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Diffusion in Mixtures of Elastic Materials.

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DIFFUSION IN MIXTURES OF ELASTIC MATERIALS

A Dissertation

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by

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ABSTRACT

A thermomechanical theory of a diffusing mixture of n constituents is presented. Equations of balance of mass, linear momentum, moment of momentum, and energy are postulated both for the mixture and for the constituents. It is suggested that in a theory with partial stresses and partial heat fluxes, the diffusive force, and thus all the constitutive functions, should be allowed to depend on density gradients.

An entropy inequality for the mixture is proposed and used to restrict the constitutive equations in a mixture of elastic materials subject to diffusion and heat conduction. The constituents are later assumed to be isotropic solids, and the constitutive equations are restricted further to satisfy the axiom of material frame-indifference and to reflect material symmetry. It is shown that the constitutive functions can be written in terms of the left Cauchy-Green tensors of the constituents.

The results are linearized by assuming that the mixture undergoes small temperature changes, small deformations, and small departures from equilibrium. The linearized stress turns out to be symmetric as a result of its dependence on the left Cauchy-Green tensors and the fact that the

constituents are isotropic. Finally, two problems are worked in which small amplitude plane waves are propagated through a binary mixture of isotropic elastic solids infinite in extent.

1. INTRODUCTION

The purposes of this work are to develop a thermo-mechanical theory of a diffusing mixture and show that the concept of a partial stress and a partial heat flux can be retained in such a theory, and more important, to apply the theory to a mixture of elastic materials subject to diffusion and heat conduction and illustrate its use in linearized form for the case of a binary mixture of two isotropic elastic solids. In recent years several thermo-mechanical theories of mixtures subject to diffusion have appeared. Eringen & Ingram have proposed one which they apply to a chemically reacting mixture of viscous fluids, each constituent having its own temperature distribution.¹ Green & Naghdi have published a paper in which they consider a binary mixture of viscous fluids.² Their entropy inequality is different from the one used here. It leads to an expression for the partial stress on the constituents which for an ideal gas mixture, will not reduce to the

¹A. C. Eringen and J. D. Ingram, "A Continuum Theory of Chemically Reacting Media - I," International Journal of Engineering Science, III(1965), 197-212.

²A. E. Green and P. M. Naghdi, "A Dynamical Theory of Interacting Continua," International Journal of Engineering Science, III(1965), 231-241.

usual expression for the partial pressure,³ while the partial stress used in this paper will yield the standard result. A theory using the idea of a chemical potential tensor instead of a partial stress tensor has been developed by Bowen, who applies it to mixtures of elastic materials.⁴ And Müller has proposed a theory in which he considers a binary mixture of viscous fluids.⁵ Two features of Müller's theory are that in his entropy inequality he leaves arbitrary the entropy flux into the mixture, and he is the first writer who consistently allows for a dependence on density gradients in his constitutive equations.⁶

These writers all rely essentially on equations of balance first proposed by Truesdell⁷ and later generalized

³Ibid., p. 240.

⁴R. M. Bowen, "Toward a Thermodynamics and Mechanics of Mixtures," Archive for Rational Mechanics and Analysis, XXIV(1967), 370-403.

⁵I. Müller, "A Thermodynamic Theory of Mixtures of Fluids," Archive for Rational Mechanics and Analysis, XXVIII(1968), 1-39.

⁶Ibid., pp. 2, 9.

⁷C. Truesdell, "Sulle Basi della Termomeccanica," Atti della Accademia Nazionale dei Lincei, XXII(1957), 36, 158, 159-160. See also C. Truesdell and R. Toupin, The Classical Field Theories (Vol. III/1 of Handbuch der Physik, ed. S. Flügge. 54 vols.; Berlin-Göttingen-Heidelberg: Springer, 1960), pp. 472, 567-568, 613-614.

by Kelly.⁸ Their theories also extend to mixtures the approach taken by Coleman & Noll, who use the entropy inequality to restrict the constitutive equations and who do not limit themselves to conditions near equilibrium.⁹ There is considerable other work on mixture theories in the literature. Truesdell has compared and improved some of the earlier mechanical and thermomechanical theories of diffusion and gives extensive references to papers on diffusion.¹⁰ Green & Naghdi have recently published a second paper on mixtures where, in contrast to their previous work, they introduce an energy equation and an entropy inequality for each constituent.¹¹ In doing so, they assign partial internal energy densities, partial heat fluxes, and partial heat supplies, which they did not do previously.

⁸P. D. Kelly, "A Reacting Continuum," International Journal of Engineering Science, II(1964), 138-142.

⁹B. D. Coleman and W. Noll, "The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity," Archive for Rational Mechanics and Analysis, XIII(1963), 167-178.

¹⁰C. Truesdell, "Mechanical Basis of Diffusion," Journal of Chemical Physics, XXXVII(1962), 2336-2344. See also Truesdell and Toupin, op. cit., pp 705-708.

¹¹A. E. Green and P. M. Naghdi, "A Theory of Mixtures," Archive for Rational Mechanics and Analysis, XXIV(1967), 244.

Adkins¹² and Green & Adkins¹³ have used Truesdell's balance equations to study various non-linear theories of diffusion. Green & Naghdi's first theory of mixtures has been applied by Green & Steel,¹⁴ Steel,¹⁵ Mills,¹⁶ Crochet & Naghdi,¹⁷

¹²J. E. Adkins, "Non-Linear Diffusion: I. Diffusion and Flow of Mixtures of Fluids," Phil. Trans. Roy. Soc. Lond., A/CCLV(1963), 607-633; J. E. Adkins, "Non-Linear Diffusion: II. Constitutive Equations for Mixtures of Isotropic Fluids," Phil. Trans. Roy. Soc. Lond., A/CCLV (1963), 635-648; J. E. Adkins, "Non-Linear Diffusion: III. Diffusion Through Isotropic Highly Elastic Solids," Phil. Trans. Roy. Soc. Lond., A/CCLVI(1964), 301-316; J. E. Adkins, "Diffusion of Fluids Through Aeolotropic Highly Elastic Solids," Archive for Rational Mechanics and Analysis, XV(1964), 222-234.

¹³A. E. Green and J. E. Adkins, "A Contribution to the Theory of Non-Linear Diffusion," Archive for Rational Mechanics and Analysis, XV(1964), 235-246.

¹⁴A. E. Green and T. R. Steel, "Constitutive Equations for Interacting Continua," International Journal of Engineering Science, IV(1966), 483-500.

¹⁵T. R. Steel, "Applications of a Theory of Interacting Continua," Quarterly Journal of Mechanics and Applied Mathematics, XX(1967), 57-72.

¹⁶N. Mills, "Incompressible Mixtures of Newtonian Fluids," International Journal of Engineering Science, IV (1966), 97-112; N. Mills, "On a Theory of Multi-Component Mixtures," Quarterly Journal of Mechanics and Applied Mathematics, XX(1967), 499-508.

¹⁷M. J. Crochet and P. M. Naghdi, "On Constitutive Equations for Flow of Fluid Through an Elastic Solid," International Journal of Engineering Science, IV(1966), 383-401; M. J. Crochet and P. M. Naghdi, "Small Motions Superposed on Large Static Deformations in Porous Media," Acta Mechanica, IV(1967), 315-335.

and Atkin.¹⁸ Also Dixon has published a paper in which he treats a binary mixture of charged fluids.¹⁹

It should be pointed out that the various theories of mixtures differ from one another somewhat, both in conception and in detail. Consequently, exact comparisons are sometimes difficult, and results in this paper will often be referred to as "equivalent to" rather than "identical to" the results of others.

In chapter 2 the kinematics of motion and the equations of balance for a mixture of n bodies are discussed. Axioms of balance of mass, linear momentum, moment of momentum, and energy are postulated both for the mixture and for the constituents; and the equations which make these postulates compatible are derived. In any mixture theory in which diffusion takes place an important quantity is the diffusive force on the constituents. This force, also called the diffusive resistance or the linear momentum supply, is a body force, additional to the external body force on each

¹⁸ R. J. Atkin, "Constitutive Theory for a Mixture of an Isotropic Elastic Solid and a Non-Newtonian Fluid," Zeitschrift für Angewandte Mathematik und Physik, XVIII (1967), 803-825.

¹⁹ R. C. Dixon, "Diffusion of a Charged Fluid," International Journal of Engineering Science, V(1967), 265-287.

constituent. It arises from the physical interaction of each constituent with all the others; that is, it occurs because diffusion is taking place. It will be shown in chapter 2, that, if the diffusive force is allowed to depend on second deformation gradients, which implies a dependence on density gradients in a nonreacting mixture of fluids, the concept of a partial stress and a partial heat flux as used by Truesdell can be retained without loss of generality. The axiom of equipresence²⁰ then requires that all the constitutive functions be allowed to depend on second deformation gradients.

A statement of the second axiom of thermodynamics is proposed in chapter 3. The statement of this axiom, the entropy inequality, is the simplest generalization of the entropy inequality for single materials.

In chapter 4, a mixture of elastic materials in which diffusion and heat conduction can take place is considered. Also, it is shown how the balance equations are satisfied and how the entropy inequality is to be used. The restrictions imposed by the entropy inequality are derived in chapter 5. In chapter 6 the constitutive equations are

²⁰M. E. Gurtin, "Thermodynamics and the Possibility of Spatial Interaction in Elastic Materials," Archive for Rational Mechanics and Analysis, XIX(1965), 340.

further restricted to satisfy the axiom of material frame-indifference and to reflect material symmetry in a mixture of isotropic solids.

The results are linearized in chapter 7 by assuming that the mixture undergoes small temperature changes, small deformations, and small departures from equilibrium. At the end of the chapter the case in which there are two constituents in the mixture is summarized briefly.

Chapter 8 deals with the propagation of small amplitude plane waves through a binary mixture of isotropic elastic solids infinite in extent. Two different problems are considered. In the first, solutions are sought for the wave number when both constituents have displacements exponential in form. In the second problem one of the constituents undergoes, instead, a homogeneous strain.

Notation

Direct tensor notation is used instead of components whenever possible. Vectors in the three-dimensional inner product space U and points in the Euclidean 3-space E (except $\tilde{x}$) are indicated by Latin minuscules with tildas ($\sim$) underneath: $\tilde{x}$, $\tilde{g}$, ---. Linear transformations are

regarded as the same as second-order tensors and are indicated by Latin majuscles with tildas underneath: $\tilde{T}$, $\tilde{I}$, ---. If $\tilde{A}$ is a linear transformation, $\tilde{A}^T$ denotes its transpose; $\tilde{A}^{-1}$, its inverse; $\text{tr } \tilde{A}$, its trace; and $\det \tilde{A}$, its determinant. The identity linear transformation is denoted by $\tilde{I}$. Components are referred to a fixed time-independent rectangular Cartesian coordinate system with basis $(\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3)$. Latin minuscule component indices indicate spatial coordinates; Latin majuscule, material coordinates. The gradient with respect to spatial coordinates is denoted by grad , and the gradient with respect to material coordinates, by ∇ . The divergence with respect to spatial coordinates is denoted by div . A quantity corresponding to a particular material in the mixture is identified by placing a Latin minuscule directly below the symbol for the quantity. The summation convention is used only for summations over coordinates. Summations over the constituents will always be indicated by the summation sign. The Latin majuscule C is used to indicate the complete contraction operator.²¹ The symbol " $\otimes$ " indicates the tensor product; " $\wedge$ ", the exterior product ($\tilde{a} \wedge \tilde{b} = \tilde{a} \otimes \tilde{b} - \tilde{b} \otimes \tilde{a}$).

²¹W. H. Greub, Linear Algebra (second edition; New York: Academic Press, 1963), p. 108.

2. KINEMATICS AND THE EQUATIONS OF BALANCE

In this chapter the kinematics of motion and the equations of balance for a mixture of n diffusing bodies B , $a = 1, \dots, n$, are discussed. Axioms of balance of mass, linear momentum, moment of momentum, and energy are postulated both for the mixture and for the constituents in the mixture; and the equations which make these postulates compatible are derived. Most of the definitions in the first part of this chapter are identical to those of Bowen and are presented here only for completeness.¹

Each body B is essentially a piece of a differentiable manifold isomorphic to the 3-dimensional Euclidean space E . The concept of a body has been made mathematically rigorous by Noll.² The elements of the body B , called particles, are denoted by X .

Definition. A configuration of B is a homeomorphism χ of B into E .

Definition. A motion of B is a one-parameter family of configurations χ_t , where t , a real number, is the time.

¹Bowen, op. cit., pp. 374-377.

²W. Noll, The Axiomatic Method, with Special Reference to Geometry and Physics (Amsterdam: North Holland Company, 1959), p. 267.

Definition. The position of the particle $\underline{x}$ at the time t is given by

$$\underline{x} = \chi_{\underline{a}t}(\underline{x}) = \chi_{\underline{a}}(\underline{x}, t). \quad (2.1)$$

It is assumed that positions in E can be occupied simultaneously by particles from each body.³

Definition. A reference configuration for $\underline{B}$ is a fixed configuration $\underline{\kappa}$.

The position of the particle $\underline{x}$ in $\underline{\kappa}$ is therefore given by

$$\underline{X} = \underline{\kappa}(\underline{x}). \quad (2.2)$$

Equations (2.1) and (2.2) can be combined to yield

$$\underline{x} = \chi_{\underline{a}}(\underline{\kappa}^{-1}(\underline{x}), t) = \chi_{\underline{a}\underline{\kappa}}(\underline{x}, t). \quad (2.3)$$

The function $\chi_{\underline{a}\underline{\kappa}}$ is called the deformation function for $\underline{B}$.

Both $\chi_{\underline{a}\underline{\kappa}}$ and $\chi_{\underline{a}\underline{\kappa}}^{-1}$ are assumed to be functions of class C^3 on their domains.

Definition. The region occupied by $\underline{B}$ in E is a compact set with piecewise smooth boundaries and is defined as

³Truesdell and Toupin, op. cit., p. 469.

$$\chi_a(B, t) = \left\{ \chi_a(X, t) : X \in B \right\}.^4 \quad (2.4)$$

Definition. The mixture at the time t is defined as

$$B_t = \bigcup_{a=1}^n \chi_a^{-1} \left(\bigcap_{a=1}^n \chi_a(B, t), t \right). \quad (2.5)$$

Definition. The region occupied by B_t in E is defined as

$$V(B_t) = \bigcap_{a=1}^n \chi_a(B, t). \quad (2.6)$$

Equation (2.6) and the requirement that each configuration be a homeomorphism imply that each point $x \in V(B_t)$ is occupied by exactly n particles, one from each body. The part of B that is in B_t is clearly $B \cap B_t$.

Definition. The velocity and acceleration of $X \in B$ are

$$\dot{x}_a = \frac{\partial \chi_a(X, t)}{\partial t}$$

and

$$\ddot{x}_a = \frac{\partial^2 \chi_a(X, t)}{\partial t^2}.$$

(2.7)

In general a backward prime with subscript a will denote the material derivative following the motion of the a^{th} constituent.

⁴Noll, The Axiomatic Method, p. 267.

Definition. The gradient of the deformation at $X \in B$ is

$$\underset{\sim}{F} = \nabla_{\underset{\sim}{X}} \underset{\sim}{X}(X, t). \quad (2.8)$$

Because of the assumptions made about the smoothness of $\underset{\sim}{X}^{-1}$, $\underset{\sim}{F}^{-1}$ exists. Therefore,

$$|\det \underset{\sim}{F}| > 0. \quad (2.9)$$

Definition. The velocity gradient of $X \in B$ is

$$\underset{\sim}{L} = \text{grad}_{\underset{\sim}{X}} \underset{\sim}{X}(x, t). \quad (2.10)$$

It is easy to show that

$$\underset{\sim}{L} = \underset{\sim}{F} \underset{\sim}{F}^{-1}. \quad (2.11)$$

Definition. If ρ denotes the density of B , the density of B_t at $\underset{\sim}{x} \in V(B_t)$ and at the time t is defined as

$$\rho = \rho(\underset{\sim}{x}, t) = \sum_{\underset{\sim}{a}} \rho(\underset{\sim}{x}, t). \quad (2.12)$$

Definition. The concentration of $B \cap B_t$ at $\underset{\sim}{x} \in V(B_t)$ and at the time t is defined as

$$\underset{\sim}{c} = \underset{\sim}{c}(\underset{\sim}{x}, t) = \rho / \rho. \quad (2.13)$$

Equations (2.12) and (2.13) imply that the concentrations are related by

$$\sum_{\underset{\sim}{a}} \underset{\sim}{c} = 1. \quad (2.14)$$

Definition. The mean velocity of B_t at $x \in V(B_t)$ and at the time t is defined as

$$\dot{\tilde{x}} = \dot{\tilde{x}}(\tilde{x}, t) = \frac{1}{\rho} \sum_{\tilde{a}} \rho_{\tilde{a}} \dot{\tilde{x}}_{\tilde{a}}. \quad (2.15)$$

Definition. The diffusion velocity of $X \in B \cap B_t$, which is at $\tilde{x} \in V(B_t)$ at the time t , is defined as

$$\underset{\tilde{a}}{u} = \underset{\tilde{a}}{u}(\tilde{x}, t) = \dot{\tilde{x}}_{\tilde{a}} - \dot{\tilde{x}}. \quad (2.16)$$

It follows from equations (2.12), (2.15), and (2.16) that the diffusion velocities are related by

$$\sum_{\tilde{a}} \rho_{\tilde{a}} \underset{\tilde{a}}{u} = \underset{\sim}{0}. \quad (2.17)$$

Definition. The gradient of the mean velocity of B_t at $\tilde{x} \in V(B_t)$ and at the time t is

$$\underset{\sim}{L} = \text{grad } \dot{\tilde{x}}(\tilde{x}, t). \quad (2.18)$$

From equations (2.10), (2.16), and (2.18) it follows that

$$\text{grad } \underset{\tilde{a}}{u} = \underset{\tilde{a}}{L} - \underset{\sim}{L}. \quad (2.19)$$

This result and the fact that

$$\sum_{\tilde{a}} \left(\underset{\tilde{a}}{u} \otimes \text{grad } \rho_{\tilde{a}} + \rho_{\tilde{a}} \text{grad } \underset{\tilde{a}}{u} \right) = \text{grad } \sum_{\tilde{a}} \rho_{\tilde{a}} \underset{\tilde{a}}{u} = \underset{\sim}{0} \quad (2.20)$$

imply that

$$\rho_{\sim} \underset{\sim}{L} = \sum_{\tilde{a}} \left(\rho_{\tilde{a}} \underset{\tilde{a}}{L} + \underset{\tilde{a}}{u} \otimes \text{grad } \rho_{\tilde{a}} \right). \quad (2.21)$$

If ψ is a differentiable function of $\underline{x}$ and t , the derivatives of ψ following the motion defined by $\dot{\underline{x}}$ and following the motion $\dot{\underline{x}}_t$ are, respectively,

$$\dot{\psi} = \frac{\partial \psi}{\partial t}(\underline{x}, t) + (\text{grad } \psi(\underline{x}, t)) \dot{\underline{x}}$$

and

(2.22)

$$\dot{\psi}_t = \frac{\partial \psi}{\partial t}(\underline{x}, t) + (\text{grad } \psi(\underline{x}, t)) \dot{\underline{x}}_t.$$

Therefore,

$$\dot{\psi}_t - \dot{\psi} = (\text{grad } \psi(\underline{x}, t)) \underline{u} = (\nabla \psi(\underline{x}, t)) \underline{F}^{-1} \underline{u}, \quad (2.23)$$

where equation (2.16) has been used. Notice that $\psi(\underline{x}, t) = \underline{x}$ in equation (2.23) yields equation (2.16) again, since $\text{grad } \underline{x}$ is $\underline{I}$, the identity tensor. If $\psi = \dot{\underline{x}}$, then

$$\dot{\dot{\underline{x}}}_t - \ddot{\underline{x}} = \underline{L} \underline{u}. \quad (2.24)$$

It follows from equation (2.22)₂ that, if ψ is a differentiable function of $\underline{x}$ and t , then

$$\rho \dot{\psi}_t = \frac{\partial \rho \psi}{\partial t} + \text{div}(\rho \psi \dot{\underline{x}}) - \left(\frac{\partial \rho}{\partial t} + \text{div}(\rho \dot{\underline{x}}) \right) \psi. \quad (2.25)$$

Summing equation (2.25) over the constituents and rearranging the result gives

$$\begin{aligned} \rho \dot{\psi} = \sum_a \left[\rho_a \dot{\psi}_t + \left(\frac{\partial \rho_a}{\partial t} + \text{div}(\rho_a \dot{\underline{x}}) \right) \psi_a - \text{div}(\rho_a \psi_a \underline{u}) \right] \\ - \left(\frac{\partial \rho}{\partial t} + \text{div}(\rho \dot{\underline{x}}) \right) \psi, \end{aligned} \quad (2.26)$$

where equations (2.16), (2.22)₁, and (2.22)₂ and the notation

$$\Psi = \frac{1}{\rho} \sum_a \rho \Psi_a \quad (2.27)$$

have been used. Also, for convenience, $\dot{\Psi}$ is written $\dot{\Psi}$ when

it occurs, since the second subscript a is redundant.

