1968

Weighted Locally Convex Spaces of Continuous Functions.

William H. Summers

Louisiana State University and Agricultural & Mechanical College

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OF CONTINUOUS FUNCTIONS

A Dissertation

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in
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by
William H. Summers
B.S., University of Texas at Arlington, 1961
M.S., Purdue University, 1963
August, 1968

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\( K^+(X) \) non-negative constant functions on \( X \)

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In this paper we will obtain a representation of the bi-
equicontinuous completed tensor product of weighted function
spaces as another weighted space, and a representation of the
topological dual spaces of weighted spaces (to do this, we
found it advantageous to make a detailed investigation of
the weighted function spaces $CV_0(X)$). Moreover, we consider
the question of full-completeness in function spaces, and
obtain several illuminating results in this direction.

The first chapter contains preliminary material (with­
out proofs) brought together for the convenience of the
reader. In Chapter II we define a Nachbin family $V$, the
weighted space $CV_0(X)$, and give several examples. These
examples show that virtually all continuous function spaces
commonly encountered in analysis are weighted spaces (e.g.,
$(C_0(X), |||·|||)$ and $(C(X), c_{op})$ are weighted spaces, as
well as the extremely important space $(C_b(X), \beta)$).

Chapter III contains a discussion of the basic properties
of the spaces $CV_0(X)$. Here we consider such questions as
completeness, the existence of approximate identities (both
bounded and unbounded), and a characterization of the bounded
subsets. The main part of this chapter, however, is devoted
to characterizing the topological dual spaces $CV_0(X)^*$ of
the spaces $CV^O(X)$ for a large class of Nachbin families $V$. This characterization leads to results on factorization of measures and a useful characterization of a base for the equicontinuous subsets of $CV^O(X)^*$, together with a characterization of the extremal points of the members of this base. We give, finally, a Stone-Weierstrass theorem for $CV^O(X)$ which includes the known results of this type.

In Chapter IV we prove our principal result: a weighted representation theorem for biequicontinuous completed tensor products (Theorem 4.8). This result is then used to obtain Grothendieck's representation of $C_b(X) \sim C_b(Y)$, an analogous representation of $(C_b(X), \beta) \sim (C_b(Y), \beta)$, and several other similar cases. In doing this we encounter some interesting new subspaces of $C(X \times Y)$.

Chapter V contains several applications of the previously developed theory to the question of full-completeness in function spaces. In particular, we obtain a condition on $X$ necessary for $CV^O(X)$ to be fully complete, and this condition is used to rule out certain properties on $X$ as being sufficient for full-completeness. Moreover, a necessary condition for the full-completeness of $CV^O(X) \sim C_b(Y)$ is obtained, and we show that a converse of this result would have useful consequences. We conclude this chapter by giving a necessary and sufficient condition (involving the notion of a Nachbin family) for $(C_b(X), \beta)$ to be $B^*_T$-complete.
INTRODUCTION

One of the earliest appearances of the concept of using weights to determine both a subspace of the space $C(X)$ of all scalar-valued continuous functions on a topological space $X$ and a locally convex topology on this subspace was the classical approximation problem of Bernstein [2]. Nachbin [19, 20] treats this problem and the more general 'weighted approximation problem'. This concept has also been used in the study of entire functions by Taylor [29].

In this paper we will take a variation of the definition of a set of weights as given by Nachbin [19, 20] which allows us to introduce a partial order on the sets of weights. The weighted spaces which arise in this way are the same as those defined by Nachbin, and by focusing our attention on sets of weights belonging to certain intervals induced by our partial order, many useful properties of weighted spaces are deduced. Among the more interesting of these results is our characterization of the topological duals of weighted spaces.

We also give several examples of weighted spaces which show that virtually all of the spaces of continuous functions usually encountered in analysis are weighted spaces. One of the most interesting of these, and one that has received much recent attention, is the space $C_b(X)$ of all bounded continuous
complex-valued functions on a locally compact Hausdorff space $X$ endowed with the strict topology $\beta$. It has been studied by Buck [4], Conway [9], Collins and Dorroh [8], and Collins [7], to mention only a few. The topology $\beta$ has also been used in the study of spaces of bounded holomorphic functions by Rubel and Shields [24], and in problems in spectral synthesis by Herz [16].

In [10], Dieudonné showed that $C(X) \oplus C(Y)$ could be embedded as a dense subspace of $C(X \times Y)$ endowed with the compact-open topology. One of the results of our investigation is to add a topological dimension, so to speak, to a generalization of this 'algebraic' theorem and obtain a representation of the biequicontinuous completed tensor product of two weighted spaces as another weighted space. From this representation we are able to give several new and interesting explicit examples including the case of $(C_b(X), \beta) \cong (C_b(Y), \beta)$.

We also apply our results to the study of full-completeness in weighted spaces. It is in this area that very little is known except when formally stronger properties (e.g., Fréchet) imply full-completeness. We are able, however, to establish a necessary condition for the full-completeness of weighted spaces whose determining set of weights belongs to a certain interval. In particular, we obtain the result for $(C_b(X), \beta)$. We also investigate the relationship between the full-completeness of the biequicontinuous completed tensor
product of weighted spaces and the full-completeness of the
weighted spaces themselves, and obtain the surprising result
that, in order for the biequicontinuous completed tensor
product to be fully complete, it is necessary for the weighted
spaces themselves to be fully complete. A converse to this
result would have considerable interest, but appears to us
at this time to be a quite difficult problem.

In order to make the reading of this paper as painless
as possible, we have tried to make it essentially self-
contained and included considerable detail in our proofs.
In addition, we have included a table of spaces and an index
of symbols and definitions. Finally, all theorems, examples,
and other such items are numbered consecutively for easy
reference, with item y in Chapter x being labelled x.y.
CHAPTER I

Preliminaries

In this chapter we will set the stage for our investigation by introducing those spaces of continuous functions which play a role in our development. In the sequel, our notation will be primarily that of [18]. We will assume a familiarity with the basic ideas of topology as found in Kelley [17], measure theory as found in Rudin [25], and the theory of locally convex topological vector spaces as found in [23].

We will let $\mathbb{R}$ denote the space of real numbers with the usual topology, while $\mathbb{C}$ will denote the complex numbers with the usual topology, and $\mathbb{N}$ will denote the positive integers with the discrete topology (when a topology is implied). Throughout the remainder of this paper, $X$ (and $Y$) will denote a completely regular $T_1$-space. Although we will sometimes find occasion to hypothesize additional and even stronger properties (e.g., locally compact and Hausdorff), complete regularity and $T_1$ will always be implicit.

We shall let $C(X)$ denote the space of all continuous complex-valued functions on $X$, while $B(X)$ will denote the space of all bounded complex-valued functions on $X$. 

4
A function $f: X \to \mathbb{C}$ vanishes at infinity if 
\[ \{ x \in X : |f(x)| \geq \epsilon \} \] is relatively compact for every $\epsilon > 0$.
Let $B_0(X)$ denote the space of all complex-valued functions on $X$ which vanish at infinity, $C_b(X) = C(X) \cap B(X)$, and $C_0(X) = C(X) \cap B_0(X)$. For a function $f: X \to \mathbb{C}$, let $N(f) = \{ x \in X : f(x) \neq 0 \}$; $N(f)$ is called the non-zero set of $f$, $\overline{N(f)}$ (the topological closure of $N(f)$ in $X$) is called the support of $f$, and this set will be denoted by $\text{spt}(f)$. Define $C_c(X)$ to be the space of all continuous complex-valued functions on $X$ which have compact support, and note that $C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq B(X)$. Frequently, it will be necessary to speak of the subspace of a space of complex-valued functions consisting of all elements which take values in the non-negative reals ($\mathbb{R}^+$). We will identify this subspace by superscripting with the symbol $^+$; e.g., $C^+(X)$ is the space of all non-negative continuous functions on $X$.

A real-valued function $f$ on $X$ will be called upper semicontinuous (u.s.c.) if $\{ x \in X : f(x) < a \}$ is open for every $a \in \mathbb{R}$ and lower semicontinuous (l.s.c.) if $\{ x \in X : f(x) > a \}$ is open for every $a \in \mathbb{R}$. We refer the reader to Rudin [25] for the properties of such functions.
Let $N(X)$ denote the space of all complex-valued functions $f$ on $X$ with the property that $|f|$ is u.s.c.

Now let $S$ be a subset of $X$, let $f$ be a $C$-valued function on $X$, and denote by $R(f; S)$ the restriction of
f to $S$. If $R(f; S) \in B(S)$, then define $||f||_S = \sup\{|f(x)|: x \in S\}$. In the case that $S = X$ we write $||f||$, and $||\cdot||$ is a norm on $B(X)$. Moreover, in the topology (called the uniform topology) induced by this norm, $C_0(X)$, $C_b(X)$, and $B(X)$ are Banach spaces while $C_c(X)$ is uniformly dense in $C_0(X)$.

By a locally convex space we will mean a locally convex topological vector space which is also Hausdorff. If $E$ is a locally convex space, then we will denote by $E^*$ the vector space of all continuous linear functionals on $E$. If $x \in E$ and if $x^* \in E^*$, then the value of $x^*$ at $x$ will be denoted by $\langle x, x^* \rangle$. The weak-* topology on $E^*$ is denoted by $\sigma(E^*, E)$ and is defined by the family $\{P_x : x \in E\}$ of semi-norms on $E^*$ where $P_x(x^*) = |\langle x, x^* \rangle|$. Similarly, the weak topology on $E$ is denoted by $\sigma(E, E^*)$ and is defined by the family $\{P_{x^*} : x^* \in E^*\}$ of semi-norms on $E$ where $P_{x^*}(x) = |\langle x, x^* \rangle|$. If $A \subseteq E$, then the polar of $A$ is defined to be $\{x^* \in E^* : |\langle x, x^* \rangle| \leq 1 \text{ for all } x \in A\}$ and will be denoted by $A^\circ$. The reader is invited to examine [23] for the basic properties of polar sets.

For the remainder of this chapter we will assume $X$ is locally compact. Now let $K$ be a compact subset of $X$ and define $C(X; K) = \{f \in C_c(X) : \text{spt}(f) \subseteq K\}$, endowed with the uniform topology; $C(X; K)$ is a Banach space. Now let $\{K_i\}$ be a base for the compact subsets of $X$ and let $\phi_i : C(X; K_i) \rightarrow C_c(X)$ be the injection map for each $i$. The
inductive limit topology on $\mathcal{C}_c(X)$ is the finest locally convex linear topology for which all of the mapping $\varphi_i$ are continuous. It is easy to see that this topology is independent of which base $\{K_i\}$ for compacta in $X$ is chosen. We will let $(\mathcal{C}_c(X), \text{ind lim})$ denote $\mathcal{C}_c(X)$ endowed with the inductive limit topology, and we will denote $(\mathcal{C}_c(X), \text{ind lim})^*$ by $M(X)$.

The elements of $M(X)$ are called complex Radon measures on $X$. Every $\mu \in M(X)$ can be expressed uniquely as $\alpha + i\beta$ where $\alpha$ and $\beta$ are real Radon measures. We have the following minimal decomposition of a real Radon measure on $X$.

1.1. Theorem ([11, p.178]). If $\lambda$ is a real Radon measure on $X$, then there exist unique positive Radon measures $\lambda^+$ and $\lambda^-$ on $X$ with the following properties:

(a) $\lambda = \lambda^+ - \lambda^-$;

(b) if $\alpha$ and $\beta$ are positive Radon measures on $X$ for which $\lambda = \alpha - \beta$, then $\alpha - \lambda^+$ and $\beta - \lambda^-$ are positive Radon measures on $X$;

(c) if $f \in \mathcal{C}_c^+(X)$, then $\lambda^+(f) = \sup \{\lambda(g) : g \in \mathcal{C}_c^+(X), g \leq f\}$.

We have the following measure theoretic characterization of $M^+(X)$ (and hence of $M(X)$ by the preceding theorem).

1.2. Theorem ([25, p.40]). $L \in M^+(X)$ if and only if there exists a $\sigma$-algebra $\mathcal{M}$ in $X$ which contains Borel $\mathcal{(X)}$.
(i.e., the σ-algebra generated by the open sets in X) and there is a unique positive measure μ on ℳ which represents L in the sense that

(a) \( L(f) = \int f \, d\mu \) for every \( f \in C_c(X) \);
(b) \( \mu(K) < \infty \) for every compact set \( K \subseteq X \);
(c) \( \mu \) is outer regular; i.e., for every \( E \in \mathcal{M} \) we have \( \mu(E) = \inf\{\mu(V) : E \subseteq V, \ V \text{ open}\} \);
(d) \( \mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\} \) holds for every open set \( E \), and for every \( E \in \mathcal{M} \) with \( \mu(E) < \infty \);
(e) if \( E \in \mathcal{M} \), \( A \subseteq E \), and \( \mu(E) = 0 \), then \( A \in \mathcal{M} \).

We define the support of a Radon measure \( \mu \in M^+(X) \) as the complement in \( X \) of the union of all open sets \( U \subseteq X \) for which \( \mu(U) = 0 \), and denote this set by \( \text{spt} \mu \).

By definition, \( \text{spt} \mu \) is a closed (possibly void) subset of \( X \). If \( \mu \in M(X) \), then the support of \( \mu \) is defined to be the support of the positive measure \( V(\mu) = \alpha^+ + \alpha^- + \beta^+ + \beta^- \) where \( \alpha \) and \( \beta \) are the real and imaginary parts of \( \mu \) and \( \alpha = \alpha^+ - \alpha^- \), \( \beta = \beta^+ - \beta^- \) are the minimal decompositions of \( \alpha \) and \( \beta \).

1.3. Theorem ([11, p.202]). Let \( \mu \in M(X) \) and let \( U \) be an open subset of \( X \). Then \( U \subseteq X \setminus \text{spt} \mu \) (the complement of \( \text{spt} \mu \) in \( X \)) if and only if \( \int f \, d\mu = 0 \) for every \( f \in C_c(X) \) for which \( \text{spt}(f) \subseteq U \).

It follows from Theorem 1.2 that a measure \( \mu \in M^+(X) \) will be inner regular (i.e., for every \( E \in \text{Borel}(X) \),

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\( \mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\} \) provided \( \mu(X) < \infty \). We will let \( M_b(X) = \{\mu \in M(X) : \forall (\mu)(X) < \infty\} \); i.e., the set of all bounded regular Borel measures on \( X \). If \( \mu \in M_b(X) \) and \( E \in \text{Borel}(X) \), then set \( |\mu|(E) = \sup\{\sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a partition of } E \text{ by Borel sets}\} \). Then \( |\mu| \), called the variation of \( \mu \), is in \( M_b(X) \), and \( |\mu| = |\mu|(X) \) defines a norm on \( M_b(X) \) making \( M_b(X) \) into a Banach space. We note that \( \text{spt} \mu = \text{spt}|\mu| \).

1.4. **Theorem (Riesz Representation Theorem [25, p.131])**. A linear functional \( L \) on \( (C_c^0(X), ||\cdot||) \) is continuous if and only if there corresponds a unique \( \mu \in M_b(X) \) such that \( L(f) = \int fd\mu \) for every \( f \in C_c^0(X) \). Moreover, \( ||\mu|| = ||L|| = \sup\{|\int fd\mu| : f \in C_c(X), ||f|| \leq 1\} \).

1.5. **Corollary.** If \( U \) is open in \( X \) and if \( \mu \in M_b(X) \), then \( |\mu|(U) = \sup\{|\int fd\mu| : f \in C_c(X), ||f|| \leq 1, \text{spt}(f) \subseteq U\} \).

The \( \beta \) or strict topology on \( C_b(X) \) is that locally convex topology on \( C_b(X) \) induced by the semi-norms \( P_\varphi(f) = ||f\varphi|| \), for every \( \varphi \in C_c^0(X) \). Conway [9] is an excellent source of information on \( (C_b(X), \beta) \).

1.6. **Theorem (Buck [4])**. A linear functional \( L \) on \( (C_b(X), \beta) \) is continuous if and only if there is a unique \( \mu \in M_b(X) \) such that \( L(f) = \int fd\mu \) for all \( f \in C_b(X) \). In particular, \( (C_b(X), \beta)^* = M_b(X) \).

Now let \( \{K_i\} \) be a base for compacta in \( X \). The compact-open (c-op) topology on \( C(X) \) is that locally convex
topology on $C(X)$ induced by the semi-norms $P_i(f) = ||f||_{K_i}$, for every $i$. Note that this topology is independent of the choice of the base for compacta, and that $(C(X), c\text{-}op)$ is a locally convex space even when $X$ is not assumed to be locally compact (recall our local compactness assumption for the balance of this chapter).

1.7. **Theorem** ([11, p.203]). A linear functional $L$ on $(C(X), c\text{-}op)$ is continuous if and only if there exists a unique $\mu \in M(X)$ such that $\text{spt} \mu$ is compact and $L(f) = \int f \text{d}\mu$, for every $f \in C(X)$.

Hence $(C(X), c\text{-}op)^* = M_c(X)$, where $M_c(X) = \{ \mu \in M(X) : \text{spt} \mu \text{ is compact} \}$. Observe that $M_c(X) \subseteq M_b(X) \subseteq M(X)$.

We conclude this chapter with some remarks on extremal points. If $a$ and $b$ are in a locally convex space $E$, then $[a, b]$ denotes $\{ \lambda a + (1-\lambda)b : \lambda \in [0, 1] \}$; if $A \subseteq E$, then $x \in A$ is called an **extremal point** of $A$ if $[a, b] \subseteq A$ and $x \in [a, b]$ implies $x = a = b$. The set of all extremal points of a subset $A \subseteq E$ will be denoted by $\mathcal{S}(A)$ (where $\mathcal{S}(A)$ is possibly void).

1.8. **Theorem** (Krein-Milman Theorem [23, p.138]). If $A$ is a convex compact subset of $E$, then $A$ is the closed convex hull of $\mathcal{S}(A)$.

If $B = \{ f \in C_0(X) : ||f|| \leq 1 \}$, then $B^\circ$ is the closed unit ball in $M_b(X)$. Conway [9] has extended a result of 

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1.9. Theorem. Let $E_1$ and $E_2$ be locally convex spaces, $K_i$ a compact convex subset of $E_i$, $i = 1, 2$, and $t: K_1 \to K_2$ a continuous onto affine map. If $x_2 \in \mathcal{S}(K_2)$, then there exists $x_1 \in \mathcal{S}(K_1)$ such that $t(x_1) = x_2$.

1.10. Corollary. Let $E$ be a locally convex space and let $x \in E$. If $A$ is an absolutely convex $\sigma(E^*, E)$-compact subset of $E^*$, then there exists $x^* \in \mathcal{S}(A)$ such that $\langle x, x^* \rangle = \sup\{|\langle x, y^* \rangle| : y^* \in A\}$.
CHAPTER II

Nachbin Families and the Space $CV_o(X)$

In this chapter we will define a Nachbin family and the locally convex space of continuous functions determined by it, and then investigate the relation between such spaces for different Nachbin families. Examples will also be given (including several new ones), in order to make clear how our results in the following chapters are connected with the familiar spaces of continuous functions.

2.1. Definition. A Nachbin family on $X$ is a set $V$ of non-negative u.s.c. functions on $X$ which satisfies the following condition:

(*) if $u, v \in V$ and if $\lambda > 0$, then there is a $w \in V$ such that $\lambda u, \lambda v \leq w$ (pointwise on $X$).

In order to define the associated space of functions, we will need the following properties of u.s.c. functions which are not among those properties usually listed, and so we provide our own proofs.

2.2. Theorem. If $u$ and $v$ are non-negative u.s.c. functions on $X$, then $uv$ is u.s.c.

Proof. Choose $\epsilon > 0$ and let $A = \{x \in X : u(x)v(x) < \epsilon\}$. It will suffice to show $A$ is open, and, to this end, we
fix $x_0 \in A$. If $u(x_0) = 0$, then

$B = \{ x \in X : u(x) < \varepsilon (v(x_0) + 1)^{-1} \} \cap \{ x \in X : v(x) < v(x_0) + 1 \}$

is a neighborhood of $x_0$ and $B \subseteq A$. Hence we may assume $u(x_0), v(x_0) > 0$. Now choose $n \in \mathbb{N}$ so that $\varepsilon \leq nu(x_0)v(x_0)$; thus $\varepsilon - u(x_0)v(x_0) \leq (n-1)u(x_0)v(x_0)$, which implies $(\varepsilon - u(x_0)v(x_0))(u(x_0)v(x_0))^{-1} \leq n - 1$. Clearly, there exists $m \in \mathbb{N}$ such that $m^2 - 2m - (n-1) \geq 0$, and hence $2m^{-1} + (n-1)m^{-2} \leq 1$. If we let $\eta = (\varepsilon - u(x_0)v(x_0))(mv(x_0))^{-1}$, $a = (\varepsilon - u(x_0)v(x_0))(mu(x_0))^{-1}$, and

$B = \{ x \in X : v(x) < v(x_0) + \sigma \} \cap \{ x \in X : u(x) < u(x_0) + \eta \}$,

then $B$ is a neighborhood of $x_0$. Also, if $x \in B$, then

$u(x_0)v(x) < u(x_0)v(x_0) + \sigma u(x_0) + \eta v(x_0) + \eta \sigma = u(x_0)v(x_0) + 2m^{-1}(\varepsilon - u(x_0)v(x_0)) + m^{-2}(\varepsilon - u(x_0)v(x_0))^2(u(x_0)v(x_0))^{-1} \leq u(x_0)v(x_0)$

$+ 2m^{-1}(\varepsilon - u(x_0)v(x_0)) + (n-1)m^{-2}(\varepsilon - u(x_0)v(x_0)) = u(x_0)v(x_0) + (2m^{-1} + (n-1)m^{-2})(\varepsilon - u(x_0)v(x_0)) \leq \varepsilon$; i.e.,

$B \subseteq A$. Thus $A$ is open and the proof is complete.

2.3. Theorem. $N(X) \cap \mathcal{B}_0(X) \subseteq \mathcal{B}(X)$.

Proof. Let $f \in N(X) \cap \mathcal{B}_0(X)$, and suppose $f \notin \mathcal{B}(X)$. Hence there exists $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $|f(x_n)| \geq n$ for $n \geq 1$. Since $K = \{ x \in X : |f(x)| \geq 1 \}$ is compact, and since $\{x_n\}_{n=1}^{\infty} \subseteq K$, $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x_0 \in K$. But $A = \{ x \in X : |f(x)| < |f(x_0)| + 1 \}$ is a neighborhood of $x_0$, which implies there is an $n \in \mathbb{N}$, $n \geq |f(x_0)| + 1$, such that $x_n \in A$. This contradicts $|f(x_n)| \geq n$, and so $f \in \mathcal{B}(X)$.
2.4. Definition. Let \( V \) be a Nachbin family on \( X \).

Then \( CV_0(X) = \{ f \in C(X) : fv \in B_0(X) \text{ for every } v \in V \} \),

endowed with the topology generated by the set of semi-norms \( \{ P_v : v \in V \} \) where \( P_v(f) = \|fv\| \) for every \( f \in CV_0(X) \).

If \( V \) is a Nachbin family on \( X \), then for \( v \in V \), \( f \in C(X) \) we have that \(fv \in N(X)\) by 2.2. Hence \( f \in CV_0(X) \)

and \( v \in V \) implies \(fv \in B(X)\) (by Theorem 2.3.), and \( P_v \)
is defined for every \( v \in V \). It is easy to see that \( CV_0(X) \)
is a linear subspace of \( C(X) \), and hence that \( CV_0(X) \) is a

locally convex topological vector space. The topology \( \omega \)
generated by \( \{ P_v : v \in V \} \) will be called the weighted topology;

if it is necessary to specify the weighted topology with respect to a particular Nachbin family \( V \), we will use the notation \( \omega_V \).

For a Nachbin family \( V \) on \( X \) and for \( v \in V \), let

\[ V_v = \{ f \in CV_0(X) : \|fv\| \leq 1 \} \].

Since for \( \epsilon > 0 \) and \( \{v_k : k = 1, \ldots, n\} \subseteq V \) there exists \( v \in V \) such that

\[ \epsilon^{-1}v_k \leq v, \quad k = 1, \ldots, n, \]

then \( f \in V_v \) implies \( \|fv_k\| \leq \epsilon, \)

\( k = 1, \ldots, n \). Consequently, \( \{V_v : v \in V\} \) forms a base of neighborhoods (of \( 0 \)) in \( CV_0(X) \) which are closed and absolutely convex.

Some remarks are in order concerning the relation between our Nachbin family \( V \) on \( X \) and the set of weights \( U \) on \( X \) considered by Nachbin [19, 20], where \( U \) is defined to be a set of non-negative u.s.c. functions on \( X \) satisfying the property that if \( u, v \in U \), then there exists
w ∈ U and λ > o such that u, v ≤ λw. Nachbin then defines
C_u^∞(X) = \{f ∈ C(X): fu ∈ B_o(X) for every u ∈ U\}, endowed
with the topology generated by the semi-norms \{P_u: u ∈ U\}
where P_u(f) = ||fu|| for every f ∈ C_u^∞(X). Clearly, a
Nachbin family on X is a set of weights in the sense of
Nachbin. Now let U be a set of weights on X in the sense
of Nachbin, and let V = \{λu: λ ≥ o, u ∈ U\}. Then V is a
Nachbin family on X. Since U ⊆ V, it is clear that
C_v^o(X) ⊆ C_u^∞(X). Conversely, if f ∈ C_u^∞(X), then, for
v ∈ V, v ≠ o, there exists λ > o and u ∈ U such that
v = λu, and thus, for ε > o, K = \{x ∈ X: |f(x)|v(x) ≥ ε\}
is compact since K ⊆ \{x ∈ X: |f(x)|u(x) ≥ λ^{-1} ε\}; i.e.,
fv ∈ B_o(X), which implies f ∈ C_v^o(X). If v ∈ V, v ≠ o,
then there exists λ > o and u ∈ U such that v = λu;
hence f ∈ C_v^o(X) with ||fu|| ≤ λ^{-1} implies ||fv|| ≤ 1
(i.e., f ∈ V_v). Since it is clear that w_v is a finer
topology on C_v^o(X) than the topology generated by
\{P_u: u ∈ U\}, we have that the two topologies coincide on
C_v^o(X) = C_u^∞(X). Thus the class of weighted spaces obtained
by Nachbin's approach is precisely that which is obtained by
our approach (via Nachbin families).

2.5. Definition. Let U and V be two Nachbin families
on X. We write U ≤ V if for every u ∈ U there is a
v ∈ V such that u ≤ v. In case U ≤ V and V ≤ U, we
write U ≈ V.
Clearly, \( \leq \) is a partial order and \( \approx \) is an equivalence relation on the class of all Nachbin families on \( X \).

2.6. **Theorem.** If \( U \) and \( V \) are Nachbin families on \( X \) with \( U \leq V \), then

\begin{align*}
&\text{(1) } CV_o(X) \subseteq CU_o(X), \text{ and} \\
&\text{(2) } r(u; CV_o(X)) \subseteq u_V.
\end{align*}

(In analogy with our symbol for the restriction of a function, \( r(\mathcal{J}; E) \) will denote the relative topology induced on the subset \( E \) of a topological space \( (F, \mathcal{J}) \) by the topology \( \mathcal{J} \).)

**Proof.** Let \( f \in CV_o(X), u \in U, \text{ and } \varepsilon > 0 \). Then there is a \( v \in V \) such that \( u \leq v \), and thus

\[ K = \{ x \in X : |f(x)|u(x) \geq \varepsilon \} \subseteq \{ x \in X : |f(x)|v(x) \geq \varepsilon \}, \]

which is compact. Since \( K \) is closed (by 2.2), \( K \) is compact and hence \( f \in CU_o(X) \).

If \( u \in U \), then there is a \( v \in V \) such that \( u \leq v \).

Thus \( f \in V_v \) implies \( |f(x)|u(x) \leq |f(x)|v(x) \leq 1 \) for every \( x \in X \), and so \( f \in V_u \cap CV_o(X) \); i.e., \( V_v \subseteq V_u \cap CV_o(X) \).

2.7. **Corollary.** If \( U \) and \( V \) are Nachbin families on \( X \) with \( U \approx V \), then \( CU_o(X) = CV_o(X) \); i.e., they are the same sets with the same topologies.

