})}$$

$$+\frac{(1-\tanh^2 y)}{4}\ |u^-|_\infty \left[||\partial_x^k w||^2_{L^2(\mathbb{R})} + ||\partial_x^{k+1} w||^2_{L^2(\mathbb{R})}\right]^{\frac{1}{2}} ||\partial_x^k u^+||_{L^2(\mathbb{R})}.$$

From which we deduce that

$$\frac{1}{2}\frac{d}{dt}||w||^2_{H^k_x(\mathbb{R})} \leq \left[||\partial_x^{k-1}\eta_{yy}||_{L^2(\mathbb{R})} + (1-\tanh^2 y) a\Big(||\partial_x^k u^+||_{H^1(\mathbb{R})}, ||\partial_x^k u^-||_{H^1(\mathbb{R})}\Big)\right]||w||_{H^k_x(\mathbb{R})}$$

$$+\frac{1}{2}\left[(1+\tanh y)||u^+||_{H^1(\mathbb{R})} + (1-\tanh y)||u^-||_{H^1(\mathbb{R})}\right]||w||^2_{H^k_x(\mathbb{R})}.$$

This leads us to the following inequality

$$\frac{d}{dt}||w||_{H^k_x(\mathbb{R})} \leq (C_\eta + C_1) + (C_+ + C_-)||w||_{H^k_x(\mathbb{R})}.$$

Proceeding with the variant of Gronwall's lemma, it follows that

$$||w||_{H^k_x(\mathbb{R})} \leq ||w(x,y,0)||_{H^k_x(\mathbb{R})} e^{(C_+ + C_-)t} + \frac{C_\eta + C_1}{C_+ + C_-}\Big(e^{(C_+ + C_-)t} - 1\Big). \tag{5.8}$$

Similarly as in the case of $H^1$, we let $y \to \pm\infty$ on both sides of inequality (2.5), to conclude

$$\lim_{y\to\pm\infty} ||w||_{H^k_x(\mathbb{R})} \leq 0 \cdot e^{C_\pm t} + 0 \cdot \Big(e^{C_\pm t} - 1\Big) = 0.$$

Where all of the constants vanish except for $C_+$ as $y \to +\infty$ also, as $y \to -\infty$, $C_-$ does not vanish. Placing this together, we conclude that

$$\lim_{y\to\pm\infty} ||w||_{H^k_x(\mathbb{R})} = \lim_{y\to\pm\infty} ||\eta(x,y,t) - u^\pm(x,t)||_{H^k_x(\mathbb{R})} = 0.$$

Which proves the theorem. □

*Remark* 5.3. This result contributes further to the understanding of two actively studied models in the field nonlinear dispersive wave propagation. An interesting problem stemming from Theorem 5.1 would be to consider the case of pure power nonlinearities and general dispersive operators, i.e. to consider the following two equations:

1. One is

$$u_t + \Big(1 - A\Big)^{-1}\partial_x\Big(u + \frac{u^{p+1}}{p+1}\Big) = 0, \quad u = u(x,t), \quad x \in \mathbb{R}, \quad t \geq 0, \tag{5.9}$$

where $\widehat{A\varphi(\xi)} = a(\xi)\hat{\varphi(\xi)}$ is a general dispersive operator and $p \geq 1$.

2. The other is

$$u_t + \Big(1 - B\Big)^{-1}\partial_x\Big(u + \frac{u^{q+1}}{q+1}\Big) = 0, \quad u = u(x,y,t), \quad (x,y) \in \mathbb{R}^2, \quad t \geq 0, \tag{5.10}$$

For (5.10), $\widehat{B\varphi(\xi,\mu)} = b(\xi,\mu)\hat{\varphi}(\xi,\mu)$ is Fourier multiplier and $q \geq 1$.

The author suspects that a similar result could be established for suitable $A, B, p$, and $q$.

### 5.3 The NON-KP model

In this chapter, we introduce a new modified Kadomstev Petviashvili equation, also referred to as the (NON-KP) model. An example of a Cauchy problem corresponding to such a system is listed below:

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} + v_y - v_{xxy} = 0, \quad \text{for } (x,y) \in \mathbb{R}^2 \text{ and } \quad t > 0, \\ v_t - v_{xxt} + u_y + uu_y = 0, \\ u(x,y,0) = u_0, \\ v(x,y,0) = v_0. \end{cases} \tag{5.11}$$

To find the dispersion relation of (5.11), we insert the ansatz

$$(u,v) = \left(Ae^{i(\xi x + \mu y - \omega t)}, Be^{i(\xi x + \mu y - \omega t)}\right) \tag{5.12}$$

into the linearized version of the system to obtain

$$\begin{cases} -\omega A + \xi A - \xi^2 \omega A + B\mu + B\xi^2 \mu = 0, \\ A\mu - \omega B - B\xi^2 \omega = 0. \end{cases} \tag{5.13}$$

As before, we obtain the following matrix equation

$$\begin{pmatrix} -\omega + \xi - \xi^2\omega & \mu + \xi^2\mu \\ \mu & -\omega - \xi^2\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If the existence of a nontrivial solution is to be guaranteed, we must have that

$$\left(-\omega - \xi^2\omega\right)^2 - \left(\xi + \xi^3\right)\omega - \mu^2\left(1 + \xi^2\right) = 0.$$

It follows that the dispersion relation for (5.11) is

$$\omega(\xi,\mu) = \frac{\xi \pm \sqrt{\xi^2 + 4\mu^2\,(1+\xi^2)}}{2\,(1+\xi^2)} \quad \text{Dispersion relation (NON-KP)} \tag{5.14}$$

*Remark* 5.4. In what follows we define the dependent variables

$$\omega_1(\xi,\mu) := \frac{\xi + \sqrt{\xi^2 + 4\mu^2\,(1+\xi^2)}}{2\,(1+\xi^2)}, \tag{5.15}$$

and

$$\omega_2(\xi,\mu) := \frac{\xi - \sqrt{\xi^2 + 4\mu^2\,(1+\xi^2)}}{2\,(1+\xi^2)}. \tag{5.16}$$

## 5.4 Variational derivation

A defining characteristic of the KP models is that the well-posedness results are subject to the restriction that at all transverse positions, the mass

$$\int u\,dx = \text{constant independent of } y.$$

Moreover, in 2006, for a rather general class of equations of KP type, Molinet et al [MST07b] , showed that the zero-mass (in $x$) constraint is satisfied at any non zero time even if it is not satisfied at initial time zero. In an effort to remedy this Jerry L. Bona and Ronald Smith suggested a modification to the Kadomstev & Petviashvilli equations which does not impose non-physical restrictions upon the initial data.

As in Chapter 2, we obtain the following Hamiltonian:

$$H = \frac{1}{2}\iint \Big[RL_2(L_1^2 + K_1 - 2UL_1)L_2R + R^3 + wK_2L_2^2W\Big]\,dX\,dz \tag{5.17}$$

Retaining the dependent variables $R$ and $W$, the Hamiltonian stated above gives rise to the coupled evolution equations

$$\begin{cases} L_1L_2^2(R_T - UR_X) + (\frac{3}{4}R^2 + \frac{L_1^2+K_1}{2}L_2^2R)_X + \frac{1}{2}(K_2L_2^2W)_z = 0, \\ L_1L_2^2W_T + (\frac{3}{4}R^2 + \frac{L_1^2+K_1}{2}L_2^2R)_z = 0. \end{cases} \tag{5.18}$$

*Remark* 5.5. Utilizing the same scheme above, one can calculate the dispersion relation for the general form of the NON-KP model. For a small amplitude Fourier component proportional to $e^{i(\xi x+\mu z-\omega t)}$ we have that

$$\begin{pmatrix} -2\omega+\xi\left(-2U\hat{L_1}+\hat{L_1^2}+\hat{K_1}\right) & \mu\hat{K_2} \\ \mu\left(\hat{L_1^2}+\hat{K_1}\right) & -2\omega \end{pmatrix}\begin{pmatrix}\hat{R}\\ \hat{W}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

For a solution to exist we must have that

$$4\omega=\xi\left(-2U\hat{L_1}+\hat{L_1^2}+\hat{K_1}\right)\pm\sqrt{\xi^2\left(-2U\hat{L_1}+\hat{L_1^2}+\hat{K_1}\right)^2+4\mu^2\hat{K_2}\left(\hat{L_1^2}+\hat{K_1}\right)} \tag{5.19}$$

