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On Matroid and Polymatroid Connectivity

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Abstract

Matroids were introduced in 1935 by Hassler Whitney to provide a way to abstractly capture the dependence properties common to graphs and matrices. One important class of matroids arises by taking as objects some finite collection of one-dimensional subspaces of a vector space. If, instead, one takes as objects some finite collection of subspaces of dimensions at most k in a vector space, one gets an example of a k-polymatroid.

Connectivity is a pivotal topic of study in the endeavor to understand the structure of matroids and polymatroids. In this dissertation, we study the notion of connectivity from several angles. It is a well-known result of Tutte that, for every element x of a connected matroid M, at least one of the deletion and contraction of x from M is connected. Our first result shows that, in a connected k-polymatroid, only two such elements are guaranteed. We show that this bound is sharp and characterize those 2-polymatroids that achieve this minimum.

It is well known that, for any integer n greater than one, there is a number r such that every 2-connected simple graph with at least r edges has a minor isomorphic to an n-edge cycle or $K_{2,n}$. This result was extended to matroids by Lovász, Schrijver, and Seymour who proved that every sufficiently large connected matroid has an n-element circuit or an n-element cocircuit as a minor. As our second result, we generalize these theorems by providing an analogous result for connected 2-polymatroids. Significant progress on the corresponding problem for k-polymatroids is also described.

Finally, we look at tangles, a tool that has been used extensively in recent results in matroid structure theory. We prove that a matroid with at least two elements is a tangle matroid if and only if it cannot be covered by three hyperplanes. Some consequences of this theorem are also noted. In particular, no binary matroid of rank at least two is a tangle matroid.

Chapter 1
Preliminaries

In this chapter, we introduce basic graph, matroid, and polymatroid terminology that will be needed throughout this dissertation. The language used for graphs, matroids, and polymatroids closely follows [3], [14], and [16].

1.1 Fundamental Graph Definitions

A *finite multigraph* (or, in this dissertation, a *graph*) G is a pair (V, E), where V is a finite set and E is a finite multiset whose elements are unordered pairs of elements in V. The elements of V are the *vertices* of G, while the elements of E are the *edges* of G. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. The number of vertices in a graph is its *order*.

In a graph G, a vertex $v \in V(G)$ is *incident* with an edge $e \in E(G)$ if $v \in e$. Two not-necessarily-distinct vertices, u and v, are *adjacent* if $\{u, v\} \in E(G)$. In such a case, we say that the edge $\{u, v\}$ has *endpoints* u and v. An edge whose endpoints are the same vertex is a *loop*. Two distinct edges e and f are *adjacent* if they share an endpoint and are *parallel* if they share two distinct endpoints. A graph that has neither loops nor parallel edges is a *simple graph*. For a vertex v, the number of edges incident with v is the *degree* of v, where loops are counted twice. A vertex of degree zero is an *isolated* vertex.

The deleting of edges and vertices, and contracting of edges from graphs are important tools when working in graph theory. Deleting an edge from a graph is achieved by removing that edge from the edge set of the graph. Deleting a vertex from a graph is done by removing the vertex from the vertex set along with deleting all edges incident with that vertex. Finally, the contraction of an edge $e = \{x, y\}$ is obtained by replacing the edge e with a vertex v_e that is adjacent to all the former neighbors of x and y. Formally, if $e \in E(G)$, then the *edge*

deletion of e from G is the graph $G\backslash e = (V, E - e)$. If $G = (V, E)$ is a graph and $v \in V(G)$, then the *vertex deletion* of v from G is the graph $G\backslash v = (V - v, \{e \in E : v \notin e\})$. The *contraction* of $e = \{x, y\}$ from G is the graph $G/e = (V', E')$, where $V' = (V\backslash\{x, y\}) \sqcup \{v_e\}$ and

$$E' = \{\{v, w\} \in E : \{v, w\} \cap \{x, y\} = \varnothing\} \cup \{\{v_e, w\} : \{x, w\} \in E\backslash\{e\} \text{ or } \{y, w\} \in E\backslash\{e\}\}.$$

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If there exists a bijection $\sigma : V \to V'$ with $\{x, y\} \in E$ if and only if $\{\sigma(x), \sigma(y)\} \in E'$ for all $x, y \in V$, then G and G' are *isomorphic*, written $G \cong G'$. In this dissertation, we will not distinguish between isomorphic graphs and will thus write $G = G'$ rather than $G \cong G'$.

Let H and G be graphs. If H can be obtained from G by a series of vertex deletions, then H is an *induced subgraph* of G. If H can be obtained from G by a series of vertex and edge deletions, then H is a *subgraph* of G. Finally, if H can be obtained from G by a series of vertex deletions, edge deletions, and contractions, then H is a *minor* of G.

1.2 Some Important Classes of Graphs

A simple graph in which every two distinct vertices are adjacent is a *complete graph*. The complete graph of order n is denoted by K_n. A graph $G = (V, E)$ is called *bipartite* if V can be partitioned into two classes such that no two vertices in the same class are adjacent. A bipartite graph in which every two vertices from different partition classes is adjacent is called a *complete bipartite graph*. If G is a complete bipartite graph whose partitions are of size m and n, then we write $G = K_{m,n}$.

A *path* is a nonempty graph $P = (V, E)$ with

$$V = \{x_0, x_1, \ldots, x_k\} \text{ and } E = \{\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}\},$$

where all the x_i are distinct. The number of edges in a path is its *length*. The path of length r is written as P_r. The vertices x_0 and x_k are called its *ends* or *end vertices*. Two or more

paths are *independent* if none of them contains a non-end vertex of another. If a graph G contains a path P as a subgraph, then we say that the ends of P are *linked* by P. If every every two distinct vertices in G are linked by a path P, then G is *connected*. Otherwise, G is *disconnected*. Given a path P with distinct end vertices x_0 and x_k, if we add the edge $\{x_0, x_k\}$, then the graph $C = (V(P), E(P) \cup \{x_0, x_k\})$ is a *cycle*.

We now look at several classes of graphs that will come up in matroid theory: *theta graphs (Θ-graphs)*, *tight handcuffs*, and *loose handcuffs*. A Θ-graph is a graph consisting entirely of three independent paths that all share the same pair of distinct end vertices. A tight-handcuffs graph consists entirely of two cycles with exactly one common vertex. Finally, a loose-handcuffs graph consists of two vertex-disjoint cycles and a path, where the path is disjoint from the cycles except that each endpoint lies in a different cycle.

1.3 Fundamental Matroid Definitions

In this section, we give basic matroid definitions. Note that we will often, in this section and throughout this dissertation, denote a singleton $\{x\}$ as x and a pair {x,y} as xy. A *matroid* M is an ordered pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E satisfying the following conditions.

(I1) $\varnothing \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| \leq |I_2|$, then there exists an element $e \in I_2 - I_1$ such that $I_1 \cup e$ is a member of $\mathcal{I}$.

The set E or $E(M)$ is the *ground set* of M, while the members of $\mathcal{I}$ are the *independent sets* of M. A subset of E that is not in $\mathcal{I}$ is called *dependent*. A minimal dependent set is a *circuit*. A maximal independent set is a *basis*. If B is a basis of a matroid M and $e \in E(M) - B$,

then the set $B \cup \{e\}$ contains a unique circuit, $C(e, B)$, called the *fundamental circuit* of e with respect to B.

For a subset X of E, all maximal independent sets in X have the same cardinality, which is the *rank*, $r(X)$, of X. If $r(X) = r(E)$, then X is a *spanning* set. A function $r : 2^E \to \mathbb{Z}$ is the *rank function* of a matroid $M = (E, \mathcal{I})$ if and only if it satisfies the following properties.

(R1) $0 \leq r(x) \leq 1$ for each $x \in E$.

(R2) $r(X) \leq r(Y)$ whenever $X \subseteq Y \subseteq E$.

(R3) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ for all $X, Y \subseteq E$.

The rank function r of a matroid with ground set E uniquely determines the matroid. We will thus often, especially in this dissertation, define a matroid in terms of the rank function rather than the independent sets. In this case, we will write a matroid as $M = (E, r)$ instead of $(E, \mathcal{I})$. We will also often write $r(M)$ for $r(E(M))$ and call $r(M)$ the matroid's rank.

We borrow much matroid language from graphs. For a matroid $M = (E, r)$, if $x \in E$ such that $r(x) = 0$, then x is a *loop*. If $x, y \in E$ are distinct elements such that $r(x) = r(y) = r(xy) = 1$, then x and y are *parallel elements*. A *parallel class* is a maximal subset X of $E(M)$ such that any two distinct members of X are parallel. A parallel class is *trivial* if it contains just one element. A matroid that has neither loops nor parallel elements is *simple*. A matroid is *connected* if and only if its ground set cannot be partitioned into two nonempty sets X and Y such that $r(X) + r(Y) = r(X \cup Y)$.

For a matroid $M = (E, r)$, the *closure* of a set $X \subseteq E$, denoted $\mathrm{cl}_M(X)$ or $\mathrm{cl}(X)$, is the set $\mathrm{cl}(X) = \{x \in E : r(X \cup x) = r(X)\}$. If a set X equals its closure, then X is a *flat*. A *line* is a rank-2 flat and a flat of rank $r(M) - 1$ is a *hyperplane*.

If $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are matroids for which there is bijection $\sigma : E_1 \to E_2$ such that $X \subseteq E_1$ is in $\mathcal{I}_1$ if and only if $\sigma(X)$ is in $\mathcal{I}_2$, then M_1 and M_2 are *isomorphic*.

1.4 Some Important Examples of Matroids

One useful class of matroids are those matroids whose independent sets are exactly the sets of at most a certain cardinality. Let m and n be nonnegative integers such that $m \leq n$. The *uniform matroid* $U_{m,n}$ is that matroid whose ground set E has n elements and whose independent sets consist of all subsets of E with at most m elements.

