2013

Large deviations for stochastic Navier-Stokes equations with nonlinear viscosities

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LARGE DEVIATIONS FOR STOCHASTIC NAVIER-STOKES EQUATIONS WITH NONLINEAR VISCOSITIES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
Ming Tao
B.S. in Math. USTC, 2004
M.S. in Math. USTC, 2007
May 2013

Acknowledgements

This dissertation would not be possible without several contributions. First of all, I am extremely grateful to my advisor, Professor Sundar, for his constant encouragement and guidance throughout this work.

Meanwhile, I really wish to express my sincere thanks to all the committee members, Professors Kuo, Richardson, Sage, Stoltzfus and the Dean's representative Professor Koppelman, for their help and suggestions on the corrections of this dissertaion.

I also take this opportunity to thank the Mathematics department of Louisiana State University for providing me with a pleasant working environment and all the necessary facilities.

This dissertation is dedicated to my parents for their unceasing support and encouragement.

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Abstract

In this work, a Wentzell-Freidlin type large deviation principle is established for the two-dimensional stochastic Navier-Stokes equations (SNSE's) with nonlinear viscosities. We first prove the existence and uniqueness of solutions to the two-dimensional stochastic Navier-Stokes equations with nonlinear viscosities using the martingale problem argument and the method of monotonicity. By the results of Varadhan and Bryc, the large deviation principle (LDP) is equivalent to the Laplace-Varadhan principle (LVP) if the underlying space is Polish. Then using the stochastic control and weak convergence approach developed by Budhiraja and Dupuis, the Laplace-Varadhan principle for solutions of stochastic Navier-Stokes equations is obtained in appropriate function spaces.

Chapter 1
Introduction

The theory of large deviations is one of the most active topics in the probability theory with many deep developments and applications to many areas including statistical mechanics, communication networks, information theory, risk-sensitive control and queueing systems. The framework for the theory along with important applications can be found in the book by Varadhan [42]. The proofs of large deviation principle (LDP) have usually relied on first approximating the original problem by time-discretization so that LDP can be shown for the resulting simpler problems via contraction principle and then showing that LDP holds in the limit. The discretization method to establish LDP was introduced by Wentzell and Freidlin [12]. Several authors have proved the Wentzell-Freidlin type large deviation results for the two-dimensional stochastic Navier-Stokes equations (SNSE's) with multiplicative noise (e.g. S.S. Sritharan, P. Sundar [36]). Most of these works are concerned with the case of Newtonian (linear) viscosity.

In this work we obtain the results for the two-dimensional stochastic Navier-Stokes equations with nonlinear and hyperviscosities. The deterministic Navier-Stokes equations with nonlinear viscosity was introdued by Ladyzhenskaya [18],[19] and hyperviscosity was introduced by Lions [21],[22] to demonstrate global unique solvability. We consider the stochastic versions of these equations in the martingale problem framework of Stroock-Varadhan [37]. The existence of admissible martingale solutions are proven by using weak limits and Galerkin approximations. Using the equivalence between the martingale solution and the weak solution, one can claim the existence of the weak solution (in the sense of being unique in law).

As another approach, the method of monotonicity is also employed to obtain the existence and uniqueness of the strong solution (in the sense of partial differential equations as well as stochastic analysis) to the two-dimensional stochastic Navier-Stokes equations with nonlinear viscosities.

Dupuis and Ellis [6] have recently combined weak convergence methods with the stochastic control approach developed earlier by Fleming [11] to the large deviations theory. This approach was motivated by a deterministic result of Laplace which states that for any given $h \in C([0,1])$,

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 e^{-nh(x)} dx = - \min_{x\in[0,1]} h(x) \tag{1.0.1}$$

Varadhan's lemma ([6],[42]) and Bryc's converse ([6]) show that the large deviation principle (LDP) is equivalent to the Laplace-Varadhan principle (LVP) if the underlying space is Polish. We refer the reader to the first part of the book by Dupuis and Ellis [6].

The main goal of this work is to prove the LDP for two-dimensional stochastic Navier-Stokes equations with nonlinear and hyperviscosities, which is based on the theory introduced by Budhiraja and Dupuis [3]. In their work, they used the stochastic control and weak convergence approach to obtain the LVP for the family $\{g^\varepsilon(W(\cdot))\}_{\varepsilon>0}$, where g^ε is an appropriate family of measurable maps from the Wiener space to some Polish space. Using this large deviation principle, we can then derive Wentzell-Freidlin type large deviation results for the stochastic Navier-Stokes equations with nonlinear viscosities.

The structure of the present work is as follows. In Chapter 2, some basic definitions and well-known results from the large deviation theory are given, and the Wentzell-Freidlin theory is briefly described. In Chapter 3, under certain functional setting, the abstract evolution equation of the two-dimensional stochastic Navier-

Stokes equations with nonlinear and hyperviscosities is formulated. In Chapter 4, some energy estimates are proved, and then the weak limit and Galerkin approximation arguments are employed to show the existence of martingale solutions. In Chapter 5, the method of monotonicity is used to obtain the existence and uniqueness of strong solutions for the SNSE's with nonlinear viscosities. In Chapter 6, a variational representaion of positive functionals of an infinite dimensional Brownian motion is proved. This representation is the crucial step in the proofs of the LDP for a wide class of stochastic dynamical systems driven by a small noise. Finally, the theory of Budhiraja and Dupuis is briefly described to set up a base for our main result, and then the Laplace-Varadhan principle for the two-dimensional stochastic Navier-Stokes equations with nonlinear viscosities is established.

Chapter 2
Large Deviations Theory

2.1 Basic Definitions and Results

Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a Polish space E (i.e. a complete separable metric space).

Definition 2.1.1. A function $I : E \to [0, \infty]$ is called a *rate function* if I is lower semicontinuous. A rate function I is called a *good rate function* if for any $M < \infty$, the level set $K_M = \{x \in E : I(x) \leq M\}$ is compact in E.

Definition 2.1.2. (**Large Deviation Principle**) We say that the sequence $\{X_n\}$ obeys the *large deviation principle* (LDP) with a good rate function I if

(i) for each closed set $F \subset E$

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \in F\} \leq -\inf_{x\in F} I(x) \tag{2.1.1}$$

(ii) for each open set $G \subset E$

$$\liminf_{n\to\infty} \frac{1}{n} \log P\{X_n \in G\} \geq -\inf_{x\in G} I(x) \tag{2.1.2}$$

If P_n denotes the distribution of X_n (i.e. $P_n(A) = P\{\omega : X_n \in A\}$ for any Borel subset A of E), then we also say that the family of probability measures $\{P_n\}$ satisfies the large deviation princeiple with rate function I.

We next recall two well-known theorems that have many important applications in the theory of large deviations.

Cramér 's Theorem [42]. Let us denote by $X_n = (\xi_1 + \xi_2 + \cdots + \xi_n)/n$, where $\xi_1, \xi_2, \cdots, \xi_n$ are n independent random variables with a common distribution α and P_n be the distribution of X_n.

Assume that, for all θ, the moment generating function

$$M(\theta) = \mathbb{E}[\exp(\theta\xi)] = \int e^{\theta x} d\alpha(x) < \infty \tag{2.1.3}$$

Then the sequence of probability measures $\{P_n\}$ satisfies the large deviation principle with a rate function $I(\cdot)$ given by

$$I(x) = \sup_{\theta}[\theta x - \log M(\theta)]. \tag{2.1.4}$$

For example, let $\xi_1, \xi_2, \cdots, \xi_n$ be n independent and identically distributed (i.i.d) $\mathcal{N}(0,1)$ random variables, then

$$P_n(A) = \sqrt{\frac{n}{2\pi}} \int_A \exp(-\frac{nx^2}{2})dx \tag{2.1.5}$$

and $\{P_n\}$ satisfies the LDP with the rate function $I(x) = \frac{x^2}{2}$.

Schilder's Theorem [6]. Let $C_0([0,1] : \mathbb{R}^d)$ denote the space consisting of all continuous functions f that map $[0,1]$ to $\mathbb{R}^d$ and vanish at the origin, i.e. any element f of $C_0([0,1] : \mathbb{R}^d)$ satisfies $f(0) = 0$. When equipped with the supremum norm, $C_0([0,1] : \mathbb{R}^d)$ is a Banach space.

Let $\{W(t) : t \in [0,1]\}$ be a standard Brownian motion in $\mathbb{R}^d$ and we define for $n \in \mathbb{N}$ the process

$$Y_n(t) := \frac{1}{\sqrt{n}} W(t)$$

Then Schilder's Theorem states that $\{Y_n\}$ satisfies the large deviation principle with a rate function $I(f)$ defined for $f \in C_0([0,1] : \mathbb{R}^d)$ as follows:

$$I(f) = \frac{1}{2} \int_0^1 (f^{'}(t))^2 dt \tag{2.1.6}$$

if $f(t)$ is absolutely continuous with a square integrable derivative $f'(t)$; Otherwise $I(f) = \infty$.

Remark: Schilder's Theorem has many important applications in large deviation theory, such as in the derivation of the Strassen's renowned Law of Iterated Logarithm, in the Wentzell-Freidlin's estimate on the large deviations of randomly perturbed dynamical systems, to name a few.

Next, we give the definiton of the Laplace-Varadhan principle which provides an evaluation, for all bounded continuous functions h mapping E into $\mathbb{R}$, of asymptotics of quantities of the form $\frac{1}{n}\log\mathbb{E}\{\exp[-nh(X_n)]\}$, as $n\to\infty$. The weak convergence approach is ideally suited to such evaluations.

Definition 2.1.3. (**Laplace-Varadhan Principle**) Let I be a rate function on the Polish space E. The sequence $\{X_n\}$ is said to satisfy the *Laplace-Varadhan principle* on E with rate function I if for all bounded continuous funtions h

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\{\exp[-nh(X_n)]\} = -\inf_{x\in E}\{h(x)+I(x)\} \tag{2.1.7}$$

In the following, Varadhan' Lemma and Bryc's converse lemma show that the large deviation principle (LDP) and the Laplace-Varadhan principle (LVP) are equivalent for Polish space valued random elements. Let E be a Polish space and $(\Omega,\mathcal{F},P)$ be a probability space.

Lemma 2.1.4. *(Varadhan's Lemma [6],[42]) Let $\{X_n\}$ be a family of E-valued random elements defined on $(\Omega,\mathcal{F},P)$ and satisfying the LDP with rate function I. Then $\{X_n\}$ satisfies the LVP with the same rate function I.*

Lemma 2.1.5. *(Bryc's converse [6]) The LVP implies the LDP with the same rate function. More precisely, if I is a rate function on E and for all bounded countinuous functions h,*

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\{\exp[-nh(X_n)]\} = -\inf_{x\in E}\{h(x)+I(x)\}$$

then $\{X_n\}$ satisfies the large deviation principle with rate function I.

2.2 The Wentzell-Freidlin Theory

Let us consider the stochastic differential equation on $\mathbb{R}^d$:

$$d\mathbf{x}_\varepsilon(t) = b(\mathbf{x}_\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(\mathbf{x}_\varepsilon(t))dW(t), \qquad \mathbf{x}_\varepsilon(0) = \xi \tag{2.2.1}$$

where the process $W(t)$ is a d-dimensional Brownian motion.

Existence and uniqueness of the solution to the stochastic differential equation (2.2.1) can be obtained if the coefficients $b(\cdot)$ and $\sigma(\cdot)$ are assumed to be globally Lipschitz and with linear growth.

The unique solution $\mathbf{x}_\varepsilon(t)$ is the diffusion process corresponding to the operator:

$$L_\varepsilon = \frac{1}{2}\varepsilon \sum_{i,j} a_{ij}(\mathbf{x}_\varepsilon)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j b_j(\mathbf{x}_\varepsilon)\frac{\partial}{\partial x_j} \tag{2.2.2}$$

where $a(\mathbf{x}_\varepsilon) = \sigma(\mathbf{x}_\varepsilon)\sigma^*(\mathbf{x}_\varepsilon)$ and we assume further that $a(\cdot)$ is uniformly elliptic and bounded.

Let $P_{\varepsilon,\xi}$ be the measure induced by $\mathbf{x}_\varepsilon(t)$ on $C[0,T]$, where $C[0,T]$ denotes the space of $\mathbb{R}^d$-valued continuous functions on some arbitrary but finite time interval $[0,T]$. The measure $P_{\varepsilon,\xi}$ is concentrated on the trajectories that start from ξ at time 0:

$$P_{\varepsilon,\xi}\{\mathbf{x}_\varepsilon(\cdot) : \mathbf{x}_\varepsilon(0) = \xi\} = 1 \tag{2.2.3}$$

Then $\{P_{\varepsilon,\xi}\}$ satisfies the large deviation principle (LDP) with a rate function $I_\xi(f)$ defined for $f \in C[0,T]$ as follows:

$$I_\xi(f) = \frac{1}{2}\int_0^T \langle f'(t) - b(f(t)), a^{-1}(f'(t) - b(f(t)))\rangle dt \tag{2.2.4}$$

if $f(0) = \xi$ and $f'(t)$ is square-integrable on $[0,T]$; Otherwise, $I_\xi(f) = \infty$.

As the simplest 1-D situation, we take $\sigma = 1$ and $b = 0$, then $a = 1$ and

$$I_\xi(f) = \frac{1}{2}\int_0^T (f'(t))^2 dt \tag{2.2.5}$$

The stochastic differential equation reduces to :

$$\mathbf{x}_\varepsilon(t) = \sqrt{\varepsilon} W(t) \tag{2.2.6}$$

and $P_{\varepsilon,\xi}$ reduces to the Wiener measure P_ε, which coincides with the previous infinite dimensional example: Schilder's Theorem.

Chapter 3
Stochastic Navier-Stokes Equations

In physics, the Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the fluid stress is the sum of a diffusing viscous term (proportional to the gradient of velocity), plus a pressure term.

The equations are useful because they describe the physics of many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier-Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with Maxwell's equations they can be used to model and study magnetohydrodynamics.

The Navier-Stokes equations are also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, mathematicians have not yet proven that in three dimensions solutions always exist (existence), or that if they do exist, then they do not contain any singularity (smoothness). These are called the Navier-Stokes existence and smoothness problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US dollars 1,000,000 prize for a solution or a counter-example.

The Navier-Stokes equations dictate not position but rather velocity. A solution of the Navier-Stokes equations is called a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. Once the velocity field is solved for, other quantities of interest (such as flow rate or drag force) may be found. This is different from what one normally sees in classical mechanics, where solutions are typically trajectories of position of a particle or deflection of a continuum. Studying velocity instead of position makes more sense for a fluid; however for visualization purposes one can compute various trajectories.

A very significant feature of the Navier-Stokes equations is the presence of convective acceleration: the effect of time independent acceleration of a fluid with respect to space. While individual fluid particles are indeed experiencing time dependent acceleration, the convective acceleration of the flow field is a spatial effect. The effect of stress in the fluid is interpretated by gradients of surface forces, analogous to stresses in a solid.

3.1 Nonlinear and Hyperviscosities Models

Let $G \subset \mathbb{R}^n$, $n = 2$, be an arbitrary bounded open domain with a smooth boundary ∂G, and $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $t \in [0, T]$, we consider the stochastic Navier-Stokes equation for a viscous impressible flow with a non-slip condition at the boundary :

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \eta(\mathbf{u}) = \mathbf{f}(t) + \sigma(t, \mathbf{u})\frac{dW_t}{dt} \tag{3.1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \forall\ (x, t, \omega) \in G \times (0, T) \times \Omega \tag{3.1.2}$$

$$\mathbf{u}(x, t, \omega) = 0, \quad \forall\ (x, t, \omega) \in \partial G \times (0, T) \times \Omega \tag{3.1.3}$$

$$\mathbf{u}(x, 0, \omega) = \mathbf{u}_0(x, \omega), \qquad \forall\ (x, \omega) \in G \times \Omega \tag{3.1.4}$$

In the above, $\mathbf{u} = (u_1, u_2)$ is the two dimensional velocity and $\eta(\mathbf{u})$ denotes the (possibly nonlinear) stress tensor. Convective acceleration is represented by the nonlinear quantity: $(\mathbf{u} \cdot \nabla)\mathbf{u}$. The vector field $\mathbf{f}(t)$ represents the external body force, and typically these consist of only gravity forces, but may include other types (such as electromagnetic forces). The process $\{W_t\}$ is an infinite-dimensional Hilbert space-valued Wiener process (Definition 3.1.1) for an appropriate Hilbert space.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0\leq t\leq T}$ of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and P-completeness. Let H be a real separable Hilbert space and Q be a strictly positive definite, symmetric, trace class operator (Definition 3.2.7) on H.

Definition 3.1.1. A stochastic process $\{W(t)\}_{0\leq t\leq T}$ is said to be an H-valued $\mathcal{F}_t$ adapted Wiener process with covariance operator Q if

(1) for each non-zero $h \in H$, $|Q^{1/2}h|_H^{-1}(W(t), h)$ is a standard one-dimensional Wiener process, and

(2) for any $h \in H$, $(W(t), h)$ is a $\mathcal{F}_t$-adapted martingale.

Throughout this paper, we will be working on the nonlinear and hyperviscosities models in the following cases for the stress tensor $\eta(\mathbf{u})$.

Case 1: Nonlinear Constitutive Relationship [18],[19],[21],[22]

$$\eta_1(\mathbf{u}) := -p\mathbf{I} + \nu_0\nabla\mathbf{u} + \nu_1|\nabla\mathbf{u}|_{\mathbb{R}^n}^{q-2}\nabla\mathbf{u} \tag{3.1.5}$$

where p denotes the pressure and is a scalar-valued function, $\nu_0, \nu_1 > 0$ and $q \geq 3$. In this case,

$$\nabla \cdot \eta_1(\mathbf{u}) = -\nabla p + \nu_0\Delta\mathbf{u} + \nu_1\nabla \cdot (|\nabla\mathbf{u}|_{\mathbb{R}^n}^{q-2}\nabla\mathbf{u}) \tag{3.1.6}$$

Case 2: Nonlinear Nonlocal Viscosity [19]

$$\eta_2(\mathbf{u}) = -p\mathbf{I} + (\nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)})\nabla\mathbf{u} \tag{3.1.7}$$

In this case, the nonlinear viscosity is given by

$$\nu(\|\nabla\mathbf{u}\|_{L^2(G)}) := \nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)} \tag{3.1.8}$$

where $\nu_0, \nu_1 > 0$ and $\|\nabla\mathbf{u}\|^2_{L^2(G)} := \int_G |\nabla\mathbf{u}(x)|^2_{\mathbb{R}^n} dx$, so that

$$\nabla \cdot \eta_2(\mathbf{u}) = -\nabla p + (\nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)})\Delta\mathbf{u} \tag{3.1.9}$$

Case 3: Hyperviscosity [22]

$$\eta_3(\mathbf{u}) = -p\mathbf{I} + \nu_0\nabla\mathbf{u} - \nu_1(-1)^m\nabla(\Delta^{m-1}\mathbf{u}) \tag{3.1.10}$$

with $m \geq 2$ and $\nu_0, \nu_1 > 0$. In this case, we prescribe additional boundary conditions $(\partial\mathbf{u}/\partial n)\mid_{\partial G} = \cdots = (\partial^{m-1}\mathbf{u}/\partial n^{m-1})\mid_{\partial G} = 0$ and we have

$$\nabla \cdot \eta_3(\mathbf{u}) = -\nabla p + \nu_0\Delta\mathbf{u} - \nu_1(-1)^m\Delta^m\mathbf{u} \tag{3.1.11}$$

This type of regularization has been used in atomspheric dynamics models and also in the study of vortex reconnections [27],[20].

3.2 Functional Setting

The stochastic Navier-Stokes equation (3.1.1) can be written in the abstract evolution form for bounded domains by introducing the following function spaces.

Let $\mathcal{V}$ denote the space of $C_c^\infty(G)$ functions which are divergence free. Define the sapces H and $V_{r,q}$ as the completion of $\mathcal{V}$ in $L^2(G)$ and in $W^{r,q}(G)$ norms respectively. For bounded domains, H and $V_{r.q}$ can be characterized as follows:

$$\begin{aligned} H &:= \{\mathbf{u} \in L^2(G) : \nabla \cdot \mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}\mid_{\partial G} = 0\} \\ V_{r,q} &:= \{\mathbf{u} \in W_0^{r,q}(G) : \nabla \cdot \mathbf{u} = 0\} \end{aligned}$$

where $\mathbf{n}$ is the outward normal, and $W_0^{r,q}(G)$ is the closure of $C_c^\infty(G)$ in $W^{r,q}(G)$. The choice for r and q in each case is given below.

Case 1, Nonlinear Constitutive Relationship: $r = 1, q \geq 3$;

Case 2, Nonlinear Nonlocal Viscosity: $r = 1, q = 2$;

Case 3, Hyperviscosity: $r = m \geq 2, q = 2$.

We define the operators $\mathcal{A}_i$ $(i = 1, 2, 3)$ as follows:

$$\langle \mathcal{A}_1(\mathbf{u}), \mathbf{v} \rangle := \int_G |\nabla \mathbf{u}|_{\mathbb{R}^n}^{q-2} \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, \qquad \forall \mathbf{u}, \mathbf{v} \in V_{1,q} \tag{3.2.1}$$

$$\langle \mathcal{A}_2(\mathbf{u}), \mathbf{v} \rangle := \|\nabla \mathbf{u}\|_{L^2(G)}^2 \int_G \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, \qquad \forall \mathbf{u}, \mathbf{v} \in V_{1,2} \tag{3.2.2}$$

$$\langle \mathcal{A}_3(\mathbf{u}), \mathbf{v} \rangle := \sum_{\alpha \in \mathbb{N}, |\alpha| = m} \int_G D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{v} dx, \qquad \forall \mathbf{u}, \mathbf{v} \in V_{m,2} \tag{3.2.3}$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing (integral) in G.

Then we have the following estimates

$$|\langle \mathcal{A}_1(\mathbf{u}), \mathbf{v} \rangle| \leq \|\nabla \mathbf{u}\|_{L^q(G)}^{q-1} \|\nabla \mathbf{v}\|_{L^q(G)} \tag{3.2.4}$$

$$\langle \mathcal{A}_1(\mathbf{u}), \mathbf{u} \rangle = \|\nabla \mathbf{u}\|_{L^q(G)}^q = \|\mathbf{u}\|_{V_{1,q}(G)}^q \tag{3.2.5}$$

$$|\langle \mathcal{A}_2(\mathbf{u}), \mathbf{v} \rangle| \leq \|\nabla \mathbf{u}\|_{L^2(G)}^3 \|\nabla \mathbf{v}\|_{L^2(G)} \tag{3.2.6}$$

$$\langle \mathcal{A}_2(\mathbf{u}), \mathbf{u} \rangle = \|\nabla \mathbf{u}\|_{L^2(G)}^4 = \|\mathbf{u}\|_{V_{1,2}(G)}^4 \tag{3.2.7}$$

$$|\langle \mathcal{A}_3(\mathbf{u}), \mathbf{v} \rangle| \leq \sum_{\alpha \in \mathbb{N}, |\alpha| = m} \|D^\alpha \mathbf{u}\|_{L^2(G)} \|D^\alpha \mathbf{v}\|_{L^2(G)} \tag{3.2.8}$$

$$\langle \mathcal{A}_3(\mathbf{u}), \mathbf{u} \rangle = \|\mathbf{u}\|_{V_{m,2}}^2 \geq C \|\mathbf{u}\|_{H^m}^2 \tag{3.2.9}$$

where the $V_{m,2}$-norm is

$$\|\mathbf{u}\|_{V_{m,2}} = \left(\sum_{\alpha\in\mathbb{N},|\alpha|=m} ||D^\alpha \mathbf{u}||^2_{L^2(G)} \right)^{1/2} \tag{3.2.10}$$

The inequality in (3.2.9) is valid due to the additional boundary conditions in the Case 3: $(\partial \mathbf{u}/\partial n)\mid_{\partial G} = \cdots = (\partial^{m-1}\mathbf{u}/\partial n^{m-1})\mid_{\partial G} = 0$.

In the following, we give some monotonicities of the operators $\mathcal{A}_i$ $(i = 1, 2, 3)$ which will be used later.

Definition 3.2.1. A mapping $\mathcal{A} : V \to V'$ is said to be monotone if for any $\mathbf{u}, \mathbf{v} \in D(\mathcal{A})$, we have $\langle \mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle \geq 0$. And $\mathcal{A}$ is called strictly monotone if $\langle \mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle = 0$ implies $\mathbf{u} = \mathbf{v}$.

By the definition of $\mathcal{A}_3$, it is linear, self-adjoint and positive definite. With $\mathbf{u} - \mathbf{v}$ replacing $\mathbf{u}$ in (3.2.9), we conclude that $\langle \mathcal{A}_3(\mathbf{u}) - \mathcal{A}_3(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle = ||\mathbf{u} - \mathbf{v}||^2_{V_{m,2}} \geq 0$ (i.e. , $\mathcal{A}_3$ is strictly monotone).

Lemma 3.2.2. *The nonlinear viscous operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are strictly monotone:* $\forall \mathbf{u}, \mathbf{v} \in \mathcal{D}(G)$

$$\langle \mathcal{A}_1(\mathbf{u}) - \mathcal{A}_1(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle \geq \gamma(n, q)\|\nabla\mathbf{u} - \nabla\mathbf{v}\|^q_{L^q(G)} \tag{3.2.11}$$

and

$$\begin{aligned} \langle \mathcal{A}_2(\mathbf{u}) - \mathcal{A}_2(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle &= \frac{1}{2}(\|\nabla\mathbf{u}\|^2_{L^2(G)} + \|\nabla\mathbf{v}\|^2_{L^2(G)})\|\nabla\mathbf{u} - \nabla\mathbf{v}\|^2_{L^2(G)} \\ &+ \frac{1}{2}(\|\nabla\mathbf{u}\|^2_{L^2(G)} - \|\nabla\mathbf{v}\|^2_{L^2(G)})^2 \end{aligned} \tag{3.2.12}$$

where $\mathcal{D}(G)$ is the class of test functions.

