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Some classes of graphs that are nearly cycle-free

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SOME CLASSES OF GRAPHS THAT ARE NEARLY CYCLE-FREE

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Abstract

A graph is almost series-parallel if there is some edge that one can add to the graph and then contract out to leave a series-parallel graph, that is, a graph with no K_4-minor. In this dissertation, we find the full list of excluded minors for the class of graphs that are almost series-parallel. We also obtain the corresponding result for the class of graphs such that uncontracting an edge and then deleting the uncontracted edge produces a series-parallel graph.

A notable feature of a 3-connected almost series-parallel graph is that it has two vertices whose removal leaves a tree. This motivates consideration of those graphs for which there are two vertices whose removal is cycle-free. We find the full list of excluded minors for the class of graphs that have a set of at most two vertices whose removal is cycle-free.

Chapter 1
Introduction

This chapter contains some basic graph theory and matroid theory terminology that will be used throughout this dissertation. The terminology used for graphs and matroids closely follows [5] and [13].

1.1 Fundamental Graph Definitions

A *multigraph* G is a pair $(V(G), E(G))$ of sets, where $V(G)$ is a set and $E(G)$ is a multiset whose elements are unordered pairs of elements in $V(G)$. We refer to $V(G)$ and $E(G)$ as the sets of vertices and edges, respectively, of G. When it is clear to which graph G we are referring, $V(G)$ and $E(G)$ are abbreviated by V and E, respectively. A *simple graph* is a multigraph in which the edges are distinct pairs of distinct vertices. Throughout this dissertation, when we refer to a graph, we assume it is a multigraph unless otherwise specified.

The number of vertices of a graph G is its *order*, written $|V(G)|$ and its number of edges is denoted by $|E(G)|$. The graphs we consider are all *finite*, that is, they all have a finite order and a finite number of edges. For the empty graph, we write $\emptyset$. A graph of order zero or one is called *trivial*. We will primarily consider non-trivial graphs.

Let e be the edge $\{v, w\}$ where v and w are in $V(G)$. In this dissertation, we denote a single edge between $\{v, w\}$ by vw or wv. The edge e is *between* v and w, and we call v and w the *endpoints* of e. An edge is *incident* with each of its endpoints. An edge whose endpoints are the same vertex is a *loop*. If e and f are edges, having the same pair of distinct endpoints, then e and f are *parallel*. For an edge e in a graph G, the *parallel class* of e is the set e together with every element parallel to e. Two distinct edges are *adjacent* if they have an endpoint in common. Two distinct vertices v and w are *adjacent*, or are *neighbors* if there is an edge between v and w when this occurs. We also say that there is an edge *joining* v and w. The set of neighbors of v in G is the *neighborhood* $N_G(v)$ of v. We abbreviate $N_G(v)$ to $N(v)$ when it is understood which graph is meant. If all vertices of G are pairwise adjacent and G is simple, then G is *complete*. A complete graph on n vertices is denoted K_n. For example, K_3 is a triangle.

The *degree*, $d_G(v)$ or $d(v)$, of a vertex v is the number of edges meeting v where a loop is counted twice. The number $\delta(G)$ is the *minimum degree* of any vertex in G, and $\Delta(G)$ is the *maximum degree* of any vertex in G. If all the vertices of G have the same degree k, then G is *k-regular*, or simply *regular*. A 3-regular graph is called *cubic*.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Graphs G and G' are *isomorphic*, written $G \cong G'$, if there are bijections $\sigma : V(G) \to V(G')$ and $\phi : E(G) \to E(G')$ such that a vertex v of G is incident with an edge e of G if and only if $\sigma(v)$ is incident with $\phi(e)$. We do not normally distinguish between isomorphic graphs. Thus, we often write $G = G'$, rather than $G \cong G'$.

A graph H is a *subgraph* of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and if each edge of H is an edge of G. We also say G is a *supergraph* of H and G *contains* H. If $V(H) = V(G)$, then we say that H *spans* G. When it is clear which graph we are referring to, we say that H is *spanning*. If V' is a non-empty subset of $V(G)$, then $G[V']$ denotes the subgraph of G whose vertex set is V' and whose edge set consists of those edges of G that have both endpoints in V'. We say that $G[V']$ is the subgraph of G *induced* by V'. Similarly, if E' is a non-empty subset of $E(G)$, then $G[E']$, the subgraph of G induced by E', has E' as its edge set and the set of endpoints of edges in E' as its vertex set.

If G and G' are graphs, their *union* $G \cup G'$ is the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. If $V(G)$ and $V(G')$ are disjoint, then so are $E(G)$ and $E(G')$, and G and G' are called *disjoint graphs*.

1.2 Deletion, Contraction, and Graph Minors

If U is any set of vertices of G, then the graph obtained by *deleting* all the vertices in U and their incident edges is denoted $G - U$. If $U = \{u\}$ is a single vertex, then we write $G - u$. If F is a set of possible edges of G, then $G \backslash F$ is the graph $(V, E \backslash F)$ obtained by *deleting* the subset F, and $G + F$ is the graph $(V, E \cup F)$ obtained by adding the edges F to the graph G. For the deletion and addition of a single edge, we write $G \backslash e$ and $G + e$, respectively.

Let e be an edge of a graph $G = (V, E)$ with endpoints x and y. We denote the *contraction* of the edge e by G/e. This is the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the former neighbors of x and y. The graph G/e is a graph with vertex set $V'(G/e) = (V \backslash \{x, y\}) \cup \{v_e\}$ where v_e is the new vertex. Let f be a function which maps every vertex in $V \backslash \{x, y\}$ to itself, and otherwise maps to the new vertex v_e. The edge set $E'(G/e) = E \backslash e$ and, for every $z \in V$, the vertex $z' = f(z) \in V'$ is incident to an edge $e' \in E'$ if and only if the corresponding edge $e \in E$ is incident to z in G. Let $H \subseteq E(G)$. Then G/H is the contraction of the set H from G.

The fundamental substructures of graphs are graph minors which can be obtained by deleting some vertices and edges, and then contracting some further edges. Formally, any sequence of deletions and contractions from G can be written in the form $(G - U) \backslash X/Y$ for some set of vertices U and some pair of disjoint sets of edges X and Y. The sets U, X, and Y may be empty. Graphs of the form $(G - U) \backslash X/Y$ are called *minors* of G. If $U \cup X \cup Y$ is non-empty, then we call $(G - U) \backslash X/Y$ a *proper minor* of G. Note that every subgraph G of a graph is also a minor of G, and G is a minor of itself. A graph G has an N-*minor* if N is a minor of G and we say G *contains* N as a minor or simply G contains N.

1.3 Several Important Classes of Graphs

A *path* is a non-empty graph $P = (V(P), E(P))$ of the form $V(P) = \{x_0, x_1, \ldots x_k\}$ and $E(P) = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where all x_i are distinct. The vertices x_0 and x_k are *linked* by P and are called its *ends*. The vertices $x_1, \ldots, x_{k-1}$ are the *inner* vertices of P.

The number of edges of the path is its *length*, and the path of length k is denoted P_k. We often refer to the path by the natural sequence of its vertices, writing $P = x_0x_1 \ldots x_k$ and calling P a path *from* x_0 to x_k or *between* x_0 and x_k.

Two or more paths are *internally disjoint* if none of them contains an inner vertex of another. Two paths from a to b, for example, are *internally disjoint* if and only if a and b are their only common vertices.

If $P = x_0 \ldots x_{k-1}$ is a path, then the graph $C = P + x_{k-1}x_0$ is called a *cycle*, often written $x_0x_1 \ldots x_{k-1}x_0$. The *length* of a cycle is its number of edges (or vertices) and the cycle of length k is called a *k-cycle* and denoted by C_k. A C_k^n is the graph obtained from a cycle of length k by replacing every edge of C_k by n parallel edges. A *wheel*, denoted $\mathcal{W}_r$, is a simple graph that is formed by taking an r-cycle and adding a vertex adjacent to every vertex of the cycle. We call the r-cycle from which $\mathcal{W}_r$ is formed, the *rim* of the wheel, and every other edge not in this cycle is a *spoke* of the wheel.

An *acyclic* graph, one not containing any cycles, is called a *forest.* A connected forest is called a *tree*. The vertices of degree 1 in a tree are its *leaves*. So every tree with at least two vertices has at least two leaves.

The class of bipartite graphs is a well-known class and has been studied extensively. A graph G is *bipartite* if its vertex set has a partition (A, B) into possibly empty sets such that each edge has one endpoint in A and one endpoint in B; that is, G has no edge having both endpoints in A or both endpoints in B. If the graph G induced on a vertex set contains no edges, then that set is *stable*. The vertex set of a bipartite graph is the union of two stable sets.

Let G be a bipartite graph with vertex partition (A, B), where $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_l\}$. The *complete bipartite graph* $K_{k,l}$ is the simple bipartite graph where each vertex in A is adjacent with every vertex in B. Note that any subgraph of G is also bipartite. Thus, the class of bipartite graphs is closed under edge and vertex deletion. Since

G contains no edge of the form $a_i a_j$, any path in G must alternate between A-vertices and B-vertices, such as $a_{n_1} b_{n_2} a_{n_3} \dots b_{n_{m-1}} a_{n_m}$. Clearly any cycle of G also alternates between A-vertices and B-vertices. Thus, a bipartite graph contains no odd cycle. The converse of this also holds, that is, a graph is bipartite if and only if it contains no odd cycles.

Another well-known class of graphs is the class of *planar graphs*, graphs that can be drawn in the Euclidean plane so that vertices correspond to points of the plane; the edges correspond to arcs connecting vertices; two distinct edges do not intersect except possibly at their endpoints; and no vertex lies in the interior of an edge. Such a drawing in the plane is called a *plane graph*. Clearly every minor of a plane graph is a plane graph. Thus, the class of planar graphs is closed under taking minors.

1.4 Graph Operations

A graph G' is a *subdivision* of a graph G if G' can be obtained from G by replacing non-loop edges of G by paths of non-zero length and replacing loop edges by cycles. The *simplification* of a graph G, denoted $\mathrm{si}(G)$, is the graph obtained from G by deleting all loops and all but one element from each parallel class. We refer to $\mathrm{si}(G)$ as the *underlying simple graph* of G.

A plane graph G has a *dual graph* G^*, that is, the graph whose vertices are the faces of G such that, for each edge $e \in E(G)$, there is an edge $e' \in E(G^*)$ whose endpoints are the faces that meet e in G.

A *clique-sum* of two graphs G_1 and G_2 is obtained from the disjoint union of G_1 and G_2 by identifying a complete subgraph of G_1 with a complete subgraph (of the same order) of G_2, and then deleting the edges of the identified subgraph. If the identified complete subgraph has order k, then the clique-sum is called a *k-sum* and is written $G_1 \oplus_k G_2$. The 0-sum is the disjoint union of G_1 and G_2 and the 1-sum consists of subgraphs G_1 and G_2 sharing exactly one vertex.

Let G_1 and G_2 be disjoint graphs and let p_i be a non-loop edge of G_i. Assign a direction to p_i labeling its tail by u_i and its head by v_i. The *series connection*, $S(G_1, G_2)$, of G_1 and

G_2 with respect to the directed edges p_1 and p_2 is formed by deleting p_1 from G_1 and p_2 from G_2, identifying u_1 and u_2 as a new vertex u, and then adding a new edge p joining v_1 and v_2. The *parallel connection*, $P(G_1, G_2)$, of G_1 and G_2 is formed by deleting p_1 from G_1 and p_2 from G_2, identifying the vertices u_1 and u_2 as the vertex u, identifying vertices v_1 and v_2 as the vertex v, and then adding a new edge p joining u and v. Thus the parallel connection is obtained by simply identifying p_1 and p_2 so that their directions agree. Notice that the 2-sum of two graphs G_1 and G_2, written $G_1 \oplus_2 G_2$ is the deletion of p from the parallel connection of G_1 and G_2 with respect to the edges p_1 of G_1 and p_2 of G_2. We call p_1 and p_2 *basepoints* of G_1 and G_2, respectively, and p the *basepoint* of the series connection, parallel connection, and 2-sum. For $n \geq 2$, let $G_1, G_2, \ldots, G_n$ be graphs whose edge sets are disjoint except that each has a non-loop directed edge labelled p. The parallel connection $P(G_1, G_2, \ldots, G_n)$ of $G_1, G_2, \ldots, G_n$ is obtained by identifying all the edges labelled by p so that their directions agree.

1.5 Graph Connectivity

A non-empty graph G is *connected* if any two of its vertices are linked by a path in G. If $U \subseteq V(G)$ and $G[U]$ is connected, we say that U is connected in G.

We call a maximal connected subgraph of a graph G a *component* of G. If $A, B \subseteq V(G)$ and $X \subseteq V(G) \cup E(G)$ are such that every path from A to B in G contains a vertex or an edge from X, we say that X *separates* the sets A and B in G. Then X is a *separating set* or *cut set* in G if X separates two vertices of $G - X$ in G. A *cut-vertex* is a vertex that separates two other vertices of the same component, and a *bridge* is an edge separating its ends. We call a set Y of vertices a *vertex cut* of G if Y separates a component of G. If $G \backslash X$ has more components than G for some set X of edges of G, then we call X an *edge cut* of G. An edge e for which $\{e\}$ is an edge cut is called a *cut-edge*. A minimal edge cut is also called a *bond* of G.

A graph G is k-connected for $k \in \mathbb{N}$ if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of G are separated by fewer than k other vertices. Every (nonempty) graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs. For a graph G, *the vertex connectivity*, $\kappa(G)$, is defined to be zero when G is disconnected. When G is connected, $\kappa(G)$ is the minimum cardinality of a vertex cut in G unless every two distinct vertices of G are adjacent, in which case, $\kappa(G) = |V(G)| - 1$.

A *block* of a graph G is a maximal 2-connected subgraph, a parallel class that is not contained in a 2-connected subgraph, a loop, or an isolated vertex. The *block-graph* of a connected graph G is a tree T whose vertex set is the disjoint union of the blocks of G and those vertices of G that belong to more than one block. The only edges in T are those that join vertices of G to blocks that contain them. We call a block that is a leaf of a connected block-graph an *end block*.

Menger's theorem [12] from 1927 establishes another characterization of k-connectivity.

Theorem 1.5.1. *Let G be a graph having at least $n+1$ vertices. Then G is n-connected if and only if all pairs of distinct vertices of G are joined by at least n internally disjoint paths.*

This theorem implies that a graph G is k-connected if and only if, for each pair v and w of distinct vertices, G contains k internally disjoint paths from v to w.

1.6 Excluded Minors

Kuratowski [10] proved the following characterization of planar graphs in 1930.

Theorem 1.6.1. *A graph G is planar if and only if it has no subdivision isomorphic to K_5 or $K_{3,3}$.*

A class of graphs is closed under the minor operation if every minor of a graph in the class is also in the class, and we say that this class is *minor-closed.* An *excluded minor* or *forbidden minor* of a minor-closed class of graphs is a graph that is not in the class, but all of whose proper minors are in the class.

Instead of dealing with subdivisions, Wagner [22] generalized Theorem 1.6.1 and gave the following excluded-minor characterization of planar graphs in 1937.

Theorem 1.6.2. *A graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

In Theorem 1.6.2, K_5 and $K_{3,3}$ are excluded minors for the class of planar graphs. Neither of these graphs is in the class of planar graphs, but every proper minor of each graph is planar. Furthermore, K_5 and $K_{3,3}$ are the only graphs fitting this description. Much work has been done characterizing various classes of graphs by their excluded or forbidden minors.

The Graph Minors Project [14] of Neil Robertson and Paul Seymour is a set of results published in a series of 23 papers starting in 1983 relating graph minors to topological embeddings. This set of results proved the Graph Structure Theorem and is regarded as some of the most important work ever done in graph theory. In particular, Robertson and Seymour proved that every class of graphs that is closed under taking minors can be characterized by a finite number of excluded minors. The results and tools developed in this series of papers have since been successfully used to attack a large number of problems in graph theory.

1.7 Fundamental Matroid Definitions

Much of the motivation of this dissertation arises from matroid theory. Throughout this dissertation, we also use matroid theory to solve several graph-theoretic problems. This section contains an introduction to some basic matroid theory terminology and follows [13] closely.

A *matroid* M is an ordered pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a collection of *independent sets* that are subsets of the finite *ground set* E and satisfy the following three conditions:

(i) $\emptyset \in \mathcal{I}$.

(ii) If $I \in \mathcal{I}$ and $I' \subseteq I$, then I' is a member of $\mathcal{I}$.

(iii) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| \leq |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e$ is a member of $\mathcal{I}$.

If M is a matroid on $(E, \mathcal{I})$, then M is called a matroid *on* E. A subset of E that is not in $\mathcal{I}$ is called *dependent*. A minimal dependent set in an arbitrary matroid M is called a *circuit* of M and we denote the set of circuits of M by $\mathcal{C}$ or $\mathcal{C}(M)$. The maximal independent sets of M are called the *bases* of M and the sets of bases is denoted by $\mathcal{B}$ or $\mathcal{B}(M)$. The bases of M all have the same cardinality, and this cardinality is equal to the *rank* of M, written $r(M)$. The rank of a subset X of $E(M)$, written $r(X)$, is the cardinality of a largest independent set of M contained in X. Clearly $X \in \mathcal{I}$ if and only if $r(X) = |X|$. The definition of matroid given above defines a matroid by its independent sets, but a matroid may also be defined in terms of its sets of bases, its set of circuits, or its rank.

The circuits of a matroid satisfy the following three conditions:

(C1) $\emptyset \in \mathcal{C}(M)$.

(C2) If C_1 and C_2 are in $\mathcal{C}(M)$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}(M)$ and $e \in C_1 \cup C_2$, then there is a member C_3 of $\mathcal{C}(M)$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

The *closure* or *span*, $cl_M(X)$ or $cl(X)$, of a subset X of $E(M)$ is the maximal set $X' \subseteq E(M)$ satisfying $X \subseteq X'$ and $r_M(X') = r_M(X)$. If $X = cl(X)$, then X is called a *flat* or *closed set* of M. A *hyperplane* of M is a flat of rank $r(M) - 1$. A subset X of $E(M)$ is a *spanning set* if $cl(X) = E(M)$. Equivalently, X is a spanning set if and only if $r(X) = r(M)$. The closure of any basis is the entire matroid, and a set X is a basis if and only if it is a minimal spanning set. Also, X is a hyperplane if and only if it is a maximal nonspanning set.

If M_1 and M_2 are the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$, then M_1 is *isomorphic* to M_2 if there is a bijection $\phi : E_1 \to E_2$ such that a subset X of E_1 is in $\mathcal{I}_1$ if and only if $\phi(X)$ is in $\mathcal{I}_2$.

1.8 Several Important Classes of Matroids

An important class of matroids are *representable matroids* or vector matroids, those matroids that can be represented by a finite collection of vectors from a vector space. Let A be an $m \times n$ matrix over a field F. The *vector matroid* of A, denoted by $M[A]$, is a matroid $(E, \mathcal{I})$, where E is the set of column labels of M, and I is the set of subsets X of E for which the multiset of columns labelled by X is a linearly independent set in the vector space $V(m, F)$, the m-dimensional vector space over F. It is easy to check that the pair $(E, \mathcal{I})$ satisfies (i), (ii), and (iii) and is therefore a matroid. A matroid is said to be *representable over* $GF(q)$, the q-element field, if it is isomorphic to the vector matroid of a matrix over $GF(q)$. The vector matroids that are representable over $GF(2)$ and $GF(3)$ are called *binary matroids* and *ternary matroids*, respectively. A matroid is *regular* if it can be represented over the real numbers as the vector matroid of a totally unimodular matrix, one for which all subdeterminants are $\{0, 1, -1\}$.

Another class of matroids is the class of *graphic matroids*, those matroids that can be realized by graphs as follows. Let G be a graph. The *cycle matroid* of a graph G, written $M(G)$, arises by taking the ground set $E = E(G)$ and the set of circuits to be the set of edge-sets of the cycles in G. Any matroid that is isomorphic to the cycle matroid of a graph is called a *graphic matroid*. Notice that the independent sets of $M(G)$ are the edge sets of forests in G. It is not hard to show that every graphic matroid is regular and every regular matroid is binary. In this dissertation we deal exclusively with graphic and binary matroids.

We say that a matroid M has a specific graph property if there is a graph with that property whose cycle matroid is M. For example we say that a matroid M is planar if there is a planar graph with cycle matroid $M(G) \cong M$.

1.9 Matroid Operations and Minors

The *dual* M^* of a matroid M is the matroid with ground set $E(M)$ whose set of bases is $\{E(M) - B : B \in \mathcal{B}(M)\}$. A basis of M^* is a *cobasis* of M, and an independent set in M^* is a *coindependent set* of M. The classes of matroids that are closed under minors and also closed under duality are usually easier to work with than those without one or both properties. In this dissertation, we will introduce and work with classes closed under both properties and consider the dual matroids.

Let e be an element in the ground set E of a matroid M. The *deletion* of a subset $T \subseteq E$, written $M \backslash T$, is the matroid with ground set $E - T$ having $\{C \in \mathcal{C}(M) : C \cap T = \emptyset\}$ as its set of circuits. For a graph G, it is easy to see that $M(G)\backslash T = M(G \backslash T)$ for any subset T of $E(G)$. The *contraction* of a subset $T \in E$, written M/T, results in a matroid with ground set $E - T$ whose circuits are the minimal non-empty members of $\{C - T : C \in \mathcal{C}(M)\}$. The contraction of a subset T of E is also given by $M/T = (M^* \backslash T)^*$. We note that if G is a graph and $T \subseteq E(G)$, then $M(G/T) = M(G)/T$.

The circuits of M^* are the *cocircuits* of M. A three-element circuit is called a *triangle* and a three-element cocircuit is called a *triad*.

A matroid N is a *minor* of a matroid M if $N = M \backslash X/Y$ for some disjoint subsets X and Y of $E(M)$. Some classes of matroids have the property that all of their minors are also in the class, and we say that such classes are *closed under minors* or *minor-closed*. It is not hard to check that the class of graphic matroids is minor-closed.

The $r \times r$ *identity matrix* is denoted I_r. Let A be a the matrix $[I_r|D]$ and let D^T be the transpose of D. The dual of $M[I_r|D]$ is equal to $M[D^T|I_{|E(M)|-r}]$. It is not difficult to see that if G is a plane graph, then $M^*(G) = M(G^*)$. A matroid that has a graphic dual is called *cographic*.

A matroid M is a *relaxation* of a matroid N if, for some circuit-hyperplane H of N, the set of bases of M is the set of bases of N together with H.

A *loop* in a matroid is an element of rank zero. In a graph, a loop edge corresponds to a loop element in the matroid. In a vector matroid, a loop element corresponds to the zero vector. Since a loop in M is in no basis of M, it is in every basis of M^*, and it is a *coloop* of M^*. A pair of elements are *parallel* if they form a circuit. In a graph, these elements are in the same parallel class. In a vector matroid, parallel elements correspond to non-zero scalar multiples of the same non-zero vector. For a non-loop element e in a matroid M, the *parallel class* of e is the set e together with every element parallel to e.

1.10 Matroid Connectivity

A matroid is *connected* if and only if for every pair of distinct elements in its ground set, there is a circuit containing both elements. If M is not connected, then M is *disconnected*.

Let M be a matroid with ground set E and rank function r. A partition (X, Y) of the ground set E is a *k-separation* if $\min\{|X|, |Y|\} \geq k$ and $r(X) + r(Y) - r(M) \leq k - 1$. If M has a k-separation, then M is called *k-separated* or *k-separable*. A matroid is 1-separated if and only if it is disconnected.

The notion of k-connectivity in matroids was introduced by Tutte in 1966 [20]. If M is k-separated for some k, then the *Tutte connectivity* $\lambda(M)$ of M is $\min\{j : M \text{ is } j\text{−separated}\}$; otherwise we take $\lambda(M)$ to be ∞. In general, when we discuss matroid connectivity, we are referring to the Tutte connectivity. If n is an integer exceeding one, we say that M is *k-connected* if $\lambda(M) \geq k$. It is not difficult to check that a matroid is k-connected if and only if its dual is k-connected, and we have that $\lambda(M) = \lambda(M^*)$. A k-connected graph G has no vertices of degree less than k and, more generally, such a graph has no bonds of size less than k.

Tutte's definition of matroid connectivity and the standard definition of graph connectivity are not equivalent and this difference is shown in the following result, which follows from Menger's Theorem 1.5.1. This shows us that an n-connected matroid cannot have small circuits, whereas an n-connected graph can.

Corollary 1.10.1. *Let G be a graph having no isolated vertices. If $V(G) \geq 3$, then $M(G)$ is 2-connected if and only if G is 2-connected and loopless.*

We define $\mathcal{W}^r$, the *rank-r whirl*, to be the matroid formed by starting with the matroid $M(\mathcal{W}_r)$, the graphic matroid obtained from the r-wheel, and then relaxing the rim of the wheel. Thus, $E(\mathcal{W}^r) = E(\mathcal{W}_r)$, while the bases of $\mathcal{W}^r$ consists of the rim together with all edge sets of spanning trees of $\mathcal{W}_r$.