Another important class of matroids are the *representable matroids*. For an $m \times n$ matrix A over a field $\mathbb{F}$, let E be the set of column labels of A, and let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labeled by X is a set and is linearly independent in the vector space $V(m, \mathbb{F})$. Then $M[A] = (E, \mathcal{I})$ is a matroid and, in particular, $M[A]$ is the *vector matroid* of A. If M is isomorphic to the vector matroid of a matrix over a field $\mathbb{F}$, then M is *representable over* $\mathbb{F}$ or is $\mathbb{F}$-*representable*. A matroid is *representable* if it is representable over some field. Let $GF(q)$ be the unique finite field on q elements. If a matroid is $GF(2)$-representable or $GF(3)$-representable, then it is *binary* or *ternary*, respectively.

Within the class of representable matroids are *projective geometries*. These matroids are often thought of as the matroid analogues of complete graphs. Let A be an $m \times n$ matrix over $GF(q)$. A loop in $M[A]$ has a corresponding column equal to the zero vector in A. A pair of elements x and y are parallel in $M[A]$ if and only if the corresponding columns of A are scalar multiples of one another and neither is zero. If we require $M[A]$ to be simple and rank r, it becomes evident, then, since $GF(q)$ has only a finite number of elements, that there is a maximum number of columns possible in A and thus a maximum number of elements possible in a simple rank-r $GF(q)$-representable matroid. Formally, if M is a rank-r simple $GF(q)$-representable matroid, then

$$|E(M)| \leq \frac{q^r - 1}{q - 1}. \tag{1.1}$$

A rank-r simple matroid that is $GF(q)$-representable and for which equality holds in Equation 1.1 is a *projective geometry* and is denoted by $PG(r-1, q)$.

Finally, we look at *bicircular matroids*. The bicircular matroid of a graph G, denoted by $B(G)$ is the matroid on $E(G)$ whose circuits consist precisely of the edge sets of all Θ-graphs, all loose handcuffs, and all tight handcuffs.

1.5 Matroid Operations

The *dual* of a matroid is a useful tool throughout matroid theory. Let $M = (E, r)$ be a matroid. Define a new matroid function r^* by

$$r^*(X) = |X| + r(E - X) - r(E),$$

for all $X \subseteq E$. It is easily checked that r^* is a matroid rank function on E. The matroid (E, r^*) is denoted by M^* and is called the *dual matroid* of M. If a set C is a circuit of M^*, then it is a *cocircuit* of M; if X is an independent set of M^*, then it is a *coindependent set* of M. Note that a set X is a hyperplane in M if and only if $E - X$ is a cocircuit of M. Also, $M = (M^*)^*$; that is, the dual of the dual of a matroid is the original matroid.

Let $M = (E, I)$ be a matroid and suppose that $X \subseteq E$. The pair

$$(X, \{I \subseteq X : I \in \mathcal{I}\})$$

is a matroid $M|X$ called the *restriction* of M to X. Similarly, we call the matroid $M\backslash X = M|(E-X)$ the *deletion* of X from M. Matroid contraction is defined as the dual of deletion. Formally, if $M = (E, r)$ is a matroid and $X \subseteq E$, then $M/X = (M^*\backslash X)^*$ is the *contraction* of X from M. The rank function $r_{M/X}$ of M/X is given by the following, where we use the notation r/X to denote the rank function of M/X.

$$r/X(A) = r(X \cup A) - r(X),$$

for all $A \subseteq E$. For a matroid N, if $N \cong M/C\backslash D$ for disjoint sets $C, D \subseteq E$, then N is a *minor* of M. Observe that the order in which deletion and contraction is done does not affect the resulting matroid. If M is a matroid, then $si(M)$ is the corresponding simple matroid in which all loops and all but one member of each parallel class in M have been deleted.

If we restrict deletion such that we are only allowed to delete one member of a two-element circuit, then we have the operation *parallel deletion.* If a matroid N can be obtained from a matroid M by a series of contractions and parallel deletions, then N is a *parallel minor* of M.

Let $\mathcal{C}$ be a class of matroids. If, for any $M \in \mathcal{C}$ we have, for all $X \subseteq E(M)$, that both $M \backslash X$ and M/X are members of $\mathcal{C}$, then $\mathcal{C}$ is a *minor-closed* class of matroids. If M is a matroid not in $\mathcal{C}$, but, for any $x \in E(M)$, we have that both M/x and $M \backslash x$ are in $\mathcal{C}$, then M is an *excluded minor* of $\mathcal{C}$. The terms *parallel-minor-closed class* and *excluded parallel minor* are defined analogously.

If a matroid Q can be obtained from a matroid M by adding new elements to Q and then contracting these new elements, then we say that Q is a *quotient* of M. Formally, a matroid Q is a quotient of M if there exists a matroid N such that $N/X = Q$ and $N \backslash X = M$ for some $X \subseteq E(N)$. If (E_1, r_2) and (E_2, r_2) are two matroids and $f : E_1 \to E_2$ is a map with the property that the preimage of each flat in M_2 is a flat in M_1, then f is a *strong map.*

1.6 Fundamental Polymatroid Definitions

Let M be a matroid with ground set E and rank function r. The pair (E, r) is an example of a 1-polymatroid. In fact, the class of 1-polymatroids is exactly the class of matroids. For an arbitrary positive integer k, we now define a k-polymatroid noting that it is very much like a matroid but allows individual elements to have ranks up to k.

Let E be a finite set and f be a function from the power set of E into the integers. We say that f is *normalized* if $f(\varnothing) = 0$; f is *submodular* if $f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y)$ for all $X, Y \subseteq E$; and f is *increasing* if $f(X) \leq f(Y)$ whenever $X \subseteq Y \subseteq E$. We call the pair (E, f) a *polymatroid* $\mathcal{Q}$ if f is normalized, submodular, and increasing. The set E is called the *ground set* of $\mathcal{Q}$ while f is the *rank function.* For a positive integer k, a polymatroid (E, f) is a *k-polymatroid* if $f(z) \leq k$ for all z in E.

For ease of notation, a rank-1 element of a k-polymatroid is called a *point*; a rank-0 element of a k-polymatroid is called a *loop*. Let a and b be elements of a k-polymatroid with rank function f. If $f(ab) = f(a) = f(b)$, then we say that a and b are *parallel*; if $f(ab) = f(a) + f(b)$, we say that a and b are *skew*; and if $f(ab) < f(a) + f(b)$, we say that a and b *intersect*.

1.7 Some Important Examples of Polymatroids

An important way to obtain a k-polymatroid from a matroid is as follows. Given a matroid M with ground set S and rank function r, we obtain a k-polymatroid $\mathcal{Q} = (E, f)$ by taking E to be some subset of the set of flats of M of rank at most k and letting $f(X) = r\left(\bigcup_{x \in X} x\right)$ for all subsets X of E. Indeed, every k-polymatroid can be obtained in this way (see, for example, [12, 14]). This fundamental fact allows us, in particular, to think of a 2-polymatroid as an arrangement of loops, points, and lines of a matroid.

Another natural class of 2-polymatroids arises from graphs. To see this, let G be a graph and set $E = E(G)$. For a subset X of E, define a function f by $f(X) = |V(X)|$ where $V(X)$ is the set of vertices of G that meet some edge of X. Then (E, f) is a 2-polymatroid. We will call the 2-polymatroids that can be represented in this way *Boolean* and note that there is a one-to-one correspondence between the class of Boolean 2-polymatroids and the class of graphs without isolated vertices [17].

Finally, we consider k-polymatroids that are derived from other polymatroids. Let $\mathcal{Q}_1 = (E, f_1)$ and $\mathcal{Q}_2 = (E, f_2)$ be k-polymatroids on the same ground set. It is not difficult to check that (E, f) is a $2k$-polymatroid where $f(Z) = f_1(Z) + f_2(Z)$ for all $Z \subseteq E$. We denote (E, f) by $\mathcal{Q}_1 + \mathcal{Q}_2$ or, when $\mathcal{Q}_1 = \mathcal{Q}_2$, by $2\mathcal{Q}_1$. We are not limited, however, to a sum of only two polymatroids. For example, the sum of k copies of the matroid $U_{n-1,n}$, denoted $kU_{n-1,n}$, is a k-polymatroid consisting of n rank-k elements placed as independently as possible in rank $kn - k$.

1.8 Polymatroid Operations

One attractive feature of k-polymatroids is that there are notions of duality, deletion, and contraction that retain many of the nice properties of the same notions in matroids. Let $\mathcal{Q} = (E, f)$ be a k-polymatroid. For all subsets X of E, let

$$f^*(X) = k|X| + f(E - X) - f(E).$$

Then (E, f^*) is a k-polymatroid $\mathcal{Q}^*$, which, following [16], we call the *k-dual* of $\mathcal{Q}$.

For a subset X of E, define $f_{\mathcal{Q}\backslash X}$ and $f_{\mathcal{Q}/X}$, for all subsets A of $E-X$, by $f_{\mathcal{Q}\backslash X}(A) = f(A)$ and $f_{\mathcal{Q}/X}(A) = f(X \cup A) - f(A)$. Let $\mathcal{Q}\backslash X = (E - X, f_{\mathcal{Q}\backslash X})$ and $\mathcal{Q}/X = (E - X, f_{\mathcal{Q}/X})$. It is common to write $f\backslash X$ instead of $f_{\mathcal{Q}\backslash X}$ and f/X instead of $f_{Q/X}$. It is easy to verify that both of $\mathcal{Q}/X$ and $\mathcal{Q}\backslash X$ are k-polymatroids, and that $\mathcal{Q}^*\backslash X = (\mathcal{Q}/X)^*$. We call $\mathcal{Q}\backslash X$ and $\mathcal{Q}/X$ the *deletion* and *contraction* of X from $\mathcal{Q}$. We note that the k-dual is the unique involutory operation on k-polymatroids that interchanges deletion and contraction (see [20]).

For a polymatroid $\mathcal{Q} = (E, f)$, the *connectivity function* λ_f of f is defined, for all subsets X of E, by $\lambda_f(X) = f(X) + f(E - X) - f(E)$; and the *local connectivity function* $\sqcap_f(X, Y)$ is defined, for a pair of subsets X and Y of E, by $f(X) + f(Y) - f(X \cup Y)$. When there is no potential for creating ambiguity, we may write $\sqcap$ and λ in place of $\sqcap_f$ and λ_f. The following properties of the connectivity function and the local connectivity function for polymatroids will be used throughout the dissertation and are provided in [1, §2].