Truesdell & Toupin refer to equation (2.26) as a fundamental identity, as it is purely kinematical and is quite useful in the development of the balance equations.⁵

Definition. If $\tilde{b}$ denotes the partial body force density of B , the body force density for B_t at $\tilde{x} \in V(B_t)$ and at the time t is

$$\tilde{b} = \frac{1}{\rho} \sum_a \rho \tilde{b}_a \quad (2.28)$$

Definition. If $\tilde{r}$ denotes the partial heat supply density of B , the heat supply density for B_t at $\tilde{x} \in V(B_t)$ and at the time t is

$$\tilde{r} = \frac{1}{\rho} \sum_a \rho \tilde{r}_a \quad (2.29)$$

Definition. If $\tilde{e}$ denotes the partial internal energy density of B , the inner part of the internal energy density for B_t at $\tilde{x} \in V(B_t)$ and at the time t is

⁵Truesdell and Toupin, op. cit., p. 471.

$$\epsilon_I = \frac{1}{\rho} \sum_a \rho_a \epsilon_a^6 \quad (2.30)$$

and the internal energy density is

$$\epsilon = \epsilon_I + \frac{1}{2\rho} \sum_a \rho_a u_a^2. \quad (2.31)$$

Let R be an arbitrary fixed region contained in $V(B_t)$ at the time t . The equations of balance of mass, linear momentum, moment of momentum, and energy for the part $P_t = \bigcup_{a=1}^n \chi_a^{-1}(R, t)$ of the mixture B_t which occupies the fixed region R at the time t are postulated to be

$$\begin{aligned} \frac{\partial}{\partial t} \int_R \rho \, dv &= - \oint_{\partial R} \rho \, \dot{\underline{x}} \cdot d\underline{s}, \\ \frac{\partial}{\partial t} \int_R \rho \, \dot{\underline{x}} \, dv &= - \oint_{\partial R} \rho \, \dot{\underline{x}} (\dot{\underline{x}} \cdot d\underline{s}) + \oint_{\partial R} \underline{T} d\underline{s} + \int_R \rho \, \underline{b} \, dv, \end{aligned} \quad (2.32)$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_R (\underline{x} - \underline{o}) \wedge \rho \, \dot{\underline{x}} \, dv &= - \oint_{\partial R} [(\underline{x} - \underline{o}) \wedge \rho \, \dot{\underline{x}}] \dot{\underline{x}} \cdot d\underline{s} \\ &\quad + \oint_{\partial R} (\underline{x} - \underline{o}) \wedge \underline{T} d\underline{s} + \int_R (\underline{x} - \underline{o}) \wedge \rho \, \underline{b} \, dv, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \int_R \rho \left(\epsilon + \frac{1}{2} \dot{\underline{x}}^2 \right) dv &= - \oint_{\partial R} \rho \left(\epsilon + \frac{1}{2} \dot{\underline{x}}^2 \right) \dot{\underline{x}} \cdot d\underline{s} \\ &\quad + \oint_{\partial R} \dot{\underline{x}} \cdot \underline{T} d\underline{s} + \int_R \sum_a \dot{\underline{x}}_a \cdot \rho_a \, \underline{b}_a \, dv - \end{aligned}$$

⁶Ibid., p. 614.

$$- \oint_{\partial R} \underline{\underline{q}} \cdot d\underline{\underline{s}} + \int_R \rho r dv.$$

In equations (2.32) ∂R is the surface of R , dv is the element of volume, $d\underline{\underline{s}}$ is the outward directed element of area, $\underline{\underline{T}}$ is the stress tensor, $\underline{\underline{o}}$ is a fixed point in E , and $\underline{\underline{q}}$ is the heat flux vector.

Similarly, the equations of balance of mass, linear momentum, moment of momentum, and energy for the part $P = \chi_{\underline{\underline{a}}}^{-1}(R, t)$ of the body $\underline{\underline{B}}$ which occupies the region R at the time t are postulated to be

$$\begin{aligned} \frac{\partial}{\partial t} \int_R \rho dv &= - \oint_{\partial R} \rho \underline{\underline{x}} \cdot d\underline{\underline{s}} + \int_R \hat{c} dv, \\ \frac{\partial}{\partial t} \int_R \rho \underline{\underline{x}} dv &= - \oint_{\partial R} \rho \underline{\underline{x}} (\underline{\underline{x}} \cdot d\underline{\underline{s}}) + \oint_{\partial R} \underline{\underline{T}} d\underline{\underline{s}} + \int_R \rho \underline{\underline{b}} dv \\ &\quad + \int_R \hat{c} \underline{\underline{x}} dv + \int_R \bar{\underline{\underline{p}}} dv, \end{aligned} \tag{2.33}$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_R (\underline{\underline{x}} - \underline{\underline{o}}) \wedge \rho \underline{\underline{x}} dv &= - \oint_{\partial R} [(\underline{\underline{x}} - \underline{\underline{o}}) \wedge \rho \underline{\underline{x}}] \underline{\underline{x}} \cdot d\underline{\underline{s}} \\ &\quad + \oint_{\partial R} (\underline{\underline{x}} - \underline{\underline{o}}) \wedge \underline{\underline{T}} d\underline{\underline{s}} + \int_R (\underline{\underline{x}} - \underline{\underline{o}}) \wedge \rho \underline{\underline{b}} dv \\ &\quad + \int_R (\underline{\underline{x}} - \underline{\underline{o}}) \wedge \hat{c} \underline{\underline{x}} dv + \int_R (\underline{\underline{x}} - \underline{\underline{o}}) \wedge \bar{\underline{\underline{p}}} dv \\ &\quad + \int_R \bar{\underline{\underline{M}}} dv, \end{aligned}$$

and

$$\begin{aligned}
 & \frac{\partial}{\partial t} \int_R \rho \left(\epsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) dv \\
 &= - \oint_{\partial R} \rho \left(\epsilon + \frac{1}{2} \dot{\mathbf{x}}^2 \right) \tilde{\mathbf{x}} \cdot d\tilde{\mathbf{s}} + \oint_{\partial R} \tilde{\mathbf{x}} \cdot \tilde{\mathbf{T}} d\tilde{\mathbf{s}} + \int_R \tilde{\mathbf{x}} \cdot \tilde{\rho} \mathbf{b} dv \\
 &+ \int_R \hat{\mathbf{c}} \epsilon dv + \int_R \frac{1}{2} \hat{\mathbf{c}} \dot{\mathbf{x}}^2 dv + \int_R \tilde{\mathbf{x}} \cdot \tilde{\hat{\mathbf{p}}} dv + \int_R \tilde{\hat{\epsilon}} dv \\
 &- \oint_{\partial R} \tilde{\mathbf{q}} \cdot d\tilde{\mathbf{s}} + \int_R \rho \mathbf{r} dv.
 \end{aligned}$$

In equations (2.33) $\hat{\mathbf{c}}$ is the mass supply of the a^{th} constituent per unit volume per unit time due to chemical reactions, $\tilde{\mathbf{T}}$ is the stress tensor for the a^{th} constituent, $\tilde{\hat{\mathbf{p}}}$ is the diffusive force of the a^{th} constituent per unit volume, $\tilde{\hat{\mathbf{M}}}$ is the body couple on the a^{th} constituent per unit volume, $\tilde{\hat{\epsilon}}$ is the energy supply of the a^{th} constituent per unit volume per unit time and includes the rate of work of the body couple $\tilde{\hat{\mathbf{M}}}$, and $\tilde{\mathbf{q}}$ is the heat flux vector for the a^{th} constituent. As the form of equations (2.32)₃ and (2.33)₃ indicate, it has been assumed that there are no surface couples and no resultant body couples on the mixture.

When sufficient assumptions of smoothness are made, equations (2.32) reduce to the local statements

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \tilde{\mathbf{x}}) = 0,$$

$$\begin{aligned}\rho \dot{\tilde{x}} &= \operatorname{div} \tilde{T} + \rho \tilde{b}, \\ \tilde{T} &= \tilde{T}^T,\end{aligned}\tag{2.34}$$

and

$$\rho \dot{\tilde{e}} = \operatorname{tr}(\tilde{T} \tilde{L}) + \sum_a \rho_a \tilde{u}_a \cdot \tilde{b}_a - \operatorname{div} \tilde{q} + \rho r.$$

Likewise, equations (2.33) reduce to

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \dot{\tilde{x}}) &= \hat{c}, \\ \rho \dot{\tilde{x}} &= \operatorname{div} \tilde{T} + \rho \tilde{b} + \tilde{p}, \\ \tilde{T} - \tilde{T}^T &= \tilde{M},\end{aligned}\tag{2.35}$$

and

$$\rho \dot{\tilde{e}} = \operatorname{tr}(\tilde{T}^T \tilde{L}) - \operatorname{div} \tilde{q} + \rho r + \hat{\tilde{e}}.$$

Substitution of equations (2.34)₁ and (2.35)₁ into equation (2.26) yields

$$\rho \dot{\tilde{\psi}} = \sum_a \left[\rho_a \dot{\tilde{\psi}} + \hat{c}_a \tilde{\psi} - \operatorname{div}(\rho_a \tilde{\psi} \tilde{u}_a) \right].\tag{2.36}$$

If $\tilde{\psi} = \psi = 1$, then equation (2.36) implies that

$$\sum_a \hat{c}_a = 0,\tag{2.37}$$

where equation (2.17) has been used. Actually, equation (2.37) is the relation that makes the balance of mass of the mixture compatible with that of the constituents. That is, summation of equation (2.35)₁ over the constituents and

comparison with equation (2.34)₁ requires that equation (2.37) be true. In words equation (2.37) states that the total mass supply is zero; hence, it is an alternate form of the balance of mass of the mixture and will be used instead of equation (2.34)₁.

If equation (2.35)₁ is multiplied by $|\det \underline{F}|$, it can be rewritten as

$$\overline{(\rho |\det \underline{F}|)} = \hat{c} |\det \underline{F}|, \quad (2.38)$$

where equation (2.22)₂, with $\psi = \rho$, and the identity

$$\overline{|\det \underline{F}|} = |\det \underline{F}| \operatorname{div} \underline{x} \quad (2.39)$$

have been used. If there are no chemical reactions, each $\hat{c}$ is zero, equation (2.37) is identically satisfied, and the right side of equation (2.38) vanishes. This latter equation can then be integrated to yield

$$\rho |\det \underline{F}| = \rho_0, \quad (2.40)$$

where ρ_0 can be interpreted as the density of B in its reference configuration and, in general, is a function of position $\underline{x}$. This result implies that in the absence of chemical reactions the density and the deformation gradient of a given constituent are no longer independent of each other. This fact will prove useful in chapter 4, when

constitutive equations are postulated for a mixture in which chemical reactions are excluded. However, throughout this chapter and the next, it will be assumed that chemical reactions can occur.

Definition. The inner part of the stress on the mixture is

$$\tilde{T}_I = \tilde{T} + \sum_a \rho_a \tilde{u}_a \otimes \tilde{u}_a. \quad (2.41)$$

By use of equations (2.16), (2.17), (2.24), (2.28), (2.34), (2.35)₂, (2.37), and (2.41), together with the identities

$$\rho_a \ddot{\tilde{x}}_a = \sum_a \left[\rho_a \ddot{\tilde{x}}_a + \hat{c}_a \tilde{u}_a - \operatorname{div}(\rho_a \tilde{u}_a \otimes \tilde{u}_a) \right], \quad (2.42)$$

$$\rho \dot{\tilde{e}}_I = \rho \dot{\tilde{e}} - \sum_a \left[\rho_a \tilde{u}_a \cdot \dot{\tilde{u}}_a + \hat{c}_a \frac{\tilde{u}_a^2}{2} - \operatorname{div} \left(\rho_a \frac{\tilde{u}_a^2}{2} \tilde{u}_a \right) \right], \quad (2.43)$$

$$\sum_a \rho_a \tilde{u}_a \cdot \tilde{L} \tilde{u}_a = \operatorname{tr} \left[\sum_a (\rho_a \tilde{u}_a \otimes \tilde{u}_a) \tilde{L} \right], \quad (2.44)$$

and

$$\operatorname{div} \sum_a \tilde{T}_a^T \tilde{u}_a = \sum_a \tilde{u}_a \cdot \operatorname{div} \tilde{T}_a + \operatorname{tr} \sum_a \tilde{T}_a^T \operatorname{grad} \tilde{u}_a, \quad (2.45)$$

the balance equations for the mixture can be written as

$$\begin{aligned} \sum_a \hat{c}_a &= 0, \\ \sum_a \rho_a (\ddot{\tilde{x}}_a - b_a) &= \operatorname{div} \tilde{T}_I - \sum_a \hat{c}_a \tilde{u}_a, \end{aligned}$$

⁷Ibid., p. 568.

$$\mathbf{T}_I = \tilde{\mathbf{T}}_I^T, \quad (2.46)$$

and

$$\begin{aligned} \rho \dot{\epsilon}_I &= \text{tr}(\mathbf{T}_I \mathbf{L}) + \text{tr} \sum_{\mathbf{a}} \tilde{\mathbf{T}}^T \text{grad}_{\mathbf{a}} u - \sum_{\mathbf{a}} \left(\mathbf{u} \cdot \hat{\mathbf{p}}_{\mathbf{a}} + \hat{c} \frac{u^2}{2} \right) \\ &\quad - \text{div} \left[\mathbf{q} - \sum_{\mathbf{a}} \rho \left(-\frac{\tilde{\mathbf{T}}^T}{\rho} + \frac{u^2}{2} \mathbf{I} \right) \mathbf{u} \right] + \rho r. \end{aligned}$$

Equations (2.42) and (2.43) come from equation (2.36) with

$\tilde{\psi} = \dot{\mathbf{x}}_{\mathbf{a}}$ and $\psi = \frac{u^2}{2}$, respectively. Equations (2.46) are equivalent to Bowen's equations (2.26).⁸

Comparison of equations (2.35)₂ and (2.46)₂ shows that the compatibility condition for linear momentum is

$$\sum_{\mathbf{a}} \left(\hat{\mathbf{p}}_{\mathbf{a}} + \hat{c} \mathbf{u} \right) = \text{div} \left(\mathbf{T}_I - \sum_{\mathbf{a}} \tilde{\mathbf{T}} \right). \quad (2.47)$$

This relation will be used in place of equation (2.46)₂.

Equation (2.35)₃ and (2.46)₃ are compatible without requiring any further restrictions on the variables. Equations (2.35)₄ and (2.46)₄ imply that

$$\begin{aligned} \sum_{\mathbf{a}} \left[\hat{c} + \mathbf{u} \cdot \hat{\mathbf{p}}_{\mathbf{a}} + \hat{c} \left(\epsilon + \frac{u^2}{2} \right) \right] \\ = \text{tr} \left[\left(\mathbf{T}_I - \sum_{\mathbf{a}} \tilde{\mathbf{T}}^T \right) \mathbf{L} \right] - \end{aligned}$$

⁸Bowen, op. cit., p. 379.

$$- \operatorname{div} \left[\underline{\underline{q}} - \sum_{\alpha} \left(\underline{\underline{\bar{q}}}_{\alpha} + \rho_{\alpha} \left(\epsilon \underline{\underline{I}} - \frac{\underline{\underline{\bar{T}}}_{\alpha}^T}{\rho_{\alpha}} + \frac{u^2}{2} \underline{\underline{I}} \right) \underline{\underline{u}}_{\alpha} \right) \right], \quad (2.48)$$

where equations (2.19) and (2.29) and the identity

$$\rho \dot{\epsilon}_I = \sum_{\alpha} \left[\rho_{\alpha} \dot{\epsilon}_{\alpha} + \hat{c}_{\epsilon} - \operatorname{div}(\rho_{\alpha} \epsilon \underline{\underline{u}}) \right] \quad (2.49)$$

have been used. Equation (2.49) follows from equation (2.36) with $\psi = \epsilon$.

The theories of mixtures proposed by Truesdell & Toupin, Kelly, Eringen & Ingram, Green & Naghdi, and Müller have the feature that $\underline{\underline{\bar{T}}}$ is a partial stress and as a result

$$\underline{\underline{T}}_I = \sum_{\alpha} \underline{\underline{\bar{T}}}_{\alpha}.^9 \quad (2.50)$$

This assumption originated in an attempt to generalize the concept of a partial pressure, which occurs in theories of dilute mixtures and mixtures of perfect gases. In addition Truesdell & Toupin have the result that

$$\underline{\underline{q}} = \sum_{\alpha} \left[\underline{\underline{\bar{q}}}_{\alpha} + \rho_{\alpha} \left(\epsilon \underline{\underline{I}} - \frac{\underline{\underline{\bar{T}}}_{\alpha}^T}{\rho_{\alpha}} + \frac{u^2}{2} \underline{\underline{I}} \right) \underline{\underline{u}}_{\alpha} \right],^{10} \quad (2.51)$$

so that the right sides of equations (2.47) and (2.48)

⁹Truesdell and Toupin, op. cit., p. 568; Kelly, op. cit., p. 140; Eringen and Ingram, op. cit., p. 205; Green and Naghdi, International Journal of Engineering Science, III, 238; Müller, op. cit., p. 5.

¹⁰Truesdell and Toupin, op. cit., p. 613.

vanish.¹¹ Essentially, equation (2.51) states that the heat flux is the sum of partial heat fluxes. Bowen questioned the validity of equations (2.50) and (2.51) for mixtures of general materials and formulated a theory of diffusion in mixtures of elastic materials without introducing partial stresses and partial heat fluxes. However, he introduced the concept of a chemical potential tensor¹² for each constituent which is unsatisfactory, since it cannot be motivated. Also, he proposed equations of motion for the constituents of an unusual type. He showed that these equations could be put formally into the same form as equation (2.35)₂, but, for a theory without chemical reactions, the second gradient of the deformation of the a^h constituent has to be included in the list of independent variables for the diffusive force of the a^h constituent.¹³

It will be shown below that, with minor qualification, partial stresses and partial heat fluxes can be defined in such a fashion that equations formally similar to equations (2.50) and (2.51) are satisfied. Bowen's results turn out to be a special case of the resulting theory and will be

¹¹Ibid., pp. 568, 614.

¹²Bowen, op. cit., p. 380.

¹³Ibid., p. 383.

discussed further in chapter 5. To prove the above assertion the stress tensor, diffusion force, body couple, heat flux, and energy supply terms for the a^{th} constituent are redefined as follows:

$$\begin{aligned}\tilde{\mathbf{T}}_a &= \bar{\mathbf{T}}_a + c_a \left(\tilde{\mathbf{T}}_I - \sum_c \tilde{\mathbf{T}}_c \right) \\ \hat{\mathbf{p}}_a &= \hat{\mathbf{p}}_a - \text{div} \left[c_a \left(\tilde{\mathbf{T}}_I - \sum_c \tilde{\mathbf{T}}_c \right) \right], \\ \hat{\mathbf{M}}_a &= \hat{\mathbf{M}}_a - c_a \sum_c \hat{\mathbf{M}}_c, \\ \tilde{\mathbf{q}}_a &= \bar{\mathbf{q}}_a + c_a \left[\tilde{\mathbf{q}}_a - \sum_c \left(\bar{\mathbf{q}}_c + \rho_c \left(\epsilon_c \mathbf{I} - \frac{\tilde{\mathbf{T}}_c^T}{\rho_c} + \frac{u_c^2}{2} \mathbf{I} \right) \tilde{\mathbf{u}}_c \right) \right],\end{aligned}\tag{2.52}$$

and

$$\begin{aligned}\hat{\mathbf{e}}_a &= \hat{\mathbf{e}}_a - c_a \text{tr} \left[\left(\tilde{\mathbf{T}}_I - \sum_c \tilde{\mathbf{T}}_c^T \right) \tilde{\mathbf{L}}_a \right] \\ &\quad + \text{div} \left[c_a \left[\tilde{\mathbf{q}}_a - \sum_c \left(\bar{\mathbf{q}}_c + \rho_c \left(\epsilon_c \mathbf{I} - \frac{\tilde{\mathbf{T}}_c^T}{\rho_c} + \frac{u_c^2}{2} \mathbf{I} \right) \tilde{\mathbf{u}}_c \right) \right] \right].\end{aligned}$$

Two results that are immediately apparent from equations (2.52)₁ and (2.52)₄ are that

$$\tilde{\mathbf{T}}_I = \sum_a \tilde{\mathbf{T}}_a \tag{2.53}$$

and

$$\tilde{\mathbf{q}}_a = \sum_c \left[\bar{\mathbf{q}}_c + \rho_c \left(\epsilon_c \mathbf{I} - \frac{\tilde{\mathbf{T}}_c^T}{\rho_c} + \frac{u_c^2}{2} \mathbf{I} \right) \tilde{\mathbf{u}}_c \right], \tag{2.54}$$

where equation (2.14) has been used. The quantity $\tilde{\mathbf{T}}_a^T$ can be used in equation (2.54) instead of $\tilde{\mathbf{T}}_a^T$ as a consequence

of the fact that

$$\sum_{\mathbf{a}} \mathbf{T}_{\mathbf{a}}^T \mathbf{u}_{\mathbf{a}} = \sum_{\mathbf{a}} \bar{\mathbf{T}}_{\mathbf{a}}^T \mathbf{u}_{\mathbf{a}}, \quad (2.55)$$

which follows from equations (2.17) and (2.52)₁. Equations (2.35)₃ and (2.52) can be used to show that

$$\operatorname{div} \mathbf{T}_{\mathbf{a}} + \hat{\mathbf{p}}_{\mathbf{a}} = \operatorname{div} \bar{\mathbf{T}}_{\mathbf{a}} + \hat{\bar{\mathbf{p}}}_{\mathbf{a}}, \quad (2.56)$$

$$\mathbf{T}_{\mathbf{a}} - \mathbf{T}_{\mathbf{a}}^T - \hat{\mathbf{M}}_{\mathbf{a}} = \bar{\mathbf{T}}_{\mathbf{a}} - \bar{\mathbf{T}}_{\mathbf{a}}^T - \hat{\bar{\mathbf{M}}}_{\mathbf{a}} \quad (2.57)$$

and

$$\operatorname{tr}(\mathbf{T}_{\mathbf{a}}^T \mathbf{L}) - \operatorname{div} \mathbf{q}_{\mathbf{a}} + \hat{\epsilon}_{\mathbf{a}} = \operatorname{tr}(\bar{\mathbf{T}}_{\mathbf{a}}^T \mathbf{L}) - \operatorname{div} \bar{\mathbf{q}}_{\mathbf{a}} + \hat{\bar{\epsilon}}_{\mathbf{a}}. \quad (2.58)$$

These last three equations imply that the equations of balance for the constituents, equations (2.35), can be rewritten as

$$\begin{aligned} \frac{\partial \rho_{\mathbf{a}}}{\partial t} + \operatorname{div} \rho_{\mathbf{a}} \dot{\mathbf{x}}_{\mathbf{a}} &= \hat{\mathbf{c}}_{\mathbf{a}}, \\ \rho_{\mathbf{a}} \ddot{\mathbf{x}}_{\mathbf{a}} &= \operatorname{div} \mathbf{T}_{\mathbf{a}} + \rho_{\mathbf{a}} \mathbf{b}_{\mathbf{a}} + \hat{\mathbf{p}}_{\mathbf{a}}, \\ \mathbf{T}_{\mathbf{a}} - \mathbf{T}_{\mathbf{a}}^T &= \hat{\mathbf{M}}_{\mathbf{a}} \end{aligned} \quad (2.59)$$

and

$$\rho_{\mathbf{a}} \dot{\epsilon}_{\mathbf{a}} = \operatorname{tr}(\mathbf{T}_{\mathbf{a}}^T \mathbf{L}) - \operatorname{div} \mathbf{q}_{\mathbf{a}} + \rho_{\mathbf{a}} r_{\mathbf{a}} + \hat{\epsilon}_{\mathbf{a}}.$$

These equations are equivalent to those proposed by Truesdell & Toupin and Kelly, with the exception that Truesdell

& Toupin assume $\hat{\underline{M}} = 0$.¹⁴

Equations (2.53), (2.54), and (2.59) are formally identical to equations (2.50), (2.51), and (2.35), respectively. If equation (2.52)₂ is expanded and equation (2.47) is used, it follows that

$$\begin{aligned} \hat{\underline{p}} = \hat{\underline{p}} - c \sum_{\underline{e}} (\hat{\underline{p}}_{\underline{e}} + \hat{c}u) \\ + \frac{1}{\rho} \left(\underline{T}_I - \sum_{\underline{e}} \bar{\underline{T}} \right) \left(\text{grad } \underline{\rho} - c \sum_{\underline{e}} \text{grad } \underline{\rho} \right). \end{aligned} \quad (2.60)$$

Equation (2.60) suggests that in a theory with partial stresses and partial heat fluxes any constitutive equation for the diffusive force, and thus all the constitutive equations, must be allowed to depend on density gradients. Essentially, what has been shown is that the original balance equations for the constituents, equations (2.35), appended with equations (2.50) and (2.51), are valid as long as the constitutive equations are allowed to depend on density gradients.