In particular, if \( V \) is a Nachbin family on \( X \), then \( V \approx U \) where \( U \) is the Nachbin family \( V U \{0\} \) on \( X \). Also, \( V \approx V^C \) where \( V^C \) is the set of all finite sums of non-negative scalar multiples of members of \( V \); i.e., the positive cone generated by \( V(V^C) \) is clearly a Nachbin family on \( X \).
Hence when convenient, we may assume without loss of generality that either \( 0 \in V \) or that \( V \) is a positive cone.

We have the following partial converse of 2.6.

2.8. Theorem. Let \( U \) and \( V \) be Nachbin families on \( X \) and assume (1) and (2) of Theorem 2.6 hold. If either

(i) \( V \subseteq B_o^+(X) \) or (ii) \( X \) is locally compact,

then \( U \leq V \).

Proof. If \( u \in U \), then there is a \( v \in V \) such that \( V_v \subseteq V_u \cap CV_o(X) \). If we set \( A = \{ x \in X: (u-v)(x) > 0 \} \), we would show \( A \) is void. If not, let \( x_0 \in A \) and let \( B = \{ x \in X : v(x) < \frac{1}{2} (v(x_0) + u(x_0)) \} \). Then \( B \) is an open set and \( x_0 \in B \), which implies there exists \( \theta \in C(X) \) such that \( 0 \leq \theta \leq 1 \), \( \theta(x_0) = 1 \), and \( \theta(X \setminus B) = 0 \). If (ii) holds, then we may assume \( \theta \in C_c^+(X) \), in which case it is clear that \( \theta \in CV_o(X) \); while if (i) holds, then for \( w \in V \) and \( \varepsilon > 0 \) we have \( K = \{ x \in X : \theta(x)w(x) \geq \varepsilon \} \)

\( \subseteq \{ x \in X : w(x) \geq \varepsilon \} \), which implies \( K \) is compact from whence it follows that \( \theta \in CV_o(X) \).

In either case, \( f = 2(v(x_0) + u(x_0))^{-1} \theta \) is in \( CV_o(X) \) and \( |f(x)|v(x) \leq 1 \) for every \( x \in X \), which implies \( f \in V_v \). But \( f(x_0)u(x_0) = 2u(x_0)(v(x_0) + u(x_0))^{-1} > 1 \), which contradicts \( f \in V_u \). Hence \( A \) is void, and the proof is complete.

2.9. Remark. In the proof of the preceding theorem, we made use of the obvious but useful fact that \( C_c(X) \subseteq CV_o(X) \) for every Nachbin family \( V \) on \( X \).
In extending Nachbin's concept of a set of weights to our Nachbin families, we have in a sense increased the number of semi-norms used in generating the weighted topology. One possible drawback to this is in recognizing when \( w \) is metrizable (since a locally convex space \( E \) whose topology is generated by a countable number of semi-norms is metrizable [23, p.17]). This difficulty is remedied by the following theorem.

2.10. **Theorem.** If \( V \) is a Nachbin family on \( X \), if \( CV_0(X) \) is Hausdorff, and if \( U \) is countable set of non-negative u.s.c. functions on \( X \) such that \( W = \{\lambda u: \lambda \geq 0, u \in U\} \) is a Nachbin family on \( X \) with \( W \sim V \), then \( CV_0(X) \) is metrizable.

**Proof.** By 2.7, it suffices to show \( CW_0(X) \) is metrizable. To do this, it suffices to show that for \( w \in W \) there is a \( u \in U \) and \( \epsilon > 0 \) such that \( f \in CV_0(X) \) with \( ||fu|| \leq \epsilon \) implies \( f \in V_w \). Since \( w \in W \) implies there exist \( u \in U \) and \( \lambda \geq 0 \) such that \( w = \lambda u \), we take \( u \) and any \( \epsilon > 0 \) if \( \lambda = 0 \), while if \( \lambda > 0 \), then we take \( u \) and \( \epsilon = \lambda^{-1} \). The result clearly follows.

2.11. **Corollary.** If in the preceding theorem \( U \) is finite, then \( CV_0(X) \) is normable.

We will, unless otherwise specified, restrict our attention to those Nachbin families \( V \) on \( X \) for which \( CV_0(X) \) is Hausdorff; i.e., \( V \) has the property that if \( f \in CV_0(X) \),
f \neq 0$, then there is a $v \in V$ such that $|f v| > 0$. $V$ has this property if and only if given any non-void open subset $A$ of $X$ for which there is an $f \in CV_0(X)$ such that $R(f; A) \neq 0$, then there is a $v \in V$ such that $R(v; A) \neq 0$. So certainly, $CV_0(X)$ is Hausdorff if for each $x \in X$ there is a $v \in V$ such that $v(x) > 0$. Requiring that $CV_0(X)$ be Hausdorff is in some cases very close to requiring that $X$ be locally compact as we show below.

2.12. Theorem. If $V$ is a Nachbin family on $X$ with $C^+_c(X) \leq V \subseteq C^+_o(X)$ (it is easy to see that $C^+_c(X)$ is a Nachbin family on $X$), then $CV_0(X)$ is Hausdorff if and only if there exists a dense locally compact (and thus also open) subspace $Y$ of $X$.

Proof. Assume $Y$ is a dense locally compact subspace of $X$; thus $Y$ is open in $X$ [13, p. 45]. If $f \in CV_0(X)$, $f \neq 0$, then there is an $x_o \in Y$ such that $f(x_o) \neq 0$, and hence there is an open neighborhood $A$ of $x_o$ with $A \subseteq Y$ and such that $\overline{A}$ is compact. Moreover, there is a $\varphi \in C(X)$ such that $0 \leq \varphi \leq 1$, $\varphi(x_o) = 1$, and $\varphi(X \setminus A) = 0$. Since $spt(\varphi) \subseteq \overline{A}$, $\varphi \in C^+_c(X)$ which implies there is a $v \in V$ such that $\varphi \leq v$, and so $|f(x_o)|v(x_o) > 0$.

Now assume $CV_0(X)$ is Hausdorff. If $x_o \in X$ and if $A$ is an open neighborhood of $x_o$, then there exists a
\(\varphi \in C(X)\) such that \(0 \leq \varphi \leq 1\), \(\varphi(x_0) = 1\), and \(\varphi(X \setminus A) = 0\).

If \(v \in V\) and \(\varepsilon > 0\), then \(K = \{x \in X : \varphi(x)v(x) \geq \varepsilon\} \subseteq \{x \in X : v(x) \geq \varepsilon\}\), which implies \(K\) is compact. Hence \(\varphi \in CV_o(X)\), and since \(\varphi \neq 0\), there exists \(v \in V\) such that \(\varphi v \neq 0\); i.e., \(N(v) \cap A\) is not void. So \(Y = \bigcup\{N(v) : v \in V\}\) is dense in \(X\), and \(y \in Y\) implies there exists \(v \in V\) such that \(y \in N(v)\) whence \(y \in B = \{x \in X : v(x) > \frac{1}{2} v(y)\}\). But \(B\) is open (since \(v \in C_o(X)\)), \(\overline{B} = \{x \in X : v(x) \geq \frac{1}{2} v(y)\}\) is compact, and \(\overline{B} \subseteq Y\); i.e., \(Y\) is locally compact.

If we let \(X\) denote the rationals with the topology \(r(\mathbb{Q}; X)\), where \(\mathbb{Q}\) is the usual topology on \(\mathbb{R}\), then \(X\) is a completely regular \(T_1\)-space which does not have a dense locally compact subspace. In particular, if \(V\) is a Nachbin family on \(X\) with \(C_c^+(X) \subseteq V \subseteq C_o(X)\), then \(V = \{0\}\) and \(CV_o(X)\) is \(C(X)\) with the indiscrete topology.

For a subset \(S\) of \(X\), we will denote the characteristic function of \(S\) by \(\chi_S\); if \(S\) is closed, then \(\chi_S\) is u.s.c. In the sequel, the set \(\chi_c(X) = \{\lambda \chi_K : \lambda \geq 0, K \subseteq X, K \text{ compact}\}\) will play a useful role (see 2.13).

2.13. Example. If \(V = \chi_c(X)\), then \(V\) is a Nachbin family on \(X\) and \(CV_o(X) = (C(X), c\text{-op})\).

We have already remarked that \(V = C_c^+(X)\) is a Nachbin family on \(X\) (in fact, if \(\mathcal{A}\) is any linear subspace of \(C(X)\), then \(\mathcal{A}^+\) is a Nachbin family on \(X\)), and have pointed
out the inadequacy of \( V \) unless \( X \) is 'close' to being locally compact. This remark, in conjunction with 2.13, helps motivate the consideration of u.s.c. functions in defining a Nachbin family on \( X \). However, if \( X \) is locally compact, then \( C_c^+(X) \) assumes some importance as a Nachbin family on \( X \), as we now demonstrate.

2.14. Theorem. If \( U = \chi_c(X) \) and \( V = C_c^+(X) \), then \( V \leq U \) (and hence by 2.6, we have that the injection map \( i:CU_0(X) \to CV_0(X) \) is a continuous isomorphism onto). Moreover, the following are equivalent:

1. \( X \) is locally compact;
2. \( U \approx V \);
3. \( i \) is a topological isomorphism (in which case \( CV_0(X) = (C(X), c\text{-op}) \)).

Proof. If (1) holds and if \( u \in U \), then there exists \( \psi \in C_c^+(X) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi(spt(u)) = 1 \). Thus \( v = \|u\|\psi \) is in \( V \) and \( u \leq v \), which implies (2) holds.

That (2) implies (3) is clear in view of 2.7, while if (3) holds, then the hypothesis of Theorem 2.8 is satisfied (since \( V \subseteq B_0(X) \)), which implies \( U \leq V \) (i.e., (2) holds).

Now assume (2) holds, and let \( x_o \in X \). Since \( \chi_{\{x_o\}} \in U \), there is a \( v \in V \) such that \( v(x_o) \geq 1 \), and thus \( A = \{x \in X : v(x) \geq \frac{1}{2} \} \) is a compact neighborhood of \( x_o \). So (1) holds, and the proof is complete.
The remainder of this chapter will be devoted to developing several more examples. For easy reference, we will display all of our examples in a chart at the end of this chapter.

2.15. Example. We will let \( K^+(X) \) denote the non-negative constant functions on \( X \). If \( V = K^+(X) \), then \( V \) is a Nachbin family on \( X \) and \( CV_0(X) = (C_0(X), ||\cdot||) \).

To obtain the next example, we make use of the following "multiplier" theorem.

2.16. Theorem (Buck [4]). Let \( X \) be locally compact. If \( f \in C(X) \) and if \( f \circ \varphi \in C_0(X) \) for every \( \varphi \in C_0(X) \), then \( f \in C_b(X) \) (and, of course, conversely).

In particular, the above theorem says that \( C_b(X) \) is the largest subalgebra of \( C(X) \) which contains \( C_0(X) \) as an ideal (when \( X \) is locally compact).

2.17. Example. Let \( X \) be locally compact and let \( V = C_0^+(X) \). Then \( V \) is a Nachbin family on \( X \), and it is an easy consequence of 2.16 together with the definition of the strict topology \( \beta \) that \( CV_0(X) = (C_b(X), \beta) \). This should be considered as one of our main motivating spaces.

We now consider the Nachbin family \( V = C^+(X) \) and investigate the corresponding space \( CV_0(X) \). We remark that Examples 2.13 and 2.15 are well-known, while Example 2.17 was observed by H. S. Collins. The following example and its extensions are new.
A subset $S$ of $X$ is called **relatively precompact** if $f \in C(X)$ implies $R(f; S) \in C_b(S)$. We will let $C_p(X) = \{f \in C(X) : N(f)$ is relatively precompact}. 

2.18. **Remark.** It is clear that $C_p(X)$ is a linear subspace of $C(X)$. In particular, since $f \in C_p(X)$ implies $f$ is bounded on $N(f)$ and hence on $X$, we have that $C_p(X) \subseteq C_b(X)$. However, $C_p(X) \subseteq C_b(X)$ if and only if $X$ is pseudo-compact.

2.19. **Example.** If $V = C^+(X)$, then $C V_0(X) = (C_p(X) \cap C_0(X), \omega)$.

**Proof.** Assume $f \in C_p(X) \cap C_0(X)$, and let $v \in V$, $\varepsilon > 0$. To show that $f \in C V_0(X)$, it will suffice to show $K = \{x \in X : |f(x)| |v(x) | \geq \varepsilon\}$ is compact. Since $K$ is void if $N(v) \cap N(f)$ is void, we assume $N(f) \cap N(v)$ is not void. In this case, $K \subseteq \{x \in X : |f(x)| \geq \varepsilon(|v|_{N(f)})^{-1}\}$, from which it follows that $K$ is compact.

Now assume $f \in C V_0(X)$. Since $K^+(X) \subseteq V$, Theorem 2.6 and Example 2.15 imply that $C V_0(X) \subseteq C_0(X)$; i.e., $f \in C_0(X)$. If $f \notin C_p(X)$, then there exists $g \in C(X)$ such that $g$ is not bounded on $N(f)$. We can thus choose {$x_n$}_{n=1}^{\infty} \subseteq N(f)$ such that $|g(x_1)| > 1$ and $|g(x_n)| > 1 + |g(x_{n-1})|$ for $n > 1$. If, for $n \in N$, we define $A_n = \{x \in X : |g(x) - g(x_n)| < \frac{1}{2}\}$, then $A_n$ is an open neighborhood of $x_n$ for each $n \in N$, while $\overline{A_n} \cap \overline{A_m}$ is void for any $m, n \in N$ with $m \neq n$; i.e., {$x_n$}_{n=1}^{\infty} is an
discrete sequence in \( X \) (which is clearly closed). By complete regularity, for each \( n \in \mathbb{N} \) there exists \( \varphi_n \in C(X) \) such that \( 0 \leq \varphi_n \leq 1, \varphi_n(x_n) = 1, \) and \( \varphi_n(X \setminus A_n) = 0. \)

Now \( v = \sum_{n=1}^{\infty} |f(x_n)|^{-1} \varphi_n \) is a well-defined function on \( X \) with \( v \geq 0. \) If \( x_0 \in X, \) then there exists a neighborhood \( A \) of \( x_0 \) such that \( A \) meets at most finitely many of \( \{ A_n \}_{n=1}^{\infty}, \) which implies \( v \in V. \) So \( K = \{ x \in X : f(x)v(x) \geq 1 \} \) is compact and \( \{ x_n \}_{n=1}^{\infty} \subseteq K; \) this implies \( \{ x_n \}_{n=1}^{\infty} \) has a cluster point in \( K \) and contradicts the choice of \( \{ x_n \}_{n=1}^{\infty}. \)

Thus \( f \in C_p(X), \) and the proof is complete.

In the course of verifying the preceding assertion, we have proved a fact on extending functions which we feel is of interest, and which we will state explicitly after introducing some terminology. A subset \( S \) of \( X \) is said to be \( C^*\)-embedded (\( C \)-embedded) in \( X \) if every \( f \in C_b(S) (C(S)) \) has an extension in \( C_b(X) (C(X)). \) We have proved the following result.

2.20. Lemma. If \( S \subseteq X \) and if there exists \( f \in C(X) \) such that \( f \) is unbounded on \( S, \) then there exists a denumerable closed and discrete subset \( D \) of \( X \) such that \( D \subseteq S \) and \( D \) is both \( C \)-embedded and \( C^* \)-embedded in \( X. \)

Recalling that for the Nachbin family \( V = C^+_c(X), \) we have \( C_{V_0}(X) = (C(X), w_V), \) it seems natural to expect \( C_{U_0}(X) = (C_c(X), w_U) \) when \( U = C^+(X). \) This is not the case! In fact, \( C_c(X) \subseteq C_p(X) \) and hence
$C_c(X) \subseteq C_p(X) \cap C_0(X)$, but we show below that this containment can be proper. To do this, we list several properties of a certain topological space, some of which are not now needed but will be useful to us in later examples. We begin by stating a result of Fine and Gillman (which depends on the continuum hypothesis). Throughout the rest of this paper, $\beta X$ will denote the Stone-Čech compactification of $X$.

2.21. Theorem (Fine and Gillman [12]). No proper dense subspace of $\beta N \setminus N$ is C*-embedded in $\beta N \setminus N$.

2.22. Theorem. Let $p \in \beta N \setminus N$ and let $X = \beta N \setminus \{p\}$, endowed with the relative topology from $\beta N$. Then $X$ is a locally compact Hausdorff space which is extremally disconnected and pseudo-compact, but neither compact nor normal.

Proof. That $X$ is locally compact and Hausdorff but not compact is clear. Since a space $Y$ is extremally disconnected if and only if $\beta Y$ is extremally disconnected [13, p. 96], and since every open subspace of an extremally disconnected space is extremally disconnected [13, p. 23], $X$ is extremally disconnected (because $N$ is). If the cardinality of $\beta Y \setminus Y$, for a space $Y$, is less than or equal one, then $Y$ is pseudo-compact [13, p. 95]; hence $X$ is pseudo-compact (for $\beta X = \beta N$). A space is normal if and only if every closed subspace is C*-embedded [13, p. 48], and thus to complete the proof it will suffice to show that the closed set $S = (\beta N \setminus N) \cap X$ is not C*-embedded in $X$. But $S$ is

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C*-embedded in $X$ if and only if $S$ is C*-embedded in $S^bX$ [13, p.89]; i.e., if and only if $S$ is C*-embedded in $S^bN = \beta N \setminus N$. However, $S$ is a proper dense subspace of $\beta N \setminus N$, and by 2.21 is not C*-embedded in $\beta N \setminus N$.

In the above theorem, the continuum hypothesis was assumed only to verify the non-normality of $X$.

2.23. Example. Let $X$ be the space of the preceding theorem and let $V = C^+(X)$. By Theorem 2.22 and Example 2.19, $C^0_v(X) = (C_p(X) \cap C_o(X), \omega) = (C_o(X), \omega)$. But for $\varphi \in C_o(N)$ with $N(\varphi) = N$; and $\tilde{\varphi}$ a continuous extension of $\varphi$ to $X$, $\tilde{\varphi} \in C_o(X) \setminus C_c(X)$ since $N(\tilde{\varphi}) = N$; i.e., $C_c(X) \not\subseteq C^0_v(X)$.

So we see that not even local compactness is enough for $C_p(X) \cap C_o(X)$ to be $C_c(X)$. However, we do have the following sufficient condition.

2.24. Theorem. If $X$ is locally compact and if every $\sigma$-compact subset of $X$ is contained in an open and closed $\sigma$-compact subset of $X$, then $C_p(X) \cap C_o(X) = C_c(X)$ (as sets).

Proof. It suffices to show $C_p(X) \cap C_o(X) \subseteq C_c(X)$, so let $f \in C_p(X) \cap C_o(X)$ and suppose $f \not\in C_c(X)$. Since $f \in C_o(X)$, $N(f)$ is $\sigma$-compact and hence there exists an open and closed $\sigma$-compact set $S \subseteq X$ such that $N(f) \subseteq S$. Thus $N(f) \subseteq S$ and $\bar{N}(f)$ is relatively precompact but not compact. Now $S = \bigcup_{n=1}^{\infty} K_n$ where $K_n$ is compact and
$K_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$ (if $A \subseteq X$, then $A^\circ$ denotes the topological interior of $A$). For each $n \in \mathbb{N}$, choose $x_n \in (X \setminus K_n) \cap N(f)$ (this is possible, since otherwise $N(f) \subseteq K_{n_0}$ for some $n_0 \in \mathbb{N}$, and $N(f)$ would be compact), and let $n_1 = 1$. Then there is a $\varphi_1 \in C_c(X)$ such that $0 \leq \varphi_1 \leq 1$, $\varphi_1(x_{n_1}) = 1$, and $\text{spt}(\varphi_1) \subseteq S \setminus K_{n_1}$. Further, there exists $n_2 \in \mathbb{N}$ and $\varphi_2 \in C_c(X)$ so that $\text{spt}(\varphi_1) \subseteq K_{n_2}^\circ$, $0 \leq \varphi_2 \leq 1$, $\varphi_2(x_{n_2}) = 1$, and $\text{spt}(\varphi_2) \subseteq S \setminus K_{n_2}$. Inductively, we obtain $\{(x_{n_k}, \varphi_k)\}_{k=1}^\infty$ such that $\varphi_k \in C_c(X)$, $0 \leq \varphi_k \leq 1$, $\varphi_k(x_{n_k}) = 1$, $\text{spt}(\varphi_k) \subseteq S \setminus K_{n_k}$, $k \geq 1$ and $\text{spt}(\varphi_j) \subseteq K_{n_k}^\circ$ for every $j \in \mathbb{N}$ with $j < k$ where $k > 1$. If we let

$$g(x) = \begin{cases} \sum_{k=1}^\infty k\varphi_k(x), & x \in S \\ 0, & x \in X \setminus S \end{cases},$$

then $g$ is a well-defined function on $X$. Since for each $x \in S$ there exists a neighborhood $A$ of $x$ for which $\overline{A}$ is compact, $A \cap N(\varphi_k)$ is void for all but at most finitely many $k \in \mathbb{N}$, which implies $g \in C(X)$. However, $g$ is not bounded on $N(f)$, and this contradicts our choice of $f$.

In particular, the hypothesis of the above theorem is satisfied if $X$ is locally compact and paracompact. We also note that (in view of 2.23) the space $X$ of 2.22 has a $\sigma$-compact subset which is not contained in any open and closed $\sigma$-compact subset of $X$ (and hence $X$ is not paracompact).
2.25. **Theorem.** If $X$ is locally compact, then the following are equivalent:

1. $X$ is pseudo-compact and every $\sigma$-compact subset of $X$ is contained in an open and closed $\sigma$-compact subset of $X$;
2. $C_0(X) = C_c(X)$ and every $\sigma$-compact subset of $X$ is contained in an open and compact subset of $X$.

**Proof.** If (1) holds, then 2.24 yields $C_c(X) = C_p(X) \cap C_0(X) = C(X) \cap C_0(X) = C_0(X)$. Let $S$ be a $\sigma$-compact subset of $X$ and let $K$ be an open and closed $\sigma$-compact subset of $X$ with $S \subseteq K$. Since $K$ is open and $\sigma$-compact, there exists $\varphi \in C_0(X)$ such that $N(\varphi) = K$ (Buck [4]); since $K$ is closed, $\overline{N(\varphi)} = K$, which implies $K$ is compact.

Now assume (2) holds, and observe it suffices to show $X$ is pseudo-compact. However, for $\nu = C_0^+(X) = C_c^+(X)$, 2.14 and 2.17 imply this result.

2.26. **Theorem.** If $\nu = C^+(X)$, then the uniform topology on $C_0^+(X)$ is contained in $\omega_\nu$, and $r(\omega_\nu; C_c(X))$ is contained in the inductive limit topology on $C_c(X)$.

**Proof.** If $U = K^+(X)$, then $U \subseteq V$, and this implies $r(\omega_U; C_0^+(X)) \subseteq \omega_\nu$ (by 2.6).

Now let $\nu \in V$, for every compact set $K$ in $X$ such that $N(\nu) \cap K$ is not void, let $B_K = \{f \in C(X; K) : |f| \leq (|\nu||_K)^{-1}\}$, and let $B_K = C(X; K)$ for each compact subset $K$ of $X$ such that $N(\nu) \cap K$ is void. Since the absolutely convex hull $A$ of $\bigcup_K B_K$ is
an inductive limit neighborhood in $C_c(X)$ [23, p. 79],
then to conclude the proof it will suffice to show that
$f \in A$ implies $f \in V \cap C_c(X)$. However, if $f \in A$, then
there exists $(\lambda_k)_{k=1}^n \subseteq \mathbb{C}$ and $(f_k)_{k=1}^n$ where $\sum_{k=1}^n |\lambda_k| \leq 1$
and $f_k \in B_{K_k}$, $k = 1, \ldots, n$ ($K_k$ is a compact subset of $X$
for $k = 1, \ldots, n$) with $f = \sum_{k=1}^n \lambda_k f_k$. So
$$|f(x)|v(x) \leq \sum_{k=1}^n |\lambda_k| |f_k(x)||v(x) \leq \sum_{k=1}^n |\lambda_k| \leq 1,$$
and this implies $||fv|| \leq 1$ (i.e., $f \in V \cap C_c(X)$).

We will conclude this chapter by establishing sufficient
conditions on $X$ in order that, for $V = C^+(X)$, $CV_0(X)$ is
$(C_c(X), \text{ind lim})$. The fact that this important function
space is a weighted space (and the proof is non-trivial!) leads us to interesting results on factorization of Radon
measures (e.g., see 3.29). In the next chapter we will give
eamples to show that the hypothesis of 2.24 is not sufficient
to obtain $(C_c(X), \text{ind lim})$ (see 3.28, (2)), and that our
sufficient conditions (in 2.28) are not necessary (see 3.33).

2.27. Lemma. If $X$ is locally compact and $\sigma$-compact
and if $A$ is an inductive limit neighborhood of zero in
$C_c(X)$, then there exists $v \in C^+(X)$ with the property that
for every $f \in C_c(X)$ with $||fv|| \leq 1$ we have $f \in A$.

Proof. Now $X = \bigcup_{n=1}^\infty K_n$ where $K_n$ is compact and
$K_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$, and $K_0^c$ is non-void. Moreover, for every $n \in \mathbb{N}$, there exists $\epsilon_n > 0$ such that for
Let $C_1 = K_1$ and let $C_n = K_n \cap (X \setminus K_{n+1}^0)$ for $n > 1$. For $n > 2$, there exists $\varphi_n \in C_c(X)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n(C_n) = 1$, and $\text{spt}(\varphi_n) \subseteq (X \setminus K_{n-2}) \cap K_{n+1}^0$, while there exist $\varphi_1, \varphi_2 \in C_c(X)$ so that $0 \leq \varphi_k \leq 1$, $\varphi_k(C_k) = 1$, and $\text{spt}(\varphi_k) \subseteq K_{k+1}$ for $k = 1, 2$. Choose $a_1 = \max\{2, \epsilon_1\}$, let $a_n = \max\{2^n, n\epsilon_n, \epsilon_{n-1}a_{n-1}\}$ for $n > 1$, and then define $v = \sum_{n=1}^{\infty} a_n \epsilon_n^{-1} \varphi_n$. Since $x \in X$ implies there exists $n \in \mathbb{N}$ such that $x \in K_n^0$, $x \notin \text{spt}(\varphi_j)$ for $j \geq n + 2$ and hence $v$ is well-defined and continuous; i.e., $v \in C^+(X)$.

Observe that if $n < m$, then $\epsilon_n a_n^{-1} \geq \epsilon_m a_m^{-1}$ since $n > 1$ implies $a_n > \epsilon_n \epsilon_{n-1} a_{n-1}$. Now let $f \in C^+(X)$ with $||fv|| \leq 1$, and note that $x \in C_n$ implies $f(x) \leq (v(x))^{-1} \leq \epsilon_n a_n^{-1}$ for $n \geq 1$. Choose $n_0 = \min\{n \in \mathbb{N}: N(f) \subseteq K_n\}$ and let $f_n = 2^n f_n'$, $n = 1, \ldots, n_0$, where $f_n' = [(f \wedge \epsilon_n a_n^{-1}) - \epsilon_{n+1} a_{n+1}^{-1}] \vee 0$, $1 \leq n < n_0$, and $f_n' = f \wedge \epsilon_{n_0} a_{n_0}^{-1}$. Clearly, $N(f_n') \subseteq K_n$; since $x \in N(f_n)$, $1 \leq n < n_0$, implies $f(x) > \epsilon_{n+1} a_{n+1}^{-1}$ while $x \in C_m$ for $m > n$ implies $f(x) \leq \epsilon_m a_m^{-1} \leq \epsilon_{n+1} a_{n+1}^{-1}$, we also have $N(f_n') \subseteq K_n$ for $1 \leq n < n_0$. Moreover, $f_n(x) = 2^n f_n'(x) \leq 2^n \epsilon_n a_n^{-1} \leq \epsilon_n$, $1 \leq n \leq n_0$, which implies $f_n \in B_n$, $1 \leq n \leq n_0$.

Consequently, if $g = \sum_{n=1}^{n_0} 2^{-n} f_n'$, then $g \in H$; the proof will then be complete when we show $f = g$. To this end, observe that $g = \sum_{n=1}^{n_0} f_n'$ and fix $x \in X$. If $f(x) = 0$,
then \( f^n_1(x) = 0, \ 1 \leq n \leq n_0 \), and this implies \( g(x) = 0 \).