Lemma 1.8.1. *If A, B, C, and D are subsets of the ground set E of a k-polymatroid (E, f), then the following hold:*

(i) $\lambda(A) + \lambda(B) \geq \lambda(A \cup B) + \lambda(A \cap B)$;

(ii) If $A \subseteq B$ and $C \subseteq D$, then $\sqcap(A, C) \leq \sqcap(B, D)$.

Following Matúš [13], we say that a k-polymatroid $\mathcal{Q} = (E, f)$ is *connected* or 2-*connected* if $\lambda_f(X) > 0$ for all nonempty proper subsets X of E; otherwise, $\mathcal{Q}$ is *disconnected.* If

$\lambda_f(X) = 0$, then X is a *separator*; it is *nontrivial* if $X \notin \{\emptyset, E\}$. When X is a nontrivial separator, $(X, E - X)$ is called a 1-*separation* of $\mathcal{Q}$. It is straightforward to check that $\mathcal{Q}$ is connected if and only if $\mathcal{Q}^*$ is connected (see [15]).

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids on disjoint ground sets. Let $\mathcal{Q}_1 \oplus \mathcal{Q}_2 = (E_1 \cup E_2, f)$ where $f(Z) = f_1(Z \cap E_1) + f_2(Z \cap E_2)$ for all $Z \subseteq E_1 \cup E_2$. It is well known and easily checked that $\mathcal{Q}_1 \oplus \mathcal{Q}_2$ is a k-polymatroid. Following [2], we call it the *direct sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. If $\mathcal{Q}_1$ is connected, we say that it is a *connected component* of $\mathcal{Q}$. In this case, we will often also refer to the ground set of $\mathcal{Q}_1$ as a connected component of $\mathcal{Q}$. Evidently, a k-polymatroid is connected if and only if it cannot be written as a direct sum of two k-polymatroids with nonempty ground sets.

Chapter 2
Non-Essential Elements in k-polymatroids

2.1 Introduction

A classical result of Tutte is that, for every element x of a connected matroid M, either $M\backslash x$ or M/x is connected. This property of being able to either delete or contract any element while maintaining connectivity, however, does not hold for k-polymatroids. We call an element x of a connected k-polymatroid *essential* if both its deletion and contraction from the k-polymatroid destroy connectivity. In this chapter, we show that every k-polymatroid has at least two elements that are non-essential, show that this bound is sharp for each integer k exceeding one, and characterize all 2-polymatroids with exactly two non-essential elements. Much of the work here appears in [9].

Additional motivation for this chapter comes from the desire to find the unavoidable minors for connected 2-polymatroids, which is done in chapter 3. This study of essential elements turns out to be a crucial step in that endeavor. In fact, one may divide the class of unavoidable minors for connected 2-polymatroids into two categories: those that resemble circuits and cocircuits in matroids, and those that have exactly two non-essential elements.

The main results, Theorems 2.3.3 and 2.3.9, are stated and proved in Section 2.3. The concepts of 2-sum and parallel connection for k-polymatroids, ideas that play an important role in the proofs of the main results, are studied in Section 2.2.

2.2 Parallel Connection and 2-Sum

Here, we expand upon the notion of parallel connection for polymatroids that is given in [13]. This operation for polymatroids is a generalization of that for matroids in that it consists

of sticking together two polymatroids as freely as possible across a designated element of each. Below, we give a formal definition that mimics the language of parallel connection for matroids.

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids on disjoint ground sets. Let $\mathcal{Q}_1 \oplus \mathcal{Q}_2 = (E_1 \cup E_2, f)$ where $f(Z) = f_1(Z \cap E_1) + f_2(Z \cap E_2)$ for all $Z \subseteq E_1 \cup E_2$. It is well known and easily checked that $\mathcal{Q}_1 \oplus \mathcal{Q}_2$ is a k-polymatroid. Following [2], we call it the *direct sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Evidently, a k-polymatroid is 2-connected if and only if it cannot be written as a direct sum of two k-polymatroids with nonempty ground sets.

Next, suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids with $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p)$. Let $P(\mathcal{Q}_1, \mathcal{Q}_2) = (E_1 \cup E_2, f)$ where, for all $A \subseteq E$, if $A_1 = A \cap E_1$ and $A_2 = A \cap E_2$, then

$$f(A) = \min\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\}.$$

A routine check shows that $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is a k-polymatroid. We call this k-polymatroid the *parallel connection* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to the *basepoint* p. When $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are matroids, this definition of parallel connection coincides with that for matroids. A limitation of our definition of $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is that it requires the basepoints to have the same rank. To rectify this, we extend the matroid operation of principal truncation (see, for example, [14, Section 7.3]).

Intuitively, the principal truncation of an element p is achieved by adding a point on p as freely as possible and then contracting the added point. To define this operation formally, let $\mathcal{Q} = (E, f)$ be a polymatroid with $p \in E$ and let f_p be the function defined, for all subsets A of E, by

$$f_p(X) = \begin{cases} f(X) - 1, & \text{if } f(X \cup p) = f(X); \\ f(X), & \text{otherwise.} \end{cases}$$

It is not difficult to check that (E, f_p) is a polymatroid. We denote it by $T_p(\mathcal{Q})$ and say that it is obtained from $\mathcal{Q}$ by *truncating* p. This operation can be repeated. For a positive

integer n, we define $T_p^n(\mathcal{Q}) = T_p(T_p^{n-1}(\mathcal{Q}))$ where $T_p^0(\mathcal{Q}) = \mathcal{Q}$. It is an easy exercise to verify that $T_p^n(\mathcal{Q})$ has rank function f_p^n defined, for all $X \subseteq E$, by

$$f_p^n(X) = \begin{cases} \max\{f(X \cup p) - n, 0\}, & \text{if } f(X \cup p) - f(X) \leq n; \\ f(X), & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are polymatroids with $E_1 \cap E_2 = \{p\}$. Let $n = f_2(p) - f_1(p) > 0$. We expand the notion of parallel connection to this case by setting $P(\mathcal{Q}_1, \mathcal{Q}_2)$ to be $P(\mathcal{Q}_1, T_p^n(\mathcal{Q}_2))$. When $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are matroids such that p is a loop of $\mathcal{Q}_1$ and a non-loop of $\mathcal{Q}_2$, this definition coincides with that for matroids.

The following familiar properties of parallel connection hold for k-polymatroids.

Proposition 2.2.1. *Let $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ be two polymatroids for which $E_1 \cap E_2 = \{p\}$. Then*

(i) $P(\mathcal{Q}_1, \mathcal{Q}_2)/p = \mathcal{Q}_1/p \oplus \mathcal{Q}_2/p$; and

(ii) for all $e \in E_1 - p$, $P(\mathcal{Q}_1, \mathcal{Q}_2)/e = P(\mathcal{Q}_1/e, \mathcal{Q}_2)$ and $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash e = P(\mathcal{Q}_1\backslash e, \mathcal{Q}_2)$.

Proof. The proof of this proposition is not significantly different from the proof of the corresponding result for matroids (see, for example, [14]) and is omitted. □

The following result of Oxley and Whittle (see [15, Theorem 3.1]) is used throughout this chapter.

Lemma 2.2.2. *Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid where $|E| \geq 2$ and let A be a nonempty proper subset of E. If*

$$f(A) + f(E - A) - f(E) < \min\{f(X) + f(E - X) - f(E) : \varnothing \neq X \subsetneq E\},$$

then at least one of $\mathcal{Q}/A$ and $\mathcal{Q}\backslash A$ is connected. □

From this lemma, we obtain the following result on non-essential elements. Recall that an element e of a connected k-polymatroid $\mathcal{Q}$ is non-essential if either $\mathcal{Q}\backslash e$ or $\mathcal{Q}/e$ is connected.

Proposition 2.2.3. *If $\mathcal{Q} = (E, f)$ is a connected k-polymatroid and $e \in E$ with $f(e) = 1$, then e is non-essential.*

Proof. This is an immediate consequence of Lemma 2.2.2. □

Theorem 2.2.4. *Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids such that $E_1 \cap E_2 = \{p\}$ where $f_1(p) = f_2(p)$. Then both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are connected if and only if $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is connected. Further, if $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$ is connected, then $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is connected.*

Proof. If $(X, Y \cup p)$ is a 1-separation of $\mathcal{Q}_1$, then it is not difficult to check that $(X, E_2 \cup Y)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)$ and that $(X, (E_2 - p) \cup Y)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$. On the other hand, suppose $(X, Y \cup p)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)$, and f_3 is the rank function for $P(\mathcal{Q}_1, \mathcal{Q}_2)$. Let $X_i = X \cap E_i$ and $Y_i = Y \cap E_i$ for each $i \in \{1, 2\}$, and observe that

$$\begin{aligned} f_3(X) &= \min\{f_1(X_1) + f_2(X_2), f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p)\}; \\ f_3(Y \cup p) &= f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p); \text{ and} \\ f_3(E_1 \cup E_2) &= f_1(E_1) + f_2(E_2) - f_1(p). \end{aligned}$$

If $f_1(X_1)+f_2(X_2) \le f_1(X_1\cup p)+f_2(X_2\cup p)-f_1(p)$, then since $f_3(X)+f_3(Y\cup p) = f_3(E_1\cup E_2)$, we have

$$f_1(X_1) + f_2(X_2) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) = f_1(E_1) + f_2(E_2).$$

As $f_i(X_i) + f_i(Y_i \cup p) \ge f_i(E_i)$ for each $i \in \{1, 2\}$, it follows that $(X_i, Y_i \cup p)$ is a 1-separation for each $i \in \{1, 2\}$. On the other hand, if $f_1(X_1) + f_2(X_2) > f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p)$, then, as $f_3(X) + f_3(Y \cup p) = f_3(E_1 \cup E_2)$, we have

$$f_1(X_1 \cup p) + f_2(X_2 \cup p) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p) = f_1(E_1) + f_2(E_2).$$

From submodularity again, it follows that $f_2(p) = f_1(p) = 0$, and thus $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are disconnected. $\square$

In addition to parallel connection, we make use of the 2-sum operation. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be k-polymatroids on ground sets E_1 and E_2, respectively, with $E_1 \cap E_2 = \{p\}$. If $f_1(p) = f_2(p) = 1$ and p is not a separator for either $\mathcal{Q}_1$ or $\mathcal{Q}_2$, then the 2-*sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is defined to be $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$ and denoted $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$. The following shows some fundamental connectivity properties of this 2-sum operation.