Making the choice of operators (4.7), a straightforward calculation shows that the above ”general” dispersion relation reduces to dispersion relation 5.14, namely

$$\begin{aligned}
4\omega&=\xi\left(\hat{L_1^2}+\hat{K_1}\right)\pm\sqrt{\xi^2\left(\hat{L_1^2}+\hat{K_1}\right)^2+4\mu^2\hat{K_2}\left(\hat{L_1^2}+\hat{K_1}\right)}\\
&=\xi\left(1+\left(\widehat{\frac{1+\frac{1}{6}\partial_x^2}{1-\frac{1}{6}\partial_x^2}}\right)\right)\pm\sqrt{\xi^2\left(1+\left(\widehat{\frac{1+\frac{1}{6}\partial_x^2}{1-\frac{1}{6}\partial_x^2}}\right)\right)^2+4\mu^2\left(1+\left(\widehat{\frac{1+\frac{1}{6}\partial_x^2}{1-\frac{1}{6}\partial_x^2}}\right)\right)}\\
&=\xi\left(\frac{2}{1+\frac{1}{6}\xi^2}\right)\pm\sqrt{\xi^2\left(\frac{2}{1+\frac{1}{6}\xi^2}\right)^2+4\mu^2\left(\frac{2}{1+\frac{1}{6}\xi^2}\right)}\\
&=\frac{2\xi}{1+\frac{1}{6}\xi^2}\pm\sqrt{\frac{4\xi^2}{\left(1+\frac{1}{6}\xi^2\right)^2}+\frac{4\cdot 2\mu^2\left(1+\frac{1}{6}\xi^2\right)}{\left(1+\frac{1}{6}\xi^2\right)^2}}\\
&=2\left(\frac{\xi\pm\sqrt{\xi^2+2\mu^2\left(1+\frac{1}{6}\xi^2\right)}}{1+\frac{1}{6}\xi^2}\right)
\end{aligned}$$

Thus, we arrive at (5.14), up to a constant, i.e.

$$\omega=\frac{1}{2}\left(\frac{\xi\pm\sqrt{\xi^2+2\mu^2\left(1+\frac{1}{6}\xi^2\right)}}{1+\frac{1}{6}\xi^2}\right). \tag{5.20}$$

Where the choice $U=0$ was made, as a straightforward calculation shows that the Hamiltonian of (5.18) is the same as the Hamiltonian of the system corresponding to $U=0$.

These are the NON-Kadomstev-Petviashvili (NON-KP) models. The selection (4.6) in (5.18) leads to a system which has the form

$$\begin{cases} R_T - UR_X + (\frac{3}{4}R^2 + \frac{L_1^2+K_1}{2}L_2^2R)_X + \frac{1}{2}W_z = 0, \\ W_T + (\frac{3}{4}R^2 + \frac{L_1^2+K_1}{2}L_2^2R)_z = 0, \end{cases} \tag{5.21}$$

while the choice of (4.7) gives results in a system resembling (5.18)

$$\begin{cases} (1 - \frac{1}{6}\partial_X^2)(R_T - UR_X) + (\frac{3}{4}R^2 + R)_X + \frac{1}{2}(1 - \frac{1}{6}\partial_X^2)W_z = 0, \\ (1 - \frac{1}{6}\partial_X^2)W_T + (\frac{3}{4}R^2 + R)_z = 0. \end{cases} \tag{5.22}$$

Dropping the various constants and taking $R$ to be $u$ and $W$ to be $v$, one observes that (5.22) is generalized version of the system 5.11. A point of departure from the usual KP models is that if $(u, v)$ is a solution to the linearized version of either model, then the mass $m(y,t) = \int_{\mathbb{R}} u dx$ is no longer independent of the transverse variable $y$. Indeed, $m$ satisfies the wave equation

$$m_{tt} = m_{zz}, \quad z \in \mathbb{R}, \quad t \geq 0.$$

### 5.5 Hamiltonian evolutionary structure

This section is dedicated to observing the Hamiltonian structure of system (5.11). We write the NON-KP model in the following form

$$\partial_t \boldsymbol{\eta} - A\boldsymbol{\eta} + N(\boldsymbol{\eta}) = 0, \tag{5.23}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} u \\ v \end{pmatrix}, \; A = \begin{pmatrix} -(1-\partial_x^2)^{-1}\partial_x & -\partial_y \\ -(1-\partial_x^2)^{-1}\partial_y & 0 \end{pmatrix} \text{ and } N(\boldsymbol{\eta}) = \begin{pmatrix} (1-\partial_x^2)^{-1}uu_x \\ (1-\partial_x^2)^{-1}uu_y \end{pmatrix}.$$

the skew adjoint matrix

$$J = \begin{pmatrix} -(1-\partial_x^2)^{-1}\partial_x & -(1-\partial_x^2)^{-1}\partial_y \\ -(1-\partial_x^2)^{-1}\partial_y & 0 \end{pmatrix}.$$

So that system (5.11) is equivalent to

$$\partial_t \boldsymbol{\eta} = J(\mathrm{grad} H)(\boldsymbol{\eta}),$$

where $H(\boldsymbol{\eta})$ is the functional given by

$$H(\boldsymbol{\eta}) = H(u,v) = \int_{\mathbb{R}^2} \left( \frac{v_x^2}{2} + \frac{v^2}{2} + \frac{u^2}{2} + \frac{u^3}{6} \right) dxdy.$$

*Remark* 5.6. Since $(1-\partial_x^2)^{-1}$ is a symmetric operator and $-\partial_x, -\partial_y$ are skew symmetric, it follows that $J$ is skew symmetric with respect to the scalar product $(\cdot,\cdot)$ on $L^2(\mathbb{R}^2;\mathbb{R}^2)$, defined as

$$(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) = \int_{\mathbb{R}^2} u\tilde{u} + v\tilde{v}\ dxdy.$$

Upon taking variational derivatives, one can observe that the above Hamiltonian gives rise to system (5.11), viz.

$$\begin{aligned} \partial_t \boldsymbol{\eta} &= J(\mathrm{grad} H)(\boldsymbol{\eta}) \\ &= J(\mathrm{grad} H)(u,v) \\ &= \begin{pmatrix} -(1-\partial_x^2)^{-1}\partial_x & -(1-\partial_x^2)^{-1}\partial_y \\ -(1-\partial_x^2)^{-1}\partial_y & 0 \end{pmatrix} \begin{pmatrix} u + \frac{u^2}{2} \\ (1-\partial_x^2)v \end{pmatrix} \\ &= \begin{pmatrix} -(1-\partial_x^2)^{-1}\partial_x \left(u + \frac{u^2}{2}\right) & -(1-\partial_x^2)^{-1}(1-\partial_x^2)\partial_y v \\ -(1-\partial_x^2)^{-1}\partial_y \left(u + \frac{u^2}{2}\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(1-\partial_x^2)^{-1}\partial_x \left(u + \frac{u^2}{2}\right) & -\partial_y v \\ -(1-\partial_x^2)^{-1}\partial_y \left(u + \frac{u^2}{2}\right) & 0 \end{pmatrix} \end{aligned}$$

Therefore, we arrive at the operator form of system (5.11), viz.

$$\begin{cases} u_t = -\partial_x(1-\partial_x^2)^{-1}\left(u + \frac{u^2}{2}\right) - v_y, \\ v_t = -\partial_y(1-\partial_x^2)^{-1}\left(u + \frac{u^2}{2}\right). \end{cases} \tag{5.24}$$

As $H$ is the Hamiltonian for system (5.11), it directly follows that $H$ is conserved by the flow of (5.11), i.e.

$$\begin{aligned}\frac{d}{dt}H(\boldsymbol{\eta}) &= H'(\boldsymbol{\eta})\boldsymbol{\eta}_t \\ &= ((\mathrm{grad}H)(\boldsymbol{\eta}), \boldsymbol{\eta}_t) \\ &= ((\mathrm{grad}H)(\boldsymbol{\eta}), J(\mathrm{grad}H)(\boldsymbol{\eta})) \\ &= 0, \qquad \text{since } J \text{ is skew-adjoint.}\end{aligned}$$

*Remark* 5.7. Provided one is presented with system 5.11, the Hamiltonian can be obtained via a straightforward calculation, i.e.