Proof. Note that, for $\mathbf{u}, \mathbf{v} \in \mathcal{D}(G)$,

$$\langle \mathcal{A}_1(\mathbf{u}) - \mathcal{A}_1(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle = \int_G (|\nabla\mathbf{u}|^{q-2}_{\mathbb{R}^n}\nabla\mathbf{u} - |\nabla\mathbf{v}|^{q-2}_{\mathbb{R}^n}\nabla\mathbf{v}) \cdot (\nabla\mathbf{u} - \nabla\mathbf{v})dx$$

The integrand is estimated by the following algebraic inequality ([10], Lemma 4.4): If $q \geq 2$, then $\forall\ \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ we have

$$(|\mathbf{y}|_{\mathbb{R}^n}^{q-2}\mathbf{y} - |\mathbf{z}|_{\mathbb{R}^n}^{q-2}\mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) \geq \gamma(n,q)|\mathbf{y} - \mathbf{z}|_{\mathbb{R}^n}^{q}$$

Thus, we get $\langle \mathcal{A}_1(\mathbf{u}) - \mathcal{A}_1(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle \geq \gamma(n,q)\|\nabla\mathbf{u} - \nabla\mathbf{v}\|_{L^q(G)}^q$.

We now consider

$$\langle \mathcal{A}_2(\mathbf{u}) - \mathcal{A}_2(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle = \int_G (\|\nabla\mathbf{u}\|_{L^2(G)}^2\nabla\mathbf{u} - \|\nabla\mathbf{v}\|_{L^2(G)}^2\nabla\mathbf{v}) \cdot (\nabla\mathbf{u} - \nabla\mathbf{v})dx$$

Denote $a = \|\nabla\mathbf{u}\|_{L^2(G)}^2$ and $b = \|\nabla\mathbf{v}\|_{L^2(G)}^2$. The following equalities hold:

$$\begin{aligned} a\nabla\mathbf{u} - b\nabla\mathbf{v} &= (\frac{a+b}{2} + \frac{a-b}{2})\nabla\mathbf{u} - (\frac{a+b}{2} - \frac{a-b}{2})\nabla\mathbf{v} \\ &= \frac{a+b}{2}(\nabla\mathbf{u} - \nabla\mathbf{v}) + \frac{a-b}{2}(\nabla\mathbf{u} + \nabla\mathbf{v}) \end{aligned}$$

Thus, the integral in Eq. (3.2) can be written as

$$\begin{aligned} &\frac{1}{2}(\|\nabla\mathbf{u}\|_{L^2(G)}^2 + \|\nabla\mathbf{v}\|_{L^2(G)}^2)\int_G (\nabla\mathbf{u} - \nabla\mathbf{v}) \cdot (\nabla\mathbf{u} - \nabla\mathbf{v})dx \\ &\quad + \frac{1}{2}(\|\nabla\mathbf{u}\|_{L^2(G)}^2 - \|\nabla\mathbf{v}\|_{L^2(G)}^2)\int_G (\nabla\mathbf{u} + \nabla\mathbf{v}) \cdot (\nabla\mathbf{u} - \nabla\mathbf{v})dx \\ &= \frac{1}{2}(\|\nabla\mathbf{u}\|_{L^2(G)}^2 + \|\nabla\mathbf{v}\|_{L^2(G)}^2)\|\nabla\mathbf{u} - \nabla\mathbf{v}\|_{L^2(G)}^2 \\ &\quad + \frac{1}{2}(\|\nabla\mathbf{u}\|_{L^2(G)}^2 - \|\nabla\mathbf{v}\|_{L^2(G)}^2)^2 \end{aligned}$$

which completes the proof. □

Next, we give two more results ([34] page 260) about the demicontinutiy and hemicontinuity of $\mathcal{A}_i$. Note that $\mathcal{A}_2 : V_{1,2} \to V_{1,2}'$ is continuous.

Lemma 3.2.3. *The operator $\mathcal{A}_1$ is demicontinuous: let $\mathbf{u}^n \to \mathbf{u}$ in $V_{1,q}$, then $\mathcal{A}_1(\mathbf{u}^n) \to \mathcal{A}_1(\mathbf{u})$ in the weak-star (weak) topology of $V_{1,q}'$.*

Lemma 3.2.4. *$\mathcal{A}_1$ and $\mathcal{A}_2$ are strongly hemicontinuous: $\forall V_N \subset V, V_N$ finite-dimensional, the maps $\mathbf{u} \to \mathcal{A}_1(\mathbf{u})$ and $\mathbf{u} \to \mathcal{A}_2(\mathbf{u})$ are continuous from $V_N \to V'$.*

Let us denote V, for ease of notation, as the space $V_{r,q}$ in Case 1, 2 and 3.

Let V' be the dual of V, we have the dense, continuous embedding $V \subset H$, then for its dual space V' it follows that $H' \subset V'$ continuously and densely. Identifying H and its dual H', we have that

$$V \subset\!\!\rightharpoondown H = H' \subset\!\!\rightharpoondown V'$$

continuously and densely. If $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V' and V(i.e. $\langle z, v \rangle := z(v)$ for $z \in V', v \in V$), then it follows that

$$\langle z, v \rangle = (z, v)_H, \qquad \forall z \in H, v \in V$$

and (V, H, V') is called a Gelfand triple.

Define the operator $\mathbf{A} : V_{1,2} \to V'_{1,2}$ by

$$\mathbf{Au} = -\Pi_H \Delta \mathbf{u}$$

for $\mathbf{u} \in D(\mathbf{A}) = W^{2,2}(G) \cap V_{1,2}$ where $\Pi_H : L^2(G) \to H$ is the Leray projector. The operator $\mathbf{A}$ is known as the Stokes operator [4] and is positive, self-adjoint.

Notation : From now on, we will use $|\mathbf{u}|$ to denote the H-norm of $\mathbf{u}$, and $||\mathbf{u}||$ to denote the $V_{1,2}$-norm of $\mathbf{u}$. That is, if $\mathbf{u} = (u_1, u_2)$, then

$$|\mathbf{u}|^2 = \iint_G \{u_1^2(x_1, x_2) + u_2^2(x_1, x_2)\} dx_1 dx_2$$

and

$$||\mathbf{u}||^2 = \sum_{i,j=1}^{2} \iint_G (\frac{\partial u_i}{\partial x_j})^2 dx_1 dx_2$$

Note that on the space H, the norm is the $L^2(G)$ norm, while on the space $V_{1,2}$, $||\mathbf{u}|| = |\nabla \mathbf{u}| = |\mathbf{A}^{1/2}\mathbf{u}|$.

Define the trilinear form $b(\cdot, \cdot, \cdot) : V_{1,2} \times V_{1,2} \times V_{1,2} \to \mathbb{R}$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_G u_i \frac{\partial v_j}{\partial x_i} w_j dx \tag{3.2.13}$$

Then we can define the bilinear operator $\mathbf{B} : V_{1,2} \times V_{1,2} \to V'$ such that

$$\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \tag{3.2.14}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_{1,2}$. We will use $\mathbf{B}(\mathbf{u})$ to denote $\mathbf{B}(\mathbf{u}, \mathbf{u})$.

Note that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \tag{3.2.15}$$

and hence $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.

Lemma 3.2.5. *The trilinear form $b(\cdot, \cdot, \cdot)$ satisfies:*

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C|\mathbf{u}| \|\mathbf{u}\| \|\mathbf{v}\| \tag{3.2.16}$$

Proof. Using the definition above and the Hölder inequality,

$$\begin{aligned} |b(\mathbf{u}, \mathbf{w}, \mathbf{v})| &= |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \\ &= \left| \sum_{i,j=1}^{2} \int_G u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \\ &\leq \left(\sum_{i=1}^{2} \int_G |u_i|^4 dx \right)^{1/4} \left(\sum_{i,j=1}^{2} \int_G (\frac{\partial v_j}{\partial x_i})^2 dx \right)^{1/2} \\ &\quad \left(\sum_{j=1}^{2} \int_G |w_j|^4 dx \right)^{1/4} \\ &= ||\mathbf{u}||_{L^4(G)} \, ||\mathbf{v}|| \, ||\mathbf{w}||_{L^4(G)} \end{aligned} \tag{3.2.17}$$

By the Sobolev embedding theorem: If $\mathbf{u} \in W^{r,q}(G)$ with $r < \frac{n}{q}$, then $\mathbf{u} \in L^p(G)$, where $\frac{1}{p} = \frac{1}{q} - \frac{r}{n}$, and in addition we have the estimate

$$||\mathbf{u}||_{L^p(G)} \leq C||\mathbf{u}||_{W^{r,q}(G)} \tag{3.2.18}$$

We take $r = 1/2, q = 2$ and $p = 4$, then

$$\begin{aligned} |b(\mathbf{u}, \mathbf{w}, \mathbf{v})| &= |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \\ &\leq C||\mathbf{u}||_{W^{1/2,2}} \, ||\mathbf{v}|| \, ||\mathbf{w}||_{W^{1/2,2}} \\ &\leq C|\mathbf{u}|^{1/2}||\mathbf{u}||^{1/2}||\mathbf{v}|| \, |\mathbf{w}|^{1/2}||\mathbf{w}||^{1/2} \end{aligned} \tag{3.2.19}$$

Letting $\mathbf{u} = \mathbf{w}$, we obtain that

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C|\mathbf{u}|\|\mathbf{u}\|\|\mathbf{v}\|$$

□

We recall that $V_{r,q} = \{\mathbf{u} \in W_0^{r,q}(G) : \nabla \cdot \mathbf{u} = 0\}$ and give a property of $\mathbf{B}(\mathbf{u})$.

Lemma 3.2.6. *In the Case 1, $r = 1, q \geq 3$: if $\mathbf{u} \in L^q(0, T; V_{1,q})$, then $\mathbf{B}(\mathbf{u}) \in L^{q'}(0, T; V'_{1,q})$. In the Case 2, $r = 1, q = 2$: if $\mathbf{u} \in L^4(0, T; V_{1,2})$, then $\mathbf{B}(\mathbf{u}) \in L^2(0, T; V'_{1,2})$. In the Case 3, $r = m, q = 2$: if $\mathbf{u} \in L^2(0, T; V_{m,2}) \cap L^\infty(0, T; H)$, then $\mathbf{B}(\mathbf{u}) \in L^2(0, T; V'_{m,2})$.*

Proof. For Case 1, we start with the observation that for $k < \frac{n}{q} < 1$

$$W^{1,q}(G) \subset W^{k,q}(G) \subset L^p(G), \qquad \text{where } \frac{1}{p} = \frac{1}{q} - \frac{k}{n} \geq 0. \tag{3.2.20}$$

Applying the Hölder inequality, we have $|b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq \|\mathbf{u}\|^2_{L^p(G)} \|\nabla \mathbf{v}\|_{L^q(G)}$, with $2/p + 1/q = 1$.

$$\begin{aligned}
\left| \int_0^T \langle \mathbf{B}(\mathbf{u}), \mathbf{v} \rangle dt \right| &\leq \int_0^T \|\mathbf{u}(t)\|^2_{L^p(G)} \|\nabla \mathbf{v}(t)\|_{L^q(G)} dt \\
&\leq C \left(\int_0^T \|\mathbf{u}\|^{2q'}_{L^p(G)} dt \right)^{1/q'} \left(\int_0^T \|\nabla \mathbf{v}\|^q_{L^q(G)} dt \right)^{1/q}
\end{aligned} \tag{3.2.21}$$

where $1/q + 1/q' = 1$.

Then using the embedding $W^{1,q}(G) \subset L^p(G)$ noted above,

$$\|\mathbf{B}(\mathbf{u})\|_{L^{q'}(0,T;V'_{1,q})} \leq C \left(\int_0^T \|\mathbf{u}\|^{2q'}_{L^p(G)} dt \right)^{1/q'} \leq C \left(\int_0^T \|\mathbf{u}\|^{2q'}_{V_{1,q}} dt \right)^{1/q'} \tag{3.2.22}$$

Taking $2q' \leq q$, again by Hölder's inequality, we obtain that

$$\|\mathbf{B}(\mathbf{u})\|_{L^{q'}(0,T;V'_{1,q})} \leq C \left(\int_0^T \|\mathbf{u}\|^q_{V_{1,q}} dt \right)^{2/q} < \infty \tag{3.2.23}$$

since $\mathbf{u} \in L^q(0, T; V_{1,q})$. The two conditions $q \geq 2q'$ and $1/q + 1/q' = 1$ indicates the condition: $q \geq 3$.

For Case 2, letting $\mathbf{u} = \mathbf{w}$ in Eq. (3.2.17), we have

$$|\langle \mathbf{B}(\mathbf{u}), \mathbf{v}\rangle| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq ||\mathbf{u}||^2_{L^4(G)}||\mathbf{v}|| \tag{3.2.24}$$

It follows that

$$\int_0^T \langle \mathbf{B}(\mathbf{u}), \mathbf{v}\rangle ds \leq \left(\int_0^T ||\mathbf{u}||^4_{L^4(G)} ds\right)^{1/2} \left(\int_0^T ||\mathbf{v}||^2 ds\right)^{1/2} \tag{3.2.25}$$

Using the embedding $W^{1,2}(G) \subset W^{1/2,2}(G) \subset L^4(G)$, we have

$$\|\mathbf{B}(\mathbf{u})\|_{L^2(0,T;V'_{1,2})} \leq C \left(\int_0^T \|\mathbf{u}\|^4_{V_{1,2}} dt\right)^{1/2} < \infty \tag{3.2.26}$$

since $\mathbf{u} \in L^4(0, T; V_{1,2})$.

For Case 3, from the estimate in Lemma 3.2.5

$$|b(\mathbf{u}, \mathbf{u}, \mathbf{v})| = |b(\mathbf{u}, \mathbf{v}, \mathbf{u})| \leq C|\mathbf{u}|\|\mathbf{u}\|\|\mathbf{v}\|$$

it follows that

$$\left|\int_0^T \langle \mathbf{B}(\mathbf{u}), \mathbf{v}\rangle dt\right| \leq C\|\mathbf{u}\|_{L^\infty(0,T;H)}\|\mathbf{u}\|_{L^2(0,T;V_{m,2})}\|\mathbf{v}\|_{L^2(0,T;V_{m,2})}. \tag{3.2.27}$$

from which the Lemma follows in Case 3. □

Next, we give some basic definitions and notations which are necessary in the sequel. Let $(U_1, (\cdot, \cdot)_{U_1})$ and $(U_2, (\cdot, \cdot)_{U_2})$ be two separable Hilbert spaces. The space of all bounded linear operators from U_1 to U_2 is denoted by $L(U_1, U_2)$; for simplicity we write $L(U_1)$ instead of $L(U_1, U_1)$.

Definition 3.2.7. (**Trace class operator**) Let $T \in L(U_1)$ and $e_k, k \in \mathbb{N}$ be an orthonormal basis of U_1. Then we define the trace of the operator T as

$$\mathbf{tr}\, T := \sum_{k\in\mathbb{N}} (Te_k, e_k)_{U_1} \tag{3.2.28}$$

and we say T is a trace class operator if the series is convergent.

Definition 3.2.8. (**Hilbert-Schmidt operator**) A bounded linear operator $T : U_1 \to U_2$ is called Hilbert-Schmidt if

$$\sum_{k\in\mathbb{N}} (Te_k, Te_k)_{U_2} < \infty \tag{3.2.29}$$

where $e_k, k \in \mathbb{N}$, is an orthonormal basis of U_1.

Definition 3.2.9. (**Pseudo inverse**) Let $T \in L(U_1, U_2)$ and $Ker(T) := \{x \in U_1 | Tx = 0\}$. The pseudo inverse of T is defined as

$$T^{-1} := (T|_{Ker(T)^\perp})^{-1} : T|_{Ker(T)^\perp} \to Ker(T)^\perp \tag{3.2.30}$$

Note that T is one-to-one on $Ker(T)^\perp$.

We state the followig result as a proposition (page 25, [32]).

Proposition 3.2.10. *If $Q \in L(U_1)$ is nonnegative and symmetric, then there exists exactly one element $Q^{1/2} \in L(U_1)$ that is nonnegative and symmetric such that $Q^{1/2} \circ Q^{1/2} = Q$.*

Now let us denote $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_0 = (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in H_0 \tag{3.2.31}$$

where $Q^{-1/2}$ is the pseudo inverse of $Q^{1/2}$.

Let $|\cdot|_0$ denote the norm in H_0. Clearly, the imbedding of H_0 in H is Hilbert-Schmidt since Q is a trace class operator, i.e. $\mathbf{tr}Q < \infty$.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H. Define the norm on the space L_Q by $|S|^2_{L_Q} = \mathbf{tr}\,(SQS^*)$. The noise coefficient $\sigma : [0, T] \times V \to L_Q(H_0; H)$ is such that it satisfies the following assumptions :

(A.1). The function $\sigma \in C([0, T] \times V; L_Q(H_0; H))$

(A.2). For all $t \in (0, T)$, there exists a positive constant K such that

$$|\sigma(t, \mathbf{u})|_{L_Q}^2 \leq K(1 + |\mathbf{u}|^2), \qquad \forall\, \mathbf{u} \in V$$

(A.3). For all $t \in (0, T)$, there exists a positive constant L such that

$$|\sigma(t, \mathbf{u}) - \sigma(t, \mathbf{v})|_{L_Q}^2 \leq L|\mathbf{u} - \mathbf{v}|^2, \qquad \forall\, \mathbf{u}, \mathbf{v} \in V$$

3.3 Abstract Formulation

Now we can formulate the abstract evolution form of the stochastic Navier-Stokes equation (3.1.1). By applying the Leray projection Π_H to each term of the stochastic Navier-Stokes system, and employing the result of Helmholtz that $L^2(G)$ admits an orthogonal decomposition into divergence free and irrotational components,

$$L^2(G) = H \oplus H^{\perp} \tag{3.3.1}$$

where the divergence free component is

$$H = \{\mathbf{u} \in L^2(G) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}\,|_{\partial G} = 0\} \tag{3.3.2}$$

and the irrotational component can be characterized by

$$H^{\perp} = \{\mathbf{g} \in L^2(G) : \mathbf{g} = \nabla h, h \in W^{1,2}(G)\} \tag{3.3.3}$$

The system (3.1.1) can be written as

$$d\mathbf{u} + [\nu_0 \mathbf{A}\mathbf{u} + \nu_1 \mathcal{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u})]dt = \mathbf{f}(t)dt + \sigma(t, \mathbf{u})dW_t \tag{3.3.4}$$

where the external body force $\mathbf{f}$ is assumed to be V'-valued for all t. The procedure of applying the Leray projection eliminates the pressure p from the equation.

If we replace the noise coefficient σ in the equation (3.3.4) by $\sqrt{\varepsilon}\sigma$ for $\varepsilon > 0$, then the resulting solution is denoted by $\mathbf{u}^{\varepsilon}$. The main goal of this work is to establish the large deviation principle (LDP), equivalently Laplace-Varadhan principle (LVP), for the family $\{\mathbf{u}^{\varepsilon}; \varepsilon > 0\}$.

Chapter 4
Martingale Solutions

The martingale problem was initiated by Stroock and Varadhan to study Markov processes. It provides us with a new concept for the solution of a stochastic differential equation. Using this approach, existence and uniqueness of solutions of stochastic differential equations can be proved under milder conditions on the coefficients, and such solutions are weak solutions that are unique in law.

4.1 Martingale Problems

For a time-homogeneous $\mathbb{R}^d$-valued Markov process $X := \{X_t\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with infinitesimal generator L, and the domain of L denoted by D, one can show that

$$M^f(t) = f(X_t) - \int_0^t Lf(X_s)ds$$

is a martingale for each $f \in D$. This important property led Stroock and Varadhan to formulate the martingale problem.

Definition 4.1.1. A process $X := \{X_t\}$ with continuous paths defined on some probability space $(\Omega, \mathcal{F}, P)$ is called a solution to the *martingale problem* for the initial distribution μ and the operator L, if the following hold:

(1) The distribution of X_0 is μ;

(2) For any $f \in D$, the process $M^f(t) := f(X_t) - \int_0^t Lf(X_s)ds$ is a $\mathcal{F}_t^X$-martingale.

In the above definition of the martingale problem, we are allowed to construct the process X on any probability space. Since X has continuous paths, let us take $\Omega = C([0,\infty);\mathbb{R}^d)$, the space of all continuous $\mathbb{R}^d$-valued functions defined on

$[0,\infty)$. Define $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. Equipped with the topology of uniform convergence on compact subsets of $[0,\infty)$, the space Ω is a complete separable metric space. Let $\mathcal{F}$ be the Borel σ-field of Ω, and $\mathcal{F}_{s,t} = \sigma\{X(r) : s \le r \le t\}$. When $s = 0$, we simply write $\mathcal{F}_t$ instead of $\mathcal{F}_{0,t}$. With such canonical choice of $(\Omega, \mathcal{F})$, and the process X, we can recast the definition of the martingale problem for (μ, L) as follows:

Definition 4.1.2. A probability measure P on $(\Omega, \mathcal{F})$ is called a solution to the *martingale problem* for the initial distribution μ and the operator L, if the following hold:

(1) $P\{\omega : X_0(\omega) \in B\} = \mu(B)$ for all Borel sets B in $\mathbb{R}^d$;

(2) For any $f \in D$, the process $M^f(t) := f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$ is a $\mathcal{F}_t$-martingale with respect to P.

The definition of time-inhomogeneous martingale problems can be analogously defined as follows. If $\{L_t : t \ge 0\}$ is a family of operators defined on a common domain $D \subset C_b(\mathbb{R}^d)$.

Definition 4.1.3. A probability measure P on $(\Omega, \mathcal{F})$ is called a solution to the *martingale problem* for the initial distribution μ and the operators L_t if

(1) $P\{\omega : X_0(\omega) \in B\} = \mu(B)$ for all Borel sets B in $\mathbb{R}^d$;

(2) For any $f \in D$, the process $M^f(t) := f(X_t) - f(X_0) - \int_0^t L_s f(X_s)ds$ is a $\mathcal{F}_t$-martingale with respect to P.

The following are equivalent forms of the infinite dimensional martingale problem, and the proof of this result is same as in the finite dimensional case [37].

Theorem 4.1.4. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t)$ be the filtered probability space, where $\mathcal{F}_t := \sigma\{\mathbf{u}(s) : 0 < s < t\}$. Then the following martingale formulations are equivalent. Find a probability measure P on $\mathcal{B}(\Omega)$ such that:*

(i) If $f(\cdot)$ is a cylindrical function defined as

$$f(\mathbf{u}) := \varphi(\langle \mathbf{u}, \theta_1 \rangle, \cdots, \langle \mathbf{u}, \theta_m \rangle), \tag{4.1.1}$$

with $\theta_i \in V_{r,q}$ and $\varphi(\cdot) \in C_0^\infty(\mathbb{R}^m)$, then $M_t^f := f(\mathbf{u}(t)) - \int_0^t Lf(\mathbf{u}(s))ds$ is an $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$-martingale where

$$\begin{aligned} Lf(\mathbf{u}) := \quad & \frac{1}{2}\mathbf{tr}\left(\sigma(\mathbf{u})Q\sigma^*(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial \mathbf{u}^2}\right) \\ & - \left(\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s), \frac{\partial f}{\partial \mathbf{u}}\right) \end{aligned} \tag{4.1.2}$$

(ii)

$$\mathbf{M}_t = \mathbf{u}(t) - \mathbf{u}_0 + \int_0^t \{\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\}ds \tag{4.1.3}$$

is a $V'_{r,q}$-valued right-continuous $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$-martingale with quadratic variation,

$$\langle\langle \mathbf{M}_t \rangle\rangle := \int_0^t \sigma(\mathbf{u}(s))Q\sigma^*(\mathbf{u}(s))ds. \tag{4.1.4}$$

(iii) $\forall \theta \in V_{r,q}, \mathbf{M}_t^\theta := \langle \mathbf{M}_t, \theta \rangle$ is a right-continuous $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$-martingale with quadratic variation,

$$\langle\langle \mathbf{M}_t^\theta \rangle\rangle := \int_0^t (\sigma(\mathbf{u}(s))\theta, Q\sigma^*(\mathbf{u}(s))\theta)ds. \tag{4.1.5}$$

Note: Let $\phi \in C_b(\Omega)$ be $\mathcal{F}_s$-measurable, then $\mathbf{M}_t^\theta$ being an $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$-martingale is the same as

$$\mathbb{E}^P[\phi(\cdot)(\mathbf{M}_t^\theta - \mathbf{M}_s^\theta)] = 0 \tag{4.1.6}$$

for all $0 < s < t$.

In this chapter, our main goal is to resolve the following martingale problem: find the probability measure P on $\mathcal{B}(\Omega)$ such that

$$\mathbf{M}_t = \mathbf{u}(t) - \mathbf{u}_0 + \int_0^t \{\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\}ds$$

is a $V'_{r,q}$-valued $(\Omega, \mathcal{F}_t, P)$-martingale.

4.2 Energy Estimates

In this section, we first state the following proposition which gives the stochastic version of a result of J.L. Lions, and then derive the energy estimates satisfied by every martingale solution.

Proposition 4.2.1. *([31], [13]) Consider the probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *and* $\mathcal{F}_t$*-adapted processes* $\mathbf{y}, \mathbf{z}, \mathbf{M}_t$ *such that* $\mathbf{M}_t$ *is an* H*-valued square integrable, right-continuous martingale with* $\mathbf{M}_0 = 0$*, and quadratic variation*

$$\langle\langle \mathbf{M}_t \rangle\rangle := \int_0^t \sigma(\mathbf{u}(s)) Q \sigma^*(\mathbf{u}(s)) ds. \tag{4.2.1}$$

and $\mathbf{y} \in L^q(0,T;V_{r,q}), \mathbf{z} \in L^{q'}(0,T;V'_{r,q})$ *a.s, and for* P *a.s,*

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_0^t \mathbf{z}(s) ds + \mathbf{M}_t, \qquad t \in [0,T] \tag{4.2.2}$$

with $\mathbf{y} \in H$*. Then the paths of* $\mathbf{y}$ *are a.s. in* $D(0,T;H)$ *(*H*-valued Skorohod space, see Appendix A) and the Itô formula applies for* $|\mathbf{y}|^2$ *: for* P *a.s.,*

$$\begin{aligned} |\mathbf{y}(t)|^2 &= |\mathbf{y}_0|^2 + 2\int_0^t \langle \mathbf{z}(s), \mathbf{y}(s) \rangle ds \\ &+ 2\int_0^t (\mathbf{y}(s), d\mathbf{M}_s) + \mathbf{tr}\langle\langle \mathbf{M}_t \rangle\rangle \end{aligned} \tag{4.2.3}$$

The next theorem gives energy estimates satisfied by martingale solutions.