The next two theorems, Tutte's Wheels and Whirls Theorem [20] and Bixby's Lemma [1] are basic structural results for 3-connected matroids.

Theorem 1.10.2 (Tutte's Wheels and Whirls Theorem)**.** *The following statements are equivalent for a 3-connected matroid M having at least one element:*

(i) For every element e of M, neither $M \backslash e$ nor M/e is 3-connected.

(ii) M has rank at least three and is isomorphic to a wheel or a whirl.

The *simplification* of a matroid M, denoted $\mathrm{si}(M)$, is the matroid obtained from M by deleting all loops and all but one element from each parallel class. The *cosimplification* of a matroid M, denoted $\mathrm{co}(M)$, is the matroid $(\mathrm{si}(M^*))^*$. In a 3-connected matroid M, an element e is *vertically contractible* if $\mathrm{si}(M/e)$ is 3-connected.

Theorem 1.10.3. *Let M be a 3-connected matroid. For every element $e \in E(M)$, either $\mathrm{co}(M \backslash e)$ or $\mathrm{si}(M/e)$ is 3-connected.*

In Tutte's matroid connectivity, a matroid has connectivity k if and only if its dual has connectivity k. However, matroid connectivity does not correspond to graph connectivity. Tutte's definition of matroid connectivity can be modified to generalize the notion of connectivity in graphs. Below we give the definition of vertical n-connectedness, however one loses invariance under duality.

For a positive integer k, we say that a matroid is *vertically k-separated* if there is a partition (X, Y) of $E(M)$ such that $\min\{r(X), r(Y)\} \geq k$ and $r(X)+r(Y)-r(M) \leq k-1$. It is easy to see that if M is vertically k-separated, then M is k-separated. The *vertical connectivity* $\kappa(M)$ of M is the least positive integer j such that M is vertically j-separated; otherwise we let $\kappa(M) = r(M)$. In general, a matroid M is called *vertically n-connected* if n is an integer for which $2 \leq n \leq \kappa(M)$. Vertical n-connectedness in matroids is a direct generalization of the notion of n-connectedness in graphs. It is not difficult to show that $\kappa(M(G)) = \kappa(G)$.

1.11 Graphic Matroid Isomorphism and Roundedness

A graph G is 2-*isomorphic* to the graph H, written $G \cong_2 H$, if H can be transformed into a graph isomorphic to G by a sequence of operations of types (a), (b), and (c), as follows:

(a) *Vertex identification.* If v and v' are vertices in distinct components of G, then modify G by identifying v and v' as a new vertex v''.

(b) *Vertex cleaving.* This is the reverse operation of vertex identification. A graph can only be cleft at a cut-vertex or at a vertex incident with a loop.

(c) *Twisting.* Let G be the graph obtained from disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and u_2 of G_2 as the vertex u of G, and identifying the vertices v_1 of G_1 and v_2 of G_2 as the vertex v of G. In a *twisting* G' of G *about* $\{u, v\}$, we instead identify u_1 with v_2 and v_1 with u_2. We call G_1 and G_2 the *pieces* of the twisting.

Whitney's 2-Isomorphism Theorem [23], stated below, identifies when two graphs have isomorphic cycle matroids. Shorter proofs have also been given by both Truemper [18] and Wagner [21].

Theorem 1.11.1 (Whitney's 2-Isomorphism Theorem)**.** *Let G and H be graphs having no isolated vertices. Then $M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

The following theorem, proved by Edmonds (see Truemper [18]) and Greene [8], implies that a 3-connected graphic matroid uniquely determines a graph.

Theorem 1.11.2. *Let G and H be 3-connected loopless graphs without isolated vertices. If $M(G) \cong M(H)$, then $G \cong H$.*

A class $\mathcal{N}$ of matroids is *t-rounded* if every member of $\mathcal{N}$ is $(t+1)$-connected and the following condition holds: If M is a $(t+1)$-connected matroid having an $\mathcal{N}$-minor and X is a subset of $E(M)$ with at most t elements, then M has an $\mathcal{N}$-minor using X.

Seymour [15,17] gave the following characterization of t-rounded classes for the case when $t = 1$ or 2.

Theorem 1.11.3. *Let t be 1 or 2 and $\mathcal{N}$ be a collection of $(t+1)$-connected matroids. Then $\mathcal{N}$ is t-rounded if and only if the following condition holds: If M is a $(t+1)$-connected matroid having an $\mathcal{N}$-minor N such that $|E(M) - E(N)| = 1$, and X is a subset of $E(M)$ with at most t elements, then M has an $\mathcal{N}$-minor using X.*

Chapter 2
Almost Series-Parallel Graphs

A *series-parallel* graph is formed recursively from a forest by the operations of adjoining a loop, subdividing an edge, or adding an edge in parallel to an existing non-loop edge. Equivalently, series-parallel graphs can be characterized as graphs having no K_4-minor [6]. We consider the class $\mathcal{S}$ of graphs that are *almost series-parallel*, graphs such that there is some edge that one can add to the graph and then contract out to leave a series-parallel graph. Notice that the operation of adding an edge e joining distinct vertices u and v and then contracting e has the effect of identifying the vertices u and v. In their description of the structure of the class of binary matroids, Geelen, Gerards, and Whittle [7] make essential use of an operation they call *perturbation* which consists of adding a set S of elements to a graphic matroid to produce a new binary matroid and then contracting out S. It is clear that all series-parallel graphs are almost series-parallel since adding a loop edge to a series-parallel graph and contracting it leaves the original series-parallel graph.

Lemma 2.0.4. *The class $\mathcal{S}$ of almost series-parallel graphs is closed under taking minors.*

Proof. As $G \in \mathcal{S}$, there is an edge e such that $(G+e)/e$ is series-parallel. Consider the graph $G+e$ and suppose $f \in E(G)$. Then clearly $(G+e)/e\backslash f = ((G\backslash f)+e)/e$. As $(G+e)/e$ is series-parallel, so is $(G+e)/e\backslash f$. Hence $G\backslash f \in \mathcal{S}$. On the other hand, $(G+e)/e/f = ((G/f)+e)/e$ where we observe that if e and f have the same ends, then e is added as a loop to G/f. In all cases, $((G/f)+e)/e$ is series-parallel, so G/f is series-parallel. We now show that for any vertex $v \in V(G)$, its deletion $G-v$ is almost series-parallel. If v is not incident with e, then $((G+e)/e)-v = ((G-v)+e)/e$ is a subgraph of $(G+e)/e$, which is a series-parallel graph, and thus $G-v$ is almost series-parallel. If v is incident with e, then $G-v$ is a subgraph of $((G+e)/e)$, which is a series-parallel graph, and any series-parallel graph is almost series-parallel. Hence, $\mathcal{S}$ is minor-closed. $\square$

Since $\mathcal{S}$ is a minor-closed class, Robertson and Seymour's Graph Minors Theorem implies $\mathcal{S}$ has a finite number of excluded minors. In this chapter, we find the full list of excluded minors for the graphs that are almost series-parallel.

2.1 Main Results

The next result, the main theorem of the chapter, gives the excluded minors for $\mathcal{S}$. These excluded minors are shown in Figure 2.1.

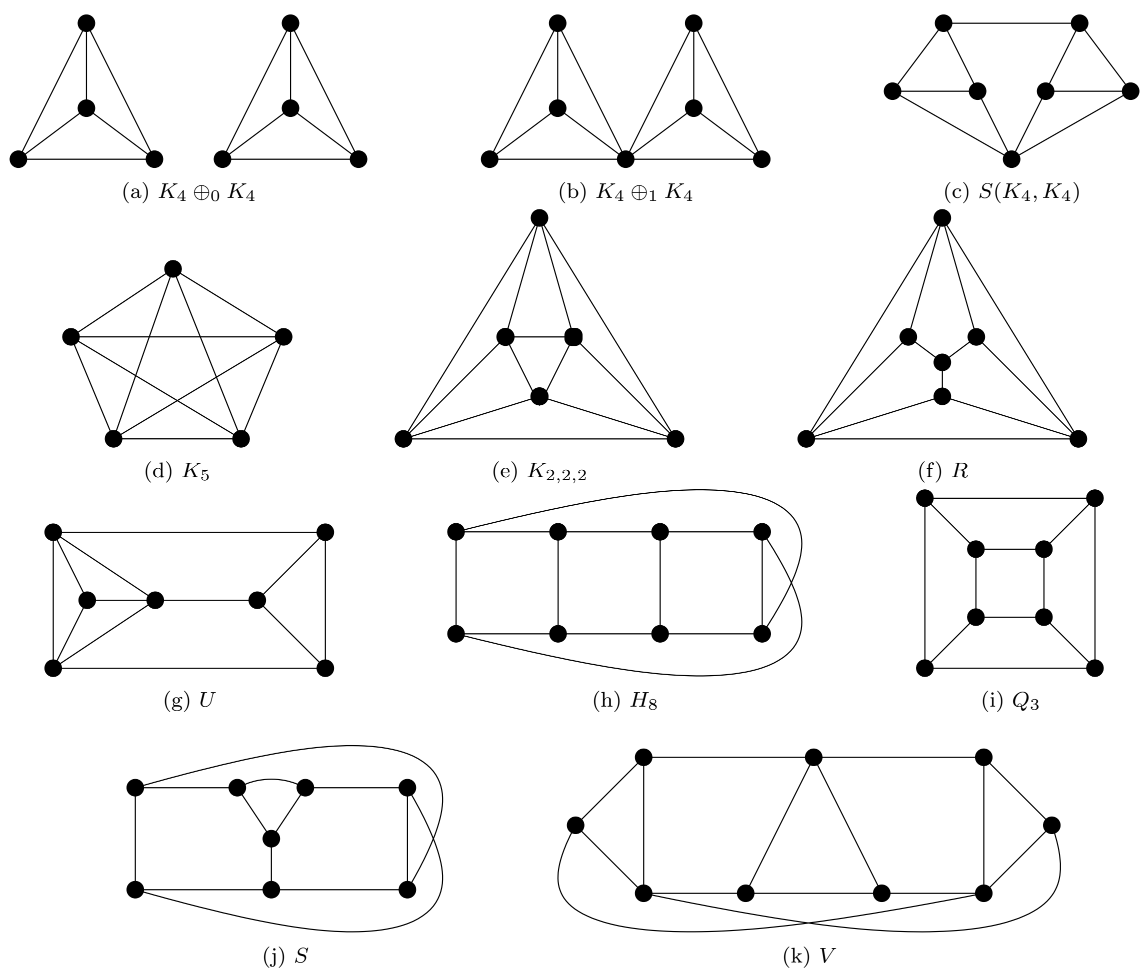


FIGURE 2.1: Excluded Minors for $\mathcal{S}$

Theorem 2.1.1. *The excluded minors for the class of almost series-parallel graphs are the following* 11 *graphs:* $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $S(K_4, K_4)$, K_5, $K_{2,2,2}$, R, U, H_8, Q_3, S, *and* V.

Clearly every excluded minor for $\mathcal{S}$ is a simple graph with no isolated vertices. To prove the theorem, we divide the argument into cases based on the vertex connectivity of an excluded minor G. When the vertex connectivity is not equal to three the excluded minors are fairly easy to determine. Most of the work arises when the vertex connectivity is three. In that case, the argument breaks into two main parts: either $\kappa(G\backslash e) = 3$ for some edge e, or G is minimally 3-connected, that is, $\kappa(G\backslash e) = 2$ for every edge e.

2.2 Preliminaries

In this section, we introduce some more terminology and results that will be used throughout this dissertation. Much of what we introduce here has to do with the connectivity of a graph G and various decompositions of a graph.

A *graph-labelled tree* [13] of a 2-connected loopless graph G is a tree T with vertex set $\{G_1, G_2, \ldots, G_k\}$ for some positive integer k such that

(i) each G_i is a 3-connected simple graph, a cycle, or a set of parallel edges;

(ii) if G_{j_1} and G_{j_2} are joined by an edge e_i of T, then $E(G_{j_1}) \cap E(G_{j_2}) = \{e_i\}$ and $\{e_i\}$ is not a bridge of G_{j_1} or G_{j_2}; and

(iii) if G_{j_1} and G_{j_2} are non-adjacent, then $E(G_{j_1}) \cap E(G_{j_2})$ is empty.

Let e be an edge of a graph-labelled tree T and suppose that e joins vertices labelled by N_1 and N_2. Suppose that we contract the edge e of the tree T and relabel by $N_1 \oplus_2 N_2$ the vertex that results by identifying the endpoints of e, leaving all other edges and vertex labels unchanged. Then it is not difficult to see that we retain a graph-labelled tree, and it is natural to denote this tree by T/e. This process can be repeated, and since the operation of 2-sum is associative, for every subset $\{e_1, e_2, \ldots, e_m\}$ of $E(T)$, the graph-labelled tree $T/e_1, e_2, \ldots, e_m$ is well-defined.

A 2-*sum decomposition* [19] is a graph-labelled tree T such that if $V(T) = \{G_1, G_2, \ldots, G_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$, then

(i) $E(G) = (E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)) - \{e_1, e_2, \ldots e_{k-1}\}$;

(ii) $|E(G_i)| \geq 3$ for all i unless $|E(G)| < 3$, in which case $k = 1$ and $G_1 = G$; and

(iii) G is the graph that labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

The following result of Cunningham and Edmonds [4] and Seymour [16] describes how every 2-connected graph can be written in terms of 2-sums of 3-connected graphs.

Proposition 2.2.1. *Let G be a 2-connected graph. Then G has a 2-sum decomposition T in which every vertex label is 3-connected, a cycle, or a parallel class.*

The next result of Tutte [19] gives a unique 2-sum decomposition called the *canonical* 2-*sum decomposition.*

Proposition 2.2.2. *Let G be a 2-connected loopless graph. Then G has a 2-sum decomposition T in which every vertex label is 3-connected, a cycle, or a bond, and there are no two adjacent vertices that are both labelled by cycles or are both labelled by bonds. Moreover, T is unique within relabeling of its edges.*

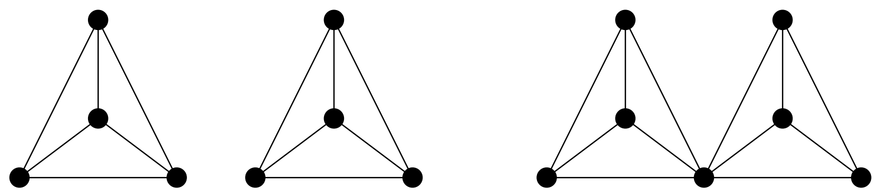
The following result of Tutte [20] plays a vital role in this dissertation.

Lemma 2.2.3 (Tutte's Triangle Lemma)**.** *Let M be a 3-connected matroid with at least four elements, and suppose that $\{e, f, g\}$ is a triad of M such that neither M/e nor M/f is 3-connected. Then M has a triangle that contains e and exactly one of f and g.*

2.3 Excluded Minors for the Class $\mathcal{S}$ with $\kappa(G) \neq 3$

Finding the excluded minors for vertex connectivity not three is straightforward. We consider separately when the vertex connectivity is zero or one, when it is two, and when it is at least four.

Proposition 2.3.1. *Let G be a simple graph with $\kappa(G) \in \{0, 1\}$. Then G is an excluded minor for the class $\mathcal{S}$ if and only if $G \cong K_4 \oplus_0 K_4$ or $K_4 \oplus_1 K_4$.*



(a) $K_4 \oplus_0 K_4$ (b) $K_4 \oplus_1 K_4$

FIGURE 2.2: $K_4 \oplus_0 K_4$ and $K_4 \oplus_1 K_4$

Proof. It is easy to check that both $K_4 \oplus_0 K_4$ and $K_4 \oplus_1 K_4$ are excluded minors for $\mathcal{S}$. Conversely, let G be an excluded minor for $\mathcal{S}$. Clearly $G \cong G_1 \oplus_{\kappa(G)} G_2$ for some graphs G_1 and G_2 each having at least one edge. If both G_1 and G_2 have a K_4-minor, then it is not difficult to check that G has $K_4 \oplus_{\kappa(G)} K_4$ as a minor. Hence we may assume that G_1 has no K_4-minor, but G_2 does have such a minor. Choose an edge f in G_1. Then $G \backslash f \in \mathcal{S}$ so there is an edge e that can be added to and contracted from $G \backslash f$ to leave a series-parallel graph. To destroy the K_4-minor in G_2, the edge e must join two vertices of G_2. Then $(G + e)/e$ is series-parallel, so $G \in \mathcal{S}$; a contradiction. □

Proposition 2.3.2. *Let G be a simple graph with $\kappa(G) = 2$. Then G is an excluded minor for the class $\mathcal{S}$ if and only if $G \cong S(K_4, K_4)$.*

Proof. It is straightforward to check that $S(K_4, K_4)$ is an excluded minor for $\mathcal{S}$. Now let G be an excluded minor for $\mathcal{S}$. As $\kappa(G) = 2$, we can write $G = G_1 \oplus_2 G_2$ where the edge e is the basepoint of the 2-sum, e is not a loop, a and b are the vertex ends of e, and G_1 and

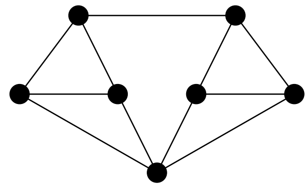


FIGURE 2.3: $S(K_4, K_4)$

G_2 are 2-connected. Since G is an excluded minor, $((G_1 \oplus_2 G_2) + e)/e$ is not series-parallel so either G_1/e or G_2/e has K_4 as a minor. Suppose G_1/e has K_4 as a minor. We show next that K_4 is also a minor of G_2. Suppose not. Then G_2 is series-parallel and has a vertex w of degree two in G. To see this, observe that the only possible non-trivial parallel class of G_2 involves e. As G is simple and G_2 is series-parallel, G_2 is forced to have a degree-2 vertex not incident with e. Let g be an edge incident with w and $G' = G/g$. Then G' is in $\mathcal{S}$ and so has an edge f so that $(G' + f)/f$ has no K_4-minor. The construction of G' means that $(G + f)/f$ has no K_4-minor; a contradiction. Thus G_2 does indeed have a K_4-minor.

We now show, since G_1/e and G_2 both have K_4 as a minor, that G contains and is isomorphic to $S(K_4, K_4)$. We first show that there is a K_4-minor in G_2 using the edge e. Consider the graphic matroid $M(G_2)$ associated with G_2. By Theorem 1.11.3, since $M(G)$ is 2-connected, $M(G)$ has a proper K_4-minor using e, otherwise $G = K_4$. Therefore, the graph G has a K_4-minor using the edge e.

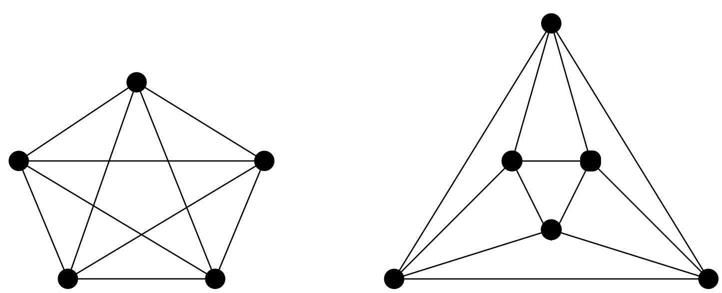
Since G_1/e has a K_4-minor, $(G_1/e)/X\backslash Y = K_4$ for some subsets X and Y of $E(G_1) - \{e\}$. By uncontracting the edge e from the K_4-minor, the graph $G_1/X\backslash Y$ has e as either a subdivided edge of K_4 or a pendant edge, that is, an edge adjacent to a vertex of degree one of K_4. In the first case, if e is a subdivided edge, then G contains $S(K_4, K_4)$, as desired.

We will show in what follows that e cannot be a pendant edge. Suppose that e is a pendant edge of $G_1/X\backslash Y$. Since every vertex of G_1 becomes a vertex of $(G_1/e)/X\backslash Y$, we label the vertices of $G_1/X\backslash Y$ with four labels: 1, 2, 3, and 4, where each label corresponds to a distinct vertex of $(G_1/e)/X\backslash Y = K_4$ to which each vertex of G_1 is identified in the contraction of the set X. Let the graph G also have the same corresponding labels for the vertices of $V(G_1)$ in $V(G)$. Since a and b are identified as a single composite vertex in $(G_1/e)/X\backslash Y$, they have the same label, say 1. Since e is a pendant edge of $G_1/X\backslash Y$, one of a or b, say a, is not adjacent to any vertices labelled by 2, 3, or 4. However, since the original graph is 2-connected, a is adjacent to at least two vertices, both of which are labelled by 1. So there is at least one edge f, not the same as e, incident with a and also to a vertex labelled by 1. Let X' and Y' be subsets of $E(G_1)$ such that $X' = X - \{e\}$ and $Y' = Y - \{f\}$. Then the graph $G_1/X'\backslash Y'$ is $K_4 \oplus_1 C_2$ where C_2 is a 2-cycle and $E(C_2) = \{e, f'\}$ for some edge f' that runs between a and b in $G_1/X'\backslash Y'$. Now, $G_1/X'\backslash Y'$ has a K_4-minor. Also, let H be the graph $G/X'\backslash Y'$. Then G_2 remains unaltered in H and $G_2\backslash e + f'$ has a K_4-minor, which gives $K_4 \oplus_1 K_4$ as a proper minor, a contradiction.

Therefore G is an excluded minor for $\mathcal{S}$ if and only if $G \cong S(K_4, K_4)$. $\square$

To find the excluded minors of connectivity at least four, we use the following result of Halin and Jung [9].

Lemma 2.3.3. *If G is a simple graph with minimum degree 4, then G contains K_5 or $K_{2,2,2}$ as a minor.*



(a) K_5 (b) $K_{2,2,2}$

FIGURE 2.4: K_5 and $K_{2,2,2}$

It is straightforward to check that both K_5 and $K_{2,2,2}$ are excluded minors for the class $\mathcal{S}$. Combining this with the last lemma, we immediately obtain the following corollary from which it follows that $\mathcal{S}$ contains no 4-connected graphs.

Corollary 2.3.4. *Let G be a simple graph with $\kappa(G) \geq 4$. Then G is an excluded minor for $\mathcal{S}$ if and only if $G \cong K_5$ or $K_{2,2,2}$.*

2.4 Members of the Class $\mathcal{S}$ with $\kappa(G) = 2$

In this section, we describe the structure of almost series-parallel graphs with vertex connectivity two.

Proposition 2.4.1. *Let G be a simple graph that is a member of $\mathcal{S}$ and suppose $\kappa(G) = 2$. Then either G is series-parallel, or G can be constructed as follows.*

(i) Take the parallel connection with basepoint p of graphs $G_1, G_2, G_3, \ldots, G_n$ where, for each $i \in [1, n]$, the graph G_i is 3-connected and simple, and if $i \geq 2$, then G_i/p is series-parallel.

(ii) Possibly delete the edge p.

(iii) At each edge of the resulting graph, attach via 2-sum a 2-connected series-parallel graph or a parallel class.

Moreover, every graph constructed using (i)-(iii) is an almost series-parallel graph with connectivity two.

Proof. As $\kappa(G) = 2$, consider the Cunningham and Edmond's canonical 2-sum decomposition T of G letting $G_1, G_2, \ldots, G_n$ label the vertices of T where by Proposition 2.2.1 each G_i is a 3-connected graph, a cycle, or a parallel class.

If some G_i is 3-connected, then it has a minor isomorphic to K_4 by Tutte's Wheels Theorem 1.10.2. Moreover, the edge e that one can add to the graph G and contract out to leave a series-parallel graph must be added so that G_i/e has no K_4-minor. Suppose two 3-connected members G_m and G_n of the 2-sum decomposition have basepoints e_m and e_n. Since $G \in \mathcal{S}$ there is one edge that one can add to the graph and then contract out to leave a graph with no K_4-minor. Since $(G + e)/e$ has no K_4-minor, and G_m has a K_4-minor, the edge e meets two vertices of G_m. Similarly e meets two vertices of G_n. However, the only edge meeting both two vertices of G_m and two vertices of G_n are the basepoints e_m and e_n, respectively. This implies that every 3-connected G_i has the same basepoint, and the edge e that one must add to the graph G_i and contract out to destroy the K_4 in each 3-connected G_i is in a 2-cycle with e. If there is more than one 3-connected G_i, then because $G \in \mathcal{S}$, for every 3-connected G_j, the contraction of the basepoint is series-parallel.