Corollary 2.2.5. *Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids such that $E_1 \cap E_2 = \{p\}$ where $f_1(p) = f_2(p) = 1$. Then the following are equivalent.*

(i) *$\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both connected;*

(ii) *$\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$ is connected;*

(iii) *$P(\mathcal{Q}_1, \mathcal{Q}_2)$ is connected.*

Proof. Using Theorem 2.2.4, we have only to show that (*iii*) implies (*ii*). From Proposition 2.2.1, we observe that $P(\mathcal{Q}_1, \mathcal{Q}_2)/p$ is disconnected. Since $f_1(p) = f_2(p) = 1$, we use Proposition 2.2.3 to see that p is non-essential and therefore that $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p = \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$ is connected. $\square$

We say that a k-polymatroid $\mathcal{Q}$ is 3-*connected* if and only if it cannot be written as a 2-sum of a pair of k-polymatroids each with fewer elements than $\mathcal{Q}$. The following proposition allows us to give an alternative definition.

Proposition 2.2.6. *Suppose $\mathcal{Q} = (E, f)$ is a k-polymatroid for which there exists a partition (X_1, X_2) of E such that $f(X_1) + f(X_2) = f(E) + 1$ and $\min\{|X_1|, |X_2|\} \geq 2$. Then there are polymatroids $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on ground sets $X_1 \cup p$ and $X_2 \cup p$, respectively, where p is a new point not in E, such that $\mathcal{Q} = \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$.*

Proof. For $(i, j) \in \{(1,2),(2,1)\}$, let $\mathcal{Q}_i = (X_i \cup p, f_i)$ where f_i is defined, for all $A \subseteq X_i \cup p$, by

$$f_i(A) = \begin{cases} f((A-p) \cup X_j) - f(X_j) + 1 & \text{if } p \in A; \\ f(A) & \text{if } p \notin A. \end{cases}$$

It is routine to check that f_i is a k-polymatroid. Let f_3 be the rank function of $P(\mathcal{Q}_1, \mathcal{Q}_2)$. Since $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 = P(\mathcal{Q}_1, \mathcal{Q}_2) \backslash p$, it suffices to show that $f_3(A) = f(A)$ for all subsets A of E. Choose such a subset A, let $A_i = A \cap X_i$ for $i \in \{1, 2\}$, and note that

$$\begin{aligned} f_3(A) &= \min\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\} \\ &= \min\{f(A_1) + f(A_2), f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E)\}. \end{aligned}$$

Observe that if U and V are disjoint subsets of E with $S \subseteq U$ and $T \subseteq V$, then

$$\begin{aligned} f(U) + f(V) + f(S \cup T) &\geq f(U) + f(S \cup V) + f(T) \\ &\geq f(U \cup V) + f(S) + f(T). \end{aligned}$$

Rearranging this inequality provides that

$$f(U) + f(V) - f(U \cup V) \geq f(S) + f(T) - f(S \cup T). \tag{2.1}$$

Since $f(X_1) + f(X_2) = f(E) + 1$, we have from (2.1) that

$$f(A_1 \cup X_2) \in \{f(A_1) + f(X_2), f(A_1) + f(X_2) - 1\},$$

with $f(A_2 \cup X_1)$ behaving similarly. If $f(A_1 \cup X_2) = f(A_1) + f(X_2)$, then another application of (2.1) shows that

$$f(A_1) + f(A_2) = f(A_1 \cup A_2).$$

From submodularity, we have that $f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E) \geq f(A_1 \cup A_2)$, and it follows that $f_3(A) = f(A_1) + f(A_2) = f(A)$, as desired. By symmetry, then, we have only to consider when $f(A_1 \cup X_2) = f(A_1) + f(X_2) - 1$ and $f(A_2 \cup X_1) = f(A_2) + f(X_1) - 1$. In

this case, we observe that

$$\begin{aligned}
f(A_1) + f(A_2) &= f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(X_1) - f(X_2) + 2 \\
&= f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) + 1 \\
&\geq f(E) + f(A) - f(E) + 1 \\
&= f(A) + 1.
\end{aligned}$$

From this, it follows that

$$f_3(A) = f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) = f(A_1) + f(A_2) - 1,$$

and with an application of (2.1), that

$$f(A_1) + f(A_2) = f(A) + 1.$$

Combining these equations yields that $f_3(A) = f(A)$ and the conclusion holds. □

Corollary 2.2.7. *A k-polymatroid $\mathcal{Q} = (E, f)$ is 3-connected if and only if for any partition (X, Y) of E with $f(X) + f(Y) = f(E) + 1$, either $|X| = 1$ or $|Y| = 1$.* □

From this, it is clear that a k-polymatroid $\mathcal{Q}$ is 3-connected if and only if $\mathcal{Q}^*$ is 3-connected. Our final result shows that 2-summing commutes for k-polymatroids. We omit the proof since it involves a routine, but tedious, exhaustive case-check.

Proposition 2.2.8. *For $i \in \{1, 2, 3\}$, let $\mathcal{Q}_i = (E_i, f_i)$ be a k-polymatroid for which $E_1 \cap E_2 = \{p_1\}$ and $E_2 \cap E_3 = \{p_2\}$ with $f_1(p_1) = f_2(p_1) = f_2(p_2) = f_3(p_2) = 1$. Then $\mathcal{Q}_1 \oplus_2 (\mathcal{Q}_2 \oplus_2 \mathcal{Q}_3) = (\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2) \oplus_2 \mathcal{Q}_3$.* □

2.3 Non-Essential Elements

Recall that an element e of a connected k-polymatroid $\mathcal{Q}$ is non-essential if either $\mathcal{Q} \backslash e$ or $\mathcal{Q}/e$ is connected. Tutte showed in [19] that every element of a connected matroid is

non-essential. We expand this result to k-polymatroids by determining the number of non-essential elements that are guaranteed to exist in any k-polymatroid. To do so, we make extensive use of the truncation operation defined in the previous section.

Lemma 2.3.1. *Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $e \in E$. Then $T_e(\mathcal{Q})$ is connected if and only if $\mathcal{Q}$ is connected with $f(e) > 1$.*

Proof. Let (A, B) be a partition of E with $e \in A$ and B nonempty. Suppose $T_e(\mathcal{Q})$ is connected. Then certainly $f(e) > 1$ or else e would be a loop in $T_e(\mathcal{Q})$. To show that $\mathcal{Q}$ is connected, observe that

$$\begin{aligned} f(A) + f(B) &= f_e(A) + f(B) + 1 \\ &\geq f_e(A) + f_e(B) + 1 \\ &> f_e(E) + 1 \\ &= f(E). \end{aligned}$$

We now assume that $\mathcal{Q}$ is connected with $f(e) > 1$. Then

$$f_e(A) + f_e(B) \geq f(A) + f(B) - 2 \geq f(E) - 1 = f_e(E),$$

and it thus suffices to consider the case when both $f_e(B) = f(B) - 1$ and $f(A) + f(B) = f(E) + 1$. From the first of these equations, we have $f(B \cup e) = f(B)$ and so, from the second equation, get $f(A) + f(B \cup e) = f(E) + 1$. It follows from submodularity that

$$f(E) + f(e) \leq f(A) + f(B \cup e) = f(E) + 1,$$

and therefore $f(e) \leq 1$. □

Lemma 2.3.2. *Let $\mathcal{Q} = (E, f)$ be a k-polymatroid with $e \in E$ and disjoint sets $C, D \subseteq E - e$ such that $f(C \cup e) > f(C)$. Then $T_e(\mathcal{Q})\backslash D/C = T_e(\mathcal{Q}\backslash D/C)$.*

Proof. Let $X \subseteq E - (C \cup D)$. It is straightforward to show that $f_e\backslash D/C(X) = (f\backslash D/C)_e(X)$. □

Theorem 2.3.3. *Every connected k-polymatroid having at least two elements has at least two non-essential elements.*

Proof. Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $|E| \geq 2$. We proceed by induction on the rank of $\mathcal{Q}$. If $f(E) = 0$, then $\mathcal{Q}$ is not connected and we are done. Thus we assume the theorem holds for polymatroids of rank less than that of $\mathcal{Q}$. If possible, choose $e \in E$ such that $f(E-e) < f(E)$. If each $e \in E$ satisfies $f(E-e) = f(E)$, then choose $e \in E$ such that $f(e) = \max\{f(x) : x \in E\}$. If $f(e) = 1$, then $\mathcal{Q}$ consists entirely of rank-1 elements and so consists entirely of non-essential elements by Proposition 2.2.3. Otherwise, we use Lemma 2.3.1 to see that $T_e(\mathcal{Q})$ is a connected k-polymatroid with at least two elements and rank one less than the rank of $\mathcal{Q}$. By induction, then, we may pick two elements $a, b \in E$ that are non-essential in $T_e(\mathcal{Q})$. By combining Lemmas 2.3.1 and 2.3.2, we note that if an element of $E - e$ is non-essential in $T_e(\mathcal{Q})$, then it is non-essential in $\mathcal{Q}$. Therefore we need only show that either e is non-essential in $\mathcal{Q}$, or there are two elements $x, y \in E - e$ that are non-essential in $T_e(\mathcal{Q})$. Clearly, if $a, b \in E - e$, then we are done. Thus assume that e is non-essential in $T_e(\mathcal{Q})$. If $f(E-e) < f(E)$, then it is not difficult to show that e is essential in $\mathcal{Q}$. Hence assume that $f(E - x) = f(E)$ for all $x \in E$. If $T_e(\mathcal{Q})/e$ is connected, then, as $T_e(\mathcal{Q})/e = \mathcal{Q}/e$, the theorem holds. Hence we may assume that $T_e(\mathcal{Q})\backslash e$ is connected. If $|E| = 2$, then the result is obvious and so $T_e(\mathcal{Q})\backslash e$ is a connected k-polymatroid with at least two elements. Let x and y be non-essential in $T_e(\mathcal{Q})\backslash e$.