Theorem 4.2.2. *Let* $\mathbf{f}$ *be in* $L^2(0,T;H) \subset L^{q'}(0,T;V'_{r,q})$*, for* $q \geq 2$*, and* P *be any probability measure on* $(\Omega, \mathcal{F}_t)$ *such that* P *is carried by the subset of paths* $\omega \in \Omega$ *with* $\mathbf{u}(\cdot, \omega) \in L^q(0,T;V_{r,q}) \cap L^\infty(0,T;H)$ *and* K *as in the assumption (A.2). We assume that*

$$\mathbf{M}_t = \mathbf{u}(t) - \mathbf{u}_0 + \int_0^t \{\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\} ds \tag{4.2.4}$$

is a $V'_{r,q}$*-valued* $(\Omega, \mathcal{F}_t, P)$*-martingale with quadratic variation*

$$\langle\langle \mathbf{M}_t \rangle\rangle := \int_0^t \sigma(\mathbf{u}(s)) Q \sigma^*(\mathbf{u}(s)) ds. \tag{4.2.5}$$

Then

$$\mathbb{E}^P \left[|\mathbf{u}(t)|^2 + \nu_0 \int_0^t |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds + \nu_1 \int_0^t \|\mathbf{u}(s)\|_{V_{r,q}}^q ds \right] \\ \leq C \left(\mathbb{E}^P[|\mathbf{u}_0|^2], \int_0^T |\mathbf{f}(s)|^2 ds, K, T \right) \tag{4.2.6}$$

and

$$\mathbb{E}^P \left[\sup_{0 \leq t \leq T} |\mathbf{u}(t)|^2 \right] \leq C \left(\mathbb{E}^P[|\mathbf{u}_0|^2], \int_0^T |\mathbf{f}(s)|^2 ds, K, T \right) \tag{4.2.7}$$

Let the initial data satisfy $\mathbb{E}|\mathbf{u}_0|^{2l} < +\infty$, *for* $1 < l < \infty$, *then*

$$\mathbb{E}^P \{ \sup_{0 \leq t \leq T} |\mathbf{u}(t)|^{2l} + \nu_0 \int_0^t |\mathbf{u}(s)|^{2l-2} |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds \\ + \nu_1 \int_0^t |\mathbf{u}(s)|^{2l-2} \|\mathbf{u}(s)\|_{V_{r,q}}^q ds \} \\ \leq C \left(\mathbb{E}^P |\mathbf{u}_0|^{2l}, \int_0^T |\mathbf{f}(s)|^{2l} ds, K, T \right) \tag{4.2.8}$$

Proof. Define a stopping time

$$\tau_N := \inf\{t; |\mathbf{u}(t,\omega)| \geq N, \omega \in \Omega\}. \tag{4.2.9}$$

Note that if $\mathbf{u} \in L^q(0,T;V_{r,q})$, then $\mathbf{A}(\mathbf{u}), \mathcal{A}(\mathbf{u})$ and $\mathbf{B}(\mathbf{u})$ (Lemma 3.2.6), all belong to $L^{q'}(0,T;V'_{r,q})$

Using the proposition above, we obtain the following energy equality

$$|\mathbf{u}(t \wedge \tau_N)|^2 + 2\nu_0 \int_0^{t \wedge \tau_N} |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds + 2\nu_1 \int_0^{t \wedge \tau_N} \langle \mathcal{A}(\mathbf{u}(s)), \mathbf{u}(s) \rangle ds \\ = |\mathbf{u}_0|^2 + 2 \int_0^{t \wedge \tau_N} (\mathbf{f}(s), \mathbf{u}(s)) ds + 2 \int_0^{t \wedge \tau_N} (\mathbf{u}(s), \sigma(\mathbf{u}(s)) dW) \\ + \int_0^{t \wedge \tau_N} \mathbf{tr}(\sigma(\mathbf{u}(s)) Q \sigma^*(\mathbf{u}(s))) ds. \tag{4.2.10}$$

Applying Young's inequality to the first integral on the right,

$$\begin{aligned}
&|\mathbf{u}(t\wedge\tau_N)|^2 + 2\int_0^{t\wedge\tau_N}\{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\langle\mathcal{A}(\mathbf{u}(s)),\mathbf{u}(s)\rangle\}ds \\
&\le |\mathbf{u}_0|^2 + \int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2ds + 2\int_0^{t\wedge\tau_N}(\mathbf{u}(s),\sigma(\mathbf{u}(s))dW) \\
&+\int_0^{t\wedge\tau_N}|\mathbf{u}(s)|^2ds + \int_0^{t\wedge\tau_N}\mathbf{tr}(\sigma(\mathbf{u}(s))Q\sigma^*(\mathbf{u}(s)))ds. \qquad (4.2.11)
\end{aligned}$$

Using estimates (3.2.5), (3.2.9) and also assumption (A.2),

$$\begin{aligned}
&|\mathbf{u}(t\wedge\tau_N)|^2 + 2\int_0^{t\wedge\tau_N}\{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\|\mathbf{u}(s)\|^q_{V_{r,q}}\}ds \\
&\le |\mathbf{u}_0|^2 + \int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2ds + 2\int_0^{t\wedge\tau_N}(\mathbf{u}(s),\sigma(\mathbf{u}(s))dW) \\
&\quad +(K+1)\int_0^{t\wedge\tau_N}|\mathbf{u}(s)|^2ds + K \qquad (4.2.12)
\end{aligned}$$

Taking expectation and noting that the stochastic integral on the right-hand side of the above estimate is a martingale, and hence has a zero mean, we get

$$\begin{aligned}
&\mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^2 + 2\int_0^{t\wedge\tau_N}\{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\|\mathbf{u}(s)\|^q_{V_{r,q}}\}ds\right] \\
&\le \mathbb{E}^P\left[|\mathbf{u}_0|^2\right] + \mathbb{E}^P\left[\int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2ds\right] \\
&\quad +(K+1)\,\mathbb{E}^P\left[\int_0^{t\wedge\tau_N}|\mathbf{u}(s)|^2ds\right] + K \qquad (4.2.13)
\end{aligned}$$

Dropping the second and third term on the left side,

$$\begin{aligned}
\mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^2\right] &\le \mathbb{E}^P\left[|\mathbf{u}_0|^2\right] + \mathbb{E}^P\left[\int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2ds\right] \\
&+ (K+1)\left[\int_0^t \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^2ds\right] + K \qquad (4.2.14)
\end{aligned}$$

Applying the Gronwall inequality,

$$\begin{aligned}
\mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^2\right] &\leq \left(\mathbb{E}^P[|\mathbf{u}_0|^2]+\mathbb{E}^P\left[\int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2ds\right]+K\right)\\
&\quad\cdot\exp\left[\int_0^t(K+1)ds\right]\\
&\leq \left(\mathbb{E}^P[|\mathbf{u}_0|^2]+\int_0^T|\mathbf{f}(s)|^2ds+K\right)\cdot e^{(K+1)T}\\
&\leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right)
\end{aligned} \tag{4.2.15}$$

Setting $\mathbf{u}_N(t)=\mathbf{u}(t)$ if $|\mathbf{u}(t)|\leq N$; Otherwise, $\mathbf{u}_N(t)=0$. Then

$$\begin{aligned}
\mathbb{E}^P[|\mathbf{u}_N(t)|^2] &= \mathbb{E}^P[|\mathbf{u}(t\wedge\tau_N)|^2]\\
&\leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right)
\end{aligned} \tag{4.2.16}$$

By the monotone convergence theorem,

$$\mathbb{E}^P[|\mathbf{u}(t)|^2]\leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right) \tag{4.2.17}$$

Similarly, using the estimate on the right-hand side of (4.2.13), we obtain

$$\begin{aligned}
&\mathbb{E}^P\left[2\int_0^{t\wedge\tau_N}\{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2+\nu_1\|\mathbf{u}(s)\|_{V_{r,q}}^q\}ds\right]\\
&\leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right)
\end{aligned} \tag{4.2.18}$$

Again, by the monotone convergence theorem,

$$\begin{aligned}
&\mathbb{E}^P\left[\int_0^t\{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2+\nu_1\|\mathbf{u}(s)\|_{V_{r,q}}^q\}ds\right]\\
&\leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right)
\end{aligned} \tag{4.2.19}$$

Hence the estimate (4.2.6) is proved.

To prove the estimate (4.2.7), we proceed in the similar way as above. We first take supremum upto time $T \wedge \tau_N$ on both sides of (4.2.12),

$$
\begin{aligned}
&\sup_{0\leq s\leq T\wedge\tau_N} |\mathbf{u}(s)|^2 + 2\int_0^{T\wedge\tau_N} \{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\|\mathbf{u}(s)\|_{V_{r,q}}^q\}ds \\
&\leq |\mathbf{u}_0|^2 + \int_0^{T\wedge\tau_N} |\mathbf{f}(s)|^2 ds + 2\sup_{0\leq s\leq T\wedge\tau_N}\int_0^s (\mathbf{u}(r), \sigma(\mathbf{u}(r))dW_r) \\
&\quad +(K+1)\int_0^{T\wedge\tau_N} |\mathbf{u}(s)|^2 ds + K
\end{aligned} \tag{4.2.20}
$$

Taking expectation, we get

$$
\begin{aligned}
&\mathbb{E}^P\left[\sup_{0\leq s\leq T\wedge\tau_N} |\mathbf{u}(s)|^2 + 2\int_0^{T\wedge\tau_N} \{\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\|\mathbf{u}(s)\|_{V_{r,q}}^q\}ds\right] \\
&\leq \mathbb{E}^P\left[|\mathbf{u}_0|^2\right] + 2\mathbb{E}^P\left[\sup_{0\leq s\leq T\wedge\tau_N}\int_0^s (\mathbf{u}(r), \sigma(\mathbf{u}(r))dW_r)\right] \\
&\quad +\mathbb{E}^P\left[\int_0^{T\wedge\tau_N} |\mathbf{f}(s)|^2 ds\right] + (K+1)\int_0^T \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^2 ds + K
\end{aligned} \tag{4.2.21}
$$

Apply the Burkholder-Davis-Gundy inequality to the second term above,

$$
\begin{aligned}
&2\mathbb{E}^P\left[\sup_{0\leq s\leq T\wedge\tau_N}\int_0^s (\mathbf{u}(r), \sigma(\mathbf{u}(r))dW_r)\right] \\
&\leq C\mathbb{E}^P\left[\left(\mathbf{tr}\langle\langle\int_0^{T\wedge\tau_N} (\mathbf{u}(s), \sigma(\mathbf{u}(s))dW_s)\rangle\rangle\right)^{1/2}\right] \\
&\leq C\mathbb{E}^P\left[\left(\int_0^{T\wedge\tau_N} |\mathbf{u}(s)|^2|\sigma(\mathbf{u}(s))|_{L_Q}^2 ds\right)^{1/2}\right] \\
&\leq C\mathbb{E}^P\left[\left(\sup_{0\leq s\leq T\wedge\tau_N} |\mathbf{u}(s)|^2\right)^{1/2}\left(\int_0^{T\wedge\tau_N} |\sigma(\mathbf{u}(s))|_{L_Q}^2 ds\right)^{1/2}\right] \\
&\leq C\left(\mathbb{E}^P\left[\sup_{0\leq s\leq T\wedge\tau_N} |\mathbf{u}(s)|^2\right]\right)^{1/2} \\
&\qquad \cdot\left(\mathbb{E}^P\left[\int_0^{T\wedge\tau_N} |\mathbf{u}(s)|^2 ds\right]\right)^{1/2} + C_K \\
&\leq C_\varepsilon\mathbb{E}^P\left[\sup_{0\leq s\leq T\wedge\tau_N} |\mathbf{u}(s)|^2\right] + \varepsilon\mathbb{E}^P\left[\int_0^{T\wedge\tau_N} |\mathbf{u}(s)|^2 ds\right] + C_K
\end{aligned} \tag{4.2.22}
$$

We have used the assumption (A.2) and Young's inequality in the last two steps of the above estimate.

It follows from (4.2.15), (4.2.21) and (4.2.22) that

$$\mathbb{E}^P\left[\sup_{0\le s\le T\wedge\tau_N}|\mathbf{u}(s)|^2\right]\le C\left(\mathbb{E}^P[|\mathbf{u}_0|^2],\int_0^T|\mathbf{f}(s)|^2ds,K,T\right) \tag{4.2.23}$$

If we define

$$\Omega_N:=\{\omega\in\Omega;\ \sup_{0\le t\le T}||\mathbf{u}(t)||<N\} \tag{4.2.24}$$

then we have

$$\int_{\Omega_N}\sup_{0\le t\le T}||\mathbf{u}(t)||^2P(d\mathbf{u})+\int_{\Omega\setminus\Omega_N}\sup_{0\le t\le T}||\mathbf{u}(t)||^2P(d\mathbf{u})\le C \tag{4.2.25}$$

Dropping the first integral and noting that, in $\Omega\setminus\Omega_N$, $\sup_{0\le t\le T}||\mathbf{u}(t)||\ge N$,

$$N^2P(\Omega\setminus\Omega_N)\le C \tag{4.2.26}$$

Since $P\{\omega\in\Omega;\tau_N<T\}\le P(\Omega\setminus\Omega_N)\le C/N^2$, we have

$$\limsup_{N\to\infty}P\{\omega\in\Omega;\tau_N<T\}=0$$

and hence $\tau_N\to T$ as $N\to\infty$.

In order to get the higher-order estimates (4.2.8), we consider the scalar-valued semi-martingale

$$h(t):=h_0+\int_0^t\phi(s)ds+N_t, \tag{4.2.27}$$

where $h(t)=|\mathbf{u}(t)|^2, h_0=|\mathbf{u}_0|^2, N_t=2\int_0^t(\mathbf{u}(s),\sigma(\mathbf{u})dW_s)$ and

$$\phi=-2\nu_0|\mathbf{A}^{1/2}\mathbf{u}|^2-2\nu_1\langle\mathcal{A}\mathbf{u},\mathbf{u}\rangle+\mathbf{tr}(\sigma(\mathbf{u})Q\sigma(\mathbf{u})^*)+2(\mathbf{f}(s),\mathbf{u}) \tag{4.2.28}$$

We now recall the scalar-valued Itô formula,

$$h(t)^l=h_0^l+l\int_0^th^{l-1}(s)\phi(s)ds+l\int_0^th^{l-1}(s)dN_s+\frac{l(l-1)}{2}\int_0^th^{l-2}(s)d\langle N\rangle_s \tag{4.2.29}$$

Then we get

$$
\begin{aligned}
&|\mathbf{u}(t)|^{2l} + 2l\int_0^t |\mathbf{u}(s)|^{2l-2}\left[\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1\langle\mathcal{A}\mathbf{u}(s),\mathbf{u}(s)\rangle\right]ds \\
&\le |\mathbf{u}_0|^{2l} + l\int_0^t |\mathbf{u}(s)|^{2l-2}\mathbf{tr}(\sigma(\mathbf{u}(s))Q\sigma(\mathbf{u}(s))^*)ds \\
&+2l\int_0^t |\mathbf{u}(s)|^{2l-2}(\mathbf{u}(s),\sigma(\mathbf{u}(s))dW_s) \\
&+2l\int_0^t |\mathbf{u}(s)|^{2l-2}(\mathbf{f}(s),\mathbf{u}(s))ds \\
&+2l(l-1)\int_0^t |\mathbf{u}(s)|^{2l-2}|\sigma(\mathbf{u}(s))|^2_{L_Q}ds
\end{aligned}
\tag{4.2.30}
$$

Taking expectation, and then using estimates (3.2.5), (3.2.9) and also assumption (A.2), we obtain

$$
\begin{aligned}
&\mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^{2l} + 2l\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-2}\left(\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1||\mathbf{u}(s)||^q_{V_{r,q}}\right)ds\right] \\
&\le \mathbb{E}^P\left[|\mathbf{u}_0|^{2l}\right] + l\,\mathbb{E}^P\left[\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-2}|\sigma(\mathbf{u}(s)|^2_{L_Q}ds\right] \\
&+2l\,\mathbb{E}^P\left[\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-1}|\mathbf{f}(s)|ds\right] \\
&+2l(l-1)\mathbb{E}^P\left[\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-2}K(1+|\mathbf{u}(s)|^2)ds\right] \\
&\le \mathbb{E}^P\left[|\mathbf{u}_0|^{2l}\right] + C(l,K)\int_0^t \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^{2l}ds \\
&+2l\,\mathbb{E}^P\left[\int_0^t |\mathbf{u}(s)|^{2l-1}|\mathbf{f}(s)|ds\right]
\end{aligned}
\tag{4.2.31}
$$

Dropping the second term on the left-hand side,

$$
\begin{aligned}
&\mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^{2l}\right] \\
&\le \mathbb{E}^P\left[|\mathbf{u}_0|^{2l}\right] + C(l,K)\int_0^t \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^{2l}ds \\
&+2l\,\mathbb{E}^P\left[\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-1}|\mathbf{f}(s)|ds\right] \\
&\le \mathbb{E}^P\left[|\mathbf{u}_0|^{2l}\right] + C(l,K)\int_0^t \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^{2l}ds \\
&+2l\int_0^t (\mathbb{E}^P|\mathbf{u}(s)|^{2l})^{1-1/2l}(\mathbb{E}^P|\mathbf{f}(s)|^{2l})^{1/2l}ds
\end{aligned}
\tag{4.2.32}
$$

Denoting $m(t) := \mathbb{E}^P|\mathbf{u}(t)|^{2l}$, we have

$$m(t \wedge \tau_N) \leq m_0 + C \int_0^{t\wedge\tau_N} [m(s) + g(s)m(s)^{1-1/2l}]ds \tag{4.2.33}$$

where $g(s) := (\mathbb{E}^P|\mathbf{f}(s)|^{2l})^{1/2l}$.

We state a result which is due to Krylov ([16], section 2.5):

Let $m(\cdot) \in C[0,T]$ satisfy $m(t) \leq m_0 + C\int_0^t [m(s) + g(s)m(s)^{1-1/2l}]ds$, then

$$m(t) \leq \left[m_0^{1/2l} + C\int_0^t \exp C(t-s)g(s)ds \right]^{2l} \tag{4.2.34}$$

It follows from (4.2.33) and (4.2.34) that

$$\begin{aligned} \mathbb{E}^P\left[|\mathbf{u}(t\wedge\tau_N)|^{2l}\right] &\leq \left[(\mathbb{E}^P|\mathbf{u}_0|^{2l})^{1/2l} + C_T\int_0^T \left(\mathbb{E}^P|\mathbf{f}(s)|^{2l}\right)^{1/2l} ds \right]^{2l} \\ &\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l}, \int_0^T |\mathbf{f}(s)|^{2l}ds, K, T\right) \end{aligned} \tag{4.2.35}$$

We use the monotone convergence theorem as before to get

$$\mathbb{E}^P\left[|\mathbf{u}(t)|^{2l}\right] \leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l}, \int_0^T |\mathbf{f}(s)|^{2l}ds, K, T\right) \tag{4.2.36}$$

Thus,

$$\mathbb{E}^P\left[\int_0^T |\mathbf{u}(t)|^{2l}dt\right] \leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l}, \int_0^T |\mathbf{f}(s)|^{2l}ds, K, T\right) \tag{4.2.37}$$

for any $t \in (0,T)$.

Dropping the first term in (4.2.31),

$$\begin{aligned} &\mathbb{E}^P\left[2l\int_0^{t\wedge\tau_N} |\mathbf{u}(s)|^{2l-2}\left(\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2 + \nu_1||\mathbf{u}(s)||_{V_{r,q}}^q\right)ds\right] \\ &\leq \mathbb{E}^P\left[|\mathbf{u}_0|^{2l}\right] + C(l,K)\int_0^t \mathbb{E}^P|\mathbf{u}(s\wedge\tau_N)|^{2l}ds \\ &\quad +2l\int_0^t (\mathbb{E}^P|\mathbf{u}(s)|^{2l})^{1-1/2l}(\mathbb{E}^P|\mathbf{f}(s)|^{2l})^{1/2l}ds \end{aligned} \tag{4.2.38}$$

Setting $G := \mathbb{E}^P|\mathbf{u}_0|^{2l} + \int_0^T |\mathbf{f}(s)|^{2l}ds$, then

$$\begin{aligned} \int_0^t (\mathbb{E}^P|\mathbf{u}(s)|^{2l})^{1-1/2l}(\mathbb{E}^P|\mathbf{f}(s)|^{2l})^{1/2l}ds &\leq \int_0^t G^{1-1/2l}G^{1/2l}ds \\ &\leq GT \end{aligned} \tag{4.2.39}$$

Using (4.2.37)-(4.2.39) and the monotone convergence theorem, we obtain

$$\mathbb{E}^P\left[\int_0^T |\mathbf{u}(s)|^{2l-2}\left(\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)|^2+\nu_1||\mathbf{u}(s)||_{V_{r,q}}^q\right)ds\right]$$
$$\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l},\int_0^T|\mathbf{f}(s)|^{2l}ds,K,T\right) \tag{4.2.40}$$

We take supremum and expectation on both sides of (4.2.30) to get

$$\begin{aligned}\mathbb{E}^P\left[\sup_{t\in[0,T]}|\mathbf{u}(t)|^{2l}\right] &\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l},\int_0^T|\mathbf{f}(s)|^{2l}ds,K,T\right)\\ &+ C\mathbb{E}^P\left[\sup_{t\in[0,T]}\int_0^t|\mathbf{u}(s)|^{2l-2}(\mathbf{u}(s),\sigma(\mathbf{u}(s))dW_s)\right]\end{aligned} \tag{4.2.41}$$

Using the Burkholder-Davis-Gundy inequality and assumption (A.2)

$$\begin{aligned}&\mathbb{E}^P\left[\sup_{t\in[0,T]}|\mathbf{u}(t)|^{2l}\right]\\ &\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l},\int_0^T|\mathbf{f}(s)|^{2l}ds,K,T\right)+C\mathbb{E}^P\left[\int_0^T|\mathbf{u}(s)|^{4l}ds)\right]^{1/2}\\ &\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l},\int_0^T|\mathbf{f}(s)|^{2l}ds,K,T\right)+C_\varepsilon\mathbb{E}^P\left[\int_0^T|\mathbf{u}(s)|^{2l}ds)\right]\\ &+\varepsilon\mathbb{E}^P\left[\sup_{t\in[0,T]}|\mathbf{u}(s)|^{2l}\right]\end{aligned} \tag{4.2.42}$$

Hence by (4.2.37) and (4.2.42), we get

$$\mathbb{E}^P\left[\sup_{t\in[0,T]}|\mathbf{u}(t)|^{2l}\right]\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^{2l},\int_0^T|\mathbf{f}(s)|^{2l}ds,K,T\right) \tag{4.2.43}$$

Finally, by (4.2.40) and (4.2.43), we obtain the higher-order estimates (4.2.8).

□

Let $\mathcal{C}=\{X_t\}$ be a class of random variables defined on a probability space $(\Omega,\mathcal{F},P)$. Suppose any $X_t\in\mathcal{C}$ satisfies the property that $\mathbb{E}\{|X_t|^{1+\varepsilon}\}$ is bounded for some $0<\varepsilon<1$, then as $K\to\infty$

$$\sup_{X_t\in\mathcal{C}}\int_{\{|X_t|\geq K\}}|X_t|dP\leq\frac{1}{K^\varepsilon}\mathbb{E}[|X_t|^{1+\varepsilon}]\to 0 \tag{4.2.44}$$

This idea leads us to show the next proposition which indicates the uniform integrability of $\mathbf{M}_t^\theta$.

Proposition 4.2.3. *If* $\mathbf{f} \in L^2(0,T;H)$ *and the initial data satisfy the condition:*

$$\mathbb{E}^P[|\mathbf{u}_0|^2] < \infty \tag{4.2.45}$$

Then for suitable $0 < \varepsilon < 1$

$$\mathbb{E}^P[|\mathbf{M}_t^\theta|^{1+\varepsilon}] \leq C\left(\mathbb{E}^P|\mathbf{u}_0|^2, \int_0^T |\mathbf{f}(s)|^2 ds\right) \tag{4.2.46}$$

where, for any $\theta \in V_{r,q}$,

$$\mathbf{M}_t^\theta = \langle \mathbf{u}(t), \theta\rangle + \int_0^t \langle \nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s), \theta\rangle ds \tag{4.2.47}$$

Proof. Since ([19], lemma 1 and 2, Chapter 1)

$$\begin{aligned} ||\mathbf{B}(\mathbf{u})||_{V'_{r,q}} &\leq C||\mathbf{B}(\mathbf{u})||_{V'_{1,2}} \\ &\leq C|\mathbf{u}|\,|\mathbf{A}^{1/2}\mathbf{u}| \end{aligned} \tag{4.2.48}$$

we have

$$\begin{aligned} |\mathbf{M}_t^\theta| &\leq C\{|\mathbf{u}(t)| + \int_0^t (\nu_0|\mathbf{A}^{1/2}\mathbf{u}(s)| + \nu_1||\mathbf{u}(s)||_{V_{r,q}}^{q-1} \\ &\quad +|\mathbf{u}(s)||\mathbf{A}^{1/2}\mathbf{u}(s)| + |\mathbf{f}(s)|)ds\} \end{aligned} \tag{4.2.49}$$

From the Jensen inequality, it follows that

$$\begin{aligned} |\mathbf{M}_t^\theta|^{1+\varepsilon} &\leq C_\varepsilon\{|\mathbf{u}(t)|^{1+\varepsilon} + (\int_0^t |\mathbf{A}^{1/2}\mathbf{u}(s)|ds)^{1+\varepsilon} + (\int_0^t ||\mathbf{u}(s)||_{V_{r,q}}^{q-1} ds)^{1+\varepsilon} \\ &\quad +(\int_0^t |\mathbf{u}(s)||\mathbf{A}^{1/2}\mathbf{u}(s)|ds)^{1+\varepsilon} + (\int_0^t |\mathbf{f}(s)|ds)^{1+\varepsilon}\} \end{aligned} \tag{4.2.50}$$

We take the expectation and consider term by term to get the estimate (4.2.46).