Summation of equation (2.52)₂ over the constituents yields

¹⁴Truesdell and Toupin, op. cit., pp. 472, 567-568, 613; Kelly, op. cit., pp. 138-143.

$$\sum_{\mathbf{a}} \hat{\mathbf{p}}_{\mathbf{a}} = \sum_{\mathbf{a}} \bar{\hat{\mathbf{p}}}_{\mathbf{a}} - \text{div}(\mathbf{T}_{\mathbf{I}} - \sum_{\mathbf{a}} \bar{\mathbf{T}}_{\mathbf{a}}) \quad (2.61)$$

where equation (2.14) has been used. Also, it follows from equations (2.17), (2.19), (2.46)₃, (2.52)₂, and (2.53) that

$$\begin{aligned} \text{tr} \sum_{\mathbf{a}} \bar{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}} \bar{\mathbf{L}}_{\mathbf{a}} - \sum_{\mathbf{a}} \mathbf{u}_{\mathbf{a}} \cdot \hat{\mathbf{p}}_{\mathbf{a}} &= \text{tr}(\mathbf{T}_{\mathbf{I}} \mathbf{L}) + \text{tr} \sum_{\mathbf{a}} \bar{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}} \text{grad} \mathbf{u}_{\mathbf{a}} \\ &\quad - \sum_{\mathbf{a}} \mathbf{u}_{\mathbf{a}} \cdot \bar{\hat{\mathbf{p}}}_{\mathbf{a}}. \end{aligned} \quad (2.62)$$

Therefore, the balance equations for the mixture, equations (2.46)₁, (2.47), (2.46)₃, and (2.46)₄, become, respectively,

$$\begin{aligned} \sum_{\mathbf{a}} \hat{\mathbf{c}}_{\mathbf{a}} &= 0, \\ \sum_{\mathbf{a}} (\hat{\mathbf{p}}_{\mathbf{a}} + \hat{\mathbf{c}}_{\mathbf{a}} \mathbf{u}_{\mathbf{a}}) &= 0, \\ \sum_{\mathbf{a}} (\mathbf{T}_{\mathbf{a}} - \mathbf{T}_{\mathbf{a}}^{\mathbf{T}}) &= 0, \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} \rho \dot{\mathbf{e}}_{\mathbf{I}} &= \text{tr} \sum_{\mathbf{a}} \bar{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}} \bar{\mathbf{L}}_{\mathbf{a}} - \sum_{\mathbf{a}} \left(\mathbf{u}_{\mathbf{a}} \cdot \hat{\mathbf{p}}_{\mathbf{a}} + \hat{\mathbf{c}}_{\mathbf{a}} \frac{\mathbf{u}_{\mathbf{a}}^2}{2} \right) \\ &\quad - \text{div} \left[\mathbf{q} - \sum_{\mathbf{a}} \rho_{\mathbf{a}} \left(- \frac{\bar{\mathbf{T}}_{\mathbf{a}}^{\mathbf{T}}}{\rho} + \frac{\mathbf{u}_{\mathbf{a}}^2}{2} \mathbf{I} \right) \mathbf{u}_{\mathbf{a}} \right] + \rho r, \end{aligned}$$

where equation (2.53) has been used. Equations (2.63) are equivalent to the equations of balance for the mixture used by Truesdell & Toupin, Kelly, Eringen & Ingram, Green

& Naghdi, Bowen, and Müller.¹⁵ However, to get Green & Naghdi's energy equation it should be noticed that their heat flux is the entire term in the brackets in equation (2.63)₄.

From equation (2.52)₃ it is obvious that

$$\sum_{\mathbf{a}} \hat{\tilde{\mathbf{M}}} = 0, \quad (2.64)$$

where equation (2.14) has been used. Equation (2.64) is actually the condition which makes equations (2.59)₃ and (2.63)₃ compatible. $\hat{\tilde{\mathbf{M}}}$ appears only in equations (2.59)₃ and (2.64); therefore, equation (2.59)₃ can be used to determine $\hat{\tilde{\mathbf{M}}}$ and equation (2.64) will be satisfied whenever equation (2.63)₃ is satisfied. By use of equations (2.14), (2.52)₂, and (2.52)₅ the compatibility equation for energy, equation (2.48) reduces to

$$\sum_{\mathbf{a}} \left[\hat{\tilde{\epsilon}} + \tilde{\mathbf{u}} \cdot \hat{\tilde{\mathbf{p}}} + \hat{\tilde{\epsilon}} \left(\epsilon + \frac{\mathbf{u}^2}{2} \right) \right] = 0. \quad (2.65)$$

Notice that the left sides of equations (2.63)₂ and (2.65) are formally the same as those of equations (2.47) and (2.48), respectively, but the right sides of the former equations

¹⁵Truesdell and Toupin, op. cit., pp. 472, 568, 614; Kelly, op. cit., pp. 139-143; Eringen and Ingram, op. cit., 203-207; Green and Naghdi, International Journal of Engineering Science, III, 234-238; Bowen, op. cit., p. 379; Müller, op. cit., pp. 4-6.

vanish. Equation (2.65), like equation (2.63)₂ is identical to the result of Truesdell and Toupin.¹⁶

To summarize the results of this chapter the equations that must be satisfied at this point in the development are equations (2.59) and (2.63). Also, equation (2.65) can be used instead of equation (2.63)₄.

¹⁶Truesdell and Toupin, op. cit., p. 614.

3. THE SECOND AXIOM OF THERMODYNAMICS

The second axiom of thermodynamics is an inequality, called the entropy inequality. Before this inequality is stated the following definition is made.

Definition. If η_a denotes the partial entropy density of B , the entropy density for B_t at $\underline{x} \in V(B_t)$ and at the time t is

$$\eta = \frac{1}{\rho} \sum_a \rho_a \eta_a. \quad (3.1)$$

The second axiom of thermodynamics for the part P_t of the mixture B_t which occupies the fixed region R at the time t is postulated to be the inequality

$$\begin{aligned} \frac{\partial}{\partial t} \int_R \rho \eta dv \geq & - \oint_{\partial R} \rho \eta \underline{\tilde{x}} \cdot d\underline{s} - \oint_{\partial R} \sum_a \frac{\underline{h}_a}{\theta} \cdot d\underline{s} \\ & + \int_R \sum_a \frac{\rho_a \underline{r}_a}{\theta} dv, \end{aligned} \quad (3.2)$$

where $\underline{h}_a$ is an influx vector for the a^{th} constituent, not as yet related to $\underline{q}_a$, and $\theta = \theta(\underline{x}, t)$ is the temperature of the a^{th} constituent and is assumed to be positive.

The local statement of the entropy inequality is

$$\rho \dot{\eta} + \operatorname{div} \sum_a \frac{\underline{h}_a}{\theta} - \sum_a \frac{\rho_a \underline{r}_a}{\theta} \geq 0. \quad (3.3)$$

This equation is a generalization of Müller's entropy inequality to multiple temperatures.¹ If equation (2.36) is used with $\psi = \eta$, the inequality (3.3) takes the form

$$\sum_{\alpha} \left[\rho_{\alpha} \dot{\eta}_{\alpha} + \operatorname{div} \left(\frac{\mathbf{h}_{\alpha}}{\theta_{\alpha}} - \rho_{\alpha} \eta_{\alpha} \mathbf{u}_{\alpha} \right) - \frac{\rho_{\alpha} r_{\alpha}}{\theta_{\alpha}} + \hat{c}_{\alpha} \eta_{\alpha} \right] \geq 0. \quad (3.4)$$

If equation (2.59)₄ is used to eliminate the term $\rho_{\alpha} r_{\alpha}$, the result is

$$\begin{aligned} \sum_{\alpha} \frac{1}{\theta_{\alpha}} \left[\rho_{\alpha} (\theta_{\alpha} \dot{\eta}_{\alpha} - \dot{\epsilon}_{\alpha}) + \operatorname{tr} (\mathbf{T}_{\alpha}^T \mathbf{L}_{\alpha}) - \frac{\mathbf{q}_{\alpha}}{\theta_{\alpha}} \cdot \mathbf{h}_{\alpha} + \operatorname{div} (\mathbf{h}_{\alpha} - \mathbf{q}_{\alpha}) + \hat{e}_{\alpha} \right. \\ \left. - \mathbf{u}_{\alpha} \cdot \hat{\mathbf{p}}_{\alpha} - \hat{c}_{\alpha} \left(\epsilon_{\alpha} - \theta_{\alpha} \eta_{\alpha} + \frac{u_{\alpha}^2}{2} \right) - \theta_{\alpha} \operatorname{div} (\rho_{\alpha} \eta_{\alpha} \mathbf{u}_{\alpha}) \right] \geq 0, \end{aligned} \quad (3.5)$$

where $\mathbf{q}_{\alpha} = \operatorname{grad} \theta_{\alpha}$ and $\hat{e}_{\alpha} = \hat{\epsilon}_{\alpha} + \mathbf{u}_{\alpha} \cdot \hat{\mathbf{p}}_{\alpha} + \hat{c}_{\alpha} \left(\epsilon_{\alpha} + \frac{u_{\alpha}^2}{2} \right)$.

At this point the assumption will be made that

$$\mathbf{h}_{\alpha} = \mathbf{q}_{\alpha} + \rho_{\alpha} \theta_{\alpha} \eta_{\alpha} \mathbf{u}_{\alpha}, \quad (3.6)$$

so that the inequality (3.2) could have been postulated as

$$\begin{aligned} \frac{\partial}{\partial t} \int_R \rho_{\alpha} \eta_{\alpha} dv \geq - \oint_{\partial R} \sum_{\alpha} \rho_{\alpha} \eta_{\alpha} \dot{\mathbf{x}}_{\alpha} \cdot d\mathbf{s} - \oint_{\partial R} \sum_{\alpha} \frac{\mathbf{q}_{\alpha}}{\theta_{\alpha}} \cdot d\mathbf{s} \\ + \int_R \sum_{\alpha} \frac{\rho_{\alpha} r_{\alpha}}{\theta_{\alpha}} dv. \end{aligned} \quad (3.7)$$

Inequality (3.7) is the simplest generalization of the

¹Müller, op. cit., p. 7.

entropy inequality for single materials, and it turns out to have the same form as Bowen's when all the constituents have the same temperature. From inequality (3.3) and equation (3.6), or from inequality (3.7), it follows that

$$\rho \dot{\eta} + \operatorname{div} \sum_{\alpha} \left(\frac{\underline{q}}{\theta} + \rho \eta \underline{u} \right) - \sum_{\alpha} \frac{\rho r}{\theta} \geq 0. \quad (3.8)$$

And the inequality (3.5) becomes

$$\begin{aligned} \sum_{\alpha} \frac{1}{\theta} \left[\rho (\theta \dot{\eta} - \dot{\epsilon}) + \operatorname{tr}(\underline{T}^T \underline{L}) - \frac{\underline{q}}{\theta} \cdot \underline{q} + \hat{\epsilon} - \underline{u} \cdot \hat{\underline{p}} \right. \\ \left. - \hat{c} \left(\epsilon - \theta \eta + \frac{u^2}{2} \right) \right] \geq 0. \end{aligned} \quad (3.9)$$

Inequality (3.9) is the local statement of the second law of thermodynamics for a theory of mixtures in which each constituent has its own temperature. The constituents will now be constrained to all have the same temperature

$$\theta = \theta(\underline{x}, t).$$

Definition. The partial free energy density at $\underline{x} \in B$ is

$$\underline{\psi} = \underline{\epsilon} - \theta \underline{\eta}. \quad (3.10)$$

Definition. The inner part of the free energy density for B_t at $\underline{x} \in V(B_t)$ and at the time t is

$$\psi_I = \frac{1}{\rho} \sum_{\alpha} \rho \psi. \quad (3.11)$$

Equation (3.11) implies that

$$\psi_I = \epsilon_I - \theta \eta, \quad (3.12)$$

where equations (2.30), (3.1), and (3.10) have been used.

Inequality (3.8) now becomes

$$\begin{aligned} \rho \theta \dot{\eta} + \operatorname{div} \left[\underline{q} - \sum_{\alpha} \rho \left(\psi_I - \frac{\underline{T}^T}{\rho} + \frac{u^2}{2} \underline{I} \right) \underline{u} \right] \\ - \frac{g}{\theta} \cdot \left[\underline{q} - \sum_{\alpha} \rho \left(\psi_I - \frac{\underline{T}^T}{\rho} + \frac{u^2}{2} \underline{I} \right) \underline{u} \right] - \rho r \geq 0 \end{aligned} \quad (3.13)$$

after using equations (2.29), (2.54), and (3.10). Inequality (3.13) is identical to Bowen's inequality (3.5), if the definition is made that

$$\underline{K} = \psi_I - \frac{\underline{T}^T}{\rho}. \quad (3.14)$$

This equation will be used in chapter 5 to show that Bowen's results follow as a special case of the theory presented here. Notice that equation (3.14) implies that

$$\sum_{\alpha} c_{\alpha} \underline{K} = \psi_I - \frac{\underline{T}^T}{\rho}, \quad (3.15)$$

where equations (2.13), (2.53), and (3.11) have been used.

The inequality (3.13) is different from Green & Naghdi's entropy inequality. Their inequality is obtained if the

²Bowen, op. cit., p. 381.

quantities $\tilde{h}_a$, $a = 1, \dots, n$, are chosen such that

$$\sum_a \tilde{h}_a = \tilde{q} - \sum_a \rho \left(-\frac{\tilde{T}^T}{\rho} + \frac{u^2}{2} \tilde{I} \right) u_a^3, \quad (3.16)$$

where the comment made following equation (2.63) should be recalled. By use of equation (2.54) it is apparent that

$$\tilde{h}_a = \tilde{q}_a + \rho \epsilon u_a \quad (3.17)$$

is the most obvious choice for $\tilde{h}_a$, though not the only choice, that satisfies equation (3.16). Defining $\tilde{h}_a$ this way is equivalent to deleting the quantity $\sum_a \rho \psi u_a$ from both terms in which it appears in the inequality (3.13). It is the absence of $\sum_a \rho \psi u_a$ from their entropy inequality that causes Green & Naghdi to obtain their unusual partial pressures, as will be demonstrated in chapter 5.

By use of equations (2.63)₄ and (3.12) the inequality (3.13) can be rewritten as

$$\begin{aligned} & -\rho (\dot{\psi}_I + \eta \dot{\theta}) + \text{tr} \sum_a \tilde{T}_a^T \tilde{L}_a - \sum_a \left(\tilde{u}_a \cdot \hat{\tilde{p}}_a + \hat{\tilde{c}}_a \frac{u_a^2}{2} \right) \\ & - \frac{\tilde{q}}{\theta} \cdot \left[\tilde{q} - \sum_a \rho \left(\tilde{\psi}_I - \frac{\tilde{T}^T}{\rho} + \frac{u^2}{2} \tilde{I} \right) u_a \right] \\ & - \text{div} \sum_a \rho \psi u_a \geq 0. \end{aligned} \quad (3.18)$$

³Green and Naghdi, International Journal of Engineering Science, III, 237.

This inequality can be further rearranged to become

$$\begin{aligned}
 - \frac{\dot{(\rho \psi_I)}}{(\rho \psi_I)} - \rho \eta \dot{\theta} - \text{tr} \sum_{\mathbf{a}} \rho_{\mathbf{a}} \mathbf{K} \mathbf{L} - \sum_{\mathbf{a}} \mathbf{u}_{\mathbf{a}} \cdot [\hat{\mathbf{p}}_{\mathbf{a}} + \text{grad}(\rho \psi)] \\
 - \sum_{\mathbf{a}} \hat{c}_{\mathbf{a}} \frac{u_{\mathbf{a}}^2}{2} - \frac{g}{\theta} \cdot \left[\mathbf{q} - \sum_{\mathbf{a}} \rho_{\mathbf{a}} \left(\mathbf{K}_{\mathbf{a}} + \frac{u_{\mathbf{a}}^2}{2} \mathbf{I} \right) \mathbf{u}_{\mathbf{a}} \right] \geq 0,
 \end{aligned} \tag{3.19}$$

where equation (3.14) and the identity

$$\begin{aligned}
 \rho \dot{\psi}_I + \text{div} \sum_{\mathbf{a}} \rho_{\mathbf{a}} \psi_{\mathbf{a}} \mathbf{u}_{\mathbf{a}} &= \frac{\dot{(\rho \psi_I)}}{(\rho \psi_I)} + \sum_{\mathbf{a}} \rho_{\mathbf{a}} \psi_{\mathbf{a}} \text{tr} \mathbf{L}_{\mathbf{a}} \\
 &+ \sum_{\mathbf{a}} \mathbf{u}_{\mathbf{a}} \cdot \text{grad}(\rho \psi)
 \end{aligned} \tag{3.20}$$

have been used. Equation (3.20) follows from equations (2.19), (2.22)₁ with $\psi = \rho$, and (2.34)₁. Inequality (3.19) is the entropy inequality for a theory of mixtures having a single temperature distribution $\theta(\mathbf{x}, t)$, and it is this form that will be used in the remaining chapters of this paper.

4. MIXTURES OF ELASTIC MATERIALS WITH DIFFUSION AND HEAT CONDUCTION

In this chapter a mixture of elastic materials in which diffusion and heat conduction can take place is considered. This case has been chosen for two reasons, first, it can be specialized to the theory of single elastic materials with heat conduction¹ and to the theory of mixtures of fluids with diffusion and heat conduction,² and second, this is the case that was studied by Bowen, which means there are some results available for direct comparison.³

Before a thermodynamic process is defined and constitutive equations are postulated, an important point needs to be made. Because the theory has been restricted to the case in which each constituent has the same temperature, it is not essential to use equations (2.59)₄ directly, and thus, constitutive equations are not needed for $\hat{\epsilon}$ and

¹C. Truesdell and W. Noll, The Non-Linear Field Theories of Mechanics (Vol. III/3 of Handbuch der Physik, ed. S. Flugge. 54 vols.; Berlin-Heidelberg-New York: Springer, 1965), pp. 294-298.

²S. R. de Groot and P. Mazur, Non-Equilibrium Thermodynamics (Amsterdam: North Holland Company, 1962), pp. 235-264.

³Bowen, op. cit., p. 385.

q. The reason for this is that the constraint of having a single temperature distribution causes $\hat{\epsilon}_a$ and $\hat{q}_a$ to become indeterminate as far as the entropy inequality is concerned. (Compare inequalities (3.9) and (3.19).) This means that $\hat{\epsilon}_a$ and $\hat{q}_a$ have to satisfy equations (2.59)₄ and (2.65) only. Therefore, $\hat{\epsilon}_a$ and $\hat{q}_a$ can be formally eliminated from the theory by allowing these quantities to take on any value necessary to satisfy equation (2.59)₄. The only restriction on these values is that equation (2.65) must be satisfied. Henceforth, equation (2.65) will be written in the form of equation (2.63)₄. A thermodynamic process can now be defined without reference to $\hat{\epsilon}_a$, $\hat{q}_a$, and, in addition, r .

Definition. A thermodynamic process for the mixture is a set of $9n + 3$ functions whose values are

$$\tilde{x}_a = \tilde{x}_{a\tilde{x}}(X, t), \quad \tilde{\rho}_a = \tilde{\rho}_a(X, t), \quad \tilde{T} = \tilde{T}(X, t),$$

$$\tilde{b}_a = \tilde{b}_a(X, t), \quad \tilde{\epsilon}_a = \tilde{\epsilon}_a(X, t), \quad \tilde{\eta} = \tilde{\eta}(X, t),$$

$$\hat{\tilde{c}}_a = \hat{\tilde{c}}_a(X, t), \quad \hat{\tilde{p}}_a = \hat{\tilde{p}}_a(X, t), \quad \hat{\tilde{M}} = \hat{\tilde{M}}(X, t),$$

for $a = 1, \dots, n$, and

$$\theta = \theta(\tilde{x}, t), \quad \tilde{q} = \tilde{q}(\tilde{x}, t), \quad r = r(\tilde{x}, t),$$

which satisfy equations (2.59)₁, (2.59)₂, and (2.63)₄.

This definition is a generalization of the definition

given by Coleman & Noll for single materials⁴ and follows very closely the one used by Bowen for mixtures.⁵

The equations of balance suggest that constitutive equations are needed for ϵ_a , η_a , T_a , $\hat{c}_a$, $\hat{p}_a$, $\hat{M}_a$, for $a = 1, \dots, n$, and for q . For a mixture of elastic materials with diffusion and heat conduction, it is assumed that $\hat{c}_a = 0$, $a = 1, \dots, n$, and the remaining quantities are functions of θ , g , $\frac{F}{I}$, $\dots$, $\frac{F}{n}$, $\frac{\nabla F}{I}$, $\dots$, $\frac{\nabla F}{n}$, $\frac{\dot{x}}{I}$, $\dots$, $\frac{\dot{x}}{n}$. That is,

$$\begin{aligned}\epsilon_a &= \epsilon_a(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}), \\ \eta_a &= \eta_a(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}), \\ T_a &= T_a(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}), \\ \hat{c}_a &= 0, \\ \hat{p}_a &= \hat{p}_a(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}), \\ \hat{M}_a &= \hat{M}_a(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}),\end{aligned}\tag{4.1}$$

and

$$q = q(\theta, g, \frac{F}{b}, \frac{\nabla F}{b}, \frac{\dot{x}}{b}),$$

where the subscript b under the last three arguments indicates a dependence on all n of each of these arguments.

⁴Coleman and Noll, op. cit., p. 169.

⁵Bowen, loc. cit.