We have already shown \( \{\varepsilon_n a_n^{-1}\}_{n=1}^{\infty} \) is monotone decreasing, while \( \varepsilon_n a_n^{-1} \leq \varepsilon_n (n\varepsilon_n)^{-1} = \frac{1}{n} \), \( n \geq 1 \), shows this sequence converges to zero, and since \( x \in C_n \) for some \( n \in \mathbb{N} \), we have \( f(x) \leq \varepsilon_n a_n^{-1} \). It now follows that if \( f(x) > 0 \), then there exists \( m \in \mathbb{N} \) so that \( \varepsilon_{m+1} a_{m+1}^{-1} < f(x) \leq \varepsilon_m a_m^{-1} \). If \( m < n_0 \), then, for \( n < m \) (if there are any such \( n \)), we have \( f^n_1(x) = 0 \) since \( f(x) \leq \varepsilon_m a_m^{-1} \leq \varepsilon_{n+1} a_{n+1}^{-1} \), while \( f^n_m(x) = f(x) - \varepsilon_{m+1} a_{m+1}^{-1} \), \( f^n_1(x) = \varepsilon_n a_n^{-1} - \varepsilon_{n+1} a_{n+1}^{-1} \) for \( m < n < n_0 \), and \( f^n_{n_0}(x) = \varepsilon_{n_0} a_{n_0}^{-1} \). If \( m \geq n_0 \), then \( f^n_1(x) = 0, \ n < n_0 \), and \( f^n_{n_0}(x) = f(x) \). In any case,

\[
g(x) = \sum_{n=1}^{n_0} f^n_1(x) = f(x), \text{ and the proof is complete.}
\]

2.28. Theorem. If \( X \) is locally compact and \( \sigma \)-compact and if \( V = C^+(X) \), then \( CV_0(X) = (C_c(X), \text{ind lim}) \).

Proof. In view of 2.19 and 2.24, it will suffice to show that \( \omega \) is the inductive limit topology on \( C_c(X) \).

By 2.26, we have that the inductive limit topology is finer than \( \omega \). Let \( A \) be an inductive limit neighborhood (which we may assume to be absolutely convex) in \( C_c(X) \). From 2.27, there is a \( v \in V \) so that if \( f \in V_v \) with \( f \geq 0 \), then \( f \in A \). Choose \( u = \frac{1}{2} v \) and note that \( u \in V \); let \( f \in V_u \) and write \( f = f^+_1 - f^-_1 + i(f^+_2 - f^-_2) \) where \( f^+_j, f^-_j \in C^+_c(X), \ j = 1,2 \); and define \( g^+_j = \frac{i}{4} f^+_j, \ g^-_j = \frac{i}{4} f^-_j \) for \( j = 1,2 \). If \( x \in X \), then \( g^+_j(x) v(x) = f^+_j(x) u(x) \leq |f(x)| u(x) \leq 1, \ j = 1,2 \), which implies \( g^+_j \in V_v, \ j = 1,2 \). Similarly,
$g_j \in V, j = 1,2,$ and hence $g_j^+, g_j^- \in A, j = 1,2$. We now have that \( \frac{1}{4} g_1^+ + (-\frac{1}{4})g_1^- + \frac{1}{4} g_2^+ + (-\frac{1}{4})g_2^- \in A; \) i.e., $f \in A$.

Thus $V_u \subseteq A$, and this concludes the proof.

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CHAPTER III

Properties of CV_o(X)

In this chapter we examine CV_o(X) as a locally convex space with particular emphasis on a characterization of the topological dual CV_o(X)*. This characterization of CV_o(X)* is crucial in obtaining our representation theorem in Chapter IV (see 4.8), and leads to many other interesting results (and new problems!).

We first consider the question of the completeness of CV_o(X). The difficulty here arose in finding the proper setting for our general situation so as to include a sufficiently broad spectrum of weighted spaces. It is here that our partial order on Nachbin families, which was introduced in Chapter II, plays a significant role, since our answer (given in 3.2 and 3.3) might properly be termed a "comparison test" for completeness.

3.1. Lemma. N(X) ∩ B_o(X) is complete in the uniform topology.

Proof. From 2.3, N(X) ∩ B_o(X) ⊆ B(X), and since B(X) is a Banach space (in the uniform topology), it will suffice to show N(X) ∩ B_o(X) is closed in B(X). To do this, let f ∈ B(X) be a limit point of N(X) ∩ B_o(X) in
B(X), let \( \varepsilon > 0 \), and let \( F = \{ x \in X : |f(x)| \geq \varepsilon \} \). Fix 
\( x_0 \in X \setminus F \), let \( \eta = \frac{1}{2}(\varepsilon - |f(x_0)|) \), and choose \( g \in N(X) \cap B_0(X) \) so that \( |g - f| < \eta \). Then \( A = \{ x \in X : |g(x)| < \frac{1}{2}(\varepsilon + |f(x_0)|) \} \) is an open set, and \( x_0 \in A \) since \( |g(x_0)| < |f(x_0)| + \eta = \frac{1}{2}(\varepsilon + |f(x_0)|) \). Moreover, if \( x \in A \), then \( |f(x)| < |g(x)| + \eta < \varepsilon \); i.e., \( A \subseteq X \setminus F \). Hence \( F \) is closed, and this implies \( f \in N(X) \). Now choose \( h \in N(X) \cap B_0(X) \) so that \( |h - f| < \varepsilon/2 \), and let \( K = \{ x \in X : |h(x)| \geq \varepsilon/2 \} \). Thus \( K \) is compact, while \( x \in F \) implies \( |h(x)| > |f(x)| - \varepsilon/2 > \varepsilon - \varepsilon/2 = \varepsilon/2 \); therefore \( F \subseteq K \), which implies \( F \) is compact (i.e., \( f \in B_0(X) \)).

3.2. Theorem. Let \( U \) be a Nachbin family on \( X \) for which the following properties hold:

(1) if \( x \in X \), then there is a \( u \in U \) so that \( u(x) > 0 \);

(2) \( C_U(X) \) is complete.

If \( V \) is a Nachbin family on \( X \) with \( U \subseteq V \), then \( C_V(X) \) is complete.

Proof. Let \( \{ f_i \} \) be an \( \omega_V \)-Cauchy net in \( C_V(X) \). It follows from 2.6 that \( \{ f_i \} \) is \( \omega_U \)-Cauchy, and hence there exists \( f \in C(X) \) such that \( f_i \rightarrow f(\omega_U) \). For \( v \in V \), \( \{ f_i v \} \) is a uniformly Cauchy net in \( N(X) \cap B_0(X) \) (by 2.2), and hence by 3.1 there exists \( f_v \in N(X) \cap B_0(X) \) so that \( f_i v \rightarrow f_v \) in the uniform topology. From (1), \( f_i v \rightarrow f_v \) pointwise, which implies \( f_v = f_v \) for every \( v \in V \). So
we have \( f \in CV_o(X) \), and it is clear that \( f_i \to f(\omega_V) \).

Since \((C(X), c\text{-}op)\) is complete whenever \( X \) is a k-space [32], we have the following result.

3.3. Corollary. If \( X \) is a k-space and if \( V \) is a Nachbin family on \( X \) with \( \chi_C(X) \leq V \), then \( CV_o(X) \) is complete.

As is well-known [4, p. 98], if \( X \) is locally compact, then the Banach space \((C_o(X), \|\cdot\|)\) has an approximate identity. In particular, there is a net \( \{\varphi_i\} \subseteq C_c^+(X) \) such that \( 0 \leq \varphi_i \leq 1 \) for every \( i \), while for any compact set \( K \) in \( X \) there is an \( i_X \) so that for every \( i \geq i_X \) we have \( \varphi_i(K) = 1 \) (and \( \lim_i \|\varphi_i\varphi - \varphi\| = 0 \) for every \( \varphi \in C_o(X) \)). Following [8], we will call such an approximate identity canonical.

3.4. Lemma. If \( X \) is locally compact, then the uniformly closed subalgebra of \( B(X) \) generated by \( N(X) \cap B_o(X) \) has a canonical approximate identity \( \{\varphi_i\} \) and \( \{\varphi_i\} \subseteq N(X) \cap B_o(X) \).

Proof. Since \( C_c^+(X) \subseteq N(X) \cap B_o(X) \), a canonical approximate identity (if one exists) will be contained in \( N(X) \cap B_o(X) \). Now let \( \mathcal{K} = \{K_i : K_i \text{ compact, } i \in I\} \) be a base for compacta in \( X \) with \( I \) partially ordered by 'if and only if \( K_i \subseteq K_j \)'. For each \( i \in I \), there is a \( \varphi_i \in C_c(X) \) such that \( 0 \leq \varphi_i \leq 1 \) and \( \varphi_i(K_i) = 1 \); we will show that \( \{\varphi_i\} \) is the desired canonical approximate identity.
Let \( f \in N(X) \cap B_0(X) \), let \( \epsilon \gg \delta \), and let
\[ K = \{ x \in X : |f(x)| \geq \frac{\epsilon}{2} \} \]. Since \( K \) is compact, there is an
\( i_K \in I \) such that \( i \geq i_K \) implies \( K \subseteq K_i \). If \( x \in K \) and
if \( i \geq i_K \), then \( |\varphi_i(x)f(x) - f(x)| = 0 \), while if \( x \in X \setminus K \)
and if \( i \geq i_K \), then \( |\varphi_i(x)f(x) - f(x)| \leq 2|f(x)| \leq \epsilon \).
Thus \( i \geq i_K \) implies \( ||\varphi_i f - f|| \leq \epsilon \); i.e., \( \lim_{i} ||\varphi_i f - f|| = 0 \).
The result now easily extends to the uniformly closed sub-
algebra of \( B(X) \) generated by \( N(X) \cap B_0(X) \), since \( \{\varphi_i\} \)
is uniformly bounded.

3.5. Theorem. If \( X \) is locally compact and if \( V \) is
any Nachbin family on \( X \), then \( CV_0(X) \) has a canonical
approximate identity.

Proof. By 3.4, there is a net \( \{\varphi_i\} \subseteq C_c(X) \), which
is a canonical approximate identity for \( N(X) \cap B_0(X) \);
since \( C_c(X) \subseteq CV_0(X) \), \( \{\varphi_i\} \) will be a canonical approx-
imate identity for \( CV_0(X) \) provided \( \varphi_i f \to f(\omega) \) for each
\( f \in CV_0(X) \). If \( f \in CV_0(X) \), however, then for any \( v \in V \)
we have \( fv \in N(X) \cap B_0(X) \), which implies \( \lim_{i} ||(\varphi_i f - f)v|| = 0 \)
(i.e., \( \varphi_i f \to f(\omega) \)).

3.6. Corollary. If \( X \) is locally compact and \( V \) is
any Nachbin family on \( X \), then \( C_c(X) \) is \( \omega \)-dense in
\( CV_0(X) \).

3.7. Corollary. The following are equivalent:
(1) \( X \) is locally compact;
(2) \((\mathcal{C}(X), \text{c-op})\) has a canonical approximate identity;

(3) there is a net \(\{\phi_n\} \subseteq \mathcal{C}_c^+(X)\) such that \(\phi_n \to 1\) (c-op).

Proof. In view of 3.5, we have (1) implies (2), while it is obvious that (2) implies (3).

Now assume (3) holds. Let \(V = \mathcal{C}_c^+(X)\), let \(U = \mathcal{X}_c(X)\), note that \(V \subseteq U\), and recall that \((\mathcal{C}(X), \text{c-op}) = \mathcal{C}_U^0(X)\).

If \(u \in U\), then there is a \(\lambda > 0\) and a compact set \(K\) in \(X\) such that \(u = \lambda \chi_K\). There exists \(i_K\) such that \(|(\phi_{i_K} - 1)\chi_K| < \frac{1}{2}\), and thus if \(x \in K\), \(-\frac{1}{2} < \phi_{i_K}(x) - 1, \frac{1}{2} < \phi_{i_K}(x)\), and \(\lambda \leq 2\lambda \phi_{i_K}(x)\); i.e., \(u \leq 2\lambda \phi_{i_K}\). But \(2\lambda \phi_{i_K} \in \mathcal{C}_c^+(X)\), and so \(U \subseteq V\). Since then \(U \approx V\), we have from 2.14 that \(X\) is locally compact, and this completes the proof.

We will return to approximate identities at the end of this chapter where we discuss and give examples illustrating some of the pathology which can occur in \(\mathcal{C}_U^0(X)\).

If \(U\) and \(V\) are Nachbin families on \(X\), then we define \(U \cdot V = \{uv: u \in U, v \in V\}\). In view of 2.2, \(U \cdot V\) is also a Nachbin family on \(X\). We will denote \(V \cdot V\) by \(V^2\).

3.8. Theorem. If \(V\) is a Nachbin family on \(X\) with \(V \subseteq V^2\), then \(\mathcal{C}_V^0(X)\) is a subalgebra of \(\mathcal{C}(X)\). Moreover, \(\mathcal{C}_V^0(X)\) is a topological algebra.

Proof. If \(f, g \in \mathcal{C}_V^0(X)\), then we must show \(fg \in \mathcal{C}_V^0(X)\). Let \(v \in V\), \(\epsilon > 0\), and
A = \{x \in X: |f(x)|v(x) > \varepsilon\}. Now there exist \( v_1, v_2 \in V \)
such that \( v \leq v_1v_2 \); if \( x \in A \), then \( |f(x)|v_1(x)v_2(x) > \varepsilon \),
which implies \( |f(x)|v_1(x) \geq \varepsilon / (|g(x)|v_2(x))^{-1} \geq \varepsilon / (|g_v^2|)^{-1} \).
Since \( A \) is closed and \( A \subseteq \{x \in X: |f(x)|v_1(x) \geq \varepsilon / (|g_v^2|)^{-1}\} \),
\( A \) is compact, from which it follows that \( fg \in CV_0(X) \).

If \( \{f_1\}, \{g_1\} \subseteq CV_0(X) \) with \( f_1 \to f(w), g_1 \to g(w) \)
where \( f, g \in CV_0(X) \), then for \( v \in V \) with \( v \leq v_1v_2 \) where
\( v_1, v_2 \in V \), we have \( ||(f_1g_1-fg)v|| =
||[(f_1-f)(g_1-g)+fg_1+gf_1-2fg]v|| \leq ||(f_1-f)v_1|| \cdot ||(g_1-g)v_2||
+ ||fv_1|| \cdot ||(g_1-g)v_2|| + ||gv_1|| \cdot ||(f_1-f)v_2|| \to 0 \). It is now
clear that multiplication is jointly continuous, and so
\( CV_0(X) \) is a topological algebra.

3.9. Theorem. Let \( X \) be locally compact and let \( U \) and \( V \) be Nachbin families on \( X \) with \( U \leq V \). If \( \alpha \) is
a subset of \( CU_0(X) \) and \( \alpha \cap CV_0(X) \) is \( \omega_U \)-dense in \( CV_0(X) \),
then \( \alpha \) is \( \omega_U \)-dense in \( CU_0(X) \).

Proof. If \( f \in CU_0(X) \) and if \( A \) is an \( \omega_U \)-neighborhood
of \( f \), then, since \( A \cap C_0(X) \) is non-void by 3.6, it
follows from 2.6 that \( \alpha \cap (A \cap CV_0(X)) \) is non-void; i.e.,
\( \alpha \cap A \) is non-void.

The preceding simple result on density yields an
embryonic form of Stone-Weierstrass theorem for \( CV_0(X) \) (when
used in conjunction with a known result for \( C_0(X) \), which
will be stated as the next theorem). However, once we have
identified \( CV_0(X)^* \), then we will be able to obtain a much
sharper Stone-Weierstrass theorem for $\mathcal{C}_0(X)$ (see 3.38 and 3.39). We note that if $V$ is any Nachbin family on $X$, then $\mathcal{C}_0(X)$ is self-adjoint; i.e., if $f \in \mathcal{C}_0(X)$, then $\overline{f} \in \mathcal{C}_0(X)$.

3.10. Theorem ([27, p.61]). If $X$ is locally compact and if $\mathcal{A} \subseteq \mathcal{C}_0(X)$ is a self-adjoint subalgebra of $\mathcal{C}_0(X)$ satisfying the two point property (i.e., if $x, y \in X$ with $x \neq y$, then there exists $f \in \mathcal{A}$ so that $f(x) = 0$ and $f(y) = 1$), then $\mathcal{A} \upharpoonright \mathcal{C}^0(X)$ is uniformly dense in $\mathcal{C}^0(X)$.

3.11. Corollary. Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $V \subseteq K^+(X)$. If $\mathcal{A}$ is a self-adjoint subalgebra of $\mathcal{C}_0(X)$ such that $\mathcal{A} \cap \mathcal{C}_0(X)$ has the two point property, then $\mathcal{A}$ is $\omega$-dense in $\mathcal{C}_0(X)$.

Proof. Since $\mathcal{A} \cap \mathcal{C}_0(X)$ is a self-adjoint subalgebra of $\mathcal{C}_0(X)$ satisfying the two point property, $\mathcal{A} \cap \mathcal{C}_0(X)$ is uniformly dense in $\mathcal{C}_0(X)$ by 3.10. That $\mathcal{A}$ is $\omega$-dense in $\mathcal{C}_0(X)$ now follows immediately from 3.9.

The bounded subsets of $\mathcal{C}_0(X)$, where $V$ is a Nachbin family on $X$, are precisely those subsets $A$ such that for every $v \in V$ there is a constant $b(v; A) > 0$ with $\|fv\| \leq b(v; A)$ for every $f \in A$; i.e., those subsets which are absorbed by every neighborhood (of zero). In the next theorem, we set forth a characterization of the $\omega$-bounded sets in $\mathcal{C}_0(X)$, and, after doing this, we extend a result for $(\mathcal{C}(X), c\text{-op})$ (due to Seth Warner) to $\mathcal{C}_0(X)$. Then
from this theorem, for example, we can obtain the character-
ization of the $\beta$-bounded sets in $(C_b(X), \beta)$ which was
originally discovered by Buck[4].

3.12. Theorem. If $V$ is a Nachbin family on $X$ with
$V \subseteq B(X)$, then $A \subseteq CV_o(X)$ is $w$-bounded if and only if the
following two properties hold:

(1) for every $v \in V \setminus \{o\}$ and for every $n \in N$, $A$ is
uniformly bounded on the sets $K_{v,n} = \{x \in X: (n+1)^{-1} \leq ||v||^{-1}v(x) \leq n^{-1}\};$

(2) for every $v \in V \setminus \{o\}$, there exists $a(v;A) > 0$
such that $\sup_{K_{v,n}} ||f|| : f \in A \leq na(v;A)$ for all $n \in N$.

Proof. Assume the subset $A$ of $CV_o(X)$ is $w$-bounded,
and let $v \in V \setminus \{o\}$. Then $|f(x)| v(x) \leq b(v; A)$ for every
$f \in A, x \in X$, and if $x \in K_{v,n}$ for some $n \in N$, we have
$|f(x)| \leq b(v; A)(n+1)||v||^{-1}$ for each $f \in A$; i.e., (1)
holds. Choose $a(v; A) = 2 ||v||^{-1}b(v; A)$, and let $n \in N$.
Now $n^{-1}\sup_{K_{v,n}} ||f|| : f \in A \leq ||v||^{-1}b(v;A)n+1 \leq a(v;A),$
and so (2) holds.

Assume a subset $A$ of $CV_o(X)$ satisfies (1) and (2),
and let $v \in V$. If $v = 0$, then choose $b(v; A) = 1$; so
we may assume $v \neq 0$. In this case, we choose $b(v; A) =
||v||a(v; A)$. If $x \in N(v)$, then there exists $n \in N$ so
that $x \in K_{v,n}$. For $f \in A$, we then have $|f(x)| v(x) \leq$
$\sup_{K_{v,n}} ||g|| : g \in A \leq \sup_{K_{v,n}} ||g|| n^{-1}||v|| \leq
||v||a(v;A) = b(v;A)$. Thus $A$ is bounded and the proof is
complete.
If we take $V = \chi_c(X)$, then the sets $K_{V,n}$ of the preceding theorem are void if $n > 1$, while $K_{V,1} = K$ when $v = \lambda \chi_K$. Therefore the above theorem says that the $w$-bounded sets are precisely those which are uniformly bounded on compacta. This, of course, is the well-known characterization of the $c$-op bounded sets in $C(X)$. At this time, we are not able to give a useful characterization of the $w$-bounded sets in case $V \not\subseteq B(X)$.

Now let $V$ be a Nachbin family on $X$. We say that $X$ is $V$-compact if $C^\circ V(X) \subseteq C_b(X)$. We now give the promised extension of a result for $(C(X), c$-op), which was due to Warner [32, p.274].

3.13. Theorem. Let $V$ be a Nachbin family on $X$ with $V \subseteq B(X)$ and so that $x \in X$ implies there is a $v \in V$ for which $v(x) > 0$. If $X$ is $V$-compact and if $C^\circ V(X)$ is sequentially complete, then $C^\circ V(X)$ has a countable base for bounded sets. In particular, $\{B_n\}_{n=1}^\infty$ form a base for the $w$-bounded sets in $C^\circ V(X)$, where $B_n = \{f \in C^\circ V(X) : \|f\| \leq n\}$ for $n \in \mathbb{N}$. Moreover, if $\{B_n\}_{n=1}^\infty$ form a base for the $w$-bounded sets in $C^\circ V(X)$, then $X$ is $V$-compact.

Proof. First assume $\{B_n\}_{n=1}^\infty$ form a base for the $w$-bounded sets in $C^\circ V(X)$. For $f \in C^\circ V(X)$, $[f]$ is clearly $w$-bounded, and so there exists $n \in \mathbb{N}$ such that $f \in B_n$. Hence $C^\circ V(X) \subseteq C_b(X)$, which says $X$ is $V$-compact.

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Now assume \( X \) is \( V \)-compact and that \( CV_0(X) \) is sequentially complete, and suppose there is an \( \omega \)-bounded set \( A \) in \( CV_0(X) \) such that \( A \) is not contained in any \( B_n \).

In this case, there exists \((f_n, x_n) \in A \times X\) such that \(|f_n(x_n)| \geq n^3\) for each \( n \in \mathbb{N} \). If \( g_n = \sum_{k=1}^{n} k^{-2} |f_k| \) for every \( n \in \mathbb{N} \), then \( \{g_n\}_{n=1}^{\infty} \subseteq CV_0(X) \). We wish to show \( \{g_n\}_{n=1}^{\infty} \) is \( \omega \)-Cauchy; to this end, we fix \( v \in V \), and recall there exists \( b(v; A) > 0 \) so that \(|f_k|v \leq b(v; A)\) for every \( f \in A \), which implies \(|f_k|v \leq b(v; A)\) for all \( k \in \mathbb{N} \).

Let \( \epsilon > 0 \) and choose \( n_0 \in \mathbb{N} \) so that \( \sum_{k=n_0}^{\infty} k^{-2} < \epsilon(b(v;A))^{-1} \).

Then \( m, n \geq n_0 \) implies \(|g_n - g_m|v \leq b(v; A) \sum_{k=n_0}^{\infty} k^{-2} < \epsilon; \) i.e., \( \{g_n\}_{n=1}^{\infty} \) is \( \omega \)-Cauchy. Hence there is a \( g \in CV_0(X) \) such that \( g_n \to g(u) \), and \( g \geq g_n \) for every \( n \in \mathbb{N} \); since \( \{g_n\}_{n=1}^{\infty} \) is a monotone increasing sequence. But this contradicts the \( V \)-compactness of \( X \), since then \( g(x_n) \geq g_n(x_n) \geq n^3n^{-2} = n \) for every \( n \in \mathbb{N} \). Because \( B_n \) is clearly \( \omega \)-bounded for each \( n \in \mathbb{N} \), the proof is complete.

If, for example, \( V = \chi_c(X) \), then \( X \) is \( V \)-compact if and only if \( X \) is pseudo-compact.

3.14. Corollary. If \( X \) is locally compact and if \( V \) is a Nachbin family on \( X \) with \( C_c^+(X) \leq V \subseteq B(X) \), then \( X \) is \( V \)-compact if and only if \( \{B_n\}_{n=1}^{\infty} \) forms a base for the \( \omega \)-bounded sets in \( CV_0(X) \).

Proof. By 3.3, \( CV_0(X) \) is complete, and the result is now immediate from 3.13.
If $X$ is locally compact and if $V$ is a Nachbin family on $X$ with $C^+_0(X) \leq V \subseteq B(X)$, then it follows from 2.6 and 2.17 that $X$ is $V$-compact. In view of 3.14, we have proved the following theorem, which includes the special case of $(C_b(X), \beta)$.

3.15. **Theorem.** Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $C^+_0(X) \leq V \subseteq B(X)$. A set $A$ in $CV_0(X)$ is $\omega$-bounded if and only if $A$ is uniformly bounded.

We now turn our attention to the most important topic in Chapter III; i.e., the characterization of the topological dual space $CV_0(X)^*$ of $CV_0(X)$. The complete picture (as we describe it) is included in 3.16, 3.21, and 3.26. Because of the generality of our approach, we have found it expedient to consider three separate classes of Nachbin families (not all necessarily disjoint), and in each case the technique of proof is quite different. We begin by considering the class of Nachbin families $V$ for which $C^+_0(X) \leq V \subseteq K^+(X)$.

3.16. **Theorem.** Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $C^+_0(X) \leq V \subseteq K^+(X)$. Then $T: M_b(X) \to CV_0(X)^*$, where $T(\mu)(f) = \int f d\mu$ for each $f \in CV_0(X)$, is a (linear) isomorphism onto, and in this case we write $CV_0(X)^* = M_b(X)$.

Proof. We must first show that $T$ is actually into. But if $\mu \in M_b(X)$, then by 1.6 $\widetilde{T}(\mu) \in (C_b(X), \beta)^*$ where
\( \tilde{T}(\mu)(f) = \int f d\mu \) for every \( f \in C_b(X) \). From 2.6 and 2.17, we have that \( CV_o(X) \subseteq C_b(X) \) and \( r(\beta; CV_o(X)) \subseteq \omega \).

Since \( R(\tilde{T}(\mu); CV_o(X)) = T(\mu) \), we thus have \( T(\mu) \in CV_o(X)^* \), and \( T \) is well-defined (\( T \) is clearly linear). To show \( T \) is onto, let \( L \in CV_o(X)^* \). From 2.6, 2.15, and 3.6, we have that \( C_o(X) \) is an \( \omega \)-dense subspace of \( CV_o(X) \) and \( r(\omega; C_o(X)) \) is weaker than the uniform topology on \( C_o(X) \).

Thus \( F \in (C_o(X), ||\cdot||)^* \), where \( F = R(L; C_o(X)) \). This, by 1.4, implies there is a \( \mu \in M_b(X) \) such that \( F(f) = \int f d\mu \) for all \( f \in C_o(X) \). Since \( T(\mu) = L \) on the \( \omega \)-dense subspace \( C_o(X) \), then \( T(\mu) = L \). If \( \mu \in M_b(X) \) and if \( T(\mu)(f) = 0 \) for each \( f \in CV_o(X) \), then \( \int f d\mu = 0 \) for each \( f \in C_o(X) \), which implies \( \mu = 0 \) by 1.4. Thus \( T \) is one-to-one and the proof is complete.

The above result naturally leads to the question posed below. If \( E \) is a locally convex space with dual \( E^* \), then the Mackey topology \( \tau(E, E^*) \) on \( E \) is the finest locally convex topology for \( E \) under which the dual is still \( E^* \).

That there is always such a topology is the content of the Mackey-Arens theorem [23, p.62]. \( E \) is called a Mackey space if the topology on \( E \) is \( \tau(E, E^*) \). Buck [4] asked if \( (C_b(X), \beta) \) is a Mackey space (here \( X \) is locally compact), and Conway [9] gave an affirmative answer in the case \( X \) was paracompact. So we have that if \( X \) is locally compact and paracompact, then \( \tau(C_b(X), M_b(X)) \) is the weighted topology \( \beta \) determined by the Nachbin family \( C_o^+(X) \).
This then raises the question of when \( \tau(\mathcal{C}_b(X), \mathcal{M}_b(X)) \) is a weighted topology determined by some Nachbin family \( V \) on \( X \). We answer this question below (see 3.19).

3.17. Lemma. If \( X \) is locally compact and if \( V = \mathcal{N}^+(X) \cap \mathcal{B}_0(X) \), then \( V \) is a Nachbin family on \( X \) with \( V \approx \mathcal{C}_o^+(X) \).