From the energy estimates in Theorem 4.2.2, for the first and second term of (4.2.50) with $0 < \varepsilon < 1$, we have,

$$\mathbb{E}^P[|\mathbf{u}(t)|^{1+\varepsilon}] \leq \mathbb{E}^P[|\mathbf{u}(t)|^2] \leq C\left(\mathbb{E}^P|\mathbf{u}_0|^2, \int_0^T |\mathbf{f}(s)|^2 ds\right) \tag{4.2.51}$$

and

$$\begin{aligned}
\mathbb{E}^P\left|\int_0^T |\mathbf{A}^{1/2}\mathbf{u}(t)|dt\right|^{1+\varepsilon} &\leq \mathbb{E}^P\left|\int_0^T |\mathbf{A}^{1/2}\mathbf{u}(t)|dt\right|^2 \\
&\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^2, \int_0^T |\mathbf{f}(s)|^2 ds\right) \qquad (4.2.52)
\end{aligned}$$

For the third term, with $\varepsilon = \frac{1}{q-1}$, we have

$$\begin{aligned}
\mathbb{E}^P\left|\int_0^T ||\mathbf{u}(t)||_{V_{r,q}}^{q-1} dt\right|^{q/(q-1)} &\leq C\mathbb{E}^P\int_0^T ||\mathbf{u}(t)||_{V_{r,q}}^q dt \\
&\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^2, \int_0^T |\mathbf{f}(s)|^2 ds\right) \qquad (4.2.53)
\end{aligned}$$

and for the term $\mathbb{E}^P\left|\int_0^T |\mathbf{f}(s)|ds\right|^{1+\varepsilon}$, since $\mathbf{f}$ is in $L^2(0,T;H)$

$$\begin{aligned}
\mathbb{E}^P\left|\int_0^T |\mathbf{f}(s)|ds\right|^{1+\varepsilon} &\leq \mathbb{E}^P\left|\int_0^T |\mathbf{f}(s)|ds\right|^2 \\
&\leq T\mathbb{E}^P\left[\int_0^T |\mathbf{f}(s)|^2 ds\right] < \infty \qquad (4.2.54)
\end{aligned}$$

Finally, we consider for $0 < \varepsilon < 1$

$$\begin{aligned}
&\mathbb{E}^P\left|\int_0^T |\mathbf{u}(s)||\mathbf{A}^{1/2}\mathbf{u}(s)|ds\right|^{1+\varepsilon} \\
&\leq \mathbb{E}^P\left|\int_0^T |\mathbf{u}(s)||\mathbf{A}^{1/2}\mathbf{u}(s)|ds\right|^2 \\
&\leq C\mathbb{E}^P\left[\sup_{0\leq t\leq T}|\mathbf{u}(t)|^2\left(\int_0^T |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds\right)\right] \\
&\leq C\left(\mathbb{E}^P|\mathbf{u}_0|^2, \int_0^T |\mathbf{f}(s)|^2 ds\right) \qquad (4.2.55)
\end{aligned}$$

In the above estimates, we have used the Hölder inequality. □

4.3 Tightness of Measures

Definition 4.3.1. Let $\mathcal{P} = \{P\}$ be a class of probability measures on a topological space X, then $\mathcal{P}$ is said to be tight in X if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset\subset X$ such that, for any $P \in \mathcal{P}$,

$$P(K_\varepsilon) \geq 1 - \varepsilon. \qquad (4.3.1)$$

We now recall that a topological space which is a bijective continuous image of a Polish space is called a Lusin sapce. The following results concerning Lusin topology can be found in [28].

Proposition 4.3.2. *Let $E_1, \cdots, E_n$ be Lusin topological spaces, with topologies denoted by $\tau_1, \cdots, \tau_n$. We assume that E_i are subsets of a topological space E such that the canonical embeddings $E_i \hookrightarrow E$ are continuous. Let $\Omega = E_1 \cap \cdots \cap E_n$ and τ^S the supremum of the topologies induced by $\tau_1, \cdots, \tau_n$ on Ω. Then:*

(1) Ω endowed with the topology τ^S is a Lusin space.

(2) Let $\{\mu_k\}_{k\in N}$ be a sequence of Borel probability measures on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ is the Borel algebra, such that the images of μ_k on $(E_i, \mathcal{B}(E_i))$ denoted by $\{\mu_k^i\}_{k\in N}$ are tight for τ_i, for all $i = 1, 2, \cdots, n$. Then $\{\mu_k\}_{k\in N}$ is tight for τ^S.

We denote by τ^S- topology the supremum of the topologies τ_1, τ_2, τ_3 and τ_4, $\tau^S = \tau_1 \vee \tau_2 \vee \tau_3 \vee \tau_4$, where $\tau_1 := L^\infty(0,T;H)$-weak-star, $\tau_2 := L^q(0,T;V_{r,q})$-weak, $\tau_3 := D(0,T;V'_{r,q})$-Skorohod J-topology (see Appendix A), and $\tau_4 := L^q(0,T;H)$-strong.

Corollary 4.3.3. *Let $\Omega = L^\infty(0,T;H)_{w^*} \cap L^q(0,T;V_{r,q})_\sigma \cap D(0,T;V'_{r,q}) \cap L^q(0,T:H)$. Then (Ω, τ^S) is a completely regular Lusin space.*

Here we note that a topological space is called completely regular if it is Hausdorff separated and its topology can be defined by a set of pseudodistances.

Proof. We first note the following continuous embeddings,

$$\begin{aligned} &L^\infty(0,T;H)_{w^*} \hookrightarrow L^q(0,T;V'_{r,q})_\sigma; \quad L^q(0,T;V_{r,q})_\sigma \hookrightarrow L^q(0,T;V'_{r,q})_\sigma \\ &D(0,T;V'_{r,q}) \hookrightarrow L^q(0,T;V'_{r,q})_\sigma; \quad L^q(0,T;H) \hookrightarrow L^q(0,T;V'_{r,q})_\sigma \end{aligned} \tag{4.3.2}$$

Then (Ω, τ^S) is Lusin due to the above proposition 4.3.2. □

Theorem 4.3.4. *The class of measures $\{P\}$ defined in the Theorem 4.2.2 on the Lusin space $(\Omega, \mathcal{B}(\Omega))$ is tight.*

Proof. From the energy estimates (4.2.6) and (4.2.7), we know that

$$\mathbb{E}^P[\sup_{0\le t\le T} |\mathbf{u}(t)|^2 + \nu_0 \int_0^T |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds + \nu_1 \int_0^T \|\mathbf{u}(s)\|^q_{V_{r,q}} ds] \le C \qquad (4.3.3)$$

From this, we deduce that P is tight in $L^\infty(0,T;H)_{w^*}$: for $u \in L^\infty(0,T;H)_{w^*}$,

$$\mathbb{E}^P[\sup_{0\le t\le T} |\mathbf{u}(t)|^2] \le C \qquad (4.3.4)$$

By the Chebyshev's inequality, for $R > 0$,

$$P\{\sup_{0\le t\le T} |\mathbf{u}(t)|^2 > R\} \le \frac{C}{R^2} \qquad (4.3.5)$$

In other words, as $R \to \infty$,

$$P\{\mathbf{u} \in (B_R(0))^c\} \le \frac{C}{R^2} \to 0 \qquad (4.3.6)$$

where $B_R(0)$ denotes the origin-centered closed ball with radius R and is compact in the weak-star topology of $L^\infty(0,T;H)_{w^*}$.

Thus, for any $\varepsilon > 0$, there exists R large enough such that $P\{\mathbf{u} \in B_R(0)\} \ge 1-\varepsilon$, which implies the tightness of P in $L^\infty(0,T;H)_{w^*}$.

Likewise, one can show that P is tight in $L^2(0,T;D(\mathbf{A}^{1/2}))_\sigma$ and $L^q(0,T;V_{r,q})_\sigma$.

Thus we conclude that P is tight in

$$L^\infty(0,T;H)_{w^*} \cap L^2(0,T;D(\mathbf{A}^{1/2}))_\sigma \cap L^q(0,T;V_{r,q})_\sigma$$

Now we deduce the tightness of P in $D(0,T;V'_{r,q})$ with J-topology. Recall the results on tightness of the laws of semimartingales of Metivier ([28], Chapter 4, Theorem 3). We just need to verify the following two facts:

(i) $\forall t \in [0,T]$, the distributions of $\mathbf{u}(t)$ are tight in $V'_{r,q}$.

(ii) $\forall \theta \in V_{r,q}, \forall N \in \mathbf{N}$, for all stopping times τ_N, we have, $\forall\, \varepsilon > 0, \exists\, \delta > 0$

$$\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)||_{V'_{r,q}} ds\right] \leq \delta \qquad (4.3.7)$$

and

$$\mathbb{E}^P[\int_{\tau_N}^{\tau_N+\varepsilon} \mathbf{tr}\langle\langle \mathbf{M}_s\rangle\rangle ds] \leq \delta \qquad (4.3.8)$$

Using the energy estimate $\mathbb{E}^P[|\mathbf{u}(t)|^2] \leq C,\ \forall t \in [0,T]$ and Chebyshev's inequality, we obtain that $\forall R > 0$,

$$P\{\mathbf{u} : |\mathbf{u}| > R\} \leq C/R^2$$

and hence the distributions of $\mathbf{u}(t)$ are tight in H_σ (weak topology).

Moreover, since $H \subset\!\!\rightharpoonup V'_{r,q}$ is a compact embedding, the distributions of $\mathbf{u}(t)$ are tight in $V'_{r,q}$.

We now verify (ii):

$$\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\nu_0 \mathbf{A}\mathbf{u}(s) + \nu_1 \mathcal{A}(\mathbf{u}(s)) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)||_{V'_{r,q}} ds\right] \leq \delta \qquad (4.3.9)$$

Note that for $r \geq 1, q \geq 2$, we have $V'_{1,2} \subset\!\!\rightharpoonup V'_{r,q}$ and thus

$$\begin{aligned}
\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\mathbf{A}\mathbf{u}(s)||_{V'_{r,q}} ds\right] &\leq C \cdot \mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\mathbf{A}\mathbf{u}(s)||_{V'_{1,2}} ds\right] \\
&\leq C \cdot \varepsilon^{1/2} \left(\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} |\mathbf{A}^{1/2}\mathbf{u}(s)|^2 ds\right]\right)^{1/2} \\
&\leq \varepsilon^{1/2} C_1 \qquad (4.3.10)
\end{aligned}$$

by the energy estimate in Theorem 4.2.2.

Similarly, again by the energy estimate and Hölder's inequality,

$$\begin{aligned}
\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\mathcal{A}(\mathbf{u}(s))||_{V'_{r,q}} ds\right] &\leq C \cdot \mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\mathbf{u}(s)||_{V_{r,q}}^{q-1} ds\right] \\
&\leq C \cdot \varepsilon^{1/q} \left(\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon} ||\mathbf{u}(s)||_{V_{r,q}}^{q} ds\right]\right)^{(q-1)/q} \\
&\leq \varepsilon^{1/q} C_2 \qquad (4.3.11)
\end{aligned}$$

Now we consider

$$\begin{aligned}\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}||\mathbf{f}(s)||_{V'_{r,q}}ds\right] &\leq C\cdot\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}|\mathbf{f}(s)|ds\right]\\ &\leq C\cdot\varepsilon^{1/2}\left(\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}|\mathbf{f}(s)|^2ds\right]\right)^{1/2}\\ &\leq \varepsilon^{1/2}C_3 \qquad (4.3.12)\end{aligned}$$

We next look at the nonlinear term and due to the energy estimate,

$$\begin{aligned}\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}||\mathbf{B}(\mathbf{u}(s))||_{V'_{r,q}}ds\right] &\leq \varepsilon^{1/q}\left(\mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}||\mathbf{B}(\mathbf{u}(s))||^{q'}_{V'_{r,q}}ds\right]\right)^{1/q'}\\ &\leq \varepsilon^{1/q}C_4 \qquad (4.3.13)\end{aligned}$$

We finally estimate

$$\begin{aligned}\mathbb{E}^P[\int_{\tau_N}^{\tau_N+\varepsilon}\mathbf{tr}\langle\langle\mathbf{M}_s\rangle\rangle ds] &= \mathbb{E}^P\left[\int_{\tau_N}^{\tau_N+\varepsilon}\mathbf{tr}\left(\int_0^s(\sigma(\mathbf{u}(r))Q\sigma^*(\mathbf{u}(r))dr\right)ds\right]\\ &\leq C\cdot\mathbb{E}^P[\int_{\tau_N}^{\tau_N+\varepsilon}\int_0^s|\mathbf{u}(r)|^2drds]\\ &\leq \varepsilon\cdot T\cdot C\cdot\mathbb{E}^P\left[\sup_{0\leq t\leq T}|\mathbf{u}(t)|^2\right]\leq\varepsilon\cdot C' \qquad (4.3.14)\end{aligned}$$

We have verified condtions (i) and (ii) and hence P is tight in the J-topology of $D(0,T;V'_{r,q})$.

Now we establish the tightness of $\{P\}$ in $L^q(0,T;H)$.

Since $\{P\}$ is tight in $D(0,T;V'_{r,q})$, $\forall\varepsilon>0$, there exists a compact set $K_\varepsilon\subset\subset D(0,T;V'_{r,q})$ such that, $\forall P\in\{P\}$,

$$P(K_\varepsilon)\geq 1-\varepsilon. \qquad (4.3.15)$$

The following lemma is useful and it can be found in [22].

Lemma 4.3.5. *Let $N\subset L^q(0,T;H)$ be included in a compact set of $L^q(0,T;V'_{r,q})$ and such that*

$$\sup_{\mathbf{u}\in N}\int_0^T||\mathbf{u}(t)||^q_{V_{r,q}}dt<\infty. \qquad (4.3.16)$$

then N is relatively compact (pre-compact) in $L^q(0,T;H)$.

We generate such a subset by taking the intersection $L^q(0,T;H) \cap D(0,T;V'_{r,q})$. In fact, $\forall \varepsilon > 0, \exists L_\varepsilon > 0$ such that $p \in \{P\}$

$$P\{\mathbf{u} \in L^q(0,T;H); \int_0^T ||\mathbf{u}(t)||^q_{V_{r,q}} dt \leq L_\varepsilon\} \geq 1 - \varepsilon, \tag{4.3.17}$$

which follows from the energy estimate $E^P[\int_0^T ||\mathbf{u}(t)||^q_{V_{r,q}} dt] \leq C$.

Define

$$N_\varepsilon = K_\varepsilon \cap \{\mathbf{u} \in L^q(0,T;H); \int_0^T ||\mathbf{u}(t)||^q_{V_{r,q}} dt \leq L_\varepsilon\}. \tag{4.3.18}$$

Then, we get that $\forall \varepsilon > 0, \forall P \in \{P\}$.

$$P(N_\varepsilon) \geq 1 - 2\varepsilon. \tag{4.3.19}$$

In other words, P is tight in N_ε.

Moreover, due to the inclusion

$$L^q(0,T;H) \cap D(0,T;V'_{r,q}) \subset\rightharpoonup L^q(0,T;V'_{r,q}) \tag{4.3.20}$$

we have that the set $N_\varepsilon \subset L^q(0,T;H)$ is inclued in the compact set K_ε of $L^q(0,T;V'_{r,q})$. Using the above lemma 4.3.5, we obtain that $N_\varepsilon \subset L^q(0,T;H)$ is relatively compact and hence the tightness of P in $L^q(0,T;H)$ is established. □

4.4 Martingale Solutions

The method of establishing the existence of a martingale solution is as follows. We construct approximate martingale solutions P^N which solve the Galerkin approximated martingale problems according to the theory of Strook and Varadhan. Then we use the tightness of these measures and take the limit to get the solution P to the martingale problem.

First we describe the Galerkin approximations. Let us recall our stocahstic Navier-Stokes equation

$$d\mathbf{u} + [\nu_0 \mathbf{A}\mathbf{u} + \nu_1 \mathcal{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u})]dt = \mathbf{f}(t)dt + \sigma(t,\mathbf{u})dW_t \tag{4.4.1}$$

Let N be a positive integer, and $w_1, w_2, \cdots, w_N$ the first N eigenfunctions of $\mathbf{A}$. Denote the projector $\Pi_N : H \to H_N$ onto the space of $w_1, w_2, \cdots, w_N$. Applying Π_N to (4.4.1) would yield

$$d\Pi_N \mathbf{u} + [\nu_0 \mathbf{A}(\Pi_N \mathbf{u}) + \nu_1 \Pi_N \mathcal{A}(\mathbf{u}) + \Pi_N \mathbf{B}(\mathbf{u})]dt = \Pi_N \mathbf{f}(t)dt + \Pi_N \sigma(t, \mathbf{u})dW_t$$

The Galerkin system of order N is the following system

$$\begin{aligned} &d\mathbf{u}_N + [\nu_0 \mathbf{A}(\mathbf{u}_N) + \nu_1 \Pi_N \mathcal{A}(\mathbf{u}_N) + \Pi_N \mathbf{B}(\mathbf{u}_N)]dt = g_N(t)dt + \Pi_N \sigma(t, \mathbf{u}_N)dW_t \\ &\mathbf{u}_N(0) = \mathbf{u}_N^0 \end{aligned} \tag{4.4.2}$$

where $\mathbf{u}_N(t) \in H_N$.

More precisely, for each $j \in \{1, 2, 3, \cdots, N\}$, let us denote by $\xi_j = \xi_j(t)$ the j^{th} component of $\mathbf{u}_N$:

$$\xi_j(t) = (\mathbf{u}_N, w_j)$$

Also, let $\eta_j(t) = (g_N(t), w_j)$ be the components of g_N, $\alpha_j(t) = (\Pi_N \mathcal{A}(\mathbf{u}_N), w_j)$ and $d\beta_j(t) = (\Pi_N \sigma(t, \mathbf{u}_N)dW_t, w_j)$.

Then the Galerkin system (4.4.2) is equivalent to

$$d\xi_j + [\nu_0 \lambda_j \xi_j + \nu_1 \alpha_j + \sum_{k,l=1}^{m} b(w_k, w_l, w_j)\xi_k \xi_l]dt = \eta_j dt + d\beta_j \tag{4.4.3}$$

The initial data $\mathbf{u}_N^0$ has coefficients $(\mathbf{u}_N^0, w_j) = \xi_j^0$, and $\xi_j(0) = \xi_j^0$.

As functions of $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$, $\nu_0 \lambda_j \xi_j$ and $\sum_{k,l=1}^{m} b(w_k, w_l, w_j)\xi_k \xi_l$ are both (locally) Lipschitz in ξ. Under suitable assumptions on $\mathbf{f}$ and the assumptions (A.1)-(A.3) on σ (section 3.2), the two terms η_j and $d\beta_j$ on the right hand side of equation (4.4.3) shouldn't bother us.

By the theory of stochastic differential equations (SDE), to guarantee the existence of the unique solution $\xi_j(t)$ to the Galerkin system (4.4.3), it suffices to show that α_j is also Lipschitz in ξ.

Let us recall the operators $\mathcal{A}_i (i = 1, 2, 3)$,

$$\begin{aligned}
\langle \mathcal{A}_1(\mathbf{u}), \mathbf{v} \rangle &:= \int_G |\nabla \mathbf{u}|_{\mathbb{R}^n}^{q-2} \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, && \forall \mathbf{u}, \mathbf{v} \in V_{1,q} \\
\langle \mathcal{A}_2(\mathbf{u}), \mathbf{v} \rangle &:= \|\nabla \mathbf{u}\|_{L^2(G)}^2 \int_G \nabla \mathbf{u} \cdot \nabla \mathbf{v} dx, && \forall \mathbf{u}, \mathbf{v} \in V_{1,2} \\
\langle \mathcal{A}_3(\mathbf{u}), \mathbf{v} \rangle &:= \sum_{\alpha \in \mathbb{N}, |\alpha| = m} \int_G D^\alpha \mathbf{u} \cdot D^\alpha \mathbf{v} dx, && \forall \mathbf{u}, \mathbf{v} \in V_{m,2}
\end{aligned}$$

Using the above definitions and $\mathbf{u}_N = \sum_{j=1}^N \xi_j(t) w_j$, we have that in the *Case 1*

$$\begin{aligned}
\alpha_j &= \langle \mathcal{A}_1(\mathbf{u}_N), w_j \rangle \\
&= \int_G |\nabla \mathbf{u}_N|_{\mathbb{R}^n}^{q-2} \nabla \mathbf{u}_N \cdot \nabla w_j dx \\
&= \int_G |\sum_{l=1}^N \xi_l(t) \nabla w_l|_{\mathbb{R}^n}^{q-2} (\sum_{k=1}^N \xi_k(t) \nabla w_k \cdot \nabla w_j) dx
\end{aligned}$$

So we obtain that α_j is a 'polynomial' function of ξ with highest power of $q - 1$, which is (locally) Lipschitz in ξ.

Likewise, in the *Case 2*

$$\begin{aligned}
\alpha_j &= \|\nabla \mathbf{u}_N\|_{L^2(G)}^2 \int_G \nabla \mathbf{u}_N \cdot \nabla w_j dx \\
&= \|\sum_{l=1}^N \xi_l(t) \nabla w_l\|_{L^2(G)}^2 \int_G \sum_{k=1}^N \xi_k(t) \nabla w_k \cdot \nabla w_j dx
\end{aligned}$$

and in the *Case 3*

$$\begin{aligned}
\alpha_j &= \sum_{\alpha \in \mathbb{N}, |\alpha| = m} \int_G D^\alpha \mathbf{u}_N \cdot D^\alpha w_j dx \\
&= \sum_{\alpha \in \mathbb{N}, |\alpha| = m} \sum_{k=1}^N \int_G \xi_k(t) D^\alpha w_k \cdot D^\alpha w_j dx
\end{aligned}$$

Therefore, we conclude that α_j is Lipschitz in ξ for all three cases, and hence the existence of the unique solution to the Galerkin system is obtained.

Now we are ready to take the limit to get the solution P to the martingale problem. we start with the following lemma.

Lemma 4.4.1. *Let* $\Omega = L^\infty(0,T;H)_{w^*} \cap L^q(0,T;V_{r,q})_\sigma \cap D(0,T;V'_{r,q}) \cap L^q(0,T;H)$, *then the mapping* $\mathbf{y}(\cdot) \mapsto \mathcal{A}(\mathbf{y}(\cdot))$ *from* $\Omega \to L^{q'}(0,T;V'_{r,q})_\sigma$ *is Borel measurable.*

Proof. For $\forall \nu(\cdot) \in L^q(0,T;V_{r,q})$, the mapping $\mathbf{y} \mapsto \int_0^T \langle \mathcal{A}(\mathbf{y}(t)), \nu(t)\rangle$ is continuous from $L^q(0,T;V_{r,q}) \to \mathbb{R}$.

In fact, consider a sequence of $\mathbf{y}^n(\cdot) \in L^q(0,T;V_{r,q})$ converging strongly to $\mathbf{y} \in L^q(0,T;V_{r,q})$. Then for t almost everywhere in $[0,T]$, $\mathbf{y}^n(t) \to \mathbf{y}(t)$ in $V_{r,q}$. By the demicontinuity of $\mathcal{A}$ (lemma 3.2.3), we have $\mathcal{A}(\mathbf{y}^n(t)) \to \mathcal{A}(\mathbf{y}(t))$ for almost every $t \in [0,T]$, in the weak-star topology of $V'_{r,q}$. In particular, $\forall \nu(t) \in V_{r,q}$,

$$\langle \mathcal{A}(\mathbf{y}^n(t)), \nu(t)\rangle \to \langle \mathcal{A}(\mathbf{y}(t)), \nu(t)\rangle, \qquad t \text{ a.e. in } [0,T] \tag{4.4.4}$$

Note also $|\langle \mathcal{A}(\mathbf{y}^n(t)), \nu(t)\rangle| \leq g^n(t)$, for $g^n(t) := C||\mathbf{y}^n(t)||^{q-1}_{V_{r,q}} ||\nu(t)||_{V_{r,q}}$ and $g^n(\cdot) \to g(\cdot)$ in $L^1(0,T)$, due to the strong convergence of $\mathbf{y}^n(\cdot)$ in $L^q(0,T;V_{r,q})$. Thus, $\int_0^T \langle \mathcal{A}(\mathbf{y}^n(t)), \nu(t)\rangle dt \to \int_0^T \langle \mathcal{A}(\mathbf{y}(t)), \nu(t)\rangle dt$, which implies that $\mathbf{y}(\cdot) \to \int_0^T \langle \mathcal{A}(\mathbf{y}(t)), \nu(t)\rangle dt$ is continuous from $L^q(0,T;V_{r,q}) \to \mathbb{R}$.

Therefore, $\mathcal{A}(\cdot) : L^q(0,T;V_{r,q}) \to L^{q'}(0,T;V'_{r,q})_\sigma$ is continuous, and hence $\mathcal{A}(\cdot)$ is Borel measurable from $\Omega \to L^{q'}(0,T;V'_{r,q})_\sigma$. ☐

We denote $\hat{\Omega} := \Omega \times L^{q'}(0,T;V'_{r,q})_\sigma$.

Define the image of P^N under the map $\mathbf{y} \to (\mathbf{y}, \mathcal{A}(\mathbf{y}))$ as $\hat{Q}^N := P^N \circ (I, \mathcal{A}(\cdot))^{-1}$, or $\hat{Q}^N(\mathbf{A}) := P^N\{\omega \in \Omega; (\omega, \mathcal{A}(\omega)) \in \mathbf{A}\}$, for $\mathbf{A} \in \mathcal{B}(\hat{\Omega})$.