Every non-3-connected G_i is a cycle or a parallel class. Notice that a cycle is series-parallel; the 2-sum of two cycles is series-parallel; the 2-sum of a series-parallel graph and a parallel class is series-parallel; and finally, that the 2-sum of two series-parallel graphs is series-parallel. Also, when a series-parallel graph is 2-summed, it creates no new K_4-minors and the same edge e that one can add to G_i to destroy each K_4 still leaves a graph with no K_4-minor. So if G is 2-connected, then it can be formed by the process stated above.

Now we show that every graph constructed by the process stated in (i)-(iii) is almost series-parallel. The graph formed by taking the parallel connection in (i) can be seen to be almost series-parallel by adding in and contracting an edge f parallel to the basepoint p. In $G \backslash e$, the addition and contraction of the edge f still leaves a series-parallel graph, so applying (i) and (ii) produces a graph in $\mathcal{S}$. In the contraction G/f, attaching a series-parallel graph to a series-parallel graph via 2-sum is again series-parallel, so we are able to attach a series-parallel graph to any edge of G except e via 2-sum. If a series-parallel graph H is attached via 2-sum to the edge e, then G/f is the 1-sum of a series-parallel graph, a loop on e, and H/e, a minor of a series-parallel graph. Thus applying (i)-(iii) produces a graph in $\mathcal{S}$. □

2.5 Members of the Class $\mathcal{S}$ with $\kappa(G) = 3$

In this section, we use the following elementary result. In this result, we begin with three distinct vertices of a graph G labelled a, b, and c, and we consider a minor of G. When an edge e incident with exactly one member d of $\{a, b, c\}$ is contracted, we label the vertex that results by identifying the ends of e by d.

Lemma 2.5.1. *If G is a simple 2-connected graph and $a, b, c \in V(G)$, then G has a minor that is a cycle through a, b, and c.*

Proof. Consider distinct edges e_a and e_b incident with a and b, respectively. Since G is 2-connected, there is a cycle C containing both e_a and e_b. Suppose the vertex c is not in this cycle. By Menger's Theorem, there are two paths, P_1 and P_2, from c to $V(C)$ that have only the vertex c in common such that each contains only a single vertex of $V(C)$. If P_1 and P_2 meet $V(C)$ at both a and b, then G clearly contains a cycle through a, b, and c. Now we may assume that some P_i meets $V(C)$ in a vertex other than a or b. If we contract this path, one of the vertices of C is relabeled c and hence, in this minor, C is a cycle through a, b, and c. □

The following lemma is the core of the theorem that we apply when finding the excluded minors with vertex connectivity three.

Lemma 2.5.2. *Let G be a simple graph with $\kappa(G) = 3$ such that G is a member of $\mathcal{S}$. Then G has two vertices u and v such that $G - \{u, v\}$ is a tree. Moreover, u and v are adjacent to all leaves of the tree.*

Proof. Suppose first that $|V(G)| = 4$. Since $\kappa(G) = 3$, we must have $G \cong K_4$. In that case, the lemma clearly holds. Thus, we may assume that $V(G) \geq 5$. If $(G+e)/e$ is 3-connected for all possible edges e, and $|V((G+e)/e)|$ is at least four, then each such graph $(G+e)/e$ has a K_4-minor; so $G \notin \mathcal{S}$. Assume $(G+e)/e$ is not 3-connected for some new edge e. We know that e joins two vertices u and v of a 3-vertex cut $\{u, v, w\}$ of G. Let t label the vertex of $(G+e)/e$ that results by identifying u and v. Now, $(G+e)/e$ is 2-connected but not 3-connected and we consider the block-graph T of $((G+e)/e) - \{t\}$, that is, of $G - \{u, v\}$. Since $G - \{u, v\}$ has w as a cut vertex, this block graph has at least two leaves.

No block of $G-\{u,v\}$ is 3-connected or $G-\{u,v\}$ would contain K_4 as a minor. We now show that each of u and v is adjacent to some vertex in each end block of $G-\{u,v\}$. If an end block is simply an edge, then one end of that edge has degree one in $G-\{u,v\}$, so that end is adjacent to both u and v in G since all vertices must have degree at least three in G. Suppose an end block B is 2-connected and is not adjacent to some vertex y of $\{u,v\}$. Let x be the vertex in $\{u,v\}-y$. Let z be the vertex of T connected to B. Then $\{x,z\}$ is a vertex cut of G, which cannot be since G is 3-connected. We conclude that each 2-connected end block of $G-\{u,v\}$ is adjacent to both u and v.

Let B_i be a block of the graph of $G-\{u,v\}$. Suppose B_i is 2-connected. Then B_i has at least three vertices. Suppose a and b are distinct cutvertices of $G-\{u,v\}$ belonging to B_i. There are paths P_a and P_b that begin at a and b, that end in $\{u,v\}$, that meet $V(B_i)$ in $\{a\}$ and $\{b\}$, that meet $\{u,v\}$ in a single vertex, and that meet each other in a subset of $\{u,v\}$. Since $\{a,b\}$ is not a vertex cut of G, there is a vertex c of $V(B_i)-\{a,b\}$ such that there is an internally disjoint path from c to $\{u,v\}$ which is internally disjoint from P_a and P_b. Call this path P_c. Note that the path may have length one. Now, B_i has a minor which is a cycle through a, b, and c. Thus, we have a K_4-minor using this cycle and the paths P_a, P_b, and P_c in $(G+e)/e$. So $G\notin\mathcal{S}$; a contradiction.

Suppose B_i contains a single cut-vertex, say a, of $G-\{u,v\}$. Then G has a path P_a that begins at a, has no other vertices in common with $V(B_i)$, and ends at u or v. As G is 3-connected, there are distinct vertices b and c of $V(B_i)-\{a\}$, each of which is adjacent to u or v. Let these edges be the paths P_b and P_c, respectively. Then again we can use the cycle through a, b, and c in a minor of B_i and the paths P_a, P_b, and P_c to get a K_4-minor in $(G+e)/e$, so $G\notin\mathcal{S}$; a contradiction.

We conclude from the last two paragraphs that each block B_i of $G-\{u,v\}$ consists of a single edge and hence $G-\{u,v\}$ is, in fact, a tree. □

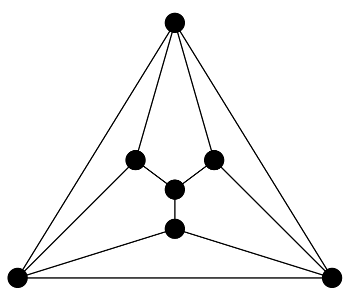
Theorem 2.5.3. *Let G be a simple graph with $\kappa(G) = 3$. Then G has two vertices u and v such that $G - \{u, v\}$ is a tree if and only if $G \in \mathcal{S}$.*

Proof. By the last lemma, if $G \in \mathcal{S}$, it has two such vertices u and v. Conversely, assume G has two such vertices u and v and consider the graph $(G + f)/f$ where f is an edge joining u and v. Suppose this graph has a K_4-minor. Removing the vertex that results from identifying u and v gives a graph isomorphic to $G - \{u, v\}$, which has no cycles. Since K_4 has no vertex whose removal has no cycles, we deduce that $(G + f)/f$ has no K_4-minor. Hence $G \in \mathcal{S}$. □

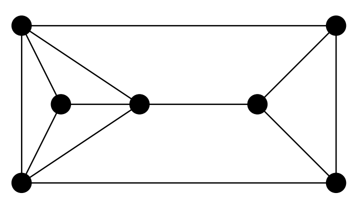
2.6 Excluded Minors for the Class $\mathcal{S}$ with $\kappa(G) = 3$

To find the excluded minors G for the class $\mathcal{S}$ with $\kappa(G) = 3$ such that there is some edge $e \in E(G)$ where $G\backslash e$ is 3-connected, we use the Theorem 2.5.3. The argument breaks into many cases but each case is straightforward.

Theorem 2.6.1. *Let G be a simple graph with $\kappa(G) = 3$ that is an excluded minor for $\mathcal{S}$ such that $G\backslash e$ is 3-connected for some edge $e \in E(G)$. Then G is one of the graphs shown in Figure 2.5.*



(a) R



(b) U

FIGURE 2.5: R and U

Proof. Since G is an excluded minor, $G\backslash e$ is a member of $\mathcal{S}$. By Lemma 2.5.2, $G\backslash e$ has two vertices u and v such that $(G\backslash e) - \{u, v\}$ is a tree T. We distinguish the cases when T is a path, when T has exactly three degree-one vertices, and when T has at least four degree-one vertices. In each case, we find that G is a member of $\mathcal{S}$, or G contains and is therefore isomorphic to one of R, U, K_5, $K_{2,2,2}$, or $S(K_4, K_4)$. But G cannot be any of these last three graphs as $\kappa(G) = 3$.

Observe that e joins two vertices of T. Otherwise $G - \{u, v\} = T$ and so $G \in \mathcal{S}$; a contradiction. Also $T + e$ has a cycle C as a subgraph; otherwise $G - \{u, v\}$ is cycle-free, a contradiction.

Lemma 2.6.2. *If T is a path, then $G \cong K_{2,2,2}$, R, or U.*

Proof. Since T is a path, exactly two vertices of $T+e$ have degree three. Each such degree-3 vertex meets a unique maximal path in $T + e$ that contains no edge of C. We call this path a *tail* of C.

2.6.3. *If T has two tails, then $G \cong R$.*

Let the vertices of the path T be in order $l_j, l_{j-1}, \ldots, l_2, l_1, r, s_1, s_2, \ldots, s_n, t, m_1, m_2, \ldots, m_k$ where $e = rt$ and the vertices of the C are $r, s_1, s_2, \ldots, s_n, t$. Since $G \backslash e$ is 3-connected, both u and v are adjacent to l_j and m_k, and we may assume that u is adjacent to some s_i where $1 \leq i \leq n$. Partition the vertices of T into three sets, the set $L = \{l_{j-1}, l_{j-2}, \ldots, l_2, l_1, r\}$, the set $M = \{m_1, m_2, \ldots, m_{k-1}, t\}$, and the set $S = \{s_1, \ldots, s_n\}$.

If v is adjacent to both a vertex in the set L and a vertex in the set M, then G contains and is isomorphic to the excluded minor R. Now, suppose v is adjacent to no vertex in $L \cup M$. Since $G \backslash e$ is 3-connected, u is adjacent to r and t. Notice that v is adjacent to no s_i for $i = 1, \ldots, n$; otherwise G contains R as a proper minor. However, if v is adjacent to no s_i for $i = 1, \ldots, n$, then $G - \{u, r\}$ has no cycles, a contradiction by Theorem 2.5.3. We may now assume that v is adjacent to exactly one of L and M, say L. This implies that u is adjacent to every vertex in M. If there is an edge from v to a vertex of S, then G contains the excluded minor K_5 as can be seen by contracting vm_k and contracting $r, l_1, l_2, \ldots, l_k$ to a single vertex. Hence, we may assume vs_i is not an edge of G for any $i = 1, \ldots, n$. If there are two or more edges from v to L, then G contains the excluded minor $S(K_4, K_4)$. So v is adjacent only to x_m, z_n, to exactly one vertex of L, and possibly u. If v is adjacent to only r, then $G - \{r, u\}$ is a tree, a contradiction. So v is not adjacent to r. Also, u is adjacent to r, and v is adjacent to some vertex in $\{l_1, l_2, \ldots, l_{j-1}\}$, which gives a proper $S(K_4, K_4)$-minor, a contradiction. Hence, 2.6.3 holds.

We now distinguish three main cases based on the length of the unique cycle C in $T + e$ containing e: either C is a 3-cycle, C is a 4-cycle, or C is a cycle of length at least 5.

Suppose first that C is a 3-cycle. If C has no tails, then G has exactly five vertices and, since G is an excluded minor for $\mathcal{S}$, it follows that G is isomorphic to K_5. Suppose C has exactly one tail. Let the vertices of the path T be, in order, $r, s, t, l_1, l_2, \ldots, l_k$, where $e = rt$. Since $G \notin \mathcal{S}$, Theorem 2.5.3 implies that the graph $G - \{r, l_1\}$ contains a cycle. Also, since G is 3-connected, u and v are adjacent to both r and l_k. Assume the tail has length one. As $G - \{u, t\}$ contains a cycle as a minor, vs is an edge of G. By symmetry, us is an edge of G. Since $G \backslash e$ is 3-connected, t has degree at least three in $G \backslash e$, so we may assume ut is an edge of G. Then $G/vl_1 \cong K_5$; a contradiction. Assume the tail has length at least two. Let the vertices of C be r, s and t where $e = rt$ and the vertices of $l_1, l_2 \ldots, l_k$ are the vertices of the tail. Without loss of generality, we may assume that u is adjacent to s. Since $G \backslash e$ is 3-connected, u or v is adjacent to t. Also, the vertices v and s are not adjacent, otherwise G has a K_5-minor. If none of $l_1, l_2, \ldots, l_{k-1}$ are adjacent to v, then $G - \{u, s\}$ has no cycles; a contradiction. We deduce that some l_i with $1 \leq i \leq k - 1$ is adjacent to v. Hence, G has as a minor one of the graphs shown in Figure 2.6 after some relabeling the l_i.

The graph in (a) has U as a minor. The graph in (b) has $S(K_4, K_4)$ as a minor, as can be seen by deleting the edge joining v and l_1.

Now, suppose C is a 4-cycle. If C has no tails, notice that u and v are not adjacent, otherwise G/uv contains a K_5-minor. Then G is isomorphic to a subgraph of $K_{2,2,2}$. Hence, G is isomorphic to $K_{2,2,2}$. If C is a 4-cycle with one tail, then let the vertices of the path T

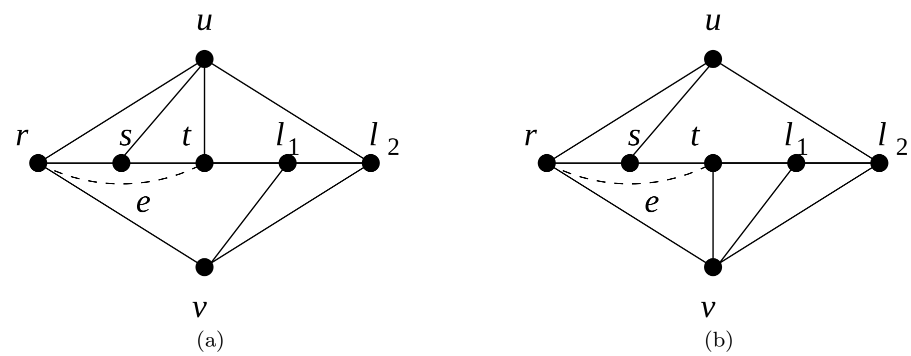


FIGURE 2.6: e lies in a 3-cycle with one tail of length two or more.

be in order $l_1, l_2, l_3, l_4, \ldots l_k$, where $e = l_1 l_4$. Since $G \backslash e$ is 3-connected, u and v are adjacent to both l_1 and l_k and we may assume u is adjacent to l_4. Also, since G is 3-connected, l_2 and l_3 have degree at least three in G so each is adjacent to either u or v. This implies that G contains contains one of the graphs shown in Figure 2.8 as a spanning subgraph, where a bold edge represents a path.

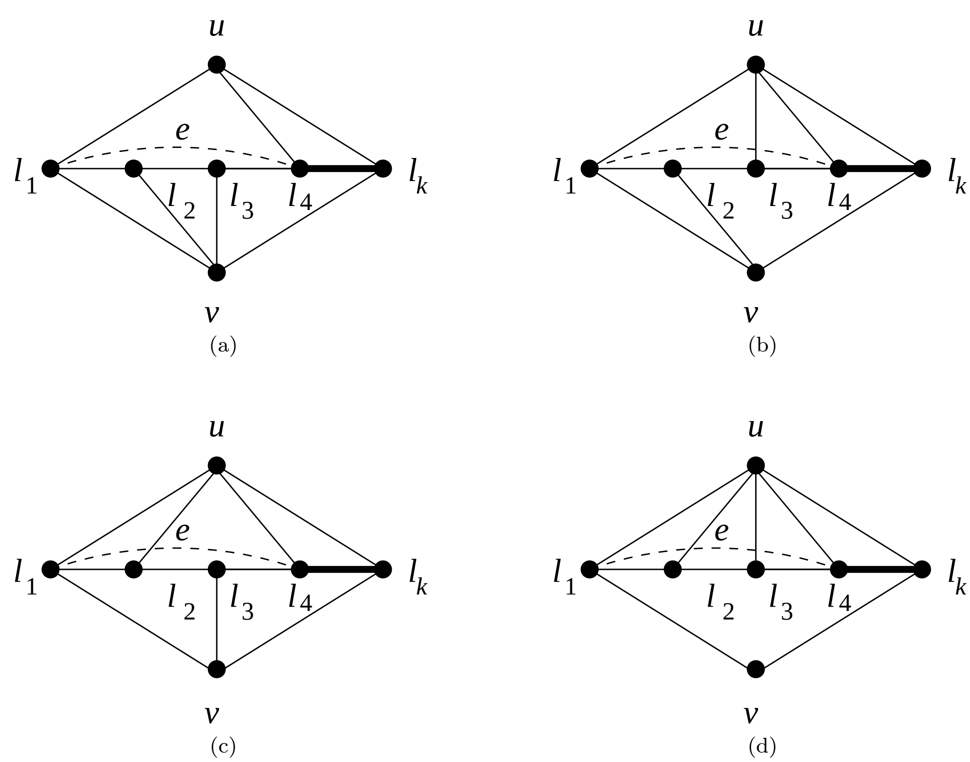


FIGURE 2.7: e lies in a 4-cycle with one tail. A bold edge represents a path.

In case (a), the vertex u is adjacent to some other vertex; otherwise $G - \{v, l_4\}$ is a tree and $G \in \mathcal{S}$, a contradiction. Now u is adjacent to l_2, l_3 or an internal vertex of the path from l_4 to l_5. If u is adjacent to l_2 or l_3, then G contains a proper K_5-minor. If u is adjacent to a vertex on the path from l_4 to l_5, then G contains an $S(K_4, K_4)$-minor. We deduce that (a) does not arise.

In cases (b) and (c), we see by contracting the edges from l_2 to l_3 and from v to l_5 that the graph G contains a K_5-minor; a contradiction.

Consider case (d). Since the deletion of u and l_1 from this graph leaves a tree, v is adjacent to a vertex on the path from l_2 to l_4, or an internal vertex of the path from l_4 to l_k. If v is adjacent to l_2 or l_3, then G has a K_5-minor as in (b) or (c). If v is adjacent to a vertex other than l_5 on the path from l_4 to l_k, then G contains the excluded minor U as in Figure 2.6 (a) when C is a 3-cycle with one tail. Therefore, the only vertex v can only be adjacent to u, l_1, l_k, and l_4 and $G - \{u, l_4\}$ is a tree. Hence $G \in \mathcal{S}$; a contradiction.

Next suppose C is a cycle of length at least 5. Label the vertices of C in order by $l_1, l_2, \ldots, l_k$, where $e = l_1 l_j$. There is at most one tail. If there is a tail, label the vertices of the tail in order by $t_1, t_2, \ldots, t_n$, where l_k is adjacent to t_1. By our labeling, the path T has the vertices in order $l_1, l_2, \ldots, l_k, t_1, t_2, \ldots, t_n$ if there is a tail, and $l_1, l_2, \ldots, l_k$ if there is no tail. Since $G \backslash e$ is 3-connected, u and v are adjacent to the ends of the path T. We first eliminate the case when every vertex in $l_2, \ldots l_{k-1}$ is adjacent to only u. If this occurs, then because $G - \{u, l_k\}$ has a cycle, there is a tail and v is adjacent to a vertex t_m of the tail for some $m = 1, \ldots n$. Now, G contains as a proper minor U, which can be seen by contracting the cycle to have length three and contracting the tail to have length two. Hence, every vertex in $l_2, \ldots, l_{k-1}$ is not adjacent only to u, and by symmetry is not adjacent only to v. Combining this with the fact that every vertex of $\{l_2, \ldots, l_k\}$ has degree at least three since G is 3-connected, we have that by symmetry at least two vertices of $\{l_1, \ldots, l_k\}$ are adjacent to u, and at least one is adjacent to v. Contract the path from l_k to t_n, so the resulting graph has u and v adjacent to l_k. Hence, up to symmetry, we have one of the following graphs shown in Figure 2.8 as a subgraph, where a bold edge represents a path.

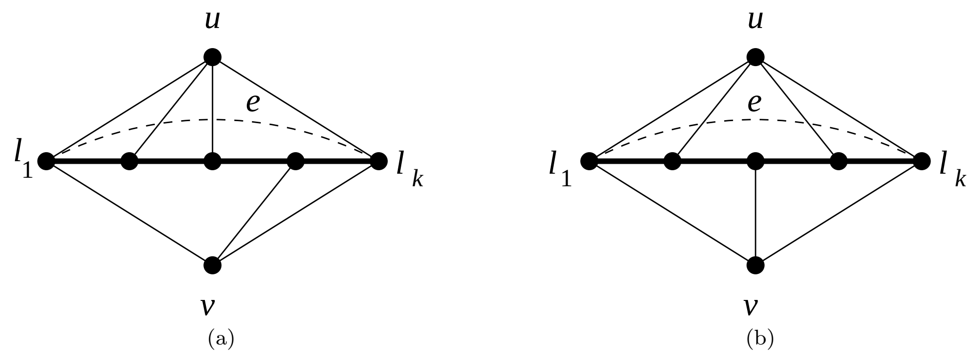


FIGURE 2.8: e lies in a cycle of length at least 5. A bold edge represents a path.

The graph in (a) has U as a proper minor, which can be seen by deleting the edge ul_k, and contracting each bold edge down to a path of length one. The graph in (b) is isomorphic to the excluded minor R.

□

Lemma 2.6.4. *Suppose T has exactly three degree-one vertices. Then G is not an excluded minor for $\mathcal{S}$.*

Proof. Since T has exactly three degree-one vertices, T has exactly one vertex of degree three, call it r. Let l_1, l_2, and l_3 be the degree-one vertices of T. By adding the edge e back into the graph $T \cong (G\backslash e) - \{u, v\}$, we see that $G - \{u, v\}$ contains one of the graphs shown in Figure 2.9 as a spanning subgraph.

Let s and t be the vertices as shown in Figure 2.9. Since $G\backslash e$ is 3-connected, u and v are adjacent to l_1, l_2, and l_3.

Suppose (a) occurs. One of u and v, say u, is adjacent to s. If there is an edge from v incident with an internal vertex on the path from r to l_1, then, by contracting the edge from v to l_2, we see that G has an $S(K_4, K_4)$-minor. If there is an edge from v to an internal vertex of the path from r to l_3, then by contracting the path from l_1 to r and the edge from

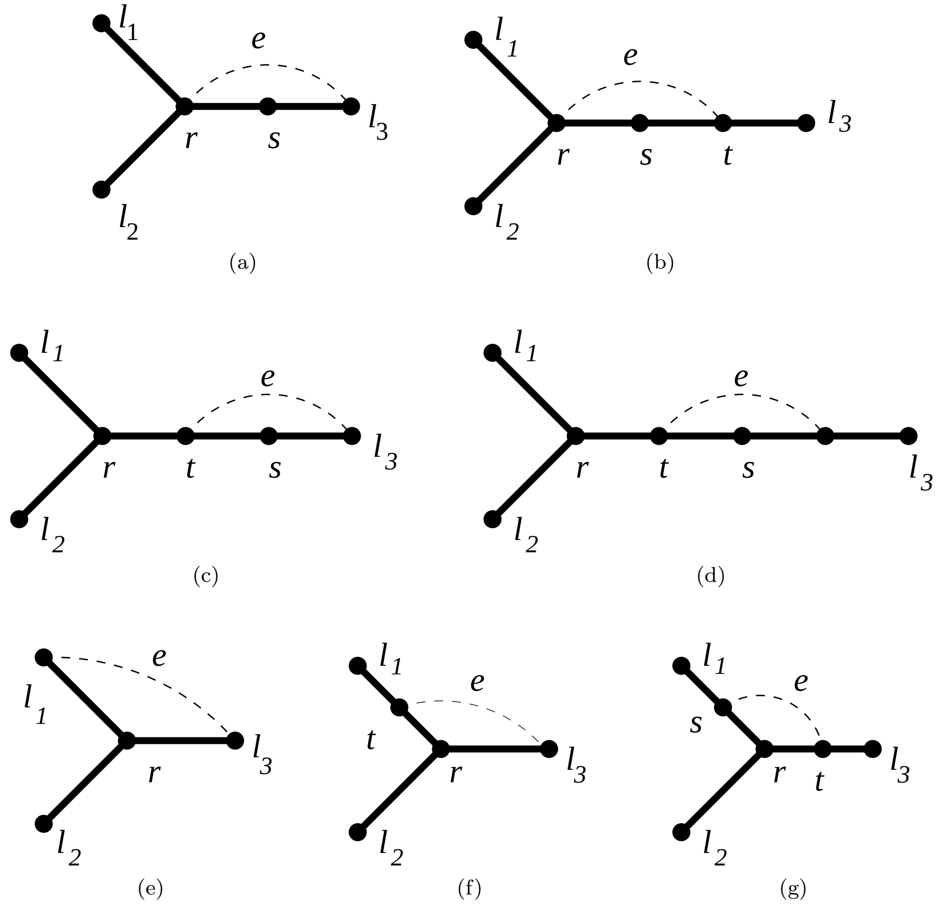


FIGURE 2.9: T has exactly three degree-one vertices. A bold edge represents a path.

l_2 to u, we see that G has a K_5-minor. Therefore, there are no other edges from v except to u or r, so $G - \{u, r\}$ is a tree and, by Theorem 2.5.3, $G \in \mathcal{S}$; a contradiction. Hence, (a) does not occur.