Proof. It is clear that \( V \) is a Nachbin family on \( X \) and that \( \mathcal{C}_o^+(X) \subseteq V \). If \( v \in V \), then \( v \in \mathcal{B}(X) \) by 2.3. Since to complete the proof it will suffice to show there is a \( \varphi \in \mathcal{C}_o^+(X) \) such that \( v \leq \varphi \), we may assume \( ||v|| = 1 \).

For each \( n \in \mathbb{N} \), define \( K_n = \{ x \in X : v(x) \geq 2^{-n} \} \). Since \( K_n \) is compact for each \( n \in \mathbb{N} \), for every \( n \in \mathbb{N} \) there is a \( \varphi_n \in \mathcal{C}(X) \) such that \( 0 \leq \varphi_n \leq 1 \) and \( \varphi_n(K_n) = 1 \).

Define \( \varphi = \sum_{n=1}^{\infty} 2^{-(n-1)} \varphi_n \), and note that (since \( \varphi \) is the uniform limit of functions in \( \mathcal{C}_c(X) \)) we have \( \varphi \in \mathcal{C}_o^+(X) \).

If \( x \in \mathcal{N}(v) \), then there exists \( n \in \mathbb{N} \) such that \( x \in K_n \), which implies \( v(x) \leq 2^{-(n_0-1)} \) where \( n_0 = \min\{n \in \mathbb{N} : x \in K_n \} \). Since \( \varphi(x) \geq 2^{-(n_0-1)} \), it follows that \( v \geq \varphi \).

3.18. Lemma. Let \( X \) be locally compact and let \( V \) be a Nachbin family on \( X \). If \( \mathcal{C}_v(X) = \mathcal{C}_b(X) \) and if \( \beta \subseteq \omega_V \), then \( V \approx \mathcal{C}_o^+(X) \).

Proof. From 2.8 we have \( \mathcal{C}_o^+(X) \subseteq V \). Since \( 1 \in \mathcal{C}_v(X) \), \( 1 \cdot v = v \in \mathcal{N}^+(X) \cap \mathcal{B}_0(X) \) for every \( v \in V \), and hence \( V \leq \mathcal{N}^+(X) \cap \mathcal{B}_0(X) \). It now follows from 3.17 that \( V \approx \mathcal{C}_o^+(X) \).
3.19. **Theorem.** Let $X$ be locally compact. $(C_b(X), \beta)$ is a Mackey space if and only if there exists a Nachbin family $V$ on $X$ so that $CV_o(X) = (C_b(X), \tau(C_b(X), M_b(X)))$.

We now return to our consideration of $CV_o(X)^*$, and this time consider the class of Nachbin families $V$ for which $C_c^+(X) \leq V \leq C_o^+(X)$.

3.20. **Lemma.** Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $C_c^+(X) \leq V \leq C_o^+(X)$. Then there is a subspace $M$ of $M_b(X)$ with $M_c(X) \subseteq M$ and a (linear) onto isomorphism $T:M \rightarrow CV_o(X)^*$ where, for $\mu \in M$,

$$T(\mu)(f) = \int fd\mu$$

for every $f \in C_b(X)$.

**Proof.** By 2.6, 2.17, and 3.6, we have that $C_b(X)$ is an $\omega$-dense subspace of $CV_o(X)$ and $\tau(\omega; C_b(X)) \subseteq \beta$. If $L \in CV_o(X)^*$, then $F \in (C_b(X), \beta)^*$ where $F = R(L; C_b(X))$. By 1.6 there is a $\mu_L \in M_b(X)$ such that $F(f) = \int fd\mu_L$ for every $f \in C_b(X)$. Now let $M = \{\mu_L \in M_b(X): L \in CV_o(X)^*\}$ and define $T:M \rightarrow CV_o(X)^*$ by $T(\mu_L) = L$. We observe at this point that $T$ is the inverse of the function $T^{-1}:CV_o(X)^* \rightarrow M$ defined by $T^{-1}(L) = \mu_L$, and we see from the Hahn-Banach theorem that $T^{-1}$ is one-to-one. Moreover, from 1.6 we have that $T^{-1}$ is linear, and so $M$ is a linear subspace of $M_b(X)$ (since $T^{-1}$ is onto $M$ by definition). Consequently, $T$ is a linear isomorphism onto $CV_o(X)^*$, and $T(\mu)(f) = \int fd\mu$ for every $f \in C_b(X)$. Thus the proof will be complete when we show $M_c(X) \subseteq M$. But if $\mu \in M_c(X)$, then by 1.7,
\[ \hat{L} \in (C(X), c\text{-}op)^* \] where \( \hat{L}(f) = \int fd\mu \) for every \( f \in C(X) \).

From 2.6 and 2.14 we have that \( CV_o(X) \subseteq C(X) \) and
\[ r(c\text{-}op; CV_o(X)) \subseteq \omega, \] and hence \( L \in CV_o(X)^* \) where
\[ L(f) = \hat{L}(f) \] for each \( f \in CV_o(X) \). Since \( L(f) = \int fd\mu \) for
every \( f \in C_b(X), \mu \in M \).

Let \( X \) be locally compact, let \( \mu \in M_b(X) \), and let \( g \) be a Borel measurable function on \( X \) which is bounded on the compact subsets of \( X \). For each \( A \in \text{Borel}(X) \), define
\[ g \cdot \mu(A) = \int A g d\mu; \] then \( g \cdot \mu \in M(X) \) [11, p.221]. In particular, if \( V \) is a Nachbin family on \( X \), then each \( v \in V \) is Borel measurable and, from the proof of 2.3, bounded on the compact subsets of \( X \). We will denote \( \{v \cdot \mu : v \in V, \mu \in M_b(X)\} \) by
\[ V \cdot M_b(X). \] As we have already remarked, \( V \cdot M_b(X) \) is always contained in \( M(X) \), and will actually be a linear subspace of \( M(X) \) for a large class of Nachbin families \( V \) on \( X \), although this is by no means obvious from the definition. In fact, \( V \cdot M_b(X) \) is our candidate for \( CV_o(X)^* \) in the case of the Nachbin families yet to be considered, but, of course, we do not expect uniqueness of representation (e.g., see 3.34).

3.21. Theorem. Let \( X \) be locally compact and let \( V \) be a Nachbin family on \( X \) with \( C^+_c(X) \leq V \leq C^+_o(X) \). Then
\[ V \cdot M_b(X) \] is a linear subspace of \( M_b(X) \) and
\[ T:V \cdot M_b(X) \to CV_o(X)^* \] is a (linear) isomorphism onto where
\[ T(\mu)(f) = \int fd\mu \] for each \( f \in CV_o(X) \). In this case we will write \( CV_o(X)^* = V \cdot M_b(X) \).
Proof. If $\mu \in V \cdot M_b(X)$, then there is a $v \in V$ and $\nu \in M_b(X)$ such that $\mu = v \cdot \nu$. If $f \in CV_0(X)$, then

$$\int fd\mu = \int fd\nu \leq ||fv|| \cdot ||v|| \leq \infty; \text{ i.e., } T(\mu) \text{ is a linear functional on } CV_0(X).$$

Moreover, if $\{f_i\}$ is a net in $CV_0(X)$ with $f_i \to \varnothing(\nu)$, then $|T(\mu)(f_i)| = \int f_i \cdot \nu \leq ||f_i|| \cdot ||\nu|| \to 0$, and we see $T$ is a well-defined function into $CV_0(X)^*$. Since $V \cdot M_b(X)$ is clearly contained in $M_b(X)$, Theorem 1.6 yields that $T$ is one-to-one. We claim that if $T$ is onto $CV_0(X)^*$, then $V \cdot M_b(X)$ is a linear subspace of $M_b(X)$. To see this, first observe that $V \cdot M_b(X)$ is closed under multiplication by scalars. If $\mu, \nu \in V \cdot M_b(X)$, then $T(\mu), T(\nu) \in CV_0(X)^*$, which implies $T(\mu) + T(\nu) \in CV_0(X)^*$. But there is a $\tau \in V \cdot M_b(X)$ such that $T(\tau) = T(\mu) + T(\nu)$, and so

$$\int fd\tau = T(\tau)(f) = T(\mu)(f) + T(\nu)(f) = \int fd\mu + \int fd\nu = \int fd(\mu + \nu) \text{ for every } f \in CV_0(X).$$

In view of 1.6, this implies $\mu + \nu = \tau$, and hence $V \cdot M_b(X)$ is a linear subspace of $M_b(X)$ and $T$ is a linear isomorphism onto. We show $T$ is onto.

Let $L \in CV_0(X)^*$. By 3.20, there exists $\mu \in M_b(X)$ such that $L(f) = \int fd\mu$ for every $f \in C_b(X)$. We may write $\mu$ uniquely as $\alpha + i\beta$ where $\alpha$ and $\beta$ are real Radon measures on $X$ with $\alpha, \beta \in M_b(X)$. If $f \in C_b(X)$ with $f:X \to \mathbb{R}$, then

$$|\int fd\alpha| = (|\int fd\alpha|^2)^{1/2} \leq (|\int fd\alpha|^2 + |\int fd\beta|^2)^{1/2} = |\int fd\mu|.\text{ Hence } f \in C_b(X) \text{ implies } |\int fd\alpha| = |\int (Re f) \alpha + i(Im f) \alpha| \leq |\int (Re f) \mu| + |\int (Im f) \mu|,$$
this it follows that \( L_\alpha \in CV_0(X)^* \) where \( L_\alpha(f) = \int f d\alpha \) for each \( f \in C_b(X) \) (recall that \( C_b(X) \) is dense in \( CV_0(X) \), and note that \( \{f_1\} \subseteq C_b(X) \) with \( f_1 \to o(w) \) implies \( \text{Ref}_i = \frac{1}{2}(f_1 + f_i) \in C_b(X) \) for every \( i \) and \( \text{Ref}_i \to o(w) \)). Similarly, \( L_\beta \in CV_0(X)^* \) where \( L_\beta(f) = \int f d\beta \) for each \( f \in C_b(X) \). If \( \alpha = \alpha^+ - \alpha^- \) is the minimal decomposition of \( \alpha \), then \( \alpha^+ , \alpha^- \in M_b(X) \). Since \( L \in CV_0(X)^* \), there is a \( \nu \in V \) such that \( |L(f)| \leq 1 \) for every \( f \in V_\nu \). If
\[
u(x) = \begin{cases} v(x)^{-1}, & x \in N(v) \\ +\infty, & x \in X \setminus N(v), \end{cases}
\]
then \( u \) is a non-negative l.s.c. function on \( X \). If \( \mathfrak{f}_u = \{ \phi \in C_c^+(X) : \phi \leq u \} \), then \( \phi \in \mathfrak{f}_u \) implies \( |\phi v| \leq 1 \); i.e., \( \mathfrak{f}_u \subseteq V_\nu \). Moreover, if \( \phi \in \mathfrak{f}_u \), then \( \int \phi d\alpha^+ = \sup \{ \int \psi d\alpha : \psi \in \mathfrak{f}_u, \psi \leq \phi \} \), while \( \psi \in \mathfrak{f}_u \) implies \( |\int \psi d\mu| \leq 1 \). So \( \int \psi d\alpha \leq |\int \phi d\alpha| \leq |\int \psi d\mu| \leq 1 \), and therefore \( \sup \{ \int \phi d\alpha^+ : \phi \in \mathfrak{f}_u \} \leq 1 \). Now \( \sup \{ \int \phi d\alpha^+ : \phi \in \mathfrak{f}_u \} \) is, by definition, the upper integral of the function \( u \).

However, the upper integral of a non-negative l.s.c. function \( u \) is finite if and only if \( u \) is integrable, in which case the upper integral and the integral agree [11, p.189]. Thus \( u \) is \( \alpha^+ \)-integrable and \( \int u d\alpha^+ = \int \frac{1}{V} |\mu| \leq 1 \). Consequently, \( \int \frac{1}{V} |\mu| \leq 4 \), and in this case \( \frac{1}{V} \mu \in M_b(X) \). Thus \( v \cdot (\frac{1}{V} \mu) = \mu \) is in \( V \cdot M_b(X) \) and \( T(\mu) \) agrees with \( L \) on an \( \omega \)-dense subspace of \( CV_0(X) \). Since \( T(\mu) \in CV_0(X)^* \), \( T(\mu) = L \) and \( T \) is onto. As we have already seen, this completes the proof.
3.22. Corollary. If $X$ is locally compact and if $V$ is a Nachbin family on $X$ with $C_c^+(X) \leq V \leq C_0^+(X)$, then $M_c(X) \subseteq V \cdot M_b(X) \subseteq M_b(X)$. Moreover, if $V \approx C_c^+(X)$, then $V \cdot M_b(X) = M_c(X)$, while if $V \approx C_0^+(X)$, then $V \cdot M_b(X) = M_b(X)$.

Proof. In view of 3.20 and 3.21, it is immediate that $M_c(X) \subseteq V \cdot M_b(X) \subseteq M_b(X)$. Clearly, if $V \approx C_c^+(X)$, then $M_c(X) = V \cdot M_b(X)$, while it follows from 1.6 that $V \cdot M_b(X) = M_b(X)$ if $V \approx C_0^+(X)$.

3.23. Corollary. Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $C_c^+(X) \leq V \leq C_0^+(X)$. If $\mu \in V \cdot M_b(X)$, then $|\mu| \in V \cdot M_b(X)$. Moreover, if we write $\mu = \alpha + i\beta$ where $\alpha, \beta$ are real Radon measures on $X$ and if $\alpha = \alpha^+ - \alpha^-$, $\beta = \beta^+ - \beta^-$ are the minimal decompositions $\alpha$ and $\beta$, respectively, then $\alpha, \beta, \alpha^+, \alpha^-, \beta^+, \beta^- \in V \cdot M_b(X)$.

Proof. In the course of the proof of 3.21, we saw that $\frac{1}{V}|\mu| \in M_b(X)$ for at least one $v \in V$. We also observed that $\alpha, \beta \in V \cdot M_b(X)$, and the result now follows immediately.

We define $\delta : X \to ((C(X), c\text{-op})., \sigma((C(X), c\text{-op})., C(X)))$ by $\delta(x)(f) = f(x)$. It is well-known that $\delta$ is a homeomorphism onto its image (e.g., see [32, p.266]). Varadarajan [31] has obtained other results for the map $\delta$ in a slightly different setting. In particular, he has shown that if $X$ is a metric space and if $\delta : X \to ((C_b(X), ||\cdot||)^*, \sigma((C_b(X), ||\cdot||)^*, C_b(X)))$ is defined by $\delta(x)(f) = f(x)$ for each $f \in C_b(X)$, then $\delta(X)$ is sequentially closed.
[31, p.197]. With only the assumption that $X$ be locally compact and Hausdorff, we obtain the following stronger result.

3.24. **Lemma.** Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $C^+_c(X) \leq V \leq C^+_0(X)$. If $\delta: X \to M_b(X)$ is defined by $\delta(x) = \delta_x$ (the point mass at $x$) for each $x \in X$, then $\delta$ is a one-to-one continuous closed map into $(C^+_0(X)^*, \sigma(C^+_0(X)^*, C^+_0(X)))$.

**Proof.** Since $\delta(X) \subseteq M_c(X)$, $\delta(X) \subseteq C^+_0(X)^*$ by 3.22. It is clear that $\delta$ is one-to-one and continuous. Let $F$ be a closed subset of $X$, and note that it will suffice to show $\delta(F)$ is $\sigma(M_b(X), C_b(X))$-closed since $C^+_0(X)^* \subseteq M_b(X)$ by 3.22, while $C_b(X) \subseteq C^+_0(X)$ by 2.6. To this end, let $\mu \in \overline{\delta(F)}$ (the $\sigma(M_b(X), C_b(X))$-closure in $M_b(X)$). Since $1 \in C_b(X)$, we have that $\mu \neq 0$ and hence $\text{spt } \mu$ is not void. Choose $x_o \in \text{spt } \mu$ and a net $\{\delta(x_i)\}$ in $\delta(F)$ so that $\delta(x_i) \rightarrow \mu(\sigma(M_b(X), C_b(X)))$, and suppose $\{x_i\}$ does not converge to $x_o$. In this case there is a neighborhood $A$ of $x_o$ for which $\{x_i\}$ is not eventually in $A$; i.e., for every $i_o$, there is an $i \geq i_o$ such that $x_i \notin A$.

Now by 1.3, there is a $\varphi \in C_c^+(X)$ such that $\text{spt } \varphi \subseteq A$ and $|\int \varphi d\mu| > 0$. Let $\varepsilon > 0$, choose $i_o$ so that $i \geq i_o$ implies $|\int \varphi d\mu - \varphi(x_i)| < \varepsilon$, and choose $j \geq i_o$ such that $x_j \notin A$. Thus $|\int \varphi d\mu| \leq |\int \varphi d\mu - \varphi(x_j)| + |\varphi(x_j)| < \varepsilon$, which contradicts the choice of $\varphi$. So $\mu = \delta(x_o) \in \delta(F)$, since $x_i \rightarrow x_o$ implies $\delta(x_i) \rightarrow \delta(x_o)$. 

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In the case where $X$ is locally compact, we now have a characterization of $CV_o(X)^*$ whenever $V$ is a Nachbin family on $X$ which satisfies either $C_c^+(X) \leq V \leq C_o^+(X)$ or $C_o^+(X) \leq V \leq K^+(X)$. Unfortunately, this does not necessarily include a characterization for all Nachbin families $V$ on $X$ which satisfy $C_c^+(X) \leq V \leq K^+(X)$ (much less, for all such $V$). For example, if we take $X = \mathbb{R}$ and let $V = \{ \varphi \in C_c^+(\mathbb{R}) : \varphi \leq \lambda \sin(\lambda) \lambda \geq 0 \}$, then $V$ is a Nachbin family on $\mathbb{R}$ and $C_c^+(\mathbb{R}) \leq V \leq K^+(\mathbb{R})$, but $V \not\subseteq C_o^+(\mathbb{R})$ nor is $C_o^+(\mathbb{R}) \leq V$. We are able, however, to give a characterization of $CV_o(X)^*$ for a large class of Nachbin families $V$ on $X$ which will include the above example. In particular, if $V$ is a Nachbin family on $X$ with $V \subseteq C^+(X)$, then we obtain a result analogous to Theorem 3.21 (although the proof in this case is entirely different) which has important consequences in our application to tensor products (see Chapter IV).

Let $E$ and $F$ be locally convex spaces and let $t : E \to F$ be linear and continuous. Then $t$ is $\sigma(E, E^*)$, $\sigma(F, F^*)$ continuous and, in this case, there is a natural linear operator $t^* : F^* \to E^*$ called the dual map of $t$, which is defined by $\langle e, t^* f^* \rangle = \langle te, f^* \rangle$ where $e \in E$ and $f^* \in F^*$ [23, p.38]. Moreover, $t^*$ is weak-* continuous (i.e., $\sigma(F^*, F), \sigma(E^*, E)$ continuous), and if $A$ is any subset of $E$, then $t(A)^0 = t^{-1}(A^0)$[23, p.39]. We will also need the fact that if $A$ is any subset of $E$, then $A^{00}$ is the closed absolutely convex hull of $A$ [23, p.36].
The technique employed in the lemma below was first used by Conway \[9\] in his characterization of the \(\beta\)-equicontinuous subsets of \(M_b(X)\).

3.25. Lemma. Let \(X\) be locally compact, let \(V\) be a Nachbin family on \(X\) with \(V \subseteq C^+(X)\) (recall we are assuming \(V\) is such that \(CV_0(X)\) is Hausdorff), and define \(T_V:CV_0(X) \to C_0(X)\) for each \(v \in V\) by \(T_V(f) = fv\) for each \(f \in CV_0(X)\). If \(v \in V\), then \(V^0_v = T_V^*(B^0)\) where
\[
B = \{f \in C_0(X) : \|f\| \leq 1\}.
\]

Proof. Since \(v \in V\) implies \(v \in C^+(X)\), \(fv \in C_0(X)\) for every \(f \in CV_0(X)\), and so \(T_V\) is a well-defined map into \(C_0(X)\) for each \(v \in V\), which is clearly linear and continuous (\(C_0(X)\) is here endowed with the uniform topology). So \(v \in V\) implies \(T_V^*:M_b(X) \to CV_0(X)^*\) is weak-*continuous.

Now \(B^0\) is \(\sigma(M_b(X), C_0(X))\)-compact by Alaoglu's theorem \[23, p.61\], and so \(T_V^*(B^0)\) is \(\sigma(CV_0(X)^*, CV_0(X))\)-compact, while \(T_V^*(B^0)\) is obviously absolutely convex. Since \(T_V(V_v) \subseteq B\) while \(T_V(f) \in B\) implies \(f \in V_v\), we have \(T_V^{-1}(B) = V_v\). Hence \(V_v = T_V^{-1}(B) = \{T_V^*(B^0)\}^0\), from which it follows that \(V_v^0 = (T_V^*(B^0))^{00} = T_V^*(B^0)\).

3.26. Theorem. Let \(X\) be locally compact and let \(V\) be a Nachbin family on \(X\) with \(V \subseteq C^+(X)\). Then \(V \cdot M_b(X)\) is a linear subspace of \(M(X)\) and \(T:V \cdot M_b(X) \to CV_0(X)^*\) is a (linear)onto isomorphism where \(T(\mu)(f) = \int fd\mu\) for each \(f \in CV_0(X)\). In this case we will write \(CV_0(X)^* = V \cdot M_b(X)\).
Proof. That \( T \) is a well-defined function into \( CV_0(X)^* \) follows exactly as in the proof of 3.21. Since \( V \cdot M_b(X) \subseteq M(X) \), we have that \( T \) is one-to-one from 1.2 (and the fact that \( C_c(X) \) is dense in \( CV_0(X) \) from 3.6). The demonstration that \( V \cdot M_b(X) \) is a linear subspace of \( M(X) \) and that \( T \) is a linear isomorphism will follow just as in the proof of 3.21 (except here we use 1.2 instead of 1.6) once we have shown \( T \) is onto.

To this end, fix \( L \in CV_0(X)^* \). Now there is a \( v \in V \) such that \( L \in V_v^O \) since \( CV_0(X)^* = \bigcup_{v \in V} V_v^O \) [23, p.47], and hence there is a \( \mu \in M_b(X) \) such that \( L = T_v^* \mu \) by 3.25. If \( f \in CV_0(X) \), then \( \langle f, L \rangle = \langle f, T_v^* \mu \rangle = \langle T_v f, \mu \rangle = \langle f v, \mu \rangle = \int f v \mu \, d\mu = \int f (v \cdot \mu) \mu = \langle f, T(v \cdot \mu) \rangle \), from which it follows that \( L = T(v \cdot \mu) \). Thus \( T \) is onto and the proof is complete.

3.27. Corollary. Let \( X \) be locally compact and let \( V \) be a Nachbin family on \( X \) with \( V \subseteq C^+(X) \). If \( v \in V \), then \( V_v^O = v \cdot \{ \mu \in M_b(X) : ||\mu|| \leq 1 \} \).

Proof. If \( B = \{ f \in C_0(X) : ||f|| \leq 1 \} \), then \( B^O = \{ \mu \in M_b(X) : ||\mu|| \leq 1 \} \). By 3.25, \( V_v^O = T_v^*(B^O) \), while from the proof of the above theorem, however, we have that \( T_v^* \mu = v \cdot \mu \) for every \( \mu \in M_b(X) \) (here, of course, the identification established in the above theorem is implicit).

As we have already seen, if \( X \) is locally compact and if \( V \) is a Nachbin family on \( X \) with \( V \subseteq C^+(X) \), then
CV_0(X)^* \subseteq M(X). Further, if K^+(X) \subseteq V, then 
M_b(X) \subseteq CV_0(X)^* . Both containments may be proper as is 
illustrated in the following example.

3.28. Example. Let X be locally compact and let 
V = C^+(X) . Consequently, CV_0(X)^* = V \cdot M_b(X) and 
M_b(X) \subseteq V \cdot M_b(X) \subseteq M(X).

(1) Let X = (0, 1) and let m denote Lebesque 
measure on X. If \nu(x) = \frac{1}{x} for x \in X, then \nu \in V, 
m \in M_b(X), \nu \cdot m \in V \cdot M_b(X), but \nu \cdot m \notin M_b(X). Hence 
M_b(X) \not\subseteq V \cdot M_b(X).

(2) Let X be an uncountable discrete space and let 
\sigma denote counting measure on X. Then \sigma \in M(X), but 
\sigma \notin V \cdot M_b(X). For suppose there exists \nu \in V, \mu \in M_b(X) 
such that \sigma = \nu \cdot \mu (i.e., \sigma \in V \cdot M_b(X)). Then \sigma(x) = 1 
= \nu(x) \mu(x) for each x \in X, and this implies \mu(x) > 0 
for every x \in X. But this contradicts the fact that 
\mu \in M_b(X), since \mu \in M_b(X) implies \mu(x) \neq 0 for at 
most countably many x \in X. Therefore V \cdot M_b(X) is a proper 
subset of M(X).

Example 3.28, (2) also serves as an example (promised 
in Chapter II) of a space X satisfying the hypothesis of 
2.24 for which CV_0(X) is not (C_c(X), \text{ind } \lim), where 
V = C^+(X) . The latter statement is, in this case, an 
immediate consequence of 1.2. It is true, however, that 
CV_0(X) = (C_c(X), w_V) where w_V is properly coarser than the 
inductive limit topology, and this follows from 2.24 and 2.26.
The preceding example leads us to an interesting question: for which locally compact Hausdorff spaces \( X \) is it true that \( M(X) = C^+(X) \cdot M_b(X) \)? We will say that a locally compact Hausdorff space \( X \) is \textbf{\textit{C-reducible}} if \( M(X) = C^+(X) \cdot M_b(X) \).

3.29. \textbf{Theorem}. If \( X \) is locally compact and \( \sigma \)-compact, then \( X \) is \( C \)-reducible.

\textbf{Proof}. This follows easily from 1.2, 2.28, and 3.26.

3.30. \textbf{Lemma}. If \( X \) is \( C \)-reducible, then every \( \mu \in M(X) \) has \( \sigma \)-compact support.

\textbf{Proof}. If \( \mu \in M(X) \), then there exist \( v \in C^+(X) \), \( v \in M_b(X) \) so that \( \mu = v \cdot v \). From 3.22 we have \( \text{spt} \ v \) is \( \sigma \)-compact, and since \( \text{spt} \ \mu \subseteq \text{spt} \ v \), \( \text{spt} \ \mu \) is \( \sigma \)-compact.

3.31. \textbf{Theorem}. If \( X \) is \( C \)-reducible, then every closed and discrete subset of \( X \) is countable.

\textbf{Proof}. Let \( S \) be a closed and discrete subset of \( X \), and note that if \( K \) is a compact subset of \( X \), then \( K \cap S \) is finite. Define the linear functional \( L \) on \( C_c(X) \) by \( L(f) = \sum_{x \in S} f(x) \). Then \( L \in (C_c(X), \text{ind lim})^* \), and by 1.2 there is a \( \mu \in M(X) \) such that \( L(f) = \int f d\mu \) for every \( f \in C_c(X) \). It now follows from 1.3 that \( S \subseteq \text{spt} \ \mu \). But \( \text{spt} \ \mu \) is \( \sigma \)-compact by 3.30, and this implies \( S \) is \( \sigma \)-compact whence countable.
3.32. Corollary. Let \( X \) be locally compact. Then \( X \) is \( \sigma \)-compact if and only if \( X \) is paracompact and \( C \)-reducible.

Proof. Assume \( X \) is paracompact and \( C \)-reducible. Since \( X \) is paracompact, \( X = \bigcup \alpha S_\alpha \) where \( S_\alpha \) is an open and closed \( \sigma \)-compact subset of \( X \) for each \( \alpha \) and \( S_\alpha \cap S_\beta \) is void whenever \( \alpha \neq \beta \) [3, p.107]. Choose \( x_\alpha \in S_\alpha \) for each \( \alpha \) and let \( S = \bigcup \alpha \{x_\alpha\} \). Then \( S \) is closed and discrete, and hence \( S \) is countable by 3.31. So \( X \) is clearly \( \sigma \)-compact. The converse is Theorem 3.29.

In the next example, we show that the converse of 3.29 is false.