By Theorem 4.3.4, P^N are tight on Ω with Lusin topology and the energy estimate (Theorem 4.2.2) gives

$$\mathbb{E}^{P^N}\Big[\int_0^T ||\mathcal{A}(\mathbf{y}(t))||^{q'}_{V'_{r,q}} dt\Big] \leq C, \tag{4.4.5}$$

thus, $\exists\ \rho_\varepsilon > 0$ such that

$$\sup_N \hat{Q}^N\big\{(\omega,\nu) \in \hat{\Omega}; \int_0^T ||\nu(t)||^{q'}_{V'_{r,q}} dt \geq \rho_\varepsilon\big\} \leq \varepsilon. \tag{4.4.6}$$

Since $\left\{\nu \in L^{q'}(0,T;V'_{r,q})_\sigma; \int_0^T ||\nu(t)||^{q'}_{V'_{r,q}} dt \le \rho_\varepsilon\right\}$ is a compact set in $L^{q'}(0,T;V'_{r,q})_\sigma$, the measures $\hat{Q}^N$ form a tight family on $\hat{\Omega}$.

We now take the limits.

Denote $\mathbf{y}(t,\omega,\nu) := \omega(t), \chi(t,\omega,\nu) := \nu(t), \forall(\omega,\nu) \in \hat{\Omega}$, and $\mathcal{G}_t$ the canonical right continuous filtration generated on $\hat{\Omega}$ by $(\mathbf{y},\chi)$.

We define the functional $\tilde{\mathbf{M}}^\theta_t$ as

$$\begin{aligned}
\tilde{\mathbf{M}}^\theta_t(\mathbf{y},\chi) \;:=\; & \langle \mathbf{y}(t),\theta\rangle - \langle \mathbf{y}(0),\theta\rangle \\
& + \int_0^t \langle \nu_0 \mathbf{A}\mathbf{y}(s) + \nu_1\chi(s) + \mathbf{B}(\mathbf{y}(s)) - \mathbf{f}(s),\theta\rangle ds, \\
& \forall\theta \in V_{r,q}, \quad t \in [0,T] \qquad (4.4.7)
\end{aligned}$$

We can connect the two martingale problems (one in terms of the measure P with martingale $\mathbf{M}^\theta_t$ and another one in terms of the measure $\hat{Q}$ with martingale $\tilde{\mathbf{M}}^\theta_t$) explicitly as

$$\mathbf{M}^\theta_t(\mathbf{y}) = \int_{L^{q'}(0,T;V'_{r,q})_\sigma} \tilde{\mathbf{M}}^\theta_t(\mathbf{y},\chi)\hat{Q}(d\chi) \qquad (4.4.8)$$

We know that $\hat{Q}^N$ satisfy

1. $\hat{Q}^N\{(\omega,\nu) \in \hat{\Omega}; \mathcal{A}(\omega) = \nu\} = 1$,

2. $\tilde{\mathbf{M}}^\theta_t(\cdot,\cdot)$ is uniformly integrable (proposition 4.2.3), continuous in the Lusin topology of $\hat{\Omega}$, and is a $(\hat{\Omega},\mathcal{G}_t,\hat{Q}^N)$-martingale with

$$\langle\langle \tilde{\mathbf{M}}^\theta_t \rangle\rangle := \int_0^t (\sigma(\mathbf{y}(s))\theta, Q\sigma^*(\mathbf{y}(s))\theta) ds. \qquad (4.4.9)$$

Thanks to the above continuity and uniform integrability properties of $\mathbf{M}^\theta_t$, $\forall\phi \in C_b(\hat{\Omega})$ which is $\mathcal{G}_s$-measurable, $\mathbb{E}^{\hat{Q}^N}[\phi(\cdot)(\tilde{\mathbf{M}}^\theta_t - \tilde{\mathbf{M}}^\theta_s)] = 0$ will produce, in the limit, $\mathbb{E}^{\hat{Q}}[\phi(\cdot)(\tilde{\mathbf{M}}^\theta_t - \tilde{\mathbf{M}}^\theta_s)] = 0$, which is the same as saying that $\tilde{\mathbf{M}}^\theta_t(\cdot,\cdot)$ is a $(\hat{\Omega},\mathcal{G}_t,\hat{Q})$-martingale.

We just need to show that

$$\hat{Q}\{(\omega,\nu)\in\hat{\Omega};\mathcal{A}(\omega)=\nu\}=1 \tag{4.4.10}$$

Consider functions of the form

$$\zeta(\omega,\nu,t)=\sum_{i=1}^{k}\varphi(\omega,\nu,t)\mathbf{e_i},\qquad \mathbf{e_i}\in\mathbf{V_{r,q}} \tag{4.4.11}$$

which form a dense set in $L^q(\hat{\Omega},\hat{Q};L^q(0,T;V_{r,q}))$. Here $\varphi(\cdot,\cdot,t)$ are continuous in $\hat{\Omega}$ with paths in $L^q(0,T)$. We restrict ourselves to the special case of $\zeta(\omega,\nu,t)=\varphi(\omega,\nu,t)\mathbf{e_0}$, $\mathbf{e_0}\in\mathbf{V_{r,q}}$.

We now define

$$\begin{aligned}\Psi(\omega,\nu)= \quad & \int_0^T\{\langle\chi(\omega,\nu,s)-\mathcal{A}(\zeta(\omega,\nu,s)),\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s)\rangle\\ & +\nu_0|\mathbf{A}^{1/2}(\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s))|^2\}ds\end{aligned} \tag{4.4.12}$$

By the definition of $\hat{Q}^N$ and the monotonicity (Lemma 3.2.2) of $\mathcal{A}(\cdot)$,

$$\begin{aligned}\mathbb{E}^{\hat{Q}^N}[\Psi] &= \mathbb{E}^{\hat{Q}^N}[\int_0^T\{\langle\chi(\omega,\nu,s)-\mathcal{A}(\zeta(\omega,\nu,s)),\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s)\rangle\\ &\quad +\nu_0|\mathbf{A}^{1/2}(\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s))|^2\}ds]\\ &= \mathbb{E}^{\hat{Q}^N}[\int_0^T\{\langle\mathcal{A}(\mathbf{y}(\omega,\nu,s))-\mathcal{A}(\zeta(\omega,\nu,s)),\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s)\rangle\\ &\quad +\nu_0|\mathbf{A}^{1/2}(\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s))|^2\}ds]\geq 0,\end{aligned} \tag{4.4.13}$$

We write Ψ as $\Psi_1+\Psi_2$ where

$$\begin{aligned}\Psi_1(\omega,\nu) &= \int_0^T\{\langle\chi(\omega,\nu,s),\mathbf{y}(\omega,s)\rangle-\frac{1}{2}\mathbf{tr}(\sigma(\mathbf{y}(\omega,\mathbf{s}))\mathbf{Q}\sigma^*(\mathbf{y}(\omega,\mathbf{s})))\\ &+ \ \nu_0|\mathbf{A}^{1/2}\mathbf{y}(\omega,s)|^2\}ds\end{aligned} \tag{4.4.14}$$

and

$$\begin{aligned}\Psi_2(\omega,\nu) &= -\int_0^T\langle\mathcal{A}(\varphi(\omega,\nu,s)\mathbf{e_0}),\mathbf{y}(\omega,\nu,\mathbf{s})\rangle\mathbf{ds}\\ &\quad -\int_0^T\{\langle\chi(\omega,\nu,s)-\mathcal{A}(\varphi(\omega,\nu,s)\mathbf{e_0}),\varphi(\omega,\nu,\mathbf{s})\mathbf{e_0}\rangle\\ &\quad -\frac{1}{2}\mathbf{tr}(\sigma(\mathbf{y}(\omega,\mathbf{s}))\mathbf{Q}\sigma^*(\mathbf{y}(\omega,\mathbf{s})))\}\mathbf{ds}+\mathbf{2}\langle\nabla\mathbf{y},\nabla\zeta\rangle\end{aligned} \tag{4.4.15}$$

here we have used that $|\mathbf{A}^{1/2}(\mathbf{y}(\omega,s)-\zeta(\omega,\nu,s))|^2 - |\mathbf{A}^{1/2}\mathbf{y}(\omega,s)|^2 = 2\langle \nabla \mathbf{y}, \nabla\zeta\rangle$

Note that $\Psi_2(\cdot,\cdot)$ is continuous in the Lusin topology of $\hat{\Omega}$. Using the strong hemicontinuity (Lemma 3.2.4) of $\mathcal{A}$ and the fact that φ is continuous, we obtained from the energy estimate and Lemma 4.4.2 below that

$$\lim_N \mathbb{E}^{\hat{Q}^N}[\Psi_2] = \mathbb{E}^{\hat{Q}}[\Psi_2] \tag{4.4.16}$$

Lemma 4.4.2. *(Lemma 15, [34]) Let Ω be a Lusin space and let $\{P^n\}$ be a tight sequence of probability measures on $\mathcal{B}(\Omega)$ converging weakly to P on $\mathcal{B}(\Omega)$. Let $f(\cdot) \in C(\Omega)$ be a possibly unbounded function such that the following uniform integrability holds: For some $\varepsilon > 0$,*

$$\sup_n \mathbb{E}^{P^n}[|f|^{1+\varepsilon}] \le C. \tag{4.4.17}$$

Then $\mathbb{E}^{P^n}[f] \to \mathbb{E}^P[f]$, as $n \to \infty$.

For Ψ_1 which is not continuous in the Lusin topology of $\hat{\Omega}$, we use the energy equality in the proposition 4.2.1 to get

$$\mathbb{E}^{\hat{Q}}[\Psi_1] = \mathbb{E}^{\hat{Q}}[\tilde{\Psi}_1] \tag{4.4.18}$$

where

$$\tilde{\Psi}_1(\omega,\nu) := \frac{1}{2}(|\mathbf{y}_0|^2 - |\mathbf{y}(T)|^2) + \int_0^T (\mathbf{f}(s), \mathbf{y}(\omega,s))ds. \tag{4.4.19}$$

Note that $\tilde{\Psi}_1$ is upper semicontinuous on the Lusin topology of $\hat{\Omega}$, and hence using the fact that if we integrate an upper semicontinuous function with a probability measure we will get an upper semicontinuous functional with respect to the measure (Theorem 55, Chapter III of [5]) we get

$$\limsup_N \mathbb{E}^{\hat{Q}^N}[\tilde{\Psi}_1(\cdot,\cdot)] \le \mathbb{E}^{\hat{Q}}[\tilde{\Psi}_1(\cdot,\cdot)] \tag{4.4.20}$$

From equations (4.4.13),(4.4.16),(4.4.18) and (4.4.20), we obtain that

$$\begin{aligned} 0 &\leq \limsup_N \mathbb{E}^{\hat{Q}^N}[\Psi] = \limsup_N \mathbb{E}^{\hat{Q}^N}[\tilde{\Psi}_1 + \Psi_2] \\ &\leq \mathbb{E}^{\hat{Q}}[\tilde{\Psi}_1 + \Psi_2] = \mathbb{E}^{\hat{Q}}[\Psi_1 + \Psi_2] \\ &= \mathbb{E}^{\hat{Q}}[\Psi] \end{aligned} \tag{4.4.21}$$

Setting $\zeta(\omega, \nu, t) = \mathbf{y}(\omega, t) - \lambda \mathbf{w}(\omega, \nu, t)$, where $\lambda > 0$ and $\mathbf{w}(\cdot, \cdot, t) : \hat{\Omega} \to L^q(0, T; V_{r,q})$ is any bounded measurable mapping,

$$\begin{aligned} \int_{\hat{\Omega}} [\int_0^T \langle \chi(\omega, \nu, s) - \mathcal{A}(\mathbf{y}(\omega, s) - \lambda \mathbf{w}(\omega, \nu, s)), \mathbf{w}(\omega, \nu, s) \rangle \\ + \nu_0 \lambda |\mathbf{A}^{1/2} \mathbf{w}(\omega, \nu, s)|^2 ds] \hat{Q}(d\omega, d\nu) \geq 0. \end{aligned} \tag{4.4.22}$$

Using the hemicontinuity (Lemma 3.2.4) of $\mathcal{A}$, as $\lambda \to 0$,

$$\begin{aligned} \langle \chi(\omega, \nu, s) - \mathcal{A}(\mathbf{y}(\omega, s) - \lambda \mathbf{w}(\omega, \nu, s)), \mathbf{w}(\omega, \nu, s) \rangle \\ \to \langle \chi(\omega, \nu, s) - \mathcal{A}(\mathbf{y}(\omega, s)), \mathbf{w}(\omega, \nu, s) \rangle \end{aligned} \tag{4.4.23}$$

with $|\langle \chi - \mathcal{A}(\mathbf{y} - \lambda \mathbf{w}), \mathbf{w} \rangle| \leq C ||\mathbf{w}||_{V_{r,q}} \{ (||\mathbf{y}||_{V_{r,q}} + ||\mathbf{w}||_{V_{r,q}})^{q-1} + ||\chi||_{V'_{r,q}} \}$.

By the dominated convergence theorem,

$$\int_{\hat{\Omega}} \left[\int_0^T \langle \chi(\omega, \nu, s) - \mathcal{A}(\mathbf{y}(\omega, s)), \mathbf{w}(\omega, \nu, s) \rangle ds \right] \hat{Q}(d\omega, d\nu) \geq 0. \tag{4.4.24}$$

Since the inequality (4.4.24) holds for each bounded measurable $\mathbf{w}$, we could take $-\mathbf{w}$ to get the opposite inequality, and hence $\hat{Q}\{(\omega, \nu) \in \hat{\Omega}; \mathcal{A}(\omega) = \nu\} = 1$.

Chapter 5
Solutions by Monotonicity Method

The method of monotonicity was first used by Krylov and Rozovskii [15] to prove the existence and uniqueness of strong solutions for a wide class of stochastic evolution equations under certain assumptions on the drift and diffusion coefficients, which is in fact a nice refinement of an important result by Pardoux [31].

In this chapter, a monotonicity lemma and certain priori estimates are given in the first section, and then the existence and uniqueness of the SNSE with nonlinear viscosities is proved in the second section.

5.1 Monotonicity and Priori Estimates

We start with the following lemma.

Lemma 5.1.1. *For a given $R > 0$, let B_R denote the $L^4(G)$-ball in $V_{r,q}$:*

$$B_R = \{\mathbf{v} \in V_{r,q} : ||\mathbf{v}||_{L^4(G)} \leq R\} \tag{5.1.1}$$

Define the nonlinear operator F on $V_{r,q}$ by

$$F(\mathbf{u}) = -\nu_0 \mathbf{A}\mathbf{u} - \nu_1 \mathcal{A}(\mathbf{u}) - \mathbf{B}(\mathbf{u}) \tag{5.1.2}$$

Then for any $0 < \varepsilon < \frac{\nu_0}{2L}$ where L is the constant that appears in the assumption (A.3) (see Section 3.2), the pair $(F, \sqrt{\varepsilon}\sigma)$ is monotone in B_R: i.e. for any $\mathbf{u} \in V_{r,q}$ and $\mathbf{v} \in B_R$, if we denote $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then

$$\langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{w}\rangle - \frac{R^4}{\nu_0^3}|\mathbf{w}|^2 + \varepsilon|\sigma(t,\mathbf{u}) - \sigma(t,\mathbf{v})|^2_{L_Q} \leq 0. \tag{5.1.3}$$

Proof. Since $\langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{w}\rangle = b(\mathbf{u},\mathbf{w},\mathbf{w}) = 0$, we obtain that

$$\langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{u}\rangle = \langle \mathbf{B}(\mathbf{u},\mathbf{w}), \mathbf{v}\rangle \tag{5.1.4}$$

then

$$\langle \mathbf{B}(\mathbf{u}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{u}\rangle = -\langle \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle \tag{5.1.5}$$

Also, $\langle \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{v}, \mathbf{w}), \mathbf{v}\rangle$

Using the above two equations, we have

$$\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -\langle \mathbf{B}(\mathbf{w}), \mathbf{v}\rangle \tag{5.1.6}$$

From the Hölder inequality, it follows that

$$\begin{aligned} |\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| &\leq ||\mathbf{w}||_{L^4(G)}\, ||\mathbf{w}||\, ||\mathbf{v}||_{L^4(G)} \\ &\leq |\mathbf{w}|^{1/2}||\mathbf{w}||^{3/2}||\mathbf{v}||_{L^4(G)} \\ &\leq \frac{\nu_0}{2}||\mathbf{w}||^2 + \frac{27}{32\nu_0^3}|\mathbf{w}|^2||\mathbf{v}||^4_{L^4(G)} \end{aligned} \tag{5.1.7}$$

where the last inequality follows from the that for any real numbers a, b and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$$

and we take $p = \frac{4}{3}, q = 4$, and $a = (\frac{2\nu_0}{3})^{3/4}||\mathbf{w}||^{3/2}, b = (\frac{3}{2\nu_0})^{3/4}|\mathbf{w}|^{1/2}||\mathbf{v}||_{L^4(G)}$.

Note that $\langle \mathbf{A}(\mathbf{w}), \mathbf{w}\rangle = ||\mathbf{w}||^2$, and by the lemma 3.2.2 we have

$$\langle \mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}), \mathbf{w}\rangle \geq 0. \tag{5.1.8}$$

Thus,

$$\begin{aligned} \langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{w}\rangle &= -\nu_0||\mathbf{w}||^2 - \nu_1\langle \mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v}), \mathbf{w}\rangle \\ &\quad -\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle \\ &\leq -\frac{\nu_0}{2}||\mathbf{w}||^2 + \frac{R^4}{\nu_0^3}|\mathbf{w}|^2 \end{aligned} \tag{5.1.9}$$

Therefore, the proof is completed upon using the assumption (A.3) and that $\varepsilon \leq \frac{\nu_0}{2L}$.

□

Remark: The above lemma holds for more general cases:

$$F(\mathbf{u}) = -\nu \mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u})$$

where $\nu = \nu(|\nabla \mathbf{u}|)$ is a continuous function of $|\nabla \mathbf{u}|$ and $\nu(|\nabla \mathbf{u}|) > \nu_0 > 0$.

For example, if $\nu = \nu_0 + \nu_1 |\nabla \mathbf{u}|^2$, then we get exactly the same as *Case 2*: Nonlinear Nonlocal Viscosity, since the operator $\mathcal{A}_2(\mathbf{u}) = |\nabla \mathbf{u}|^2 \mathbf{A}\mathbf{u}$.

Now we consider $H_n := span\{e_1, e_2, \cdots, e_n\}$, where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in D(A)$. Let P_n denote the orthogonal projection of H to H_n. Define $W_n = P_n W$. Let $\sigma_n = P_n \sigma$. Define $\mathbf{u}_n^\varepsilon$ as the solution of the following stochastic differential equation: For each $\mathbf{v} \in H_n$,

$$d(\mathbf{u}_n^\varepsilon(t), \mathbf{v}) = \{\langle \mathbf{f}(t), \mathbf{v}\rangle + (F(\mathbf{u}_n^\varepsilon(t)), \mathbf{v})\}dt + \sqrt{\varepsilon}(\sigma_n(t, \mathbf{u}_n^\varepsilon(t))dW_n(t), \mathbf{v}) \quad (5.1.10)$$

with $\mathbf{u}_n^\varepsilon(0) = P_n \mathbf{u}(0)$.

The standard theory of finite-dimensional stochastic differential equations [14] guarantees the existence of a unique solution to (5.1.10) under the assumptions (A.1)-(A.3) (see Section 3.2) if $\mathbf{f} \in L^2(0, T; H)$ and $\mathbf{u}(0)$ is $\mathcal{F}_0$-measurable in $L^2(P)$.

The following theorem gives a priori estimates and its proof is almost the same as that of Theorem 4.2.2.

Theorem 5.1.2. *Let* $\mathbf{f}$ *be in* $L^2(0, T; H)$ *and let* $\mathbb{E}^P(|\mathbf{u}(0)|^2) < \infty$. *Let* $\mathbf{u}_n^\varepsilon$ *denote the unique strong solution of the finite system of Eq.* (5.1.10) *in* $C(0, T; H_n)$. *Then, with* K *as in the assumption (A.2), the following estimates hold:*

$$\mathbb{E}^P \left[|\mathbf{u}_n^\varepsilon(t)|^2 + \nu_0 \int_0^t |\mathbf{A}^{1/2}\mathbf{u}_n^\varepsilon(s)|^2 ds + \nu_1 \int_0^t \|\mathbf{u}_n^\varepsilon(s)\|_{V_{r,q}}^q ds \right] \leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2], \int_0^T |\mathbf{f}(s)|^2 ds, K, T \right) \quad (5.1.11)$$

and

$$\mathbb{E}^P \left[\sup_{0 \leq t \leq T} |\mathbf{u}_n^\varepsilon(t)|^2 \right] \leq C\left(\mathbb{E}^P[|\mathbf{u}_0|^2], \int_0^T |\mathbf{f}(s)|^2 ds, K, T \right) \quad (5.1.12)$$

If $\mathbb{E}|\mathbf{u}_0|^4 < \infty$ *and* $\mathbf{f}$ *is in* $L^4(0,T;H)$*, then*

$$\mathbb{E}^P[\sup_{0\le t\le T}|\mathbf{u}_n^\varepsilon(t)|^4 + \nu_0\int_0^t |\mathbf{u}_n^\varepsilon(s)|^2|\mathbf{A}^{1/2}\mathbf{u}_n^\varepsilon(s)|^2 ds$$
$$+\nu_1\int_0^t |\mathbf{u}_n^\varepsilon(s)|^2\|\mathbf{u}_n^\varepsilon(s)\|_{V_{r,q}}^q ds]$$
$$\le C\left(\mathbb{E}^P|\mathbf{u}_0|^4, \int_0^T |\mathbf{f}(s)|^4 ds, K, T\right) \tag{5.1.13}$$

5.2 Existence and Uniqueness

Definition 5.2.1. (Strong Solution) A strong solution $\mathbf{u}^\varepsilon$ is defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as a $C([0,T];H)\cap L^q(\Omega\times(0,T);V_{r,q})$ valued adapted process which satisfies

$$d\mathbf{u}^\varepsilon + [\nu_0\mathbf{A}\mathbf{u}^\varepsilon + \nu_1\mathcal{A}(\mathbf{u}^\varepsilon) + \mathbf{B}(\mathbf{u}^\varepsilon)]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t,\mathbf{u}^\varepsilon)dW(t)$$
$$\mathbf{u}^\varepsilon(0) = \mathbf{u}_0, \tag{5.2.1}$$

in the weak sense and also the estimates in Theorem 5.1.2.

Theorem 5.2.2. *Let* $\mathbb{E}^P|\mathbf{u}_0|^4 < \infty$ *and* $\mathbf{f} \in L^4(0,T;H)$*. If* ε *is small enough, then under the assumptions (A.1)-(A.3) on the diffusion coefficient* σ*, there exists a strong solution to the following stochastic Navier-Stokes equation:*

$$d\mathbf{u}^\varepsilon + [\nu_0\mathbf{A}\mathbf{u}^\varepsilon + \nu_1\mathcal{A}(\mathbf{u}^\varepsilon) + \mathbf{B}(\mathbf{u}^\varepsilon)]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t,\mathbf{u}^\varepsilon)dW(t) \tag{5.2.2}$$

and the solution is pathwise unique.

Proof. Part I:(Existence)

Let $\Omega_T := \Omega \times [0,T]$. Using the priori estimates in the above Theorem 5.1.2, it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\mathbf{u}_n$ have the following limits:

$$\begin{aligned}
&\mathbf{u}_n \to \mathbf{u} \qquad \text{weakly in } L^q(\Omega_T, V_{r,q}) \\
&\mathbf{u}_n \to \mathbf{u} \qquad \text{weak-star in } L^4(\Omega, L^\infty(0,T;H)) \\
&\mathbf{u}_n(T) \to \eta \quad \text{weakly in } L^2(\Omega; H)
\end{aligned} \tag{5.2.3}$$

For ease of notation, we have defined $\mathbf{u} := \mathbf{u}^\varepsilon$ and $\mathbf{u}_n := \mathbf{u}_n^\varepsilon$.

Recall that

$$F(\mathbf{u}) = -\nu_0 \mathbf{A}\mathbf{u} - \nu_1 \mathcal{A}(\mathbf{u}) - \mathbf{B}(\mathbf{u})$$

.

Since $F(\mathbf{u}_n)$ is bounded in $L^{q'}(0,T;V'_{r,q})$, we have

$$F(\mathbf{u}_n) \to F_0 \quad \text{weakly in } L^{q'}(\Omega_T, V'_{r,q}) \tag{5.2.4}$$

Likewise,

$$\sigma_n(\cdot, \mathbf{u}_n) \to S \quad \text{weakly in } L^2(\Omega_T, L_Q). \tag{5.2.5}$$

since σ has linear growth (assumption A.2) and $\mathbf{u}_n$ is bounded in $L^2(0,T;H)$ uniformly in n by the priori estimates.

We extend the Eq. (5.1.10) to an open interval $(-\delta, T+\delta)$ by setting the terms equal to 0 outside of the interval $[0,T]$. Let $\phi(t)$ be a function in $H^1(-\delta, T+\delta)$ with $\phi(0) = 1$.

Define for all integers $j \geq 1$,

$$e_j(t) = \phi(t) e_j \tag{5.2.6}$$

where $\{e_j\}$ is the fixed orthonormal sequence for H.

Applying the Itô formula to the function $(\mathbf{u}_n(t), e_j(t))$, we obtain that

$$\begin{aligned}(\mathbf{u}_n(T), e_j(T)) &= (\mathbf{u}_n(0), e_j) + \int_0^T \left(\mathbf{u}_n(s), \frac{de_j(s)}{ds}\right) ds \\ &+ \int_0^T \langle F(s, \mathbf{u}_n(s)), e_j(s)\rangle ds \\ &+ \sqrt{\varepsilon} \int_0^T (\sigma_n(s, \mathbf{u}_n(s)) dW_n(s), e_j(s)) \end{aligned} \tag{5.2.7}$$

Let $\mathcal{P}_T$ denote the class of predictable processes taking values in $L^2(\Omega_T; L_Q(H_0; H))$ with inner product defined by

$$(G, J)_{\mathcal{P}_T} = \mathbb{E} \int_0^T \mathbf{tr}\{G(s) Q J^*(s)\} ds, \quad \forall G, J \in \mathcal{P}_T \tag{5.2.8}$$

The map $J : \mathcal{P}_T \to L^2(\Omega_T)$ defined by the real-valued integral

$$J_t(G) = \int_0^t (G(s) dW(s), e_j(s)) \tag{5.2.9}$$

is linear and continuous.