The response functions in equations (4.1) are assumed to be of class C^1 in each of their arguments. The statement that $\hat{c} = 0$, $a = 1$, ---, n , means that there are no chemical reactions; therefore, equation $(2.63)_1$ is identically satisfied, and equation $(2.59)_1$ can be replaced by equation (2.40), as was shown in chapter 2. And because of equation (2.40) a dependence on density gradients is assured in equations (4.1) through the dependence on second deformation gradients. Equations (4.1), in general, will depend on all of the reference configurations κ_1 , ---, κ_n , but it is assumed that they do not depend on the positions X_1 , ---, X_n . Also ρ_0 is assumed not to depend on $\tilde{x}$ and is, therefore, a known constant.

Equations (3.10) , $(4.1)_1$, and $(4.1)_2$ imply that

$$\psi = \psi(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \dot{\underline{x}}). \quad (4.2)$$

By use of equations (2.30) , (3.1) , (3.11) , $(4.1)_1$, $(4.1)_2$, and (4.2) it follows that

$$\begin{aligned} \epsilon_I &= \epsilon_I(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \dot{\underline{x}}), \\ \eta &= \eta(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \dot{\underline{x}}), \end{aligned} \quad (4.3)$$

and

$$\psi_I = \psi_I(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \dot{\underline{x}}).$$

The response functions in equations (4.2) and (4.3) will be C^1 functions of their arguments also.

Definition. An admissible thermodynamic process for the mixture is one that is consistent with the constitutive equations (4.1).

Theorem. For every choice of the n deformation functions $\chi_{\tilde{a}}$, $a = 1, \dots, n$, and the temperature distribution Θ , there exists a unique admissible thermodynamic process.

Proof. F , ∇F , and $\dot{\chi}$ can be calculated from $\chi_{\tilde{a}}$, $a = 1, \dots, n$; $\underline{g}$ can be found from Θ . Therefore ϵ , η , $\underline{T}$, $\hat{\underline{p}}$, $\hat{\underline{M}}$, for $a = 1, \dots, n$, and $\underline{q}$ can be determined from the constitutive equations (4.1); and $\hat{\underline{c}} = 0$, $a = 1, \dots, n$. Finally, ρ , $\underline{b}$, for $a = 1, \dots, n$, and r can be computed by use of equations (2.40), (2.59)₂, and (2.63)₄, respectively. Thus, an admissible thermodynamic process exists, and from the procedure used, it is obviously unique.

This theorem was first proved for single materials by Coleman & Noll⁶ and was later generalized to mixtures by Bowen.⁷ No use has yet been made of equations (2.59)₃, (2.63)₂, and (2.63)₃, or of the entropy inequality (3.19).

⁶Coleman and Noll, op. cit., p. 171.

⁷Bowen, op. cit., p. 387.

They can be regarded either as restrictions on the constitutive equations or as restrictions on the independent variables χ_a , $a = 1, \dots, n$, and θ . The former point of view will be adopted here, thus generalizing the logic of Coleman & Noll.⁸ Hence, it will now be required that equations $(2.59)_3$, $(2.63)_2$, and $(2.63)_3$, and inequality (3.19) be satisfied for every admissible thermodynamic process. With the exception of inequality (3.19) all of these equations are easily satisfied by including the restrictions implied by equations $(2.59)_3$, $(2.63)_2$, and $(2.63)_3$ as part of the defining constitutive assumptions. It will be assumed in the following chapters that this has been done. The only restrictions of consequence then come from the entropy inequality (3.19).

⁸Coleman and Noll, op. cit., p. 174.

5. RESTRICTIONS IMPOSED BY THE SECOND AXIOM OF THERMODYNAMICS

The purpose of this chapter is to derive the restrictions which must be placed on the constitutive equations (4.1) so that the entropy inequality (3.19) is satisfied. Equations (4.2) and (4.3)₃ and the assumed smoothness of $\chi_{\tilde{\mathbf{x}}}, \psi$, for $a = 1, \dots, n$, and Θ imply that

$$\begin{aligned} \frac{\dot{\rho\psi}}{\rho\psi} = & \frac{\partial \rho\psi}{\partial \theta} \dot{\theta} + \frac{\partial \rho\psi}{\partial \underline{g}} \cdot \dot{\underline{g}} + \text{tr} \sum_{\tilde{\mathbf{e}}} \frac{\partial \rho\psi}{\partial \underline{F}_{\tilde{\mathbf{e}}}}^T \dot{\underline{F}}_{\tilde{\mathbf{e}}} \\ & + c \sum_{\tilde{\mathbf{e}}} \frac{\partial \rho\psi}{\partial \nabla \underline{F}_{\tilde{\mathbf{e}}}} \dot{\nabla \underline{F}}_{\tilde{\mathbf{e}}} + \sum_{\tilde{\mathbf{x}}} \frac{\partial \rho\psi}{\partial \dot{\underline{x}}_{\tilde{\mathbf{x}}}} \cdot \dot{\underline{x}}_{\tilde{\mathbf{x}}} \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} \text{grad}(\rho\psi) = & \frac{\partial \rho\psi}{\partial \theta} \underline{g} + (\text{grad } \underline{g}) \frac{\partial \rho\psi}{\partial \underline{g}} + \sum_{\tilde{\mathbf{e}}} \underline{F}_{\tilde{\mathbf{e}}}^{-1T} \left(\frac{\partial \rho\psi}{\partial \underline{F}_{\tilde{\mathbf{e}}}} [\nabla \underline{F}_{\tilde{\mathbf{e}}}] \right) \\ & + \sum_{\tilde{\mathbf{e}}} \underline{F}_{\tilde{\mathbf{e}}}^{-1T} \left(\frac{\partial \rho\psi}{\partial \nabla \underline{F}_{\tilde{\mathbf{e}}}} [\nabla \nabla \underline{F}_{\tilde{\mathbf{e}}}] \right) + \sum_{\tilde{\mathbf{x}}} \underline{F}_{\tilde{\mathbf{x}}}^{-1T} \frac{\partial \rho\psi}{\partial \dot{\underline{x}}_{\tilde{\mathbf{x}}}} \end{aligned} \quad (5.2)$$

where the notation has been used that

$$\frac{\partial \rho\psi}{\partial \underline{F}_{\tilde{\mathbf{e}}}} [\nabla \underline{F}_{\tilde{\mathbf{e}}}] = \frac{\partial \rho\psi}{\partial x_{1,I}} x_{1,IJ} \delta_J$$

and

$$\frac{\partial \rho\psi}{\partial \nabla \underline{F}_{\tilde{\mathbf{e}}}} [\nabla \nabla \underline{F}_{\tilde{\mathbf{e}}}] = \frac{\partial \rho\psi}{\partial x_{1,IJ}} x_{1,IJK} \delta_K$$

If equations (5.1) and (5.2) are substituted into inequality (3.19), it follows that

$$\begin{aligned}
& - \left(\frac{\partial \rho \psi}{\partial \theta} + \rho \eta \right) \dot{\theta} - \frac{\partial \rho \psi}{\partial \underline{g}} \cdot \dot{\underline{g}} - c \sum_{\underline{a}} \frac{\partial \rho \psi}{\partial \nabla \underline{F}} \frac{\dot{\underline{a}}}{\nabla \underline{F}} - \sum_{\underline{a}} \frac{\partial \rho \psi}{\partial \dot{\underline{x}}} \cdot \frac{\dot{\underline{x}}}{\underline{a}} \\
& - \text{tr} \sum_{\underline{a}} \left[\rho_{\underline{a} \underline{a} \underline{a}}^{KL} + \frac{\partial \rho \psi}{\partial \underline{F}} \frac{\dot{\underline{F}}}{\underline{a}} \right] - \frac{g}{\theta} \cdot \left[\underline{g} - \sum_{\underline{a}} \rho \left(\underline{K} + \frac{\underline{u}^2}{2} \underline{I} \right) \underline{u} \right] \\
& - \sum_{\underline{a}} \underline{u} \cdot \left[\hat{\underline{p}} + \frac{\partial \rho \psi}{\partial \theta} \underline{g} + (\text{grad } \underline{g}) \frac{\partial \rho \psi}{\partial \underline{g}} + \sum_{\underline{e}} \underline{F}^{-1} \left(\frac{\partial \rho \psi}{\partial \underline{F}} [\nabla \underline{F}] \right) \right. \\
& \left. + \sum_{\underline{e}} \underline{F}^{-1} \left(\frac{\partial \rho \psi}{\partial \nabla \underline{F}} [\nabla \nabla \underline{F}] \right) + \sum_{\underline{e}} \underline{F}^{-1} \underline{F}^T \frac{\partial \rho \psi}{\partial \dot{\underline{x}}} \right] \geq 0, \quad (5.4)
\end{aligned}$$

where equation (4.1)₄ has been used. By use of equation (2.11) and the identity

$$\dot{\underline{F}} = \underline{\dot{F}} - (\nabla \underline{F}) \underline{F}^{-1} \underline{u}, \quad (5.5)$$

which follows from equation (2.23) with $\psi = \underline{F}$, it can be shown that

$$\begin{aligned}
\text{tr} \sum_{\underline{a}} \left[\rho_{\underline{a} \underline{a} \underline{a}}^{KL} + \frac{\partial \rho \psi}{\partial \underline{F}} \frac{\dot{\underline{F}}}{\underline{a}} \right] &= \text{tr} \sum_{\underline{a}} \underline{F}^{-1} \left(\rho_{\underline{a} \underline{a}}^K + \underline{F} \frac{\partial \rho \psi}{\partial \underline{F}} \right) \underline{\dot{F}} \\
&= \text{tr} \sum_{\underline{a}} \frac{\partial \rho \psi}{\partial \underline{F}} (\nabla \underline{F}) \underline{F}^{-1} \underline{u}. \quad (5.6)
\end{aligned}$$

Furthermore, it can be seen by simply expanding the terms that the following identities hold:

$$\text{tr} \sum_{\underline{a}} \frac{\partial \rho \psi}{\partial \underline{F}} (\nabla \underline{F}) \underline{F}^{-1} \underline{u} = \sum_{\underline{a}} \underline{u} \cdot \underline{F}^{-1} \left(\frac{\partial \rho \psi}{\partial \underline{F}} [\nabla \underline{F}] \right)$$

and

(5.7)

$$\sum_a \underline{u} \cdot \sum_c \underline{F}^{-1} \underline{F}^T \frac{\partial \rho \underline{\psi}}{\partial \dot{\underline{x}}_c} = \text{tr} \sum_a \underline{F}^{-1} \sum_c \left(\underline{u} \otimes \frac{\partial \rho \underline{\psi}}{\partial \dot{\underline{x}}_c} \right) \underline{F}.$$

The meaning of the term on the right side of equation (5.7)₁ should be clear from examination of equation (5.3)₁. If equations (5.6) and (5.7)₁ are combined, and if the result, along with equation (5.7)₂, is substituted into inequality (5.4), it follows that

$$\begin{aligned} & - \rho \left(\frac{\partial \psi}{\partial \theta} \underline{I} + \eta \right) \dot{\theta} - \rho \frac{\partial \psi}{\partial \underline{g}} \cdot \dot{\underline{g}} - \rho C \sum_a \frac{\partial \psi}{\partial \nabla \underline{F}} \otimes \frac{\dot{\underline{F}}}{\underline{F}} - \rho \sum_a \frac{\partial \psi}{\partial \dot{\underline{x}}_a} \cdot \dot{\underline{x}}_a \\ & - \text{tr} \sum_a \underline{F}^{-1} \left(\rho \underline{K} + \underline{F} \frac{\partial \rho \underline{\psi}}{\partial \underline{F}} + \sum_c \rho \underline{u} \otimes \frac{\partial \rho \underline{\psi}}{\partial \dot{\underline{x}}_c} \right) \underline{F} \\ & - \frac{\underline{g}}{\theta} \cdot \left[\underline{q} - \sum_a \rho \left(\underline{K} + \frac{\underline{u}^2}{2} \underline{I} - \theta \frac{\partial \psi}{\partial \theta} \underline{I} \right) \underline{u} \right] \\ & - \sum_a \underline{u} \cdot \left[\hat{\underline{p}} - \underline{F}^{-1} \left(\frac{\partial \rho \underline{\psi}}{\partial \underline{F}} [\nabla \underline{F}] \right) + \sum_c \underline{F}^{-1} \left(\frac{\partial \rho \underline{\psi}}{\partial \underline{F}} [\nabla \underline{F}] \right) \right] \\ & - \sum_a \underline{u} \cdot \left[(\text{grad } \underline{g}) \frac{\partial \rho \underline{\psi}}{\partial \underline{g}} + \sum_c \underline{F}^{-1} \left(\frac{\partial \rho \underline{\psi}}{\partial \nabla \underline{F}} [\nabla \underline{F}] \right) \right] \geq 0, \end{aligned} \quad (5.8)$$

where the fact that ρ is a function of $\underline{F}$ only has been used.

By application of a standard argument developed by Coleman and Noll¹ to the inequality (5.8), it can be seen immediately that for $a = 1, \dots, n$,

¹Coleman and Noll, op. cit., pp. 174-176.

$$\begin{aligned}
\eta &= - \frac{\partial \psi_I}{\partial \theta}, \\
\frac{\partial \psi_I}{\partial \underline{g}} &= \underline{0}, \\
\frac{\partial \psi_I}{\partial \nabla \underline{F}} &= \underline{0}, \\
\frac{\partial \psi_I}{\partial \dot{\underline{x}}} &= \underline{0},
\end{aligned} \tag{5.9}$$

$$\underline{\kappa} = - \frac{1}{\underline{\rho}} \left(\underline{F} \frac{\partial \rho \psi_I}{\partial \underline{F}}^T + \sum_{\underline{e}} \underline{\rho} \underline{u}_{\underline{e}} \frac{\partial \psi}{\partial \dot{\underline{x}}} \right),$$

$$\sum_{\underline{a}} \underline{u}_{\underline{a}} (\text{grad } \underline{g}) \frac{\partial \rho \psi}{\partial \underline{g}} = 0,$$

and

$$\sum_{\underline{a}} \underline{u}_{\underline{a}} \sum_{\underline{e}} \underline{F}^{-1} \left(\frac{\partial \rho \psi}{\partial \nabla \underline{F}} [\nabla \nabla \underline{F}] \right) = 0.$$

Equations (5.9)₁ through (5.9)₇ follow directly from the fact that $\dot{\theta}$, $\dot{\underline{g}}$, each $\dot{\nabla \underline{F}}$, each $\dot{\underline{x}}$, each $\dot{\underline{F}}$, $\text{grad } \underline{g}$, and each $\nabla \nabla \underline{F}$ can be assigned arbitrary values. Equations (5.9)₂, (5.9)₃, and (5.9)₄ show that

$$\psi_I = \psi_I(\theta, \underline{F}), \tag{5.10}$$

so that equation (5.9)₁ implies that

$$\eta = \eta(\theta, \underline{F}) = - \frac{\partial \psi_I(\theta, \underline{F})}{\partial \theta}. \tag{5.11}$$

From equations (3.14) and (5.9)₅ it follows that

$$\tilde{\mathbf{T}}^T = \rho \psi \tilde{\mathbf{I}} + \tilde{\mathbf{F}} \frac{\partial \rho \psi \tilde{\mathbf{I}}}{\partial \tilde{\mathbf{F}}} + \sum_c \rho \tilde{\mathbf{u}}_c \frac{\partial \psi}{\partial \dot{\tilde{\mathbf{x}}}_c}, \quad (5.12)$$

which can be rewritten as

$$\tilde{\mathbf{T}}^T = \rho (\psi - \psi_I) \tilde{\mathbf{I}} + \rho \tilde{\mathbf{F}} \frac{\partial \psi \tilde{\mathbf{I}}}{\partial \tilde{\mathbf{F}}} + \sum_c \rho \tilde{\mathbf{u}}_c \frac{\partial \psi}{\partial \dot{\tilde{\mathbf{x}}}_c}, \quad (5.13)$$

$$\tilde{\mathbf{T}}^T = \rho \tilde{\mathbf{F}} \frac{\partial \psi \tilde{\mathbf{I}}}{\partial \tilde{\mathbf{F}}} + \frac{\partial}{\partial \dot{\tilde{\mathbf{x}}}} \left(\sum_c \rho \psi \tilde{\mathbf{u}}_c \right), \quad (5.14)$$

or

$$\tilde{\mathbf{T}}^T = \sum_c \rho \left[\tilde{\mathbf{F}} \frac{\partial \psi}{\partial \tilde{\mathbf{F}}} + \tilde{\mathbf{u}}_c \frac{\partial \psi}{\partial \dot{\tilde{\mathbf{x}}}_c} \right]. \quad (5.15)$$

The derivation of equations (5.13), (5.14), and (5.15) makes use of the identities

$$\tilde{\mathbf{F}} \frac{\partial \rho \psi \tilde{\mathbf{I}}}{\partial \tilde{\mathbf{F}}} = \rho \tilde{\mathbf{F}} \frac{\partial \psi \tilde{\mathbf{I}}}{\partial \tilde{\mathbf{F}}} - \rho \psi \tilde{\mathbf{I}} = \sum_c \rho \tilde{\mathbf{F}} \frac{\partial \psi}{\partial \tilde{\mathbf{F}}} - \rho \psi \tilde{\mathbf{I}}$$

and (5.16)

$$\rho (\psi - \psi_I) \tilde{\mathbf{I}} + \sum_c \rho \tilde{\mathbf{u}}_c \frac{\partial \psi}{\partial \dot{\tilde{\mathbf{x}}}_c} = \frac{\partial}{\partial \dot{\tilde{\mathbf{x}}}} \left(\sum_c \rho \psi \tilde{\mathbf{u}}_c \right).$$

As will be shown in the next chapter, as a result of the axiom of material frame-indifference,

$$\sum_c \frac{\partial \psi}{\partial \dot{\tilde{\mathbf{x}}}_c} = 0. \quad (5.17)$$

Therefore, from equations (2.12), (3.11), (5.16)₂, and (5.17) it follows that

$$\sum_a \sum_c \rho_c \tilde{u}_c \frac{\partial \psi}{\partial \tilde{x}_a} = \sum_a \frac{\partial}{\partial \tilde{x}_a} \left(\sum_c \rho_c \tilde{\psi}_c \tilde{u}_c \right) = 0. \quad (5.18)$$

As a result

$$\tilde{T}_I = \tilde{T}_I(\theta, \tilde{F}) = \rho \sum_a \tilde{F}_a \frac{\partial \psi_I}{\partial \tilde{F}_a} = \sum_a \sum_c \rho_c \tilde{F}_a \frac{\partial \psi}{\partial \tilde{F}_a}, \quad (5.19)$$

where equations (2.12), (2.53), (3.11), (5.13) or (5.14), (5.15) and (5.18) have been used. Equations (5.10), (5.11), and (5.19)₁ are identical to the results of Bowen² and generalize the classical results for single hyperelastic materials.³ In order that equation (2.46)₃ be satisfied, it is necessary that

$$\sum_a \tilde{F}_a \frac{\partial \psi_I}{\partial \tilde{F}_a} = \sum_a \frac{\partial \psi_I}{\partial \tilde{F}_a} \tilde{F}_a^T. \quad (5.20)$$

It will be shown in the next chapter that equation (5.20) is automatically satisfied if ψ_I satisfies the axiom of material frame-indifference. Equation (5.9)₆ can be used to show that

$$\frac{\partial}{\partial \tilde{g}} \left(\sum_a \rho_a \tilde{\psi}_a \tilde{u}_a \right) + \left[\frac{\partial}{\partial \tilde{g}} \left(\sum_a \rho_a \tilde{\psi}_a \tilde{u}_a \right) \right]^T = 0, \quad (5.21)$$

which, in turn, implies that

²Bowen, op. cit., p. 389.

³Truesdell and Noll, op. cit., pp. 301-302.

$$\sum_a \rho_a \psi_a u_a = a(\theta, F, \nabla F, \dot{x}) + g(\theta, F, \nabla F, \dot{x}) q, \quad (5.22)$$

where $g = -g^T$. Equation (5.9)₇ imposes restrictions on the way in which $\sum_a \rho_a \psi_a u_a$ can depend on ∇F , but these are complicated to derive and are of no interest here. Because of equations (5.9) the entropy inequality (5.8) reduces to

$$\begin{aligned} & -\frac{g}{\theta} \cdot \left[q - \sum_a \rho_a \left(K_a + \frac{u_a^2}{2} I - \theta \frac{\partial \psi_a}{\partial \theta} I \right) u_a \right] \\ & - \sum_a u_a \cdot \left[\hat{p}_a - F_a^{-1} \left(\frac{\partial \rho_a \psi_a}{\partial F} I [\nabla F] \right) + \sum_c F_c^{-1} \left(\frac{\partial \rho_c \psi_c}{\partial F} I [\nabla F] \right) \right] \geq 0. \end{aligned} \quad (5.23)$$

Definition. The mixture is in equilibrium at $x \in V(B_t)$ and at the time t if $g = 0$ and $\dot{x} = 0$, for $a = 1, \dots, n$.