3.33. Example. Let \( X \) denote the set of all ordinals less than the first uncountable ordinal \( \Omega \) with the order topology. For \( \mu \in M^+(X) \), define \( g:X \rightarrow \mathbb{R}^+ \) by \( g(x) = \int_{x[1,x]} \mu \). Then \( g \) is a monotone increasing function on \( X \), and so the set of points of discontinuity \( D \) of \( g \) is countable. Since every \( \sigma \)-compact subset of \( X \) is relatively compact [30], there exists \( x_0 \in X \) so that \( x \leq x_0 \) for every \( x \in D \). If we define \( f(x) = \begin{cases} g(x_0), & x \leq x_0 \\ g(x), & x > x_0 \end{cases} \) then \( f \in C^+(X) \). But every \( h \in C(X) \) is eventually constant [30], and hence there is a \( y_0 \in X \) such that \( f(x) = f(y_0) \) for every \( x \geq y_0 \). Consequently, if \( z_0 = \max(x_0, y_0) \), then \( \text{spt} \mu \subseteq [1, z_0] \); i.e., \( \mu \in M_c^+(X) \), and we have \( M(X) \subseteq M_c(X) \subseteq C^+(X) \cdot M_c(X) \subseteq C^+(X) \cdot M_b(X) \). This implies \( M(X) = C^+(X) \cdot M_b(X) \), and \( X \) is \( C \)-reducible. \( X \) is certainly locally compact, but not \( \sigma \)-compact.
Example 3.33 also shows that the converse of 2.28 is false. For let $X$ be the space described in 3.33. It is clear from the properties of $X$ used in 3.33 that (2) of 2.25 holds for $X$. Therefore $X$ is pseudo-compact, and the hypothesis of 2.24 is satisfied. In view of 2.19, we have that $CV_o(X) = C_c(X) = C_o(X)$, where $V = C^+(X) = C_b^+(X)$, and hence it will suffice to show $w = \text{ind lim}$. The following general theory will be used to do this.

A barrel $A$ in a locally convex space $E$ is an absolutely convex, closed, and absorbent (i.e., if $x \in E$, then there is a $\lambda > 0$ so that $x \in \lambda A$) subset in $E$; $E$ is called barrelled if every barrel in $E$ is a neighborhood; and if $E$ is barrelled, then $E$ is a Mackey space [23, p.66]. Moreover, an inductive limit of barrelled spaces is barrelled [23, p.81], and every Banach space is barrelled [23, p.67]. Since $V \approx K^+(X)$, $CV_o(X)$ is a Banach space, and hence $w = \tau(C_c(X), M(X))$. But $(C_c(X), \text{ind lim})$ is the inductive limit of the Banach spaces $\{C(X; K) : K \subseteq X$, $K$ compact$, \}$, and therefore $(C_c(X), \text{ind lim})$ is barrelled, which implies $\text{ind lim} = \tau(C_c(X), M(X))$. Thus $w = \text{ind lim}$, and the proof is complete.

As we have seen, for certain Nachbin families $V$ on a locally compact space $X$, $V \cdot M_b(X)$ is a linear subspace of $M(X)$. If $\mu \in V \cdot M_b(X)$, then there exist $v \in V$, $v \in M_b(X)$ so that $\mu = v \cdot v$ (a factorization of $\mu$!). In general, this factorization is not unique as we show in the following example.
3.34. Example. Let \( m \) denote Lebesgue measure on \( \mathbb{R} = X \), let \( u(x) = \begin{cases} e^x, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases} \) and let \( v(x) = \begin{cases} e^{2x}, & x \geq 0 \\ e^{-2x}, & x < 0 \end{cases} \). Then \( u, v, \frac{1}{u}, \frac{1}{v} \in C^+(X) \), and hence \( \mu, v \in M(X) \) where \( \mu = \frac{1}{u} \cdot m \) and \( v = \frac{1}{v} \cdot m \). For \( a < 0 \) and \( b > 0 \), we have
\[
\mu([a,b]) = \int_a^b \frac{1}{u(x)} dm(x) = \int_a^0 e^x dm(x) + \int_b^0 e^{-x} dm(x) = 1 - e^a - (e^{-b} - 1) = 2 - (e^a + e^{-b}), \quad \text{while} \quad v([a,b]) = \int_a^b \frac{1}{v(x)} dm(x) = \int_a^0 e^{2x} dm(x) + \int_b^0 e^{-2x} dm(x) = \frac{1}{2}(1 - e^{2a}) - \frac{1}{2}(e^{-2b} - 1)
\]
Thus we have \( \mu, v \in M_0(X) \) and \( m = u \cdot \mu = v \cdot v \).

We now consider the problem of characterizing the extremal points of \( V_0 \) where \( V \) is a Nachbin family on a locally compact space \( X \) with \( V \subseteq C^+(X) \) and \( v \in V \). Here again, the technique to be employed was used by Conway [9] to obtain this result in the special case when \( V = C_0^+(X) \). As yet, we have not been able to remove the continuity assumption on \( V \) although we conjecture the characterization would be the same if, for example, \( C_c^+(X) \subseteq V \subseteq C_0^+(X) \). In view of the forthcoming applications of the following theorem, we feel it would be desirable to remove the continuity assumption.

3.35. Lemma. Let \( X \) be locally compact, let \( V \) be a Nachbin family on \( X \) with \( V \subseteq C^+(X) \), let \( B = \{ f \in C_0(X) : ||f|| \leq 1 \} \), and let \( \mu \in B^0 \). If \( v \in V \) and
if $T^*_V \mu = T^*_V \delta(x)$ where $x \in N(v)$ and $T_V : C^*_V(X) \to C(X)$ is the map defined in 3.25, then $\mu = \delta(x)$.

Proof. We wish to show $\nu = 0$ where $\nu = \mu - \delta(x)$. To this end, let $\phi \in C^+(X)$ with $\text{spt}(\phi) \subseteq N(v)$, and define

$$\phi_V(y) = \begin{cases} \phi(y), & y \in N(v) \\ 0, & y \in X \setminus \text{spt}(\phi). \end{cases}$$

Then $\phi_V \in C^+(X)$ and $\phi_V \nu = \phi$. So

$$\int \phi_V \nu dv = \int \phi_V d(T^*_V \nu) = 0, \text{ since } T^*_V \nu = 0, \text{ and by } 1.3, |\nu|(N(v)) = 0. \text{ Thus } |\nu([x])| \leq |\nu|([x]) \leq |\nu|(N(v)) = 0, \text{ so } \mu([x]) = (\nu + \delta(x))([x]) = \nu([x]) + 1 = 1. \text{ If } A \text{ is any Borel set in } X \text{ and } x \notin A, \text{ then } 1 \leq |\mu|(A) + 1 = |\mu|(A) + |\mu|([x]) = |\mu|(A \cup [x]) \leq 1, \text{ which implies } |\mu|(A) = 0. \text{ Hence } \mu = \delta(x) \text{ (and } \nu = 0).$

3.36. **Theorem.** If $X$ is locally compact and if $V$ is a Nachbin family on $X$ with $V \subseteq C^+(X)$, then

$$\delta(V^O_v) = \{ \lambda \nu(x) \delta(x) : x \in N(v), |\lambda| = 1 \}, \text{ for every } v \in V.$$

Proof. Let $v \in V$ and let $T_V : C^*_V(X) \to C(X)$ be the map defined in 3.25. If $B = \{ f \in C(X) : |f| \leq 1 \}$, then $B^O$ and $V^O_v$ are both absolutely convex (a polar set is always absolutely convex [23, p.34]). Moreover, $B^O$ and $V^O_v$ are both weak-*compact by Alaoglu's theorem [23, p.61]. Now $T^*_V M_b(X) \to V \cdot M_b(X)$ is linear, and hence $R(T^*_V ; B^O)$ is affine. By 3.25, $T^*_V(B^O) = V^O_v$, and (from the proof of 3.25) $T^*_V$ is weak-*continuous. Therefore, we have from 1.9 that $\mu \in \delta(V^O_v)$ implies there is a $v \in \delta(B^O)$ so that $\mu = T^*_V v = v \cdot v$. As we previously remarked, $\nu \in \delta(B^O)$.
implies there exist \( x \in X, \lambda \in \mathcal{F}, |\lambda| = 1 \) such that 
\[ \nu = \lambda \delta(x). \]
Thus \( \mu = \lambda \nu \cdot \delta(x) = \lambda \nu(x) \delta(x), \) and since \( \mu \neq 0, \)
\( x \in N(\nu). \)

Now let \( x \in N(\nu) \) and let \( \mu = \nu(x) \delta(x) \). Clearly, 
\( \mu \in V^o \), and to complete the proof it will suffice to show 
\( \mu \in \delta(V^o_v). \) If \( \mu = \frac{1}{2}(\sigma + \tau) \) where \( \sigma, \tau \in V^o_v \), then there 
are \( \alpha, \beta \in \mathcal{B}^o \) such that \( T^*_v \alpha = \sigma \) and \( T^*_v \beta = \tau \). Thus 
\( \mu = T^*_v \delta(x) = T^*_v \left( \frac{1}{2}(\alpha + \beta) \right), \) and by 3.35, \( \delta(x) = \frac{1}{2}(\alpha + \beta). \)
However, \( \delta(x) \in \delta(B^o) \), which implies \( \delta(x) = \alpha = \beta, \) and 
hence \( \mu = \sigma = \tau; \) i.e., \( \mu \in \delta(V^o_v). \)

We now obtain a Stone-Weierstrass theorem for a class 
of Nachbin families on a locally compact space \( X \) which 
subsumes 3.10 (the Stone-Weierstrass theorem for \( (C^0(X), ||-||) \)) 
and the Stone-Weierstrass theorem for \( (C^b(X), \beta) \) obtained 
by Glicksberg [14], as well as improves our 3.11. To do 
this, we adapt the approach used by de Branges for the \( C^0(X) \) 
case (e.g., see [11]) to our more general situation. This 
is done by the following lemma.

3.37. Lemma. Let \( X \) be locally compact, let \( V \) be a 
Nachbin family on \( X \) with \( V \subseteq C^+(X) \), let \( A \) be a linear 
subspace of \( CV^o(X) \), and let \( \mu \in \delta(A^o \cap V^o_v) \), where \( v \in V. \)
If, for \( g \in C(X), R(g; \text{spt } \mu) \) is bounded and \( \mathbb{R} \)-valued 
while \( \int fg \mu = 0 \) for every \( f \in A \), then there is an \( \alpha \in \mathbb{R} \) 
such that \( g \cdot \mu = \alpha \mu, \) and in this case \( R(g; \text{spt } \mu) = \alpha \) 
(\( \text{spt } \mu \) not void).
Proof. If \( \mu = 0 \), then \( g \cdot \mu = 0 \cdot \mu \) and we are done, and so we may assume \( \mu \neq 0 \). By 3.27, there is a \( v \in M_b(X) \), \( ||v|| \leq 1 \) such that \( \mu = v \cdot v \). If \( ||v|| < 1 \), then
\[
\mu = ||v||v \cdot (||v||^{-1}v) + (1 - ||v||)0
\]
which contradicts the choice of \( \mu \), and hence \( ||v|| = 1 \). We may assume without loss of generality that \( 0 \leq R(g; \text{spt } \mu) \leq 1 \). To see this, let \( h = R(g; \text{spt } \mu) \), and note there is a \( b \in \mathbb{R} \) so that \( 0 \leq h + b \). Thus \( 0 \leq ||h+b||^{-1}(h+b) \leq 1 \), while
\[
\int ||h+b||^{-1}(h+b)f\mu = 0 \text{ for every } f \in A, \text{ and if } ||h+b|| (g+b) \cdot \mu = \alpha \mu, \text{ then } g \cdot \mu = (\alpha ||h+b||^{-1}b) \cdot \mu.
\]
We may also assume without loss of generality that \( \text{spt } v = \text{spt } \mu \) (for consider the Radon measure \( v' \) defined on Borel \( (X) \) by \( v'(E) = v(E \cap \text{spt } \mu) \) for each \( E \in \text{Borel } (X) \)). Now let \( \rho = g \cdot v \), and observe that
\[
||\rho|| = \int |g|d|v| = \int_{\text{spt } \mu} h d|v| \leq ||v|| = 1.
\]
If \( ||\rho|| = 0 \), then \( g \cdot \mu = v \cdot \rho = 0 = 0 \cdot \mu \).
If \( ||\rho|| = 1 \), then \( 1 = ||\rho|| = \int |g|d|v| = \int_{\text{spt } \mu} h d|v| \), which implies \( h = 1 \); i.e., \( g \cdot \mu = 1 \cdot \mu \). So we assume
\( 0 < ||\rho|| < 1 \) and let \( \sigma = ||\rho||^{-1} \rho \), \( \tau = (1-||\rho||)^{-1}(v-\rho) \).
Clearly, \( v \cdot \sigma \in A^0 \cap V^0_V \), while \( f \in A \) implies
\[
\int fvd\tau = (1-||\rho||)^{-1}(\int fd\mu - \int fgd\mu) = 0, \text{ and hence } v \cdot \tau \in A^0.
\]
Moreover, \( ||\tau|| = (1-||\rho||)^{-1} \int \ |1-g|d|v| \)
\[
= (1-||\rho||)^{-1} \int_{\text{spt } \mu} (1-h)d|v| = (1-||\rho||)^{-1}(||v||-||\rho||) = 1,
\]
so that \( v \cdot \tau \in V^0_V \). But \( \mu = (1-||\rho||)v \cdot \tau + ||\rho||v \cdot \sigma \), and hence \( \mu = v \cdot \sigma = v \cdot \tau \), from which it follows that \( ||\rho|| \mu = v \cdot \rho = g \cdot \mu \). The proof is now complete.
3.38. Theorem. Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $V \subseteq c^+(X)$ and so that $CV_0(X)$ is an algebra (e.g., if $V \subseteq V^2$). If $\mathcal{A}$ is a self-adjoint subalgebra of $CV_0(X)$ which has the two-point property and if either

(1) for each $\mu \in CV_0(X)^*$, $F_\mu = \{R(f; spt \mu) : f \in \mathcal{A}\} \cap C_b(spt \mu)$ is a separating family for $spt \mu$; i.e., $x, y \in spt \mu$ with $x \neq y$ implies there is an $f \in F_\mu$ such that $f(x) \neq f(y)$; or

(2) there is a Nachbin family $U$ on $X$ such that $U \sim V$, $U \subseteq c^+(X)$, and $\mathcal{A}$ is a module over $U$; i.e., $U \cdot \mathcal{A} \subseteq \mathcal{A}$;

then $\mathcal{A}$ is $w$-dense in $CV_0(X)$.

Proof. Suppose $\mathcal{A}$ is not $w$-dense in $CV_0(X)$. Then there is a $\rho \in \mathcal{A}$ such that $\rho \neq 0$, and hence there is a $v \in V$ such that $\rho \in \mathcal{A} \cap V_0$. By the Krein-Milman theorem (1.8), there is a $\mu \in \mathcal{A} \cap V_0$ such that $\mu \neq 0$. We wish to show that $spt \mu = \{x\}$ for some $x \in X$. If this is not the case, then there are $x, y \in spt \mu$ such that $x \neq y$. If (1) holds, then there exists $g \in \mathcal{A}$ such that $g(x) \neq g(y)$ and $R(g; spt \mu)$ is $R$-valued and bounded. Moreover, $\int fgdu = 0$ for every $f \in \mathcal{A}$, which by 3.37 implies $R(g; spt \mu)$ is constant. This is an obvious contradiction, and so in this case there exists $x \in X$ such that $spt \mu = \{x\}$. We now show the same result is obtained if (2) is assumed to hold.
In this case, we may assume there is a \( u \in U \) such that \( \mu \in \delta(\sigma^0 \cap V_u^0) \). Define \( T_u : C_0^o(X) \to C_0(X) \) by \( T_u(f) = fu \) for \( f \in C_0^o(X) \). We previously established (see the proof of 3.25) that \( T_u^* \) is a weak-*continuous linear map. Now \( K = T_u(\sigma^0 \cap B^0) = \sigma(M_b(X), C_0(X)) \)-compact and convex (here \( B \) still denotes the closed unit ball in \( C_0(X) \)), and if \( \sigma \in K \), then \( T_u^*(\sigma) = u \cdot \sigma \in V_u^0 \), while the fact that \( \int f u d\sigma = 0 \) for every \( f \in \sigma \) implies \( u \cdot \sigma \in \sigma^0 \).

Moreover, if \( v \in \sigma^0 \cap V_u^0 \), then there exists \( \sigma \in B^0 \) such that \( u \cdot \sigma = v \), and \( \int f u d\sigma = 0 \) for every \( f \in \sigma \), which implies \( \sigma \in T_u(\sigma^0) \). Hence \( T_u^*(K) = \sigma^0 \cap V_u^0 \), and by 1.9 there is a \( v \in \delta(K) \) such that \( \mu = T_u^* v = u \cdot v \). Now there is a \( g \in \sigma \) so that \( g(x) = 0 \) and \( g(y) = 1 \), and we may assume \( g \) is \( \mathbb{R} \)-valued. Thus \( gu \) is bounded and \( \mathbb{R} \)-valued, while \( \int (fu)(gu)dv = \int (fg)ud\mu = 0 \) for every \( f \in \sigma \). By 3.37, \( R(gu; \text{spt } v) \) is constant, and so \( 0 = g(x)u(x) = g(y)u(y) = u(y) \). It follows easily that \( R(u; \text{spt } \mu) = 0 \), and hence \( \mu = 0 \), which contradicts our choice of \( \mu \). Thus there is an \( x \in X \) so that \( \text{spt } \mu = [x] \).

Since \( \text{spt } \mu = [x] \), there exists \( \lambda \in \mathbb{C} \) such that \( \mu = \lambda \delta(x) \), and this implies \( f(x) = \lambda^{-1} \int fd\mu = 0 \) for every \( f \in \sigma \). But this is clearly impossible, and therefore \( \sigma \) is \( \omega \)-dense in \( C_0^o(X) \). This, of course, completes the proof of the theorem.
3.39. Corollary. Let $X$ be locally compact and let $V$ be a Nachbin family on $X$ with $V \subseteq C^+(X)$ and so that $CV_0(X)$ is an algebra. If $\mathcal{A}$ is a self-adjoint subalgebra of $CV_o(X)$ which has the two-point property and if either

(1) $C^+_o(X) \subseteq V$, or (2) $v \in V$ implies $\text{spt}(v)$ is compact,

then $\mathcal{A}$ is $w$-dense in $CV_0(X)$.

Proof. In either case, (1) of 3.38 holds.

In particular, the above corollary includes the cases of $(C(X), c\text{-}op)$, $(C_b(X), \beta)$, and $(C_o(X), ||\cdot||)$. Note that, for $(C_o(X), ||\cdot||)$, (2) of 3.38 would also apply.

In view of condition (1) of 3.38, the question arises as to whether there is a Nachbin family $V$ on a locally compact space $X$ and measure $\mu \in CV_o(X)^*$ for which not every $f \in CV_o(X)$ is bounded on $\text{spt} \mu$. The answer is yes, and, although there are easier examples than the one we give, the following example will also give some insight into the pathology which can arise in weighted spaces.

3.40. Example. Let $X = \mathbb{R}^+$, and define the function $\psi$ on $X$ by $\psi(x) = \begin{cases} x, & x \in [0,1) \\ 2n-x, & x \in [2n-1,2n), \ n \in \mathbb{N} \\ x-2n, & x \in [2n,2n+1], \ n \in \mathbb{N}. \end{cases}$

If $V = \{ \phi \in C^+_c(X) : \phi \geq 0 \}$, then $V$ is a Nachbin family on $X$ and $C^+_c(X) \subseteq V \subseteq K^+(X)$. Consequently, $C^+_o(X) \subseteq CV_0(X) \subseteq C(X)$. Observe that $V$ is not comparable.
to $C^+(X)$, and that $K^+(X) \cap CV_0(X) = \{0\}$. We now show that $CV_0(X)$ contains unbounded functions. Define the function $g$ on $X$ by

$$g(x) = \begin{cases} n^3(x-2n)+n, & x \in [2n-n^{-2}, 2n), n \in \mathbb{N} \\ n^3(2n-x)+n, & x \in [2n, 2n+n^{-2}], n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Since $g \in C(X)$, it will suffice to show $g \psi \in C_0(X)$ in order to show $g \in \mathcal{CV}_0(X)$. To do this we consider $g \psi(x)$ for $x \in [2n-n^{-2}, 2n]$, where $n \in \mathbb{N}$. Here $g \psi(x) = (n^3(x-2n)+n)(2n-x)$, and $(g \psi)'(x) = -2n^3x+4n^4-n$, and this implies $(g \psi)'(x) = 0$ if and only if $x = 2n - \frac{1}{2}n^{-2}$. But this is clearly a maximum of $g \psi$ on $[2n-n^{-2}, 2n]$, $g \psi(2n - \frac{1}{2}n^{-2}) = (4n)^{-1}$, and it now follows that $g \psi \in C_0(X)$. Thus $\mathcal{CV}_0(X)$ contains some unbounded functions, but not all bounded ones. Since $V \subseteq C^+(X)$, 3.26 implies $\mathcal{CV}_0(X)^* = V \cdot M_b(X)$. Let $m$ denote Lebesque measure on $\mathbb{R}^+$ and let $\nu = e^{-x}m$. Then $\nu \in M_b(X)$, as we saw in 3.34, and so $\mu = \nu \cdot \nu \in \mathcal{CV}_0(X)^*$. But $\text{spt} \mu = \mathbb{R}^+$, and hence $g$ is unbounded on $\text{spt} \mu$.

We will close this chapter by giving an example of a weighted space $\mathcal{CV}_0(X)$, where $K^+(X) \leq V \leq C^+(X)$, which is a Banach space (actually a commutative semi-simple Banach-* algebra) in which not every uniformly bounded set is bounded, and which has an approximate identity, but no bounded approximate identity.

3.41. Example. Let $X = \mathbb{R}$ and let $V = \{\lambda \rho : \lambda \geq 0, \rho(x) = x^2 + 1\}$. Then $V$ is a Nachbin family on $X$ with
$K^+(X) \leq V \subseteq C^+(X)$, and thus $C_c(X) \subseteq CV_o(X) \subseteq C_o(X)$ (and both inclusions are proper). It follows from 2.11 that $CV_o(X)$ is a normed space, while 3.3 implies $CV_o(X)$ is complete. Since $\rho \leq \rho^2$, 3.8 yields that $CV_o(X)$ is a topological algebra (in fact, for $f, g \in CV_o(X)$, $||fg|| \leq ||f|| \cdot ||g||$). By 3.5, $CV_o(X)$ contains a canonical approximate identity, which is uniformly bounded by definition. Hence it will suffice to prove the following theorem.

3.42. Theorem. Let $X$ be locally compact and let $V$ be any Nachbin family on $X$. Then the following are equivalent:

1. no approximate identity in $CV_o(X)$ is bounded;
2. there is a canonical approximate identity in $CV_o(X)$ which is not bounded;
3. there is a $v \in V$ which is unbounded.

Proof. Since $CV_o(X)$ has a canonical approximate identity by 3.5, (1) certainly implies (2). Now assume that $(\varphi_i)$ is an unbounded canonical approximate identity in $CV_o(X)$. Hence there is a $v \in V$ such that $||\varphi_i v||$ is unbounded. So $n \in \mathbb{N}$ implies there is an $i_n$ such that $||\varphi_i v|| > n$, and this implies there is an $x_n \in X$ such that $\varphi_{i_n} (x_n) v(x_n) > n$. Since $0 \leq \varphi_i \leq 1$, $v(x_n) > n$, which implies $v$ is unbounded; i.e., (2) implies (3).

Now assume there is a $v \in V$ such that $v$ is unbounded, and let $(\varphi_i)$ be any approximate identity in $CV_o(X)$. Now
for each \( n \in \mathbb{N} \) there exist \( x_n \in X \), \( \theta_n \in C_c(X) \) so that
\[
v(x_n) \geq n, \quad 0 \leq \theta_n \leq 1, \quad \text{and} \quad \theta_n(x_n) = 1.
\]
Since \( \{\theta_n\}_{n=1}^\infty \subseteq C_0(X) \), for each \( n \in \mathbb{N} \) there is an \( i_n \) such that
\[
\| (\varphi_i \theta_n - \theta_n) v \| < 1,
\]
this implies \( |1 - \varphi_i (x_n)|v(x_n) < 1 \), and hence \( v(x_n) - 1 < |\varphi_i (x_n)|v(x_n) \leq |\varphi_i v| \). But
\[
n - 1 < v(x_n) - 1,
\]
from which it follows that \( \{\varphi_i\} \) is not bounded, and so (3) implies (1).
CHAPTER IV

Tensor Products: The Weighted Representation Theorem

The primary purpose of this chapter is to establish a topological isomorphism between the biequicontinuous completed tensor product $CU_0(X) \overset{\sim}{\otimes} CV_0(Y)$ of two weighted spaces and another weighted space $CW_0(X \times Y)$, where $X$ and $Y$ are locally compact. By doing so, we obtain an extension of the classical Dieudonné density theorem [10]. This representation yields several corollaries involving well-known spaces, including a theorem of Grothendieck [15, p.90] for $C_0(X) \overset{\sim}{\otimes} C_0(Y)$ and our analogous result for $(C_b(X), \beta) \overset{\sim}{\otimes} (C_b(Y), \beta)$, as well as leads to the discovery of new subspaces of $C(X \times Y)$.

We will begin by presenting a brief introduction to tensor products. For a more extensive treatment the reader is invited to examine [26]. Let $E$ and $F$ be vector spaces, let $E'$ and $F'$ be their algebraic duals, and let $B(E', F')$ be the space of bilinear functionals on $E' \times F'$. If $x \in E$ and $y \in F$, we define the element $x \otimes y$ of $B(E', F')$ by $x \otimes y(x', y') = \langle x, x' \rangle \langle y, y' \rangle$, for all $(x', y') \in E' \times F'$; $x \otimes y$ is called an elementary tensor. The tensor product $E \otimes F$ of $E$ and $F$ is the linear span of all elementary
tensors in \( B(E', F') \). If \( b \in E \otimes F \), then \( b \) may have several representations as linear combinations of elementary tensors. However, if \( b \neq 0 \), then there is an \( n \in \mathbb{N} \) and linearly independent subsets \( \{x_i\}_{i=1}^n \subseteq E \), \( \{y_i\}_{i=1}^n \subseteq F \) such that

\[
b = \sum_{i=1}^n x_i \otimes y_i.
\]

Now assume \( E \) and \( F \) are locally convex spaces. If \( b \in E \otimes F \) is such that \( b(x^*, y^*) = 0 \) for every \((x^*, y^*) \in E^* \times F^*\), then it follows from the Hahn-Banach theorem and the representation for \( b \) as \( \sum_{i=1}^n x_i \otimes y_i \), where \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) are linearly independent in \( E \) and \( F \) respectively, that \( b = 0 \). Thus it suffices to consider \( E \otimes F \) as a subspace of the space of bilinear functionals on \( E^* \times F^* \). If \( b \in E \otimes F \), then \( b : E^* \times F^* \to \mathbb{C} \) is separately continuous where \( E^* \) and \( F^* \) have their weak-* topologies.

The biequicontinuous topology \( \mathcal{J}_e \) on \( E \otimes F \) is defined by the semi-norms \( P(b) = \sup\{|b(x^*, y^*)| : x^* \in P, y^* \in Q\} \), where \( P \) and \( Q \) are arbitrary weak-*closed and equicontinuous subsets of \( E^* \) and \( F^* \), respectively; i.e., \( \mathcal{J}_e \) is the topology of uniform convergence on \( \{A_i^0 \times B_j^0 : \{A_i\} \) and \( \{B_j\} \) are neighborhood bases in \( E \) and \( F \), respectively\}. We will denote the completion of \( (E \otimes F, \mathcal{J}_e) \) by \( E \hat{\otimes} F \).

If \( X_i, \ i = 1,2 \) are topological spaces, we will denote the projection map of \( X_1 \times X_2 \) onto \( X_i \), by \( \Pi_{X_i} \), \( i = 1,2 \).
\( \Pi_{x_1} \) is both continuous and open. If \( f \) is a function on \( X \) and if \( g \) is a function on \( Y \), we will denote by \( f \times g \) the function on \( X \times Y \) defined by \( f \times g(x, y) = f(x)g(y) \) for all \( (x, y) \in X \times Y \). In particular, if \( U \) is a Nachbin family on \( X \) and \( V \) is a Nachbin family on \( Y \), then 
\[
U \times V = \{ u \times v : u \in U, v \in V \}. 
\]
Recall that we are assuming \( C^{U}(X) \) and \( C^{V}(Y) \) are Hausdorff spaces.