$$u_t + v_y + Qu_x + Quu_x = 0, \quad \text{for} \quad Q := (1 - \partial_x^2)^{-1}, \tag{5.25}$$

or, equivalently

$$uu_t + uv_y + uQu_x + uQuu_x = 0. \tag{5.26}$$

We integrate over $\mathbb{R}^2$ to obtain the following string of implications:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} \frac{u^2}{2}\,dxdy - \int_{\mathbb{R}^2} u_y v\,dxdy + \int_{\mathbb{R}^2} uQuu_x\,dxdy = 0. \\ &\frac{d}{dt}\int_{\mathbb{R}^2} \frac{v^2}{2}\,dxdy + \frac{d}{dt}\int_{\mathbb{R}^2} \frac{v_x^2}{2}\,dxdy + \int_{\mathbb{R}^2} vuu_y\,dxdy = -\int_{\mathbb{R}^2} u_y v\,dxdy. \\ &\frac{d}{dt}\int_{\mathbb{R}^2} \frac{u^2}{2}\,dxdy + \frac{d}{dt}\int_{\mathbb{R}^2} \frac{v^2}{2}\,dxdy + \frac{d}{dt}\int_{\mathbb{R}^2} \frac{v_x^2}{2}\,dxdy + \int_{\mathbb{R}^2} vuu_y\,dxdy \\ &+ \int_{\mathbb{R}^2} uQuu_x\,dxdy = 0.\end{aligned}$$

And we also have

$$\frac{d}{dt}\int_{\mathbb{R}^2}\frac{u^2}{2}\,dxdy+\frac{d}{dt}\int_{\mathbb{R}^2}\frac{v^2}{2}\,dxdy+\frac{d}{dt}\int_{\mathbb{R}^2}\frac{v_x^2}{2}\,dxdy-\int_{\mathbb{R}^2}\frac{u^2}{2}v_y\,dxdy$$
$$+\int_{\mathbb{R}^2}uQuu_x\,dxdy=0.$$
$$\frac{d}{dt}\int_{\mathbb{R}^2}\frac{u^2}{2}\,dxdy+\frac{d}{dt}\int_{\mathbb{R}^2}\frac{v^2}{2}\,dxdy+\frac{d}{dt}\int_{\mathbb{R}^2}\frac{v_x^2}{2}\,dxdy$$
$$-\int_{\mathbb{R}^2}\frac{u^2}{2}(-u_t-Qu_x-Quu_x)\,dxdy+\int_{\mathbb{R}^2}uQuu_x\,dxdy=0.$$
$$\frac{d}{dt}\int_{\mathbb{R}^2}\left[\frac{u^2}{2}+\frac{v^2}{2}+\frac{v_x^2}{2}+\frac{u^3}{6}\right]dxdy+\int_{\mathbb{R}^2}\frac{u^2}{2}Qu_x\,dxdy$$
$$+\int_{\mathbb{R}^2}\frac{u^2}{2}Quu_x\,dxdy+\int_{\mathbb{R}^2}uQuu_x\,dxdy=0.$$
$$\frac{d}{dt}\int_{\mathbb{R}^2}\left[\frac{u^2}{2}+\frac{v^2}{2}+\frac{v_x^2}{2}+\frac{u^3}{6}\right]dxdy$$
$$+\int_{\mathbb{R}^2}\frac{u^2}{2}Qu_x\,dxdy+\int_{\mathbb{R}^2}\frac{u^2}{2}Q\frac{\partial}{\partial_x}\frac{u^2}{2}\,dxdy-\int_{\mathbb{R}^2}u_xQ\frac{u^2}{2}\,dxdy=0.$$
$$\frac{d}{dt}\int_{\mathbb{R}^2}\left[\frac{u^2}{2}+\frac{v^2}{2}+\frac{v_x^2}{2}+\frac{u^3}{6}\right]dxdy=0.$$

### 5.6 Diagonalization

Before venturing into the task of obtaining linear estimates, we transform 5.11 into an equivalent system with a diagonal linear part. First, observe that the Fourier transform of $A$ is

$$\hat{A}(\xi,\mu)=i\begin{pmatrix}-\frac{\xi}{1+\xi^2} & -\mu\\ -\frac{\mu}{1+\xi^2} & 0\end{pmatrix}.$$

Now we calculate the characteristic polynomial of $\hat{A}(\xi,\mu)$. Denote the identity matrix by $\mathcal{I}$, so that

$$\begin{aligned}\hat{A}(\xi,\mu)&=\det\left(\hat{A}-\lambda\mathcal{I}\right)\\&=\lambda^2+\frac{\xi i}{1+\xi^2}\lambda+\frac{\mu^2}{1+\xi^2}\end{aligned}$$

So that its eigenvalues are given by

$$(\lambda_1,\lambda_2)=(-\omega_1 i,-\omega_2 i)\,,$$

as expected. Next, we search for a nonzero $\boldsymbol{\eta}$, such that

$$\left(\hat{A} - \lambda_1 \mathcal{I}\right) \boldsymbol{\eta} = \mathbf{0}.$$

More explicitly, we need to solve the following matrix equation.

$$\begin{pmatrix} \frac{\xi}{1+\xi^2} - \omega_1 & \mu \\ \\ \frac{\mu}{1+\xi^2} & -\omega_1 \end{pmatrix} \begin{pmatrix} u \\ \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}.$$

The solution that produces the first eigenvector, coresponding to eigenvalue $\lambda_1$ is

$$\begin{aligned} u &= \frac{\left(1+\xi^2\right)\omega_1}{\mu} v \\ &= \frac{\xi + \sqrt{\xi^2 + 4\mu^2\left(1+\xi^2\right)}}{2\mu} v. \end{aligned}$$

Thus,

$$E_1 = \begin{pmatrix} 1 \\ \\ \frac{\mu}{(1+\xi^2)\omega_1} \end{pmatrix}.$$

A similar calculation for $\lambda_2$ yields

$$E_2 = \begin{pmatrix} 1 \\ \\ \frac{\mu}{(1+\xi^2)\omega_2} \end{pmatrix}.$$

Define $\hat{P}$ to be the matrix $(E_1, E_2)$, in the canonical basis, so that

$$\hat{P} = \begin{pmatrix} 1 & 1 \\ \\ \frac{\mu}{(1+\xi^2)\omega_1} & \frac{\mu}{(1+\xi^2)\omega_2} \end{pmatrix}$$

and

$$\hat{P}^{-1} = \begin{pmatrix} \frac{\omega_1}{\omega_1-\omega_2} & \frac{-\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \\ \\ \frac{-\omega_2}{\omega_1-\omega_2} & \frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \end{pmatrix}.$$

If we let $\hat{D}$ be the diagonal matrix of eigenvalues, i.e.

$$\hat{D} = \begin{pmatrix} \lambda_1 & 0 \\ \\ 0 & \lambda_2 \end{pmatrix},$$

it follows that

$$
\begin{aligned}
\hat{A} &= \begin{pmatrix} 1 & 1 \\ \\ \frac{\mu}{(1+\xi^2)\omega_1} & \frac{\mu}{(1+\xi^2)\omega_2} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{\omega_1}{\omega_1-\omega_2} & \frac{-\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \\ \\ \frac{-\omega_2}{\omega_1-\omega_2} & \frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \end{pmatrix} \\
&= \hat{P}\hat{D}\hat{P}^{-1}.
\end{aligned}
$$

Placing this together, we write

$$
\partial_t \hat{\boldsymbol{\eta}} - \hat{A}\hat{\boldsymbol{\eta}} + \hat{N}(\hat{\boldsymbol{\eta}}) = 0, \tag{5.27}
$$

and make the change of variables $\hat{\boldsymbol{\eta}} = \hat{P}\hat{\boldsymbol{w}}$, for

$$
\boldsymbol{w} = \begin{pmatrix} w_1 \\ \\ w_2 \end{pmatrix}.
$$

As a result, we can rewrite (5.27) in the following form

$$
\partial_t \hat{\boldsymbol{w}} - \hat{P}^{-1}\hat{A}\hat{P}\hat{\boldsymbol{w}} + \hat{P}^{-1}\hat{N}(\hat{P}\hat{\boldsymbol{w}}) = 0, \tag{5.28}
$$

or equivalently,

$$
\partial_t \hat{\boldsymbol{w}} - \hat{D}\hat{\boldsymbol{w}} + \hat{P}^{-1}\hat{N}(\hat{P}\hat{\boldsymbol{w}}) = 0. \tag{5.29}
$$