Next, suppose that (b) occurs. Without loss of generality, u is adjacent to s. If there is an edge from v to an internal vertex of the path from t to l_3, then, G has an $S(K_4, K_4)$-minor. This can be seen by deleting all edges of the path in T from r to l_2 and contracting the edges from l_1 to u and from l_2 to v. If there is an edge from v to an internal vertex on the path from r to l_1, from v to an internal vertex on the path from r to l_2, or from v to an internal vertex on the path from r to l_t, then G has an $S(K_4, K_4)$-minor or K_5-minor as in (a). So v is adjacent to only the vertices l_1, l_2, l_3, and possibly the vertex u. Hence, $G - \{u, r\}$ is a tree and, by Theorem 2.5.3, $G \in \mathcal{S}$; a contradiction. Thus, (b) does not occur.

Next assume that (c) occurs. Without loss of generality, u is adjacent to s. Since $G\backslash e$ is 3-connected, at least one of u or v is adjacent to t. If there is an edge from v to t, then G has an $S(K_4, K_4)$-minor, which can be seen by deleting the edge from l_3 to v, deleting the path from r to t, and also contracting the path from l_2 to r down to a vertex. Hence, v is not adjacent to t, so u is adjacent to t. Now, G contains as a proper minor the excluded minor U, which can be seen by deleting the path from r to t, deleting the edge from l_1 to u, deleting the edge from l_3 to v, and contracting the edge from l_2 to v. We deduce that (c) does not arise.

By the same argument that excludes (c), we see that (d) does not occur.

Now assume that (e) occurs. Since $G - \{l_1, l_3\}$ has a cycle, we have without loss of generality that there is an edge from u to another vertex u' of T. By contracting the edge between l_2 and v and also contracting the path from u' to r to a single vertex, the graph G contains a K_5-minor, a contradiction. Hence, (e) does not occur.

Suppose (f) occurs. Since $G - \{l_1, l_3\}$ has a cycle, there is an edge from u to another vertex of T. However, this edge is not incident with r and one of: an internal vertex on the path from r to t, an internal vertex on the path from r to l_3, or an internal vertex on the path from t to l_2; otherwise G contains a K_5-minor as in case (e). Therefore, there is an edge between u and t or u and an internal vertex on the path from l_1 to t. The resulting graph has a K_5-minor, which can be seen by contracting the edge between l_1 and v, and contracting the path from r to l_2 down to a vertex. Hence, f does not occur.

If (g) occurs, then because both u and v are connected to l_1, l_2, and l_3, the graph G contains a K_5-minor; a contradiction. In every case G either either in the class $\mathcal{S}$ or contains as a proper minor an excluded minor for $\mathcal{S}$. □

Hence, we may assume T has at least four degree-one vertices.

Lemma 2.6.5. *Suppose T has at least four degree-one vertices. Then G is not an excluded minor for $\mathcal{S}$.*

Proof. Let l_1, l_2, l_3, and l_4 be degree-one vertices of T. If there is a vertex of degree at least four in T, call it r. Since $G\backslash e$ is 3-connected, both u and v are adjacent all leaves of the tree T.

2.6.6. *The edge e is incident with a vertex of degree at least three of T.*

Suppose that e is incident only with vertices of degree one or two in T. Then $G - \{u, v\}$ has as a subgraph one of the graphs shown in Figure 2.10.

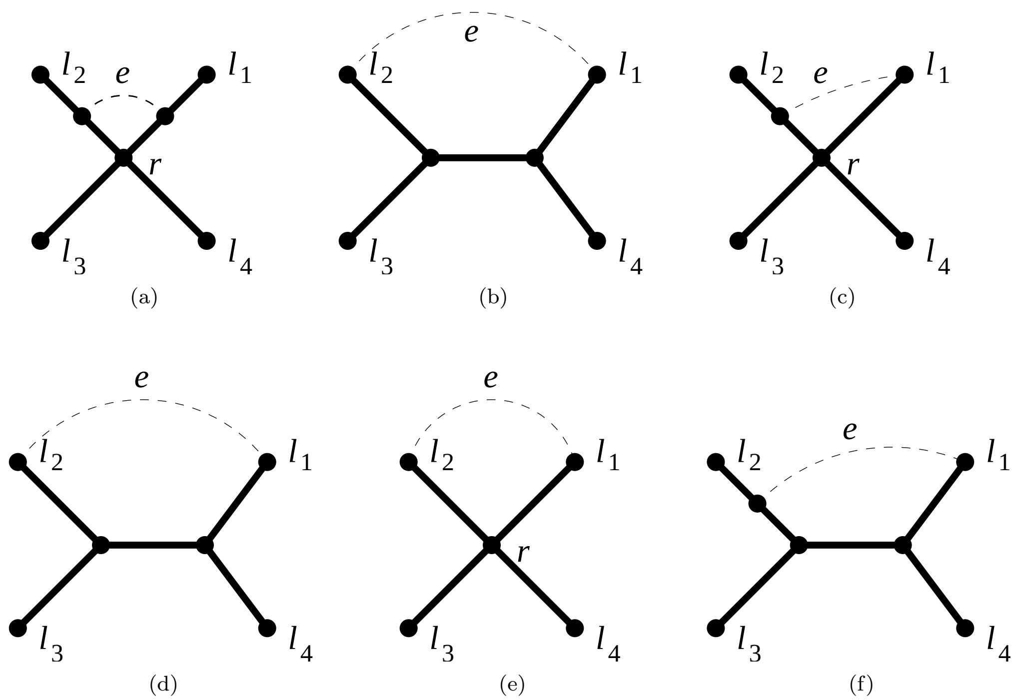


FIGURE 2.10: e is incident only with degree-2 vertices of T. A bold edge represents a path.

In each of the cases shown, G has a K_5-minor, which can be seen by contracting the edge from l_3 to u and the edge from l_3 to v; a contradiction. Hence 2.6.6 holds.

2.6.7. *At most one edge of the path P between two vertices of degree greater than two is contained in the cycle C in $G - \{u, v\}$*

Let r_1 and r_2 be vertices of degree at least three in T. Suppose two edges of a path between r_1 and r_2 are contained in the cycle C and s is a vertex, distinct from r_1 and r_2 on the path contained in the cycle in $G - \{u, v\}$. Then $G - \{u, v\}$ has one of the graphs shown in Figure 2.11 as a subgraph.

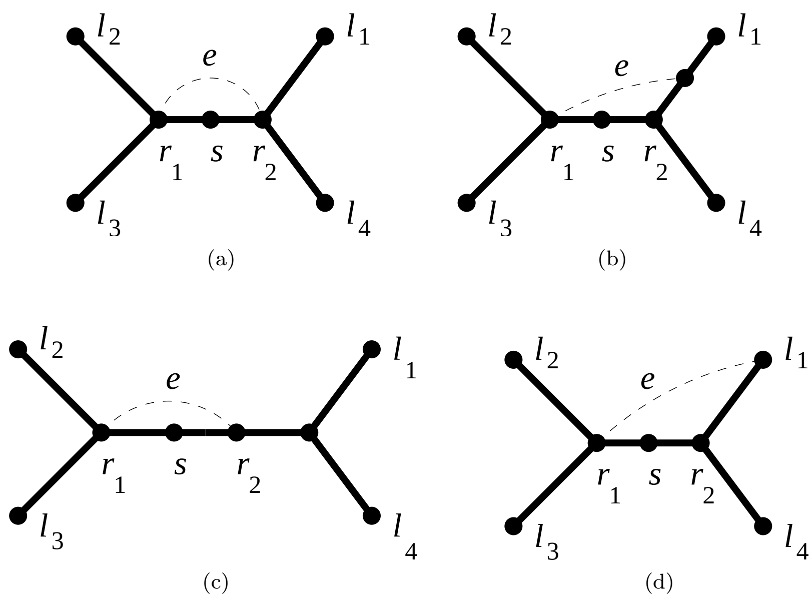


FIGURE 2.11: C contains at least two edges of the path between r_1 and r_2. A bold edge represents a path.

In each case, either s has degree greater than two in T, in which case there is a path from s to some leaf l_5. Since $G \backslash e$ is 3-connected, either u is adjacent to s or u is adjacent to l_5. By deleting the edges ul_2 and ul_1, contracting the path between r_1 and l_2, and contracting the path between r_2 and l_1, we see that the excluded minor R is a proper minor. Hence, 2.6.7 holds.

2.6.8. *There is no cycle in $T + e$ containing two vertices having degree at least three in T.*

Let r_1 and r_2 be vertices of degree at least three in T. Suppose there is a cycle in $G - \{u, v\}$ containing r_1 and r_2. Then $G - \{u, v\}$ has one of the following graphs shown in Figure 2.12 as a subgraph.

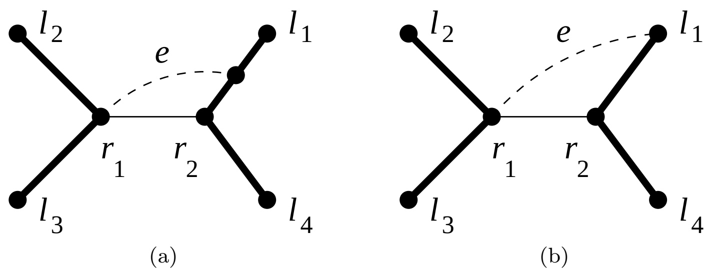


FIGURE 2.12: C contains two vertices of degree at least three in T. A bold edge represents a path.

By contracting the edge between u and l_2, contracting the edge between v and l_3, and contracting the path between r_1 and l_4, the graph G has a K_5-minor; a contradiction. Hence 2.6.8 holds.

By eliminating all subgraphs of $G-\{u,v\}$ in 2.6.6, 2.6.7, and 2.6.8, we see that the graph $G-\{u,v\}$ has one of the graphs shown in Figure 2.13 as a subgraph.

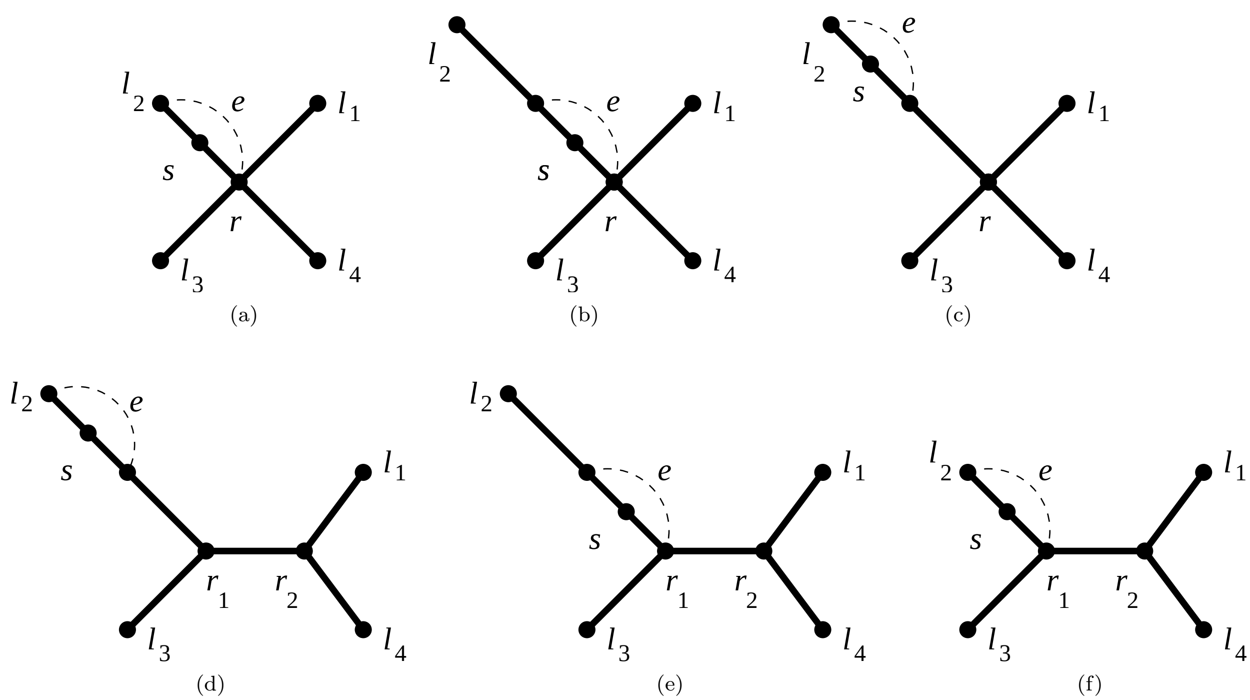


FIGURE 2.13: C contains two vertices of degree at least three in T. A bold edge represents a path.

Suppose $T + e$ is one of (d), (e), and (f). One of u and v, say u is adjacent to s. By deleting the edge from u to l_4, contracting the path from l_4 to r_2, and contracting the edge from v to l_3, we see that G has an $S(K_4, K_4)$-minor; a contradiction.

Now, suppose that $T + e$ is (c). Then G contains the excluded minor $S(K_4, K_4)$, which can be seen by deleting the edge from u to l_3, deleting the edge from v to l_2, deleting the path from r to l_1, contracting the edge from l_1 to u, and contracting the path from l_3 to t. Hence (c) does not occur.

We are left with the cases when $T+e$ is one of (a) or (b). In case (b), notice that there are no branches of the tree T adjacent to an internal vertex of the path from l_2 to r; otherwise G has a K_5-minor, which can be seen by contracting the path from l_1 to r and contracting the edge between l_3 and u. In both (a) and (b), the vertex v is adjacent to only leaves of the tree, and possibly u and r. Now, by the same argument as in Lemma 2.6.4 (a) and (b), this does not arise. $\square$

Therefore, T does not have four or more degree-one vertices.

The case analysis included in the previous Lemmas 2.6.2, 2.6.4, and 2.6.5 finishes the proof of Theorem 2.6.1. $\square$

2.7 Minimally 3-Connected Excluded Minors for the Class $\mathcal{S}$

In an excluded minor G, if there is an edge e such that $G\backslash e$ is 3-connected, then G is one of the excluded minors in Theorem 2.6.1. Now we assume that G is minimally 3-connected, that is, there is no edge e such that $G\backslash e$ is 3-connected.

Before we prove the main theorem of this section, we find the structure of a minimally 3-connected excluded minor G. The following lemmas characterize the structure of minimally 3-connected excluded minors and play a vital role in finding such excluded minors for the class $\mathcal{S}$. We begin by showing in Lemma 2.7.1 that for any edge e, in $G\backslash e$ no two minors of any two distinct components of the 2-sum decomposition are 3-connected. We use Lemma 2.7.2 to prove Lemma 2.7.3, where we show that for any edge e, the deletion $G\backslash e$ is not series-parallel. Then, in Lemma 2.7.4 we show that every edge of G is incident with a vertex of degree three.

Lemma 2.7.1. *Let G be a simple minimally 3-connected excluded minor for $\mathcal{S}$ such that $G\backslash e \cong G_1 \oplus_2 G_2$ for some edge e adjacent to two vertices a and b and graphs G_1 and G_2. Let H_1 be minor of G_1 and H_2 be a minor of G_2. If H_1 is 3-connected, then H_2 is not 3-connected.*

Proof. Since $G\backslash e \in \mathcal{S}$, there is an edge f joining two vertices x and y such that $((G\backslash e) + f)/f$ has no K_4-minor. Suppose both G_1 and G_2 have 3-connected minors, H_1 and H_2, respectively.

By Proposition 2.4.1, since $G\backslash e \in \mathcal{S}$, this implies that G can be formed by taking the parallel connection of H_1 and H_2, with basepoint f that may or may not be deleted. Then at each edge of the resulting graph, attach via 2-sum, a 2-connected series-parallel graph, or a parallel class. If G_1 and G_2 are exactly H_1 and H_2, then because each of H_1 and H_2 is 3-connected, there are two vertices whose removal is cycle-free, and those vertices are the vertices a and b, so $H_1 - \{a, b\}$ and $H_2 - \{a, b\}$ are cycle-free. Hence, $G - \{a, b\}$ is cycle-free and $G \in \mathcal{S}$; a contradiction. As G is simple, this implies that at least one 2-connected series-parallel graph is 2-summed onto some edge. Now, $G - \{a, b\}$ has a cycle, C, as a minor, which lies entirely in G_1 or G_2. Also, C does not lie entirely in H_1; otherwise $H_1 - \{a, b\}$ has a cycle, a contradiction. So, C is in a 2-connected series-parallel graph that is 2-summed onto some edge adjacent to neither a nor b of $H_1 \oplus_2 H_2$. Every 2-connected series-parallel graph has in a vertex of degree two, which is not destroyed by 2-summing the graph onto $H_1 \oplus_2 H_2$, and since $\kappa(G) = 3$, this vertex is incident with f. Combining the cycle C in the 2-connected series-parallel graph in $G - \{a, b\}$ with the K_4-minor in both H_1 and H_2, the graph G has $S(K_4, K_4)$ as a proper minor, which can be seen by contracting the edge e and also contracting an edge adjacent to b in G_2; a contradiction. Hence, both H_1 and H_2 are not 3-connected

□

Lemma 2.7.2. *Let G be a simple 3-connected graph. If G is an excluded minor for the class $\mathcal{S}$, then G does not contains an exact copy of the fan with three spokes shown in Figure 2.14 as a minor where no edge adjacent to v_a, v_b or v_c was deleted or contracted to form such a minor.*

Proof. Suppose G contains the 3-spoke fan shown in Figure 2.14. Let v_a, v_b, v_c, v_d, w, and u, be the vertices shown and a be the edge joining w to v_a.

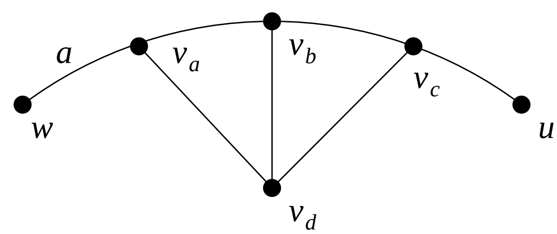


FIGURE 2.14: 3-spoke fan.

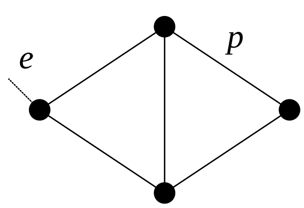
Since $G/a \in \mathcal{S}$ is 3-connected, by Proposition 2.5.2 there are two vertices whose deletion is a tree. Suppose the composite formed after contracting a, call it v', is not one of the vertices of G/a whose removal is a tree. Then the deletion of the two vertices x and y that leave a tree in G/a also leave a tree in G since contracting a creates no new cycles in $G - \{x, y\}$. By Theorem 2.5.3, G is a member of $\mathcal{S}$; a contradiction. Hence, the composite vertex v' is one of the vertices of G/a whose removal is cycle-free.

Now, v_b, v_c, and v_d are in a triangle, so the other vertex of G/a whose removal is cycle-free is one of v_b, v_c, or v_d. If $G/a - \{v', v_b\}$ is a tree, Menger's theorem implies that there is a path from u to v_d in G avoiding v_a and v_c. Therefore $G/a - \{v', v_b\}$ contains a cycle; a contradiction. If $G/a - \{v', v_d\}$ is a tree, then $G - \{v_d, w\}$ contains no cycles; a contradiction. Similarly, if $G/a - \{v', v_c\}$ is a tree, then $G - \{v_d, w\}$ contains no cycles; a contradiction.

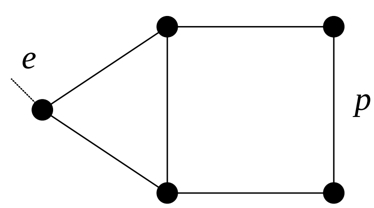
□

Lemma 2.7.3. *If G is an excluded minor, then $G\backslash e$ is not series parallel.*

Proof. Suppose $G\backslash e$ is series parallel. Using Cunningham and Edmonds's canonical 2-sum decomposition [4], we see that $G\backslash e$ can be decomposed into a path in which the vertices are labelled alternately by cycles and parallel classes. Each parallel class has three elements and each cycle has three or four elements. The path must have triangles on both ends since $G\backslash e$ is series parallel. Hence, our canonical 2-sum decomposition is: triangle, 3-element parallel class, triangle or square, 3-element parallel class on both ends of the path ending in one of the following two graphs and the next vertex of the 2-sum decomposition will be attached via 2-sum onto the edge p shown in the Figure 2.15.



(a) two triangles

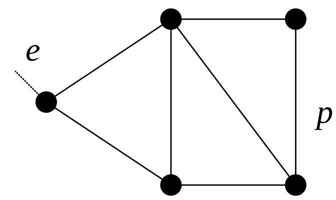


(b) a triangle and a 4-cycle

FIGURE 2.15: $G\backslash e$ Decomposition: Path Ends.

Suppose first that G ends in (a), two triangles. Now, p is not adjacent to a cycle of length greater than four; otherwise G would contain a 3-spoke fan as in Lemma 2.7.2. Hence, p is adjacent to another triangle, attached via 2-sum with basepoint p; otherwise G contains a 3-spoke fan as in Lemma 2.7.2, a contradiction.

Now, G must end in either three triangles as in Figure 2.16 (a) or a triangle and a 4-cycle as in Figure 2.15 (b). If G ends in two copies Figure 2.16 (a), one copy of Figure 2.16 (a) and one copy of Figure 2.15 (b), or two copies of Figure 2.15 (b), then G contains an



(a) 3 triangles

FIGURE 2.16: $G\backslash e$ Decomposition: Path Ends.

$S(K_4, K_4)$-minor; a contradiction. It is easy to see that smaller cases are contained in $\mathcal{S}$. Therefore, $G\backslash e$ is not series parallel. □

Lemma 2.7.4. *If e is an edge of a minimally 3-connected excluded minor, then e is incident with a vertex of degree three.*

Proof. Suppose G is a minimally 3-connected excluded minor. Combining Proposition 2.4.1, Lemma 2.7.1, and Lemma 2.7.3 we know that the deletion of any edge x is the 2-sum of a 3-connected graph G_1 and a series-parallel graph G_2 where exactly one other series-parallel graph may be 2-summed onto any edge of G_1. Also, since G_2 is a series-parallel graph, it has a vertex of degree two that is not destroyed by 2-summing G_1, which is why only one other series-parallel graph besides G_2 can be 2-summed onto G_1. Since G is 3-connected, this vertex is incident with e and becomes a vertex of degree three in G. Recall that we picked the deleted edge x arbitrarily so every edge of G is incident with a vertex of degree three. □

In the following theorem, we find the minimally 3-connected excluded minors for $\mathcal{S}$. This completes the list of excluded minors for $\mathcal{S}$.