If $T_e(\mathcal{Q})\backslash\{e, x\}$ is connected, then $T_e(\mathcal{Q})\backslash\{x\}$ is connected unless

$$f_e(e) + f_e(E - \{e, x\}) = f_e(E - x) = f_e(E). \tag{2.2}$$

In this case, suppose $(A \cup x, B)$ partitions $E - e$ non-trivially such that

$$f/e(A \cup x) + f/e(B) = f/e(E - e).$$

Then

$$f(A \cup \{e, x\}) + f(B \cup e) = f(E) + f(e). \tag{2.3}$$

Observe, however, that (2.2) implies that $f(e) + f(E - \{e, x\}) = f(E)$ and thus, since $B \subseteq E - \{e, x\}$, that $f(e) + f(B) = f(B \cup e)$. Applying this to (2.3) shows that $(A \cup \{e, x\}, B)$ is a 1-separation of $\mathcal{Q}$, a contradiction. It remains to consider the case when $T_e(\mathcal{Q})\backslash e/x$ is connected. By a similar argument to the above, we have that $(e, E - x)$ is the only possible 1-separation of $T_e(\mathcal{Q})/x$. If $\mathcal{Q}/x$ is connected, we are done. Thus assume that $(A \cup e, B)$ is a 1-separation of $\mathcal{Q}/x$. Since $(f/x)_e(A \cup e) = f/x(A \cup e) - 1$ and $(f/x)_e(E - x) = f/x(E - x) - 1$, it follows that

$$(f/x)_e(A \cup e) + f/x(B) = (f/x)_e(E - x). \tag{2.4}$$

Now, either $f/x(B) = (f/x)_e(B)$ or $f/x(B) = (f/x)_e(B) + 1$. Observe that if $f(\{x, e\}) = f(x)$, then $\mathcal{Q}\backslash e$ is connected and we are done. Thus $f(\{x, e\}) > f(x)$ and we have, from Lemma 2.3.2, that $T_e(\mathcal{Q}/x) = T_e(\mathcal{Q})/x$. Thus if $f/x(B) = (f/x)_e(B) + 1$, then (2.4) becomes

$$f_e/x(A \cup e) + f_e/x(B) = f_e/x(E - x) - 1,$$

contradicting the submodularity of $T_e(\mathcal{Q})/x$. On the other hand, if $f/x(B) = f_e/x(B)$, then

$$f_e/x(A \cup e) + f_e/x(B) = f_e/x(E - x).$$

As $(e, E - x)$ is the only possible 1-separation of $T_e(\mathcal{Q})/x$, it follows that $A = \varnothing$. Then, since $(A \cup e, B)$ is a 1-separation of $\mathcal{Q}/x$,

$$f/x(e) + f/x(E - \{e, x\}) = f/x(E - x).$$

Since $f/x(E - \{e, x\}) = f/x(E - x)$, it follows that $f(\{x, e\}) = f(e)$ and thus $\mathcal{Q}\backslash x$ is connected. $\square$

We now know that every connected k-polymatroid has at least two non-essential elements. The next example shows that this bound is sharp.

Example 2.3.4. Choose integers $k \geq 1$ and $n \geq 1$. Let E be a set with $|E| = k$ and choose distinct elements $a, b \notin E$. Take $M = (E \cup \{a, b\}, r)$ to be a matroid isomorphic to

$U_{1,k+1} \oplus U_{0,1}$ where b is the loop and $\mathcal{Q} = (E \cup \{a, b\}, f)$ to be a n-polymatroid isomorphic to $nU_{k,k+1} \oplus U_{0,1}$ where a is the loop. Then the $(n+1)$-polymatroid $M + \mathcal{Q}$ has a and b as its only non-essential elements. If $n = 1$, then we denote $M + \mathcal{Q}$ by $\mathcal{S}_k$ for each k. The 2-polymatroid $\mathcal{S}_k$ is shown geometrically in Figure 2.1 for $k \in \{1, 2, 3\}$.

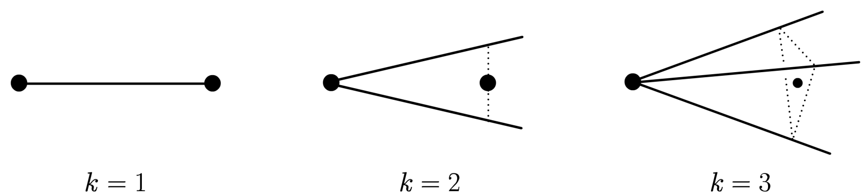

Figure 2.1: $\mathcal{S}_k$ for $k \in \{1, 2, 3\}$

Lemma 2.3.5. *If $\mathcal{Q} = (E, f)$ is a connected 2-polymatroid and, for some $e \in E$, both $f \backslash e$ and f/e are not connected, then $f(E) = f(E - e)$, $f(e) = 2$, and $f(X) + f(E - X) - f(E) = 1$ for some set $X \subsetneq E$.*

Proof. This is an immediate consequence of Lemma 2.2.2. □

If $\mathcal{Q} = (E, f)$ is a 2-polymatroid and $x \in E$ such that $f(E - x) = f(E) - 1$, then $f^*(e) = 1$ and we say that e is a *copoint*. In the following theorem, we show that the polymatroids given in Example 2.3.4 when $n = 1$ are the only 3-connected 2-polymatroids with exactly 2 non-essential elements and no copoints. After obtaining this result, it is not difficult to remove the no-copoints requirement, which is done in Corollary 2.3.7.

Theorem 2.3.6. *If $\mathcal{Q}$ is a 3-connected 2-polymatroid with at least three elements, no copoints, and exactly two non-essential elements, then $\mathcal{Q}$ is isomorphic to $\mathcal{S}_k$ for some k.*

Proof. Let $\mathcal{Q} = (E, f)$ be a 3-connected 2-polymatroid with an essential element a. Choose a nontrivial partition (X, Y) of $E - a$ with $|X|$ maximal such that $f(X \cup a) + f(Y \cup a) = f(E) + 2$. A partition of this type is a 1-separation of $\mathcal{Q}/a$ and thus exists. Similarly, choose a partition

(A, B) of $E - a$ with $|A|$ maximal such that $f(A) + f(B) = f(E)$. Then

$$\begin{aligned} 2f(E) + 2 &= f(A) + f(B) + f(X \cup a) + f(Y \cup a) \\ &\geq f(A \cup X \cup a) + f(B \cap Y) + f(B \cup Y \cup a) + f(A \cap X) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} 2f(E) + 2 &= f(A) + f(B) + f(X \cup a) + f(Y \cup a) \\ &\geq f(A \cup Y \cup a) + f(B \cap X) + f(B \cup X \cup a) + f(A \cap Y). \end{aligned} \tag{2.6}$$

Since $\mathcal{Q}$ is 3-connected, we get from (2.5) that at least one of $|A \cap X|$ and $|B \cap Y|$ is less than 2. In fact, for some $k \geq 1$,

$$(|A \cap X|, |B \cap Y|) \in \{(0, k), (0, 0), (1, 1), (k, 0)\}.$$

If $|A \cap X| = 0$ and $|B \cap Y|$ is nonzero, then (2.6) tells us that, since neither $A \cap Y$ nor $B \cap X$ may be empty, both must be singletons. Thus $A = (X \cap A) \cup (Y \cap A) = A \cap Y$ and A is a singleton. However, B then contains at least two elements, contradicting the maximality of A.

Next, we assume both $A \cap X$ and $B \cap Y$ are empty. Again, from (2.6), we get that $|A \cap Y| = |B \cap X| = 1$. Let $x \in B \cap X$ and $y \in A \cap Y$. Since $f(A \cup Y \cup a) + f(B \cap X) = f(E) + 1$, we have that $f(\{a, y\}) + f(x) = f(E) + 1$. As $\mathcal{Q}$ has no copoints, it follows that $f(x) = 1$ and similarly that $f(y) = 1$. It follows, since $f(A) + f(B) = f(E)$, that $f(E) = 2$, so $f(\{a, y\}) = f(\{a, x\}) = 2$. If $f(\{x, y\}) = 1$, then $f(\{x, y\}) + f(a) = f(E) + 1$, which is impossible since $f(a) = 2$. Therefore, $\mathcal{Q} \cong \mathcal{S}_1$.

Now, we assume $|A \cap X| = |B \cap Y| = 1$ and let $A \cap X = \{x\}$; $B \cap Y = \{y\}$. From (2.6) and the maximality of A and X, we have $|A \cap Y| = |B \cap X| \leq 1$. If $|A \cap Y| = |B \cap X| = 0$, then, similarly to the previous case, we have that $\mathcal{Q} \cong \mathcal{S}_1$. We thus assume $|A \cap Y| = |B \cap X| = 1$ and let $\{w\} = B \cap X$ and $\{z\} = A \cap Y$. From this, we may use (2.5) and (2.6) to get $f(w) = f(z) = f(x) = f(y) = 1$. As rank-1 elements are always non-essential, this contradicts that $\mathcal{Q}$ has exactly two non-essential elements.

Finally, we consider the case when $|B \cap Y| = 0$ and $A \cap X$ is nonempty. Arguing as above, we find that each of $B \cap X$ and $A \cap Y$ consists of a single rank-one element, which we call x and y, respectively. Using (2.5), (2.6), and the 3-connectedness of $\mathcal{Q}$, we are able to find that $f(\{a, y\}) = 2$, $f(\{a, x, y\}) = 3$, $f(E - \{a, x\}) = f(E) - 1$, $f(E - \{a, y\}) = f(E)$, $f(E - \{a, x, y\}) = f(E) - 1$, and $f(\{a, x\}) = 3$. Indeed, as x and y are points, they are the sole non-essential elements of $\mathcal{Q}$. Thus we may choose $b \in E - \{a, x, y\}$ and note that b must be essential. If we repeat the previous steps of this proof using b instead of a, we come to the conclusion that b satisfies $f(\{b, y\}) = 2$, $f(\{b, x, y\}) = 3$, $f(E - \{b, x\}) = f(E) - 1$, $f(E - \{b, y\}) = f(E)$, $f(E - \{b, x, y\}) = f(E) - 1$, and $f(\{b, x\}) = 3$. As b was chosen arbitrarily, we have that these equations are satisfied for all $p \in E - \{x, y\}$.