If each $\dot{x}_a$ is set equal to zero, then $\dot{x} = 0$ and each $u_a = 0$. Therefore, the inequality (5.23) reduces to

$$-\frac{g}{\theta} \cdot q(\theta, g, F, \nabla F, 0) \geq 0. \quad (5.24)$$

Allowing g to approach zero yields

$$q(\theta, 0, F, \nabla F, 0) = 0; \quad (5.25)$$

that is, the heat flux vanishes in equilibrium. If instead g is set equal to zero, the inequality (5.23) becomes

$$-\sum_a \dot{x}_a \cdot \left[\hat{p}_a(\theta, 0, F, \nabla F, \dot{x}) - F_a^{-1} \left(\frac{\partial \rho_a \psi_a}{\partial F} I^{(\theta, F)} [\nabla F] \right) + \right.$$

$$+ \sum_{\mathbf{e}} \mathbf{F}_{\mathbf{e}}^{-1T} \left(\frac{\partial \rho \psi}{\partial \mathbf{F}_{\mathbf{e}}} (\theta, \mathbf{0}, \mathbf{F}_{\mathbf{e}}, \nabla \mathbf{F}_{\mathbf{e}}, \hat{\mathbf{x}}_{\mathbf{e}}) [\nabla \mathbf{F}_{\mathbf{e}}] \right) \geq 0, \quad (5.26)$$

where equations (2.16), (2.63)₂, and (4.1)₄ have been used. The purpose of using $\hat{\mathbf{x}}_{\mathbf{e}}$ instead of $\mathbf{u}_{\mathbf{e}}$ is that the former quantities are independent, whereas the latter must satisfy equation (2.17). Now as each $\hat{\mathbf{x}}_{\mathbf{e}}$ approaches zero, it follows that

$$\begin{aligned} \hat{\mathbf{p}}_{\mathbf{e}}(\theta, \mathbf{0}, \mathbf{F}_{\mathbf{e}}, \nabla \mathbf{F}_{\mathbf{e}}, \mathbf{0}) &= \mathbf{F}_{\mathbf{e}}^{-1T} \left(\frac{\partial \rho \psi}{\partial \mathbf{F}_{\mathbf{e}}} (\theta, \mathbf{F}_{\mathbf{e}}) [\nabla \mathbf{F}_{\mathbf{e}}] \right) \\ &- \sum_{\mathbf{e}} \mathbf{F}_{\mathbf{e}}^{-1T} \left(\frac{\partial \rho \psi}{\partial \mathbf{F}_{\mathbf{e}}} (\theta, \mathbf{0}, \mathbf{F}_{\mathbf{e}}, \nabla \mathbf{F}_{\mathbf{e}}, \mathbf{0}) [\nabla \mathbf{F}_{\mathbf{e}}] \right). \end{aligned} \quad (5.27)$$

Notice that equation (5.27) satisfies equation (2.63)₂, $\sum_{\mathbf{e}} \hat{\mathbf{p}}_{\mathbf{e}} = \mathbf{0}$. Equations (5.25) and (5.27) result from taking the differential of inequalities (5.24) and (5.26), respectively, and observing that both of these differentials must vanish in equilibrium. This is equivalent to setting the differential of inequality (5.23) to zero in equilibrium. It is true also that the second differential of inequality (5.23) must be nonnegative in equilibrium. Therefore,

$$\begin{aligned} &\frac{\mathbf{v}}{\theta} \cdot \left[\frac{\partial \mathbf{q}}{\partial \mathbf{g}} \mathbf{v} + \sum_{\mathbf{d}} \left(\frac{\partial \mathbf{q}}{\partial \hat{\mathbf{x}}_{\mathbf{d}}} - \sum_{\mathbf{a}} \rho_{\mathbf{a}} \left[\mathbf{K}_{\mathbf{a}} - \theta \frac{\partial \psi}{\partial \theta} \mathbf{I} \right] (\delta_{\mathbf{a}\mathbf{d}} - \mathbf{c}_{\mathbf{d}}) \right) \frac{\mathbf{v}}{\mathbf{d}} \right] \\ &+ \sum_{\mathbf{e}} \frac{\mathbf{v}}{\mathbf{e}} \cdot \left[\frac{\partial}{\partial \mathbf{g}} \left(\hat{\mathbf{p}}_{\mathbf{e}} + \sum_{\mathbf{e}} \mathbf{F}_{\mathbf{e}}^{-1T} \left(\frac{\partial \rho \psi}{\partial \mathbf{F}_{\mathbf{e}}} [\nabla \mathbf{F}_{\mathbf{e}}] \right) \right) \right] \mathbf{v} + \end{aligned}$$

$$+ \sum_d \frac{\partial}{\partial \tilde{x}_d} \left(\hat{p}_a + \sum_c F_c^{-1T} \left(\frac{\partial \rho_a \psi}{\partial F_c} [\nabla F] \right) \right) \tilde{v}_d \leq 0, \quad (5.28)$$

for all vectors $\tilde{v}$ and $\tilde{v}$, $a = 1, \dots, n$, where all the variables are evaluated at the equilibrium state $(\theta, 0, F, \nabla F, 0)$.

Next will be considered the case in which the constitutive equations (4.1) do not depend on ∇F , $b = 1, \dots, n$. In this event if equation (5.27) is rewritten as

$$\begin{aligned} \hat{p}_a(\theta, 0, F, 0) = \sum_{\substack{c \\ c \neq a}} \left[\rho_a F_c^{-1T} \left(\frac{\partial \psi(\theta, 0, F, 0)}{\partial F_c} [\nabla F] \right) \right. \\ \left. - \rho_a F_c^{-1T} \left(\frac{\partial \psi(\theta, 0, F, 0)}{\partial F_c} [\nabla F] \right) \right], \quad (5.29) \end{aligned}$$

where equation (3.11) has been used, it follows immediately that

$$\frac{\partial \psi(\theta, 0, F, 0)}{\partial F_c} = 0, \quad (5.30)$$

for $c \neq a$. This means that, in equilibrium,

$$\psi_a = \psi(\theta, 0, F, 0). \quad (5.31)$$

For a mixture of fluids it can be shown that equation (5.31) becomes

$$\psi_a = \psi(\theta, 0, 1/\rho_a, 0), \quad (5.32)$$

which is similar to the result that Müller has obtained for

the special case he calls a mixture of simple fluids.⁴ Equation (5.32) is known not to be true in general; thus, it is essential in the present formulation to allow for a dependence on $\nabla \mathbf{F}$, $b = 1, \dots, n$. Equations (5.29) and (5.30) also imply that

$$\hat{\mathbf{p}}_a(\theta, 0, \mathbf{F}, 0) = 0. \quad (5.33)$$

That is, in a theory of a mixture of elastic materials with no chemical reactions in which a dependence on second deformation gradients is excluded, the diffusive forces vanish in equilibrium.

It is difficult to make an exact comparison of the results here to those of Bowen because Bowen did not allow for a dependence on the second deformation gradients. However, with equation (3.14), it is possible to define vectors $\hat{\mathbf{f}}_a$, $b = 1, \dots, n$, which correspond to Bowen's diffusion forces. The result is

$$\hat{\mathbf{f}}_a = \hat{\mathbf{p}}_a + \text{grad}(\rho_a \psi) + \rho_a \eta \mathbf{g} - K_a \text{grad} \rho_a + \mathbf{s}_a, \quad (5.34)$$

where $\mathbf{s}_a$ is given by

$$\mathbf{s}_a = \rho_a X_{I,1} (|\det \mathbf{F}|^{-1} x_{j,I})_{,1} K_{a,j} \delta_1. \quad (5.35)$$

⁴Müller, op. cit., p. 27.

If equations (5.34), (5.35) and (3.14) are used, then the equation of motion (2.59)₂ can be written formally in the form postulated by Bowen. An examination of equation (5.34) shows that, because of the term $\text{grad}(\rho \psi)$, $\hat{f}$ will depend upon $(\theta, \underline{g}, \underline{F}, \underline{\nabla F}, \underline{\dot{x}}, \text{grad } \underline{g}, \underline{\nabla \nabla F}, \underline{\dot{F}})$. It is easily seen that $\hat{f}$ depends upon the same quantities as does $\hat{p}$, η and $\underline{K}$ if and only if

$$\psi = \psi(\theta, \underline{F}). \quad (5.36)$$

Thus a theory formally similar to Bowen's can be obtained as a special case of the theory presented here if equation (5.36) is assumed.

From equations (5.9)₅ and (5.16)₁ it follows that

$$\underline{K} = \psi \underline{I} - \frac{\rho}{\rho} \underline{F} \frac{\partial \psi}{\partial \underline{F}} - \frac{1}{\rho} \sum_c \rho_c u_c \frac{\partial \psi}{\partial \underline{x}_c}. \quad (5.37)$$

In the special case where ψ is given by equation (5.36), equation (5.37) becomes

$$\underline{K} = \psi \underline{I} - \frac{\rho}{\rho} \underline{F} \frac{\partial \psi}{\partial \underline{F}}; \quad (5.38)$$

however, equation (3.15) is unaffected. Equation (5.38) is Bowen's expression for the chemical potential tensor.⁵

⁵Bowen, op. cit., p. 393.

As was pointed out in chapter 3, Green & Naghdi use a different entropy inequality. Use of their entropy inequality to restrict the constitutive equations (4.1) gives

$$\tilde{T}^T = \rho_{\tilde{c}}^F \frac{\partial \psi_I(\theta, F)}{\partial F}^T$$

and

(5.39)

$$\begin{aligned} \hat{p}(\theta, \underline{\tilde{c}}, \underline{\tilde{F}}, \underline{\nabla \tilde{F}}, \underline{\tilde{c}}) = \sum_{\substack{c \\ c \neq a}} \left[\rho_{\tilde{c}}^F^{-1} \left(\frac{\partial \psi_I(\theta, F)}{\partial F} \right) [\underline{\nabla \tilde{F}}] \right. \\ \left. - \rho_{\tilde{c}}^F^{-1} \left(\frac{\partial \psi_I(\theta, F)}{\partial F} \right) [\underline{\nabla \tilde{F}}] \right]. \end{aligned}$$

Notice that even though $\tilde{T}^T$ in equation (5.39)₁ differs from $\tilde{T}^T$ in equations (5.13) and (5.14), $\tilde{T}_I$ is still given by equation (5.19)₁. For the case of a binary mixture of fluids equation (5.39)₂ reduces to Green & Naghdi's equation (5.16)₂.⁶ It is a simple matter to verify that, for an ideal gas mixture, this equation will not produce the usual quantity called partial pressure, whereas equation (5.13) or (5.14) will.

⁶Green and Naghdi, International Journal of Engineering Science, III, 240.

In the special case when the constitutive equations are independent of ∇F , $b = 1$, ---, n , equation (5.39)₂ implies that

$$\hat{p}(\theta, \underline{0}, F, \underline{0}) = \underline{0}$$

and

(5.40)

$$\psi_I = \psi_I(\theta).$$

If Green & Naghdi had not modified their original constitutive equations to allow for a special dependence on density gradients, they would have been forced to the unusual result of equation (5.40)₂. Equation (5.40)₂ is completely unacceptable whereas equation (5.31) corresponds to known special types of mixtures.⁷ The unusual nature of equation (5.40)₂ adds weight to the argument that Green & Naghdi's entropy inequality is too special.

⁷I. Prigogine and R. Defay, Chemical Thermodynamics, trans. D. H. Everett (London-New York-Toronto: Longmans Green, 1954), pp. 137-155.

6. MATERIAL FRAME-INDIFFERENCE AND MATERIAL SYMMETRY

The constitutive equations not only have to satisfy the restrictions imposed by the entropy inequality but also must satisfy the axiom of material frame-indifference¹ and must reflect any symmetry that the materials possess.

Definition. A change of frame (of reference) is a time-dependent mapping of E into E defined by

$$\underline{\tilde{x}}^* = \underline{\tilde{c}} + \underline{\tilde{Q}}(\underline{\tilde{x}} - \underline{\tilde{o}}), \quad (6.1)$$

where $\underline{\tilde{c}} = \underline{\tilde{c}}(t) \in E$ and $\underline{\tilde{Q}} = \underline{\tilde{Q}}(t)$ is an orthogonal linear transformation. The axiom of material frame-indifference states that an admissible thermodynamic process must remain admissible after a change of frame. By use of a method developed by Noll² it follows that necessary and sufficient conditions for the axiom of material frame-indifference to be satisfied are the following:

$$\begin{aligned} & \psi(\theta, \underline{\tilde{g}}, \underline{\tilde{F}}, \underline{\tilde{\nabla F}}, \underline{\tilde{x}}) \\ &= \psi(\theta, \underline{\tilde{Qg}}, \underline{\tilde{QF}}, \underline{\tilde{Q\nabla F}}, \underline{\dot{\tilde{c}}} + \underline{\tilde{Q}\dot{\tilde{x}}} + \underline{\dot{\tilde{Q}}}(\underline{\tilde{x}} - \underline{\tilde{o}})), \end{aligned}$$

¹Truesdell and Noll, op. cit., pp. 41-44.

²W. Noll, "On the Continuity of the Solid and Fluid States," Journal of Rational Mechanics and Analysis, IV (1955), 19.

$$\begin{aligned}
& \eta(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) \\
&= \eta(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{\dot{c}} + \underline{Q\dot{x}} + \underline{\dot{Q}}(\underline{x} - \underline{c})), \\
& \underline{Q}^T \eta(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) \underline{Q}^T \\
&= \underline{T}(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{\dot{c}} + \underline{Q\dot{x}} + \underline{\dot{Q}}(\underline{x} - \underline{c})), \quad (6.2)
\end{aligned}$$

$$\begin{aligned}
& \underline{Q}\hat{p}(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) \\
&= \hat{p}(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{\dot{c}} + \underline{Q\dot{x}} + \underline{\dot{Q}}(\underline{x} - \underline{c})),
\end{aligned}$$

$$\begin{aligned}
& \underline{Q}\hat{M}(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) \underline{Q}^T \\
&= \hat{M}(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{\dot{c}} + \underline{Q\dot{x}} + \underline{\dot{Q}}(\underline{x} - \underline{c})),
\end{aligned}$$

and

$$\begin{aligned}
& \underline{Qq}(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) \\
&= \underline{q}(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{\dot{c}} + \underline{Q\dot{x}} + \underline{\dot{Q}}(\underline{x} - \underline{c})),
\end{aligned}$$

for $a = 1, \dots, n$, and for all orthogonal linear transformations $\underline{Q}(t)$, all $\underline{c} \in E$, all $\underline{\dot{Q}}(t)$ for which $\underline{\dot{Q}}\underline{Q}^T = -\underline{Q}\underline{\dot{Q}}^T$, and all $(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}})$ in the domain of the response functions. The quantity $\underline{Q\nabla F}$ is defined as $\underline{Q\nabla F} = \nabla(\underline{QF})$ for all $\underline{Q}$ independent of $\underline{x}$. An immediate consequence of equation (6.2)₁ is that

$$\underline{\psi}(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{x}}) = \underline{\psi}(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{\dot{u}}), \quad (6.3)$$

where $\underline{\dot{u}} = \underline{\dot{x}} - \underline{\dot{x}}_d$ and for a given value of a , d is a fixed positive integer less than or equal to n . Equation (6.3)

follows by taking $\dot{\underline{c}} = -\dot{\underline{x}}$, $\underline{Q} = \underline{I}$, and $\dot{\underline{Q}} = \underline{0}$ in equation (6.2)₁. Similar results hold for the other response functions. In words this means that the response functions depend on velocity only through the $n - 1$ velocity differences. It can also be shown that equations (6.2) are satisfied if and only if

$$\begin{aligned}
 \underline{\psi} &= \underline{\psi}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}), \\
 \underline{\eta} &= \underline{\eta}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}), \\
 \underline{T} &= \underline{F}_{\underline{c}}^T \underline{T}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}) \underline{F}_{\underline{c}}^T, \\
 \underline{\hat{p}} &= \underline{F}_{\underline{c}}^T \underline{\hat{p}}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}), \\
 \underline{\hat{M}} &= \underline{F}_{\underline{c}}^T \underline{\hat{M}}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}) \underline{F}_{\underline{c}}^T,
 \end{aligned} \tag{6.4}$$

and

$$\underline{q} = \underline{F}_{\underline{c}} \underline{q}(\theta, \underline{F}_{\underline{c}}^T \underline{q}, \underline{F}_{\underline{c}}^T \underline{F}, \nabla \underline{C}_{\underline{b}}, \underline{F}_{\underline{b}}^T \underline{u}_{\underline{d}}),$$

where, for a given value of a , c and d are fixed integers such that $1 \leq c, d \leq n$. Also, $\underline{C}_{\underline{b}} = \underline{F}_{\underline{b}}^T \underline{F}_{\underline{b}}$ is the right Cauchy-Green tensor.

Equations (3.1), (3.11), (5.10), (5.11), (6.4)₁, and (6.4)₂ imply that

$$\begin{aligned}
 \underline{\psi}_{\underline{I}} &= \underline{\psi}_{\underline{I}}(\theta, \underline{F}_{\underline{c}}^T \underline{F}) \\
 \text{and} \\
 \underline{\eta} &= \underline{\eta}(\theta, \underline{F}_{\underline{c}}^T \underline{F}) = - \frac{\partial \underline{\psi}_{\underline{I}}(\theta, \underline{F}_{\underline{c}}^T \underline{F})}{\partial \theta}
 \end{aligned} \tag{6.5}$$

It follows from equation (6.5)₁ that

$$\frac{\partial \psi_I(\theta, \tilde{F})^T}{\partial \tilde{F}_a} = \frac{\partial \psi_I(\theta, \tilde{F}_c^T \tilde{F}_b)^T}{\partial \tilde{F}_c^T \tilde{F}_a} \tilde{F}_c^T, \quad a \neq c,$$

and

$$\frac{\partial \psi_I(\theta, \tilde{F})^T}{\partial \tilde{F}_c} = \frac{\partial \psi_I(\theta, \tilde{F}_c^T \tilde{F}_b)^T}{\partial \tilde{F}_c^T \tilde{F}_c} \tilde{F}_c^T + \sum_b \frac{\partial \psi_I(\theta, \tilde{F}_c^T \tilde{F}_b)^T}{\partial \tilde{F}_c^T \tilde{F}_b} \tilde{F}_b^T. \quad (6.6)$$

Equations (5.19) and (6.6) yield

$$\begin{aligned} \tilde{T}_I = \tilde{T}_I(\theta, \tilde{F}_c^T \tilde{F}_b) &= \rho \sum_a \left[\tilde{F}_a \frac{\partial \psi_I(\theta, \tilde{F}_c^T \tilde{F}_b)^T}{\partial \tilde{F}_c^T \tilde{F}_a} \tilde{F}_c^T \right. \\ &\quad \left. + \tilde{F}_c \frac{\partial \psi_I(\theta, \tilde{F}_c^T \tilde{F}_b)^T}{\partial \tilde{F}_c^T \tilde{F}_c} \tilde{F}_a^T \right]. \end{aligned} \quad (6.7)$$

Equations (5.19) and (6.7) show that the symmetry condition, equation (5.20), is satisfied. From equations (6.3) and (6.4)₁ it can be seen that

$$\frac{\partial \psi(\theta, \tilde{g}, \tilde{F}, \nabla \tilde{F}, \dot{\tilde{x}})}{\partial \dot{\tilde{x}}_a} = \tilde{F}_c \frac{\partial \psi(\theta, \tilde{F}_c^T \tilde{g}, \tilde{F}_c^T \tilde{F}_b, \nabla \tilde{C}, \tilde{F}_c^T \tilde{u}_d)}{\partial \tilde{F}_c^T \tilde{u}_d}, \quad a \neq d,$$

and

$$\frac{\partial \psi(\theta, \tilde{g}, \tilde{F}, \nabla \tilde{F}, \dot{\tilde{x}})}{\partial \dot{\tilde{x}}_a} = - \sum_{\substack{f \\ f \neq d}} \tilde{F}_c \frac{\partial \psi(\theta, \tilde{F}_c^T \tilde{g}, \tilde{F}_c^T \tilde{F}_b, \nabla \tilde{C}, \tilde{F}_c^T \tilde{u}_d)}{\partial \tilde{F}_c^T \tilde{u}_d}.$$

An immediate consequence of equations (6.8) is

$$\sum_i \frac{\partial \psi(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \dot{\underline{x}})}{\partial \dot{\underline{x}}_i} = \underline{0}, \quad (6.9)$$

which was used in the last chapter. With equations (6.6) and (6.8) it is now possible to rewrite equation (5.13), the expression for the stress on the constituents. However, the resulting equations are quite complicated. Furthermore, nothing has yet been said about material symmetry. It turns out that it is convenient to satisfy the requirements of material symmetry before those of material frame-indifference. Toward this end it is helpful to define a contraction operator $\hat{C}$ as follows. If $\underline{\Gamma}$ is any third-order tensor and $\underline{A}$ is any second-order tensor (linear transformation), then

$$\hat{C}(\underline{\Gamma} \otimes \underline{A} \otimes \underline{A}) = \Gamma_{i,j,k} A_{j,l} A_{k,m} \delta_i \otimes \delta_l \otimes \delta_m. \quad (6.10)$$

The mixture is characterized by $5n + 1$ response functions each of which, in general, depends on all n reference configurations. This means that n different isotropy groups would exist for each response function. To avoid this situation the approach taken here will be analogous to the one developed by Bowen.³

³Bowen, op. cit., pp. 399-400.

Definition. The isotropy group $\mathcal{L}_c$ for the c^{th} constituent in a mixture of elastic materials with diffusion and heat conduction is the set of all unimodular linear transformations $\underline{H}$ such that the following identities hold:

$$\begin{aligned}
 \psi_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= \psi_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}), \\
 \eta_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= \eta_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}), \\
 T_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= T_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}), \\
 \hat{p}_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= \hat{p}_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}), \\
 \hat{M}_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= \hat{M}_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}),
 \end{aligned} \tag{6.11}$$

and

$$\begin{aligned}
 \underline{q}_a(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) &= \underline{q}_a(\theta, \underline{g}, \underline{F}, \text{---}, \underline{FH}, \text{---}, \underline{F}, \nabla \underline{F}, \text{---}, \hat{C}(\nabla \underline{F} \underline{\Theta} \underline{H} \underline{\Theta} \underline{H}), \text{---}, \nabla \underline{F}, \underline{u}),
 \end{aligned}$$

for $a = 1, \text{---}, n$, and for all $(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u})$ in the domain of the response functions.

To reflect material symmetry the constitutive equations (4.1)₂, (4.1)₃, (4.1)₅, (4.1)₆, (4.1)₇, and (4.2) must satisfy equations (6.11), for $a = 1, \dots, n$, for each $\tilde{H} \in \mathcal{L}_c$, $c = 1, \dots, n$.

At this point it will be assumed that the mixture consists of n isotropic solids; so that $\mathcal{L}_c = \mathcal{O}$, for $c = 1, \dots, n$, where $\mathcal{O}$ is the orthogonal group of linear transformations.

Theorem. For a mixture of isotropic solids ψ satisfies the requirements of material symmetry if and only if

$\psi = \psi(\theta, \underline{g}, \underline{B}, \underline{M}, \underline{u})$, where $\underline{B} = \underline{F}\underline{F}^T$ is the left Cauchy-Green tensor and $\underline{M} = \hat{C}(\nabla \underline{F} \underline{Q} \underline{F}^{-1} \underline{Q} \underline{F}^{-1})$.