Now we explicitly calculate $\hat{P}^{-1}\hat{N}(\hat{P}\hat{\boldsymbol{w}})$. Let

$$
\begin{aligned}
K(\hat{w}_1, \hat{w}_2) &:= \left(\widehat{w_1 + w_2}\right) * \left(\widehat{w_1 + w_2}\right) \\
&= (\hat{w}_1)^2 + (\hat{w}_2)^2 + 2\hat{w}_1\hat{w}_2 \\
&= \widehat{w_1 * w_1} + \widehat{w_2 * w_2} + 2\widehat{w_1 * w_2}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\hat{P}^{-1}\hat{N}(\hat{P}\hat{\boldsymbol{w}}) &= \frac{1}{2}\begin{pmatrix} \frac{\omega_1}{\omega_1-\omega_2} & \frac{-\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \\ \frac{-\omega_2}{\omega_1-\omega_2} & \frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \end{pmatrix} \begin{pmatrix} \frac{\xi i}{(1+\xi^2)}\left(\widehat{w_1+w_2}\right) * \left(\widehat{w_1+w_2}\right) \\ \frac{\mu i}{(1+\xi^2)}\left(\widehat{w_1+w_2}\right) * \left(\widehat{w_1+w_2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\omega_1}{\omega_1-\omega_2} & \frac{-\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \\ \frac{-\omega_2}{\omega_1-\omega_2} & \frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \end{pmatrix} \begin{pmatrix} \frac{\xi i}{(1+\xi^2)} K(\hat{w}_1,\hat{w}_2) \\ \frac{\mu i}{(1+\xi^2)} K(\hat{w}_1,\hat{w}_2) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\xi i}{(1+\xi^2)}\frac{\omega_1}{(\omega_1-\omega_2)} - \frac{\mu i}{(1+\xi^2)}\frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \\ -\frac{\xi i}{(1+\xi^2)}\frac{\omega_2}{(\omega_1-\omega_2)} + \frac{\mu i}{(1+\xi^2)}\frac{\omega_1\omega_2\left(1+\xi^2\right)}{\mu(\omega_1-\omega_2)} \end{pmatrix} \begin{pmatrix} K(\hat{w}_1,\hat{w}_2) \\ K(\hat{w}_1,\hat{w}_2) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\xi i}{(1+\xi^2)}\frac{\omega_1}{(\omega_1-\omega_2)} - i\frac{\omega_1\omega_2}{(\omega_1-\omega_2)} \\ -\frac{\xi i}{(1+\xi^2)}\frac{\omega_2}{(\omega_1-\omega_2)} + i\frac{\omega_1\omega_2}{(\omega_1-\omega_2)} \end{pmatrix} \begin{pmatrix} K(\hat{w}_1,\hat{w}_2) \\ K(\hat{w}_1,\hat{w}_2) \end{pmatrix} \\
&= \begin{pmatrix} i\frac{\xi\omega_1}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} + i\frac{\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} \\ -i\frac{\xi\omega_2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} - i\frac{\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} \end{pmatrix} \begin{pmatrix} K(\hat{w}_1,\hat{w}_2) \\ K(\hat{w}_1,\hat{w}_2) \end{pmatrix} \\
&= \begin{pmatrix} i\frac{\xi\omega_1+\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} \\ -i\frac{\xi\omega_2+\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}} \end{pmatrix} \begin{pmatrix} K(\hat{w}_1,\hat{w}_2) \\ K(\hat{w}_1,\hat{w}_2) \end{pmatrix}.
\end{aligned}
$$

Summarizing, we have that (5.11) is equivalent to

$$
\begin{cases} \partial_t\hat{w}_1 - i\omega_1\hat{w}_1 + \hat{M}_1(\xi,\mu)K(\hat{w}_1,\hat{w}_2) = 0, \\ \partial_t\hat{w}_2 - i\omega_2\hat{w}_2 + \hat{M}_2(\xi,\mu)K(\hat{w}_1,\hat{w}_2) = 0, \\ \hat{w}_1(\xi,\mu,0) = \hat{w}_{10}, \\ \hat{w}_2(\xi,\mu,0) = \hat{w}_{20}. \end{cases} \tag{5.30}
$$

Where $M_1(\partial_x,\partial_y)$ and $M_2(\partial_x,\partial_y)$ are Fourier multipliers with symbols $i\frac{\xi\omega_1+\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}}$ and $-i\frac{\xi\omega_2+\mu^2}{\sqrt{\xi^2+4\mu^2(1+\xi^2)}}$, respectively. An application of Duhamel's formula allows us to write (5.30) in integral form.

$$\begin{cases} w_1(t) = S_1(t)w_{10} - \displaystyle\int_0^t S_1(t-t')M_1(\partial_x, \partial_y)K(w_1, w_2)dt', \\ w_2(t) = S_2(t)w_{20} - \displaystyle\int_0^t S_2(t-t')M_2(\partial_x, \partial_y)K(w_1, w_2)dt'. \end{cases}$$

Where $\mathcal{F}_{x,y}\left[S_i(t)w_{i0}\right] := e^{\omega_i t}\hat{w}_{i0}$, $i = 1, 2.$, denotes the solution of the free NON-KP evolution.

*Remark* 5.8. If $(w_1, w_2)$ solves (5.6) locally, then it also solves (5.30) in a distributional sense. As a result, it also solves (5.11), in lieu of the change of variables.

### 5.7 Linear estimates

As previously mentioned, the first step to obtain linear estimates involves a localization in time. To this end, let $\psi(t) \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class on the real line, supp $\psi \subset [-2, 2]$, and $\psi = 1$ on $[-1, 1]$. For $T > 0$, let $\psi_T(t) = \psi(\frac{t}{T})$ and define the temporally truncated operator

$$\mathbf{\Omega}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \psi_T(t)\begin{pmatrix} S_1(t)w_{10} \\ S_2(t)w_{20} \end{pmatrix} - \psi_T(t)\int_0^t \begin{pmatrix} S_1(t-t')M_1(\partial_x, \partial_y)K(w_1, w_2) \\ S_2(t-t')M_2(\partial_x, \partial_y)K(w_1, w_2) \end{pmatrix} dt'.$$

The problem of existence now reduces to finding a fixed point for the above operator equation (5.7) in a suitable Banach space.

*Remark* 5.9. Since supp $\psi \subset [-2, 2]$, it is a direct consequence that supp $\psi_T \subset [-2T, 2T]$.

**Lemma 5.10.** *For $s \in \mathbb{R}$, there exists a universal constant $C$ such that*

$$\|\psi_T(t)S_i(t)w_{i0}\|_{X^{b,s}} \leq CT^{\frac{1}{2}-b}\|w_{i0}\|_{Y^s} \quad \textit{for} \quad i = 1, 2.$$

*Proof.* We first estimate $\|\psi(t)S_i(t)w_{i0}\|_{X_1^{b,s}}$, then make use of a simple calculation to arrive at the desired conclusion.

$$\begin{aligned}
\|\psi(t)S_i(t)w_{i0}\|_{X^{b,s}} &= \left\| \langle\xi\rangle \langle\tau-\omega_i(\xi,\mu)\rangle^b \langle|\xi|+|\mu|\rangle^s \mathcal{F}_t\left(\psi(t)e^{\omega_i t}\hat{w}_{i0}(\xi,\mu)\right)\right\|_{L^2_{\xi,\mu,\tau}} \\
&= \left\| \langle\xi\rangle \langle\tau-\omega_i(\xi,\mu)\rangle^b \langle|\xi|+|\mu|\rangle^s \hat{w}_{i0}(\xi,\mu)\mathcal{F}_t\left(\psi(t)e^{\omega_i t}\right)\right\|_{L^2_{\xi,\mu,\tau}} \\
&= \left\| \langle\xi\rangle \langle\tau-\omega_i(\xi,\mu)\rangle^b \langle|\xi|+|\mu|\rangle^s \hat{w}_{i0}(\xi,\mu)\hat{\psi}(\tau-\omega_i(\xi,\mu)\right\|_{L^2_{\xi,\mu,\tau}} \\
&= \left\| \langle\xi\rangle \langle|\xi|+|\mu|\rangle^s \hat{w}_{i0}(\xi,\mu) \left\|\langle\beta\rangle^b \hat{\psi}(\beta)\right\|_{L^2_\beta}\right\|_{L^2_{\xi,\mu}} \\
&= \|w_{i0}\|_{Y^s}\cdot\|\psi\|_{H^b} \\
&\leq C\,\|w_{i0}\|_{Y^s}
\end{aligned}$$