Since, for each $p \in E - \{x, y\}$, we have that $f(\{p, y\}) = 2$, it follows that $f(E - x) \leq |E| - 1$ and thus $f(E) \leq |E| - 1$. If possible, choose a minimal set $P \subseteq E - \{x, y\}$ for which $f(P) \leq |P|$ and let $b \in P$. By the minimality of P, we have $f(P - b) \geq |P - b| + 1 = |P| \geq f(P)$ and thus $f(P - b) = f(P)$. Recall, however, that $f(x) + f(E - \{b, x\}) = f(E)$. Since $P - b \subseteq E - \{b, x\}$, it follows that $f(E - x) = f(E - \{b, x\})$, contradicting the connectivity of $\mathcal{Q}$. Therefore, for all $L \subseteq E - \{x, y\}$, we have $f(L) = |L| + 1$ and thus $f(E) = |E| - 1$. It then follows that $\mathcal{Q} \cong \mathcal{S}_{|E|-2}$, as desired. □

In a connected 2-polymatroid $\mathcal{Q} = (E, f)$, if $x \in E$ has $f(x) = 1$, then we may turn x into a copoint by defining $\mathcal{Q}^e = (E, f^e)$ where, for all $X \subseteq E$, we have

$$f^e(X) = \begin{cases} f(X) + 1 & \text{if } e \in X; \\ f(X) & \text{otherwise.} \end{cases}$$

We call this operation *element expansion.*

Corollary 2.3.7. *Every 3-connected 2-polymatroid on at least three elements with exactly two non-essential elements can be obtained from some $\mathcal{S}_n$ by performing a sequence of element expansions.*

Proof. Suppose $\mathcal{Q} = (E, f)$ is such a 2-polymatroid having $\{x_1, x_2, \ldots, x_n\}$ as its set of copoints. Let $\mathcal{R} = T_{x_1}(T_{x_2}(\cdots T_{x_n}(\mathcal{Q}))\cdots)$. It is not difficult to check that $\mathcal{R}$ is 3-connected and we can use Lemmas 2.3.1 and 2.3.2 to see that $\mathcal{R}$ has exactly two non-essential elements. From Theorem 2.3.6, we have that $\mathcal{R}$ is isomorphic to $\mathcal{S}_n$ for some n. The conclusion follows. □

We conclude by characterizing all those 2-polymatroids with exactly two non-essential elements. The following proposition will be helpful to this end.

Proposition 2.3.8. *Let $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ be connected k-polymatroids such that $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p) = 1$. An element x in $(E_1 \cup E_2) - p$ is non-essential in either $\mathcal{Q}_1$ or $\mathcal{Q}_2$ if and only if x is non-essential in $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$.*

Proof. From Proposition 2.2.1,

$$(\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2)\backslash x = P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash\{x, p\} = P(\mathcal{Q}_1\backslash x, \mathcal{Q}_2)\backslash p = (\mathcal{Q}_1\backslash x) \oplus_2 \mathcal{Q}_2.$$

Similarly, $(\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2)/x = (\mathcal{Q}_1/x) \oplus_2 \mathcal{Q}_2$. By combining these equations with Corollary 2.2.5, we obtain the proposition. □

The connected 2-polymatroids with exactly two non-essential elements consist of the members of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$ along with paths of 2-sums of such 2-polymatroids where the base-points of the 2-sums are non-essential in both summands.

Theorem 2.3.9. *Let $\mathcal{Q}$ be a connected 2-polymatroid with at least three elements. Then $\mathcal{Q}$ has exactly two non-essential elements if and only if, for some $n \geq 1$, there is a sequence $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ of 2-polymatroids such that*

(i) each $\mathcal{Q}_i$ is isomorphic to some member of $\{U_{1,2} + U_{1,1}, \mathcal{S}_1, \mathcal{S}_2, \ldots\}$;

(ii) if either $n = 1$ or $2 \leq i \leq n-1$, then $\mathcal{Q}_i$ is isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$;

(iii) the ground sets of $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ are disjoint except that, for each i in $\{1, 2, \ldots, n-1\}$, the sets $E(\mathcal{Q}_i)$ and $E(\mathcal{Q}_{i+1})$ meet in a single rank-1 element; and

(iv) $\mathcal{Q} \cong \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 \oplus_2 ... \oplus_2 \mathcal{Q}_n$.

Proof. If we have a sequence satisfying the four conditions, Proposition 2.3.8 implies that $\mathcal{Q}$ has exactly two non-essential elements. For the converse, we proceed by induction on the rank of E. If $f(E) = 1$, then, since $|E| > 2$ and $\mathcal{Q}$ is connected, it follows that $\mathcal{Q}$ consists of $|E|$ points, each of which must be non-essential, a contradiction. Thus assume $f(E) > 1$ and that the conclusion holds for 2-polymatroids of rank less than $f(E)$. If $\mathcal{Q}$ is 3-connected, then, from Corollary 2.3.7, there are three possibilities: $n = 1$ with $\mathcal{Q}_1$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$; $n = 2$ with $\mathcal{Q}_1$ isomorphic to $U_{1,2} + U_{1,1}$ and $\mathcal{Q}_2$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$; or $n = 3$ with both $\mathcal{Q}_1$ and $\mathcal{Q}_3$ isomorphic to $U_{1,2} + U_{1,1}$ and $\mathcal{Q}_2$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$. We thus assume that $\mathcal{Q}$ is not 3-connected. Choose a non-trivial partition (X, Y) of E such that $f(X) + f(Y) = f(E) + 1$ and $2 \leq |X| \leq |Y|$. If $f(X) = 1$, then each member of X is a point and is thus non-essential. As $\mathcal{Q}$ has exactly two non-essential elements, X consists of two points which are necessarily parallel. However, $\mathcal{Q} \backslash x$, where $x \in X$, is connected with two non-essential elements. Clearly these two non-essential elements are also non-essential in $\mathcal{Q}$, a contradiction. Therefore $f(X) > 1$ and thus $f(Y) < f(E)$. Similarly, $f(X) < f(E)$. We now use Proposition 2.2.6 to choose 2-polymatroids $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on ground sets $X \cup p$ and $Y \cup p$, respectively, where p is a point not in E and $\mathcal{Q} = \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$. Moreover, the ranks of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are each less than that of $\mathcal{Q}$. If x is a non-essential element of $\mathcal{Q}_1$ that meets E, then, by using Proposition 2.3.8, x is a non-essential element of $\mathcal{Q}$. Thus each of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ has p as a non-essential element and has exactly one other non-essential element. By induction, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy the four conditions in the theorem. It follows immediately that the 2-sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, that is $\mathcal{Q}$, satisfies the four conditions. □

Chapter 3
Unavoidable Minors for Connected k-polymatroids

3.1 Introduction

The core principle of Ramsey theory is that sufficiently large objects within some class must exhibit some structure. Ramsey's seminal result of this type (see [18]) may be stated as follows.

Theorem 3.1.1. *For all positive integers n, there is an integer r such that every graph of order at least r contains as an induced subgraph either the complete graph on n vertices or n independent vertices.*

Another well-known graph-theoretic result of this type (see, for example, [3]), is the following theorem, which will be used in the proof of the main result.

Theorem 3.1.2. *For all positive integers n, there is an integer r such that every connected graph of order at least r contains K_n, $K_{1,n}$, or P_r as an induced subgraph.*

At the 1992 Seattle Graph Minors Conference, Robin Thomas informally asked for an analogue of Theorem 3.1.2 for connected matroids. This was quickly found by Lovász, Schrijver, and Seymour, who proved the following result.

Theorem 3.1.3. *For all positive integers n, there is an integer r such that every connected matroid with at least r elements has a minor isomorphic to either $U_{1,n}$ or $U_{n-1,n}$.*

In this chapter, we prove analogues of Theorem 3.1.1 and Theorem 3.1.3 for k-polymatroids and 2-polymatroids, respectively. These, the main results of this chapter, are proved in Section 3.3; and several technical lemmas needed for the main results are proved in Section 3.2. Much of the work here appears in [10].

The following proposition, which will be used later, shows an important property of separators and minors.

Proposition 3.1.4. *Let $\mathcal{Q} = (E, f)$ be a k-polymatroid with separator A and let $\mathcal{R}$ be a minor of $\mathcal{Q}$ with separator X. Then $A \cap X$ is a separator of $\mathcal{R}$.*

Proof. Let g be the rank function of $\mathcal{R}$ and choose disjoint sets C and D of E such that $\mathcal{R} = \mathcal{Q}\backslash D/C$. Set $B = E - A$ and $Y = E - (X \cup C \cup D)$. Since $\lambda_g(X) = 0$, it follows that $f/C(X) + f/C(Y) = f/C(X \cup Y)$ and thus $f(C \cup X) + f(C \cup Y) = f(C \cup X \cup Y) + f(C)$. Since $(C \cup X) \cap A \subseteq A$ and $(C \cup X) \cap B \subseteq B$, Lemma 1.8.1 shows that

$$0 \leq \sqcap_f((C \cup X) \cap A, (C \cup X) \cap B)) \leq \sqcap_f(A, B) = 0.$$

By also applying this technique to Y, we have

$$\begin{aligned} f((C \cup X) \cap A) + f((C \cup X) \cap B) + f((C \cup Y) \cap A) &+ f((C \cup Y) \cap B) \\ &= f(X \cup Y \cup C) + f(C). \end{aligned}$$

Using the same method, we obtain

$$f((C \cup X) \cap A) + f(C \cap B) = f(C \cup (X \cap A)).$$

Thus

$$\begin{aligned} f(C \cup (X \cap A)) + f(C \cup (X \cap B)) + f(C \cup (Y \cap A)) &+ f(C \cup (Y \cap B)) \\ &= f(X \cup Y \cup C) + 3f(C), \end{aligned}$$

from which it follows that

$$f/C(X \cap A) + f/C(X \cap B) + f/C(Y \cap A) + f/C(Y \cap B) = f/C(X \cup Y).$$

Therefore, $\lambda_g(X \cap A) = 0$. $\square$

We now give the definition of 2-sum, but only in the case when the elements to be connected are of the same rank. The reader is referred to [9] for a more thorough treatment. The 2-sum for polymatroids is a generalization of that for matroids in that it consists of sticking together two polymatroids as freely as possible across a designated element of each, followed by a deletion of the element. Below, we give a formal definition that mimics the language of 2-sum for matroids.