Theorem 2.7.5. *Let G be a simple minimally 3-connected graph. Then G is an excluded minor for $\mathcal{S}$ if and only if G is one of the graphs shown in Figure 2.17.*

Proof. We use the matroid of a graph G to show that a minimally 3-connected excluded minor G is close to being 3-connected and can be formed from a 3-connected graph as

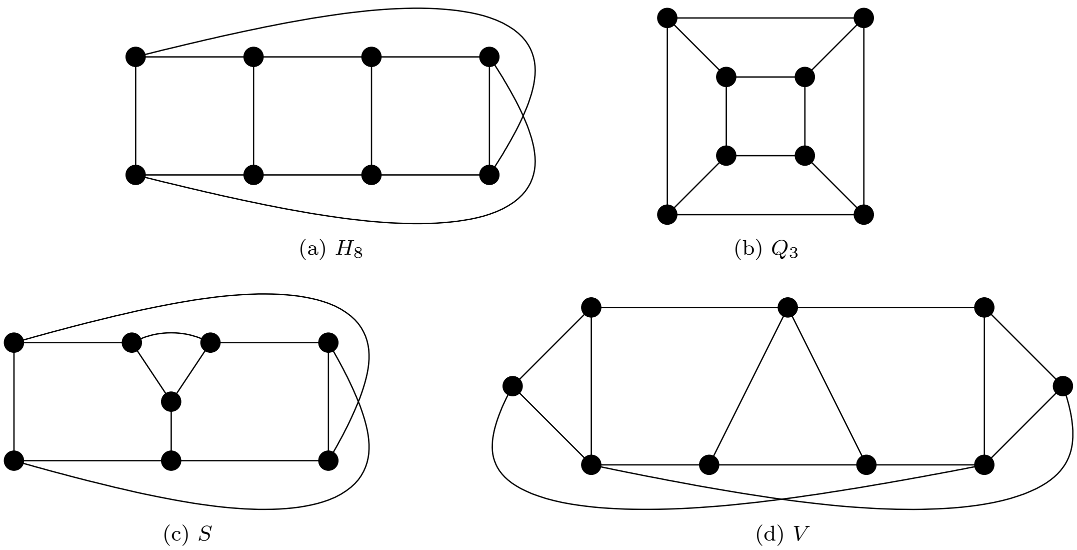


FIGURE 2.17: H_8, Q_3, S, and V

stated at the end of this paragraph. By a result of Cunningham [3] and Seymour [16], there is a vertically contractible edge in the dual $M^*(G)$, call it e. Now, the simplification, $\mathrm{si}(M^*(G)/e) = \mathrm{si}(M^*(G/e))$ is 3-connected for some edge $e \in M^*(G)$. In the dual, the cosimplification $\mathrm{co}(M(G\backslash e)) = \mathrm{co}(M(G)\backslash e)$ is 3-connected for some edge $e \in M(G)$. Now, $\mathrm{co}(M(G)\backslash e)$, has no coloops since $G\backslash e$ is 2-connected. Also, $G\backslash e$ has at most two series classes since G is 3-connected. This means that G can be formed from a 3-connected graph H by either subdividing an edge and adding e from the newly created vertex to another vertex of H; or subdividing two edges and adding e joining the two newly created vertices.

We distinguish two main cases determined by whether or not e lies in a triangle. Let u and v be the vertices incident with e. By Lemma 2.7.4, the edge e is incident with a vertex of degree three, say v.

First, suppose e is in a triangle with edges $\{e, g, h\}$, and let w be the third vertex of the triangle where g is incident with vertices v and w, and h is incident with vertices u and w. Suppose e is in two triangles. However, $\mathrm{co}(M(G\backslash e))$ is simple and 3-connected, and if e is in two triangles, then is a parallel pair of edges in $G\backslash e$; a contradiction. Hence, e is in at most one triangle. By Lemma 2.2.3 (Tutte's Triangle Lemma) [20], since $G\backslash e$ and $G\backslash g$ are not 3-connected and $\{e, g, h\}$ is a triangle, we have that h is in a triad with either e or g. Hence, either v and w have degree three, or v and u have degree three.

Since u and v have degree three, the vertices u and v are incident with two other edges, call them f and k, respectively. Let x be the other vertex incident with f and y the other vertex incident with k. Since G is an excluded minor, $G/f \in \mathcal{S}$, so by Theorem 2.5.3 there are two vertices of G/f, call them v_f and v'_f, whose removal leaves a tree. Without loss of generality v_f is the composite vertex formed in the contraction; otherwise $G - \{v_f, v'_f\}$ is a tree, a contradiction. We divide our argument into cases based on the degrees of u, v, and w.

2.7.6. *If $d(u) = d(v)$ and $d(w) \geq 4$, then $G \cong V$.*

Suppose first that $d(u) = d(v) = 3$ and $d(w)$ is at least four. Notice that this is equivalent to the case when $d(u) = d(w) = 3$ and $d(v)$ is at least four. Since $d(w)$ is at least four, w is adjacent to two other vertices, call them w_1, and w_2. Now, we have the following graph shown in Figure 2.18 is a subgraph of G.

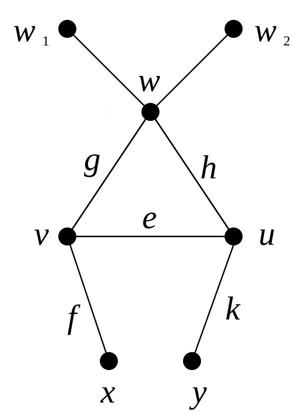


FIGURE 2.18: e is added in a triangle.

Consider the vertex v'_f. Notice that $v'_f \neq u$; otherwise $G - \{u, x\}$ is a tree, a contradiction. Similarly, $v'_f \neq w$; otherwise $G - \{w, x\}$ is a tree, a contradiction. If $v'_f = w_1$, then w_1 is adjacent to all leaves of the tree. Also, w_1 must have degree exactly three since w_1 is adjacent to w and $d(w) \geq 4$ and there are no two adjacent vertices of degree at least four in a minimally 3-connected excluded minor by Lemma 2.7.4. This means that $(G/f)-\{v_f, v'_f\}$ is a path, and hence, $G - \{v_f, w\}$ has no cycles; a contradiction. Similarly, $v'_f \neq w_2$.

Suppose that $v'_f = y$. Now, both $d(x)$ and $d(y)$ are at least four; otherwise $G - \{x, w\}$ or $G - \{y, w\}$ has no cycles, a contradiction. Notice that x and y are not adjacent to w; otherwise there would be two adjacent vertices of degree at least four, a contradiction by Lemma 2.7.4. Either $(G/f) - \{v_f, y\}$ is a path or it has at least three degree-one vertices. However, $(G/f)-\{v_f, y\}$ is not a path since $(G/f)-\{v_f, y\} = G-\{v, x, y\}$, and $G-\{v, x, y\}$ has a vertex of degree three, but a path has no such vertex. Hence, $(G/f) - \{v_f, y\}$ has at least three degree-one vertices. Since no two adjacent vertices have degree exceeding three, but $\kappa(G) = 3$, every nonleaf vertex other than w and u of the tree $(G/f) - \{v_f, y\}$ is adjacent to exactly one of x and y. Also, as every leaf vertex is adjacent to both x and y, by examining the possible graphs, we find that the graph G is isomorphic to the excluded minor V.

Now, we may assume that $v'_f \notin \{u, w, w_1, w_2, y\}$. If v'_f is the only other vertex besides u, v, x, w, w_1, w_2, and y, then v'_f can be adjacent to only w_1, w_2, x, and y. Hence, $G - \{v, v'_f\}$ is a tree; a contradiction. This implies that there is some other vertex $v''_f \notin \{u, v, w, x, y, w_1, w_2, v''_f\}$. We consider separately when w_1, w_2, and y are leaves of the tree, and when they are not. If w_1 is not a leaf of the tree, then G contains $S(K_4, K_4)$ as a proper minor; a contradiction. If y is not a leaf of the tree, then, again G contains a proper $S(K_4, K_4)$-minor; a contradiction. Notice that there are no edges from v'_f to $v_f = x$ because no two adjacent vertices of G have degree exceeding three. Hence, $G - \{v_f, w\}$ is a tree; a contradiction. Therefore, 2.7.6 holds.

2.7.7. *If $d(u) = d(v) = d(w) = 3$, then $G \cong S$.*

Now, suppose that $d(u) = d(v) = d(w) = 3$. There are two paths from w to w' and u to y', where w' and y' are vertices adjacent to leaves of the tree. We have the graph shown in Figure 2.19 is a subgraph of G, where a dotted line represents a path.

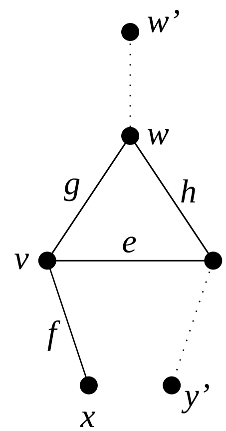


FIGURE 2.19: e is added in a triangle. A dotted line represents a path.

If $v'_f = u$, then $(G/f) - \{v_f, v'_f\}$ is a path since u is adjacent to the ends of the path, has degree three, and is adjacent to v. This implies that w and y' are the ends of the path. Hence, $G - \{x, w\}$ is a path; a contradiction. By symmetry, $v'_f \neq w$. Suppose that $v'_f = y'$ or by symmetry w'. Either $G/f - \{v_f, v'_f\}$ is a path or has at least three degree one vertices. If $G/f - \{v_f, v'_f\}$ is a path, then either G contains $S(K_4, K_4)$ as a minor or $G - \{v, w'\}$ is a path; both of which are contradictions. If $G/f - \{v_f, v'_f\}$ has at least three degree-one vertices, then G contains and is isomorphic to S.

We may now assume $v'_f \neq \{u, w, x, y'\}$. If $(G/f) - \{v_f, v'_f\}$ is not a path and there is an edge from v'_f to the path from u to y', then G has a K_5-minor, so there is no such edge. If there is an edge from v'_f to the path from w to w', then G contains $S(K_4, K_4)$ as a proper minor; a contradiction.

Now, assume $(G/f) - \{v_f, v'_f\}$ is a path. If v_f and v'_f are not adjacent, then without loss of generality the path from w to w' has a third vertex. Hence G has an $S(K_4, K_4)$-minor; a contradiction. If v_f and v'_f are adjacent, then $G - \{x, w\}$ is a tree since there are no other edges from v'_f besides the edges from v'_f to two leaves of the tree, otherwise there are two adjacent vertices in G of degree at least four. By Lemma 2.7.4 this is a contradiction. Therefore 2.7.8 holds.

We may now assume that e is not in a triangle.

2.7.8. *If e is not in a triangle, then $G \cong H_8$, Q_3, or S.*

Notice that u and v are incident with at least two other edges. Let f and g be the two other edges incident with v; let u_f the other vertex incident with f; and let u_g be the other vertex incident with g. Now, G has the following graph shown in Figure 2.20 as a subgraph.

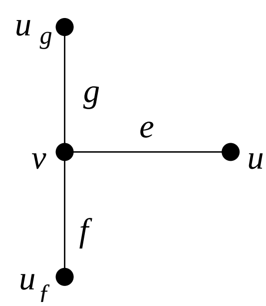


FIGURE 2.20: e is not in a triangle.

There is no triangle containing the edges g and e, so G/g is 3-connected and Theorem 2.5.3 has two vertices v_g and v'_g whose deletion is a tree. We may assume v_g is the composite vertex that is obtained by contracting g. Otherwise the same two vertices in G/g whose removal from G/g leave a tree, also leave a tree when removed from G; a contradiction.

Suppose $G/g - \{v_g, v'_g\}$ has at least three degree-one vertices. If v_g is adjacent to a non-leaf vertex of the tree and is also adjacent to a vertex of degree two of the tree, then the resulting graph obtained by adding all such adjacencies contains a H_8-minor. We may now assume that v_g is only adjacent to vertices of degree at least three. Now, if the tree has at least two vertices of degree at least three, then G contains the graph H_8 as a minor. If the tree contains exactly one vertex of degree at least three, then the removal of that vertex and the vertex v'_g is a tree; a contradiction.

Now, we may assume that $G/g - \{v_g, v'_g\}$ is a path. Likewise, $G/f - \{v_f, v'_f\}$ is a path. Now v'_g is neither u_f nor u, otherwise the deletion of u_g and v'_g is a tree. Let x_1 and x_2 be the leaves of the path $G/g - \{v_g, v'_g\}$. Notice that f is adjacent to either x_1 or x_2; otherwise x_1 and x_2 have degree three in G/f and $G/f - \{v_f, v'_f\}$ is not a path, so without loss of generality suppose f is adjacent to x_1. Also, $d(u_f) = d(v) = d(u_g) = 3$ since $\mathrm{co}(M(G\backslash e))$ is 3-connected. So both g and f are not in a triangle in G which implies that both ends of f, namely v and u_f, have degree three. By symmetry u_g has degree three. We consider separately when u_g is adjacent to x_2 of the path $G/g - \{v_g, v'_g\}$, and when it is not.

Suppose u_g is adjacent to x_2. If u_g is adjacent to u'_g, then $G - \{v'_g, v\}$ is a path; a contradiction. If u_g is not adjacent to v'_g, then it is adjacent to another vertex w of the path $G/g - \{v_g, v'_g\}$. If w is closer to x_2 than u on the path $G/g - \{v_g, v'_g\}$, then suppose there is another vertex on the path $G/g - \{v_g, v'_g\}$, call it a. If a is on the path from w to u, then G contains and is isomorphic to H_8. If a is on the path from either x_1 to w or u to x_2, then G contains and is isomorphic to S. If there are no other vertices on the path, then v'_g is adjacent to u or w since G is 3-connected. If v'_g is only adjacent to one of u or w, then the vertex to which it is adjacent, and x_1 or x_2, is a two vertex cut of G; a contradiction. Hence, v'_g is adjacent to both u and w and has a K_5-minor. If u is closer on the path $G/g - \{v_g, v'_g\}$ to x_2, then there is another vertex on the path between x_2 and u since e is not in a triangle in G. The previously mentioned vertex on the path between x_2 and u is adjacent to v'_g which gives a proper $S(K_4, K_4)$-minor; a contradiction. Notice that $u \neq w$, otherwise e would be in a triangle.

We may now assume that u_g is not adjacent to x_2. If u_g is also adjacent to v'_g, then $G - \{v, v'_g\}$ is a tree, a contradiction. If u_g is not adjacent to v'_g, then it is adjacent to two intermediate vertices of the path $G/g - \{v_g, v'_g\}$, call them c and d. If there is another vertex on the path from c to d, then v_g is adjacent to that vertex and G contains and is isomorphic to Q_3. If there is another vertex on $G/g - \{v_g, v'_g\}$ minus the path from c to

d, then G contains a proper $S(K_4, K_4)$-minor; a contradiction. So we may assume that the only vertices on the path $G/g - \{v_g, v'_g\}$ are u_f, c, d and u_e. If v_g is adjacent to exactly one of the intermediate vertices c or d, then the removal of that vertex and u_f or u_e is a tree; a contradiction. If v_g is adjacent to both vertices, then G has an S-minor, and is isomorphic to S.

□

We restate the main theorem of this chapter, Theorem 2.1.1, for convenience giving the excluded minors for $\mathcal{S}$ which follow from Proposition 2.3.1, Proposition 2.3.2, Proposition 2.3.4, Theorem 2.6.1, and Theorem 2.7.5.

Main Theorem. *The excluded minors for $\mathcal{S}$ are the following 11 graphs: $K_4 \oplus_0 K_4$, $K_4 \oplus_1 K_4$, $S(K_4, K_4)$, K_5, $K_{2,2,2}$, R, U, H_8, Q_3, S, and V.*

2.8 The Dual Operation

We extend the excluded minors for $\mathcal{S}$ to a related class of graphs, $\mathcal{S}^*$. The class $\mathcal{S}$ consists of those graphs G such that there is a graph H for which $H\backslash e = G$ and H/e is series-parallel for some edge $e \in E(H)$. Consider the class $\mathcal{S}^*$ of graphs G such that there is a graph H for which $H/e = G$ and $H\backslash e$ is series-parallel for some edge $e \in E(H)$. Then $G \in \mathcal{S}^*$ if G has a vertex v that can be replaced by two vertices v_1 and v_2 so that each edge of G incident with v is incident with exactly one of v_1 and v_2 and the graph obtained by this operation is series-parallel. In general, we refer to this operation as *splitting the vertex v*.

It is easy to check that the class $\mathcal{S}^*$ is closed under taking minors. By Robertson and Seymour's Graph Minors Theorem, it has a finite number of excluded minors.

In the proof of the excluded minors for $\mathcal{S}^*$ it is useful to consider the dual of the graph and we use the following proposition given in [13].

Proposition 2.8.1. *Let G be a graph having no isolated vertices. If G is series-parallel, then its dual G^* is series-parallel.*

Lemma 2.8.2. *A graph G is series-parallel if and only if its associated graphic matroid $M(G)$ is series-parallel.*

Proof. If G is series-parallel, then clearly $M(G)$ is series-parallel by taking $M(G)$ to be the cycle matroid of the graph G. Let $M(G)$ be a series-parallel matroid. By Whitney's 2-Isomorphism Theorem 1.11.1, the graph $G \cong_2 H$ for some series-parallel graph H. However, H can be transformed into a graph isomorphic to G by the operations of vertex identification, vertex cleaving, and twisting, and none of these operations create a K_4-minor. Hence, G is series-parallel. □

Proposition 2.8.3. *If $M\backslash f$ is series-parallel and M/f is planar, then M is planar.*

Proof. Suppose M is nonplanar. Then $M\backslash X/Y \cong M(K_5)$ or $M(K_{3,3})$ for some subsets X and Y of $E(M)$. Suppose first, that $f \notin X \cup Y$. We have that $M\backslash f$ contains $(M\backslash X/Y)\backslash f$. The previous matroid is $M(K_5)\backslash f$ or $M(K_{3,3})\backslash f$. However, the deletion of any edge from both $M(K_5)$ and $M(K_{3,3})$ contains $M(K_4)$. Hence, $M\backslash f$ is series-parallel; a contradiction. Next, suppose that $f \in X$. The matroid $M\backslash f$ contains $M\backslash X/Y$, which is isomorphic to either $M(K_5)$ or $M(K_{3,3})$, and both of those matroids contain $M(K_4)$ as a minor. Hence, $M\backslash f$ is series-parallel; a contradiction. Finally, suppose $f \in Y$. Then M/f contains $M\backslash X/Y$ which is isomorphic to either $M(K_5)$ or $M(K_{3,3})$. By Wagner's Theorem 1.6.2, the matroid M/f is nonplanar; a contradiction. Therefore, M is a planar, and 2.8.3 holds. □

Lemma 2.8.4. *Let G be a connected planar graph. If G is a member of $\mathcal{S}^*$, then G^* is a member of $\mathcal{S}$*

Proof. Since $G \in \mathcal{S}^*$, there is graph H and an edge $e \in E(H)$ for which $H/e = G$ and $H\backslash e$ is series-parallel. Since the dual of a planar graph is planar, G^* is planar. Also, by Proposition 2.8.3, the graph H is planar. Consider the graphic matroid associated with G^*. Now, $M(G^*) = M(G)^* = M(H/e)^* = M(H)^*\backslash e = M(H^*)\backslash e$. Since the dual of the series-parallel matroid is series-parallel, the matroid $M(H\backslash e)^* = M(H)^*/e = M(H^*)/e$ is series-parallel. Therefore, there is a graph H^*, for which $H^*\backslash e \cong_2 G^*$ and H^*/e is series-parallel. By Whitney's 2-Isomorphism Theorem, this 2-isomorphism is an isomorphism for some graph K^* and some edge $e \in K^*$ for which $K^*\backslash e = G^*$ and K^*/e is series-parallel, thus $G^* \in \mathcal{S}$. □

Lemma 2.8.5. *Let G be a connected planar graph. If G is an excluded minor for $\mathcal{S}^*$, then G^* is an excluded minor for $\mathcal{S}$.*

Proof. First, we show that $G^* \notin \mathcal{S}$. Suppose $G^* \in \mathcal{S}$. Then there is a graph H and an edge $e \in E(H)$ for which $H\backslash e = G^*$ and H/e is series-parallel. By taking the dual of Proposition 2.8.3, the graph H is planar. Since G and H are planar graphs, consider the matroids associated with them, and $M(G) = M(G^*)^* = M(H\backslash e)^* = M(H)^*/e = M(H^*)/e$. Also, since series-parallel graphs are closed under duality, $M(H/e)^*$ is series-parallel, but $M(H/e)^* = M(H)^*\backslash e = M(H^*)\backslash e$. Now, we have a graph $H^*/e \cong_2 G$, for which $H^*\backslash e$ is series-parallel. By Whitney's 2-Isomorphism Theorem, there is some graph J and edge $e \in E(J)$ satisfying $J^*/e = G$ for which $J^*\backslash e = G^*$ is series-parallel, a contradiction.

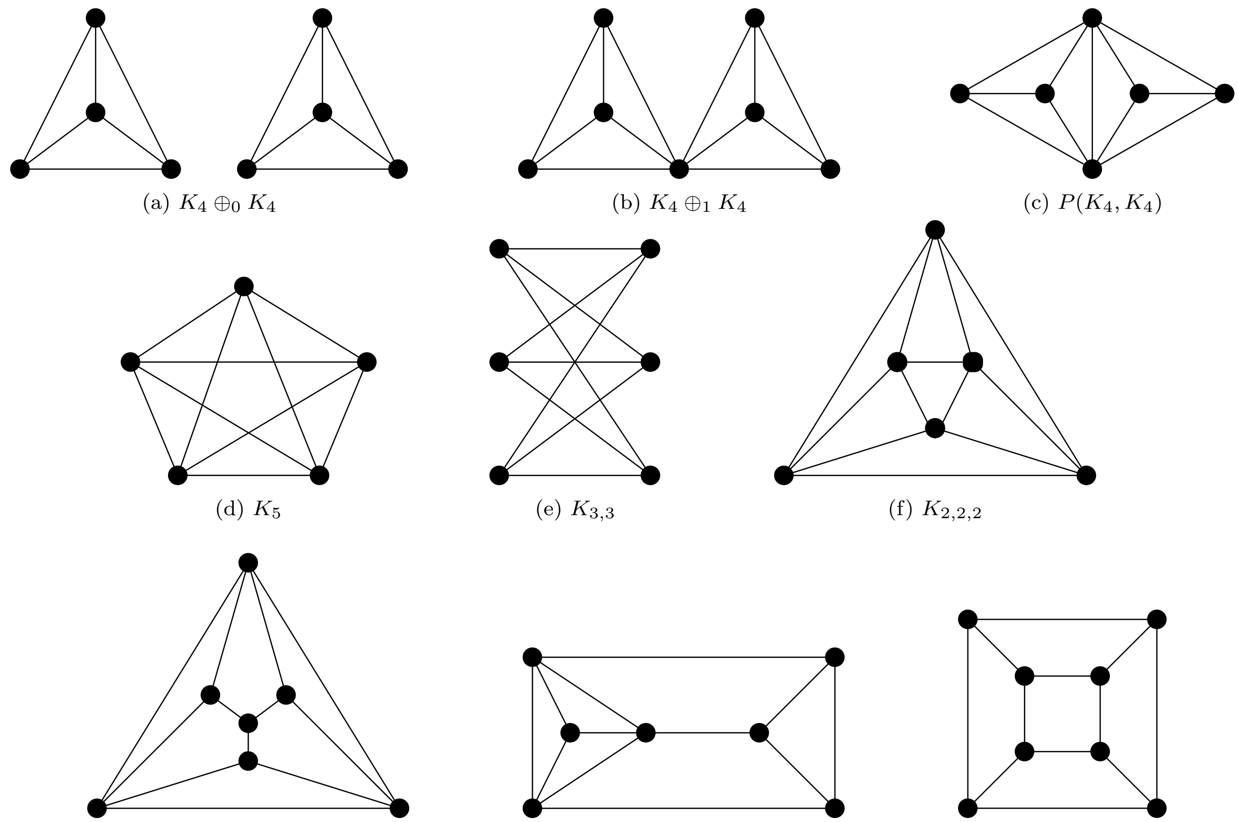
Next, we show that every proper minor of G^* is a member of $\mathcal{S}$. If F is a proper minor of G^*, then F^*, is a proper minor of G. Since G is an excluded minor for $\mathcal{S}^*$, the graph $F^* \in \mathcal{S}^*$. By Lemma 2.8.4, the dual, F is contained in $\mathcal{S}$.

Therefore, since G^* is not in $\mathcal{S}$, but every proper minor is in $\mathcal{S}$, we have shown that G^* is an excluded minor for $\mathcal{S}$.

□

The following theorem gives the excluded minors for $\mathcal{S}^*$ and follows from Theorem 2.1.1 and Lemma 2.8.5.

Theorem 2.8.6. *The excluded minors for $\mathcal{S}^*$, the class of graphs that have a vertex splitting that is series-parallel, consist of the nine graphs shown in Figure 2.21.*



(a) $K_4 \oplus_0 K_4$ (b) $K_4 \oplus_1 K_4$ (c) $P(K_4, K_4)$

(d) K_5 (e) $K_{3,3}$ (f) $K_{2,2,2}$

(g) R (h) U (i) Q_3

FIGURE 2.21: Excluded Minors for $\mathcal{S}^*$

Proof. Let G be an excluded minor for $\mathcal{S}^*$. If G is nonplanar, then by Kuratowski's Theorem [10], G has a $K_{3,3}$ or K_5-minor, both of which are excluded minors for $\mathcal{S}^*$. If G is planar and disconnected, then G has a $K_4 \oplus_0 K_4$-minor, which is also an excluded minor for $\mathcal{S}^*$. We may now assume that G is planar and connected. Since G is an excluded minor for $\mathcal{S}^*$, by Lemma 2.8.5, G^* is an excluded minor for $\mathcal{S}$. Thus, by taking the duals of the planar excluded minors for $\mathcal{S}$, we find the planar excluded minors for $\mathcal{S}^*$, shown in Figure 2.21. It is easy to check that each of these graphs is an excluded minor for $\mathcal{S}^*$. □

Chapter 3
Vertex Removal

Recall from Chapter 2 that a feature of a 3-connected almost series-parallel graph is that there are two vertices of the graph whose deletion is a tree. This idea of removing vertices from a graph and destroying all of its cycles gives rise to several new classes of graphs. Let $\mathcal{V}_n$ be the class of graphs such that the deletion of at most n vertices from G produces a graph with no cycles. In this chapter, we find the full list of excluded minors for $\mathcal{V}_1$ and $\mathcal{V}_2$, the classes of graphs such that there are, respectively, at most one vertex and at most two vertices whose removal from the graph gives a cycle-free graph.