Now, for $\psi\in\mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\|\psi_T\|_{H^b} &= \left(\int_{\mathbb{R}} \langle\tau\rangle^b T^2\hat{\psi}^2(T\tau)d\tau\right)^{\frac{1}{2}}, \\
&= \left(\int_{\mathbb{R}} \left(1+\tau^2\right)^b T^2\hat{\psi}^2(T\tau)d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

We make the change of variables $\gamma=T\tau$, so that $d\gamma=Td\tau$, to obtain

$$\begin{aligned}
&= \left(\int_{\mathbb{R}} \left(1+\frac{\gamma^2}{T^2}\right)^b T\hat{\psi}^2(\gamma)d\gamma\right)^{\frac{1}{2}} \\
&\leq C\left(\int_{\mathbb{R}} \frac{\gamma^{2b}}{T^{2b}}T\hat{\psi}^2(\gamma)d\gamma\right)^{\frac{1}{2}} \\
&= C\left(\int_{\mathbb{R}} \gamma^{2b}T^{-2b+1}\hat{\psi}^2(\gamma)d\gamma\right)^{\frac{1}{2}} \\
&= CT^{\frac{1}{2}-b}\left(\int_{\mathbb{R}} \gamma^{2b}\hat{\psi}^2(\gamma)d\gamma\right)^{\frac{1}{2}} \\
&\leq CT^{\frac{1}{2}-b} \quad \text{since } \psi\in\mathcal{S}(\mathbb{R}).
\end{aligned}$$

Placing this together, it follows that

$$\begin{aligned}\|\psi_T(t)S_i(t)w_{i0}\|_{X^{b,s}} &\le \|w_{i0}\|_{Y^s}\cdot\|\psi_T\|_{H^b}\\ &\le CT^{\frac{1}{2}-b}\|w_{i0}\|_{Y^s}.\end{aligned}$$

□

For the nonlinear part of (5.7), we have the following estimate

**Lemma 5.11.** *For $\epsilon \in \left(0,\frac{1}{4}\right)$, $b=\frac{1}{2}+\epsilon$ and $b'=\frac{1}{2}-2\epsilon$, we have that*

$$\left\|\psi_T\int_0^t S_i(t-t')F_i(w_1(t'),w_2(t'))dt'\right\|_{X^{b,s}} \le CT^{\epsilon}\|F_i(w_1,w_2)\|_{X^{-b',s}}$$

*where $F_i(w_1(t'),w_2(t')) := M_i(\partial_x,\partial_y)K(w_1,w_2)$ for $i=1,2$.*

This is accomplished by the following lemma and a trick. To this end we will prove the following lemma.

**Lemma 5.12.** *For $\epsilon \in \left(0,\frac{1}{4}\right)$, $b=\frac{1}{2}+\epsilon$ and $b'=\frac{1}{2}-2\epsilon$, we have that*

$$\left\|\psi_T\int_0^t f(t')dt'\right\|_{H^b} \le CT^{\epsilon}\|f\|_{H^{-b'}}$$

*Proof.* Firstly,

$$\begin{aligned}\psi_T\int_0^t f(t')dt' &= \psi_T\int_0^t\int_{\mathbb{R}} e^{it\tau}\hat{f}(\tau)d\tau dt'\\ &= \psi_T\int_{\mathbb{R}}\int_0^t e^{it\tau}\hat{f}(\tau)dt'd\tau\\ &= \psi_T\int_{\mathbb{R}}\hat{f}(\tau)\int_0^t e^{it\tau}dt'd\tau\\ &= \psi_T\int_{\mathbb{R}}\frac{e^{it\tau}-1}{i\tau}\hat{f}(\tau)d\tau\\ &= \psi_T\int_{|\tau|\le\frac{1}{T}}\frac{e^{it\tau}-1}{i\tau}\hat{f}(\tau)d\tau - \psi_T\int_{|\tau|\ge\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}d\tau + \psi_T\int_{|\tau|\ge\frac{1}{T}}\frac{e^{it\tau}}{i\tau}\hat{f}(\tau)d\tau\\ &=: I_1+I_2+I_3\end{aligned}$$

We proceed by estimating each integral $I_j$ for $j = 1, 2, 3$. Since

$$\begin{aligned}\frac{e^{it\tau}-1}{i\tau} &= \frac{e^{it\tau}}{i\tau} - \frac{1}{i\tau} = \sum_{k=0}^{\infty} \frac{(it\tau)^k}{k!\,(i\tau)} - \frac{1}{i\tau} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} (i\tau)^{k-1} - \frac{1}{i\tau} = \sum_{k=1}^{\infty} \frac{t^k}{k!} (i\tau)^{k-1},\end{aligned}$$

it follows that

$$I_1 = \psi_T \int_{|\tau|\le\frac{1}{T}} \sum_{k=1}^{\infty} \frac{t^k}{k!} (i\tau)^{k-1} \hat{f}(\tau) d\tau.$$

Accordingly, for $I_1$ we have

$$\begin{aligned}\|I_1\|_{H^b} &= \left\| \langle\tau\rangle^b \mathcal{F}_t \left( \int_{|\tau|\le\frac{1}{T}} \sum_{k=1}^{\infty} \frac{t^k \psi_T}{k!} (i\tau)^{k-1} \hat{f}(\tau) d\tau \right) \right\|_{L^2_\tau} \\ &= \left\| \langle\tau\rangle^b \int_{\mathbb{R}} e^{-it\tau} \left( \sum_{k=1}^{\infty} \frac{t^k \psi_T}{k!} \int_{|\tau'|\le\frac{1}{T}} (i\tau')^{k-1} \hat{f}(\tau') d\tau' \right) dt \right\|_{L^2_\tau} \\ &= \left\| \langle\tau\rangle^b \mathcal{F}_t \left( \sum_{k=1}^{\infty} \frac{t^k \psi_T}{k!} \right) \right\|_{L^2_\tau} \cdot \left| \int_{|\tau'|\le\frac{1}{T}} (i\tau')^{k-1} \hat{f}(\tau') d\tau' \right| \\ &\le \sum_{k=1}^{\infty} \left\| \frac{t^k \psi_T}{k!} \right\|_{H^b} \cdot \int_{|\tau|\le\frac{1}{T}} |i\tau|^{k-1} \left| \hat{f}(\tau) \right| d\tau =: A \cdot B.\end{aligned}$$

For $B$ we have

$$\begin{aligned}B &= \int_{|\tau|\le\frac{1}{T}} |i\tau|^{k-1} \left| \hat{f}(\tau) \right| d\tau \le T^{1-k} \int_{|\tau|\le\frac{1}{T}} \left| \hat{f}(\tau) \right| \frac{\langle\tau\rangle^{-b'}}{\langle\tau\rangle^{-b'}} d\tau \\ &\le T^{1-k} \|f\|_{H^{-b'}} \cdot \left( \int_{|\tau|\le\frac{1}{T}} \left(1+\tau^2\right)^{b'} d\tau \right)^{\frac{1}{2}} \\ &\le \sqrt{\frac{2}{T}} \sup_{|\tau|\le\frac{1}{T}} \tau^{b'} T^{1-k} \|f\|_{H^{-b'}} \\ &\le C T^{-\frac{1}{2}-b'} T^{1-k} \|f\|_{H^{-b'}} \\ &= C T^{\frac{1}{2}-k-b'} \|f\|_{H^{-b'}}.\end{aligned}$$

Since $\left\|t^k\psi_T\right\|_{H^b} \le CT^{k-b+\frac{1}{2}}$ for $A$, we have that

$$\sum_{k=1}^{\infty}\left\|\frac{t^k\psi_T}{k!}\right\|_{H^b} \le \sum_{k=1}^{\infty} T^{k-b+\frac{1}{2}}\frac{C}{k!}.$$