4.1. \textbf{Lemma.} If \( U \) is a Nachbin family on \( X \), if \( V \) is a Nachbin family on \( Y \), and if \( W = U \times V \), then \( W \) is a Nachbin family on \( X \times Y \). Moreover, \( C^{W}(X \times Y) \) is Hausdorff if for every open set \( A \) in \( Y \) (or \( X \)) there is a \( v \in V(u \in U) \) such that \( R(v; A) \neq 0 \) (\( R(u; A) \neq 0 \)) (this would be the case, of course, if \( Y \) (or \( X \)) were locally compact).

\textbf{Proof.} Let \( u \in U, v \in V \), and note that \( u \times v \geq 0 \).

To show \( u \times v \) is u.s.c., we show the set \( A = \{(x, y) : u \times v(x, y) < \epsilon \} \) is open, where \( \epsilon > 0 \). If 
\( (x_0, y_0) \in A \), then there is an \( \eta > 0 \) such that 
\[
\eta v(y_0) < \epsilon - u(x_0)v(y_0). 
\]
If we let \( B = \{ x \in X : u(x) < u(x_0) + \eta \} \) and \( C = \{ y \in Y : v(y) < \epsilon (u(x_0) + \eta)^{-1} \} \), then \( B \) is an open neighborhood of \( x_0 \), while \( C \) is an open neighborhood of \( y_0 \) since 
\[
(u(x_0) + \eta)v(y_0) = u(x_0)v(y_0) + \eta v(y_0) < \epsilon. 
\]
Now if \( (x, y) \in B \times C \), then \( u \times v(x, y) < \epsilon \), which implies \( B \times C \subseteq A \); hence \( A \) is open. Now let \( u_1 \times v_1, u_2 \times v_2 \in W \) and let \( \lambda > 0 \). Then there exist \( u \in U, v \in V \) such that 
\[
\lambda u_1, \lambda u_2 \leq u \text{ and } v_1, v_2 \leq v, \text{ and this implies } \lambda u_1 \times v_1, \lambda u_2 \times v_2 \leq u \times v. 
\]
Consequently, \( W \) is a Nachbin family on \( X \times Y \).
If \( f \in \mathcal{C}_W(X \times Y) \) with \( f \neq 0 \), then there is an open set \( A \) in \( X \times Y \) so that \( f(x, y) \neq 0 \) for every \((x, y) \in A\). Since \( \Pi_Y(A) \) is open, there exists \((v, y_o) \in V \times \Pi_Y(A)\) such that \( v(y_o) > 0 \). Define \( g: X \to \mathbb{C} \) by \( g(x) = f(x, y_o) \), and observe that \( g \in C(X) \). If \( u \in U \), \( \varepsilon > 0 \), then

\[
\{ x \in X : |g(x)u(x) \geq \varepsilon \} \subseteq \Pi_X(K) \text{ where } K = \{(x, y): |f(x, y)|u(x)v(y) \geq \varepsilon v(y_o)\}.
\]

Since \( K \) is compact, \( \Pi_X(K) \) is compact, and it follows that \( g \in \mathcal{C}_U(X) \). But \( g \neq 0 \), and hence there exists \((u, x_o) \in U \times X\) such that \( g(x_o)u(x_o) \neq 0 \). Therefore

\[
f(x_o, y_o)u(x_o)v(y_o) \neq 0, \text{ and } \mathcal{C}_W(X \times Y) \text{ is Hausdorff}.
\]

In the course of the proof that \( \mathcal{C}_W(X \times Y) \) is Hausdorff we have obtained the following result.

4.2. Corollary. Let \( U \) be a Nachbin family on \( X \), let \( V \) be a Nachbin family on \( Y \), and let \( W = U \times V \). If \( f \in \mathcal{C}_W(X \times Y) \) and if there is a \((v, y_o) \in V \times Y((u, x_o) \in U \times X)\) such that \( v(y_o) > 0 \) \((u(x_o) > 0)\), then the function \( g: X \to \mathbb{C} \) \((g: Y \to \mathbb{C})\) defined by \( g(x) = f(x, y_o)(g(y) = f(x_o, y)) \) is in \( \mathcal{C}_U(X)(\mathcal{C}_V(Y)) \).

We define \( T: C(X) \otimes C(Y) \to C(X \times Y) \) by first defining \( T \) on the elementary tensors \( f \otimes g \) by \( T(f \otimes g) = f \times g \) and then extending \( T \) linearly. To see that \( T \) is well-defined, let \( \sum_{i=1}^{n} f_i \otimes g_i = 0 \); if \( x \in X \) and \( y \in Y \), then

\[
0 = \sum_{i=1}^{n} <f_i, \delta(x)> <g_i, \delta(y)> = \sum_{i=1}^{n} f_i(x)g_i(y) = \left( \sum_{i=1}^{n} f_i \times g_i \right)(x, y)
\]

\[
= T(\sum_{i=1}^{n} f_i \otimes g_i)(x, y). \text{ } T \text{ is clearly linear and into. If}
\]
\[
T(\sum_{i=1}^{n} f_i \otimes g_i) = 0 \quad \text{and if} \quad \sum_{i=1}^{n} f_i \otimes g_i \neq 0, \quad \text{then since we may assume} \quad (f_i)_{i=1}^{n} \quad \text{and} \quad (g_i)_{i=1}^{n} \quad \text{are linearly independent in} \quad C(X) \quad \text{and} \quad C(Y), \quad \text{respectively,} \quad \sum_{i=1}^{n} f_i(x)g_i = 0 \quad \text{for any} \quad x \in X. \quad \text{Hence} \quad f_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad x \in X, \quad \text{from which it follows that} \quad \sum_{i=1}^{n} f_i \otimes g_i = 0 \quad \text{and} \quad T \quad \text{is one-to-one.}
\]

We will refer to \( T \) as the canonical embedding map.

4.3. **Lemma.** If \( U \) is a Nachbin family on \( X \), if \( V \)

is a Nachbin family on \( Y \), if \( W = U \times V \), and if \( T \) is the canonical embedding map, then \( T \) maps \( C(U(X)) \otimes C(V(Y)) \)

into \( C(W(X \times Y)) \).

**Proof.** Let \( F = T(\sum_{k=1}^{n} f_k \otimes g_k) \) where \( (f_k)_{k=1}^{n} \subseteq C(U(X)) \)

and \( (g_k)_{k=1}^{n} \subseteq C(V(X)) \), let \( w = u \times v \) where \( u \in U \) and \( v \in V \), let \( M_1 = \max\{||g_k v|| : k = 1, \ldots, n\} \), let \( M_2 = \max\{||f_k u|| : k = 1, \ldots, n\} \), and let \( \epsilon > 0 \). We need only show that the set \( A = \{(x, y) : |F(x, y)|w(x, y) \geq \epsilon\} \) is compact. If either \( M_1 = 0 \) or \( M_2 = 0 \), then \( A \) is void (and thus compact), and so we may assume \( M_1, M_2 > 0 \). Now let \( \alpha_1 = \epsilon(M_1 n)^{-1}, \alpha_2 = \epsilon(M_2 n)^{-1} \), let \( K_{1,k} = \{x \in X : |f_k(x)|u(x) \geq \alpha_1\} \), \( K_{2,k} = \{y \in Y : |g_k(y)|v(y) \geq \alpha_2\} \)

for \( k = 1, \ldots, n \), and let \( K_1 = \bigcup_{k=1}^{n} K_{1,k}, K_2 = \bigcup_{k=1}^{n} K_{2,k} \). Since \( K_1 \times K_2 \) is compact, it suffices to show \( A \subseteq K_1 \times K_2 \). If \( (x, y) \notin K_1 \times K_2 \), then either \( x \notin K_1 \) or \( y \notin K_2 \); say \( x \notin K_1 \). In this case, \( |f_k(x)|u(x) < \alpha_1 \) for \( k = 1, \ldots, n \), which implies \( |F(x, y)|w(x, y) \leq \sum_{k=1}^{n} |f_k(x)|u(x)|g_k(y)|v(y) \)

\[ \leq M_1 \sum_{k=1}^{n} |f_k(x)|u(x) < M_1 n \alpha_1 = \epsilon; \quad \text{i.e.,} \quad (x, y) \notin A. \quad \text{Hence} \quad A \subseteq K_1 \times K_2 \quad \text{and the proof is complete.} \]
Nachbin, in his book [20] which first appeared in print early this year, has also considered a part of the problem to which we have addressed ourselves. The overlap is of little consequence except for the result which we next state, where Nachbin's version was slightly more general than our own. Since this generalization will allow us to strengthen our main theorem (of this chapter), we have chosen to use Nachbin's result instead of our own. For completeness, we will include a proof (which is essentially the proof given by Nachbin).

4.4. Lemma (The Weighted Dieudonné Density Theorem [20, p.68]). If $U$ is a Nachbin family on $X$, if $V$ is a Nachbin family on $Y$, if $W = U \times V$, and if $T$ is the canonical embedding map, then $T(CU_0(X) \times CV_0(X))$ is $W$-dense in $CW_0(X \times Y)$.

For the proof, we will first need the following lemma.

4.5. Lemma. Let $V$ be a Nachbin family on $X$, let $K$ be a compact subset of $X$, and let $\{A_i\}_{i=1}^n$ be an open covering of $K$. If for every $x \in K$ there is an $f \in CV_0(X)$ such that $f(x) \neq 0$, then there exists $\{\varphi_i\}_{i=1}^n \subseteq CV_0(X)$ so that $\varphi_i \geq 0$ and $\varphi_i(X \setminus A_i) = 0$ for each $i = 1, \ldots, n$, while $\varphi = \sum_{i=1}^n \varphi_i$ satisfies $0 \leq \varphi \leq 1$ and $\varphi(K) = 1$.

Proof. We first note that if $f \in CV_0(X)$ and if $g \in C_b(X)$, then $fg \in CV_0(X)$. This is clear if $g = 0$,
while if $g \neq 0$, then \( \{x \in X : |fg(x)|v(x) \geq \epsilon\} \subseteq \{x \in X : |f(x)|v(x) \geq \epsilon\} \) for any $v \in V$ and $\epsilon > 0$, and the claim follows. If $x \in K$, then there exists $f_x \in CV_o(X)$ so that $f_x(x) \neq 0$, there is an $i_x \in [1, \ldots, n]$ such that $x \in A_{i_x}$, and hence there exists $\phi_x \in C(X)$ so that $0 \leq \phi_x \leq 1$, $\phi_x(X \setminus A_{i_x}) = 0$, and $\phi_x(x) = 1$. Since $\phi_x f_x \in CV_o(X)$, we may assume that $f_x \geq 0$ and $f_x(X \setminus A_{i_x}) = 0$ for some $i \in \{1, \ldots, n\}$.

Since $K$ is compact, there exists $[f_j^m_{j=1}] \subseteq CV_o(X)$ such that, for each $j = 1, \ldots, m$, $f_j \geq 0$ and $f_j(X \setminus A_{i_j}) = 0$ for some $i \in \{1, \ldots, n\}$, while $\sum f_j(x) > 0$ for every $x \in K$. For each $i = 1, \ldots, n$, let $\theta_i$ be the sum of all $f_j$ for which $f_j(X \setminus A_{i_j}) = 0$ (here $\theta_i = 0$ if $f_j(X \setminus A_{i_j}) \neq 0$ for every $j = 1, \ldots, m$). Then, for each $i = 1, \ldots, n$, $\theta_i \in CV_o(X)$, $\theta_i \geq 0$, and $\theta_i(X \setminus A_{i_i}) = 0$, while, for $\theta = \sum \theta_i$, $\theta(x) > 0$ for every $x \in K$. Let $\alpha = \inf\{\theta(x) : x \in K\}$ and define $\tilde{\theta} = \theta \vee \alpha$, $g = \frac{1}{\tilde{\theta}}$. Then $g \in C_b(X)$, which implies $[g^\theta_i]_{i=1}^n \subseteq CV_o(X)$. If we set $\phi_i = g^\theta_i$ for $i = 1, \ldots, n$, then $\phi_i \geq 0$ and $\phi_i(X \setminus A_{i_i}) = 0$ for each $i = 1, \ldots, n$. Moreover, if $\varphi = \sum \phi_i = g \sum \theta_i = g^\theta$, then $\varphi \geq 0$, while $x \in X$, $\theta(x) \geq \alpha$ implies $\varphi(x) = g(x) \theta(x) = \frac{\theta(x)}{\tilde{\theta}(x)} = \frac{\theta(x)}{\varphi(x)} = 1$, and $x \in X$, $\theta(x) \leq \alpha$ implies $\varphi(x) = \frac{\theta(x)}{\tilde{\theta}(x)} = \alpha^{-1} \theta(x) \leq 1$; i.e., $0 \leq \varphi \leq 1$ and $\varphi(K) = 1$.

Proof of 4.4. Let $f \in CW_o(X \times Y)$, let $u \in U$, $v \in V$, and let $\epsilon > 0$. Then $K = \{(x, y) : |f(x, y)|u(x)v(y) \geq \epsilon/4\}$ is compact, and so $K_1 = \Pi_X(K)$, $K_2 = \Pi_Y(K)$ are compact.
If \( x_0 \in K_2 \), then \( u(x_0) > 0 \); choose \( \delta = 3\varepsilon (8u(x_0)\|v\|_{K_2})^{-1} \). If we let \( A_1 = \{ x \in X : u(x) < 2u(x_0) \} \), then \( A_1 \) is an open neighborhood of \( x_0 \). Since \( K_2 \) is compact and since \( f \in C(X \times Y) \), we can find an open neighborhood \( A_2 \) of \( x_0 \) such that \( (x, y) \in A_2 \times K_2 \) implies \(|f(x, y) - f(x_0, y)| < \delta \). Then \( A = A_1 \cap A_2 \) is an open neighborhood of \( x_0 \). Choose \((x, y) \in A \times Y \): if \( y \not\in K_2 \), then \((x, y) \not\in K \), which implies \(|f(x, y)|u(x)v(y) < \varepsilon/4 \), and hence \(|f(x, y) - f(x_0, y)|u(x)v(y) \leq |f(x, y)|u(x)v(y) + |f(x_0, y)|u(x)v(y) < \varepsilon/4 + 2|f(x_0, y)|u(x_0)v(y) < \varepsilon/4 + \varepsilon/2 = 3\varepsilon/4 \); while \( y \in K_2 \) implies \(|f(x, y) - f(x_0, y)|u(x)v(y) < 2\delta u(x_0)\|v\|_{K_2} = 3\varepsilon/4 \). Thus \(|f(x, y) - f(x_0, y)|u(x)v(y) < 3\varepsilon/4 \) whenever \((x, y) \in A \times Y \). Because \( K_1 \) is compact, there exists \( \{x_i\}_{i=1}^n \subseteq K_1 \) and open sets \( \{A_i\}_{i=1}^n \) in \( X \) such that \( x_i \in A_i \), \( i = 1, \ldots, n \), \( K_1 \subseteq \bigcup_{i=1}^n A_i \), and \(|f(x, y) - f(x_i, y)|u(x)v(y) < 3\varepsilon/4 \) whenever \((x, y) \in A_i \times Y \) for all \( i = 1, \ldots, n \).

If \( y_0 \in K_2 \), then \( v(y_0) > 0 \), and hence the mapping \( g:X \to \mathbb{C} \) defined by \( g(x) = f(x, y_0) \) is in \( C^0_X(X) \) by 4.2.
If \( x \in K_1 \), then \( g(x) \neq 0 \), and so 4.5 applies. Therefore, there exists \( \{\varphi_i\}_{i=1}^n \subseteq C^0_X(X) \) such that \( \varphi_i > 0 \) and \( \varphi_i(X \setminus A_i) = 0 \) for each \( i = 1, \ldots, n \), while for \( \varphi = \sum_{i=1}^n \varphi_i \), \( \varphi(K_1) = 1 \) and \( 0 \leq \varphi \leq 1 \). Since \( u(x_i) > 0 \) for \( i = 1, \ldots, n \), the functions \( \theta_i:Y \to \mathbb{C} \) defined by \( \theta_i(y) = f(x_i, y) \) are in \( C^0_Y(Y) \) for each \( i = 1, \ldots, n \) by 4.2. Consequently,
\[ \sum_{i=1}^{n} \varphi_i \times \theta_i \text{ is in } T(\mathcal{CU}_o(X) \otimes \mathcal{CV}_o(Y)). \] Moreover,

\[ |f(x, y) - \sum_{i=1}^{n} \varphi_i(x)f(x, y)|u(x)v(y) < \varepsilon/4 \text{ for } (x, y) \in X \times Y, \]

while

\[ \sum_{i=1}^{n} \varphi_i(x)|f(x, y) - f(x_i, y)|u(x)v(y) < \frac{3\varepsilon}{4} \sum_{i=1}^{n} \varphi_i \leq 3\varepsilon/4 \text{ for } (x, y) \in X \times Y. \] Hence

\[ |f(x, y) - \sum_{i=1}^{n} \varphi_i(x)\theta_i(y)|u(x)v(y) \leq \sum_{i=1}^{n} \varphi_i(x)f(x, y)|u(x)v(y) + \]

\[ |\sum_{i=1}^{n} \varphi_i(x)f(x, y) - \sum_{i=1}^{n} \varphi_i(x)f(x_i, y)|u(x)v(y) < \varepsilon^2 + \frac{3\varepsilon}{4} = \varepsilon, \]

and the proof of 4.4 is complete.

4.6. Lemma. Let \( X \) and \( Y \) be locally compact, let \( U \) be a Nachbin family on \( X \) with \( U \subseteq C^+(X) \), let \( V \) be a Nachbin family on \( Y \) with \( V \subseteq C^+(Y) \), and let \( W = U \times V \). If \( T \) is the canonical embedding map, then \( T \) establishes a topological isomorphism between

\( (\mathcal{CU}_o(X) \otimes \mathcal{CV}_o(Y), \mathcal{F}_e) \) and \( T(\mathcal{CU}_o(X) \otimes \mathcal{CV}_o(Y)) \) endowed with the relative \( \mu_W \)-topology.

Proof. We have seen that \( T \) is a linear isomorphism, and, by 4.3, \( T \) maps \( \mathcal{CU}_o(X) \otimes \mathcal{CV}_o(Y) \) into \( \mathcal{CW}_o(X \times Y) \). It thus suffices to show \( T \) is bicontinuous. To do this, let \( F \in \mathcal{CU}_o(X) \otimes \mathcal{CV}_o(Y) \), let \( u \in U \), and let \( v \in V \); there exists \( \{f_k\}_{k=1}^{n} \subseteq \mathcal{CU}_o(X), \{g_k\}_{k=1}^{n} \subseteq \mathcal{CV}_o(Y) \) such that

\[ F = \sum_{k=1}^{n} f_k \otimes g_k. \] For \( y \in N(v) \), \( h_y = \sum_{k=1}^{n} g_k(y)v(y)f_k \) is in \( \mathcal{CU}_o(X) \), and so by 1.10 there is a \( \mu \in \mathcal{F}(\mathcal{V}_u^o) \) such that...
\[ \langle h_y, \mu \rangle = \sup \{ |\langle h_y, \nu \rangle| : \nu \in \mathcal{V}_u \}. \] Similarly, for \( \mu \in \mathcal{V}_u \), \( G_\mu = \sum_{k=1}^n f_k, \mu \rangle g_k \) is in \( \mathcal{O}\mathcal{V}_o(Y) \), which implies (again by 1.10) there is a \( \nu \in \mathcal{G}(\mathcal{V}_v) \) so that \( \langle G_\mu, \nu \rangle = \sup \{ |\langle G_\mu, \sigma \rangle| : \sigma \in \mathcal{V}_v \} \). By 3.36, \( \mu \in \mathcal{G}(\mathcal{V}_u) \), \( \nu \in \mathcal{G}(\mathcal{V}_v) \) if and only if there exists \( x \in N(u) \), \( y \in N(v) \) and \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| = |\beta| = 1 \) such that \( \mu = \alpha u(x) \delta(x) \), \( \nu = \beta v(y) \delta(y) \). Therefore for \( w = u \times v \) we have \( |T(F)w| = \sup \{ |\sum_{k=1}^n f_k(x) g_k(y) u(x) v(y)| : x \in N(u), y \in N(v) \} \)
4.8. Theorem. Let $X$ and $Y$ be locally compact, let $U$ be a Nachbin family on $X$ with $C_c^+(X) \subseteq U \subseteq C_+^+(X)$, let $V$ be a Nachbin family on $Y$ with $C_c^+(Y) \subseteq V \subseteq C_+^+(Y)$, and let $W = U \times V$. Then $C_{U^0}(X) \simeq C_{V^0}(Y)$ is topologically isomorphic to $C_{W^0}(X \times Y)$.

Proof. By 4.6, the canonical embedding map $T$ establishes a topological isomorphism between $(C_{U^0}(X) \times C_{V^0}(Y), J_e)$ and $T(C_{U^0}(X) \times C_{V^0}(Y))$ endowed with the relative $w_w$-topology. Now 4.4 implies $T(C_{U^0}(X) \times C_{V^0}(Y))$ is $w_w$-dense in $C_{W^0}(X \times Y)$, while 4.7 together with the local compactness of $X$, $Y$, and $X \times Y$ yields $C_c^+(X \times Y) \subseteq W$, and hence $C_{W^0}(X \times Y)$ is complete by 3.3. Since a topological isomorphism from a locally convex space $E$ onto a locally convex space $F$ has a (unique) extension to a topological isomorphism from the completion of $E$ onto the completion of $F$ [23, p.107], the result is now immediate.

4.9. Corollary (Grothendieck [15, p.90]). If $X$ and $Y$ are locally compact, then $(C_0(X), ||\cdot||) \cong (C_0(Y), ||\cdot||)$ is topologically isomorphic to $(C_0(X \times Y), ||\cdot||)$.

Proof. If $U = K^+(X)$, $V = K^+(Y)$, then the hypothesis of the preceding theorem is satisfied. Since $C_{U^0}(X) = (C_0(X), ||\cdot||)$ and $C_{V^0}(Y) = (C_0(Y), ||\cdot||)$, it will suffice, in view of 2.7, to show $W \cong K^+(X \times Y)$ where $W = U \times V$ (since in this case $C_{W^0}(X \times Y) = (C_0(X \times Y), ||\cdot||)$), and this is clear from the definition of $W$. 

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4.10. **Corollary.** If $X$ and $Y$ are locally compact, then $(C(X), \text{c-op}) \cong (C(Y), \text{c-op})$ is topologically isomorphic to $(C(X \times Y), \text{c-op})$.

Proof. If $U = C^+_c(X)$, $V = C^+_c(Y)$, then the hypothesis of 4.8 is satisfied. If $W = U \times V$, then $W \subseteq C^+_c(X \times Y)$. It follows from 4.7 and the local compactness of $X$ and $Y$ that $C^+_c(X \times Y) \leq W$, and $W \cong C^+_c(X \times Y)$. The result now follows from 2.14.

4.11. **Lemma.** Let $X$ and $Y$ be locally compact, let $U = C^+_c(X)$, and let $V = C^+_c(Y)$. If $W = U \times V$, then $W \cong C^+_c(X \times Y)$.

Proof. If $u \in U$, $v \in V$, then $w \in C^+_c(X \times Y)$ where $w = u \times v$. Since $w \in C^+_c(X \times Y)$ whenever either $u = 0$ or $v = 0$, we may assume $u, v \neq 0$. Now, for $\varepsilon > 0$, $A = \{ (x, y) : w(x, y) \geq \varepsilon \}$ is closed, and $A \subseteq \{ x \in X : u(x) \geq \varepsilon \|v\|^{-1} \} \times \{ y \in Y : v(y) \geq \varepsilon \|u\|^{-1} \}$ from whence it follows that $A$ is compact (i.e., $w \in C^+_c(X \times Y)$). Therefore $W \subseteq C^+_c(X \times Y)$.

Now let $\varphi \in C^+_c(X \times Y)$, and assume $\|\varphi\| \leq 1$. For each $n \in \mathbb{N}$, let $K_n = \{(x,y) : 4^{-(n-1)} \geq \varphi(x,y) \geq 4^{-n}\}$, let $F_n = \Pi_X K_n$, and let $G_n = \Pi_Y K_n$. Since $K_n$ is compact for each $n \in \mathbb{N}$, both $F_n$ and $G_n$ are also compact for each $n \in \mathbb{N}$, and hence for every $n \in \mathbb{N}$ there exist $u_n \in C^+_c(X)$, $v_n \in C^+_c(Y)$ so that $0 \leq u_n, v_n \leq 1$ and $u_n(F_n) = v_n(G_n) = 1$ (we assume, of course, that $u_n = v_n = 0$ whenever $K_n$ is
void). Define $u = \sum_{n=1}^{\infty} 2^{-(n-1)} u_n$, and observe that $u \in C^+_o(X)$ since it is the uniform limit of functions in $C^+_c(X)$. Similarly, $v = \sum_{n=1}^{\infty} 2^{-(n-1)} v_n$ is in $C^+_o(Y)$. If $(x,y) \in N(\varphi)$, then there is an $n \in \mathbb{N}$ such that $(x,y) \in K_n$. Let $n_0 = \min\{n \in \mathbb{N} : (x,y) \in K_n\}$, and note that $x \in F_{n_0}$, $y \in G_{n_0}$. Thus $u(x) \geq 2^{-(n_0-1)}$ while $v(x) \geq 2^{-(n_0-1)}$, which implies $u \times v(x,y) \geq 2^{-(n_0-1)} \geq \varphi(x,y)$. So $C^+_o(X \times Y) \subseteq W$ and the proof is complete.

4.12. Theorem. If $X$ and $Y$ are locally compact, then $(C_b(X), \beta) \cong (C_b(Y), \beta)$ is topologically isomorphic to $(C_b(X \times Y), \beta)$.

Proof. The hypothesis of 4.8 is satisfied for $U = C^+_b(X)$ and $V = C^+_b(Y)$. Since 4.11 shows that $W \cong C^+_o(X \times Y)$ where $W = U \times V$, the result now follows easily in view of 2.17.

In addition to the preceding applications of 4.8, we are able to apply it below to other combinations of $U$ and $V$ to obtain certain interesting new subspaces of $C(X \times Y)$. Moreover, in view of our characterization of $C^+_O(X)$ (Theorem 3.26), 4.8 yields a simple characterization of $[C^+_O(X) \otimes C^+_O(Y)]^*$ whenever it applies (for example, compare Grothendieck [15, p.124]).

We will say a $C$-valued function $f$ on $X \times Y$ is compact column bounded if $R(f; K \times Y) \in C_b(K \times Y)$ for
every compact subset $K$ in $X$. Note that a compact column bounded function $f$ is in $C(X \times Y)$ whenever $X$ is locally compact. We will denote by $CC_b(X \times Y)$ the set of all $f \in C(X \times Y)$ such that $f$ is compact column bounded. Clearly, $CC_b(X \times Y)$ is a subalgebra of $C(X \times Y)$. Now $C_b(X \times Y) \subseteq CC_b(X \times Y) \subseteq C(X \times Y)$ and the following example shows that both containments may be proper.

4.13. **Example.**

(1) If $f: \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is defined by $f(m,n) = m + n$, then $f \in C(\mathbb{N} \times \mathbb{N})$, but $f \notin CC_b(\mathbb{N} \times \mathbb{N})$.

(2) If $f: \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ is defined by $f(m,n) = m$, then $f \in CC_b(\mathbb{N} \times \mathbb{N})$, but $f \notin C(\mathbb{N} \times \mathbb{N})$.

4.14. **Lemma.** If $X$ is locally compact, then a $\Phi$-valued function $f$ on $X \times Y$ is in $CC_b(X \times Y)$ if and only if $f(\Phi \times 1) \in C_b(X \times Y)$, for every $\Phi \in C^+_c(X)$.

**Proof.** Assume $f \in CC_b(X \times Y)$, and let $\Phi \in C^+_c(X)$. Now there is an $m \in \mathbb{N}$ such that $|f(x,y)| \leq m$ for every $(x,y) \in \text{spt}(\Phi) \times Y$, and hence $|f(x,y)(\Phi \times 1)(x,y)| = |f(x,y)|\Phi(x) \leq m||\Phi||$ for every $(x,y) \in X \times Y$; i.e., $f(\Phi \times 1) \in C_b(X \times Y)$.