Proof. First suppose that

$$\begin{aligned} \psi(\theta, \underline{g}, \underline{F}, \nabla \underline{F}, \underline{u}) \\ = \psi(\theta, \underline{g}, \underline{F}, \dots, \underline{F}\underline{Q}, \dots, \underline{F}, \nabla \underline{F}, \dots, \hat{C}(\nabla \underline{F} \underline{Q} \underline{Q} \underline{Q}), \dots, \nabla \underline{F}, \underline{u}), \end{aligned} \quad (6.12)$$

For all $\underline{Q} \in \mathcal{O} = \mathcal{L}_c$. Let $c = 1$ and take $\underline{Q} = \underline{R}^T$, where $\underline{R} = \underline{F}(\underline{F}^T \underline{F})^{-\frac{1}{2}}$. Therefore,

$$\psi = \psi(\theta, \underline{g}, \underline{F}\underline{R}^T, \underline{F}, \dots, \underline{F}, \hat{C}(\nabla \underline{F} \underline{R}^T \underline{R}^T), \nabla \underline{F}, \dots, \nabla \underline{F}, \underline{u}). \quad (6.13)$$

Substituting equation (6.13) into equation (6.12), with $c = 2$ and $\underline{Q} = \underline{R}^T$, shows that

$$\begin{aligned} \psi = \psi(\theta, \underline{g}, \underline{\underline{FR}}^T, \underline{\underline{FR}}^T, \underline{\underline{F}}, \dots, \underline{\underline{F}}, \hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{R}}^T \underline{\underline{\otimes}} \underline{\underline{R}}^T), \\ \hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{R}}^T \underline{\underline{\otimes}} \underline{\underline{R}}^T), \nabla \underline{\underline{F}}, \dots, \nabla \underline{\underline{F}}, \underline{\underline{u}}). \end{aligned} \quad (6.14)$$

Repeating this last step $n - 2$ times incrementing c by one each time and taking $\underline{\underline{Q}} = \underline{\underline{R}}^T$ yields

$$\psi = \psi(\theta, \underline{g}, \underline{\underline{FR}}^T, \hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{R}}^T \underline{\underline{\otimes}} \underline{\underline{R}}^T), \underline{\underline{u}}). \quad (6.15)$$

But $\underline{\underline{FR}}^T = \underline{\underline{B}}^{\frac{1}{2}}$ and $\hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{R}}^T \underline{\underline{\otimes}} \underline{\underline{R}}^T) = \hat{C}(\underline{\underline{M}} \underline{\underline{\otimes}} \underline{\underline{B}}^{\frac{1}{2}} \underline{\underline{\otimes}} \underline{\underline{B}}^{\frac{1}{2}})$, so that

$$\psi = \psi(\theta, \underline{g}, \underline{\underline{B}}, \underline{\underline{M}}, \underline{\underline{u}}). \quad (6.16)$$

Now suppose that equation (6.16) is valid. It is easy to show that $\underline{\underline{B}} = \underline{\underline{FF}}^T = (\underline{\underline{FQ}})(\underline{\underline{FQ}})^T$ and $\underline{\underline{M}} = \hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{F}}^{-1} \underline{\underline{\otimes}} \underline{\underline{F}}^{-1}) = \hat{C}[\hat{C}(\nabla \underline{\underline{F}} \underline{\underline{\otimes}} \underline{\underline{Q}} \underline{\underline{\otimes}} \underline{\underline{Q}}) \underline{\underline{\otimes}} (\underline{\underline{FQ}})^{-1} \underline{\underline{\otimes}} (\underline{\underline{FQ}})^{-1}]$. These two results show that if

$$\psi(\theta, \underline{g}, \underline{\underline{F}}, \nabla \underline{\underline{F}}, \underline{\underline{u}}) = \psi(\theta, \underline{g}, \underline{\underline{B}}, \underline{\underline{M}}, \underline{\underline{u}}), \quad (6.17)$$

then equation (6.12) holds.

Now the axiom of material frame-indifference will be applied to ψ .

Theorem. For a mixture of isotropic solids ψ satisfies the axiom of material frame-indifference if and only if ψ is an isotropic function; i.e., for all orthogonal $\underline{\underline{Q}}$,

$$\psi(\theta, \underline{g}, \underline{B}, \underline{M}, \underline{u}) = \psi(\theta, \underline{Qg}, \underline{QBQ}^T, \underline{Q}\hat{C}(\underline{MQ}^T \underline{Q}^T), \underline{Qu}) \quad (6.18)$$

Proof. First suppose that equation (6.2)₁ is valid.

This is equivalent to the statement that

$$\psi(\theta, \underline{g}, \underline{F}, \underline{\nabla F}, \underline{u}) = \psi(\theta, \underline{Qg}, \underline{QF}, \underline{Q\nabla F}, \underline{Qu}) \quad (6.19)$$

for all $\underline{Q}(t) \in \mathcal{O}$. When $\underline{F} \rightarrow \underline{QF}$ and $\underline{\nabla F} \rightarrow \underline{Q\nabla F}$, then $\underline{B} \rightarrow \underline{QBQ}^T$ and $\underline{M} \rightarrow \hat{C}[\underline{Q\nabla F} \underline{Q}(\underline{QF})^{-1} \underline{Q}(\underline{QF})^{-1}] = \underline{Q}\hat{C}(\underline{MQ}^T \underline{Q}^T)$. Therefore, equation (6.18) holds. Conversely, suppose that equation (6.18) is valid. Then

$$\begin{aligned} \psi(\theta, \underline{g}, \underline{FF}^T, \hat{C}(\underline{\nabla F} \underline{QF}^{-1} \underline{QF}^{-1}), \underline{u}) \\ = \psi(\theta, \underline{Qg}, (\underline{QF})(\underline{QF})^T, \hat{C}[\underline{Q\nabla F} \underline{Q}(\underline{QF})^{-1} \underline{Q}(\underline{QF})^{-1}], \underline{Qu}) \end{aligned} \quad (6.20)$$

which means that equation (6.19) is true.

These two theorems complete the restrictions on ψ . The stress on the constituents will now be expressed in terms of the variables $\underline{B}$ and $\underline{M}$. From equation (6.17) it follows, after a somewhat lengthy calculation, that

$$\underline{F} \frac{\partial \psi}{\partial \underline{F}}^T = 2 \left[\underline{B} \frac{\partial \psi}{\partial \underline{B}} - \bar{C} \left(\frac{\partial \psi}{\partial \underline{M}} \underline{Q} \underline{M} \right) \right] \quad (6.21)$$

where the contraction operator $\bar{C}$ is defined by

$$\bar{C}(\underline{\Gamma} \underline{\otimes} \underline{\Phi}) = \Gamma_{ijk} \Phi_{ijk} \delta_k \delta_k \quad (6.22)$$

$\underline{\Gamma}$ and $\underline{\Phi}$ being arbitrary third-order tensors.

Also,

$$\sum_c \rho_c u_c \frac{\partial \psi}{\partial \dot{x}_c} = \sum_c \rho_c u_c \frac{\partial \psi}{\partial \dot{x}_c} , \quad (6.23)$$

since

$$\sum_c \rho_c u_c \frac{\partial \psi}{\partial \dot{x}_c} = u_c \frac{\partial \rho \psi}{\partial \dot{x}_c} = 0. \quad (6.24)$$

Equation (6.23) and the identity

$$\frac{\partial \psi}{\partial \dot{x}_c} = \frac{\partial \psi}{\partial u_c} - \delta_{cd} \sum_e \frac{\partial \psi}{\partial u_e} \quad (6.25)$$

imply that

$$\sum_c \rho_c u_c \frac{\partial \psi}{\partial \dot{x}_c} = \sum_c \rho_c u_c \left(\frac{\partial \psi}{\partial u_c} - \delta_{cd} \sum_e \frac{\partial \psi}{\partial u_e} \right). \quad (6.26)$$

It should be pointed out that equation (6.25) immediately implies equation (5.17), $\sum_c \frac{\partial \psi}{\partial \dot{x}_c} = 0$. Equations (5.15), (6.21),

and (6.26) yield the result that

$$\mathbf{T}^T = 2 \sum_c \rho_c \mathbf{B}_c \frac{\partial \psi}{\partial \mathbf{B}_c} + \sum_c \rho_c u_c \left(\frac{\partial \psi}{\partial u_c} - \delta_{cd} \sum_e \frac{\partial \psi}{\partial u_e} \right), \quad (6.27)$$

since

$$\frac{\partial \psi}{\partial \mathbf{M}} = 0. \quad (6.28)$$

If the notation is used that

$$\Psi = \rho \psi$$

and

(6.29)

$$\Psi = \sum_a \Psi_a = \rho \psi_I,$$

it is easy to show that

$$\sum_a \rho_a B_a \frac{\partial \Psi_a}{\partial B_a} = B \frac{\partial \Psi}{\partial B} + \frac{1}{2} \Psi I, \quad (6.30)$$

from which it follows that equation (6.27) becomes

$$\mathbf{T}_a^T = 2B_a \frac{\partial \Psi}{\partial B_a} + \Psi_a \mathbf{I} + \sum_c \frac{u_c}{\partial u_c} \left(\frac{\partial \Psi}{\partial u_c} - \delta_{cd} \sum_a \frac{\partial \Psi}{\partial u_d} \right). \quad (6.31)$$

It should be clear that $\mathbf{T}_a^T$ satisfies the requirements of material symmetry, since ψ does, and thus Ψ and Ψ_a do also. It is a simple matter to verify that $\mathbf{T}_a^T$ satisfies the axiom of material frame-indifference, also. For completeness it should be noticed that because of equations (5.9)₁ and (2.59)₃ the entropy of the mixture and the body couple of each constituent satisfy material symmetry and material frame-indifference.

In a manner completely analogous to the two theorems about ψ , it can be shown that the following theorems are true.

Theorem. For a mixture of isotropic solids $\hat{\mathbf{p}}_a$ and $\hat{\mathbf{q}}_a$ satisfy the restrictions imposed by material symmetry if and only if $\hat{\mathbf{p}}_a = \hat{\mathbf{p}}_a(\theta, \underline{\mathbf{g}}, B_a, M_a, \underline{\mathbf{u}}_a)$ and $\hat{\mathbf{q}}_a = \hat{\mathbf{q}}_a(\theta, \underline{\mathbf{g}}, B_a, M_a, \underline{\mathbf{u}}_a)$.

Theorem. For a mixture of isotropic solids $\hat{p}$ and $\underline{q}$ satisfy the axiom of material frame-indifference if and only if they are isotropic functions; that is they must satisfy, for all orthogonal $\underline{Q}$, the relations

$$\hat{p}(\theta, \underline{g}, \underline{B}, \underline{M}, \underline{u}) = \underline{Q}^T \hat{p}(\theta, \underline{Qg}, \underline{QBQ}^T, \underline{Q}\hat{C}(\underline{M}\underline{Q}\underline{Q}^T), \underline{Qu}) \quad (6.32)$$

and

$$\underline{q}(\theta, \underline{g}, \underline{B}, \underline{M}, \underline{u}) = \underline{Q}^T \underline{q}(\theta, \underline{Qg}, \underline{QBQ}^T, \underline{Q}\hat{C}(\underline{M}\underline{Q}\underline{Q}^T), \underline{Qu}).$$

To conclude this chapter equation (5.27), the diffusive force on the constituents evaluated at equilibrium, will be expressed in terms of $\underline{B}$ and $\underline{M}$. Equation (6.21) implies that

$$\frac{\partial \psi}{\partial \underline{F}} = 2 \left[\frac{\partial \psi}{\partial \underline{B}} \underline{F} - \left(\bar{C} \left(\underline{M} \frac{\partial \psi}{\partial \underline{M}} \right) \right) \underline{F}^{-1T} \right]. \quad (6.33)$$

It is convenient to define still another contraction operator $\tilde{C}$ between any linear transformation $\underline{A}$ and any third-order tensor $\underline{\Gamma}$ as follows:

$$\tilde{C}(\underline{A}\underline{\Gamma}) = A_{ij} \Gamma_{jkl} \delta_k. \quad (6.34)$$

Then, equations (5.3)₁, (5.27), (6.28), (6.33), and (6.34) imply that

$$\begin{aligned} \hat{p}(\theta, \underline{0}, \underline{B}, \underline{M}, \underline{0}) &= 2\tilde{C} \left\{ \left[\underline{B} \frac{\partial \psi}{\partial \underline{B}} \right] \underline{\Theta M} \right\} \\ &\quad - 2\tilde{C} \sum_{\underline{c}} \left\{ \left[\underline{B} \frac{\partial \psi}{\partial \underline{B}} - \bar{C} \left(\underline{B} \frac{\partial \psi}{\partial \underline{M}} \right) \right] \underline{\Theta M} \right\}. \end{aligned} \quad (6.35)$$

7. LINEARIZATION

Linearized expressions for the stress, diffusive force, heat flux, and entropy, valid for a mixture of isotropic elastic solids undergoing small temperature changes, small deformations, and small departures from equilibrium, are derived in this chapter. With these expressions it is then possible to develop the linearized equations of motion for the constituents, the linearized energy equation for the mixture, and the linearized entropy inequality. Finally, these results are restricted to binary mixtures and compared to those obtained by Steel,¹ who uses Green & Naghdi's formulation.

Definition. The displacement of $X \in B$ relative to the configuration $\tilde{\kappa}$ is defined as

$$\tilde{w} = \tilde{x} - \tilde{X}. \quad (7.1)$$

Definition. The gradient of the displacement at $X \in B$

$$\tilde{H} = \nabla_{\tilde{\kappa}} \tilde{w}(X, t). \quad (7.2)$$

It follows from equations (2.3), (2.8), (7.1), and (7.2) that

$$\tilde{F} = \tilde{I} + \tilde{H}. \quad (7.3)$$

¹Steel, op. cit., pp. 57-72.

What is desired is an expansion of the stress, diffusive force, heat flux, and entropy about the reference state $(\theta, \underline{g}, B, M, u) = (\theta_0, \underline{0}, \underline{I}, \underline{0}, \underline{0})$. Toward this end the following definitions are made;

$$\begin{aligned} \underline{\text{Definition.}} \quad \epsilon^2 = (\theta - \theta_0)^2 + \underline{g} \cdot \underline{g} + \text{tr} \sum_{\underline{a}} \underline{H} \underline{H}^T \\ + c \sum_{\underline{a}} (\nabla \underline{H} \otimes \nabla \underline{H}) + \sum_{\substack{\underline{a}, \underline{d} \\ \underline{a} \neq \underline{d}}} \underline{u}_{\underline{a}} \cdot \underline{u}_{\underline{d}}. \end{aligned} \quad (7.4)$$

Definition. The temperature change, the deformation, and the departure from equilibrium are small if $\epsilon < 1$.

Definition. A quantity of order ϵ^k is any scalar, vector, or tensor, denoted $O(\epsilon^k)$, with the property that there exists a real number N such that

$$\|O(\epsilon^k)\| < N\epsilon^k$$

as $\epsilon \rightarrow 0$.

It should be obvious that $O(\epsilon^{k_1})O(\epsilon^{k_2}) = O(\epsilon^{k_1+k_2})$.

From this point on it will be assumed that $\epsilon < 1$. It follows from the above definitions that

$$\theta - \theta_0 = O(\epsilon),$$

$$\underline{g} = O(\epsilon),$$

$$\underline{H} = O(\epsilon),$$

$$\underset{\sim}{F} = \underset{\sim}{I} + O(\epsilon),$$

$$\underset{\sim}{F}^{-1} = \underset{\sim}{I} - \underset{\sim}{H} + O(\epsilon^2) = \underset{\sim}{I} + O(\epsilon), \quad (7.5)$$

$$\underset{\sim}{B} = \underset{\sim}{I} + \underset{\sim}{H} + \underset{\sim}{H}^T + O(\epsilon^2) = \underset{\sim}{I} + O(\epsilon),$$

$$\underset{\sim}{\nabla H} = O(\epsilon),$$

$$\underset{\sim}{\nabla F} = O(\epsilon),$$

$$\underset{\sim}{M} = \underset{\sim}{\nabla H} + O(\epsilon^2) = O(\epsilon),$$

$$\underset{\sim}{u_d} = O(\epsilon),$$

and

$$\underset{\sim}{u} = \underset{\sim}{u_d} - \frac{1}{\rho} \sum_c \rho_c \underset{\sim}{u_c} = O(\epsilon).$$

To obtain an expression for the stress including terms of first order it is necessary to include second-order terms in the expression for $\underset{\sim}{\psi}$, since $\underset{\sim}{\psi}$ must be differentiated to obtain the stress. If σ is any function of $(\theta, \underset{\sim}{g}, \underset{\sim}{B}, \underset{\sim}{M}, \underset{\sim}{u})$, then the notation will be used that $\sigma(0) = \sigma(\theta_0, \underset{\sim}{0}, \underset{\sim}{I}, \underset{\sim}{0}, \underset{\sim}{0})$. Since $\underset{\sim}{\psi}$ is an isotropic function it follows that all the coefficients in the expansion of $\underset{\sim}{\psi}$ must be isotropic tensors. The same statement can be made regarding $\underset{\sim}{\hat{p}}$, $\underset{\sim}{q}$, and η . As a result all coefficients that are odd-order tensors must vanish. Also, the even-order tensor coefficients have known representations. For example, if $\underset{\sim}{A}$ is an isotropic second-order tensor and $\underset{\sim}{\Lambda}$ an isotropic fourth-order tensor,

then

$$\underline{\underline{A}} = a \underline{\underline{I}}$$

and

(7.6)

$$\underline{\underline{A}} = \lambda \underline{\underline{I}} \otimes \underline{\underline{I}} + \mu \underline{\underline{\delta}}_1 \otimes \underline{\underline{\delta}}_2 \otimes \underline{\underline{\delta}}_3 \otimes \underline{\underline{\delta}}_4 + \nu \underline{\underline{\delta}}_1 \otimes \underline{\underline{\delta}}_2 \otimes \underline{\underline{\delta}}_3 \otimes \underline{\underline{\delta}}_4,$$

where a, λ, μ , and ν are scalars.² Therefore, Ψ can be expanded about $(\theta_0, 0, \underline{\underline{I}}, 0, 0)$ to yield

$$\begin{aligned} \Psi &= \Psi(0) + \frac{\partial \Psi(0)}{\partial \theta} (\theta - \theta_0) + \text{tr} \sum_c \frac{\partial \Psi(0)}{\partial \underline{\underline{B}}_c} (\underline{\underline{B}}_c - \underline{\underline{I}}) \\ &+ \frac{1}{2} \frac{\partial^2 \Psi(0)}{\partial \theta^2} (\theta - \theta_0)^2 + \frac{1}{2} \text{tr} \left[\frac{\partial^2 \Psi(0)}{\partial \underline{\underline{g}} \partial \underline{\underline{g}}} (\underline{\underline{g}} \otimes \underline{\underline{g}}) \right] \\ &+ \frac{1}{2} c \sum_c \sum_{\bullet} \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{B}}_c \partial \underline{\underline{B}}_{\bullet}} \otimes [(\underline{\underline{B}}_c - \underline{\underline{I}}) \otimes (\underline{\underline{B}}_{\bullet} - \underline{\underline{I}})] \\ &+ \frac{1}{2} c \sum_c \sum_{\bullet} \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{M}}_c \partial \underline{\underline{M}}_{\bullet}} \otimes (\underline{\underline{M}}_c \otimes \underline{\underline{M}}_{\bullet}) + \frac{1}{2} \text{tr} \sum_c \sum_{\substack{\bullet \\ c \neq d \neq \bullet}} \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{u}}_c \partial \underline{\underline{u}}_{\bullet}} (\underline{\underline{u}}_c \otimes \underline{\underline{u}}_{\bullet}) \\ &+ \text{tr} \sum_c \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{B}}_c \partial \theta} (\theta - \theta_0) (\underline{\underline{B}}_c - \underline{\underline{I}}) + c \sum_c \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{M}}_c \partial \underline{\underline{g}}} \otimes (\underline{\underline{g}} \otimes \underline{\underline{M}}_c) \\ &+ \text{tr} \sum_c \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{u}}_c \partial \underline{\underline{g}}} (\underline{\underline{g}} \otimes \underline{\underline{u}}_c)^T + c \sum_c \sum_{\substack{\bullet \\ \bullet \neq d}} \frac{\partial^2 \Psi(0)}{\partial \underline{\underline{u}}_c \partial \underline{\underline{M}}_{\bullet}} \otimes (\underline{\underline{M}}_{\bullet} \otimes \underline{\underline{u}}_c) \\ &+ O(\epsilon^3). \end{aligned} \tag{7.7}$$

²Harold Jeffreys, Cartesian Tensors (Cambridge, England: Cambridge University Press, 1961), pp. 66-70.