Combining the above, it transpires that

$$\begin{aligned}
\|I_1\|_{H^b} &\le A\cdot B\\
&\le \sum_{k=1}^{\infty} T^{k-b+\frac{1}{2}}\frac{C}{k!}\cdot CT^{\frac{1}{2}-k-b'}\|f\|_{H^{-b'}}\\
&= \sum_{k=1}^{\infty}\frac{C}{k!}T^{1-b'-b}\|f\|_{H^{-b'}}\\
&= (e-1)\,CT^{1-b'-b}\|f\|_{H^{-b'}}\\
&= C_1T^{\epsilon}\|f\|_{H^{-b'}}.
\end{aligned}$$

For $I_2$, we have

$$\begin{aligned}
\|I_2\|_{H^b} &= \left\|\langle\tau\rangle^b\mathcal{F}_t\left(\psi_T\int_{|\tau|\ge\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}d\tau\right)\right\|_{L^2_\tau}\\
&= \left\|\langle\tau\rangle^b\int_{\mathbb{R}}e^{-it\tau}\psi_T\int_{|\tau|\ge\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}d\tau dt\right\|_{L^2_\tau}\\
&= \left\|\langle\tau\rangle^b\hat{\psi}_T\int_{|\tau|\ge\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}d\tau\right\|_{L^2_\tau}\\
&\le \|\psi_T\|_{H^b}\cdot\int_{|\tau|\ge\frac{1}{T}}\frac{\left|\hat{f}(\tau)\right|}{|\tau|}\langle\tau\rangle^{-b'}\langle\tau\rangle^{b'}d\tau\\
&\le \|\psi_T\|_{H^b}\cdot\|f\|_{H^{-b'}}\cdot\left(\int_{|\tau|\ge\frac{1}{T}}\frac{1}{T^2}\left(1+\tau^2\right)^{b'}d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

However, if one lets $\beta = \tau T$, then

$$\begin{aligned}\left(\int_{|\tau|\geq\frac{1}{T}} \frac{1}{T^2}\left(1+\tau^2\right)^{b'} d\tau\right)^{\frac{1}{2}} &= \left(\int_{|\beta|\geq 1} \frac{T^2}{\beta^2}\left(1+\frac{\beta^2}{T^2}\right)^{b'} \frac{1}{T} d\beta\right)^{\frac{1}{2}} \\ &\leq CT^{\frac{1}{2}}\left(\int_{|\beta|\geq 1} T^{-2b'}\frac{\beta^{2b'}}{\beta^2} d\beta\right)^{\frac{1}{2}} \\ &= CT^{\frac{1}{2}-b'}\left(\int_{|\beta|\geq 1} \frac{\beta^{1-4\epsilon}}{\beta^2} d\beta\right)^{\frac{1}{2}} \\ &= CT^{\frac{1}{2}-b'}\left(\int_{|\beta|\geq 1} \frac{1}{\beta^{1+4\epsilon}} d\beta\right)^{\frac{1}{2}} \\ &= CT^{\frac{1}{2}-b'}.\end{aligned}$$

Since $\epsilon \in \left(0, \frac{1}{4}\right)$ implies that $\displaystyle\int_{|\beta|\geq 1} \frac{1}{\beta^{1+4\epsilon}} d\beta < \infty$ is finite , we have

$$\begin{aligned}\|I_2\|_{H^b} &\leq C\|\psi_T\|_{H^b} \cdot \|f\|_{H^{-b'}} \cdot T^{\frac{1}{2}-b'} \leq CT^{\frac{1}{2}-b} \cdot \|f\|_{H^{-b'}} \cdot T^{\frac{1}{2}-b'} \\ &= CT^{1-b-b'}\|f\|_{H^{-b'}} = C_2 T^{\epsilon}\|f\|_{H^{-b'}}.\end{aligned}$$

For $I_3$ it follows that

$$\begin{aligned}\|I_3\|_{H^b} &= \left\|\langle\tau\rangle^b \mathcal{F}_t\left(\psi_T \int_{|\tau|\geq\frac{1}{T}} \frac{e^{it\tau}}{i\tau}\hat{f}(\tau)d\tau\right)\right\|_{L^2_\tau} = \left\|\langle\tau\rangle^b \mathcal{F}_t\left[\psi_T \cdot \mathcal{F}_\tau^{-1}\left(\mathbf{1}_{|\tau|\geq\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}\right)(t)\right]\right\|_{L^2_\tau} \\ &= \left\|\langle\tau\rangle^b \cdot \hat{\psi}_T * \left(\mathbf{1}_{|\tau|\geq\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}\right)\right\|_{L^2_\tau}.\end{aligned}$$

Define $\hat{B}(\tau) =: \mathbf{1}_{|\tau|\geq\frac{1}{T}}\frac{\hat{f}(\tau)}{i\tau}$. Then

$$\begin{aligned}\|B\|_{H^b} &= \left\|\langle\tau\rangle^b \hat{B}\right\|_{L^2_\tau} = \left\|\langle\tau\rangle^b \frac{\langle\tau\rangle^{b'}}{\langle\tau\rangle^{b'}}\hat{B}\right\|_{L^2_\tau} = \left\|\hat{f}(\tau)\frac{\langle\tau\rangle^{-b'}}{i\tau}\langle\tau\rangle^{b+b'}\mathbf{1}_{|\tau|\geq\frac{1}{T}}\right\|_{L^2_\tau} \\ &\leq \|f\|_{H^{-b'}} \cdot \sup_{|\tau|\geq\frac{1}{T}} \frac{\langle\tau\rangle^{b+b'}}{|\tau|}.\end{aligned}$$

However,

$$\sup_{|\tau|\geq\frac{1}{T}} \frac{\langle\tau\rangle^{b+b'}}{|\tau|} \leq C \sup_{|\tau|\geq\frac{1}{T}} |\tau|^{b'+b-1} = C \sup_{|\tau|\geq\frac{1}{T}} \frac{1}{|\tau|^{1-b-b'}} \leq CT^{1-b-b'} = CT^{\epsilon}.$$

As a result,

$$\|B\|_{H^b} \leq \|f\|_{H^{-b'}} \cdot CT^{1-b-b'} = CT^{\epsilon} \|f\|_{H^{-b'}}.$$

Hence,

$$\begin{aligned}
&\left\| \langle\tau\rangle^b \cdot \hat{\psi}_T * \left( \mathbf{1}_{|\tau|\geq\frac{1}{T}} \frac{\hat{f}(\tau)}{i\tau} \right) \right\|_{L^2_\tau} = \left\| \langle\tau\rangle^b \cdot \hat{\psi}_T * \hat{B} \right\|_{L^2_\tau} \\
=& \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle\tau\rangle^b \hat{\psi}_T(\tau-\tau')\hat{B}(\tau')d\tau' \right)^2 d\tau \right)^{\frac{1}{2}} \\
\leq& \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \langle\tau-\tau'\rangle^b + \langle\tau'\rangle^b \right) \hat{\psi}_T(\tau-\tau')\hat{B}(\tau')d\tau' \right)^2 d\tau \right)^{\frac{1}{2}} \\
=& \left\| \langle\tau\rangle^b \hat{\psi}_T * \hat{B} + \hat{\psi}_T * \langle\tau\rangle^b \hat{B} \right\|_{L^2_\tau} \leq \left\| \langle\tau\rangle^b \hat{\psi}_T * \hat{B} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T * \langle\tau\rangle^b \hat{B} \right\|_{L^2_\tau} \\
\leq& \left\| \langle\tau\rangle^b \hat{\psi}_T \right\|_{L^1_\tau} \left\| \hat{B} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
=& \left\| \langle\tau\rangle^b \hat{\psi}_T \right\|_{L^1_\tau} \left\| \mathbf{1}_{|\tau|\geq\frac{1}{T}} \frac{\hat{f}(\tau)}{i\tau} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
=& \left\| \langle\tau\rangle^b \hat{\psi}_T \right\|_{L^1_\tau} \left\| \langle\tau\rangle^{b'} \langle\tau\rangle^{-b'} \mathbf{1}_{|\tau|\geq\frac{1}{T}} \frac{\hat{f}(\tau)}{i\tau} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
\leq& C \sup_{|\tau|\geq\frac{1}{T}} |\tau|^{b+b'-1} \left\| \hat{\psi}_T \right\|_{L^1_\tau} \left\| \langle\tau\rangle^{-b'} \hat{f} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
=& C \sup_{|\tau|\geq\frac{1}{T}} \frac{1}{|\tau|^{1-b-b'}} \left\| \hat{\psi}_T \right\|_{L^1_\tau} \left\| \langle\tau\rangle^{-b'} \hat{f} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
\leq& CT^{1-b-b'} \left\| \hat{\psi}_T \right\|_{L^1_\tau} \left\| \langle\tau\rangle^{-b'} \hat{f} \right\|_{L^2_\tau} + \left\| \hat{\psi}_T \right\|_{L^1_\tau} \|B\|_{H^b} \\
\leq& CT^{1-b'-b} \|f\|_{H^{-b'}} + CT^{1-b-b'} \|f\|_{H^{-b'}} \\
=& CT^{1-b-b'} \|f\|_{H^{-b'}} \\
=& C_3 T^{\epsilon} \|f\|_{H^{-b'}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\psi_T\int_0^t f(t')dt'\right\|_{H^b} &\leq \|I_1\|_{H^b}+\|I_2\|_{H^b}+\|I_3\|_{H^b}\\ &\leq (C_1+C_2+C_3)T^{\epsilon}\,\|f\|_{H^{-b'}}\\ &= CT^{\epsilon}\,\|f\|_{H^{-b'}}\end{aligned}$$