By the weak convergence of

$$\sigma_n(s, \mathbf{u}_n(s)) P_n \to S \text{ in } L^2_Q(H_0; H) \tag{5.2.10}$$

we obtain that

$$(\sigma_n(s, \mathbf{u}_n(s)) P_n, R)_{\mathcal{P}_T} \to (S, R)_{\mathcal{P}_T} \tag{5.2.11}$$

for any $R \in \mathcal{P}_T$ as $n \to \infty$.

Thus we conclude that

$$\int_0^T (\sigma_n(s, \mathbf{u}_n(s)) dW_n(s), e_j(s)) \to \int_0^T \phi(s)(S(s) dW(s), e_j) \tag{5.2.12}$$

as $n \to \infty$ for each j.

Taking the limit termwise in (5.2.7),

$$\begin{aligned} -\int_0^T \left(\mathbf{u}(s), \frac{de_j(s)}{ds}\right) ds &= (\mathbf{u}_0, e_j) + \int_0^T \langle F_0(s), e_j\rangle \phi(s) ds \\ &+ \sqrt{\varepsilon} \int_0^T \phi(s)(S(s) dW(s), e_j) - (\eta, e_j)\phi(T) \end{aligned} \tag{5.2.13}$$

For any $0 < t \leq T$, choose a sequence of functions $\{\phi_k\}$ in place of ϕ such that $\phi_k \to \mathbf{1}_{[0,t]}$ and the time derivative of ϕ_k converges weakly to $-\delta_t$ as $k \to \infty$.

Then

$$(\mathbf{u}(t), e_j) = (\mathbf{u}_0, e_j) + \int_0^t \langle F_0(s), e_j \rangle ds + \sqrt{\varepsilon} \int_0^t (S(s)dW(s), e_j) \tag{5.2.14}$$

Thus,

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t F_0(s)ds + \sqrt{\varepsilon} \int_0^t S(s)dW(s) \tag{5.2.15}$$

with $\mathbf{u}(T) = \eta$.

Define $r(t) = \frac{2}{\nu_0^3} \int_0^t ||\mathbf{v}(s)||^4_{L^4(G)}$ and apply the Itô formula to $e^{-r(t)}|\mathbf{u}_n(t)|^2$,

$$\begin{aligned} e^{-r(T)}|\mathbf{u}_n(T)|^2 &= |\mathbf{u}_n(0)|^2 + \int_0^T e^{-r(s)}(2F(\mathbf{u}_n(s)) - \frac{dr(s)}{ds}\mathbf{u}_n(s), \mathbf{u}_n(s))ds \\ &\quad + \varepsilon \int_0^T e^{-r(s)}|\sigma_n(s, \mathbf{u}_n(s))|^2_{L_Q} ds \\ &\quad + 2\sqrt{\varepsilon} \int_0^T e^{-r(s)}(\mathbf{u}_n(s), \sigma_n(s, \mathbf{u}_n(s))dW(s)) \end{aligned} \tag{5.2.16}$$

and then taking expectation

$$\begin{aligned} &\mathbb{E}\left[e^{-r(T)}|\mathbf{u}_n(T)|^2 - |\mathbf{u}_n(0)|^2\right] \\ &= \mathbb{E}[\int_0^T e^{-r(s)}(2F(\mathbf{u}_n(s)) - \frac{dr(s)}{ds}\mathbf{u}_n(s), \mathbf{u}_n(s))ds] \\ &+ \varepsilon\mathbb{E}[\int_0^T e^{-r(s)}|\sigma_n(s, \mathbf{u}_n(s))|^2_{L_Q} ds] \end{aligned} \tag{5.2.17}$$

Then by the lower semi-continuity property of weak convergence,

$$\begin{aligned} &\liminf_n \mathbb{E}[\int_0^T e^{-r(s)}(2F(\mathbf{u}_n(s)) - \frac{dr(s)}{ds}\mathbf{u}_n(s), \mathbf{u}_n(s)) \\ &+ \varepsilon \int_0^T e^{-r(s)}|\sigma_n(s, \mathbf{u}_n(s))|^2_{L_Q} ds] \\ &= \liminf_n \mathbb{E}[e^{-r(T)}|\mathbf{u}_n(T)|^2 - |\mathbf{u}_n(0)|^2] \\ &\geq \mathbb{E}[e^{-r(T)}|\mathbf{u}(T)|^2 - |\mathbf{u}(0)|^2] \\ &= \mathbb{E}[\int_0^T e^{-r(s)}(2F_0(s) - \frac{dr(s)}{ds}\mathbf{u}(s), \mathbf{u}(s))ds \\ &\quad + \varepsilon \int_0^T e^{-r(s)}|S|^2_{L_Q} ds] \end{aligned} \tag{5.2.18}$$

On the other hand, by the monotonicity lemma 5.1.1, we have

$$
\begin{aligned}
& 2\mathbb{E}\left[\int_0^T \langle F(\mathbf{u}_n(s)) - F(\mathbf{v}(s)), \mathbf{u}_n(s) - \mathbf{v}(s)\rangle e^{-r(s)} ds\right] \\
& -\mathbb{E}\left[\int_0^T \frac{dr(s)}{ds} e^{-r(s)} |\mathbf{u}_n(s) - \mathbf{v}(s)|^2 ds\right] \\
& +\varepsilon\mathbb{E}\left[\int_0^T e^{-r(s)} |\sigma_n(s, \mathbf{u}_n(s)) - \sigma_n(s, \mathbf{v}(s))|_{L_Q}^2 ds\right] \\
\leq \; & 0. \qquad (5.2.19)
\end{aligned}
$$

Rearranging the terms in (5.2.19)

$$
\begin{aligned}
& \mathbb{E}[\int_0^T e^{-r(s)} (2F(\mathbf{u}_n(s)) - \frac{dr(s)}{ds} \mathbf{u}_n(s), \mathbf{u}_n(s)) \\
& \quad +\varepsilon \int_0^T e^{-r(s)} |\sigma_n(s, \mathbf{u}_n(s))|_{L_Q}^2 ds] \\
\leq \; & \mathbb{E}\left[\int_0^T e^{-r(s)} (2F(\mathbf{u}_n(s)) - \frac{dr(s)}{ds}(2\mathbf{u}_n(s) - \mathbf{v}(s)), \mathbf{v}(s)) ds\right] \\
& \quad +\mathbb{E}\left[\int_0^T e^{-r(s)} (2F(\mathbf{v}(s)), \mathbf{u}_n(s) - \mathbf{v}(s)) ds\right] \\
& \quad +\varepsilon\mathbb{E}\left[\int_0^T e^{-r(s)} (2\sigma_n(s, \mathbf{u}_n(s)) - \sigma_n(s, \mathbf{v}(s)), \sigma_n(s, \mathbf{v}(s)))_{L_Q} ds\right] \\
& \qquad (5.2.20)
\end{aligned}
$$

Using Eq. (5.2.18) and rearranging, as $n \to \infty$, we obtain

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{-r(s)} (2F_0(s) - 2F(\mathbf{v}(s)), \mathbf{u}(s) - \mathbf{v}(s)) ds\right] \\
& \quad +\mathbb{E}\left[\int_0^T e^{-r(s)} \frac{dr(s)}{ds} |\mathbf{u}(s) - \mathbf{v}(s)|^2 ds\right] \\
& \quad +\varepsilon\mathbb{E}\left[\int_0^T e^{-r(s)} |S(s) - \sigma(s, \mathbf{v}(s))|_{L_Q}^2\right] \\
& \leq 0. \qquad (5.2.21)
\end{aligned}
$$

Setting $\mathbf{v} = \mathbf{u}$, we get $S(s) = \sigma(s, \mathbf{u}(s))$.

Taking $\mathbf{v} = \mathbf{u} - \lambda\mathbf{w}$ with $\lambda > 0$, then

$$\begin{aligned}
&\lambda\mathbb{E}\left[\int_0^T e^{-r(s)}(2F_0(s) - 2F(\mathbf{u} - \lambda\mathbf{w})(s), \mathbf{w}(s))ds\right] \\
&+\lambda^2\mathbb{E}\left[\int_0^T e^{-r(s)}\frac{dr(s)}{ds}|\mathbf{w}(s)|^2 ds\right] \\
&\leq 0. \qquad (5.2.22)
\end{aligned}$$

Dividing by λ on both sides of the inequality above and letting $\lambda \to 0$,

$$\mathbb{E}\left[\int_0^T e^{-r(s)}(F_0(s) - F(\mathbf{u}(s)), \mathbf{w}(s))ds\right] \leq 0 \qquad (5.2.23)$$

Since $\mathbf{w}$ is arbitrary, we conclude that $F_0(s) = F(\mathbf{u}(s))$, thus the existence of a strong solution of Eq. (5.2.2) has been proved.

Part II: (Uniqueness)

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of the equation (5.2.2). Denote $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ and $\sigma_d = \sigma(t, \mathbf{u}_1(t)) - \sigma(t, \mathbf{u}_2(t))$. Then $\mathbf{w}$ solves the following stochastic differential equation:

$$d\mathbf{w}(t) = (F(\mathbf{u}_1(t)) - F(\mathbf{u}_2(t)))dt + \sqrt{\varepsilon}\sigma_d dW(t) \qquad (5.2.24)$$

Applying the Itô formula to the function $e^{-k(t)}|\mathbf{w}(t)|^2$, we obtain

$$\begin{aligned}
d\left[e^{-k(t)}|\mathbf{w}(t)|^2\right] &= e^{-k(t)}\left[2(F(\mathbf{u}_1(t)) - F(\mathbf{u}_2(t)), \mathbf{w}(t)) + \varepsilon|\sigma_d|^2_{L_Q}\right]dt \\
&\quad -e^{-k(t)}\frac{dr(t)}{dt}(\mathbf{w}(t), \mathbf{w}(t))dt + 2\sqrt{\varepsilon}e^{-k(t)}(\mathbf{w}(t), \sigma_d dW(t))
\end{aligned} \qquad (5.2.25)$$

Take $k(t) = k$ to be a constant function, then

$$\begin{aligned}
d\left[|\mathbf{w}(t)|^2\right] &= \left[2(F(\mathbf{u}_1(t)) - F(\mathbf{u}_2(t)), \mathbf{w}(t)) + \varepsilon|\sigma_d|^2_{L_Q}\right]dt \\
&\quad +2\sqrt{\varepsilon}(\mathbf{w}(t), \sigma_d dW(t))
\end{aligned} \qquad (5.2.26)$$

It follows from the monotonicity lemma 5.1.1 that

$$d\left[|\mathbf{w}(t)|^2\right] \leq \frac{2r^4}{\nu_0^3}|\mathbf{w}(t)|^2 dt + 2\sqrt{\varepsilon}(\mathbf{w}(t), \sigma_d dW(t)) \qquad (5.2.27)$$

Integrating over the time interval $[0, t]$, for all $0 < t \le T$,

$$|\mathbf{w}(t)|^2 \le \frac{2r^4}{\nu_0^3} \int_0^t |\mathbf{w}(s)|^2 ds + 2\sqrt{\varepsilon} \int_0^t (\mathbf{w}(s), \sigma_d dW(s)) \tag{5.2.28}$$

Taking expectation on both sides and noticing that $\int_0^T (\mathbf{w}(t), \sigma_d dW(t))$ is a martingale and hence has a zero mean, we have

$$\mathbb{E}[|\mathbf{w}(t)|^2] \le \frac{2r^4}{\nu_0^3} \mathbb{E}\left[\int_0^t |\mathbf{w}(s)|^2 ds\right] \tag{5.2.29}$$

Thus, $\mathbf{w}(s) = 0$ a.s. due to the Gronwall inequality, and hence $\mathbf{u}_1 = \mathbf{u}_2$ for all $t \in [0, T]$ a.s.

□

Chapter 6
Large Deviations Result

In this chapter, we will use the stochastic control and weak convergence approach to obtain the large deviation principle (LDP) for solutions of the two-dimensional stochastic Navier-Stokes equations with nonlinear viscosities driven by a small noise, infinite dimensional Wiener process. The starting point of this approach is the fact that the large deviation principle is equivalent to the Laplace-Varadhan principle if the underlying space is Polish. This fact is due to Varadhan's Lemma 2.1.4 and Bryc's converse Lemma 2.1.5.

Our main interest in the current work is the study of large deviations for the stochastic Navier-Stokes equations with nonlinear viscosities. We will be working with the Laplace-Varadhan principle rather than the LDP, because the Laplace-Varadhan principle transforms the problem of obtaining exponential bounds for probabilities of certain events to that in terms of expectations of continuous functionals and the latter are much more suitable for the application of weak convergence methods.

6.1 A Variational Representation

In this section, we prove a variational representation for positive functionals of a Hilbert space valued Wiener process $W(\cdot)$. This representation can be used to prove a large deviation principle for the family $\{g^\varepsilon(W(\cdot))\}$ (see Theorem 6.2.2 in the next section), where $\{g^\varepsilon\}$ is an appropriate family of measurable maps from the Wiener space to some Polish space. Using this large deviation principle, one can then derive the Wentzell-Freidlin type large deviation results for the stochastic Navier-Stokes equations with nonlinear and hyperviscosities.

We recall the definition of Wiener processes that take values in a separable Hilbert space H. Let $(\cdot,\cdot)$ denote the inner product for H. Let Q be a strictly positive, symmetric, trace class operator on H. Define $H_0 = Q^{1/2}H$ with inner product $(\mathbf{u},\mathbf{v})_0 = (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v})$, the norm in H_0 is denoted as $|\cdot|_0$, and H_0 is compactly embedded in H.

Let $(\Omega,\mathcal{F},\mu)$ be the Wiener space, where $\Omega = C([0,T];H)$, μ is the Wiener measure and $\{\mathcal{F}_t\}$ is the augmented (by subsets of μ-null sets of $\mathcal{F}$) filtration.

Definition 6.1.1. A stochastic process $\{W(t)\}_{0\le t\le T}$ on $(\Omega,\mathcal{F},\mu)$ is said to be an H-valued $\mathcal{F}_t$-adapted Wiener process with covariance operator Q if

(1) for each non-zero $h \in H$, $|Q^{1/2}h|_H^{-1}(W(t),h)$ is a standard one-dimensional Wiener process, and

(2) for any $h \in H$, $(W(t),h)$ is a $\mathcal{F}_t$-adapted martingale.

Let $\mathcal{O}$ denote the class of H_0-valued $\mathcal{F}_t$-predictable processes ϕ which satisfy

$$\mu\{\int_0^T |\phi(s)|_0^2 ds < \infty\} = 1 \tag{6.1.1}$$

Let us denote by $\mathcal{O}_b$ the subset of bounded elements of $\mathcal{O}$, and $\mathcal{O}_s$ the class of bounded simple processes of $\mathcal{O}$ respectively.

From the construction of stochastic integrals, we know that if $X := \{X_t\}$ belongs to $\mathcal{O}_b$, then there exists a sequence $X^n := \{X_t^n\}$ of processes from $\mathcal{O}_s$ that are bounded uniformly in n by the bound for X, and

$$\lim_{n\to\infty} \mathbb{E}\int_0^1 |X_s^n - X_s|_0^2 ds = 0. \tag{6.1.2}$$

It is also useful to recall the basic fact that a real valued function f on a probability space $(E,\mathcal{E},\eta)$, where E is a Polish space, can be approximated by a sequence of continuous functions $\{f_n\}$ in the almost sure sense. If $|f| \le K$, then, for all n, we can take $|f_n| \le K$.

We define the relative entrop which plays a key role in the sequel.

Definition 6.1.2. (**Relative Entropy**) Let $(E, \mathcal{E})$ be as above with $\mathcal{P}(E)$ as the class of probability measures defined on it. For $\eta \in \mathcal{P}(E)$, the *relative entropy* $R(\cdot \,||\, \eta)$ is the mapping from $\mathcal{P}(E)$ into the extended real numbers given by

$$R(\lambda \,||\, \eta) = \int_E \log \frac{d\lambda}{d\eta}(x)\lambda(dx) \tag{6.1.3}$$

if $\lambda \ll \eta$ and $\log \frac{d\lambda}{d\eta}(x)$ is λ-integrable; Otherwise, define $R(\lambda \,||\, \eta)$ to be infinity.

The following simple and elegant result gives an abstract variational representation using the relative entropy function. A proof of it can be found in [6].

Proposition 6.1.3. *Let $(E, \mathcal{E})$ be a measurable space, and f, a bounded measurable function from E to $\mathbb{R}$. Suppose that η is a probability measure on E. Then,*

(i) $-\log \int_E e^{-f(x)}\eta(dx) = \inf_{\lambda \in \mathcal{P}(E)} \left\{ R(\lambda \,||\, \eta) + \int_E f(x)\lambda(dx) \right\}$.

(ii) The infimum in the above equation is reached at a probability measure λ^* where

$$\frac{d\lambda^*}{d\eta}(x) = Ce^{-f(x)}$$

with C as the normalizing constant.

The next result is quite useful in the sequel, and its proof can be found in [2].

Proposition 6.1.4. *Consider the probability space $(E, \mathcal{E}, \eta)$, where E is a Polish space and $\mathcal{E}$ its Borel σ-field. Let f be a real-valued, bounded Borel-measurable function on E. Suppose that $\{\lambda_n\}$ is a sequence in $\mathcal{P}(E)$ such that there exists a constant C satisfying*

$$\sup_n R(\lambda_n \,||\, \eta) \leq C < \infty$$

and $\lambda_n \to \lambda$ weakly as $n \to \infty$. Then the following hold:

(i) $\lim_{n\to\infty} \int_E f d\lambda_n = \int_E f d\lambda$, *and*

(ii) if $\{f_n\}$ is a sequence of uniformly bounded functions that converges to f η-a.s., then

$$\lim_{n\to\infty}\int_E f_n d\lambda_n = \int_E f d\lambda.$$

Next, we state the Girsanov theorem in infinite dimensions.

Theorem 6.1.5. ***(Girsanov)*** *Let h be an H_0-valued $\mathcal{F}_t$-predictable process with $\int_0^T |h(s)|_0^2 ds < \infty$ a.s. for some fixed T, and*

$$\mathbb{E}\big(\exp\big\{\int_0^T h(s)dW(s) - \frac{1}{2}\int_0^T |h(s)|_0^2 ds\big\}\big) = 1.$$

Then the process $\tilde{W}(t) := W(t) - \int_0^t h(s)ds$ for $t \in [0,T]$ is a Wiener process with covariance operator Q on $(\Omega, \mathcal{F}, \eta)$ where η is the probability measure given by

$$\frac{d\eta}{d\mu} = \exp\big\{\int_0^T h(s)dW(s) - \frac{1}{2}\int_0^T |h(s)|_0^2 ds\big\}$$

A variational representation of positive functionals of a Brownian motion is stated below (W.L.O.G, we take $T = 1$).

Theorem 6.1.6. ***(Variational Representation)*** *Let f be a bounded, Borel-measurable function mapping $C([0,1]:H)$ into $\mathbb{R}$. Then*

$$-\log \mathbb{E}e^{-f(W)} = \inf_{v\in\mathcal{O}} \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v(s)|_0^2 ds + f(W + \int_0^\cdot v(s)ds)\right\} \tag{6.1.4}$$

proof of the upper bound: We first prove the upper bound which refers to replacing the equality sign in (6.1.4) by $\leq$ sign.

Consider any v in $\mathcal{O}_b$. Since v is bounded, the stochastic integral $\int_0^t v_s dW_s$ is well defined and is a square integrable martingale. If we define R_t by

$$R_t = \exp\left[\int_0^t v_s dW_s - \frac{1}{2}\int_0^t |v_s|_0^2 ds\right] \tag{6.1.5}$$

then R_t is a martingale. We defined a probability measure η_v on $\mathcal{F}_1$ by

$$\eta_v(A) = \int_A R_1 d\mu \quad \text{for } A \in \mathcal{F}_1. \tag{6.1.6}$$

By the Girsanov theorem, the process $\tilde{W}_t = W_t - \int_0^t v_s ds$ is a Brownian motion under η_v. Let T_v be the operator defined on $C([0,1];H)$ by

$$T_v(\phi)_t = \phi_t - \int_0^t v_s(\phi)ds \tag{6.1.7}$$

Then for any Borel set $A \subset C([0,1]:H), \mu(A) = \eta_v(T_v^{-1}(A))$.

Using the definition of $R(\eta_v||\mu)$ and substituting (6.1.5) and (6.1.6), we obtain

$$\begin{aligned} R(\eta_v||\mu) &= \int \left(\log \frac{d\eta_v}{d\mu}\right) d\eta_v \\ &= \int \left\{ \int_0^1 v_s(\phi)dW_s - \frac{1}{2}\int_0^1 |v_s(\phi)|_0^2 ds \right\} \eta_v(d\phi) \end{aligned} \tag{6.1.8}$$

Since $W_t = \tilde{W}_t + \int_0^t v_s ds$, we have

$$\begin{aligned} R(\eta_v||\mu) &= \mathbb{E}^v \left\{ \int_0^1 v_s d\tilde{W}_s + \int_0^1 |v_s|_0^2 ds - \frac{1}{2}\int_0^1 |v_s|_0^2 ds \right\} \\ &= \mathbb{E}^v \left\{ \frac{1}{2}\int_0^1 |v_s|_0^2 ds \right\} \end{aligned} \tag{6.1.9}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability measure η_v and we have used the martingale property of the stochastic integral in the last equality. Thus,

$$R(\eta_v||\mu) + \int f(\phi)\eta_v(d\phi) = \mathbb{E}^v \left\{ \frac{1}{2}\int_0^1 |v_s|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s ds\right) \right\} \tag{6.1.10}$$

and from [6] Proposition 2.4.2, we obtain

$$-\log \mathbb{E}e^{-f(W)} \leq \inf_{v\in\mathcal{O}_b} \mathbb{E}^v \left\{ \frac{1}{2}\int_0^1 |v_s|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s ds\right) \right\} \tag{6.1.11}$$

Now we use (6.1.11) to show that, for any $v \in \mathcal{O}$,

$$-\log \mathbb{E}e^{-f(W)} \leq \mathbb{E} \left\{ \frac{1}{2}\int_0^1 |v_s|_0^2 ds + f\left(W + \int_0^{\cdot} v_s ds\right) \right\} \tag{6.1.12}$$

where expectation is with respect to Wiener measure μ.

Step 1 : Suppose that v is in $\mathcal{O}_s$. Then $\tilde{v}$ can be recursively constructed such that $\tilde{v} \in \mathcal{O}_s$ and for $\phi \in C([0,1]:H)$, $\tilde{v}(\phi) = v(T_{\tilde{v}}(\phi))$ with probability 1.

This implies that, for $\tilde{W}(\phi) = W(\phi) - \int_0^\cdot \tilde{v}_s(\phi)ds$ and $A \in \mathcal{B}(C([0,1];H))$, $B \in \mathcal{B}(L^2([0,1];H))$

$$\begin{aligned}\eta_{\tilde{v}}(\tilde{W} \in A, \tilde{v} \in B) &= \eta_{\tilde{v}}\left(\left\{\phi : \phi - \int_0^\cdot \tilde{v}_s(\phi)ds \in A, \tilde{v}(\phi) \in B\right\}\right) \\ &= \eta_{\tilde{v}}(\{\phi : T_{\tilde{v}}(\phi) \in A, v(T_{\tilde{v}}(\phi)) \in B\}) \\ &= \mu(\{\psi : \psi \in A, v(\psi) \in B\}) \\ &= \mu(W \in A, v \in B) \qquad (6.1.13)\end{aligned}$$

which shows that the distribution of $(\tilde{W}, \tilde{v})$ under the measure $\eta_{\tilde{v}}$ is the same as the the distribution of (W, v) under μ. Using this equivalence and (6.1.11), we obtain

$$\begin{aligned}-\log \mathbb{E}^{-f(W)} &\leq \mathbb{E}^{\tilde{v}}\left\{\frac{1}{2}\int_0^1 |\tilde{v}_s|_0^2 ds + f\left(\tilde{W} + \int_0^\cdot \tilde{v}_s ds\right)\right\} \\ &= \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|_0^2 ds + f\left(W + \int_0^\cdot v_s ds\right)\right\} \qquad (6.1.14)\end{aligned}$$

which implies (6.1.12) for all $v \in \mathcal{O}_s$.

Let $L_\mu(W + \int_0^\cdot v_s ds)$ denote the measure on $C([0,1] : H)$ that is induced by $W + \int_0^\cdot v_s ds$ under μ. For $A \in \mathcal{B}(C([0,1];H))$, we have

$$\begin{aligned}\mu(W(\phi) + \int_0^\cdot v_s(\phi)ds \in A) &= \eta_{\tilde{v}}(\tilde{W}(\phi) + \int_0^\cdot \tilde{v}_s(\phi)ds \in A) \\ &= \eta_{\tilde{v}}(W(\phi) \in A) \qquad (6.1.15)\end{aligned}$$

which implies that $L_\mu(W + \int_0^\cdot v_s ds) = \eta_{\tilde{v}}$.

Using (6.1.9) and taking $f = 0$ in the equality of (6.1.14), we have for all $v \in \mathcal{O}_s$,

$$\begin{aligned}R\left(L_\mu(W + \int_0^\cdot v_s ds)||\mu\right) &= R(\eta_{\tilde{v}}||\mu) \\ &= \mathbb{E}^{\tilde{v}}\left\{\frac{1}{2}\int_0^1 |\tilde{v}_s|_0^2 ds\right\} \\ &= \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|_0^2 ds\right\} \qquad (6.1.16)\end{aligned}$$

Step 2 : Bounded v. Let $v \in \mathcal{O}_b$, so that $|v_s(\omega)|_0 \leq M < \infty$ for $0 \leq s \leq 1$, $\omega \in \Omega$. According to [14] Lemma 3.2.4, there exists a sequence of simple processes

$\{v^n, n \in \mathbb{N}\}$ such that $|v_s^n(\omega)|_0 \le M < \infty$ for all $0 \le s \le 1$, $\omega \in \Omega$, and

$$\lim_{n\to\infty} \mathbb{E} \int_0^1 |v_s^n - v_s|_0^2 ds = 0. \tag{6.1.17}$$

Thus $(W, \int_0^\cdot v_s^n ds)$ converges in distribution to $(W, \int_0^\cdot v_s ds)$ in $(C([0,1]; H))^2$.