Equations (6.29)₂ and (7.7) imply that

$$\begin{aligned} \frac{\partial \Psi}{\partial \tilde{B}} = \frac{\partial \Psi}{\partial \tilde{B}}(0) + \tilde{C} \sum_{\alpha} \frac{\partial^2 \Psi(0)}{\partial \tilde{B} \partial \tilde{B}} \otimes (\tilde{B} - \tilde{I}) + \frac{\partial^2 \Psi(0)}{\partial \tilde{B} \partial \theta} (\theta - \theta_0) \\ + O(\epsilon^2), \end{aligned} \quad (7.8)$$

where, if $\tilde{A}$ is a linear transformation and $\tilde{\Omega}$ a fourth-order tensor, the contraction operator $\tilde{C}$ is defined by

$$\tilde{C}(\tilde{\Omega} \otimes \tilde{A}) = \Omega_{ijkl} A_{kl} \delta_i \delta_j \otimes \delta_k \delta_l. \quad (7.9)$$

If equation (7.5)₆ is rewritten as

$$\tilde{B} = \tilde{I} + (\tilde{B} - \tilde{I}) = \tilde{I} + O(\epsilon), \quad (7.10)$$

then it can be combined with equation (7.8) to yield

$$\begin{aligned} \tilde{B} \frac{\partial \Psi}{\partial \tilde{B}} = \frac{\partial \Psi}{\partial \tilde{B}}(0) + \frac{\partial \Psi}{\partial \tilde{B}}(0) (\tilde{B} - \tilde{I}) + \tilde{C} \sum_{\alpha} \frac{\partial^2 \Psi(0)}{\partial \tilde{B} \partial \tilde{B}} \otimes (\tilde{B} - \tilde{I}) \\ + \frac{\partial^2 \Psi(0)}{\partial \tilde{B} \partial \theta} (\theta - \theta_0) + O(\epsilon^2). \end{aligned} \quad (7.11)$$

Similarly, equation (7.7) implies

$$\frac{\partial \Psi}{\partial \tilde{u}} = O(\epsilon). \quad (7.12)$$

Equations (7.5)₉, (7.5)₁₀ and (7.12) can be used to show that

$$\sum_{\alpha} \tilde{u}_{\alpha} \otimes \left(\frac{\partial \Psi}{\partial \tilde{u}_{\alpha}} - \delta_{\alpha} \sum_{\beta} \frac{\partial \Psi}{\partial \tilde{u}_{\beta}} \right) = O(\epsilon^2). \quad (7.13)$$

Therefore, from equations (6.31), (7.7), (7.11), and (7.13) it follows that

$$\begin{aligned} \tilde{T}^T = & 2 \left[\frac{\partial \Psi}{\partial \tilde{B}}(0) + \frac{\partial \Psi}{\partial \tilde{B}}(0) (\tilde{B} - \tilde{I}) + \tilde{C} \sum_{\alpha} \frac{\partial^2 \Psi}{\partial \tilde{B} \partial \tilde{B}}(0) \otimes (\tilde{B} - \tilde{I}) \right. \\ & \left. + \frac{\partial^2 \Psi}{\partial \tilde{B} \partial \theta}(0) (\theta - \theta_0) \right] + \left[\Psi(0) + \frac{\partial \Psi}{\partial \theta}(0) (\theta - \theta_0) \right. \\ & \left. + \operatorname{tr} \sum_{\alpha} \frac{\partial \Psi}{\partial \tilde{B}}(0) (\tilde{B} - \tilde{I}) \right] \tilde{I} + O(\epsilon^2). \end{aligned} \quad (7.14)$$

From equation (7.5)₆ it follows that

$$\tilde{B} = \tilde{I} + 2\tilde{E} + O(\epsilon^2), \quad (7.15)$$

and

$$\tilde{E} = \frac{1}{2}(\tilde{H} + \tilde{H}^T) = \tilde{E}^T \quad (7.16)$$

is the infinitesimal strain measure. In terms of $\tilde{E}$ the stress can be expressed as

$$\begin{aligned} \tilde{T}^T = & \Psi(0) \tilde{I} + 2 \frac{\partial \Psi}{\partial \tilde{B}}(0) + 2 \left(\operatorname{tr} \sum_{\alpha} \frac{\partial \Psi}{\partial \tilde{B}}(0) \tilde{E} \right) \tilde{I} + 4 \frac{\partial \Psi}{\partial \tilde{B}}(0) \tilde{E} \\ & + 4 \tilde{C} \sum_{\alpha} \frac{\partial^2 \Psi}{\partial \tilde{B} \partial \tilde{B}}(0) \otimes \tilde{E} + \frac{\partial \Psi}{\partial \theta}(0) (\theta - \theta_0) \tilde{I} \\ & + 2 \frac{\partial^2 \Psi}{\partial \tilde{B} \partial \theta}(0) (\theta - \theta_0) + O(\epsilon^2), \end{aligned} \quad (7.17)$$

where all the coefficients are isotropic, since $\tilde{T}$ is an isotropic function. Equation (7.6) implies that there exist scalar constants p, β, φ, μ , and α , for each a and c , such that

$$\begin{aligned} \Psi(0) \tilde{I} + 2 \frac{\partial \Psi(0)}{\partial B} &= -p \tilde{I}, \\ 2 \frac{\partial \Psi(0)}{\partial B} &= \beta \tilde{I}, \end{aligned} \quad (7.18)$$

$$\begin{aligned} 4 \left(\delta_{ac} \delta_j \cdot \frac{\partial \Psi(0)}{\partial B_{1k}} + \frac{\partial^2 \Psi(0)}{\partial B_{1j} \partial B_{k1}} \right) \tilde{\delta}_1 \otimes \tilde{\delta}_j \otimes \tilde{\delta}_k \otimes \tilde{\delta}_1 \\ = \lambda \tilde{I} \otimes \tilde{I} + \mu \tilde{\delta}_1 \otimes \tilde{\delta}_j \otimes (\tilde{\delta}_1 \otimes \tilde{\delta}_j + \tilde{\delta}_j \otimes \tilde{\delta}_1), \end{aligned}$$

and

$$\frac{\partial \Psi(0)}{\partial \theta} \tilde{I} + 2 \frac{\partial^2 \Psi(0)}{\partial B \partial \theta} = \alpha \tilde{I},$$

where equation (7.9) and the symmetry of $\tilde{\mathbb{E}}$ have been used.

The fact that

$$\frac{\partial^2 \Psi(0)}{\partial B_{1j} \partial B_{k1}} = \frac{\partial^2 \Psi(0)}{\partial B_{k1} \partial B_{1j}} \quad (7.19)$$

implies that

$$\lambda_{ac} = \lambda_{ca}$$

and

$$\mu_{ac} = \mu_{ca}$$

(7.20)

in equation (7.18)₃. From equations (7.9), (7.17), and

(7.18) it follows that

$$\begin{aligned} \underline{\underline{T}} = \underline{\underline{T}}^T = - p \underline{\underline{I}} + \sum_{\alpha} \left[(\lambda_{\alpha} + \beta_{\alpha}) (\text{tr } \underline{\underline{E}}) \underline{\underline{I}} + 2\mu_{\alpha} \underline{\underline{E}} \right] \\ + \alpha (\theta - \theta_0) \underline{\underline{I}} + O(\epsilon^2). \end{aligned} \quad (7.21)$$

The scalar p can be interpreted as the residual stress on the a^{th} constituent.

It is the dependence on the symmetric quantities $\underline{\underline{B}}$, as was allowed by equation (6.12), together with the isotropic representations that causes the linearized stress in equation (7.21) to be symmetric. Steel's expression for the stress in the linear theory is not symmetric. Although he does not specify the conditions he imposed on the mixture to make it reflect material symmetry, Steel apparently required the constituents to be rotated together. This is equivalent to requiring ψ , for example, to satisfy

$$\psi(\theta, \underline{\underline{g}}, \underline{\underline{F}}, \nabla \underline{\underline{F}}, \underline{\underline{u}}) = \psi(\theta, \underline{\underline{g}}, \underline{\underline{FQ}}, \hat{C}(\nabla \underline{\underline{FQ}} \underline{\underline{Q}} \underline{\underline{Q}} \underline{\underline{Q}}), \underline{\underline{u}}), \quad (7.22)$$

for all constant linear transformations $\underline{\underline{Q}} \in \mathcal{O}$. This condition is necessary for equation (6.12) to hold but is not sufficient. Statements analogous to these can be made for the other constitutive functions. These weaker conditions would allow a dependence on mixed quantities such as $\underline{\underline{F}}^T \underline{\underline{F}}$. These tensors, which Steel uses, are not symmetric

and prevent his expression for the stress in the linear case from being symmetric.

Before $\hat{\underline{p}}_{\underline{\hat{\epsilon}}}$ is expanded, there are several results that follow from the equilibrium expression for $\hat{\underline{p}}_{\underline{\hat{\epsilon}}}$ that need to be pointed out. From equation (6.35) it follows that

$$\begin{aligned}\hat{\underline{p}}_{\underline{\hat{\epsilon}}}(\theta, \underline{\hat{\epsilon}}, \underline{\hat{B}}, \underline{\hat{\epsilon}}, \underline{\hat{\epsilon}}) &= \underline{\hat{\epsilon}}, \\ \frac{\partial \hat{\underline{p}}_{\underline{\hat{\epsilon}}}}{\partial \theta}(\theta, \underline{\hat{\epsilon}}, \underline{\hat{B}}, \underline{\hat{\epsilon}}, \underline{\hat{\epsilon}}) &= \underline{\hat{\epsilon}},\end{aligned}$$

and

$$\frac{\partial \hat{\underline{p}}_{\underline{\hat{\epsilon}}}}{\partial \underline{\hat{B}}}(\theta, \underline{\hat{\epsilon}}, \underline{\hat{B}}, \underline{\hat{\epsilon}}, \underline{\hat{\epsilon}}) = \underline{\hat{\epsilon}}.$$

(7.23)

The domain in which these equations are valid obviously includes the reference state $(\theta_0, \underline{\hat{\epsilon}}, \underline{\hat{I}}, \underline{\hat{\epsilon}}, \underline{\hat{\epsilon}})$. Also, the linearized version of equation (6.35) is

$$\hat{\underline{p}}_{\underline{\hat{\epsilon}}}(\theta, \underline{\hat{\epsilon}}, \underline{\hat{B}}, \underline{\hat{M}}, \underline{\hat{\epsilon}}) = 2\tilde{C}\left[\frac{\partial \Psi(0)}{\partial \underline{\hat{B}}}\right]_{\underline{\hat{\epsilon}}} \underline{\hat{M}}_{\underline{\hat{\epsilon}}} - \sum_{\underline{\hat{\epsilon}}} \frac{\partial \Psi(0)}{\partial \underline{\hat{B}}}\bigg|_{\underline{\hat{\epsilon}}} \underline{\hat{M}}_{\underline{\hat{\epsilon}}}, \quad (7.24)$$

where equations (7.5)₉ and (7.11) have been used. If $\hat{\underline{p}}_{\underline{\hat{\epsilon}}}$ is expanded about the reference state and equations (7.23) and (7.24) are used, the result is

$$\begin{aligned}\hat{\underline{p}}_{\underline{\hat{\epsilon}}} &= \frac{\partial \hat{\underline{p}}_{\underline{\hat{\epsilon}}}(0)}{\partial \underline{\hat{g}}}\underline{\hat{g}} + 2\tilde{C}\left[\frac{\partial \Psi(0)}{\partial \underline{\hat{B}}}\right]_{\underline{\hat{\epsilon}}} \underline{\hat{M}}_{\underline{\hat{\epsilon}}} - \sum_{\underline{\hat{\epsilon}}} \frac{\partial \Psi(0)}{\partial \underline{\hat{B}}}\bigg|_{\underline{\hat{\epsilon}}} \underline{\hat{M}}_{\underline{\hat{\epsilon}}} + \sum_{\substack{\underline{\hat{\epsilon}} \\ \underline{\hat{\epsilon}} \neq \underline{\hat{\epsilon}}}} \frac{\partial \hat{\underline{p}}_{\underline{\hat{\epsilon}}}(0)}{\partial \underline{\hat{u}}_{\underline{\hat{\epsilon}}}} \underline{\hat{u}}_{\underline{\hat{\epsilon}}} \\ &\quad + O(\epsilon^2). \quad (7.25)\end{aligned}$$

The fact that $\hat{\underline{p}}_{\underline{\hat{\epsilon}}}$ is an isotropic function makes all the

coefficients isotropic tensors. Equation (7.6) implies there exist scalar constants γ_a and ξ_{ac} , for each a and c ($c \neq d$), such that

$$\frac{\partial \hat{p}_a(0)}{\partial g_a} = -\gamma_a I$$

and

$$\frac{\partial \hat{p}_a(0)}{\partial u_{ac}} = -\xi_{ac} I$$

(7.26)

so that equation (7.25) reduces to

$$\begin{aligned} \hat{p}_a = & -\gamma_a g_a - \sum_c \left[\beta_{ac} \text{grad}(\text{tr } \tilde{E}) - \beta_{ca} \text{grad}(\text{tr } \tilde{E}) \right] \\ & - \sum_{\substack{c \\ c \neq d}} \xi_{ac} u_{cd} + O(\epsilon^2), \end{aligned} \quad (7.27)$$

where equations (7.5)₉, (7.16), (7.18)₂, and (7.26) have been used. From equations (2.63)₂ and (4.1)₄ it follows that

$$\sum_a \hat{p}_a = 0. \quad (7.28)$$

Equations (7.27) and (7.28) imply that

$$\sum_a \gamma_a = 0,$$

and

$$\sum_a \xi_{ac} = 0.$$

(7.29)

The heat flux $\tilde{q}$ is linearized in the same manner as $\hat{p}$. First, equation (5.25) implies that

$$\tilde{q}(\theta, \tilde{0}, \tilde{B}, \tilde{M}, \tilde{0}) = \tilde{0},$$

$$\frac{\partial \tilde{q}(\theta, \tilde{0}, \tilde{B}, \tilde{M}, \tilde{0})}{\partial \theta} = \tilde{0}, \quad (7.30)$$

$$\frac{\partial \tilde{q}(\theta, \tilde{0}, \tilde{B}, \tilde{M}, \tilde{0})}{\partial \tilde{B}} = \tilde{0},$$

and

$$\frac{\partial \tilde{q}(\theta, \tilde{0}, \tilde{B}, \tilde{M}, \tilde{0})}{\partial \tilde{M}} = \tilde{0},$$

Therefore, $\tilde{q}$ can be expanded to give

$$\tilde{q} = \frac{\partial \tilde{q}(0)}{\partial \tilde{g}} \tilde{g} + \sum_{a \neq d} \frac{\partial \tilde{q}(0)}{\partial \tilde{u}_a} \tilde{u}_a + o(\epsilon^2). \quad (7.31)$$

From equation (7.6) it follows that there exist scalar constants k and ζ_a , for each a ($a \neq d$), such that

$$\frac{\partial \tilde{q}(0)}{\partial \tilde{g}} = -k \tilde{I}$$

and

$$\frac{\partial \tilde{q}(0)}{\partial \tilde{u}_a} = -\zeta_a \tilde{I}. \quad (7.32)$$

Then equation (7.35) becomes

$$\tilde{q} = -k\tilde{g} - \sum_{a \neq d} \zeta_a \tilde{u}_a + o(\epsilon^2). \quad (7.33)$$

From equations (5.11), (6.29)₂, (7.7), and (7.15) it follows that

$$\rho \eta = - \frac{\partial \Psi}{\partial \theta} = - \left[\frac{\partial \Psi(0)}{\partial \theta} + \frac{\partial^2 \Psi(0)}{\partial \theta^2} (\theta - \theta_0) + 2 \operatorname{tr} \sum_a \frac{\partial^2 \Psi(0)}{\partial B \partial \theta} \tilde{E}_a + o(\epsilon^2) \right], \quad (7.34)$$

If the notation is used that $\rho_0 = \sum_a \rho_a$, then equation

(7.6) implies that there exists scalar constants η_0 , m , and π , for each a , such that

$$\frac{\partial \Psi(0)}{\partial \theta} = - \rho_0 \eta_0$$

$$\frac{\partial^2 \Psi(0)}{\partial \theta^2} = - m, \quad (7.35)$$

and

$$2 \frac{\partial^2 \Psi(0)}{\partial B \partial \theta} = - \pi \mathbf{I},$$

where from equation (7.18)₄ it follows that

$$- \rho_0 \eta_0 - \sum_a \pi = \sum_a \alpha. \quad (7.36)$$

Equations (7.34) and (7.35) show that

$$\rho_0 \eta = \rho_0 \eta_0 + m(\theta - \theta_0) + \sum_a (\rho_a \eta_0 + \pi) \operatorname{tr} \tilde{E}_a + o(\epsilon^2). \quad (7.37)$$

For completeness, it should be stated that equation (5.22), an expression for the quantity $\sum_a \rho_a \psi_a u_a$, provides no

new information or restrictions on the material constants introduced thus far. Next the equations of motion, the energy equation, and the entropy inequality will be derived for the linear case.

From equation (2.22)₂ it follows that

$$\ddot{\underline{w}} = \frac{\partial^2 \underline{w}}{\partial t^2} + O(\epsilon^2). \quad (7.38)$$

Also equations (7.2) and (7.16) and the identity

$$\operatorname{div}[(\operatorname{grad} \underline{w})^T] = \operatorname{grad}(\operatorname{div} \underline{w}) \quad (7.39)$$

imply that

$$\operatorname{div} \tilde{\underline{E}} = \frac{1}{2} [\operatorname{div}(\operatorname{grad} \underline{w}) + \operatorname{grad}(\operatorname{div} \underline{w})]. \quad (7.40)$$

With equations (7.1), (7.21), (7.27), (7.38), and (7.40), the equation of motion for the constituents, equation (2.59)₂, becomes

$$\begin{aligned} \rho_o \frac{\partial^2 \underline{w}}{\partial t^2} + \sum_{a \neq 1} \xi_{aa} \left(\frac{\partial \underline{w}}{\partial t} - \frac{\partial \underline{w}}{\partial t} \right) \\ = \sum_a \left[(\lambda_{aa} + \mu_{aa}) \operatorname{grad}(\operatorname{div} \underline{w}) + \beta_{aa} \operatorname{grad}(\operatorname{div} \underline{w}) \right. \\ \left. + \mu_{aa} \operatorname{div}(\operatorname{grad} \underline{w}) \right] + (\alpha - \gamma) \underline{g} + \rho_o \underline{b} + O(\epsilon^2). \end{aligned} \quad (7.41)$$

In the derivation of equation (7.41) the result that

$$\underline{\tilde{u}}_d = \frac{\partial \underline{\tilde{w}}}{\partial t} - \frac{\partial \underline{\tilde{x}}}{\partial t} + O(\epsilon^2) \quad (7.42)$$

has also been used. Equation (7.41) is written in terms of displacements instead of strains simply for convenience.

Equations (3.12), (5.10), (5.11), and (5.13) can be combined to give

$$\rho \dot{\underline{\epsilon}}_I = \rho \theta \dot{\eta} + \text{tr} \sum_a \left[\left(\underline{\tilde{T}}^T - \rho (\underline{\tilde{\psi}} - \underline{\tilde{\psi}}_I) \underline{\tilde{I}} - \sum_c \rho u_c \frac{\partial \underline{\tilde{\psi}}}{\partial \underline{\tilde{x}}} \dot{\underline{\tilde{F}}}^{-1} \right) \underline{\tilde{F}} \right]. \quad (7.43)$$

From equations (2.63)₄, (5.5), (7.1), (7.2), (7.16), (7.21), (7.33), (7.37), and (7.43), it follows that the linearized energy equation for the mixture is

$$\begin{aligned} k \operatorname{div} \underline{\tilde{g}} - \theta_0 m \frac{\partial \theta}{\partial t} - \theta_0 \sum_a \pi \operatorname{tr} \frac{\partial \underline{\tilde{E}}}{\partial t} + \sum_{a \neq d} (\zeta_a + \omega_a) \operatorname{tr} \left(\frac{\partial \underline{\tilde{E}}}{\partial t} - \frac{\partial \underline{\tilde{E}}}{\partial t} \right) \\ + \rho_0 r + O(\epsilon^2) = 0, \end{aligned} \quad (7.44)$$

where

$$\omega_a = \sum_c \left(\delta_{ac} - c_{0c} \right) \left(\underline{\tilde{\psi}}(0) + \underline{\tilde{p}} \right). \quad (7.45)$$

In equation (7.45) $c_{0c} = \rho_{0c} / \rho_0$.

The restrictions imposed by the entropy inequality can be obtained in two different ways. In the first the linearized entropy inequality is developed from inequality

(5.23); all restrictions then being derived from this inequality. The other method is to use inequality (5.28), the second differential of inequality (5.23). Taking the second differential of the entropy inequality is equivalent to discarding terms of the inequality higher than second order. This means the inequality (5.28) will yield the restrictions directly. However, it is more convenient to use the former method here.

From equations (3.14), (6.29)₁, (7.5)₁, (7.5)₁₁, (7.7), (7.21), (7.27), (7.29), and (7.33), it follows that entropy inequality (5.23) reduces to

$$\begin{aligned} & \underline{g} \cdot \left[k \underline{g} + \sum_{\substack{\mathbf{a} \\ \mathbf{a} \neq \mathbf{d}}} \zeta_{\mathbf{a} \mathbf{d}} \underline{u} + \sum_{\mathbf{a}} \left(\Psi(\mathbf{0}) + \underline{p} - \theta_0 \frac{\partial \Psi(\mathbf{0})}{\partial \theta} \right) \underline{u}_{\mathbf{a}} + O(\epsilon^2) \right] \\ & + \sum_{\mathbf{a}} \underline{u}_{\mathbf{a}} \cdot \left[\theta_0 \gamma \underline{g} + \sum_{\substack{\mathbf{c} \\ \mathbf{c} \neq \mathbf{d}}} \theta_0 \xi_{\mathbf{a} \mathbf{c} \mathbf{c} \mathbf{d}} \underline{u} + O(\epsilon^2) \right] \geq 0. \end{aligned} \quad (7.46)$$

Equations (2.14), (2.17), (7.5)₂, and (7.5)₁₁, inequality (7.46), and the notation that

$$\varphi_{\mathbf{a}} = \sum_{\mathbf{c}} \left(\delta_{\mathbf{a} \mathbf{c}} - c_{\mathbf{a} \mathbf{0}} \right) \left(\Psi(\mathbf{0}) + \underline{p} - \theta_0 \frac{\partial \Psi(\mathbf{0})}{\partial \theta} \right), \quad (7.47)$$

imply that

$$\begin{aligned} & k \underline{g} \cdot \underline{g} + \sum_{\substack{\mathbf{a} \\ \mathbf{a} \neq \mathbf{d}}} (\theta_0 \gamma + \zeta_{\mathbf{a}} + \varphi_{\mathbf{a}}) \underline{g} \cdot \underline{u}_{\mathbf{a}} + \sum_{\substack{\mathbf{a} \\ \mathbf{a} \neq \mathbf{d}}} \sum_{\substack{\mathbf{c} \\ \mathbf{c} \neq \mathbf{d}}} \theta_0 \xi_{\mathbf{a} \mathbf{c} \mathbf{c} \mathbf{d}} \underline{u}_{\mathbf{a}} \cdot \underline{u}_{\mathbf{c}} \\ & + O(\epsilon^3) \geq 0. \end{aligned} \quad (7.48)$$

Inequality (7.48) can be used to restrict the coefficients that appear in it; however, it is easier to do this after the number of constituents has been specified.

At this point it will be assumed that $n = 2$ (and $d = 2$) so that the equations can be compared to Steel's results. Equations (7.21), (7.27), (7.33), and (7.37) reduce to

$$\begin{aligned}
 \underline{T} &= - \underline{p} \underline{I} + (\lambda_1 + \beta_1) (\text{tr } \underline{\tilde{E}}_1) \underline{I} + (\lambda_2 + \beta_2) (\text{tr } \underline{\tilde{E}}_2) \underline{I} + 2\mu_1 \underline{\tilde{E}}_1 \\
 &\quad + 2\mu_2 \underline{\tilde{E}}_2 + \alpha(\theta - \theta_0) \underline{I}, \\
 \underline{T} &= - \underline{p} \underline{I} + (\lambda_2 + \beta_3) (\text{tr } \underline{\tilde{E}}_1) \underline{I} + (\lambda_3 + \beta_4) (\text{tr } \underline{\tilde{E}}_2) \underline{I} + 2\mu_2 \underline{\tilde{E}}_1 \\
 &\quad + 2\mu_3 \underline{\tilde{E}}_2 + \alpha(\theta - \theta_0) \underline{I}, \\
 \underline{\hat{p}} &= - \gamma \underline{g} + \beta_3 \text{grad}(\text{tr } \underline{\tilde{E}}_1) - \beta_2 \text{grad}(\text{tr } \underline{\tilde{E}}_2) - \xi_1 \underline{u}_2, \\
 \underline{q} &= - k \underline{g} - \zeta_1 \underline{u}_2,
 \end{aligned} \tag{7.49}$$

and

$$\rho_0 \eta = \rho_0 \eta_0 + m(\theta - \theta_0) + \pi_1 \text{tr } \underline{\tilde{E}}_1 + \pi_2 \text{tr } \underline{\tilde{E}}_2,$$

where the notation has been used that

$$\lambda_1 = \lambda_{11}, \lambda_2 = \lambda_{12}, \lambda_3 = \lambda_{22},$$

$$\mu_1 = \mu_{11}, \mu_2 = \mu_{12}, \mu_3 = \mu_{22},$$

$$\gamma = \gamma_1 = -\gamma_2,$$

(7.50)

$$\beta_1 = \beta_{11}, \beta_2 = \beta_{12}, \beta_3 = \beta_{21}, \beta_4 = \beta_{22},$$

$$\xi = \xi_{11} = -\xi_{21},$$

and

$$\zeta = \zeta_1.$$

Notice that $\hat{\underline{p}} = \hat{\underline{p}}_1 = -\hat{\underline{p}}_2$. Also, the term $O(\epsilon^2)$ has been suppressed.