and (5.12) is proved. □

We will obtain (5.11) from (5.12). Firstly, a straightforward calculation shows that $\|u\|_{X^{b,s}} = \|S_i(-t)u\|_{H^{b,s}}$. Indeed, we have that

$$\begin{aligned}\|S_i(-t)u\|_{H^{b,s}} &= \left\|\langle\xi\rangle\langle\tau\rangle^b\langle|\xi|+|\mu|\rangle^s\,\mathcal{F}_{x,y,t}\left[S_i(-t)u\right]\right\|_{L^2_{\xi,\mu,\tau}}\\ &= \left\|\langle\xi\rangle\langle\tau\rangle^b\langle|\xi|+|\mu|\rangle^s\,e^{-it\omega_i}\hat{u}\left(\xi,\mu,\tau\right)\right\|_{L^2_{\xi,\mu,\tau}}\\ &= \left\|\langle\xi\rangle\langle\tau\rangle^b\langle|\xi|+|\mu|\rangle^s\,\hat{u}\left(\xi,\mu,\tau+\omega_i\right)\right\|_{L^2_{\xi,\mu,\tau}}\end{aligned}$$

If we let $\beta=\tau+\omega_i \quad$ then $\quad d\beta=d\tau$ and the above equals

$$\begin{aligned}\left\|\langle\xi\rangle\langle\beta-\omega_i\rangle^b\langle|\xi|+|\mu|\rangle^s\,\hat{u}\left(\xi,\mu,\beta\right)\right\|_{L^2_{\xi,\mu,\beta}} &= \left\|\langle\xi\rangle\langle\tau-\omega_i\rangle^b\langle|\xi|+|\mu|\rangle^s\,\hat{u}\left(\xi,\mu,\tau\right)\right\|_{L^2_{\xi,\mu,\tau}}\\ &= \|u\|_{X^{b,s}}\,.\end{aligned}$$

Now we are ready to prove Lemma 5.11.

*Proof.* By the above arguments, the left hand side

$$\left\|\psi_T\int_0^t S_i(t-t')F_i(w_1(t'),w_2(t'))dt'\right\|_{X^{b,s}}$$

is equal to

$$
\begin{aligned}
&\left\| S\left(-t\right)\psi_T \int_0^t S_i(t-t')F_i(w_1(t'),w_2(t'))dt' \right\|_{H^{b,s}} \\
=&\left\| \psi_T \int_0^t S_i(-t')F_i(w_1(t'),w_2(t'))dt' \right\|_{H^{b,s}} \\
=&\left\| \langle\xi\rangle \langle\tau\rangle^b \langle|\xi|+|\mu|\rangle^s \mathcal{F}_{x,y,t}\left[ \psi_T \int_0^t S_i(-t')F_i(w_1(t'),w_2(t'))dt' \right] \right\|_{L^2_{\xi,\mu,\tau}} \\
=&\left\| \langle\xi\rangle \langle|\xi|+|\mu|\rangle^s \mathcal{F}_{x,y}\left[ \langle\tau\rangle^b \mathcal{F}_t\left( \psi_T \int_0^t S_i(-t')F_i(w_1(t'),w_2(t'))dt' \right) \right] \right\|_{L^2_{\xi,\mu,\tau}} \\
\leq&\left\| \langle\xi\rangle \langle|\xi|+|\mu|\rangle^s \left\| \mathcal{F}_{x,y}\left[ \psi_T \int_0^t S_i(-t')F_i(w_1(t'),w_2(t'))dt' \right] \right\|_{H^b} \right\|_{L^2_{\xi,\mu,\tau}} \\
\leq& CT^{1-b-b'} \left\| \langle\xi\rangle \langle|\xi|+|\mu|\rangle^s \left\| \mathcal{F}_{x,y}\left[ S_i(-t')F_i(w_1(t'),w_2(t')) \right] \right\|_{H^{-b'}} \right\|_{L^2_{\xi,\mu,\tau}} \\
=& CT^{1-b-b'} \left\| S_i(-t')F_i(w_1(t'),w_2(t')) \right\|_{H^{-b',s}} \\
=& CT^{1-b-b'} \left\| F_i(w_1(t'),w_2(t')) \right\|_{X^{-b',s}} \\
=& CT^{\epsilon} \left\| F_i(w_1(t'),w_2(t')) \right\|_{X^{-b',s}}
\end{aligned}
$$

□

## 5.8 Future work

The linear estimates obtained in the previous section indicate that system 5.11 is at least locally well-posed in the following space.

$$Z := \{(w_1,w_2) \in X_1^{b,s} \times X_2^{b,s} : \|(w_1,w_2)\|_Z = \|w_1\|_{X_1^{b,0}} + \|w_2\|_{X_2^{b,0}} + \alpha \|w_1\|_{X_1^{b,s}} + \beta \|w_2\|_{X_2^{b,s}} < \infty\}$$

As a result, it should be possible to obtain the following $X^{b,s}$ energy estimates

$$\|\mathbf{\Omega}(w_1,w_2)\|_{X_1^{b,0}} \leq C\left( \|w_{10}\|_{H_x^1} + \|w_{20}\|_{H_x^1} \right) + CT^{\epsilon} \|(w_1,w_2)\|^2_{X_1^{b,0}}$$

$$\|\mathbf{\Omega}(w_1,w_2)\|_{X_1^{b,s}} \leq C\left( \|w_{10}\|_{Y^s} + \|w_{20}\|_{Y^s} \right) + CT^{\epsilon} \|(w_1,w_2)\|^2_{X_1^{b,s}} \|(w_1,w_2)\|_{X_1^{b,0}}$$

$$\|\mathbf{\Omega}(w_1,w_2)\|_Z \leq C\left( \|w_{10}\|_{H_x^1} + \|w_{20}\|_{H_x^1} \right) + \alpha \|w_{10}\|_{Y^s} + \beta \|w_{20}\|_{Y^s} + CT^{\epsilon} \|\Omega(w_1,w_2)\|_Z$$

Then by choosing appropriate parameter values for $\alpha$ and $\beta$, i.e.

$$\alpha = \frac{\|w_{10}\|_{H_x^1}}{\|w_{10}\|_{Y^s}} \qquad \beta = \frac{\|w_{20}\|_{H_x^1}}{\|w_{20}\|_{Y^s}}.$$

For $i = 1, 2$, we define the restriction space $X_T^{b,s}$, equipped with the norm $\|u\|_{X_i^{b,s}} = \inf \|v\|_{X_i^{b,s}}$. Where the infimum is taken over all $v \in X_i^{b,s}$ such that $v = u$ on $[-T, T] \times \mathbb{R}^2$.

Placing the above together, it should be possible to at least develop a local existence theory for the model (5.11). The author intends to continue in this direction. Now that the linear problem is finished. The problem is reduced to obtaining appropriate Bilinear estimates on the nonlinearity. This method involves partitioning the frequency space into its high and low components.

One should observe that the dispersion relation for the NON-KP model contains a term which depends on both spatial Fourier modes and is raised to the power $\frac{1}{2}$. This complicates the matter at hand. However, the driving force behind many successful well-posedness results pertaining to nonlinear dispersive evolution equations lies in the dispersive smoothing effect. For example, this phenomena can be seen in: [Bou93b, Bou93a, Bou93c, BT09, BLT02, ST04, IKD08, GPW10, MST02, MST07a, Che08].