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids with $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p)$. Let $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 = ((E_1 \cup E_2) - p, f)$ where, for all $A \subseteq (E_1 \cup E_2) - p$, if $A_1 = A \cap E_1$ and $A_2 = A \cap E_2$, then

$$f(A) = \min\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\}.$$

A routine check shows that $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$ is a k-polymatroid. We call this k-polymatroid the 2-*sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to the *basepoint* p.

Now, we look at a concept of polymatroids that is somewhat similar to fundamental circuits in matroids. If $\mathcal{Q} = (E, f)$ is a k-polymatroid and $S \subseteq E$ is loopless with

$$f(S) = \sum_{s \in S} f(s),$$

then we call S a *matching*. Given such a matching S and $X \subseteq E - S$, we denote, by $C(X, S)$, the minimal subset C of S for which

$$f(S \cup X) = f(C \cup X) + f(S - C).$$

It is not difficult to show that $C(X, S)$ exists and is well-defined for any matching S and set X disjoint from S. If $\mathcal{Q}$ is a matroid, S is a basis of $\mathcal{Q}$, and $x \in E$, then $C(x, S)$ is the familiar fundamental circuit of x in the basis S. The following proposition concerning $C(X, S)$ will be useful throughout this chapter.

Proposition 3.1.5. *Let $\mathcal{Q} = (E, f)$ be a k-polymatroid with a matching S and $A, B \subseteq E - S$. If $A \subseteq B$, then $C(A, S) \subseteq C(B, S)$. If $A \cap B = \varnothing$ and $f/S(A) + f/S(B) = f/S(A \cup B)$, then $C(A \cup B, S) = C(A, S) \cup C(B, S)$.*

Proof. Set $S_A = C(A, S)$, $S_B = C(B, S)$, and suppose first that $A \subseteq B$. Using Lemma 1.8.1 (ii), we observe that

$$0 \leq \sqcap(A \cup S_B, S - S_B) \leq \sqcap(B \cup S_B, S - S_B) = 0.$$

Thus

$$0 = \sqcap(A \cup S_B, S - S_B) = f(A \cup S_B) + f(S - S_B) - f(A \cup S).$$

Since S_A is the unique minimal subset of S with the property that $f(A \cup S_A) + f(S - S_A) - f(A \cup S) = 0$, it follows that $S_A \subseteq S_B$.

Next, we assume $A \cap B = \varnothing$ and $f(S \cup A) + f(S \cup B) = f(S \cup A \cup B) + f(S)$, and set $S_{A \cup B} = C(A \cup B, S)$. Note that, from the above argument, we have $S_A \subseteq S_{A \cup B}$ and $S_B \subseteq S_{A \cup B}$, and thus need only show

$$f(S \cup A \cup B) = f(S_A \cup S_B \cup A \cup B) + f(S - (S_A \cup S_B)).$$

For this, we observe that

$$\begin{aligned}
f(S_A \cup A) &+ f(S_B \cup B) + f(S) - f(S_A) - f(S_B) \\
&= f(S_A \cup A) + f(S_B \cup B) + f(S - S_A) + f(S - S_B) - f(S) \\
&= f(S \cup A) + f(S \cup B) - f(S) \\
&= f(S \cup A \cup B) \\
&\leq f(S_A \cup S_B \cup A \cup B) + f(S - (S_A \cup S_B)) \\
&\leq f(S_A \cup A) + f(S_B \cup B) + f(S - (S_A \cup S_B)) - f(S_A \cap S_B) \\
&= f(S_A \cup A) + f(S_B \cup B) + f(S) - f(S_A \cup S_B) - f(S_A \cap S_B) \\
&= f(S_A \cup A) + f(S_B \cup B) + f(S) - f(S_A) - f(S_B),
\end{aligned}$$

from which our conclusion follows. □

3.2 Some Lemmas

In this section, we prove the more technical lemmas needed in the proofs of the main results, which appear in Section 3.3.

Our first lemma, in which we identify the minors required to exist within a k-polymatroid of sufficiently high rank, provides much of the work needed for Theorem 3.3.1.

Lemma 3.2.1. *For all positive integers n, there is an integer r such that every k-polymatroid with rank at least r has a minor isomorphic to one of $U_{n,n}$; $2U_{n,n}$; $\ldots$; or $kU_{n,n}$.*

Proof. Choose $n \in \mathbb{N}$ and let $\mathcal{Q} = (E, f)$ be a k-polymatroid. We show that if $r = (2^{k+1} - k - 2)n - (2^k - 2)$, then the theorem is satisfied. If $k = 1$, then $\mathcal{Q}$ is a matroid of rank $r = n$. A restriction of $\mathcal{Q}$ to any basis gives a matroid isomorphic to $U_{n,n}$, as desired. We thus assume the conclusion holds for every j-polymatroid where $j < k$.

Let $P = \{x \in E : f(x) = k\}$ and $I = E - P$. If $P = \varnothing$, then we are done by induction. In fact, if $f(I) \geq (2^{(k-1)+1} - (k-1) - 2)n - 2^{k-1} + 2$, then $\mathcal{Q}|I$ is a $(k-1)$-polymatroid with sufficient rank and we are, again, done by induction. We thus assume $f(I) \leq (2^k - k - 1)n - 2^{k-1} + 1$ and note that

$$\begin{aligned} f(P) = f(E - I) &\geq f(E) - f(I) \\ &\geq [(2^{k+1} - k - 2)n - (2^k - 2)] - [(2^k - k - 1)n - 2^{k-1} + 1] \\ &= (2^k - 1)n - 2^{k-1} + 1. \end{aligned}$$

Let S be a maximal set of lines in P for which $f(S) = k|S|$. If $|S| \geq n$, then, as $\mathcal{Q}|S \cong kU_{n,n}$, we are done. Otherwise,

$$\begin{aligned} f/S(P - S) &= f(P) - f(S) \\ &\geq (2^k - 1)n - 2^{k-1} + 1 - (kn - 1) \\ &= (2^{(k-1)+1} - (k-1) - 2)n - 2^{k-1} + 2 \end{aligned}$$

and, for any $x \in E - S$, we have

$$\begin{aligned} f/S(x) &= f(S \cup x) - f(S) \\ &\leq f(S) + (k-1) - f(S) \\ &= k - 1. \end{aligned}$$

Thus $\mathcal{Q}/S\backslash I$ is a $(k-1)$-polymatroid of sufficiently high rank and, by the induction hypothesis, we are done. $\square$

We are now ready to classify the unavoidable minors for connected k-polymatroids. Here, we give an outline of the proof before providing the details in Theorem 3.3.2. We start with a connected k-polymatroid $\mathcal{Q} = (E, f)$, where $|E|$ is large, and use induction on k. We thus wish to show that either $\mathcal{Q}$ has a desired minor, or that $\mathcal{Q}$ has a large connected $(k-1)$-polymatroid minor.

To this end, we choose a maximal matching S of $\mathcal{Q}$. Note that every element in $\mathcal{Q}/S$ has rank at most $k-1$. If $\mathcal{Q}/S$ has any large connected component, then this component is a large connected $(k-1)$-polymatroid and we are done. We thus assume $\mathcal{Q}/S$ has no such connected component. It follows that $\mathcal{Q}/S$ has a large number of connected components, which we label as $X_1, X_2, \ldots, X_n$.

Next, we construct a bipartite graph with the vertices on one side labeled by elements of S and the vertices on the other side labeled by the X_i. Two vertices are adjacent if and only if one represents an element s of S, one represents a component X_i, and $s \in C(X_i, S)$. This graph turns out to be a large connected bipartite graph and thus has, by Theorem 3.1.2, either a long path or a large star ($K_{1,n}$) as an induced minor. If our graph has a large star as an induced minor, we have two cases. First, if the vertex of high degree is labeled by a component X_i, this translates into $|C(X_i, S)|$ being very large. We use Lemmas 3.2.2, 3.2.3, and 3.2.8, together with Corollary 3.2.9 to show that either $\mathcal{Q}$ has one of the desired minors, or $\mathcal{Q}$ has a large $(k-1)$-polymatroid as a minor.

Lemma 3.2.2. *If $\mathcal{Q} = (E, f)$ is a k-polymatroid with matching S and non-empty subset X of $E - S$ for which $(\mathcal{Q}/S)|_X$ is connected, then $\mathcal{Q}|_{X \cup C(X,S)}$ is connected.*

Proof. Set $S_X = C(X, S)$ and let X_A, X_B, S_A, S_B be pairwise-disjoint subsets of E such that $X_A \cup X_B = X$; $S_A \cup S_B = S_X$; and $f(X_A \cup S_A) + f(X_B \cup S_B) = f(X \cup S_X)$. It suffices to show

that either $X_A \cup S_A$ or $X_B \cup S_B$ is empty. If X_B is empty, then $f(X_A \cup S_A) + f(S_B) = f(X \cup S_X)$ tells us that

$$\begin{aligned} f(X \cup S) &= f(X \cup S_X) + f(S - S_X) \\ &= f(X \cup S_A) + f(S_B) + f(S - S_X) \\ &= f(X \cup S_A) + f(S - S_A), \end{aligned}$$

which would contradict the minimality of S_X unless S_B was empty. Similarly, if X_A is empty, then so is S_A. We thus assume that both X_A and X_B are non-empty. In this case, we get, from the connectivity of $(\mathcal{Q}/S)|_X$, that $f(X_A \cup S) + f(X_B \cup S) > f(X \cup S) + f(S)$. Further, since $f(X \cup S) = f(X \cup S_X) + f(S - S_X)$, we have that $f(X_A \cup S) = f(X_A \cup S_X) + f(S - S_X)$ and $f(X_B \cup S) = f(X_B \cup S_X) + f(S - S_X)$. Combining this information and performing cancellations gives us that

$$\begin{aligned} f(X_A \cup S_X) + f(X_B \cup S_X) &= f(X_A \cup S) + f(X_B \cup S) - 2f(S - S_X) \\ &> f(X \cup S) + f(S) - 2f(S - S_X) \\ &= f(X \cup S_X) + f(S_X) \\ &= f(X_A \cup S_A) + f(X_B \cup S_B) + f(S_A) + f(S_B) \\ &\geq f(X_A \cup S_X) + f(X_B \cup S_X), \end{aligned}$$

a contradiction. The theorem follows. □

Lemma 3.2.3. *If $m \in \mathbb{N}$ and a connected k-polymatroid $\mathcal{Q} = (E, f)$ has a matching S of size at least $(m+1)|E-S|^{2|E-S|}$, then $\mathcal{Q}$ has a connected minor with a matching $T \subseteq S$ and element $x \in E - S$ for which $|C(x, T)| > m$.*