For the binary mixture equation (7.41), the equation of motion for the constituents, yields

$$\begin{aligned} \rho_0 \frac{\partial^2 \underline{w}_1}{\partial t^2} + \xi \left(\frac{\partial \underline{w}_1}{\partial t} - \frac{\partial \underline{w}_2}{\partial t} \right) \\ = \mu_1 \operatorname{div}(\operatorname{grad} \underline{w}_1) + \nu_1 \operatorname{grad}(\operatorname{div} \underline{w}_1) \\ + \mu_2 \operatorname{div}(\operatorname{grad} \underline{w}_2) + \nu_2 \operatorname{grad}(\operatorname{div} \underline{w}_2) \\ + \sigma \underline{g} + \rho_0 \underline{b}_1 \end{aligned}$$

and

$$\begin{aligned} \rho_0 \frac{\partial^2 \underline{w}_2}{\partial t^2} - \xi \left(\frac{\partial \underline{w}_1}{\partial t} - \frac{\partial \underline{w}_2}{\partial t} \right) \\ = \mu_2 \operatorname{div}(\operatorname{grad} \underline{w}_1) + \nu_2 \operatorname{grad}(\operatorname{div} \underline{w}_1) \\ + \mu_3 \operatorname{div}(\operatorname{grad} \underline{w}_2) + \nu_3 \operatorname{grad}(\operatorname{div} \underline{w}_2) \\ + \sigma \underline{g} + \rho_0 \underline{b}_2, \end{aligned} \tag{7.51}$$

where

$$\nu_1 = \lambda_{11} + \beta_{11} + \beta_{21} + \mu_{11},$$

$$v = \lambda_{12} + \mu_{12},$$

$$v = \lambda_{22} + \beta_{12} + \beta_{22} + \mu_{22}, \quad (7.52)$$

$$\sigma_1 = \alpha_1 - \gamma_1,$$

and

$$\sigma_2 = \alpha_2 + \gamma_1.$$

The energy equation, equation (7.44), becomes

$$k \operatorname{div} \underline{g} - \theta_0 m \frac{\partial \theta}{\partial t} + \tau_1 \operatorname{tr} \frac{\partial \tilde{\mathbb{E}}_1}{\partial t} + \tau_2 \operatorname{tr} \frac{\partial \tilde{\mathbb{E}}_2}{\partial t} + \rho_0 r = 0, \quad (7.53)$$

where

$$\tau_1 = \zeta_1 + \omega_1 - \theta_0 \pi_1$$

and (7.54)

$$\tau_2 = -\zeta_1 - \omega_1 - \theta_0 \pi_2.$$

The entropy inequality (7.48) can be written as

$$k \underline{g} \cdot \underline{g} + \varphi \underline{g} \cdot \underline{u} + \theta_0 \xi_{12} \underline{u} \cdot \underline{u} \geq 0, \quad (7.55)$$

where the notation has been used that

$$\varphi = \theta_0 \gamma_1 + \zeta_1 + \varphi_1. \quad (7.56)$$

Inequality (7.55) can be used to show that

$$k \geq 0$$

and

$$4\theta_0 \xi k \geq \varphi^2. \quad (7.57)$$

These inequalities, in turn, imply that

$$\xi \geq 0. \quad (7.58)$$

Again in equation (7.51) and (7.53) and in inequality (7.55) the order term has been suppressed. It must be emphasized that inequalities (7.57) and (7.58) are necessary conditions that follow from the entropy inequality (5.23), but they are not sufficient to imply the validity of inequality (5.23), since higher order terms have been neglected.

Steel does not include an expanded expression for the equations of motion of the constituents such as equation (7.51); however, equations (7.49), (7.51), (7.53), and inequality (7.55) are formally equivalent to his results, provided the skew-symmetric part of his expression for the stress is set to zero.³ It must be pointed out though that most of the coefficients in these equations have different meanings. As mentioned earlier, the primary reasons for the differences are that Steel apparently rotates the constituents together when satisfying material symmetry and that he has adopted the formulation of Green & Naghdi.

³ Steel, op. cit., pp. 67-68.

8. WAVE PROPAGATION

In this chapter the propagation of small amplitude plane waves through a binary mixture of isotropic elastic solids infinite in extent is considered. Steel has studied this problem, so that there are some results available for comparison.¹ It is assumed that there is no heat supply or residual stress and that the free energy of both solids vanishes in the reference state. Furthermore, the entropy is taken to be a function of the temperature only, and the heat flux a function of the temperature gradient only. With these assumptions the energy equation, equation (7.53), has the solution $\theta = \theta_0$, for all $\underline{x}$ and all t . Also, it is supposed that there are no body forces. The waves are propagated in the X_1 direction ($\underline{X} = X_1 \underline{\delta}_1$) with frequency Ω and wave number K .

In the first case to be considered, solutions for K are sought where the displacement of each constituent is assumed to be proportional to $\exp[i(KX_1 - \Omega t)]$, i.e.,

$$\underline{w}_1 = \underline{a} \underline{\tilde{e}}^{i(KX_1 - \Omega t)}$$

and

$$\underline{w}_2 = \underline{d} \underline{\tilde{e}}^{i(KX_1 - \Omega t)},$$

(8.1)

¹Ibid., pp. 68-72.

where $\underline{a}$ and $\underline{d}$ are constant vectors. From equations (7.51) and (8.1) it follows that

$$\left[i\rho_0\Omega - \xi - \frac{iK^2}{\Omega}(\mu_1 + \nu_1) \right] a_1 + \left[\xi - \frac{iK^2}{\Omega}(\mu_2 + \nu_2) \right] d_1 = 0, \quad (8.2)$$

$$\left[i\rho_0\Omega - \xi - \frac{iK^2}{\Omega}\mu_1 \right] a_m + \left[\xi - \frac{iK^2}{\Omega}\mu_2 \right] d_m = 0, \quad m = 2, 3,$$

and

$$\left[\xi - \frac{iK^2}{\Omega}(\mu_2 + \nu_2) \right] a_1 + \left[i\rho_0\Omega - \xi - \frac{iK^2}{\Omega}(\mu_3 + \nu_3) \right] d_1 = 0, \quad (8.3)$$

$$\left[\xi - \frac{iK^2}{\Omega}\mu_2 \right] a_m + \left[i\rho_0\Omega - \xi - \frac{iK^2}{\Omega}\mu_3 \right] d_m = 0, \quad m = 2, 3,$$

where now $\nu_1 = \lambda_{11} + \mu_{11}$ and $\nu_2 = \lambda_{22} + \mu_{22}$. This

simplification of equations (7.52)₁ and (7.52)₃ follows

from equations (7.18)₁ and (7.18)₂ and the fact that

$$p_1 = p_2 = \psi_1(0) = \psi_2(0) = 0.$$

For transverse plane waves $a_1 = d_1 = 0$, so that equations (8.2)₁ and (8.3)₁ are automatically satisfied, and equations (8.2)₂ and (8.3)₂ imply that

$$\begin{aligned} \frac{1}{\Omega^2}(\mu_{13}\mu_2 - \mu_2^2)K^4 - \left[(\rho_{20}\mu_1 + \rho_{10}\mu_2) + \frac{i\xi}{\Omega}(\mu_1 + \mu_2 + 2\mu_3) \right] K^2 \\ + \rho_{10}\rho_{20}\Omega^2 + i\rho_0\Omega\xi = 0. \end{aligned} \quad (8.4)$$

This equation is almost identical in form to the corresponding equation in Steel's paper; however, the coefficients have different meaning.² Equation (8.4) yields four values of K , two whose real part is positive and two whose

²Ibid., p. 69.

real part is negative, meaning there are waves travelling in both the positive and negative X_1 directions.

In the ideal mixture there is no dependence of the theory on second deformation gradients and the partial stress on each solid is independent of the variables specifying the other solid, all interaction between the constituents being included in the diffusive force; therefore in equations (8.2) and (8.3)

$$\frac{v}{2} = 0$$

and

$$(8.5)$$

$$\frac{\mu}{2} = 0.$$

If the assumptions are made that ξ is small and $\rho_{01}\mu_1 \neq \rho_{03}\mu_3$, equations (8.4) and (8.5) imply

$$K = \pm \Omega \left(\frac{\rho_0}{\mu_3} \right)^{1/2} \pm \frac{i\hat{\xi}}{2} c_0 \left(\frac{\rho_0}{\mu_3} \right)^{1/2}$$

and

$$K = \pm \Omega \left(\frac{\rho_0}{\mu_1} \right)^{1/2} \pm \frac{i\hat{\xi}}{2} c_0 \left(\frac{\rho_0}{\mu_1} \right)^{1/2},$$

$$(8.6)$$

where, following Steel, the definition is made that

$$\hat{\xi} = \frac{\rho_0 \rho_0}{\rho_0} \xi. \quad (8.7)$$

Equations (8.6) are identical in form to Steel's results.³ Steel's other results for transverse plane waves in an ideal mixture can be used in this paper simply by changing notation; consequently they will not be restated.

For longitudinal plane waves $a_2 = a_3 = d_2 = d_3 = 0$; therefore, equations $(8.2)_2$ and $(8.3)_2$ are identically satisfied, and equations $(8.2)_1$ and $(8.3)_1$ give

$$\begin{aligned} \frac{1}{\Omega^2} (\delta \delta_{13} - \delta_2^2) K^4 - \left(\rho_0 \delta_{21} + \rho_0 \delta_{13} + \frac{i\xi}{\Omega} \delta \right) K^2 \\ + \rho_0 \rho_0 \Omega^2 + i \rho_0 \Omega \xi = 0, \end{aligned} \quad (8.8)$$

where

$$\begin{aligned} \delta_1 &= \mu_1 + \nu_1, \\ \delta_2 &= \mu_2 + \nu_2, \\ \delta_3 &= \mu_3 + \nu_3, \end{aligned} \quad (8.9)$$

and

$$\delta = \delta_1 + 2\delta_2 + \delta_3.$$

³Ibid.

Equation (8.8) is equivalent to Steel's equation (10.1),⁴ however, as before, the coefficients have different meaning. Since K is related to the wave speeds, different results are expected if wave speeds are calculated. Solving equation (8.8) results in four values for K , two whose real part is positive and two whose real part is negative, indicating that there are waves traveling in both the positive and negative X_1 directions.

If ξ is large, equation (8.8) gives

$$K^2 = \frac{i\delta\Omega\xi}{\delta\delta_{13} - \delta_{12}^2}$$

and

(8.10)

$$K = \pm \Omega \left(\frac{\rho_0}{\delta} \right)^{1/2} \left\{ 1 - \frac{i\Omega}{2\rho_0\xi\delta} \left[\left(\rho_0\delta_1 - \rho_0\delta \right) \left(\rho_0\delta_3 - \rho_0\delta \right) - \rho_0\delta_{12}^2 \right] \right\}.$$

The first solution is valid only if $\delta\delta_{13} \neq \delta_{12}^2$. The wave determined by equation (8.10)₁ is damped out rapidly, since ξ is large. The second solution, equation (8.10)₂, represents a wave whose velocity is that of one through a

⁴Ibid., p. 70.

single solid of density ρ_0 and which is lightly damped. Equation (8.10)₁ is equivalent to Steel's results, but equation (8.10)₂ is not, as it has not been possible to verify Steel's equation (10.4).⁵

If $\delta\delta_{13} = \delta_a^2$, then equation (8.8) yields the general solution

$$K^2 = \frac{\Omega^2 (\rho_{01} \rho_{02} \Omega + i \rho_0 \xi) [\Omega (\rho_{02} \delta_{13} + \rho_{01} \delta_a) - i \xi \delta_a]}{\Omega^2 (\rho_{02} \delta_{13} + \rho_{01} \delta_a)^2 + \xi^2 \delta_a^2} \quad (8.11)$$

When ξ is large, equation (8.11) can be shown to reduce to

$$K = \pm \Omega \left(\frac{\rho_0}{\delta} \right)^{1/2} \left\{ 1 - \frac{i \Omega}{2 \rho_0 \xi \delta} \left[\rho_{01} \rho_{02} \delta_{13} - \rho_0 (\rho_{02} \delta_{13} + \rho_{01} \delta_a) \right] \right\}, \quad (8.12)$$

which is exactly equation (8.10)₂ with $\delta\delta_{13} = \delta_a^2$. Steel obtains results formally identical to equations (8.11) and (8.12).⁶

If ξ is small and $\delta\delta_{13} \neq \delta_a^2$, equation (8.8) implies that

$$K^2 = \frac{\Omega^2}{2 (\delta\delta_{13} - \delta_a^2)} \left\{ \left(\rho_{02} \delta_{13} + \rho_{01} \delta_a \right) \pm \left[\left(\rho_{02} \delta_{13} - \rho_{01} \delta_a \right)^2 + 4 \rho_{01} \rho_{02} \delta_a^2 \right]^{1/2} \right\}, \quad (8.13)$$

⁵Ibid., p. 71.

⁶Ibid.

where terms involving ξ have been ignored. Since the radical in equation (8.13) is positive, Ω and the other coefficients being real, then K^2 is real and K is either real or imaginary. The case when both choices of K^2 are negative occurs only when $\rho_{02}\delta_1 + \rho_{01}\delta_3 < 0$ and $\delta_{13}\delta_2 - \delta_2^2 > 0$; then both waves travel infinitely fast and are damped. If $\delta_{13}\delta_2 - \delta_2^2 < 0$, one wave travels infinitely fast and is damped, and the other has finite velocity and is undamped. If $\delta_{13}\delta_2 - \delta_2^2 > 0$ and $\rho_{02}\delta_1 + \rho_{01}\delta_3 > 0$, both waves travel with finite velocity and are undamped.⁷ If ξ is small and $\delta_{13}\delta_2 = \delta_2^2$, equation (8.11) yields

$$K = \pm \Omega \left(\frac{\rho_{01}\rho_{02}}{\rho_{02}\delta_1 + \rho_{01}\delta_3} \right)^{1/2} \left\{ 1 - \frac{i\xi}{2\Omega} \left[\frac{\rho_{01}\rho_{02}\delta_2 - \rho_{02}(\rho_{02}\delta_1 + \rho_{01}\delta_3)}{\rho_{01}\rho_{02}(\rho_{02}\delta_1 + \rho_{01}\delta_3)} \right] \right\}. \quad (8.14)$$

Finally, it should be noticed that, when there is no interaction between the solids, the diffusive force is zero; so that $\xi = \delta_2 = 0$ in equation (8.13). Hence the solutions for K are

$$K = \pm \Omega \left(\frac{\rho_{01}}{\mu_1 + \nu_1} \right)^{1/2}$$

⁷Ibid.

and

$$K = \pm \Omega \left(\frac{\rho_0}{\mu + \nu} \right)^{1/2}, \quad (8.15)$$

the values that would exist in a single material.

In the other case to be considered, solutions for K are sought where, in contrast to the previous case, one of the constituents is assumed to have undergone a homogeneous strain. Thus, the displacements have the form

$$\tilde{w}_1 = \tilde{a} e^{i(KX_1 - \Omega t)},$$

as before, and (8.16)

$$\tilde{w}_2 = \tilde{H}_0 \tilde{X} + \tilde{d}$$

where $\tilde{H}_0$ is a constant linear transformation and $\tilde{a}$ and $\tilde{d}$ are constant vectors. Substitution of equation (8.16)₂ into equations (7.51) yields

$$\rho_0 \frac{\partial^2 \tilde{w}_1}{\partial t^2} + \xi \frac{\partial \tilde{w}_1}{\partial t} = \mu_1 \operatorname{div}(\operatorname{grad} \tilde{w}_1) + \nu_1 \operatorname{grad}(\operatorname{div} \tilde{w}_1)$$

and (8.17)

$$- \xi \frac{\partial \tilde{w}_1}{\partial t} = \mu_2 \operatorname{div}(\operatorname{grad} \tilde{w}_1) + \nu_2 \operatorname{grad}(\operatorname{div} \tilde{w}_1).$$

These two equations can be combined to give

$$\rho_0 \frac{\partial^2 \tilde{w}_1}{\partial t^2} = \mu \operatorname{div}(\operatorname{grad} \tilde{w}_1) + \nu \operatorname{grad}(\operatorname{div} \tilde{w}_1), \quad (8.18)$$

where

$$\mu = \mu_1 + \mu_2$$

and (8.19)

$$\nu = \nu_1 + \nu_2.$$

From equations (8.16)₁ and (8.18) it follows that

$$[\rho_0 \Omega^2 - (\mu + \nu) K^2] a_1 = 0$$

and (8.20)

$$[\rho_0 \Omega^2 - \mu K^2] a_m = 0, \quad m = 2, 3.$$

For transverse plane waves equation (8.20)₂ implies that

$$K = \pm \Omega \left(\frac{\rho_0}{\mu} \right)^{1/2}, \quad (8.21)$$

and for longitudinal plane waves equation (8.20)₁ shows that

$$K = \pm \Omega \left(\frac{\rho_0}{\mu + \nu} \right)^{1/2}. \quad (8.22)$$

Thus, the body undergoing homogeneous strain has no effect on the wave propagation. Both of these types of waves are undamped. It should be noticed that equation (8.18) is identically the equation of motion for a single material that occurs in classical elasticity. Thus the whole body of solutions for the displacement developed in that field are available for use here.

SELECTED BIBLIOGRAPHY

- Adkins, J. E. "Non-Linear Diffusion: I. Diffusion and Flow of Fluids," Phil. Trans. Roy. Soc. Lond., A/CCLV (1963), 607-633.
- _____. "Non-Linear Diffusion: II. Constitutive Equations for Mixtures of Isotropic Fluids," Phil. Trans. Roy. Soc. Lond., A/CCLV(1963), 635-648.
- _____. "Non-Linear Diffusion: III. Diffusion Through Isotropic Highly Elastic Solids," Phil. Trans. Roy. Soc. Lond., A/CCLVI(1964), 301-316.
- _____. "Diffusion of Fluids Through Aeolotropic Highly Elastic Solids," Archive for Rational Mechanics and Analysis, XV(1964), 222-234.
- Atkin, R. J. "Constitutive Theory for a Mixture of an Isotropic Elastic Solid and a Non-Newtonian Fluid," Zeitschrift für Angewandte Mathematik und Physik, XVIII(1967), 803-825.
- Bowen, R. M. "Toward a Thermodynamics and Mechanics of Mixtures," Archive for Rational Mechanics and Analysis, XXIV(1967), 370-403.
- Coleman, B. D., and W. Noll. "The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity," Archive for Rational Mechanics and Analysis, XIII(1963), 167-178.
- Crochet, M. J., and P. M. Naghdi. "On Constitutive Equations for Flow of Fluid Through an Elastic Solid," International Journal of Engineering Science, IV(1966), 383-401.
- _____. "Small Motions Superposed on Large Static Deformations in Porous Media," Acta Mechanica, IV(1967), 315-335.
- de Groot, S. R., and P. Mazur. Non-Equilibrium Thermodynamics. Amsterdam: North Holland Company, 1962.

- Dixon, R. C. "Diffusion of a Charged Fluid," International Journal of Engineering Science, V(1967), 265-287.
- Eringen, A. C., and J. D. Ingram. "A Continuum Theory of Chemically Reacting Media - I," International Journal of Engineering Science, III(1965), 197-212.
- Flügge, S. (ed.). Handbuch der Physik. 54 vols. Berlin-Göttingen-Heidelberg: Springer, 1955-1967.
- Green, A. E., and J. E. Adkins. "A Contribution to the Theory of Non-Linear Diffusion," Archive for Rational Mechanics and Analysis, XV(1964), 235-246.
- Green, A. E., and P. M. Naghdi. "A Dynamical Theory of Interacting Continua," International Journal of Engineering Science, III(1965), 231-241.
- _____. "A Theory of Mixtures," Archive for Rational Mechanics and Analysis, XXIV(1967), 243-263.
- Green, A. E., and T. R. Steel. "Constitutive Equations for Interacting Continua," International Journal of Engineering Science, IV(1966), 483-500.
- Greub, W. H. Linear Algebra. Second edition. New York: Academic Press, 1963.
- Gurtin, M. E. "Thermodynamics and the Possibility of Spatial Interaction in Elastic Materials," Archive for Rational Mechanics and Analysis, XIX(1965), 339-352.
- Jeffreys, H. Cartesian Tensors. Cambridge, England: Cambridge University Press, 1961.
- Kelly, P. D. "A Reacting Continuum," International Journal of Engineering Science, II(1964), 129-153.
- Mills, N. "Incompressible Mixtures of Newtonian Fluids," International Journal of Engineering Science, IV(1966), 97-112.
- _____. "On a Theory of Multi-Component Mixtures," Quarterly Journal of Mechanics and Applied Mathematics, XX(1967), 499-508.

- Müller, I. "A Thermodynamic Theory of Mixtures of Fluids," Archive for Rational Mechanics and Analysis, XXVIII (1968), 1-39.
- Noll, W. "On the Continuity of the Solid and Fluid States," Journal of Rational Mechanics and Analysis, IV(1955), 3-81.
- _____. The Axiomatic Method, with Special Reference to Geometry and Physics. Amsterdam: North Holland Company, 1959.
- Prigogine, I., and R. Defay. Chemical Thermodynamics. Translated by D. H. Everett. London-New York-Toronto: Longmans Green, 1954.
- Steel, T. R. "Applications of a Theory of Interacting Continua," Quarterly Journal of Mechanics and Applied Mathematics, XX(1967), 57-72.
- Truesdell, C. "Sulle Basi della Termomeccanica," Atti della Accademia Nazionale dei Lincei, XXII(1957), 33-38, 158-166.
- _____. "Mechanical Basis of Diffusion," Journal of Chemical Physics, XXXVII(1962), 2336-2344.
- _____, and W. Noll. The Non-Linear Field Theories of Mechanics. Vol. III/3 of Handbuch der Physik. Edited by S. Flügge. 54 vols. Berlin-Heidelberg-New York: Springer, 1965.
- _____, and R. Toupin. The Classical Field Theories. Vol. III/1 of Handbuch der Physik. Edited by S. Flügge. 54 vols. Berlin-Göttingen-Heidelberg: Springer, 1960.

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