Now assume $f(\Phi \times 1) \in C_b(X \times Y)$ for every $\Phi \in C^+_c(X)$, and let $K$ be a compact subset of $X$. Choose $\Phi \in C^+_c(X)$ so that $\Phi(K) = 1$, in which case $f(x,y) = f(x,y)\Phi(x) = f(\Phi \times 1)(x,y)$ whenever $(x,y) \in K \times Y$. Therefore $R(f; K \times Y) \in C_b(K \times Y)$ from which it follows that $f \in CC_b(X \times Y)$.
4.15. **Theorem.** If \( X \) and \( Y \) are locally compact, if \( U = C_c^+(X) \), \( V = C_c^+(Y) \), and if \( W = U \times V \), then \( C_{W}^{b}(X \times Y) = C_{C}^{b}(X \times Y) \).

For the proof of 4.15, we make use of the following lemma.

4.16. **Lemma.** Let \( X \) be compact and let \( \phi \in C(X \times Y) \). If \( \theta(y) = \sup\{|\phi(x,y)| : x \in X\} \), then \( \theta \in C^+(Y) \). Moreover, if \( \phi \in C_b(X \times Y) \), then \( \theta \in C_b^+(Y) \); if \( \phi \in C_0(X \times Y) \), then \( \theta \in C_0^+(Y) \); and if \( \phi \in C_c(X \times Y) \), then \( \theta \in C_c^+(Y) \).

**Proof.** Clearly, \( \theta \) is well-defined and \( \theta \geq 0 \). Fix \( y_o \in Y \) and let \( \epsilon > 0 \). For each \( x \in X \) there is an open neighborhood \( A(x) \) of \( x \) and an open neighborhood \( B(x) \) of \( y_o \) such that \((t,y) \in A(x) \times B(x) \) implies

\[ ||\phi(t,y)| - |\phi(x,y_o)|| < \epsilon/8. \]

Since \( X \times \{y_o\} \) is compact, there exists \( \{x_k\}_{k=1}^n \subseteq X \) such that \( \{A(x_k) \times B(x_k)\}_{k=1}^n \) covers \( X \times \{y_o\} \). Let \( B = \bigcap_{k=1}^n B(x_k) \) and let \( y \in B \). Now \( B \) is an open neighborhood of \( y_o \), and if \( (x,y_o) \in A(x_k) \times B(x_k) \) for some \( k \in \{1, \ldots, n\} \), then \( (x,y) \in A(x_k) \times B(x_k) \). Also observe that, since \( X \) is compact, for every \( p \in Y \) there is an \( x_p \in X \) so that \( \theta(p) = |\phi(x_p,p)| \). If \( |\theta(y_o) - |\phi(x,y_o)|| > \epsilon/2 \), then, since there exist \( k_1 \) and \( k_2 \) such that

\[ \theta(y) = |\phi(x_{k_1},y)| = \theta(y) - |\phi(x_{k_2},y_o)| + |\phi(x_{k_2},y_o)| - \theta(y_o) + |\theta(y_o) - |\phi(x_{k_2},y_o)|| < \epsilon/4 + (-\epsilon/2) + \epsilon/4 = 0, \]

which is
impossible. Hence $|\theta(y_0) - |\varphi(x_y, y_0)|| \leq \varepsilon/2$ and

$|\theta(y_0) - \theta(y)| \leq |\theta(y_0) - |\varphi(x_y, y_0)|| + |\varphi(x_y, y_0)| - \theta(y) < \varepsilon/2 + \varepsilon/4 < \varepsilon$, from which it follows that $\theta \in C^+(Y)$.

Now if $\varphi \in C_b(X \times Y)$, then $\theta(y) = |\varphi(x_y, y)| \leq ||\varphi||$ for every $y \in Y$; i.e., $\theta \in C^+_b(Y)$. Moreover, if $\varphi \in C_c(X \times Y)$ and if $y \in N(\theta)$, then $0 < \theta(y) = |\varphi(x_y, y)|$.

This implies $(x_y, y) \in N(\varphi)$ and therefore $y \in \Pi_Y(N(\varphi))$, which is compact. Hence $spt(\theta)$ is compact and $\theta \in C^+_c(Y)$.

Now assume $\varphi \in C_0(X \times Y)$, let $\varepsilon > 0$, and let $F = \{y \in Y : \theta(y) \geq \varepsilon\}$. If $y \in F$, then $\varepsilon \leq \theta(y) = |\varphi(x_y, y)|$, which implies $y \in \Pi_Y\{(x, y) : |\varphi(x, y)| \geq \varepsilon\}$. Since $F$ is closed and contained in a compact set, $F$ is compact, and from this we have $\theta \in C^+_0(Y)$.

Proof of 4.15. Assume first that $f \in C_{C_b}(X \times Y)$. Let $u \in U$, $v \in V$, let $w = u \times v$, and choose $\varphi \in C^+_c(X)$ so that $\varphi(spt(u)) = 1$. From the proof of 4.11, $\varphi \times v \in C^+_c(X \times Y)$, while $f(u \times 1) \in C_b(X \times Y)$ by 4.14.

Since $C_0(X \times Y)$ is an ideal in $C_b(X \times Y)$ and since $fw = f(u \times v) = f(u \times 1)(1 \times v) = f(u \times 1)(\varphi \times v)$, we have $fw \in C_0(X \times Y)$; i.e., $f \in CW_0(X \times Y)$.

Now assume $f \in CW_0(X \times Y)$, fix $\varphi \in C^+_c(X)$, and let $\psi \in C_0(X \times Y)$. Since $R(\psi; spt(\varphi) \times Y) \in C_0(spt(\varphi) \times Y)$, we have from 4.16 that $\theta \in C^+_0(X)$, where

$\theta(y) = \sup\{|R(\psi; spt(\varphi) \times Y)(x, y)| : x \in spt(\varphi)\}$. Therefore $\varphi \times \theta \in W$, which implies $f(\varphi \times \theta) \in C_0(X \times Y)$. Choose $\varepsilon > 0$ and let $A = \{(x, y) : |f(x, y)\psi(x, y)|(\varphi \times 1)(x, y) \geq \varepsilon\}$. 

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Now $\theta \geq |\psi(x, \cdot)|$ for each $x \in \text{spt}(\phi)$, and so

$|f(x, y)|((\phi \times \theta)(x, y) \geq |f(x, y)\psi(x, y)|((\phi \times 1)(x, y)$ for all $(x, y) \in X \times Y$. Thus $A \subseteq \{x, y): |f(x, y)|((\phi \times \theta)(x, y) \geq \epsilon\}$,

and since $A$ is closed, we have that $A$ is compact. But this says $f((\phi \times 1)\psi \in C_o(X \times Y)$, and so $f((\phi \times 1) \in C_b(X \times Y)$ by 2.16. It now follows from 4.14 that $f \in C_{b}(X \times Y)$.

4.17. Theorem. If $X$ and $Y$ are locally compact, then

$(C(X), c\text{-op}) \cong (C(Y), \beta)$ is topologically isomorphic to

$CW_o(X \times Y)$ where $W = C^+_c(X) \times C^+_o(Y)$ and $CW_o(X \times Y) = CC_b(X \times Y)$.

Proof. This is an immediate consequence of 4.8 and 4.15.

We will say a $\mathbb{C}$-valued function $f$ on $X \times Y$ is compact column vanishing at infinity if $R(f; K \times Y) \in C_o(K \times Y)$, for every compact subset $K$ of $X$.

The set of all $f \in C(X \times Y)$ such that $f$ is compact column vanishing at infinity will be denoted by $CC_o(X \times Y)$.

$CC_o(X \times Y)$ is a subalgebra of $C(X \times Y)$ and $C_o(X \times Y) \subseteq CC_o(X \times Y) \subseteq CC_b(X \times Y)$, where both inclusions may be proper. That the second inclusion may be proper follows from (2) of 4.13, while the following easily verifiable example gives the result for the first inclusion.

4.18. Example. Define $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C}$ by $f(m, n) = \frac{m}{n}$

Then $f \in CC_o(N \times N)$, but $f \notin C_b(N \times N)$. 

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4.19. **Lemma.** If $X$ is locally compact, then a $\mathbb{C}$-valued function $f$ on $X \times Y$ is in $C^*_c(X \times Y)$ if and only if $f(\phi \times 1) \in C^*_c(X \times Y)$, for every $\phi \in C^*_c(X)$.

**Proof.** Assume $f \in C^*_c(X \times Y)$, let $\phi \in C^*_c(X)$, and let $\epsilon > 0$. Since we may assume $\phi \neq 0$, we have that $A = \{(x,y) : f(x,y)(\phi \times 1)(x,y) \geq \epsilon\}$ is a closed set and $A \subseteq \{(x,y) \in \text{spt}(\phi) \times Y : |f(x,y)| \geq \epsilon \|\phi\|^{-1}\}$. So $A$ is compact, and this implies $f(\phi \times 1) \in C^*_c(X \times Y)$, for every $\phi \in C^*_c(X)$.

Now assume $f(\phi \times 1) \in C^*_c(X \times Y)$ for every $\phi \in C^*_c(X)$. If $K$ is a compact subset of $X$, then there is a $\phi \in C^*_c(X)$ such that $\phi(K) = 1$. In this case, $f(x,y) = f(x,y)\phi(x) = f(\phi \times 1)(x,y)$ for each $(x,y) \in K \times Y$, and since $R(f(\phi \times 1); K \times Y) \in C^*_c(K \times Y)$, we have $R(f; K \times Y) \in C^*_c(K \times Y)$; i.e., $f \in C^*_c(X \times Y)$.

4.20. **Theorem.** If $X$ and $Y$ are locally compact, if $U = C^*_c(X)$, $V = K^+(Y)$, and if $W = U \times V$, then $C^*_c(W) = C^*_c(X \times Y)$.

**Proof.** First assume $f \in C^*_c(X \times Y)$, let $u \in U$, $v \in V$, and let $w = u \times v$. Now there is a $c \in R^+$ such that $v(y) = c$ for every $y \in Y$, and so $fw = (cf)(u \times 1)$. Since $cf \in C^*_c(X \times Y)$, $fw \in C^*_c(X \times Y)$ by 4.19, which implies $f \in C^*_c(W, X \times Y)$.

If $f \in C^*_c(W, X \times Y)$ and if $\phi \in C^*_c(X)$, then $f(\phi \times 1) \in C^*_c(X \times Y)$, since $\phi \times 1 \in W$. That $f \in C^*_c(X \times Y)$ now follows from 4.19.
4.21. Theorem. If $X$ and $Y$ are locally compact, then $(C(X), \text{c-op}) \overset{\cong}{\to} (C(Y), \|\cdot\|)$ is topologically isomorphic to $CW^+_0(X \times Y)$ where $W = C^+_c(X) \times K^+(Y)$ and $CW^+_0(X \times Y) = CC^+_0(X \times Y)$.

Proof. This is an immediate consequence of 4.8 and 4.20.

4.22. Theorem. If $X$ and $Y$ are locally compact, if $U = C^+_0(X)$, $V = K^+(Y)$, and if $W = U \times V$, then $CW^+_0(X \times Y) = CC^+_0(X \times Y) \cap C_b(X \times Y)$.

Proof. Assume $f \in CC^+_0(X \times Y) \cap C_b(X \times Y)$, let $u \in U$, $v \in V$, and let $w = u \times v$. If either $f = o$, $u = o$, or $v = o$, then $fw = 0$ and we are done. So we may assume $f \neq o$, $u \neq o$, and $v \neq o$, and in this case there is a $c > 0$ such that $v(y) = c$ for each $y \in Y$. Choose $\epsilon > 0$ and let $A = \{(x,y): |f(x,y)|w(x,y) \geq \epsilon\}$. Now $A$ is closed and we wish to show $A$ is compact. Since $K = \{x \in X: u(x) \geq \epsilon(c||f||)^{-1}\}$ is compact, $B = \{(x,y) \in K \times Y: |f(x,y)| \geq \epsilon(c||u||)^{-1}\}$ is compact. If $(x,y) \in A$, then $|f(x,y)| \geq \epsilon(c||u||)^{-1} \geq \epsilon(c||u||)^{-1}$ and $u(x) \geq \epsilon(c||f(x,y)||)^{-1} \geq \epsilon(c||f||)^{-1}$, which implies $(x,y) \in B$. Therefore $A$ is compact, and it follows that $f \in CW^+_0(X \times Y)$.

Now assume $f \in CW^+_0(X \times Y)$. Consequently, for each $u \in U$ (and hence for each $u \in C^+_c(X)$), $f(u \times 1) \in C^+_0(X \times Y)$, and so $f \in CC^+_0(X \times Y)$ by 4.19. By 4.11, $C^+_0(X) \times C^+_0(Y) \approx C^+_0(X \times Y)$, while it is clear that $C^+_0(X) \times C^+_0(Y) \leq W$. 

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Thus \( C^+_o(X \times Y) \leq W \), and 2.6 and 2.17 now yield that
\( CW_o(X \times Y) \subseteq C_b(X \times Y) \), which completes the proof.

4.23. **Theorem.** If \( X \) and \( Y \) are locally compact, then \( (C_b(X), \beta) \overset{\approx}{\sim} (C_c(Y), || \cdot ||) \) is topologically isomorphic to \( CW_o(X \times Y) \) where \( W = C^+_o(X) \times K^+(Y) \) and
\( CW_o(X \times Y) = CC_o(X \times Y) \cap C_b(X \times Y) \).

**Proof.** This is an immediate consequence of 4.8 and 4.22.

4.24. **Example.** If we define \( f:N \times N \to \emptyset \) by
\( f(m,n) = \frac{1}{n} \), then \( f \in CC_o(N \times N) \cap C_b(N \times N) \), but
\( f \notin C_o(N \times N) \).

Clearly, \( CC_o(X \times Y) \cap C_b(X \times Y) \) may be properly contained in \( C_b(X \times Y) \), while the above example shows
\( C_o(X \times Y) \) may be properly contained in \( CC_o(X \times Y) \cap C_b(X \times Y) \).

We summarize the relationships between known spaces and the subspaces of \( C(X \times Y) \) introduced in this chapter in the following diagram, where each map is inclusion which is in general proper. It should be remarked at this point, that our "column" spaces give rise, of course, to analogous "row" spaces.

\[
\begin{array}{ccc}
C_b(X \times Y) & \rightarrow & CC_b(X \times Y) & \rightarrow & C(X \times Y) \\
\uparrow & & & & \uparrow \\
CC_o(X \times Y) \cap C_b(X \times Y) & \rightarrow & CC_o(X \times Y) \\
\uparrow & & \\
C_o(X \times Y)
\end{array}
\]
4.25. Theorem. If $X$ and $Y$ are locally compact and if $W_1 = C^+_c(X) \times C^+_c(Y)$, $W_2 = C^+_c(X) \times K^+(Y)$, and $W_3 = C^+_o(X) \times K^+(Y)$, then

1. $C^+_c(X \times Y) \leq W_1 \leq W_2 \leq W_3 \leq K^+(X \times Y)$;

2. $W_1 \leq C^+_o(X \times Y) \leq W_3$.

Proof. Clearly, $C^+_c(X) \times C^+_c(Y) \leq W_1$, while it follows easily from 4.7 that $C^+_c(X \times Y) \leq C^+_c(X) \times C^+_c(Y)$; i.e., $C^+_c(X \times Y) \leq W_1$. If $u \in C^+_c(X)$, $v \in C^+_o(Y)$, then $u \times v \leq u \times k$, where $k(y) = ||v||$ for each $y \in Y$; i.e., $W_1 \leq W_2$. It is obvious that $W_2 \leq W_3$, while an argument similar to that for $W_1 \leq W_2$ shows $W_3 \leq K^+(X \times Y)$.

Since $W_1 \leq C^+_c(X) \times C^+_c(Y)$ and since $C^+_o(X \times Y) \approx C^+_c(X) \times C^+_c(Y)$ by 4.11, we have $W_1 \leq C^+_o(X \times Y)$. From the proof of 4.22, we see that $C^+_o(X \times Y) \leq W_3$, and so (2) is valid.

In view of 2.6, Theorem 4.25 yields an idea of how the weighted topologies encountered in 4.9, 4.10, 4.12, 4.17, 4.21, and 4.23 compare. We summarize this in the following diagram, where each map is considered to be an inclusion map for the relativized topology.

```
\[ \begin{array}{c}
\text{uniform} \\
\downarrow \\
\text{uniform} \\
\downarrow \\
\text{c-op} \\
\downarrow \\
W_3 \\
\downarrow \\
W_1 \\
\downarrow \\
\text{c-op} \\
\end{array} \]
```
We conclude this chapter with a chart which summarizes our applications of Theorem 4.8. The spaces $X$ and $Y$ below are assumed to be locally compact.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$V$</th>
<th>$CU^0(X)$</th>
<th>$CV^0(Y)$</th>
<th>$W = U \times V$</th>
<th>$CW^0(X \times Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^+_c(X)$</td>
<td>$C^+_c(Y)$</td>
<td>$(C(X), c\text{-}op)$</td>
<td>$(C(Y), c\text{-}op)$</td>
<td>$(C(X \times Y), c\text{-}op)$</td>
<td></td>
</tr>
<tr>
<td>$C^+_c(X)$</td>
<td>$C^+_o(Y)$</td>
<td>$(C(X), c\text{-}op)$</td>
<td>$(C_b(Y), \beta)$</td>
<td>$W_1$</td>
<td>$(C_b(X \times Y), \omega)$</td>
</tr>
<tr>
<td>$C^+_c(X)$</td>
<td>$K^+(Y)$</td>
<td>$(C(X), c\text{-}op)$</td>
<td>$(C_0(Y), | \cdot |)$</td>
<td>$W_2$</td>
<td>$(C_0(X \times Y), \omega)$</td>
</tr>
<tr>
<td>$C^+_o(X)$</td>
<td>$C^+_o(Y)$</td>
<td>$(C_b(X), \beta)$</td>
<td>$(C_b(Y), \beta)$</td>
<td></td>
<td>$(C_b(X \times Y), \beta)$</td>
</tr>
<tr>
<td>$C^+_o(X)$</td>
<td>$K^+(Y)$</td>
<td>$(C_b(X), \beta)$</td>
<td>$(C_0(Y), | \cdot |)$</td>
<td>$W_3$</td>
<td>$(C_0(X \times Y) \cap C_b(X \times Y), \omega)$</td>
</tr>
<tr>
<td>$K^+(X)$</td>
<td>$K^+(Y)$</td>
<td>$(C_0(X), | \cdot |)$</td>
<td>$(C_0(Y), | \cdot |)$</td>
<td></td>
<td>$(C_0(X \times Y), | \cdot |)$</td>
</tr>
</tbody>
</table>
CHAPTER V

Full-Completeness in Weighted Spaces

The purpose of this chapter is to exploit the previously developed theory and techniques in an investigation of full-completeness in weighted spaces. In particular, we establish a necessary condition on $X$ in order that $CV_0(X)$ be fully complete whenever $V$ satisfies certain minor restrictions (our result includes the case of $(C_b(X), \beta)$), and we use our weighted representation theorem (4.8) established in the preceding chapter to obtain a necessary condition for certain biequicontinuous completed tensor products of weighted spaces to be fully complete. Moreover, several conditions which are necessary and sufficient for $(C_b(X), \beta)$ to be $B_r$-complete are given. We begin by giving a brief introduction to the theory of fully complete spaces.

Let $E$ be a locally convex space. A subset $A$ of $E^*$ is said to be nearly closed if $A \cap U^o$ is $\sigma(E^*,E)$-closed for each neighborhood $U$ in $E$. $E$ is said to be fully complete if every nearly closed linear subspace of $E^*$ is $\sigma(E^*,E)$-closed. We will say $E$ has the Krein-Smulian property if every nearly closed convex subset of $E^*$ is $\sigma(E^*,E)$-closed, and that $E$ is $B_r$-complete if every $\sigma(E^*,E)$-dense nearly closed linear subspace of $E^*$ is
σ(E^*,E)-closed. Observe that the Krein-Smulian property implies full-completeness, that full-completeness implies B_r-completeness, and this, in turn, implies completeness. Now assume F is also a locally convex space, and that t:E → F is linear and onto. Then t is said to be nearly open if t(U) is a neighborhood in F for every neighborhood U in E. The following theorem yields one of the more interesting properties of fully complete spaces.

5.1. Theorem (Ptáč [22]). Let E and F be locally convex spaces, let E_o be a dense linear subspace of E, and let t:E_o → F be linear and onto with the graph of t closed in E x F. If E is fully complete and if t is nearly open, then t is open. If, in addition, t is one-to-one, then it suffices for the conclusion that E be B_r-complete.

For our purposes, we need a variation of the preceding theorem. Assume that E and F are locally convex spaces and that t:E → F is linear (not necessarily onto). We will say that t is nearly open into if t(U) is a neighborhood in F for every neighborhood U in E.

5.2. Lemma. Let E and F be locally convex spaces. If t:E → F is linear and nearly open into, then t(E) is dense in F.

Proof. If y ∈ F, then, since t(E) is a neighborhood in F, there exists α ∈ C so that y ∈ α t(E). But αt(E) = α t(E) = t(αE) ⊆ t(E), and this implies t(E) = F.
5.3. **Theorem.** Let $E$ and $F$ be locally convex spaces and let $t: E \to F$ be linear with the graph $G_t$ of $t$ closed in $E \times F$. If $E$ is fully complete and if $t$ is nearly open into, then $t$ is open onto $F$. If, in addition, $t$ is one-to-one, then it suffices for the conclusion that $E$ be $B_r$-complete.

**Proof.** Since $G_t$ is closed in $E \times F$, we have that $t^{-1}(0)$ is closed in $E$ [23, p.115]. Because the quotient of a fully complete space by a closed linear subspace is again fully complete [5], $H = E/t^{-1}(0)$ is fully complete. If $\tilde{t}: H \to F$ is the map induced by $t$, then $\tilde{t}$ is one-to-one and nearly open into, while $G_{\tilde{t}}$ is closed in $H \times F$.

By 5.1, $\tilde{t}$ is open onto $\tilde{t}(H)$, where $\tilde{t}(H)$ is endowed with the relative topology, while $\tilde{t}(H)$ is dense in $F$ by 5.2. If $y \in F$, then there is a net $\{y_i\}$ in $\tilde{t}(H)$ such that $y_i \to y$. Thus $\{y_i\}$ is a Cauchy net in $\tilde{t}(H)$, and this implies $(\tilde{t}(y_i))$ is a Cauchy net in $H$. Consequently, there is an $x \in H$ such that $\tilde{t}(y_i) \to x$. Since $G_{\tilde{t}}$ is closed and since $y_i = \tilde{t}(\tilde{t}^{-1}(y_i)) \to y$, we have that $\tilde{t}(x) = y$. Therefore $\tilde{t}$ is onto $F$, and it now follows that $t$ is open onto $F$.

We will also make use of the following generalization of a result due to Collins [5].

5.4. **Theorem.** Let $E$ and $F$ be locally convex spaces and let $t: E \to F$ be linear and nearly open into. If $E$ is
fully complete and if $t$ is $\sigma(E,E^*) - \sigma(F,F^*)$ continuous, then $F$ is fully complete.

Proof. Let $L$ be a nearly closed linear subspace of $F^*$. If $U$ is a neighborhood in $E$, then $t^*(L) \cap \overline{U^0} = t^*(L \cap t^{-1}(U^0))$, while $t^{-1}(U^0) = \overline{t(U)^0}$. Since $t$ is nearly open into $F$, $L \cap \overline{t(U)^0}$ is $\sigma(F^*,F)$-closed, and hence $\sigma(F^*,F)$-compact (using Alaoglu's theorem). But $t^*$ is weak-$*$continuous, and so it follows that $t^*(L) \cap \overline{U^0}$ is $\sigma(E^*,E)$-compact; i.e., $t^*(L)$ is a nearly closed linear subspace of $E^*$. Consequently, $t^*(L)$ is $\sigma(E^*,E)$-closed, and this implies $t^{-1}(t^*(L))$ is $\sigma(F^*,F)$-closed. By 5.2, $t(E)$ is dense in $F$, which implies $t^*$ is one-to-one; i.e., $L = T^{-1}(t^*(L))$.

5.5. Lemma. Let $F$ be a closed subset of $X$ and let $U$ be a Nachbin family on $X$. If $V = \{R(u;F) : u \in U\}$, then $V$ is a Nachbin family on $F$. Moreover, if $U \leq U^2$, then $V \leq V^2$.

Proof. If $v \in V$, then there is a $u \in U$ such that $R(u;F) = v$. Therefore $v \geq 0$, and, for $\epsilon > 0$,

$$\{x \in F : v(x) \geq \epsilon\} = \{x \in X : u(x) \geq \epsilon\} \cap F$$

is closed, which implies $v$ is u.s.c. If $u, u_1 \in U$ and $\lambda > 0$, then there exists $w \in U$ so that $\lambda u, \lambda u_1 \leq w$, and this implies $\lambda R(u;F), \lambda R(u_1;F) \leq R(w;F)$. Thus $V$ is a Nachbin family on $F$, and it is clear that $V \leq V^2$ whenever $U \leq U^2$.

We will assume in all of our applications of 5.5 that $U$ is such that $CV_o(F)$ is Hausdorff. This would be the
case, for example, if there is a $u \in U$ such that $u(x) > 0$ whenever $x \in F$. In the sequel, we will denote the mapping $f \mapsto R(f;F)$ by $R(\cdot;F)$.

5.6. Lemma. Let $F$ be a closed subset of $X$ and let $U$ be a Nachbin family on $X$. If $V = \{R(u;F) : u \in U\}$, then $R(\cdot;F)$ is a continuous linear mapping of $C^*_0(X)$ into $CV_0(F)$.

Proof. It is clear that $R(\cdot;F)$ is a well-defined linear map of $C^*_0(X)$ into $C(F)$. Now let $f \in C^*_0(X)$, let $u \in U$, and let $\varepsilon > 0$. If $v = R(u;F)$, then $\{x \in F : |R(f;F)(x)|v(x) \geq \varepsilon\} = \{x \in X : |f(x)|u(x) \geq \varepsilon\} \cap F$ is compact; i.e., $R(f;F) \in CV_0(F)$.

To show $R(\cdot;F)$ is continuous, let $(f_i)$ be a net in $C^*_0(X)$ with $f_i \to 0$ (wot) and let $u \in U$. If $v = R(u;F)$, and if $x \in F$, then $|R(f_i;F)(x)|v(x) = |f_i(x)|u(x) \leq ||f_i||u|$. This implies $||R(f_i;F)v||_F \leq ||f_i||u| \to 0$, and so $R(\cdot;F)$ is continuous.

Before continuing our development, we will need the following general results.

5.7. Lemma (Stone's Theorem [13, p.90]). Every compact subset of $X$ is $C^*$-embedded in $X$.

5.8. Lemma. Let $F$ be a subset of $X$ and let $f \in C_b(F)$. If $f$ has an extension $g \in C(X)$, then $f$ has an extension $\tilde{f} \in C_b(X)$ with $||\tilde{f}|| = ||f||_F$. Moreover, we may assume $N(\tilde{f}) = N(g)$ whenever $f \neq 0$.
Proof. If \( f = 0 \), then choose \( \tilde{f} = 0 \). Now assume \( f \neq 0 \), let \( A = \{ x \in X : |g(x)| \leq \|f\|_p \} \), and let \( B = \{ x \in X : |g(x)| \geq \|f\|_p \} \). Then \( A \) and \( B \) are closed, \( X = A \cup B \), and \( x \in B \) implies \( |g(x)| > 0 \). Define
\[
\tilde{f}(x) = \begin{cases} 
g(x), & x \in A \\
\|f\|_p g(x) (|g(x)|)^{-1}, & x \in B,
\end{cases}
\]
and note that \( x \in A \cap B \) implies \( |g(x)| = \|f\|_p \). It now follows that \( R(\tilde{f}; F) = f \) and that \( \tilde{f} \in C_b(X) \) with \( \|\tilde{f}\| = \|f\|_p \).
Moreover, \( \tilde{f}(x) = 0 \) if and only if \( g(x) = 0 \); i.e., \( N(\tilde{f}) = N(g) \).

5.9. Lemma. The following are equivalent (here it suffices to assume only that \( X \) is Hausdorff):

(1) \( X \) is locally compact;

(2) for each closed set \( F \) in \( X \) and for every \( \varphi \in C_c^+(F) \) there is a \( \tilde{\varphi} \in C_c^+(X) \) such that \( R(\varphi; \text{spt} (\varphi)) = R(\tilde{\varphi}; \text{spt} (\varphi)) \);

(3) for each closed set \( F \) in \( X \) and for every \( \varphi \in C_o^+(F) \) there is a \( \tilde{\varphi} \in C_c^+(X) \) such that \( \varphi \leq R(\tilde{\varphi}; F) \).