3.1 Preliminaries

The following notation closely follows [2]. Denote by Ω_k the family of graphs containing k vertex-disjoint cycles. The family of graphs not belonging to Ω_k is denoted by $\overline{\Omega_k}$. Note that $\overline{\Omega_1}$ is just the family of forests.

We repeatedly use the following theorems when finding the excluded minors for $\mathcal{V}_1$ and $\mathcal{V}_2$. The first theorem was proved by Lovász [11] in 1965 and gives a characterization of those graphs that have no two vertex-disjoint cycles, and have minimum degree at least three. The six possibilities that arise in the theorem are illustrated in Figure 3.1.

Theorem 3.1.1. *Let G be a graph without two vertex-disjoint cycles. Suppose that $\delta(G) \geq 3$ and there is no vertex meeting all the cycles. Then one of the following six assertions holds.*

(i) G has three vertices and multiple edges joining every pair of vertices.

(ii) G is a K_4 in which one of the triangles may have multiple edges.

(iii) $G = K_5$.

(iv) G is a K_5^-, the graph obtained from K_5 by deleting an edge, such that some of the edges not adjacent to the missing edge may have multiple edges.

(v) G is a wheel whose spokes may have multiple edges.

(vi) G is obtained from $K_{3,p}$ by adding edges or multiple edges joining vertices in the first class.

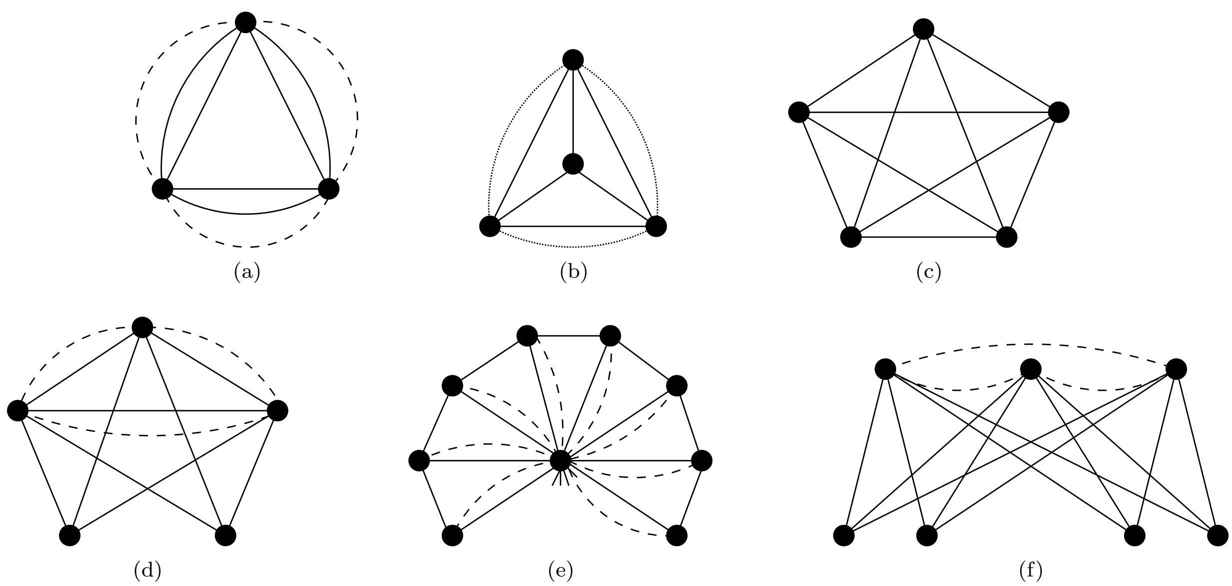


FIGURE 3.1: A dotted line shows that there may be multiple edges joining the end vertices.

Bollobás [2] generalized the previous result of Lovász in the following theorem and corollary. In the theorem, we consider the empty graph to be a forest.

Theorem 3.1.2. *A graph G does not contain two vertex-disjoint cycles if and only if either it contains a vertex x_0 such that $G - x_0$ is a forest, or it can be obtained from a subdivision G_0 of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to G_0.*

The following is an immediate consequence of the last result.

Corollary 3.1.3. *A 2-connected graph G has no two vertex-disjoint cycles if and only if G can be obtained from a subdivision G_0 of a graph listed in Theorem 3.1.1.*

In the contraction of an edge from an excluded minor for $\mathcal{V}_n$ there are n vertices whose removal leaves a graph that is cycle-free. We show that one of those vertices must be the composite vertex formed in the contraction.

Lemma 3.1.4. *Let G be an excluded minor for $\mathcal{V}_n$. For every non-loop edge $e \in E(G)$, the contraction G/e has n vertices whose removal is cycle-free, one of which is the composite vertex that results from identifying the ends of e.*

Proof. Since G is an excluded minor for $\mathcal{V}_n$, the contraction G/e has n vertices, $u_1, \ldots, u_n$, whose removal is cycle-free. Suppose the composite vertex formed in the contraction is not one of the vertices. Then $G/e - \{u_1, \ldots, u_n\}$ is cycle-free and uncontracting the edge e creates no new cycles in G. Hence $G - \{u_1, \ldots, u_n\}$ is cycle-free, a contradiction as $G \notin \mathcal{V}_n$. □

Lemma 3.1.5. *Let G be an excluded minor for $\mathcal{V}_n$. If v_1 and v_2 are adjacent vertices of G, then $G - \{v_1, v_2\}$ has $n - 1$ vertices whose removal is cycle-free.*

Proof. Let e be an edge joining v_1 and v_2. By Lemma 3.1.4 G/e has n vertices, $u_1, \ldots, u_n$, whose removal is cycle-free, one of which is the composite vertex, v, that results by identifying v_1 and v_2. Hence, $(G/e) - v = G - \{v_1, v_2\}$ has $n - 1$ vertices whose removal is cycle-free. □

The following lemma is used repeatedly to find the excluded minors for $\mathcal{V}_1$ and $\mathcal{V}_2$.

Lemma 3.1.6. *For a positive integer n, let G be an excluded minor for $\mathcal{V}_n$. Then*

(i) each component of G contains a cycle;

(ii) G has no vertices of degree one;

(iii) if v is a degree-two vertex of G, then G has a loop incident with v and this loop is a component of G;

(iv) G has no cut edge.

Proof. Each component of G contains a cycle; otherwise the deletion of an edge in a component without a cycle would be a member of the class $\mathcal{V}_n$. However, adding the edge back creates no cycles; a contradiction.

Suppose e is an edge incident with a degree-one vertex in G. Then e is in no cycles. Since G is an excluded minor, $G\backslash e$ has n vertices, $u_1, \ldots u_n$, whose removal is cycle-free. Thus, $G - \{u_1, \ldots, u_n\}$ is cycle-free; a contradiction. Therefore G has no degree-one vertices.

Suppose v is a degree-two vertex of G. Clearly either v is incident to two distinct edges or G has a loop incident with v and this loop is a component of G. First, suppose that f joins vertices v and w where v has degree two in G. Since G is an excluded minor, G/f has n vertices, $x_1, \ldots, x_n$ whose removal is cycle-free. By Lemma 3.1.4, we may assume that x_1 is the composite vertex that results from identifying v and w. But now $(G/f) - \{x_1, \ldots, x_n\} = G - \{v, w, x_2, \ldots, x_n\}$. However, since v had degree two in G, it has degree one in $G - \{w, x_2, \ldots, x_n\}$ and is in no cycles; a contradiction. Therefore, G has a loop incident with v and this loop is a component of G.

Suppose G has a cut edge, call it g. Then, $G\backslash g$ has n vertices whose removal is cycle-free. However, since g is a cut edge, it has no cycles and the removal of those same n vertices from G is cycle-free; a contradiction. Hence, G has no cut edge. □

3.2 Excluded Minors for $\mathcal{V}_1$

The next theorem follows easily from Theorem 3.1.1 and gives the excluded minors for $\mathcal{V}_1$.

Theorem 3.2.1. *A graph G is an excluded minor for the class $\mathcal{V}_1$ if and only if G is isomorphic to one of the three graphs shown in Figure 3.2.*

Proof. If G is disconnected, then, by Lemma 3.1.6 (i), each component of G contains a cycle. Thus G has a minor isomorphic to the disjoint union of two loops, $2L$. Since this graph is easily seen to be an excluded minor for $\mathcal{V}_1$, it is the only disconnected excluded minor.

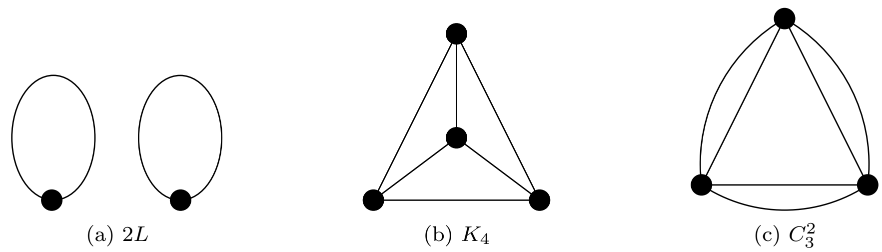


FIGURE 3.2: Excluded Minors for $\mathcal{V}_1$

Suppose G is connected. Then G does not have $2L$ as a minor, so G has no two vertex-disjoint cycles. By Lemma 3.1.6 (ii) and (iii), G has no vertices of degree one and two. Using Theorem 3.1.1, it is now a straightforward exercise to check that G is either K_4 or C_3^2, a doubled triangle. □

3.3 Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) \in \{0, 1\}$ and $\kappa(G) \geq 4$

Finding the excluded minors G for the class $\mathcal{V}_2$ with $\kappa(G) \in \{0, 1\}$ and $\kappa(G) \geq 4$ is not complicated. We consider separately when the connectivity is zero, one, and at least four.

Theorem 3.3.1. *Let G be an excluded minor for $\mathcal{V}_2$ with $\kappa(G) = 0$. If G is disconnected, then G is isomorphic to one of the three graphs $3L$, $K_4 \oplus_0 L$, or $C_3^2 \oplus_0 L$ as shown in Figure 3.3.*

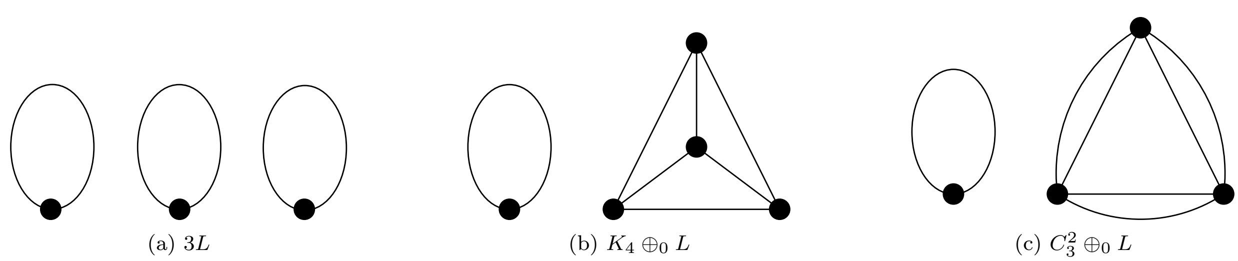


FIGURE 3.3: Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) = 0, 1$

Proof. By Lemma 3.1.6 each component of G contains a cycle. Hence, if the number of components of G is at least three, then G is isomorphic to three disjoint loops, $3L$. Suppose the number of components of G is exactly two. No component C_i of G has two vertex-disjoint cycles; otherwise G has $3L$ as a proper minor. In each C_i, either there is one vertex meeting all of the cycles, or C_i contains a C_3^2-minor or a K_4-minor by Theorem 3.1.2. It follows that G must be isomorphic to the 0-sum of a loop and either C_3^2 or K_4. It is easy to check that each of these graphs is an excluded minor for the class $\mathcal{V}_2$. □

Theorem 3.3.2. *If a graph G is an excluded minor for $\mathcal{V}_2$, then $\kappa(G) \neq 1$.*

Proof. Suppose $\kappa(G) = 1$. Then there is some vertex v whose deletion disconnects G and $G = G_1 \oplus_1 G_2$ for some connected subgraphs G_1 and G_2.

We show in what follows that exactly one of G_1 and G_2 has a vertex meeting all cycles. Moreover, this vertex is v. If G_1 and G_2 both have single vertices meeting all cycles, then $G \in \mathcal{V}_2$; a contradiction. Hence, either G_1 or G_2 has no single vertex meeting all cycles. Suppose neither G_1 nor G_2 has a single vertex meeting all cycles. Then both $G_1 - v$ and $G_2 - v$ contain cycle minors. If either G_1 or G_2 contains two vertex-disjoint cycles, then G contains $3L$ as a proper minor; a contradiction. Hence, either G_1 or G_2 has no two vertex disjoint cycles. Moreover, by Lemma 3.1.6, we have that $\delta(G) \geq 3$. Now, by Theorem 3.1.2, either G_1 or G_2, say G_1, contains a minor shown in Figure 3.1. Combining this minor with the cycle in $G_2 - v$, the graph G has as a proper minor either $C_3^2 \oplus_0 L$ or $K_4 \oplus_0 L$; a contradiction. Hence, exactly one of G_1 and G_2 has a vertex meeting all cycles.

We may now assume that G_2 has a vertex meeting all of its cycles and that G_1 has no single vertex meeting all of its cycles. Suppose G_2 has at least two edges, call one e. Then G/e has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other is a vertex w of G_1, and $G_1 - w$ is cycle-free. However, since $G_2 - v$ is also cycle-free, $G - \{u, w\} = (G_1 - u) \oplus_1 (G_2 - w)$ is cycle-free; a contradiction. Hence, G_2 has exactly one edge. By Lemma 3.1.6, G_2 has no cut edges, so G_2 is a loop.

Now $G_1 - v$ has no two vertex disjoint cycles; otherwise G contains $3L$ as a proper minor, a contradiction. Also, $G_1 - v$ has no single vertex meeting all cycles, otherwise $G \in \mathcal{V}_2$. Therefore $G_1 - v$ has no two vertex disjoint cycles and no single vertex meeting all cycles and, by Theorem 3.1.2, $G_1 - v$ contains a minor given in Figure 3.1. This, combined with G_2, the loop, gives either $C_3^2 \oplus_0 L$ or $K_4 \oplus_0 L$ as a proper minor of G; a contradiction. Hence if G is an excluded minor for $\mathcal{V}_2$, then $\kappa(G) \neq 1$.

□

To find the excluded minors of connectivity at least four, we use the result of Halin and Jung, Lemma 2.3.3. It is straightforward to check that both K_5 and $K_{2,2,2}$ are excluded minors for the class $\mathcal{V}_2$. Combining this with Lemma 2.3.3, we immediately obtain the following corollary from which it follows that $\mathcal{V}_2$ contains no 4-connected graphs.

Corollary 3.3.3. *Let G be a simple graph with $\kappa(G) \geq 4$. Then G is an excluded minor for $\mathcal{V}_2$ if and only if $G \cong K_5$.*

In order to prove the main theorem of this section, we will use the following two lemmas. The next results help characterize the structure of the connected excluded minors for $\mathcal{V}_2$.

Lemma 3.3.4. *If G is a connected excluded minor for $\mathcal{V}_2$, then G has no loops.*

Proof. Suppose G has a loop f meeting a vertex w. Notice that $G - w$ has no two vertex-disjoint cycles; otherwise G would contain $3L$ as a proper minor, a contradiction. Also, there is no single vertex of $G - w$ meeting all cycles of $G - w$; otherwise $G \in \mathcal{V}_2$. By Theorem 3.1.2, $G - w$ can be obtained from a subdivision G_0 of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to G_0. Therefore, G contains either $K_4 \oplus_0 L$ or $C_3^2 \oplus_0 L$ as a proper minor; a contradiction. Hence, G has no loops. □

Lemma 3.3.5. *If G is a connected excluded minor for $\mathcal{V}_2$, then G has no three parallel edges with the same end vertices.*

Proof. If G had three parallel edges $\{f, g, h\}$ with the same end vertices, then $G\backslash f$ has two vertices whose removal is cycle-free, one of which is incident with f. This is a contradiction since the removal of those same two vertices from G is cycle-free. □

3.4 Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) = 2$

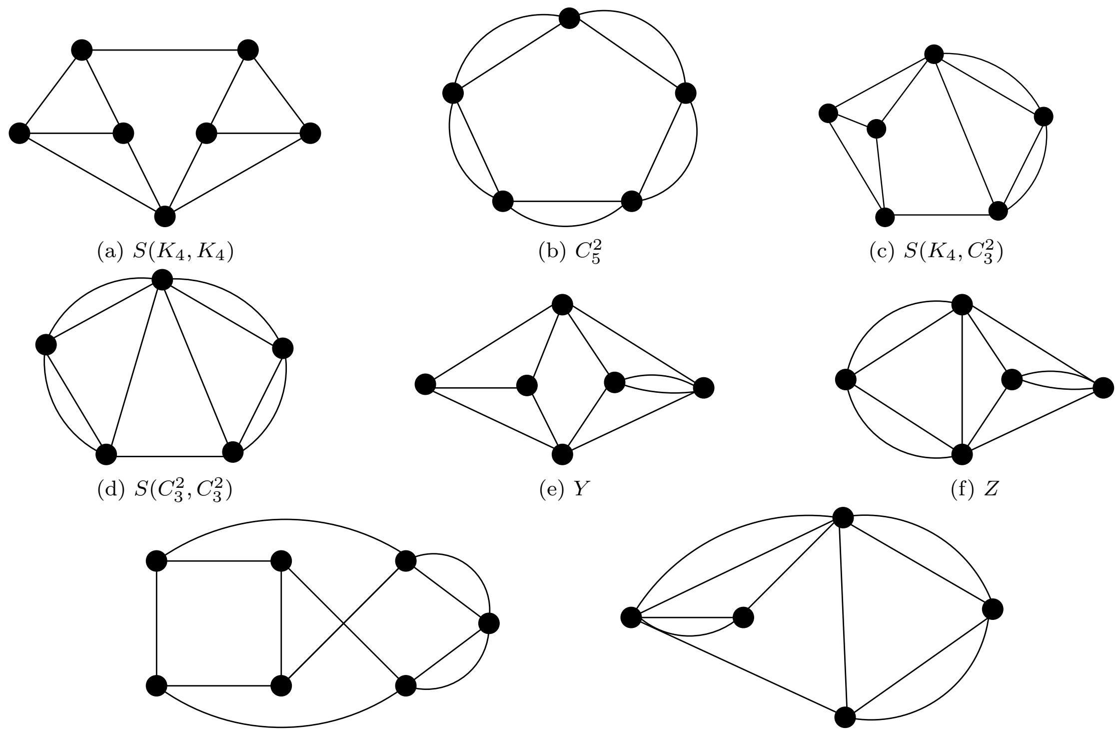
The following theorem gives the excluded minors with $\kappa(G) = 2$ and relies heavily on Theorem 3.1.2. We use this characterization when we decompose the graph along a 2-separation. The argument itself breaks into two main parts, when the basepoint of the 2-separation is an edge of the graph, and when it is not. We then find the structure of the components of the 2-sum.

Theorem 3.4.1. *If G is an excluded minor for $\mathcal{V}_2$ with $\kappa(G) = 2$, then G is one of the graphs shown in Figure 3.4.*

Proof. As $\kappa(G) = 2$, we can write $G = \bigoplus_{i=1}^{m}{}_2\, G_i$ where there is a single edge e that is the basepoint of this 2-sum, a and b are the vertex ends of e, and each G_i and each G_i/e is 2-connected. Since G is an excluded minor, $G - \{u, v\}$ has a cycle for every two vertices $u, v \in V(G)$. Now $G - \{a, b\}$ has a cycle C with vertex set $w_1, w_2, \ldots, w_k$ in, without loss of generality, G_1. If G_2 has two vertex-disjoint cycles, then G contains $3L$ as a minor, so G_2 contains no two vertex-disjoint cycles.

3.4.2. $m = 2$.

Suppose that $m \geq 3$. Since each G_i is 2-connected, there are cycles in G_2 and G_3 through a and b. Let P_i be a path from a to b in $G_i\backslash e$ for $i \in \{2, 3\}$. Let $P_2 = au_1u_2u_3 \ldots u_jb$ in G_2 and $P_3 = av_1v_2 \ldots v_qb$ in G_3. Then G/au_1 has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these is the composite vertex formed when contracting au_1 and



(a) $S(K_4, K_4)$ (b) C_5^2 (c) $S(K_4, C_3^2)$ (d) $S(C_3^2, C_3^2)$ (e) Y (f) Z (g) N (h) D

FIGURE 3.4: Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) = 2$

the other is a vertex w_l of C in G_1. However, $G - \{a, w_l\}$ has a cycle C_2 which lies in G_2 since $G - \{u_1, a, w_l\}$ has no cycles. Similarly, $G - \{b, w_m\}$ has a cycle C_3 which lies in G_3 for some vertex w_m of C. Therefore G has three vertex-disjoint cycles as a proper minor; a contradiction. Thus, 3.4.2 holds.

The remainder of the proof of the theorem decomposes into two main cases: when G has an edge joining a and b, and when G has no such edge. The next lemmas treats these cases in order.

Lemma 3.4.3. *If G has an edge h joining a and b, then G is isomorphic to one of $S(K_4, C_3^2)$, $S(C_3^2, C_3^2)$, or Z.*

Proof. The excluded minors containing an edge joining a and b are found by decomposing the graph along the 2-sum as given above, and then finding the structure of both G_1 and G_2.

In 3.4.5, it is shown that G_2 has exactly three vertices, and it follows that G_2 is isomorphic to C_3^2.

The graph G_2 has a vertex v meeting all cycles. Otherwise, Corollary 3.1.3 implies that since G_2 does not contain two vertex-disjoint cycles. Since $G_1 - \{a, b\}$ contains a cycle C, it follows that G contains either a $C_3^2 \oplus_0 L$, or a $K_4 \oplus_0 L$ as a proper minor. This is a contradiction.

3.4.4. *Neither a nor b meets every cycle of G_2.*

To see this, suppose a meets all cycles of G_2. Since G is an excluded minor, G/h has two vertices whose removal is cycle-free, one of which is the composite vertex, and the other a vertex w_p of the cycle C. However, $G - \{w_p, a\}$ is cycle-free; a contradiction. Therefore, 3.4.4 holds.

3.4.5. $G_2 \cong C_3^2$.

We now show that every edge of G_2 is incident with a or b. Suppose G_2 has an edge f not incident with a or b. By Lemma 3.1.4, G/f has two vertices whose removal is cycle-free, one of which is the composite vertex v_f. The other vertex is one of the vertices w_q in the cycle C in G_2. Let v'_f and v''_f be the ends of f. Since $G - \{w_q, v'_f, v''_f\}$ is cycle-free, $G_1 - w_q$ is cycle-free. We know that G_2 has a vertex $v \notin \{a, b\}$ such that $G_2 - v$ is cycle-free. However, $G - \{w_q, v\}$ is cycle-free as this graph is obtained from the two forests $G_1 - w_q$ and $G_2 - v$ by identifying the edge h in each. This contradiction implies that every edge of G_2 meets a or b.

It follows from the fact that G_2/e is 2-connected that G_2 has exactly three distinct vertices, a, b, and v. Since $\kappa(G) = 2$, the vertex $v \in G_2$ is adjacent to both a and b. Moreover, by Lemma 3.1.6 (iii), $d(v) \neq 2$, so v is in a 2-cycle. By symmetry, suppose a and v are in a 2-cycle. If b and v are not in a 2-cycle, then a meets every cycle of G_2, contradicting 3.4.4.

Therefore, bv is also in a 2-cycle and the exact structure of G_2 is C_3^2, a doubled triangle. Hence, we have $G_2 \cong C_3^2$, that is, 3.4.5 holds.

We now know the structure of G_2. In what follows, we find the structure of G_1 to obtain the remaining excluded minors for $\mathcal{V}_2$ when there is an edge joining a and b.

3.4.6. *$V(G_1) = \{a,b\} \cup V(C)$. Moreover, $G_1 - \{a,b\} = C$, and every vertex of C is adjacent to a or b.*

Suppose there is a vertex s of G_1 that is not in $\{a,b\} \cup C$. Also suppose an edge f joins s to some vertex t where t is not in a cycle of C. Since G is an excluded minor, after contracting the edge f, there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex, s'. However, in $(G/f) - s'$, there are two vertex-disjoint cycles: the cycle C, and a cycle of G_2/e. This is a contradiction. We now know that all neighbors of s are in $V(C)$.