By virtue of Step 1, for each $n \in \mathbb{N}$,

$$-\log \mathbb{E} e^{-f(W)} \le \mathbb{E} \left\{ \frac{1}{2} \int_0^1 |v_s^n|_0^2 ds + f\left(W + \int_0^\cdot v_s^n ds \right) \right\} \tag{6.1.18}$$

It remains to show that the inequality above continues to hold in the limit as $n \to \infty$. Let $\mu_n = L_\mu(W + \int_0^\cdot v_s^n ds)$, the (6.1.16) implies that

$$\sup_{n\in\mathbb{N}} R(\mu_n || \mu) = \sup_{n\in\mathbb{N}} \mathbb{E} \left\{ \frac{1}{2} \int_0^1 |v_s^n|_0^2 ds \right\} \le \frac{M^2}{2} < \infty \tag{6.1.19}$$

Hence we can apply Proposition 6.1.4 to obtain

$$\lim_{n\to\infty} \mathbb{E} f \left(W + \int_0^\cdot v_s^n ds \right) = \mathbb{E} f \left(W + \int_0^\cdot v_s ds \right) \tag{6.1.20}$$

Letting $n \to \infty$ in (6.1.18) we conclude that (6.1.12) is still valid for the limit process v, and thus for any $v \in \mathcal{O}_b$.

Step 3: General $v \in \mathcal{O}$. We define

$$v_s^n(\phi) = v_s(\phi) 1_{\{|v_s(\phi)|_0 \le n\}}, \quad 0 \le s \le 1, \quad \phi \in C([0,1]; H) \tag{6.1.21}$$

Then v^n is bounded for every $n \in \mathbb{N}$ and thus Step 2 guarantees that (6.1.18) holds for each v^n. Let $\mu_n = L_\mu(W + \int_0^\cdot v_s^n ds)$, then (6.1.16) implies that

$$\sup_{n\in\mathbb{N}} R(\mu_n || \mu) = \sup_{n\in\mathbb{N}} \mathbb{E} \left\{ \frac{1}{2} \int_0^1 |v_s^n|_0^2 ds \right\} \le \mathbb{E} \left\{ \frac{1}{2} \int_0^1 |v_s|_0^2 ds \right\} < \infty \tag{6.1.22}$$

As in Step 2, Proposition 6.1.4 and dominated convergence theorem yield (6.1.12) for any $v \in \mathcal{O}$, which implies the desired upper bound.

proof of the lower bound: Next, we will give a proof of the lower bound in the variational representation formula. That is,

$$-\log \mathbb{E} e^{-f(W)} \ge \inf_{v\in\mathcal{O}} \mathbb{E} \left\{ \frac{1}{2} \int_0^1 |v_s|_0^2 ds + f(W + \int_0^\cdot v_s ds) \right\}.$$

STEP 1: Let $(\Omega, \mathcal{F})$ be the Wiener space and f be a bounded measurable function mapping Ω into $\mathbb{R}$. Let μ be the Wiener measure on Ω and $\Pi(\Omega)$ be the set of probabilities on Ω. Consider the measure η_0 where infimum is attained in the variational formula

$$-\log \int_\Omega e^{-f(x)} d\mu = \inf_{\eta \in \Pi(\Omega)} \{R(\eta || \mu) + \int_\Omega f(x) d\eta\} \tag{6.1.23}$$

Then η_0 is not only absolutely continuous with respect to μ, but it is in fact equivalent to μ on $\mathcal{F}$. It follows that, for each $t \in [0,1]$, the restriction of η_0 to $\mathcal{F}_t$ is a probability measure which is equivalent to the restriction of μ to $\mathcal{F}_t$. Let R_t be the corresponding Radon-Nikodym derivative

$$R_t = \mathbb{E}\left[\frac{d\eta_0}{d\mu} | \mathcal{F}_t\right] = \mathbb{E}\left[\frac{e^{-f(x)}}{\int_\Omega e^{-f(x)} \mu(dx)} | \mathcal{F}_t\right] \tag{6.1.24}$$

Then $\{R_t; 0 \leq t \leq 1\}$ forms a μ-martingale that is bounded from below and above μ-a.s. respectively by constants $\exp(-2||f||_\infty)$ and $\exp(2||f||_\infty)$. Moreover, since R_t is a martingale with respect to the augmentation under μ of the filtration generated by a Brownian motion, it can be represented as a stochastic integral $R_t = 1 + \int_0^t u_s dW_s$, where u_s is progressively measurable.

Since R_t is bounded from below, we can define $v_t = u_t / R_t$ and write

$$R_t = 1 + \int_0^t v_s R_s dW_s \tag{6.1.25}$$

The random variable R_1 is bounded by a constant, and hence $\mathbb{E}(R_1^2) < \infty$. This observation and Eq. (6.1.25) yield $\mathbb{E} \int_0^1 |v_s|_0^2 R_s^2 ds < \infty$. Since R_t is bounded below by a constant, we have $\mathbb{E} \int_0^1 |v_s|_0^2 ds < \infty$. Also, $d\eta_0 / d\mu$ is bounded so that one obtains

$$\int_{C([0,1]:H)} \int_0^1 |v_s|_0^2 ds d\eta_0 < \infty. \tag{6.1.26}$$

These bounds and Eq. (6.1.25) allow us to write

$$R_t = \exp\left[\int_0^t v_s dW_s - \frac{1}{2} \int_0^t |v_s|_0^2 ds\right] \tag{6.1.27}$$

Since R_t is a martingale, the Girsanov theorem identifies η_0 as the measure under which the process $\tilde{W} := W - \int_0^\cdot v_s ds$ is a Brownian motion. Analogous to the derivation of Eq. (6.1.10) as in the proof of the upper bound to evaluate $R(\eta_0||\mu)$, we obtain

$$-\log \mathbb{E}e^{-f(W)} = \mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |v_s|_0^2 ds + f(\tilde{W} + \int_0^\cdot v_s)\right\} \tag{6.1.28}$$

Step 2: Let us first assume that f is continuous. Since progressively measurable processes can be approximated by bounded, simple processes in the L^2-sense, given $\varepsilon > 0$, there exists a process v^* be a bounded, simple process such that

$$\mathbb{E}^{\eta_0}\left\{\int_0^1 |v_s^* - v_s|_0^2 ds\right\} < \frac{\varepsilon}{2} \tag{6.1.29}$$

Let us write the process v^* in the form

$$v_t^*(\omega) = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{l-1}\xi_i(\omega)1_{(t_i,t_{i+1}]}(t), \qquad 0 \le t \le 1, \ \ \omega \in \Omega \tag{6.1.30}$$

where $0 = t_0 < t_1 < \cdots < t_l = 1$ and ξ_i is $\mathcal{F}_{t_i}$-measurable for each $i = 0, \cdots, l-1$. Each ξ_i can be approximated in $L^2(\mu)$ (and hence equivalently in $L^2(\eta_0)$ as well) by a smooth cylindrical functional with compact support, namely, $g_i(\omega_{s_1}, \cdots, \omega_{s_n})$, where $s_1 < s_2 < \cdots s_n \le t_i$ (see Nualart [30], Page 24). Replacing each ξ_i by g_i, and then using polygonalization in the time variable s, we can find a smooth progressively measurable functional z with continuous sample paths which approximates v^* in the sense that

$$\mathbb{E}^{\eta_0}\left\{\int_0^1 |z_s - v_s^*|_0^2 ds\right\} < \frac{\varepsilon}{2}$$

It follows that given $\epsilon > 0$, we can choose a progressively measurable process z as constructed above such that

$$\begin{aligned}&\mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |v_s|_0^2 ds + f(\tilde{W} + \int_0^\cdot v_s ds)\right\}\\ &\ge \mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |z_s|_0^2 ds + f(\tilde{W} + \int_0^\cdot z_s ds)\right\} - \varepsilon\end{aligned} \tag{6.1.31}$$

Consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \eta_0)$, under the measure η_0, $\tilde{W}(\omega) = \omega - \int_0^{\cdot} v_s(\omega)ds$ is a Brownian motion. Denote $X(\omega) := \tilde{W}(\omega) + \int_0^{\cdot} v_s(\omega)ds$, and note that $X(\omega) = \omega$. The process X_t solves

$$X_t = \tilde{W}_t + \int_0^t v_s(X)ds. \tag{6.1.32}$$

In general, we can only assert that Eq. (6.1.32) has a weak solution which is unique in law.

Define a probability measure η_1 on $(\Omega, \mathcal{F})$ by

$$\frac{d\eta_1}{d\eta_0} = \exp\{\int_0^1 (z_s - v_s)d\tilde{W}_s - \frac{1}{2}\int_0^1 |z_s - v_s|_0^2 ds\}.$$

Then $\eta_1 \equiv \eta_0$, and η_1 a.s., we can write, for all $0 \leq t \leq 1$,

$$\tilde{W}_t = \hat{W}_t + \int_0^t (z_s - v_s)ds$$

where $\hat{W}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \eta_1)$. Thus η_1 a.s., we have

$$W_t = \hat{W}_t + \int_0^t z_s ds. \tag{6.1.33}$$

We can rewrite Eq. (6.1.33) as

$$Y_t = \hat{W}_t + \int_0^t z_s(Y)ds \tag{6.1.34}$$

Eq. (6.1.34) has a strong pathwise unique solution by the choice of z. Therefore $Y = h(\hat{W})$ for some Borel measurable function h.

Note that η_1 depends on ϵ. Taking $\epsilon = 1/n$, let us denote the corresponding sequence of probability measures by $\eta^{(n)}$. Then $\eta^{(n)}$ is the law of the solution of the equation

$$Y_t^{(n)} = \hat{W}_t^{(n)} + \int_0^t z_s^{(n)}(Y^{(n)})ds$$

where $\hat{W}^{(n)}$ is a Wiener process with respect to $\eta^{(n)}$. Also, $\eta^{(n)} \to \eta_0$ weakly as $n \to \infty$. Thus, for any fixed constant $K > 0$ and any given $\epsilon > 0$, there exists an

n such that

$$\mathbb{E}^{\eta_0}\left\{\frac{1}{2}[K \wedge \int_0^1 |v_s(X)|_0^2 ds] + f(X)\right\}$$
$$\geq \mathbb{E}^{\eta^{(n)}}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(Y_n)|_0^2 ds] + f(Y)\right\} - \varepsilon \tag{6.1.35}$$

Recalling that that $Y_n = h_n(\hat{W}^{(n)})$, let us introduce the following notation:

$$\mathcal{L}_n(\cdot) := \frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(h_n(\cdot))|_0^2 ds] + f(h_n(\cdot))$$

Since $\hat{W}^{(n)}$ is a $\eta^{(n)}$-Brownian motion and W is a μ-Brownian motion, we have $\mathbb{E}^{\eta^{(n)}}(\mathcal{L}_n(\hat{W}^{(n)})) = E(\mathcal{L}_n(W))$. Then

$$\begin{aligned}
& \mathbb{E}^{\eta_0}\{\frac{1}{2}\int_0^1 |v_s(X)|_0^2 ds + f(X)\} \\
\geq\ & \mathbb{E}^{\eta^{(n)}}(\mathcal{L}(\hat{W}^{(n)})) - \varepsilon \\
=\ & \mathbb{E}(\mathcal{L}_n(W)) - \varepsilon \\
=\ & \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(h_n(W))|_0^2 ds] + f(h_n(W))\right\} - \varepsilon \\
=\ & \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^*(W)|_0^2 ds] + f(h_n(W))\right\} - \varepsilon \\
=\ & \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^*|_0^2 ds] + f(W + \int_0^\cdot z_s^* ds)\right\} - \varepsilon
\end{aligned} \tag{6.1.36}$$

where $z_s^* := z_s^{(n)} \circ h_n$ is progressively measurable. Now allow $K \to \infty$ using monotone convergence. Recalling equation (6.1.28), the inequality (6.1.36) yields the lower bound for continuous f.

Step 3: If f is not continuous, let $\{f_j\}$ be a sequence of bounded continuous functions such that $||f_j||_\infty \leq ||f||_\infty < \infty$ and $\lim_{j\to\infty} f_j = f$, μ-a.s. The proceeding argument applied to each of the functions f_j implies that there exists a sequence of progressively measurable processes $\{z^{j*}, j \in \mathbb{N}\}$ satisfies (6.1.36) for each j but with f replaced by f_j, that is

$$-\log \mathbb{E}e^{-f_j(W)} \geq \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}(W))|_0^2 ds + f_j(W + \int_0^\cdot z_s^{j*} ds)\right\} - \varepsilon. \tag{6.1.37}$$

Thanks to (6.1.16), we have

$$\sup_j R\left(L_\mu(W+\int_0^\cdot z_s^{j*}ds)||\mu\right)=\sup_j \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}|_0^2 ds\right\}\leq ||f||_\infty \quad (6.1.38)$$

It follows from this bound that the pair $(\int_0^\cdot z_s^{j*}, W)$ is tight,and hence there exists a subsequence such that $(\int_0^\cdot z_s^{j*}, W)$ converges in distribution to $(\int_0^\cdot z_s^*, W)$. It follows from (6.1.37), the dominated convergence theorem and Proposition 6.1.4 that, for all sufficiently large j,

$$-\log \mathbb{E}e^{-f(W)}\geq \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}(W))|_0^2 ds+f(W+\int_0^\cdot z_s^{j*}ds)\right\}-2\varepsilon. \quad (6.1.39)$$

this completes the proof of the lower bound.

6.2 Larger Deviation Principle

Let $(\Omega,\mathcal{F},P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and P-completeness. For example, we can take the Wiener space $(\Omega,\mathcal{F},\mu)$ as in the first section.

In what follows, the notation and terminology are built in order to state the large deviations result of Budhiraja and Dupuis [2] for Polish space valued random elements.

Let us denote

$$S_N=\left\{v\in L^2(0,T;H_0):\int_0^T |v(s)|_0^2 ds\leq N\right\} \quad (6.2.1)$$

Then S_N endowed with the weak topology on $L^2(0,T;H_0)$ is a Polish space [8].

Define $\mathcal{O}_N$ as the set of bounded stochastic controls by

$$\mathcal{O}_N=\{\phi\in\mathcal{O}:\phi(\omega)\in S_N, P\text{-a.s.}\} \quad (6.2.2)$$

Let E denote a Polish space, and for $\varepsilon>0$ let $g^\varepsilon: C([0,T];H)\to E$ be a measurable map. Define

$$X^\varepsilon=g^\varepsilon(W(\cdot)) \quad (6.2.3)$$

We are interested in the large deviation principle for X^ε as $\varepsilon \to 0$. Since $\{X^\epsilon\}$ are Polish space valued random elements, the Laplace-Vradhan principle (Definition 2.1.3) and the large deviation principle are equivalent.

Hypothesis 6.2.1. *There exists a measurable map* $g^0 : C([0,T];H) \to E$ *such that the following hold:*

(1). Let $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{O}_M$ *for some* $M < \infty$. *Let* v^ε *converge in distribution as* S_M*-valued random elements to* v. *Then* $g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s)ds)$ *converges in distribution to* $g^0(\int_0^\cdot v(s)ds)$.

(2). For every $M < \infty$, *the set* $K_M = \{g^0(\int_0^\cdot v(s)ds) : v \in S_M\}$ *is a compact subset of* E.

For each $f \in E$, define

$$I(f) = \inf_{\{v \in L^2(0,T;H_0) : f = g^0(\int_0^\cdot v(s)ds)\}} \left\{ \frac{1}{2} \int_0^T |v(s)|_0^2 ds \right\} \tag{6.2.4}$$

where infimum over an empty set is taken as ∞.

The following theorem was proven by Budhiraja and Dupuis [3]. The variational representation (Theorem 6.1.6) allows one to prove the sufficiency of Hypotheses 6.2.1 to establish the Laplace-Varadhan principle.

Theorem 6.2.2. *Let* $X^\varepsilon = g^\varepsilon(W(\cdot))$. *If* g^ε *satisfies (H.1)-(H.2) in the Hypothesis 6.2.1, then the family* $\{X^\varepsilon : \varepsilon > 0\}$ *satisfies the Laplace-Varadhan principle in* E *with rate function* I *given by Eq.* (6.2.4).

6.3 Laplace-Varadhan Principle for SNSE's

Let us recall the Navier-Stokes equations with small noise diffusions

$$\begin{aligned} &d\mathbf{u}^\varepsilon + [\nu_0 \mathbf{A}\mathbf{u}^\varepsilon + \nu_1 \mathcal{A}(\mathbf{u}^\varepsilon) + \mathbf{B}(\mathbf{u}^\varepsilon)]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}^\varepsilon)dW(t) \\ &\mathbf{u}^\varepsilon(0) = \xi \in H \end{aligned} \tag{6.3.1}$$

Theorem 5.2.2 shows that there exists a strong solution of Eq. (6.3.1) with values in the Polish space $C([0,T];H) \cap L^q(0,T;V_{r,q})$, and it is pathwise unique.

It follows that (see [3]) there exists a Borel-measurable function

$$g^\varepsilon : C([0,T];H) \to C([0,T];H) \cap L^q(0,T;V_{r,q})$$

such that $\mathbf{u}^\varepsilon(\cdot) = g^\varepsilon(W(\cdot))$ a.s.

Our aim is to verify that the family $\{g^\varepsilon\}$ satisfies Hypothesis 6.2.1 so that Theorem 6.2.2 can be invoked to prove the LVP for $\{\mathbf{u}^\varepsilon\}$ in $C([0,T];H) \cap L^q(0,T;V_{r,q})$.

In the following Lemma and its corollary, we show certain results which help to prove the last two main propositions on the compactness of the level sets and weak convergence as stated in the Hypothesis 6.2.1.

Lemma 6.3.1. *Let $\{g^\varepsilon\}$ be defined as above. For any $\mathbf{v} \in \mathcal{O}_M$ where $0 < M < \infty$, let $g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot \mathbf{v}(s)ds)$ be denoted by $\mathbf{u}^\varepsilon_\mathbf{v}$. Then $\mathbf{u}^\varepsilon_\mathbf{v}$ is the unique strong solution of the stochastic control equation*

$$\begin{aligned} &d\mathbf{u}^\varepsilon_\mathbf{v} + [\nu_0 \mathbf{A}\mathbf{u}^\varepsilon_\mathbf{v} + \nu_1 \mathcal{A}(\mathbf{u}^\varepsilon_\mathbf{v}) + \mathbf{B}(\mathbf{u}^\varepsilon_\mathbf{v})]dt = [\mathbf{f}(t) + \sigma(t, \mathbf{u}^\varepsilon_\mathbf{v})\mathbf{v}]dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}^\varepsilon_\mathbf{v})dW_t \\ &\mathbf{u}^\varepsilon_\mathbf{v}(0) = \xi \in H \end{aligned} \tag{6.3.2}$$

Proof. Since $\mathbf{v} \in \mathcal{O}_M$, we have $\int_0^T |\mathbf{v}(s)|_0^2 ds < M$ a.s., and

$$\tilde{W}(\cdot) := W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \mathbf{v}(s)ds \tag{6.3.3}$$

is a Wiener process with covariance form Q under the probability measure

$$d\tilde{P}^\varepsilon_\mathbf{v} := \exp\left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^T \mathbf{v}(s)dW(s) - \frac{1}{2\varepsilon}\int_0^T |\mathbf{v}(s)|_0^2 ds \right\} dP \tag{6.3.4}$$

Applying the Girsanov argument: let $\mathbf{u}^\varepsilon_\mathbf{v}$ be the unique solution of Eq. (6.3.1) on $(\Omega, \mathcal{F}, \tilde{P}^\varepsilon_\mathbf{v})$ with $\tilde{W}$ in place of W. Then $\mathbf{u}^\varepsilon_\mathbf{v}$ solves Eq. (6.3.2) P-a.s., and $\mathbf{u}^\varepsilon_\mathbf{v}(\cdot) = g^\varepsilon(\tilde{W}(\cdot))$.

If $\mathbf{u}^\varepsilon_\mathbf{v}$ and $\mathbf{w}$ are solutions of Eq. (6.3.2) on $(\Omega, \mathcal{F}, P)$, then $\mathbf{u}^\varepsilon_\mathbf{v}$ and $\mathbf{w}$ would solve Eq. (6.3.1) on $(\Omega, \mathcal{F}, \tilde{P}^\varepsilon_\mathbf{v})$ with $\tilde{W}$ in place of W. Thus $\mathbf{u}^\varepsilon_\mathbf{v} = \mathbf{w}$ $\tilde{P}^\varepsilon_\mathbf{v}$-a.s. so that $\mathbf{u}^\varepsilon_\mathbf{v} = \mathbf{w}$ P-a.s., and hence uniqueness of solutions to Eq. (6.3.2) is obtained. □

Corollary 6.3.2. *Let $\mathbf{v} \in L^2(0,T;H_0)$ and $\mathbf{f} \in L^4(0,T;H)$ and σ satisfies (A.1)-(A.3). Then the equation*

$$\begin{aligned} &d\mathbf{u}_\mathbf{v} + [\nu_0 \mathbf{A}\mathbf{u}_\mathbf{v} + \nu_1 \mathcal{A}(\mathbf{u}_\mathbf{v}) + \mathbf{B}(\mathbf{u}_\mathbf{v})]dt = [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\mathbf{v})\mathbf{v}]dt \\ &\mathbf{u}_\mathbf{v}(0) = \xi \in H \end{aligned} \tag{6.3.5}$$

has a unique strong solution in $C([0,T];H) \cap L^q(0,T;V_{r,q})$.

Proof. This result can be considered as a particular case of lemma 6.3.1, where the diffusion coefficient is absent. □

Next we state another lemma from Budhiraja and Dupuis (Lemma 3.2, [3]), which will be used in the proof of the compactness proposition.

Lemma 6.3.3. *Let $\{\mathbf{v}_n\}$ be a sequence of elements from $\mathcal{O}_M$ for some $M > 0$. Let $\mathbf{v}_n \to \mathbf{v}$ in distribution as S_M-valued random elements. Then $\int_0^\cdot \mathbf{v}_n(s)ds$ converges in distribution as $C([0,T];H)$-valued processes to $\int_0^\cdot \mathbf{v}(s)ds$ as $n \to \infty$.*

Now we are ready to verify the Hypothesis 6.2.1.

Proposition 6.3.4. *(**Compactness**) For any fixed positive number $M < \infty$, let*

$$K_M := \{\mathbf{u}_\mathbf{v} \in C([0,T];H) \cap L^q(0,T;V_{r,q}) : \mathbf{v} \in S_M\} \tag{6.3.6}$$

where $\mathbf{u}_\mathbf{v}$ is the unique solution in $C([0,T];H) \cap L^q(0,T;V_{r,q})$ of the equation

$$\begin{aligned} &d\mathbf{u}_\mathbf{v} + [\nu_0 \mathbf{A}\mathbf{u}_\mathbf{v} + \nu_1 \mathcal{A}(\mathbf{u}_\mathbf{v}) + \mathbf{B}(\mathbf{u}_\mathbf{v})]dt = [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\mathbf{v})\mathbf{v}]dt \\ &\mathbf{u}_\mathbf{v}(0) = \xi \in H \end{aligned} \tag{6.3.7}$$

Then K_M is compact in $C([0,T];H) \cap L^q(0,T;V_{r,q})$.

Proof. Let us consider a sequence $\{\mathbf{u}_n\}$ in K_M where $\mathbf{u}_n$ corresponds to the solution of (6.3.7) with $\mathbf{v}_n$ in place of $\mathbf{v}$. By the weak compactness of S_M, there exists a subsequence of $\{\mathbf{v}_n\}$ which converges to a limit $\mathbf{v}$ weakly in $L^2(0,T;H_0)$. The subsequence is indexed by n for ease of notation.

Denote $\mathbf{u}_\mathbf{v}$ as $\mathbf{u}$, i.e., $\mathbf{u}$ is the solution of the equation

$$d\mathbf{u} + [\nu_0 \mathbf{A}\mathbf{u} + \nu_1 \mathcal{A}(\mathbf{u}) + \mathbf{B}(\mathbf{u})]dt = [\mathbf{f}(t) + \sigma(t,\mathbf{u})\mathbf{v}]dt \tag{6.3.8}$$

We need to show $\mathbf{u}_n \to \mathbf{u}$ in $C([0,T];H) \cap L^q(0,T;V_{r,q})$ as $n \to \infty$, i.e.,

$$\sup_{0\le t\le T} |\mathbf{u}_n(t) - \mathbf{u}(t)| + \int_0^T ||\mathbf{u}_n(t) - \mathbf{u}(t)||_{V_{r,q}}^q dt \to 0 \tag{6.3.9}$$

By the energy equality (Proposition 4.2.1),

$$\begin{aligned} &|\mathbf{u}(t)|^2 + 2\nu_0 \int_0^t ||\mathbf{u}(s)||^2 ds + 2\nu_1 \int_0^t \langle \mathcal{A}(\mathbf{u}(s)), \mathbf{u}(s)\rangle ds \\ &= |\xi|^2 + 2\int_0^t \{(\mathbf{f}(s), \mathbf{u}(s)) + (\sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{u}(s))\} ds \end{aligned} \tag{6.3.10}$$

From the estimates (3.2.5) and (3.2.9),

$$\begin{aligned} &|\mathbf{u}(t)|^2 + 2\nu_0 \int_0^t ||\mathbf{u}(s)||^2 ds + 2\nu_1 \int_0^t ||\mathbf{u}(s)||_{V_{r,q}}^q ds \\ &\le |\xi|^2 + 2\int_0^t \{(\mathbf{f}(s), \mathbf{u}(s)) + (\sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{u}(s))\} ds \end{aligned} \tag{6.3.11}$$

Dropping the second term on the left hand side,

$$
\begin{aligned}
&|\mathbf{u}(t)|^2 + 2\nu_1 \int_0^t ||\mathbf{u}(s)||_{V_{r,q}}^q ds \\
&\leq |\xi|^2 + 2\int_0^t (|\mathbf{f}(s)|\,|\mathbf{u}(s)| + |\sigma(s,\mathbf{u}(s))\mathbf{v}(s)|\,|\mathbf{u}(s)|)ds \\
&\leq |\xi|^2 + \int_0^t |\mathbf{f}(s)|^2 ds + \int_0^t |\mathbf{u}(s)|^2 ds \\
&\quad + 2\int_0^t |\sigma(\mathbf{u}(s))|_{L_Q} |\mathbf{v}(s)|_0 |\mathbf{u}(s)| ds \\
&\leq |\xi|^2 + \int_0^t |\mathbf{f}(s)|^2 ds + \int_0^t |\mathbf{u}(s)|^2 ds \\
&\quad + K\int_0^t (1 + |\mathbf{u}(t)|^2) ds + \int_0^t |\mathbf{v}(s)|_0^2 |\mathbf{u}(s)|^2 ds \\
&= |\xi|^2 + \int_0^t |\mathbf{f}(s)|^2 ds + Kt + \int_0^t (K + 1 + |\mathbf{v}(s)|_0^2)|\mathbf{u}(s)|^2 ds \quad (6.3.12)
\end{aligned}
$$

we have used the assumption (A.2) in the second last step above.