Proof. Assume (1) holds, let \( \overline{F} \) be a closed subset of \( X \), and let \( \varphi \in C_c^+(F) \). By 5.7, there exists \( \tilde{\varphi} \in C_b(X) \) so that \( R(\tilde{\varphi}; \text{spt} (\varphi)) = R(\varphi; \text{spt} (\varphi)) \). Since \( \varphi \geq 0 \), we may assume \( \tilde{\varphi} \geq 0 \); since \( X \) is locally compact, it easily follows from Urysohn's lemma that we can assume \( \tilde{\varphi} \in C_c^+(X) \); i.e., (2) holds.
Assume (2) holds, let \( F \) be a closed subset of \( X \), and let \( \phi \in C^+_c(F) \). We may assume \( ||\phi||_F = 1 \). Define \( K_n = \{ x \in F : 2^{-n} \leq \phi(x) \leq 2^{-(n-1)} \} \) for each \( n \in \mathbb{N} \) and choose \( \phi_n = 0 \) for each \( n \in \mathbb{N} \) for which \( K_n \) is void. For those \( n \in \mathbb{N} \) for which \( K_n \) is not void, \( R(x_K_n;K_n) \in C^+_c(K_n) \), and hence there is a \( \phi_n \in C^+_c(X) \) such that \( R(x_K_n;K_n) = R(\phi_n;K_n) \). By 5.8, we may assume \( ||\phi_n|| \leq 1 \) for every \( n \in \mathbb{N} \), and therefore \( \bar{\phi} = \sum_{n=1}^{\infty} 2^{-(n-1)} \phi_n \) is in \( C^+_c(X) \). If \( x \in \mathbb{N}(\phi) \), then there is an \( n_0 \in \mathbb{N} \) such that \( x \in K_{n_0} \). Then \( \phi(x) \leq 2^{-(n_0-1)} \), \( \tilde{\phi}(x) \geq 2^{-(n_0-1)} \), and \( \phi \leq R(\tilde{\phi};F) \); i.e., (3) holds.

Now assume (3) holds. If \( x_0 \in X \), then \( F = \{ x_0 \} \) is a closed subset of \( X \) and \( R(x_{F};F) \in C^+_c(F) \). Hence there exists \( \tilde{\phi} \in C^+_c(X) \) so that \( \tilde{\phi}(x_0) \geq 1 \). It follows that \( \{ x \in X : \tilde{\phi}(x) \geq \frac{1}{2} \} \) is a compact neighborhood of \( x_0 \), and so (1) holds.

5.10. Lemma. Let \( X \) be locally compact, let \( U \) be a Nachbin family on \( X \), let \( F \) be a closed subset of \( X \), and let \( V = \{ R(u;F) : u \in U \} \). If \( C^+_c(X) \leq U \leq C^+_c(X) \), then \( C^+_c(F) \leq V \leq C^+_c(F) \).

Proof. If \( \phi \in C^+_c(F) \), then there is a \( \tilde{\phi} \in C^+_c(X) \) such that \( R(\phi;\text{spt}(\phi)) = R(\tilde{\phi};\text{spt}(\phi)) \) by 5.9, and there is a \( u \in U \) such that \( \tilde{\phi} \leq u \). Consequently, \( \phi \leq R(\tilde{\phi};F) \leq R(u;F) \); i.e., \( C^+_c(F) \leq V \).
If \( u \in U \), then there is a \( \phi \in C^+_o(X) \) such that \( u \leq \phi \). If \( \epsilon > 0 \), then \( \{ x \in F : R(\phi; F)(x) \geq \epsilon \} = \{ x \in X : \phi(x) \geq \epsilon \} \cap F \), which is compact. Hence \( R(\phi; F) \in C^+_o(F) \), \( \nu = R(u; F) \leq R(\phi; F) \), and we have \( \nu \leq C^+_o(F) \).

5.11. **Lemma.** Let \( U \) be a Nachbin family on \( X \), let \( F \) be a closed subset of \( X \), and let \( V = \{ R(u; F) : u \in U \} \). If either

1. \( u \in U \) implies \( \text{spt}(u) \) is compact; or
2. \( X \) is locally compact and \( CV_o(F)^* = M_b(F) \);

then \( \{ R(f; F) : f \in CU_o(X) \} \) is dense in \( CV_o(F) \).

**Proof.** By 5.6, \( R(\cdot; F) \) is a well-defined linear mapping of \( CU_o(X) \) into \( CV_o(F) \).

Assume (1) holds, let \( f \in CV_o(F) \), let \( u \in U \), and let \( v = R(u; F) \). In this case, \( \overline{N(v)} \) is compact, and so by 5.7 there is a \( g \in C_b(X) \) such that \( R(g; \overline{N(v)}) = R(f; \overline{N(v)}) \). Clearly, \( g \in CU_o(X) \), and \( |(R(g; F) - f)(x)| y(x) = 0 \) whenever \( x \in F \). The result now follows.

Now assume (2) holds, let \( \mu \in CV_o(F)^* \), and suppose \( \langle R(f; F), \mu \rangle = 0 \) for every \( f \in CU_o(X) \). We would show \( \mu = 0 \), and since \( C_c(F) \) is \( \omega_Y \)-dense in \( CV_o(F) \) by 3.6, then in view of the Hahn-Banach theorem, it will suffice to show \( \langle \phi, \mu \rangle = 0 \) for every \( \phi \in C_c(F) \). To this end, let \( \phi \in C_c(F) \) and let \( \epsilon > 0 \). We may assume, of course, that \( \phi \neq 0 \).

Since \( \mu \in M_b(F) \), there is a compact set \( K \) in \( F \) such that \( |\mu|(F \setminus K) < \epsilon(\|\phi\|_F)^{-1} \). If \( G = K \cup \text{spt}(\phi) \), then

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by 5.7 there is an \( f \in C_b(X) \) such that \( R(f;G) = R(\varphi;G) \), and we may assume \( ||f|| = ||\varphi||_F \) by 5.8. Moreover, the fact that \( X \) is locally compact (using Urysohn's lemma) allows us to assume \( f \in C_c(X) \), and hence that \( f \in CU_o(X) \).

Since
\[
|\langle \varphi, \mu \rangle| = |\int_{F \cap G} \varphi \, d\mu| = |\int_{F \cap G} R(f;F)\, d\mu - \int_{F \setminus G} R(f;F)\, d\mu| = |\int_{F \cap G} R(f;F)\, d\mu| - |\int_{F \setminus G} R(f;F)\, d\mu| = |\int_{F \cap G} R(f;F)\, d\mu| \leq |R(f;F)|_F |\mu|(F \cap G) < ||\varphi||_F \varepsilon(||\varphi||_F)^{-1} = \varepsilon
\]

\( \langle \varphi, \mu \rangle = 0 \) and the proof is complete.

5.12. Lemma. Let \( X \) be locally compact, let \( U \) be a Nachbin family on \( X \), let \( F \) be a closed subset of \( X \), and let \( V = \{R(u;F) : u \in U\} \). If either

1. \( C_c(X) \leq U \leq C_c^+(X) \); or
2. \( U \subseteq C_c^+(X) \); or
3. \( C_c^+(X) \leq U \leq K^+(X) \);

then \( CV_o(F)^* \subseteq M_b(F) \) and hence \( \{R(f;F) : f \in CU_o(X)\} \) is \( \omega_V \)-dense in \( CV_o(F) \).

Proof. If (1) holds, then \( C_o^+(F) \leq V \leq C_o^+(F) \) by 5.10. Theorem 3.21 then implies \( CV_o(F)^* = V \cdot M_b(F) \), and \( V \cdot M_b(F) \subseteq M_b(F) \) by 3.22.

If (2) holds, then \( V \subseteq C_b^+(F) \). By 3.26, \( CV_o(F)^* = V \cdot M_b(F) \), while \( V \cdot M_b(F) \) is clearly contained in \( M_b(F) \).

Finally, if (3) holds, then \( V \leq K^+(F) \). If \( \varphi \in C_o^+(F) \), then by (3) of 5.9 there is a \( \tilde{\varphi} \in C_o^+(X) \) such that...
\( \varphi \leq R(\varphi;F) \). But there is a \( u \in U \) such that \( \varphi \leq u \), and hence \( \varphi \leq R(\varphi;F) \leq R(u;F) \). So \( C^+_0(F) \leq V \leq K^+(F) \), and \( CV_0(F)^* = M_b(F) \) by 3.16.

Consequently, any of the cases (1), (2), or (3) imply (2) of 5.11, and so \( \{R(f;F): f \in C_U(X)\} \) is \( \omega_V \)-dense in \( CV_0(F) \).

5.13. Lemma. Let \( U \) be a Nachbin family on \( X \), let \( F \) be a closed subset of \( X \), and let \( V = \{R(u;F): u \in U\} \). If either

1. \( u \in U \) implies \( \text{spt}(u) \) is compact; or
2. \( X \) is locally compact and \( CV_0(F)^* \subseteq M_b(F) \); then

\( R(\cdot;F):C_U(X) \to CV_0(F) \) is nearly open into.

Proof. By 5.6, \( R(\cdot;F) \) is a well-defined linear mapping of \( C_U(X) \) into \( CV_0(F) \), while \( \{R(f;F): f \in C_U(X)\} \) is \( \omega_V \)-dense in \( CV_0(F) \) by 5.11. Let \( u \in U \) and let \( v = R(u;F) \). Thus \( v \in V \) and \( V_v^* = \{f \in CV_0(F): |fv|_F < 1\} \) is an open neighborhood of zero in \( CV_0(F) \), so that

\( \{R(f;F): f \in C_U(X)\} \cap V_v^* \) is non-void. If \( f \in \{R(g;F): g \in C_U(X)\} \cap V_v^* \), then there is an \( \tilde{f} \in C_U(X) \) such that \( R(\tilde{f};F) = f \), and we will show that we may assume \( \tilde{f} \in V_u^* \). To do this, let \( K = \{x \in X: |\tilde{f}(x)|u(x) \geq 1\} \). If \( K \) is void, then \( \tilde{f} \in V_u^* \), so suppose \( K \) is not void. Now \( K \) is compact, and if \( x \in F \), then \( |\tilde{f}(x)|u(x) = |f(x)|v(x) < 1 \), which implies \( K \) and \( F \) are disjoint. It follows from 4.5 that there is a \( \sigma \in C_b(X) \) with \( 0 \leq \sigma \leq 1 \),

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\[\sigma(K) = 0, \quad \text{and} \quad \sigma(F) = 1. \]  If \( g = \tilde{f}\sigma \), then \( g \in C_{0}^{+}(X) \) and \( R(g;F) = f \). Moreover, if \( x \in K \), then \( |g(x)|u(x) = 0 \), while if \( x \notin K \), then \( |g(x)|u(x) = |\tilde{f}(x)|\sigma(x)u(x) \leq |\tilde{f}(x)|u(x) < 1 \). Consequently, \( g \in V_u \), and we may assume \( \tilde{f} \in V_u \). Thus \( \{R(f;F): f \in C_{0}^{+}(X)\} \cap V'_v \subseteq \{R(f;F): f \in V_u\} \), from which it follows that \( V'_v \subseteq \{R(f;F): f \in C_{0}^{+}(X)\} \cap V'_v \subseteq \{R(f;F): f \in V_u\} \); i.e., \( R(\cdot;F) \) is nearly open into \( C_{0}^{+}(F) \).

5.14. Theorem. Let \( X \) and \( Y \) be locally compact, let \( U \) be a Nachbin family on \( X \) with \( C_{c}^{+}(X) \leq U \subseteq C_{b}^{+}(X) \), and let \( V \) be a Nachbin family on \( Y \) with \( C_{c}^{+}(Y) \leq V \subseteq C_{b}^{+}(Y) \). If \( C_{0}^{+}(X) \cong C_{0}^{+}(Y) \) is fully complete, then both \( C_{0}^{+}(X) \) and \( C_{0}^{+}(Y) \) are fully complete.

Proof. Let \( y_o \in Y \) and let \( F = X \times \{y_o\} \), a closed subset of \( X \times Y \). From 4.1, \( W = U \times V \) is a Nachbin family on \( X \times Y \), while \( T = \{R(w;F): w \in W\} \) is a Nachbin family on \( F \) by 5.5. Lemma 5.6 gives us that \( R(\cdot;F) \) is a well-defined continuous linear mapping of \( C_{W}^{+}(X \times Y) \) into \( C_{T}^{+}(F) \). Since \( W \subseteq C_{b}(X \times Y) \), 5.12 yields that \( C_{T}^{+}(F)^{*} \subseteq M_{b}(F) \), and so \( R(\cdot;F) \) is nearly open into \( C_{T}^{+}(F) \) by 5.13. In view of 4.8, there is a continuous linear mapping of the fully complete space \( C_{0}^{+}(X) \cong C_{0}^{+}(Y) \) into \( C_{0}^{+}(F) \) which is nearly open into, and hence \( C_{0}^{+}(F) \) is fully complete by 5.4.

Now define \( \alpha:C_{0}^{+}(F) \rightarrow C(X) \) by \( \alpha(f)(x) = f(x, y_o) \) for each \( f \in C_{0}^{+}(F) \). It is clear that \( \alpha \) is a well-defined
linear map. Since \( C^+_c(Y) \leq V \), there is a \( v \in V \) such that \( v(y_o) \geq 1 \). Since \( T = U \times \{ R(v; \{ y_o \}) : v \in V \} \), it follows from 4.2 that \( \alpha \) maps \( C^+_o(F) \) into \( C^+_o(X) \). If \( f \in C^+_o(X) \), then define \( g \in C(F) \) by \( g(x, y_o) = f(x) \). We would show \( g \in C^+_o(F) \), and to this end, let \( u \in U \), \( v \in V \), and let \( \epsilon > 0 \). Because \( U \) is a Nachbin family, there exists \( u' \in U \) so that \( v(y_o)u \leq u' \). If \( G = \{(x, y_o) : |g(x, y_o)|u(x)v(y_o) \geq \epsilon \} \), then \( (x, y_o) \in G \) implies \( |f(x)|u'(x) \geq |f(x)|u(x)v(y_o) \geq \epsilon \), and this implies \( G \subseteq \{(x \in X : |f(x)|u'(x) \geq \epsilon \} \times \{ y_o \} \); i.e., \( G \) is compact. Thus \( g \in C^+_o(F) \) and \( \alpha \) is a linear map of \( C^+_o(F) \) onto \( C^+_o(X) \) which is clearly one-to-one. If \( \{ f_i \} \) is a net in \( C^+_o(F) \) such that \( f_i \to o(\omega_T) \), then, for \( u \in U \) and \( v \in V \) with \( v(y_o) \geq 1 \), \( |\alpha(f_i)(x)|u(x) \leq |f_i(x, y_o)|u(x)v(y_o) \leq ||f_i(u \times v)||_F \to o \), and \( \alpha \) is continuous. If \( \{ f_i \} \) is a net in \( C^+_o(X) \) such that \( f_i \to o(\omega_U) \), then, for \( u \in U \), \( v \in V \), and \( u' \in U \) with \( v(y_o)u \leq u' \), \( |\alpha^{-1}(f_i)(x, y_o)|u(x)v(y_o) \leq |f_i(x)u'(x)| \leq ||f_iu'||_{-1} \to o \), and \( \alpha^{-1} \) is continuous. By 5.4, \( C^+_o(X) \) is fully complete, and similarly, \( C^+_o(Y) \) is fully complete.

It is not known whether or not the converse of the preceding theorem is true. However, it is our conjecture that it, as well as the analogous question about cartesian products of fully complete spaces, is false.

5.15. Theorem. Let \( U \) be a Nachbin family on \( X \). If \( C^+_o(X) \) is fully complete and if either
(1) \( u \in U \) implies \( \text{spt}(u) \) is compact; or
(2) \( X \) is locally compact and \( C_c^+(X) \subseteq U \subseteq C_0^+(X) \);
then \( X \) is normal.

Proof. Let \( F \) be a closed subset of \( X \), and recall that it will suffice to show \( F \) is \( C^* \)-embedded in \( X \).
\( V = \{ R(u;F) : u \in U \} \) is a Nachbin family on \( F \) by 5.5, while \( R(\cdot;F) \) is a continuous linear mapping of \( C_0(U(X)) \) into \( C_0^*(F) \) by 5.6. If (2) holds, then by (1) of 5.12 we have \( C_0^*(F)^* \subseteq M_b(F) \), so 5.13 implies \( R(\cdot;F) \) is nearly open into \( C_0^*(F) \). Hence \( R(\cdot;F) \) is open onto \( C_0^*(F) \) by 5.3.

Now let \( f \in C_b(F) \). If (1) holds, then \( C_0^*(F) = C(F) \), while if (2) holds, then 5.10 implies \( V \leq C_0^*(F) \), and from this we have \( C_b(F) \subseteq C_0^*(F) \) by 2.6 and 2.17. In either case, \( f \in C_0^*(F) \), and thus there is a \( g \in C_0(U(X)) \) such that \( R(g;F) = f \). By 5.8, we may assume \( g \in C^*_b \), and this completes the proof.

5.16. Theorem. Let \( U \) be a Nachbin family on \( X \), let \( F \) be a closed subset of \( X \), and let \( V = \{ R(u;F) : u \in U \} \).
If \( C_0(U(X)) \) is fully complete and if either
(1) \( u \in U \) implies \( \text{spt}(u) \) is compact; or
(2) \( X \) is locally compact and \( C_0^*(F)^* \subseteq M_b(F) \);
then \( C_0^*(F) \) is fully complete.

Proof. From 5.6 and 5.13, we have that \( R(\cdot;F) \) is a continuous linear mapping of \( C_0(U(X)) \) into \( C_0^*(F) \) which is nearly open into, and so \( C_0^*(F) \) is fully complete by 5.4.
5.17. Corollary. If \((C(X), \text{c-op})\) is fully complete, then \(X\) is normal and \((C(F), \text{c-op})\) is fully complete for every closed subset \(F\) in \(X\).

Proof. If \(U = \chi_c(X)\), then \(CU_0(X) = (C(X), \text{c-op})\). If \(u \in U\), then \(\text{spt}(u)\) is compact, so by 5.15, \(X\) is normal. If \(F\) is a closed subset of \(X\) and if \(V = \{R(u;F): u \in U\}\), then 5.16 implies \(CV_0(F)\) is fully complete. But \(V = \chi_c(F)\), and thus \(CV_0(F) = (C(F), \text{c-op})\).

The preceding corollary is a result implicit in a paper by Pták [21]. Collins [6] (and somewhat later Warner [32]) has shown that if \(X\) is a hemi-compact \(k\)-space, then \((C(X), \text{c-op})\) is fully complete. However, there was (before this paper) little else known about full-completeness in function spaces.

5.18. Corollary. If \(X\) is locally compact and if \((C_b(X), \beta)\) is fully complete, then \(X\) is normal and \((C_b(F), \beta)\) if fully complete for every closed subset \(F\) in \(X\).

Proof. If \(U = C^+_o(X)\), then \(CU_0(X) = (C_b(X), \beta)\). Also, \(X\) and \(U\) satisfy (2) of 5.15, so \(X\) is normal. Now let \(F\) be a closed subset of \(X\) and let \(V = \{R(u;F): u \in U\}\). By 5.12, we have \(CV_0(F)^* \subseteq M_b(F)\), and hence \(CV_0(F)\) is fully complete by 5.16. Now \(V \leq C^+_o(F)\) by 5.10, while (3) of 5.9 implies \(C^+_o(F) \leq V\); i.e., \(V \approx C^+_o(F)\). It now follows from 2.7 that \((C_b(F), \beta)\) is fully complete.
If \( X \) is compact, then \((C^b(X), \beta)\) is a Banach space and hence fully complete (see 5.20). However, if \( X \) is locally compact but not compact, \( \beta \) is never metrizable \([4]\), and the only known example in this case where \((C^b(X), \beta)\) is fully complete is when \( X \) is discrete \([7]\), in which case \((C^b(X), \beta)\) actually has the Krein-Smulian property. Other examples would be of interest.

5.19. Corollary. If a topological property for \( X \) is necessary in order that \((G(X), c\text{-}op)\) (respectively, \((C^b(X), \beta)\), where \( X \) is locally compact) be fully complete, then this property is hereditary with respect to closed subsets of \( X \).

Proof. This is an immediate consequence of 5.17 (respectively, 5.18).

A result of some interest would be to determine a class of locally compact spaces for which \((C^b(X), \beta)\) is fully complete (other than the discrete spaces). In view of the result for discrete spaces, it has been conjectured that the class of locally compact \((T_2)\) and extremally disconnected spaces would be such a class. Assuming the continuum hypothesis, this is not the case since our 2.22 gives an example of such a space which is not normal. This same example also rules out the class of pseudo-compact, locally compact spaces.

Let us denote the class of all fully complete spaces by \( \mathfrak{F} \), and assume it is true that whenever \( E, F \in \mathfrak{F} \),
then $E \cong F \in \mathcal{E}$ (for the conclusion, we may even assume $F$ is a Banach space). Then if $X$ is locally compact, if $V$ is a Nachbin family on $X$ with $C_o^+(X) \subseteq V \subseteq C_o^+(X)$, and if $C_{V_0}(X) \in \mathcal{E}$, it would follow that $X$ is paracompact. This is so since, in this case, $CW_0(X \times \beta X) \in \mathcal{E}$ by 4.8 and 5.4, where $W = V \times K^{+}(\beta X)$; from 4.7 and the proof of 4.11, we have $C_c^+(X \times \beta X) \subseteq W \subseteq C_c^+(X \times \beta X)$, and so 5.15 implies $X \times \beta X$ is normal, which is equivalent to $X$ being paracompact [28]. This remark could prove useful in showing the converse of 5.14 is false.

For our next result we will need the following general theorem.

5.20. Theorem ([18, p.212]). A metrizable locally convex space $E$ is complete if and only if $E$ has the Krein-Smulian property.

5.21. Theorem. If $X$ is locally compact and $\sigma$-compact, then the following are equivalent:

1. $(C_b(X), \beta)$ is $B_r$-complete;

2. if $V = \{ \varphi \in C_o^+(X) : N(\varphi) = X \}$, then each weak-* dense nearly closed linear subspace of $M_b(X)$ is a module over $V$;

3. each weak-*dense nearly closed linear subspace of $M_b(X)$ is module over $C_o^+(X)$ (or $C_o(X)$);

4. each weak-*dense nearly closed linear subspace of $M_b(X)$ is variation norm dense in $M_b(X)$.
Proof. We first show (1) and (4) are equivalent (recall $(C_b(X), \beta)^* = M_b(X)$). To do this, let $L$ be a weak-*dense nearly closed linear subspace of $M_b(X)$, and assume $L$ is variation norm dense in $M_b(X)$. Let $\mu \in M_b(X)$ and choose $(\mu_n)_{n=1}^\infty \subseteq L$ so that $\mu_n \to \mu$ in the variation norm. Hence $\mu_n \to \mu$ in the $\sigma(M_b(X), C_b(X))$-topology, and this implies there exists $\varphi \in C_o^+(X)$ such that $(\mu_n)_{n=1}^\infty \subseteq V^\circ [8, p.161]$. Therefore $\mu \in L \cap V^\circ (\sigma(M_b(X), C_b(X))-\text{closure}) = L \cap V^\circ$; i.e., $\mu \in L$. Consequently, $L = M_b(X)$ and we have (4) implies (1). Since (1) obviously implies (4), the claim is verified.

It is clear that (1) implies (3) and (3) implies (2). Therefore the proof will be complete when we show (2) implies (1). Since $X$ is an open $\sigma$-compact subset of $X$, there is a $\varphi \in C_o^+(X)$ such that $N(\varphi) = X$ [4]; i.e., $V$ is non-void. It easily follows that $V$ is a Nachbin family on $X$ with $V \approx C_o^+(X)$, and hence it will suffice to show $CV_o(X)$ is $B_\tau$-complete.

Let $L$ be a weak-*dense nearly closed linear subspace of $CV_o(X)^*$ and assume $L$ is a module over $V$. If we fix $v \in V$, then $A = \{\lambda v : \lambda \geq 0\}$ is a Nachbin family on $X$. Since $N(v) = X$, it follows from 2.11 and 3.3 that $CA_o(X)$ is a Banach space, and hence has the Krein-Smulian property by 5.20. From 3.26, we have $CA_o(X)^* = A \cdot M_b(X) \subseteq CV_o(X)^* = V \cdot M_b(X)$, while 2.6 implies $CV_o(X) \subseteq CA_o(X)$. If $L_v = L \cap A \cdot M_b(X)$, then $L_v$ is a linear subspace of $CA_o(X)^*$. 

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If \( \mu \in A \cdot M_b(X) \), then there exist \( \lambda > 0, v \in M_b(X) \) so that
\( \mu = (\lambda v) \cdot v, \) and we may assume \( v \in V \cdot M_b(X) \) by 3.22. Now
let \( f \in CA_0(X) \) and let \( \varepsilon > 0. \) Since \( fv \in CV_0(X), \) there
is a \( \sigma \in L \) such that \( |\int fvdv - \int fd\sigma| < \lambda^{-1}\varepsilon, \) while
\( \sigma \in M_b(X) \) implies \( (\lambda v) \cdot \sigma \in L_v. \) Thus \( L_v \) is
\( \sigma(CA_0(X)^*, CA_0(X)) \)-dense in \( A \cdot M_b(X) \) since
\( |\int fd\mu - \int fd((\lambda v) \cdot \sigma)| < \varepsilon. \) We next show \( L_v \) is nearly closed
(in \( CA_0(X)^* \), and to do this it will suffice to consider
\( t = \lambda v \) with \( \lambda > 0. \) Since \( t \in V \cap A, \) we will denote the
neighborhood defined by \( t \) in \( CA_0(X) \) by \( V^t; \) i.e.,
\( V^t = \{ f \in CA_0(X) : |ft| \leq 1 \}. \) Then \( V_t = V^t \cap CV_0(X), \)
and it follows that \( V^t_0 \) (polar in \( CV_0(X)^* \)) contains \( V^0_t \)
(polar in \( CA_0(X)^* \)). If \( \mu \in V^0_t, \) then 3.27 implies there
is a \( v \in M_b(X) \) with \( ||v|| \leq 1 \) such that \( \mu = t \cdot v, \) and
this implies \( \mu \in A \cdot M_b(X). \) Moreover, if \( f \in V^t, \) then
\( |\int fd\mu| = |\int ftvdv| \leq |ft||v|| \leq 1; \) i.e., \( \mu \in V^0_t. \) Hence
\( V^0_t = V^0_t \) as sets, and
\( L_v \cap V^0_t = L_v \cap V^0_t =
(L \cap A \cdot M_b(X)) \cap V^0_t \subseteq L \cap V^0_t \subseteq L_v \cap V^0_t. \) If \( \mu \in A \cdot M_b(X) \)
is a \( \sigma(CA_0(X)^*, CA_0(X)) \) limit point of \( L_v \cap V^0_t, \) then
\( \mu \) is a \( \sigma(CV_0(X)^*, CV_0(X)) \) limit point of \( L \cap V^0_t, \) which
implies \( \mu \in L \cap V^0_t; \) i.e., \( \mu \in L_v \cap V^0_t. \) Hence \( L_v \) is a
nearly closed linear subspace of \( A \cdot M_b(X) \) which is also
\( \sigma(CA_0(X)^*, CA_0(X)) \)-dense, and so \( L_v = A \cdot M_b(X). \) Therefore
\( V \cdot M_b(X) \subseteq \bigcup_{v \in V} L_v \subseteq L, \) and \( L = V \cdot M_b(X). \)
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BIOGRAPHY

William Hunley Summers was born in Dallas, Texas on February 5, 1936. He attended schools in Dallas and graduated from North Dallas High School in 1953. In 1959, he enrolled at Arlington State College in Arlington, Texas (now The University of Texas at Arlington) from which he received the Bachelor of Science degree in May, 1961. In the fall of 1961 he went to Purdue University where he held a teaching assistantship, receiving the Master of Science degree in June, 1963. In the fall of 1964, he went to Louisiana State University in New Orleans as an instructor. In the fall of 1965, he came to Louisiana State University as an instructor, where he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.
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Candidate: William H. Summers

Major Field: Mathematics

Title of Thesis: WEIGHTED LOCALLY CONVEX SPACES OF CONTINUOUS FUNCTIONS

Date of Examination: July 17, 1968

Approved:

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Major Professor and Chairman

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