The vertex s is adjacent to neither a nor b, since by the above argument every edge of G_1 is incident with a vertex in $V(C)$. Now, notice that s is adjacent to at most two vertices of $V(C)$; otherwise $G_1 - \{a,b\}$ contains a K_4-minor and so G has a $K_4 \oplus_0 L$-minor, a contradiction. Since $\kappa(G) = 2$, we see that s is adjacent to exactly two vertices of C, call them w_i' and w_i''. If the edge joining s and w_i' is in a 2-cycle and the edge joining s and w_i'' is in a 2-cycle, then G contains a $C_3^2 \oplus_0 L$-minor; a contradiction Hence, we may assume that there is no 2-cycle with vertex set $\{s, w_i''\}$. However, by Lemma 3.1.6, we have that (iii), $d(s) \geq 3$ so there is a 2-cycle with vertex set $\{s, w_i'\}$. By contracting an edge joining s and w_i' and then removing the newly created composite vertex s' and v, we get a cycle-free graph. However, $G - \{w_i', v\}$ also has no cycles since s has degree one in $G - \{w_i', v\}$; a contradiction.

If there was an edge f in $\mathrm{si}(G_1) - \{a,b\}$ not contained in C, then this edge f runs between two vertices of C since $V(G_1) = \{a,b\} \cup V(C)$. However, now C has a smaller cycle, say C'

containing the edge f and a vertex not contained in C'. Since C was picked arbitrarily, this is a contradiction 3.4.6. Hence, $\mathrm{si}(G_1) - \{a, b\}$ is a cycle C. Now, if $\mathrm{si}(G_1) - \{a, b\}$ is not simple, it has a 2-cycle. Let C'' be the 2-cycle and there is a vertex v of C'' not contained in that 2-cycle and since C was picked arbitrarily, by 3.4.6, this is a contradiction. By Lemma 3.1.6 $\delta(G) \geq 3$, so if C is not a 2-cycle, then every vertex of C is adjacent to a or b. If C is a 2-cycle, again every vertex of C is adjacent to a or b since $\kappa(G) = 2$. Thus, 3.4.6 holds.

First, we suppose C has three distinct vertices. Since every vertex of C is adjacent to either a or b, we may assume two vertices of C are adjacent to a and a third vertex of C is adjacent to b. Then G is $S(K_4, C_3^2)$.

We are left with the case when C is a 2-cycle. Let $V(C) = \{w_1, w_2\}$. If C is the only pair of parallel edges of G_1, then w_1 and w_2 are adjacent to both a and b. Hence, G_2 is Z. We may now suppose there is 2-cycle between w_1 and a. Suppose first that w_2 is adjacent to b. There is no 2-cycle from w_2 to b; otherwise G contains a C_5^2-minor, a 5-cycle where every edge is replaced with a pair of parallel edges, and C_5^2 is an excluded minor for $\mathcal{V}_2$. If w_2 is adjacent to a, then G is the series connection of copies of C_3^2, an excluded minor for $\mathcal{V}_2$. So, w_2 is adjacent to only w_1 and b. Hence, $G - \{w_1, v\}$ is cycle-free; a contradiction. We may now suppose that w_2 is not adjacent to b. Since $\kappa(G) = 2$, w_2 is adjacent to a and w_1 is adjacent to b. Therefore, G is isomorphic to D. This concludes the argument when G has an edge joining a and b.

□

Lemma 3.4.7. *If G has no edge joining a and b, then G is isomorphic to one of $S(K_4, K_4)$, C_5^2, $S(K_4, C_3^2)$, Y, or N.*

Proof. We begin by showing that

3.4.8. *$G_2 \backslash e$ has a vertex meeting all cycles. Moreover, this vertex is neither a nor b.*

Suppose $G_2\backslash e$ has no vertex meeting all cycles. Then, by Corollary 3.1.2, since $G_2\backslash e$ has no two vertex-disjoint cycles, $G_2\backslash e$ can be obtained from a subdivision G_0 of a graph listed in Theorem 3.1.1 by adding a forest and at most one edge joining each tree of the forest to G_0. Therefore $G_2\backslash e$ has either a C_3^2- or K_4-minor. By deleting all edges other than the cycle C in G_1 and contracting the cycle down to a loop, we have that G contains either a $C_3^2 \oplus_0 L$-minor or a $K_4 \oplus_0 L$-minor; a contradiction. Therefore 3.4.8 holds.

We find the excluded minors when there is no edge joining a and b in two main cases: when G_2 has a vertex meeting all cycles and when G_2 has no such vertex.

First, we suppose that G_2 has a vertex, v, meeting all cycles and find the structure of G_2 in 3.4.9. We proceed by finding the structure of G_1, and show in 3.4.10 that every vertex of G_1 is either a, b, or a vertex of the cycle C of $G_1 - \{a, b\}$, and that every vertex of C is adjacent to a or b. Using this structure, we extract the excluded minors for $\mathcal{V}_2$ when C has size greater than four in 3.4.11, when C has size three in 3.4.12, and finally when C has size two in 3.4.13.

3.4.9. *If G_2 has a vertex meeting all cycles, then $G_2 \cong C_3^{2^-}$, a triangle with two doubled edges and one single edge, the basepoint e.*

We begin by showing that every edge of G_2 is incident with a or b. Suppose some edge e of G_2 is incident with neither a nor b. Then G/e has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other vertex is a vertex $w_i \in V(C)$ for some $i = 1, \dots k$. Now, $G_1 - w_i$ is cycle-free and $G_2 - v$ is cycle-free, which implies that $G - \{w_i, v\}$ is cycle-free; a contradiction.

Now, we show that every vertex of G_2 is either a, b, or is adjacent to both a and b. Let u be a vertex of G_2, distinct from both a and b. By the above argument, u is adjacent to, without loss of generality, a. Suppose u is not adjacent to b. Since G_2 is 2-connected, u is adjacent to a vertex u' of G_2. However, the edge joining u and u' is adjacent to neither a nor b; a contradiction. Hence every vertex of G_2 is either a, b, or adjacent to both a and b.

Suppose u and u' are two arbitrary vertices of G_2, both of which are neither a nor b. Notice that u and u' are not adjacent; otherwise the edge joining u and u' is incident with neither a nor b, a contradiction. Also, since G_2/e is 2-connected, there is at most one vertex distinct from a and b in G_2. Hence, G_2 has exactly three vertices, a, b, and a third vertex, call it u.

By Lemma 3.1.6, u has degree at least three, so u is in a 2-cycle, $\{f, g\}$, incident to, without loss of generality, a. Suppose the edge from u to b is not in a 2-cycle. Then u has degree three in G. Since G is an excluded minor for $\mathcal{V}_2$, the graph G/f has two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex formed in the contraction. The other vertex is a vertex $w_i \in V(C)$ for some $i = 1, \dots k$. So $G - \{a, u, w_i\}$ is cycle-free. However, as u is only adjacent to a and b and there is only a single edge joining u and b, we have that $G - \{w_i, a\}$ is cycle-free; a contradiction. Therefore, the edge from u to b is in a 2-cycle. Therefore G has no loops or parallel classes of size greater than 2. Hence, 3.4.9 gives the structure of G_2.

3.4.10. *If G_2 has a vertex meeting all cycles, then $V(G_1) = \{a, b\} \cup V(C)$. Moreover, $G_1 - \{a, b\} = C$, and every vertex of C is adjacent to a or b.*

The previous statement is an immediate consequence of the argument given in 3.4.6, in this case where there is an edge e joining a and b. We note G_2 is now C_3^{2-}, the deletion of the edge e, while in the previous argument G_2 was C_3^2. However, in this proof, the edge e is never used. Thus, 3.4.10 holds.

3.4.11. *If G_2 has a vertex meeting all cycles and $|C| \geq 4$, then G is isomorphic to N.*

No three vertices of C are adjacent to a; otherwise G has $K_4 \oplus_0 L$ as a minor. By symmetry, no three vertices of C are adjacent to b. Therefore C has length exactly four. Let w_1, w_2, w_3, and w_4 be the vertices of C, labelled cyclically. Up to symmetry, either w_1 and w_2 are adjacent to a, and w_3 and w_4 are adjacent to b; or w_1 and w_3 are adjacent to a, and w_2 and w_4 are adjacent to b. But, if w_1 and w_2 are adjacent to a and w_3 and w_4 are adjacent to b, then G contains a C_5^2, an excluded minor for $\mathcal{V}_2$, as a proper minor; a contradiction. Hence, w_1 and w_3 are adjacent to a and w_2 and w_4 are adjacent to b, and G is isomorphic to N. So 3.4.11 holds.

3.4.12. *If G_2 has a vertex meeting all cycles and $|C| = 3$, then $G \in \mathcal{V}_2$.*

Let w_1, w_2 and w_3 be the vertices of C. Up to symmetry, we will assume w_1 and w_2 are adjacent to a, and w_3 is adjacent to b. Notice that w_3 is not adjacent to a; otherwise G contains $K_4 \oplus_0 L$ as a proper minor, a contradiction. Notice also that w_3 is not in a 2-cycle with b; otherwise G contains C_5^2 as a proper minor, a contradiction. Therefore w_3 is adjacent to only the vertices w_1, w_2, and b.

If both w_1 and w_2 are adjacent to b, then G contains $K_4 \oplus_0 L$ as a proper minor; a contradiction. Hence, at most one of w_1 and w_2 is adjacent to both a and b. If there are no 2-cycles in G_1 incident with a or b, then $G - \{w_1, v\}$ or $G - \{w_2, v\}$ is cycle-free.

Therefore we may assume that there is a 2-cycle with vertex set $\{a, w_1\}$. If w_2 is adjacent to both a and b, then G contains C_5^2 as a proper minor; a contradiction. This can be seen by deleting the edge joining w_2 and a, and contracting the edge joining w_2 and w_3. If both w_1 and a are in a 2-cycle and w_2 and a are in 2-cycles, then G contains $C_3^2 \oplus_0 L$ as a proper minor; a contradiction. This can be seen by deleting the edge w_3b and contracting the edge w_3w_2. Therefore G_1 is exactly a triangle $C = w_1w_2w_3$ with a 2-cycle joining w_1 and a, a single edge w_2a, and a single edge w_3b. However, in this case $G - \{w_1, v\}$ is cycle-free so $G \in \mathcal{V}_2$; a contradiction. Therefore, 3.4.12 holds.

3.4.13. *If G_2 has a vertex meeting all cycles and $|C| = 2$, then G is isomorphic to C_5^2.*

Let w_1 and w_2 be the vertices of the 2-cycle C. Since $G - \{w_1, v\}$ has a cycle, without loss of generality, w_2 and a are in a 2-cycle. Similarly, either w_1 and a, or w_1 and b are in a 2-cycle. If w_1 and a are in a 2-cycle, then G contains $C_3^2 \oplus_o L$ as a proper minor; a contradiction. This can be seen by deleting the 2-cycle from a to v. Therefore, w_1 and b are in a 2-cycle, and G is isomorphic to the excluded minor C_5^2. Hence, 3.4.13 holds.

We are now left with the case when G_2 has no vertex meeting all cycles and can be obtained from a subdivision G_0 of a graph listed in Theorem 3.1.1 (a)-(f). However, G_2 is not C_3^2; otherwise G has an edge joining a and b. Also, G_2 is not one of the graphs listed in (c), (d), or (f); otherwise $G_2 \backslash e$ contains a K_4-minor, and so G contains $K_4 \oplus_0 L$ as a proper minor, a contradiction. Therefore, G_2 is one of the graphs in (b) or (e).

3.4.14. *If G_2 has no vertex meeting all cycles, then $V(G_1) = \{a, b\} \cup V(C)$. Moreover, $\mathrm{si}(G_1 - \{a, b\}) = C$, and every vertex of C is adjacent to a or b.*

The previous statement is an immediate consequence of the argument given in 3.4.6, when there is an edge e joining a and b. We note that G_2 now contains a K_4-minor and hence a C_3^{2-}-minor, whereas in 3.4.6, we have that G_2 is C_3^2. However, the proof of 3.4.6, the edge e is never used. Thus, 3.4.14 holds.

If G_2 is one of the graphs in (e), then the underlying simple graph is a wheel. If this wheel has at least four spokes, then the basepoint e of the 2-sum is a rim edge and not a spoke; otherwise $G_2 \backslash e$ contains a K_4-minor and G contains $K_4 \oplus_0 L$ as a proper minor, a contradiction. If G_2 is (e) a wheel of size at least four and the basepoint is a rim edge, then by contracting an edge of the wheel not adjacent to the center vertex, v_c, or the vertices a or b, we see that G has two vertices whose removal is cycle-free. One of these vertices is the composite vertex, and the other is a vertex w_i of the cycle C. However, the removal of v_c and w_i is still cycle-free in G as $G_2 \backslash e - \{v_c\}$ is cycle-free and $G \in \mathcal{V}_2$; a contradiction.

Hence, $\mathrm{si}(G_2)$ is K_4. We now consider the structure of G_1. Suppose the cycle C in G_1 has at least four vertices. Since each vertex of C is adjacent to either a or b, it is easy to check that G contains either a $K_4 \oplus_0 L$ as a minor, a C_5^2-minor, or an N-minor. Therefore, we may assume C is either a 2-cycle or a 3-cycle. If C is a 3-cycle, then, by symmetry, we may assume two vertices of C are adjacent to a and one is adjacent to b, which implies that G is $S(K_4, K_4)$.

If C is a 2-cycle, then let $w_1, w_2 \in V(C)$ and, by symmetry, we may assume there is an edge from w_1 to a and from w_2 to b. Let $v_1 \neq a$ or b, where v_1 is a vertex of G_2. Notice that there cannot be 2-cycles with vertex sets $\{w_1, a\}$ and $\{w_2, b\}$; otherwise G contains C_5^2 as a proper minor, a contradiction. If G has both edges from w_1 to b and w_2 to a, then G is the excluded minor Y. Also, if G has w_1 and a in a 2-cycle and an edge from w_2 to a, then G is $S(K_4, C_3^2)$. Now, we are left with G_1 having edges from w_1 to a and from w_2 to b. There may also be either there a 2-cycle between w_1 and a and no other additional edges in G_1; or there may be an edge from w_2 to a and no other additional edges in G_1.

Now we consider G_2. Suppose G_2 is one of the graphs in (e) and the basepoint e of the 2-sum is a rim edge and not a spoke. Then $G \backslash e - v$ is cycle-free, where v is the center vertex and $v \neq a$ or b. Recall that G_1 is a 2-cycle with vertices w_1 and w_2 with two additional adjacent vertices a and b where w_1 is adjacent to a and w_2 is adjacent to b. Also, there may be either there a 2-cycle between w_1 and a and no other additional edges in G_1; or there may be an edge from w_2 to a and no other additional edges in G_1. In the first case, $G - \{w_1, v\}$ is cycle-free; and in the second case, $G - \{w_2, v\}$ is cycle-free; a contradiction. Hence, the basepoint e is a spoke edge, and not a rim edge, which means that either a or b is the center vertex. The graph G_2 has at most three 2-cycles, all adjacent to the center vertex. Since there is no edge joining a and b, the basepoint e not in a 2-cycle. If there are no 2-cycles in G_2 or exactly one 2-cycle in G_2, then it is easy to see that G_2 is isomorphic to one of the graphs in (e), with basepoint e being a rim edge. Hence, G_2 has exactly two 2-cycles, both of which are adjacent to a or b. If two 2-cycles are adjacent to a or two 2-cycles are adjacent to b, then G has $C_3^2 \oplus_0 L$ as a proper minor; a contradiction. This can be seen by deleting the edge joining a and w_1 and the edge joining a and w_2. □

This case analysis completes the determination of the excluded minors G of $\mathcal{V}_2$ with $\kappa(G) = 2$.

□

3.5 Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) = 3$

In this section, we find the excluded minors G for $\mathcal{V}_2$ with $\kappa(G) = 3$. The simple 3-connected excluded minors for $\mathcal{V}_2$ can be determined using the 3-connected excluded minors for $\mathcal{S}$. To obtain the non-simple excluded minors, we show that a non-simple excluded minor consists of a 2-cycle and another cycle, vertex disjoint from the first, with all other edges joining a vertex of the 2-cycle to vertex of the other cycle.

Theorem 3.5.1. *Let G be a simple graph with $\kappa(G) = 3$. Then G is an excluded minor for the class $\mathcal{V}_2$ if and only if G is isomorphic to one of the graphs shown in Figure 3.5.*

To prove this theorem, we will use the following two lemmas.

Lemma 3.5.2. *Let G be a simple excluded minor for $\mathcal{V}_2$ with $\kappa(G) = 3$. Then G is isomorphic to R, H_8, Q_3, or S.*

Proof. By Lemma 2.5.2, the graph $G \notin \mathcal{S}$, but every proper minor is a member of $\mathcal{S}$. Hence, G is a 3-connected excluded minor for $\mathcal{S}$. Notice that U has $K_4 \oplus_0 L$ as a minor, while V has $3L$ as a minor. However, it is straightforward to check that the remaining excluded minors for $\mathcal{S}$ with $\kappa(G) = 3$, shown in Figure 3.5 (a)-(d), are also excluded minors for $\mathcal{V}_2$. □

Lemma 3.5.3. *Let G be a non-simple excluded minor for $\mathcal{V}_2$ with $\kappa(G) = 3$. Then G is isomorphic to X, K_4^2 or W.*

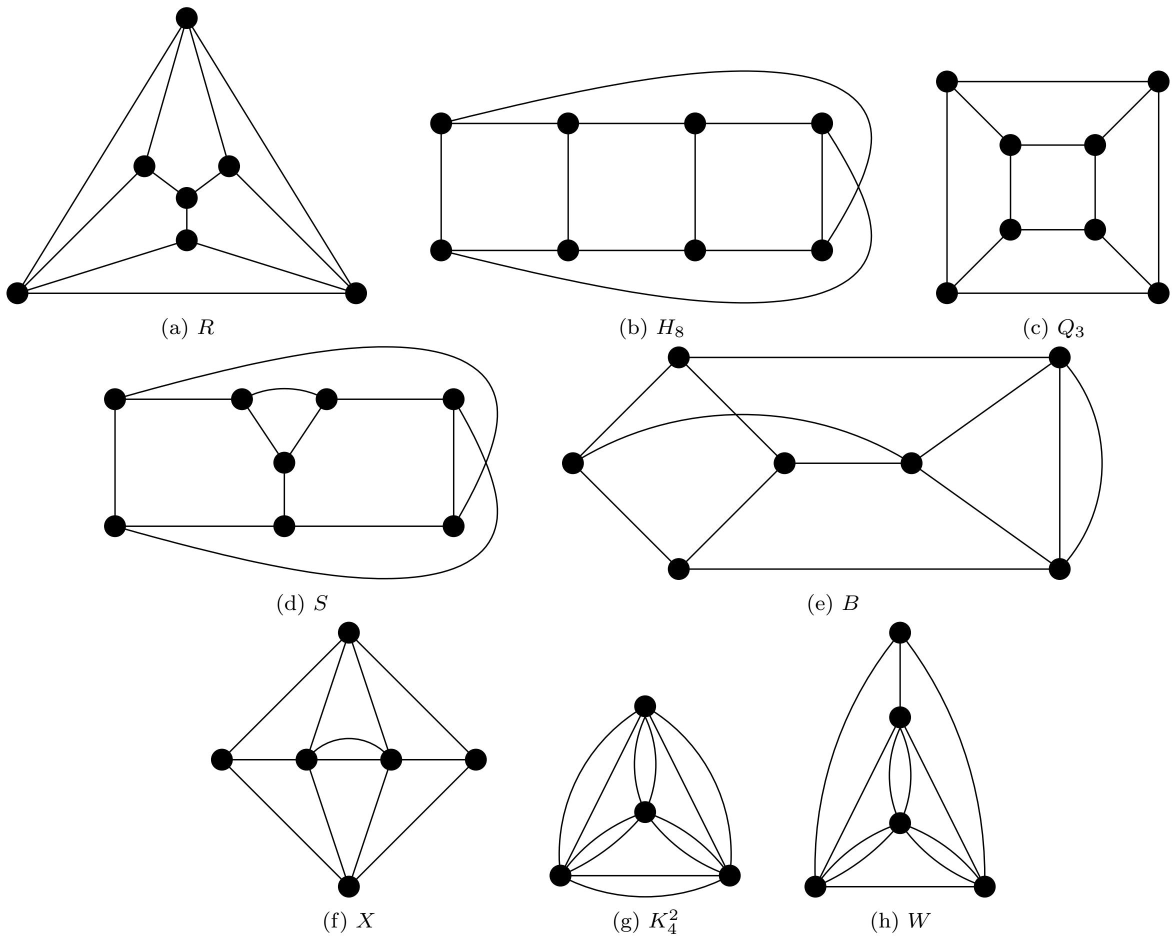


FIGURE 3.5: Excluded Minors for $\mathcal{V}_2$ with $\kappa(G) = 3$

Proof. By Lemma 3.3.4, G has no loops. Thus, G has a pair $\{f, g\}$ of parallel edges with end vertices u and v. By Lemma 3.3.5, no two vertices of G are joined by three or more parallel edges. As G is an excluded minor for $\mathcal{V}_2$, the graph $G - \{u, v\}$ has a cycle.

3.5.4. *Let C be a cycle in $G - \{u, v\}$. Every edge of G is incident with u, v, or a vertex of $V(C)$.*

Suppose there is an edge e that is not incident with u, v, or a vertex of C. Now, in G/e there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of these vertices is the composite vertex w resulting from contracting e. But $(G\backslash e) - w$ has vertex disjoint cycles on $\{u, v\}$ and $V(C)$, so no single vertex deletion from $(G\backslash e) - w$ is cycle-free, a contradiction. Thus, 3.5.4 holds.

Let $V(G) = \{u, v\} \cup V(C) \cup X$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $V(C) = \{w_1, \ldots, w_k\}$ for some $n \geq 1$ and $k \geq 2$. Since si(G) is 3-connected, each x_i for $1 \leq i \leq n$ is adjacent to at least three distinct vertices. No two distinct x_i and x_j are adjacent; otherwise the edge joining x_i to x_j is not incident with u, v, or a vertex of $V(C)$, contradicting 3.5.4. Also, no x_i is adjacent to three or more distinct vertices of $V(C)$; otherwise G contains $K_4 \oplus_0 L$ as a proper minor, a contradiction. Hence, each x_i can only be adjacent to u, v, and two distinct vertices of $V(C)$. Since G is 3-connected, each x_i has either three or four neighbors.

3.5.5. *x_i is not in a 2-cycle containing u or v.*

Suppose x_i is in a 2-cycle containing, without loss of generality u. Since si(G) is 3-connected, u is adjacent to a vertex y that is different from v and x_i. Let h be an edge joining u and y. Upon deleting h, there are two vertices, v_1 and v_2 say, whose removal leaves a cycle-free graph. Clearly $\{v_1, v_2\} \cap \{u, y\} = \emptyset$; otherwise there are two vertices of G whose removal is cycle-free, a contradiction. Since u and v are in a 2-cycle and u and x_i are in a 2-cycle, but $u \notin \{v_1, v_2\}$, we must have that $\{v_1, v_2\} = \{v, x_i\}$. But $(G \backslash h) - \{v_1, v_2\}$ has C as a cycle; a contradiction. Hence, 3.5.5 holds.

3.5.6. *If x_i is adjacent to u and v, then it is also adjacent to exactly two vertices of $V(C)$.*

Suppose that some x_i is adjacent to u, v, and exactly one vertex of $V(C)$, say w_1. After contracting an edge joining x_i to w_1, there are two vertices whose removal is cycle-free. By Lemma 3.1.4, one of them is the composite vertex formed in the contraction. The other is either u or v, say u, since u and v are in a 2-cycle. This also implies that $G - \{u, x_i, w_1\}$ is cycle-free. However, since G is an excluded minor, $G - \{u, w_1\}$ has a cycle and this cycle contains x_i, but x_i is only adjacent to v in $G - \{u, w_1\}$. Therefore x_i is in a 2-cycle with v, but this contradicts 3.5.5. Therefore, x_i is adjacent to some other vertex, and by 3.5.4 it must be a vertex of $V(C)$. Since x_i is adjacent to at most four distinct vertices, it is adjacent to exactly four vertices, and 3.5.6 holds.

Henceforth we may assume that, for any cycle C in $G - \{u, v\}$, any $x_i \in X$ is adjacent to exactly two vertices of $V(C)$ and one or two vertices of $\{u, v\}$. Choose the vertices u and v in a 2-cycle and the cycle C of $G - \{u, v\}$ so that $|V(C)|$ is a minimum. Then C has no chords.