By the Gronwall inequality, for any $T > 0$,

$$\sup_{0\leq t\leq T} |\mathbf{u}(t)|^2 \leq C\left(|\xi|^2, \int_0^T |\mathbf{f}|^2 ds, T, K, M\right) \quad (6.3.13)$$

Using the above bound in (6.3.12), it follows that

$$\int_0^T ||\mathbf{u}(s)||_{V_{r,q}}^q ds \leq C\left(|\xi|^2, \int_0^T |\mathbf{f}|^2 ds, T, K, M\right) \quad (6.3.14)$$

so the bound is uniform in n.

Let $\mathbf{w}_n = \mathbf{u}_n - \mathbf{u}$. Then $\mathbf{w}_n$ satisfies the following differential equation

$$
\begin{aligned}
&d\mathbf{w}_n(t) + [\nu_0 \mathbf{A}\mathbf{w}_n(t) + \nu_1(\mathcal{A}(\mathbf{u}_n(t)) - \mathcal{A}(\mathbf{u}(t))) + \mathbf{B}(\mathbf{u}_n(t)) - \mathbf{B}(\mathbf{u}(t))]dt \\
&= [\sigma(t,\mathbf{u}_n(t))\mathbf{v}_n(t) - \sigma(t,\mathbf{u}(t))\mathbf{v}(t)]dt \quad (6.3.15)
\end{aligned}
$$

which yields

$$
\begin{aligned}
&|\mathbf{w}_n|^2 + 2\nu_0 \int_0^t ||\mathbf{w}_n(s)||^2 ds + 2\nu_1 \int_0^t \langle \mathcal{A}(\mathbf{u}_n(s)) - \mathcal{A}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle ds \\
&+ 2\int_0^t \langle \mathbf{B}(\mathbf{u}_n(s)) - \mathbf{B}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle ds \\
&= 2\int_0^t (\sigma(s,\mathbf{u}_n(s))\mathbf{v}_n(s) - \sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{w}_n(s)) ds \quad (6.3.16)
\end{aligned}
$$

Then

$$
\begin{aligned}
&|\mathbf{w}_n(t)|^2 + \frac{1}{2}\nu_0 \int_0^t ||\mathbf{w}_n(s)||^2 ds + 2\nu_1 \int_0^t \langle \mathcal{A}(\mathbf{u}_n(s)) - \mathcal{A}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle ds \\
&+2\int_0^t \langle \mathbf{B}(\mathbf{u}_n(s)) - \mathbf{B}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle ds \\
&\leq 2\int_0^t (\sigma(s,\mathbf{u}_n(s))\mathbf{v}_n(s) - \sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{w}_n(s)) ds \qquad (6.3.17)
\end{aligned}
$$

By the lemma 3.2.2, we know that

$$
\langle \mathcal{A}(\mathbf{u}_n(s)) - \mathcal{A}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle \geq C||\mathbf{w}_n(s)||^q_{V_{r,q}} \qquad (6.3.18)
$$

and then it follows that

$$
\begin{aligned}
&|\mathbf{w}_n(t)|^2 + \frac{1}{2}\nu_0 \int_0^t ||\mathbf{w}_n(s)||^2 ds + 2C\nu_1 \int_0^t ||\mathbf{w}_n(s)||^q_{V_{r,q}} ds \\
&+2\int_0^t \langle \mathbf{B}(\mathbf{u}_n(s)) - \mathbf{B}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle ds \\
&\leq 2\int_0^t (\sigma(s,\mathbf{u}_n(s))\mathbf{v}_n(s) - \sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{w}_n(s)) ds \qquad (6.3.19)
\end{aligned}
$$

Using the properties of the function b

$$
\begin{aligned}
b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{w}_n) - b(\mathbf{u}, \mathbf{u}, \mathbf{w}_n) &= b(\mathbf{w}_n, \mathbf{u}_n, \mathbf{w}_n) + b(\mathbf{u}, \mathbf{w}_n, \mathbf{w}_n) \\
&= b(\mathbf{w}_n, \mathbf{u}, \mathbf{w}_n) \qquad (6.3.20)
\end{aligned}
$$

We obtain that

$$
\begin{aligned}
2|\langle \mathbf{B}(\mathbf{u}_n(s)) - \mathbf{B}(\mathbf{u}(s)), \mathbf{w}_n(s)\rangle| &= 2|b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{w}_n) - b(\mathbf{u}, \mathbf{u}, \mathbf{w}_n)| \\
&\leq 2|\mathbf{w}_n| \; ||\mathbf{w}_n|| \; ||\mathbf{u}|| \\
&\leq \frac{\nu_0}{2}||\mathbf{w}_n||^2 + \frac{2}{\nu_0}|\mathbf{w}_n|^2||\mathbf{u}||^2 \qquad (6.3.21)
\end{aligned}
$$

Notice that by the assumption (A.3)

$$\begin{aligned}
&\left|\int_0^t (\sigma(s,\mathbf{u}_n(s))\mathbf{v}_n(s) - \sigma(s,\mathbf{u}(s))\mathbf{v}(s), \mathbf{w}_n(s))ds\right| \\
&\leq \int_0^t |(\sigma(s,\mathbf{u}_n(s))\mathbf{v}_n(s) - \sigma(s,\mathbf{u}(s))\mathbf{v}_n(s), \mathbf{w}_n(s))|ds \\
&+ \left|\int_0^t (\sigma(s,\mathbf{u}(s))(\mathbf{v}_n(s) - \mathbf{v}(s)), \mathbf{w}_n(s))ds\right| \\
&\leq \sqrt{L}\int_0^t |\mathbf{w}_n(s)|^2 |\mathbf{v}_n(s)|_0 ds \\
&+ \sup_{0\leq t\leq T}\left|\int_0^t (\sigma(s,\mathbf{u}(s))(\mathbf{v}_n(s) - \mathbf{v}(s)), \mathbf{w}_n(s))ds\right|
\end{aligned} \tag{6.3.22}$$

By the boundedness of $\{|\mathbf{w}_n(s)|^2\}$ in $C([0,T];H)$, and using the lemma 6.3.3, the second integral on the right hand side of the above ineuqality goes to 0 as $n\to\infty$. Therefore, given any $\varepsilon > 0$, there exists an integer N large so that for all $n \geq N$,

$$\sup_{0\leq t\leq T}\left|\int_0^t (\sigma(s,\mathbf{u}(s))(\mathbf{v}_n(s) - \mathbf{v}(s)), \mathbf{w}_n(s))ds\right| < \varepsilon/2 \tag{6.3.23}$$

Let us consider

$$C_{L,\nu_0} = \max\left\{\frac{2}{\nu_0}, 2\sqrt{L}\right\}$$

Applying (6.3.21), (6.3.22) and (6.3.23) in (6.3.19), we obtain that for $n \geq N$

$$\begin{aligned}
&|\mathbf{w}_n(t)|^2 + 2C\nu_1\int_0^t ||\mathbf{w}_n(s)||^q_{V_{r,q}} ds \\
&\leq C_{L,\nu_0}\int_0^t |\mathbf{w}_n(s)|^2(||\mathbf{u}(s)||^2 + |\mathbf{v}(s)|_0^2)ds + \varepsilon
\end{aligned} \tag{6.3.24}$$

Hence by the Gronwall inequality,

$$\begin{aligned}
&\sup_{0\leq t\leq T}|\mathbf{w}_n(t)|^2 + 2C\nu_1\int_0^T ||\mathbf{w}_n(t)||^q_{V_{r,q}} dt \\
&\leq \varepsilon\exp\{C_{L,\nu_0}\int_0^T (||\mathbf{u}(t)||^2 + |\mathbf{v}(t)|_0^2 + 1)dt\}
\end{aligned} \tag{6.3.25}$$

The arbitrariness of ε finishes the proof. □

Let $\mathbf{v}_\varepsilon$ converge to $\mathbf{v}$ in distribution as random elements taking values in S_M where S_M is equipped with the weak topology.

Let $\mathbf{u}_\varepsilon$ solve the stochastic control equation

$$\begin{aligned} &d\mathbf{u}_\varepsilon + [\nu_0 \mathbf{A}\mathbf{u}_\varepsilon + \nu_1 \mathcal{A}(\mathbf{u}_\varepsilon) + \mathbf{B}(\mathbf{u}_\varepsilon)]dt \\ &= [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\varepsilon(t))\mathbf{v}_\varepsilon(t)]dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}_\varepsilon)dW_t \end{aligned} \tag{6.3.26}$$

with $\mathbf{u}_\varepsilon(0) = \xi \in H$.

Let $\mathbf{u}_\mathbf{v}$ be the solution of

$$\begin{aligned} d\mathbf{u}_\mathbf{v} + [\nu_0 \mathbf{A}\mathbf{u}_\mathbf{v} + \nu_1 \mathcal{A}(\mathbf{u}_\mathbf{v}) + \mathbf{B}(\mathbf{u}_\mathbf{v})]dt \\ = [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\mathbf{v}(t))\mathbf{v}(t)]dt \end{aligned} \tag{6.3.27}$$

with $\mathbf{u}_\mathbf{v}(0) = \xi \in H$.

Since pathwise unique strong solutions exist for the above two equations (Lemma 6.3.1 and Corollary 6.3.2), we know that the Borel-measurable function g^ε mentioned earlier satisfies the equality

$$g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \mathbf{v}_\varepsilon(s)ds) = \mathbf{u}_\varepsilon \tag{6.3.28}$$

Note that for all $\mathbf{v} \in L^2(0, T; H_0)$, we have

$$\int_0^\cdot \mathbf{v}(s)ds \in C([0, T]; H_0) \tag{6.3.29}$$

Define $g^0 : C([0, T]; H_0) \to C([0, T]; H) \cap L^q(0, T; V_{r,q})$ by

$$g^0(h) = \mathbf{u}_\mathbf{v}, \text{ if } h = \int_0^\cdot \mathbf{v}(s)ds \text{ for some } \mathbf{v} \in L^2(0, T; H_0); \tag{6.3.30}$$

If h cannot be represented as above, then define $g^0(h) = 0$.

Now we prove the following weak convergence proposition which verifies the first part of the Hypothesis 6.2.1.

Proposition 6.3.5. *(**Weak Convergence**) Let $\{\mathbf{v}_\epsilon : \epsilon > 0\} \subset \mathcal{O}_M$, for some $M < \infty$, converge in distribution as S_M-valued random elements to $\mathbf{v}$. Then $g^\epsilon(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot \mathbf{v}_\epsilon(s)\, ds)$ converges in distribution to $g^0(\int_0^\cdot \mathbf{v}(s)\, ds)$.*

Proof. Since S_M endowed with the weak topology is Polish, the Skorokhod representation theorem can be invoked to construct processes $(\tilde{\mathbf{v}}_\varepsilon, \tilde{\mathbf{v}}, \tilde{W}_\varepsilon)$ such that the joint distribution of $(\tilde{\mathbf{v}}_\varepsilon, \tilde{W}_\varepsilon)$ is the same as that of $(\mathbf{v}_\varepsilon, W)$, and the distribution of $\tilde{\mathbf{v}}$ coincides with that of $\mathbf{v}$, and $\tilde{\mathbf{v}}_\varepsilon \to \tilde{\mathbf{v}}$ a.s. in the topology.

Define $\mathbf{w}(t) := \mathbf{u}_\varepsilon(t) - \mathbf{u}_\mathbf{v}(t)$. The notation $|\cdot|_{HS}$ will denote the Hilbert-Schmidt norm in what follows. By the Itô Lemma due to Gyongy and Krylov [13], we have

$$
\begin{aligned}
&\frac{1}{2}|\mathbf{w}(t)|^2 + \nu_0 \int_0^t ||\mathbf{w}(s)||^2 ds + \nu_1 \int_0^t ||\mathbf{w}(s)||^q_{V_{r,q}} ds \\
&\leq -\int_0^t b(\mathbf{w}(s), \mathbf{u}_\mathbf{v}(s), \mathbf{w}(s)) ds \\
&\quad + \int_0^t (\sigma(s, \mathbf{u}_\varepsilon(s))\mathbf{v}_\varepsilon(s) - \sigma(s, \mathbf{u}_\mathbf{v}(s))\mathbf{v}(s), \mathbf{w}(s)) \\
&\quad + \sqrt{\varepsilon} \int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s))) dW(s) \\
&\quad + \frac{\varepsilon}{2} \int_0^t |\sigma(s, \mathbf{u}_\varepsilon(s)) Q^{1/2}|^2_{HS} ds \\
&\leq \int_0^t |\mathbf{w}(s)|\; ||\mathbf{w}(s)||\; ||\mathbf{u}_\mathbf{v}(s)|| ds \\
&\quad + \int_0^t |\sigma(s, \mathbf{u}_\varepsilon(s)) - \sigma(s, \mathbf{u}_\mathbf{v}(s)) Q^{1/2}|_{HS} |\mathbf{v}_\varepsilon(s)|_0 |\mathbf{w}(s)| ds \\
&\quad + \int_0^t |\sigma(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\varepsilon(s) - \mathbf{v}(s))|\; |\mathbf{w}(s)| ds \\
&\quad + \sqrt{\varepsilon} \int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s))) dW(s) \\
&\quad + \frac{\varepsilon}{2} \int_0^t |\sigma(s, \mathbf{u}_\varepsilon(s)) Q^{1/2}|^2_{HS} ds
\end{aligned}
\tag{6.3.31}
$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}|\mathbf{w}(t)|^2 + \frac{3\nu_0}{4}\nu_0 \int_0^t ||\mathbf{w}(s)||^2 ds + \nu_1 \int_0^t ||\mathbf{w}(s)||^q_{V_{r,q}} ds \\
&\le \frac{1}{\nu_0}\int_0^t |\mathbf{w}(s)|^2 ||\mathbf{u}_{\mathbf{v}}(s)||^2 ds + \frac{\nu_0}{4}\int_0^t ||\mathbf{w}(s)||^2 ds \\
&\quad + \sqrt{L}\int_0^t ||\mathbf{w}(s)||\, |\mathbf{v}_\varepsilon(s)|_0 |\mathbf{w}(s)| ds \\
&\quad + \int_0^t |\sigma(s, \mathbf{u}_{\mathbf{v}}(s))(\mathbf{v}_\varepsilon(s) - \mathbf{v}(s))|\ |\mathbf{w}(s)| ds \\
&\quad + \frac{\varepsilon}{2} K \int_0^t (1 + |\mathbf{u}_\varepsilon(s)|^2) ds \\
&\quad + \sqrt{\varepsilon}\left|\int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s)) dW(s))\right| \\
&\le \frac{1}{\nu_0}\int_0^t |\mathbf{w}(s)|^2 ||\mathbf{u}_{\mathbf{v}}(s)||^2 ds + \frac{\nu_0}{2}\int_0^t ||\mathbf{w}(s)||^2 ds \\
&\quad + \frac{L}{\nu_0}\int_0^t |\mathbf{v}_\varepsilon(s)|_0^2 |\mathbf{w}(s)|^2 ds + \frac{1}{\nu_0}\int_0^t |\sigma(s, \mathbf{u}_{\mathbf{v}}(s))(\mathbf{v}_\varepsilon - \mathbf{v}(s))|^2 ds \\
&\quad + \frac{\nu_0}{4}\int_0^t ||\mathbf{w}(s)||^2 ds + \frac{\varepsilon}{2} K \int_0^t (1 + |\mathbf{u}_\varepsilon(s)|^2) ds \\
&\quad + \sqrt{\varepsilon}\left|\int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s)) dW(s))\right| \qquad (6.3.32)
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{2}|\mathbf{w}(t)|^2 + \nu_1 \int_0^t ||\mathbf{w}(s)||^q_{V_{r,q}} ds \\
&\le \frac{1}{\nu_0}\int_0^t |\mathbf{w}(s)|^2 ||\mathbf{u}_{\mathbf{v}}(s)||^2 ds + \frac{L}{\nu_0}\int_0^t |\mathbf{v}_\varepsilon(s)|_0^2 |\mathbf{w}(s)|^2 ds \\
&\quad + \frac{1}{\nu_0}\int_0^t |\sigma(s, \mathbf{u}_{\mathbf{v}}(s))(\mathbf{v}_\varepsilon - \mathbf{v}(s))|^2 ds + \frac{\varepsilon}{2} K \int_0^t (1 + |\mathbf{u}_\varepsilon(s)|^2) ds \\
&\quad + \sqrt{\varepsilon}\left|\int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s)) dW(s))\right| \qquad (6.3.33)
\end{aligned}
$$

Define

$$
\begin{aligned}
\tau_{N,\varepsilon} \quad := \quad & T \wedge \inf\{t : \int_0^t (||\mathbf{u}_{\mathbf{v}}(s)||^2 + |\mathbf{u}_\varepsilon(s)|^2) ds > N \\
& \text{or } \sup_{0\le s\le t} |\mathbf{u}_{\mathbf{v}}(s)|^2 > N \text{ or } \sup_{0\le s\le t} |\mathbf{u}_\varepsilon(s)|^2 > N\} \qquad (6.3.34)
\end{aligned}
$$

Let T_0 be any number in $[0, T]$. Taking the supremum in (6.3.33) over the interval $[0, T_0 \wedge \tau_{N,\varepsilon}]$ yields that

$$
\begin{aligned}
&\frac{1}{2}\Big(\sup_{0\le t\le T_0\wedge\tau_{N,\varepsilon}} |\mathbf{w}(t)|^2\Big) + \nu_1 \int_0^t ||\mathbf{w}(s)||_{V_{r,q}}^q ds \\
&\le \frac{1}{\nu_0}\int_0^{T_0\wedge\tau_{N,\varepsilon}} |\mathbf{w}(s)|^2(||\mathbf{u}_{\mathbf{v}}(s)||^2 + L|\mathbf{v}_\varepsilon(s)|_0^2)ds \\
&+\frac{1}{\nu_0}\int_0^{T\wedge\tau_{N,\varepsilon}} |\sigma(s,\mathbf{u}_{\mathbf{v}}(s))(\mathbf{v}_\varepsilon(s)-\mathbf{v}(s))|^2 ds + \frac{\varepsilon}{2}K(T+N) \\
&+\sqrt{\varepsilon}\left\{\sup_{0\le t\le T\wedge\tau_{N,\varepsilon}}\left|\int_0^t (\mathbf{w}(s), \sigma(s,\mathbf{u}_\varepsilon(s))dW(s))\right|\right\} \qquad (6.3.35)
\end{aligned}
$$

The Burkhölder-Davis-Gundy inequality allows us to bound the expectation of the last term on the right side of (6.3.35) by

$$
\begin{aligned}
&2\sqrt{\varepsilon}E\left[\left\{\int_0^{T\wedge\tau_{N,\varepsilon}} |\mathbf{w}(s)|^2|\sigma(s,\mathbf{u}_\varepsilon(s))|_{L_Q}^2 ds\right\}^{1/2}\right] \\
&\le 2\sqrt{\varepsilon}E\left[\sup_{0\le s\le T\wedge\tau_{N,\varepsilon}}|\mathbf{w}(s)|\left\{\int_0^{T\wedge\tau_{N,\varepsilon}} |\sigma(s,\mathbf{u}_\varepsilon(s))|_{L_Q}^2 ds\right\}^{1/2}\right] \\
&\le \sqrt{\varepsilon}E\left[\sup_{0\le s\le T\wedge\tau_{N,\varepsilon}}|\mathbf{w}(s)|^2 + \int_0^{T\wedge\tau_{N,\varepsilon}} |\sigma(s,\mathbf{u}_\varepsilon(s))|_{L_Q}^2 ds\right] \\
&\le \sqrt{\varepsilon}\left(E\left[\sup_{0\le s\le T\wedge\tau_{N,\varepsilon}}|\mathbf{w}(s)|^2\right] + 2K^2(T+N)\right) < \infty \qquad (6.3.36)
\end{aligned}
$$

where the assumption (A.2) has been used in the last step. Using the Gronwall inequality and the definition of $\tau_{N,\varepsilon}$, we get

$$
\begin{aligned}
&\frac{1}{2}\Big(\sup_{0\le t\le T_0\wedge\tau_{N,\varepsilon}} |\mathbf{w}(t)|^2\Big) + \nu_1 \int_0^t ||\mathbf{w}(s)||_{V_{r,q}}^q ds \\
&\le \Big\{\frac{1}{\nu_0}\int_0^{T\wedge\tau_{N,\varepsilon}} |\sigma(s,\mathbf{u}_{\mathbf{v}}(s))(\mathbf{v}_\varepsilon(s)-\mathbf{v}(s))|^2 ds \\
&+\frac{\varepsilon}{2}K\int_0^{T\wedge\tau_{N,\varepsilon}}(1+|\mathbf{u}_\varepsilon(s)|^2)ds \\
&+\sqrt{\varepsilon}\Big(\sup_{0\le t\le T\wedge\tau_{N,\varepsilon}}\Big|\int_0^t (\mathbf{w}(s), \sigma(s,\mathbf{u}_\varepsilon(s))dW(s))\Big|\Big)\Big\} e^{\frac{N}{\nu_0}+LM}
\end{aligned}
$$

(6.3.37)

Let N be fixed, then for suitable constant C,

$$\liminf_{\varepsilon \to 0} P\{\tau_{N,\varepsilon} = T\} \geq 1 - \frac{C}{N} \tag{6.3.38}$$

Note that Eq. (6.3.36) shows that

$$\sqrt{\varepsilon} \sup_{0 \leq t \leq T \wedge \tau_{N,\varepsilon}} \left| \int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s) dW(s))) \right| \to 0 \quad \text{in probability}$$

as ε tends to zero. These two observations along with the weak convergence of $\mathbf{v}_\varepsilon \to \mathbf{v}$ in S_M, we obtain that

$$\frac{1}{2}(\sup_{0 \leq t \leq T_0 \wedge \tau_{N,\varepsilon}} |\mathbf{w}(t)|^2) + \nu_1 \int_0^t ||\mathbf{w}(s)||_{V_{r,q}}^q ds \to 0 \quad \text{in probability} \tag{6.3.39}$$

as $\varepsilon \to 0$, which completes the proof. □

The above two propositions 6.3.4 and 6.3.5 show that the family $\{g^\varepsilon\}$ satisfies the Hypothesis 6.2.1, so we can apply Theorem 6.2.2 to obtain the Laplace-Varadhan principle for $\{\mathbf{u}^\varepsilon : \varepsilon > 0\}$ in $C([0,T]; H) \cap L^q(0,T; V_{r,q})$.

Theorem 6.3.6. *Let $\{\mathbf{u}^\varepsilon(\cdot)\}$ be the solution of the equation*

$$d\mathbf{u}^\varepsilon + [\nu_0 \mathbf{A}\mathbf{u}^\varepsilon + \nu_1 \mathcal{A}(\mathbf{u}^\varepsilon) + \mathbf{B}(\mathbf{u}^\varepsilon)]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}^\varepsilon)dW_t$$

$$\mathbf{u}^\varepsilon(0) = \xi \in H \tag{6.3.40}$$

Then $\{\mathbf{u}^\varepsilon\}$ satisfies the Laplace-Varadhan principle in $C([0,T]; H) \cap L^q(0,T; V_{r,q})$ with good rate function

$$I(f) = \inf_{\{v \in L^2(0,T;H_0): f = g^0(\int_0^\cdot v(s)ds)\}} \left\{ \frac{1}{2} \int_0^T |v(s)|_0^2 ds \right\} \tag{6.3.41}$$

with the convention that the infimum of an empty set is infinity.

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Appendix: Skorohod Space

Let $D[0,1]$ denote the space of real-valued functions on $[0,1]$ that are right-continuous and have left-hand limits. In probabilistic literature, such a function is also said to be a cadlag function. Equipped with the norm

$$||x|| = \sup_{0<t<1} |x(t)|, \qquad \text{for any } x \in D[0,1]$$

$D[0,1]$becomes a Banach space, but it is non-separable. This non-separability causes well-known problems of measurability in the theory of weak convergence of measures on the space.

To overcome this inconvenience, A.V. Skorokhod introduced a metric (and topology) under which the space $D[0,1]$ becomes a separable metric space. Although the original metric introduced by Skorokhod has a drawback in the sense that the metric space obtained is not complete, it turned out that it is possible to construct an equivalent metric (i.e., giving the same topology) under which the space becomes a complete separable metric space (Polish space). This metric is defined as follows:

Let Λ denote the class of strictly increasing continuous mappings of $[0,1]$onto itself. For $\lambda \in \Lambda$, let

$$||\lambda|| = \sup_{0\le s<t\le 1} \Big| \log \frac{\lambda(t) - \lambda(s)}{t-s} \Big|$$

Then for any $x, y \in D[0,1]$, define a metric

$$d(x,y) = \inf_{\lambda\in\Lambda} \max\{||\lambda||, \sup_{0\le t\le 1} |x(t) - y(\lambda(t))|\}$$

The topology generated by this metric is called the Skorokhod topology and the complete separable metric space is called the Skorokhod space.

Generalizations of the Skorokhod space are worth mentioning. Instead of real-valued functions on $[0,1]$ it is possible to consider functions defined on $[0,\infty)$ and taking values in a metric space E. The space of cadlag functions obtained in this way is denoted by $D([0,\infty),E)$ and if E is a Polish space, then $D([0,\infty),E)$, with the appropriate topology, is also a Polish space.

Vita

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