Provided the systems were linearly well-posed, i.e. infinitesimal waves are only being considered. Upon considering the nonlinear terms, i.e. larger amplitude waves. The authors were able to obtain appropriate bilinear estimates. Assuming that it is possible to obtain such estimates, one arrives at local well-posedness for the system in question. The,n by making use of conserved quantities of the system, e.g. the Hamiltonian. One obtains global well-posedness, in a suitable Banach space.

The delicate interplay between the dispersive and nonlinear terms and how they relate to the dispersive smoothing effect, is the main reason the author chose to study Nonlinear Dispersive PDE. Since, dispersion physically means that waves of different amplitudes travel at different velocities. Unlike waves traveling through a nondispersive medium, the waves get deformed as they evolve. In some prescribed sense, if there is enough dispersion to counter act the nonlinear effects, a balance occurs and we arrive at a well behaved system which models physical phenomena occurring in nature. However, everything hinges upon a phenomena that isn't completely understood yet, the dispersive smoothing effect. Many materials occurring in nature exhibit dispersive behavor, to name a few: water, light and surface waves. This fact necessitates the mathematical analysis of these equations, which provides a precise, quantifiable understanding of the model.

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# Appendix: The Dirichlet-Neumann Operator

This Appendix is aimed at deriving a recursion formula for the terms in the Taylor expansion of the Dirichlet-Neumann operator. To accomplish this we follow the work of Craig et al. [CG94]. The method utilized in this thesis hinges on the following theorem, for a proof we refer the reader to [CM85].

**Theorem 5.13.** *There exists a constant $C = O(h_0)$ such that the operator $G(\eta)$ is analytic for $\eta \in C^\infty$ in a ball of radius $C$ about the point $\eta = 0$.*

As a direct result, the Taylor series expansion of $G(\eta)$ about 0 takes the form

$$G(\eta) = \sum_{j\geq 0} G_j(\eta), \tag{5.31}$$

for spatial dimensions $n = 2, 3$, where each $G_j(\eta)$ is homogeneous of degree $j$ in $n$. We then proceed to systematically compute the terms $G_j(\eta)$ via a recursive formula introduced in [CS93]. In what follows we define the fluid domain to be the region

$$S(\eta) = \{(x, y) : x \in S_0, -h_0 \leq y \leq \eta(x)\}.$$

*Remark* 5.14. One should observe that $S(\eta)$ is bounded below by the rigid bottom $y = -h_0$ and above by the graph $S_g = \{(x, y) : x \in S_0, y = \eta(x)\}$,
where $S_0 = \{(x, 0, z) : (x, z) \in \mathbb{R}^2\}$ is the undisturbed free surface.

The elliptic boundary value problem to be solved is

$$\begin{cases} \Delta\phi = 0 \quad \text{in} \quad S(\eta) \\ \phi = \Phi \quad \text{on} \quad S_G \\ \frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad y = -h_0 \end{cases} \tag{5.32}$$

coupled with the appropriate periodic or asymptotic conditions on $\phi$. It is well known that there exists a family of solutions to the problem for $k \in \mathbb{R}^{n-1}$. In accordance with the above discussion, we define

$$\phi_k(x, y) = e^{i(x\cdot k)} \cosh(|k|\,(y + h_0)). \tag{5.33}$$

These functions have the boundary values $\Phi_k(x, y) = e^{i(x\cdot k)} \cosh(|k|\,(\eta(x)+h_0))$, and are compatible with the asymptotic conditions as $|x| \to \infty$, or with conditions of periodicity for appropriately chosen wavevectors $k$. Now since the family of functions $\phi_k$ satisfy the equation

$$G(\eta)\Phi_k = \left[\frac{\partial\phi_k}{\partial y} - \frac{\partial\eta}{\partial x} \cdot \frac{\partial\phi_k}{\partial x}\right]_{y=\eta}. \tag{5.34}$$

Now, we substitute (5.33) into (5.34) and expand the Dirichlet-Neumann operator and hyperbolic functions to obtain

$$\sum_{j\text{ even}} \frac{(|k|\,\eta)^j}{j!}\left(\sinh(|k|\,h_0) - i\frac{\partial \eta}{\partial x}\cdot k\cosh(|k|\,h_0)\right)e^{i(x\cdot k)}$$
$$+\sum_{j\text{ odd}} \frac{(|k|\,\eta)^j}{j!}\left(\cosh(|k|\,h_0) - i\frac{\partial \eta}{\partial x}\cdot k\sinh(|k|\,h_0)\right)e^{i(x\cdot k)}$$
$$=\left(\sum_{m\geq 0} G_m(\eta)\right)\left(\sum_{j\text{ even}}\frac{(|k|\,\eta)^j}{j!}\cosh(|k|\,h_0)e^{i(x\cdot k)} + \sum_{j\text{ odd}}\frac{(|k|\,\eta)^j}{j!}\sinh(|k|\,h_0)e^{i(x\cdot k)}\right).$$

We then proceed to obtain the recursive formula by equating terms of the same degree in $\eta$. For $j = 0$ it follows that

$$G_0 e^{i(x\cdot k)} = |k|\tanh(|k|\,h_0)e^{i(x\cdot k)}. \tag{5.35}$$

By utilizing Fourier analytic techniques, one obtains a forumula for $G_j\Phi$ , valid for sufficient $\Phi$ and $j = 0, 1, 2$.

$$G_0(\eta)\Phi\,(x) = |D|\tanh(|D|\,h_0)\Phi\,(x)\,,$$

For convenience, we suppress the $\Phi$ dependency and write

$$\begin{aligned} G_0(\eta) &= |D|\tanh(|D|\,h_0), \\ G_1(\eta) &= |D|\,\eta\,|D| - |D|\tanh(|D|\,h_0)\eta\,|D|\tanh(|D|\,h_0), \\ G_2(\eta) &= -\frac{1}{2}\Big[\,|D|^2\,\eta^2\,|D|\tanh(|D|\,h_0) \\ &\quad + |D|\tanh(|D|\,h_0)\eta^2\,|D|^2 \\ &\quad - 2\,|D|\tanh(|D|\,h_0)\eta\,|D|\tanh(|D|\,h_0)\eta\,|D|\tanh(|D|\,h_0)\Big]. \end{aligned} \tag{5.36, 5.37}$$

More generally, we obtain

$$G_j(\eta) = \begin{cases} \frac{1}{j!}\Big[\eta^j\,|D|^{j+1}\tanh(|D|\,h_0) - i\frac{\partial}{\partial x}\left(\eta^j\right)\cdot D\,|D|^{j-1}\tanh(|D|\,h_0)\Big] \\ \quad - \sum_{\nu<j,\nu\text{ even}}\frac{1}{(j-\nu)!}G_\nu(\eta)\eta^{j-\nu}\,|D|^{j-\nu} \\ \quad - \sum_{\nu<j,\nu\text{ odd}}\frac{1}{(j-\nu)!}G_\nu(\eta)\eta^{j-\nu}\,|D|^{j-\nu}\tanh(|D|\,h_0) & \text{if } j \text{ is even} \\ \frac{1}{j!}\Big[\eta^j\,|D|^{j+1} - i\frac{\partial}{\partial x}\left(\eta^j\right)\cdot D\,|D|^{j-1}\Big] \\ \quad - \sum_{\nu<j,\nu\text{ even}}\frac{1}{(j-\nu)!}G_\nu(\eta)\eta^{j-\nu}\,|D|^{j-\nu}\tanh(|D|\,h_0) \\ \quad - \sum_{\nu<j,\nu\text{ odd}}\frac{1}{(j-\nu)!}G_\nu(\eta)\eta^{j-\nu}\,|D|^{j-\nu} & \text{if } j \text{ is odd} \end{cases} \tag{5.38}$$

Equations (5.36) and (5.38) constitute the recursive formula for the terms $G_j(\eta)$, as shown in [CG94].

# Vita

After studying classical composition at Southeastern Louisiana University, Jacob Grey finished his undergraduate studies in mathematics at Louisiana State University in May 2008. In August 2008 he came to Louisiana State University to pursue graduate studies in mathematics. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2015.