Since each x_i is adjacent to exactly two vertices of $V(C)$, there is a cycle of $G - \{u, v\}$ through x_i that has at most $2 + \lfloor \frac{|V(C)|}{2} \rfloor$ edges. Since $|V(C)|$ is a minimum, it follows that $V(C) \leq 4$. Hence, if $X \neq \emptyset$, then $|V(C)| = 2, 3$ or 4.

3.5.7. *If* $|V(C)| = 2$, *then* $G \cong K_4^2$.

Let $V(C) = \{w_1, w_2\}$. Suppose $X \neq \emptyset$. Since every x_i is adjacent to exactly two vertices of $V(C)$, the vertex x_i is adjacent to both w_1 and w_2. By switching the roles of the 2-cycle containing $\{u, v\}$ and the 2-cycle containing $\{w_1, w_2\}$, we see that similarly x_i is adjacent to both u and v. Hence, every x_i is adjacent to w_1, w_2, u, and v. If there are two or more distinct x_i, then by deleting an edge joining u and v, we see that the resulting graph has Y as a proper minor; a contradiction. Therefore, there is at most one x_i. Suppose that there is exactly one x_i, implying $|V(G)| = 5$. By 3.5.5, the vertex x_i is not in a 2-cycle with u or v. Similarly, by switching the roles of the 2-cycle containing $\{u, v\}$ and the 2-cycle containing $\{w_1, w_2\}$, we see that x_i is not in a 2-cycle with w_1 or w_2. Therefore, x_i is not in a 2-cycle. Since $G - \{u, w_1\}$ has a cycle, there is an edge joining v and w_2. By symmetry there is an edge joining v and w_1, an edge joining u and w_1, and an edge joining u and w_2. However, this graph has a proper K_5 minor, which can be seen by deleting an edge joining u to v and an edge joining w_1 to w_2; a contradiction. Now, X is empty and $|V(G)| = 4$. Therefore, $G \cong K_4^2$. Thus 3.5.7 holds.

Henceforth, we may assume that if $X \neq \emptyset$, then $|V(C)| \in \{3, 4\}$. We may also assume that G has no two vertex-disjoint 2-cycles; otherwise the choice of C is violated.

3.5.8. *If* $|V(C)| = 4$ *and* $X \neq \emptyset$, *then* $G \cong B$.

Let the vertices of C, in cycle order, be w_1, w_2, w_3, and w_4. Since $|V(C)|$ is a minimum, there is no x_i joining two adjacent vertices of C; otherwise there is a smaller cycle of size three through x_i in $G - \{u, v\}$. Therefore x_i is adjacent to two nonadjacent vertices of C.

Assume $X \neq \emptyset$. Let x_i be a vertex adjacent to w_1 and w_3. Suppose some x_j different from x_i is adjacent to w_2. Since x_j is also adjacent to two nonadjacent vertices by the argument in the previous paragraph, it is adjacent to w_4. However, since we picked C to be an arbitrary cycle of minimum size, by taking C to be the cycle through $\{w_1, w_2, x_j, w_4\}$, there is an edge from x_i to w_3 that is not adjacent to C or $\{u, v\}$; a contradiction. Hence, every $x_i \in X$ is adjacent to the same two nonadjacent vertices of C. Say every x_i is adjacent to w_1 and w_3. Since $\kappa(\text{si}(G)) = 3$, each vertex in $\{w_2, w_4\} \cup X$ is adjacent to u or v. Thus, without loss of generality, there are at least two vertices of $\{w_2, w_4\} \cup X$ adjacent to v.

We show in what follows that $\{w_2, w_4\} \cup X$ contains a vertex adjacent to u and a vertex adjacent to v. Suppose no vertex in $\{w_2, w_4\} \cup X$ is adjacent to u. Then u is adjacent to only v, w_1, and w_3. Moreover, v is adjacent to all vertices in $\{u, w_2, w_4\} \cup X$ as well as possibly w_1 or w_3. Now $G - \{v, w_1\}$ has a cycle. This cycle, call it D, must be a 2-cycle. If D does not meet u, then the choice of C is contradicted. Thus, D meets u and so has vertex set $\{u, w_3\}$. By symmetry, G has a cycle with vertex set $\{u, w_1\}$. But then deleting the composite vertex after contracting an edge joining v and w_2 leaves a graph without a single vertex whose removal is cycle-free; a contradiction.

We may not suppose that there are two vertices, z_1 and z_2, in $\{w_2, w_4\} \cup X$ that are adjacent to v and a different vertex, say z_3 in $\{w_2, w_4\} \cup X$ that is adjacent to u. Let h be an edge joining u and z_3. As G is an excluded minor, upon contracting h, there are two vertices whose removal is cycle-free. By Lemma 3.1.5, the graph $G - \{u, x_3\}$ has a single vertex whose removal is cycle-free. If $|X| \geq 2$, then $G - \{u, z_3\}$ has a K_4-minor a contradiction. Thus, $|X| = 1$.

Suppose $\{w_2, w_4\} \cup X$ contains distinct vertices z_1 and z_2, that are both adjacent to u and v. Take $z_3 \in (\{w_2, w_4\} \cup X) - \{z_1, z_2\}$. Since w_1 and z_3 are adjacent, $G - \{w_1, z_3\}$ has a single vertex whose removal is cycle-free. But there is no such vertex. Hence, at most one vertex in $\{w_2, w_4\} \cup X$ is adjacent to both u and v.

Next assume that v is adjacent to z_3. By the previous argument, u is adjacent to neither z_1 nor z_2. Thus, without loss of generality, u is adjacent to w_3. By Lemma 3.1.5, $G - \{w_1, z_2\}$ has a single vertex whose removal is cycle-free. But $G - \{w_1, z_1\}$ has no such vertex. We deduce that v is not adjacent to z_3.

If u is adjacent to z_1 or z_2, then G has B as a subgraph, so $G \cong B$. Thus, we may assume that u is adjacent to neither z_1 nor z_2. Without loss of generality, u is adjacent to w_3. If $d(v) > 4$, consider the graph obtained by removing an edge s incident with v, leaving edges joining v to z_1 and z_2 and a 2-cycle on $\{u, v\}$. Then $G \backslash s$ has two vertices, v_1 and v_2 say, whose removal is cycle-free, neither of which is v. Then $u \in \{v_1, v_2\}$. But $(G \backslash s) - u$ has no single vertex whose removal is cycle-free. Hence $d(v) = 4$. Now, $G - \{v, w_3\}$ has a cycle, so either $\{u, z_3\}$ is in a 2-cycle, or u is adjacent to w_1. Assume that $\{u, z_3\}$ is in a 2-cycle. Upon deleting an edge t joining u and w_3 gives a graph having an N-minor. We deduce that $\{u, z_3\}$ is not in a 2-cycle, so u is adjacent to w_1. By Lemma 3.1.5, the graph $G - \{v, z_2\}$ has a single vertex whose removal is cycle-free, however, there is no such vertex; a contradiction.

3.5.9. *If* $|V(C)| = 3$, *then* $X = \emptyset$.

Suppose first that $|X| \geq 2$. If every x_i is adjacent to the same two vertices of $V(C)$, call them w_1 and w_2, then delete the edge joining w_1 and w_2. In $G \backslash w_1 w_2$, there are two vertices whose removal leaves a cycle-free graph, neither of which is w_1 or w_2; otherwise $G \in \mathcal{V}_2$, a contradiction. Since $\{u, v\}$ is in a 2-cycle, one of the vertices is either u or v, say u. However, $(G \backslash w_1 w_2) - u$ has no vertex, in $V(G) - \{w_1, w_2\}$ whose removal is cycle-free; a contradiction. So, if x_i is adjacent to w_1 and w_2, then, without loss of generality, x_j is adjacent to w_1 and

w_3. Since si(G) is 3-connected, $G - \{x_i, x_j\}$ is connected. Consider a path of minimum length in $G - \{x_i, x_j\}$, from $\{u, v\}$ to $V(C)$. Suppose this path is incident to v and one of w_2 or w_3, say w_3. Then, upon contracting an edge joining x_i and w_2, there are two vertices whose removal leaves a cycle-free graph. One of these is the composite vertex formed in the contraction. However, the removal of this vertex has two vertex-disjoint cycles, and there is no other vertex whose removal is cycle-free; a contradiction. Hence, we may assume this path is incident to u and w_1. Upon deleting an edge t of this path incident to w_1, there are two vertices whose removal is cycle-free. Again, w_1 is not one of these vertices; otherwise $G \in \mathcal{V}_2$, a contradiction. However, the removal of no two vertices of $V(C) - w_1$ from $G \backslash t$ leaves a cycle-free graph; a contradiction. Therefore there is at most one x_i.

We may now assume there is exactly one x_i, so the graph G has exactly six vertices. Suppose x_i is adjacent to u, v, w_1, and w_2, and, without loss of generality, w_3 is adjacent to v. Now, w_3 has only three neighbors as there is no edge joining w_3 to u; otherwise G has a proper Y-minor. Suppose $d(v) \geq 5$. Then there is an edge q incident with v after whose removal from G, there is still the 2-cycle through $\{u, v\}$, an edge from v to x_i, and an edge from v to w_3. However, $G \backslash q$ has two vertices whose removal is cycle-free, one of which is u since $\{u, v\}$ is in a 2-cycle. However, $(G \backslash q) - u$ has a K_4-minor, and there is no single vertex whose removal is cycle-free; a contradiction. Therefore, $d(v) = 4$. The only other possible edges are incident with u. As $G - \{v, x_i\}$ has a path from $\{w_1, w_2\}$ to u, we may assume that u is adjacent to w_1. If u is also adjacent to w_2, then G has K_5 as a proper minor, which can be seen by contracting the edge vw_3 and deleting one of the edges joining u and v. Therefore u is only adjacent to v, w_1, and x_i, and by 3.5.5, the only other possible edge of G creates a 2-cycle containing $\{u, w_1\}$. It follows that $G - \{v, w_1\}$ is cycle-free, a contradiction. Therefore, we may assume x_i is adjacent to only u, w_1, and w_2.

Suppose u is also adjacent to w_1. If u is adjacent to w_3, then $(G \backslash uw_1) - v$ has a vertex whose removal is cycle-free. However, $(G \backslash uw_1) - v$ has a K_4-minor, so there is no such vertex; a contradiction. Therefore, v is adjacent to w_3, but u is not. If u is adjacent to w_2, then upon contracting an edge joining w_3 and v, there are two vertices whose removal leaves a cycle-free graph. By Lemma 3.1.4, one of those vertices is the composite vertex formed in the contraction. However, after removing the composite vertex, there is still a K_4-minor so there is no other vertex whose removal is cycle-free; a contradiction. Therefore, if u is adjacent to w_1, then u is adjacent to only v, x_i, and w_1. Now, $G - \{v, w_1\}$ is cycle-free as the only 2-cycles of G are incident to u or v and incident with no x_i. Hence $G \in \mathcal{V}_2$; a contradiction. Thus, u is not adjacent to w_1. By symmetry, u is not adjacent to w_2. However, u is adjacent to three distinct vertices, so u is adjacent to w_3. Hence, u is adjacent to only v, x_i, and w_3.

Since v is adjacent to at least three distinct vertices, it is adjacent to one of w_1 and w_2, say w_2. Also, v is adjacent to either w_1 or w_3. If v is adjacent to both w_1 and w_3, after contracting the edge ux_i, there are two vertices whose removal is cycle-free, one of which is the composite vertex r by Lemma 3.1.4. So $(G/ux_i) - r$ has a K_4-minor and there is no other vertex whose removal is cycle-free; a contradiction. Hence, v is adjacent to exactly one of w_1 and w_3. If v is adjacent to w_1, then, since $G - \{u, w_2\}$ has a cycle, $\{v, w_1\}$ is the vertex set of a 2-cycle. By symmetry, $\{v, w_2\}$ is the vertex set of a 2-cycle. However, now G has W as a proper minor, which can be seen by contracting the edge ux_i; a contradiction. Hence, v is adjacent to only u, w_1, and w_3. Since $G - \{u, w_1\}$ has a cycle, $\{v, w_3\}$ is the vertex set of a 2-cycle as we have eliminated all other possibilities. However, this graph has $S(K_4, C_3^2)$ as a proper minor, which can be seen by deleting an edge joining v to w_1; a contradiction. Therefore, 3.5.9 holds.

Combining 3.5.7-3.5.9, we may assume $|V(C)| \geq 3$ and $X = \emptyset$. Recall that, since $|V(C)|$ is minimal, G has no two vertex-disjoint 2-cycles. Also, C has no chords so every vertex of C is adjacent to u or v. Recall that f and g are the edges joining u and v and the vertices of C, in cyclic order are $w_1, w_2, \ldots, w_k$.

Since G is an excluded-minor for $\mathcal{V}_2$ and $\kappa(\text{si}(G)) = 3$, the graph $G\backslash f$ has two vertices whose removal is a tree, call it T_f. Since neither u nor v is one of these vertices, they are w_a and w_b for some integers a and b. Observe the following.

3.5.10. *Every leaf of T_f is adjacent to both w_a and w_b.*

Since $G\backslash f$ is 3-connected and every vertex $w_i \in V(C)$ is adjacent to exactly two other vertices of $V(C)$ and possibly u and v, every w_i is adjacent to exactly 3 or 4 vertices for every $i \in [k]$. Hence, the tree $T_f = (G\backslash f) - \{w_a, w_b\}$ has at most four leaves by 3.5.10.

Suppose T_f has four leaves. Then every leaf of the tree is adjacent to both w_a and w_b. Therefore, the leaves of the tree are u, v, w_c, and w_d for some integers c and d. However, u and v are both leaves of the tree, and are also adjacent in $G\backslash f$; a contradiction as no two leaves are adjacent. Thus, T_f does not have four leaves. Therefore, either the tree T_f is a path, or it has exactly three leaves.

3.5.11. *If T_f is a path, then G is isomorphic to K_4^2, X, or W.*

If the path has length one, then clearly G is isomorphic to K_4^2 so we will assume the path has length at least two.

Since u and v are adjacent in T_f, they are not both leaves of T_f. Suppose one of u or v, say u, is a leaf of the path T_f. Then, the other leaf is w_j for some integer j. As w_a and w_b are adjacent to the ends of the path, w_j is adjacent to both w_a and w_b. Thus the neighbor of w_j on the path T_f must be v. Hence, T_f has two edges, so $|V(G)| = 5$ and w_a and w_b are adjacent. Since $G - \{v, w_b\}$ has a cycle and also every 2-cycle of G meets $\{u, v\}$, the vertices $\{u, w_a\}$ are in a 2-cycle. By symmetry, $\{u, w_b\}$ is in a 2-cycle. Since G has no two vertex-disjoint 2-cycles, it follows that G is a subgraph of W. As W is an excluded minor, we conclude that $G = W$.

Now, we may assume that neither u nor v is an end of the path. This implies that the edge g lies in the interior of the path T_f, so the path must have length at least three. In fact, the path must have length exactly three because, if the path has length at least four, it has an end w_l that is not adjacent to either u or v; a contradiction. Thus $|V(G)| = 6$. Let w_c and w_d be the ends of the path for some integers c and d.

By 3.5.10, w_a and w_b are adjacent to the ends of the path, and both w_a and w_b are adjacent to u or v. If both w_a and w_b are adjacent to u and v, then G is isomorphic to the excluded minor X. Therefore $\text{si}(G) + f$ is as shown in Figure 3.6, where the dotted edge may or may not be present. We may assume that there is another parallel class other than $\{f, g\}$, otherwise $G \in \mathcal{V}_2$.

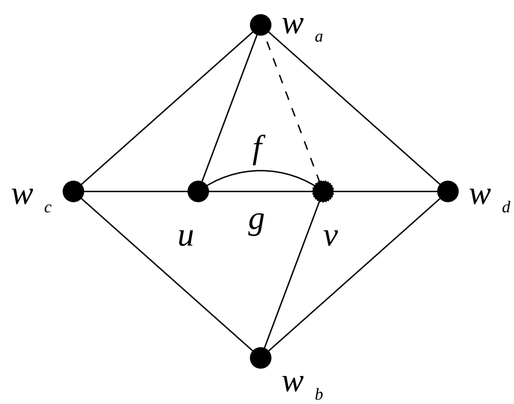


FIGURE 3.6: Illustration of $\text{si}(G) + f$ when T_f is a path. A dotted edge may or may not be present in the graph.

Since G is an excluded minor for $\mathcal{V}_2$, the graph obtained by deleting $\{v, w_a\}$ from G has a cycle, so either $\{w_d, w_b\}$, $\{w_c, u\}$, or $\{w_c, w_b\}$ is in a 2-cycle. As G has no two vertex-disjoint 2-cycles, $\{w_c, u\}$ is in a 2-cycle. If the dotted edge is absent, then, by symmetry, $\{v, w_d\}$ is in a 2-cycle and we have two vertex-disjoint 2-cycles; a contradiction. Thus, the dotted edge e is present. After deleting $\{v, w_c\}$, the resulting graph has $\{w_a, u\}$ in a 2-cycle. Now, $G/w_c w_b$ has W as a minor, a contradiction. Thus, 3.5.11 holds.

3.5.12. *T_f does not have exactly three leaves.*

Suppose the tree T_f has exactly three leaves. Since $\text{si}(G\backslash f)$ is 3-connected, each vertex of $\text{si}(G\backslash f)$ has degree at least three. By 3.5.10, w_a and w_b are adjacent to every leaf of the tree.

Thus, at most two of these leaves are in $V(C)$. Therefore u or v, say u, is a leaf of the tree. Hence u is adjacent to only v, w_a, and w_b. Since $\kappa(\mathrm{si}(G)) = 3$, the vertex v is adjacent to every w_i where $i \notin \{a, b\}$. If there were two adjacent w_i and w_j for $i, j \notin \{a, b\}$, then, since v is adjacent to both w_i and w_j, there is a cycle through v, w_i, and w_j in $(G\backslash f) - \{w_a, w_b\}$, a contradiction. Hence, $|V(C)| = 4$ and $|V(G)| = 6$. There are two w_i that are different from w_a and w_b, call them w_c and w_d for some integers c and d. Both w_c and w_d are adjacent to both w_a and w_b as they are nonadjacent vertices.

Since u is a leaf of T_f, its neighbor in T_f is v, and v is also adjacent to both w_c and w_d in T_f. Then $\mathrm{si}(G) + f$ is the graph shown in Figure 3.7 where the dotted edge may or may not be present.

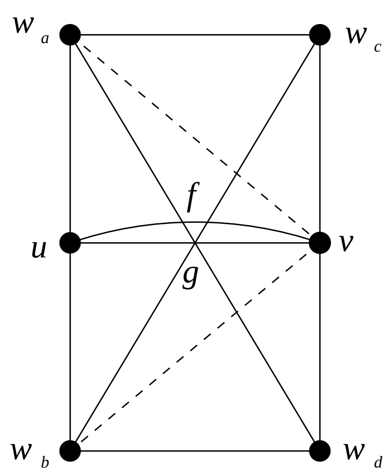


FIGURE 3.7: Illustration of $\mathrm{si}(G) + f$ when T_f has three leaves. A dotted edge may or may not be present in the graph.

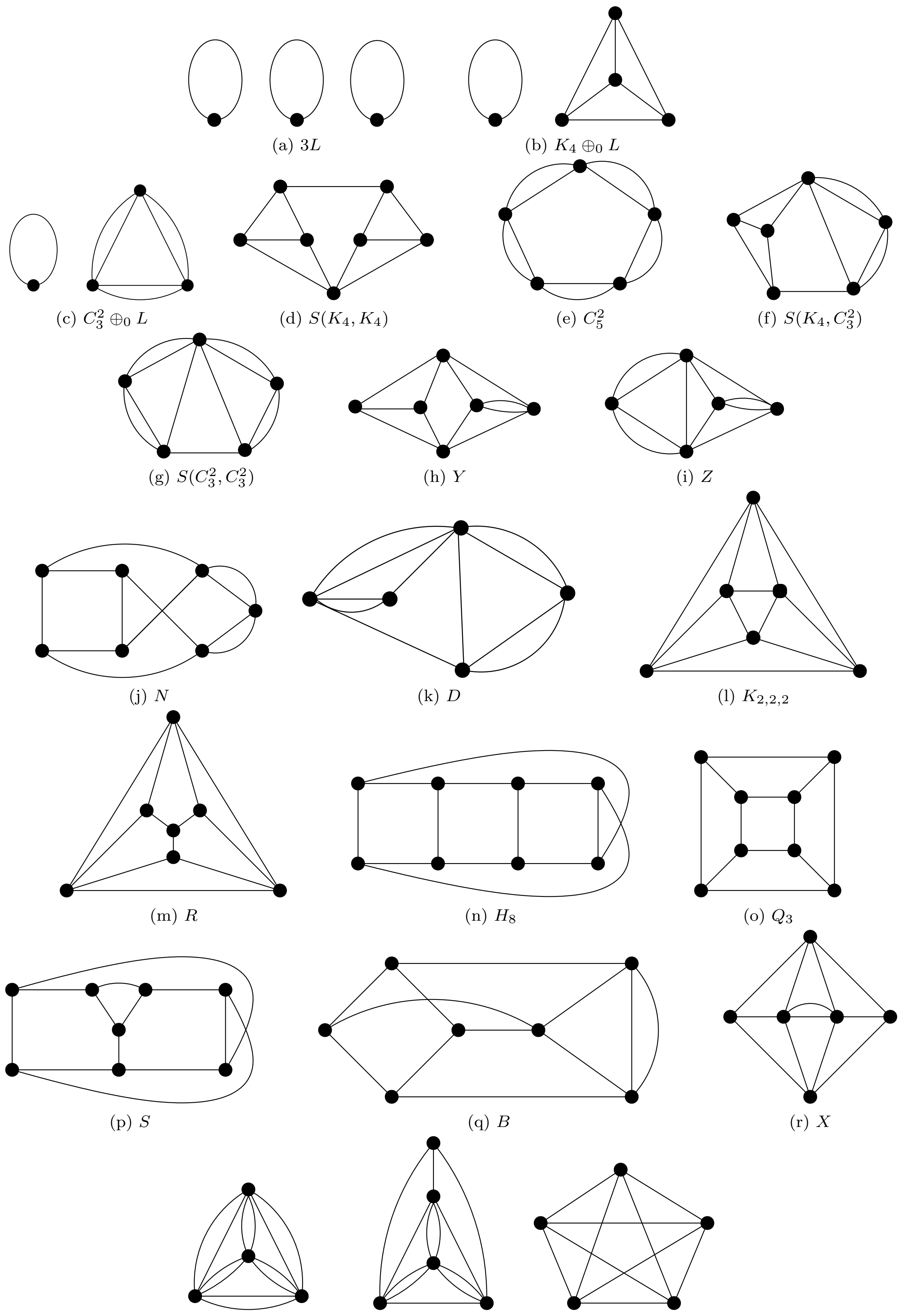
Since $G - \{v, w_b\}$ has a cycle, $\{u, w_a\}$ is in a 2-cycle. By symmetry, $\{u, w_b\}$ is in a 2-cycle. If there is an edge joining v to w_b, then G has X as a proper minor, which can be seen by contracting $w_a w_c$; a contradiction. Therefore there is no edge joining v to w_b. By symmetry there is no edge joining v to w_a. Hence, the graph $\mathrm{si}(G) + f$ is the graph shown in Figure 3.7 with no dotted edges. Now, since $G - \{u, w_c\}$ has a cycle, $\{v, w_d\}$ is in a 2-cycle, so G has two vertex-disjoint 2-cycles; a contradiction. Hence, 3.5.12 holds.

This concludes the proof of Lemma 3.5.3, finishing the classification of the excluded minors for $\mathcal{V}_2$ with $\kappa(G) = 3$ for Theorem 3.5.1.

□

By combining Theorems 3.3.1, 3.3.2, 3.4.1, and 3.5.1, we get the main result of this chapter, the excluded minor characterization for $\mathcal{V}_2$.

Theorem 3.5.13. *The excluded minors for* $\mathcal{V}_2$*, the class of graphs* G *such that* $G - \{u_1, u_2\}$ *has no cycles for some* $u_1, u_2 \in V(G)$*, are the twenty-one graphs shown in Figure 3.8.*



(a) $3L$ (b) $K_4 \oplus_0 L$

(c) $C_3^2 \oplus_0 L$ (d) $S(K_4, K_4)$ (e) C_5^2 (f) $S(K_4, C_3^2)$

(g) $S(C_3^2, C_3^2)$ (h) Y (i) Z

(j) N (k) D (l) $K_{2,2,2}$

(m) R (n) H_8 (o) Q_3

(p) S (q) B (r) X

(s) K_4^2 (t) W (u) K_5

FIGURE 3.8: Full List of Excluded Minors for $\mathcal{V}_2$

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Vita

Lisa Warshauer was born on June 20 1983, in San Marcos, Texas. She finished her undergraduate studies at Rice University in May 2005. In August 2005, she came to Louisiana State University to pursue graduate studies in mathematics. She earned a master of science degree in mathematics from Louisiana State University in May 2007. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2011.