Proof. Set $X = E - S$ and $S_x = C(x, S)$ for each $x \in X$. If $|X| = 1$, then $C(X, S) = S$ by connectivity. Since $|S| \geq (m+1)|X|^{2|X|} = m + 1 > m$, we are done. We thus assume that $|X| > 1$ and that the conclusion holds for all k-polymatroids having a matching such

that the complement of the matching is of cardinality less than $|X|$. Choose $x \in X$ and suppose $\mathcal{Q}/(S_x \cup x)$ has at least $|X|$ nonempty connected components. We may then find a component C of $\mathcal{Q}/(S_x \cup x)$ which consists only of elements in S. However, this would mean that

$$\begin{aligned} f(E) &= f/_{x \cup S_x}(E - (x \cup S_x)) + f(x \cup S_x) \\ &= f/_{x \cup S_x}(C) + f/_{x \cup S_x}(E - (C \cup x \cup S_x)) + f(x \cup S_x) \\ &= f(C \cup S_x \cup x) + f(E - C) - f(x \cup S_x) \\ &= f(C) + f(S_x \cup x) + f(E - C) - f(x \cup S_x) \\ &= f(C) + f(E - C), \end{aligned}$$

which would contradict the connectivity of $\mathcal{Q}$. Thus $\mathcal{Q}/(S_x \cup x)$ has fewer than $|X|$ connected components. It follows, then, that there must be some component R which has at least

$$\begin{aligned} \frac{|E| - |S_x| - 1}{|X| - 1} &\geq \frac{|E| - |S_x|}{|X|} - 1 \\ &= \frac{|S|}{|X|} + \frac{|X|}{|X|} - \frac{|S_x|}{|X|} - 1 \\ &\geq \frac{(m+1)|X|^{2|X|}}{|X|} - \frac{m}{|X|} \\ &= (m+1)|X|^{2|X|-1} - \frac{m}{|X|} \\ &= |X|(m+1)|X|^{2(|X|-1)} - \frac{m}{|X|} \\ &\geq (m+1)|X|^{2(|X|-1)} + (m+1)|X| - \frac{m}{|X|} \\ &\geq (m+1)|X|^{2(|X|-1)} + |X| \end{aligned}$$

elements. Since $|R| \geq (m+1)|X|^{2(|X|-1)} + |X|$, we have that

$$|R \cap S| \geq (m+1)|X|^{2(|X|-1)} \geq (m+1)|R \cap X|^{2(|R \cap X|-1)}.$$

As $R \cap S$ is a matching of $\mathcal{Q}/(S_x \cup x)$ and $|R - S| < |X|$, it follows by induction that $\mathcal{Q}/(S_x \cup x)$ has a connected minor with a matching $T \subseteq S$ and element $x \in E - S$ for which $|C(x,T)| > m$. The theorem follows. $\square$

Next, we consider the case when the vertex of high degree is labeled by an element s of S. This means that a large number of the X_i have $s \in C(X_i, S)$. Using Lemmas 3.2.4–3.2.6, we show that $\mathcal{Q}$ has either a desired minor or a large connected $(k-1)$-polymatroid minor.

Lemma 3.2.4. *Choose a positive integer n and let $\mathcal{Q} = (E, f)$ be a loopless k-polymatroid with $f(E) = k$ and $|E| \geq kn$. If there is an element e in E for which $f(e) = k$, then $\mathcal{Q}$ has a $pU_{1,n}$-minor for some p in $\{1, \dots, k\}$.*

Proof. We proceed by induction on k. If $k = 1$, then $\mathcal{Q}$ is a loopless rank-1 matroid on n elements. Thus the only possibility is $U_{1,n}$ and we are done. We then assume that the theorem holds for any j-polymatroid with $j < k$. Let a be an element in $\mathcal{Q}$ of minimal rank. Let A be a maximal subset of $E - a$ for which $f(A \cup a) = f(a)$. Necessarily, every element of A is of the same rank as a. It follows, then, that $\mathcal{Q}|(A \cup a) \cong f(a)U_{1,|A|+1}$. If $|A| \geq n - 1$, then we are done and thus assume $|A| < n - 1$. In this case, observe that $\mathcal{Q}\backslash A/a$ has no loops and satisfies

$$f/a(E - (A \cup a)) = f(E - A) - f(a) = k - f(a).$$

Further, since $f/a(e) = f(\{a, e\}) - f(a) = k - f(a)$ and $|E - (A \cup a)| \geq kn - n = (k-1)n \geq f/a(e)n$, it follows, by the induction hypothesis, that $\mathcal{Q}\backslash A/a$ has a $pU_{1,n}$-minor for some p in $\{1, \dots, f/a(e)\} \subseteq \{1, \dots, k\}$ and we are done. $\square$

Lemma 3.2.5. *Let $\mathcal{Q} = (E, f)$ be a k-polymatroid with matching S. Let the connected components of $\mathcal{Q}/S$ be $X_1, \dots, X_n$. If $s \in S$ such that $s \in C(X_i, S)$ for each $i \in \{1, \dots, n\}$, then $\mathcal{Q}$ has a connected minor with at least n elements and $\{s\}$ as a maximal matching.*

Proof. For each $i \in \{1, \ldots, n\}$, let K_i be a maximal, possibly empty, subset of X_i such that $s \notin C(K_i, S)$. For each i, choose $x_i \in X_i - K_i$. Set $K = K_1 \cup K_2 \cup \cdots \cup K_n$ and $X = \{x_1, x_2, \ldots, x_n\}$. We show that $\mathcal{R} = (\mathcal{Q}/((S - s) \cup K))|(X \cup s)$ is a connected minor with $\{s\}$ as a maximal matching.

Choose $x \in X$ and observe

$$\begin{aligned}
f_{\mathcal{R}}(\{s, x\}) &= f(S \cup K \cup x) - f(K \cup (S - s)) \\
&< f((S - s) \cup K \cup x) + f(s) - f(K \cup (S - s)) \\
&= f_{\mathcal{R}}(x) + f(s) \\
&= f_{\mathcal{R}}(x) + f_{\mathcal{R}}(s),
\end{aligned}$$

where the inequality comes from the maximality of K. From this, it follows that $\mathcal{R}$ has no loops and that each element of X intersects s. The theorem follows. $\square$

Lemma 3.2.6. *Choose an integer $n \geq 1$ and let $\mathcal{Q} = (E, f)$ be a k-polymatroid with maximal matching $\{s\}$. If $|E| \geq k^2n^2$, then either $\mathcal{Q}$ has a minor isomorphic to $aU_{1,n} + bU_{n,n}$, where $a \geq 1$ and $k \geq a + b \geq 1$, or $\mathcal{Q}/s$ has a connected component of size greater than n.*

Proof. We assume that $\mathcal{Q}/s$ has no connected component of size n and choose representatives $x_1, \ldots, x_m$ from each of these components. Since $|E| \geq k^2n^2$, it follows that $m \geq k^2n$. For each $i \in \{0, \ldots, k-1\}$, set

$$X_i = \{x_j : f/s(x_j) = i\}.$$

Since $\sum |X_i| = m \geq k^2n$, we are able to find $b \in \{0, \ldots, k-1\}$ for which $|X_b| \geq kn$. Let $\mathcal{R} = (s \cup X_b, g)$ where g is defined, for all $U \subseteq s \cup X_b$, by $g(U) = \sqcap_f(s, U)$. Using that $f/s(X_b) = b|X_b|$, it is not difficult to check that $\mathcal{R}$ is a k-polymatroid satisfying the hypotheses of Lemma 3.2.4. We may thus find subsets D and C of $s \cup X_b$ for which $\mathcal{R}\backslash D/C \cong aU_{1,n}$ for some a in $\{1, \ldots, k\}$. A quick check reveals that $\mathcal{Q}|(s \cup X_b) = \mathcal{R} + (\mathcal{Q}/s)|X_b$ and, further, that

$$(\mathcal{Q}|(s \cup X_b))\backslash D/C = \mathcal{R}\backslash D/C + ((\mathcal{Q}/s)|X_b)\backslash D/C \cong aU_{1,n} + bU_{n,n}.$$

$\square$

This result from [9] will be used in the following lemma.

Proposition 3.2.7. *Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $e \in E$. Then $T_e(\mathcal{Q})$ is connected if and only if $\mathcal{Q}$ is connected with $f(e) > 1$.*

Lemma 3.2.8. *Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $x \in E$ such that $f(E-x) = j|E-x|$ for some $j \leq k$. Choose n to be a positive integer and suppose $|E| \geq 2kn$. If $f(x) > j$ and $f(E-x) = f(E)$, then $\mathcal{Q}$ has a connected $(f(x)-1)$-polymatroid on at least n elements. Otherwise, $\mathcal{Q}$ has $aU_{n-1,n} + bU_{n,n}$ as a minor, for some integers $a \geq 1$ and $b \geq 0$ such that $a + b \leq k$.*

Proof. Suppose first that $f(E-x) = f(E)$. If $f(x) > j$, then observe that if $\mathcal{Q}/x$ is connected, then $\mathcal{Q}/x$ is a j-polymatroid with $j \leq f(x) - 1$ and we are done. If $\mathcal{Q}/x$ is disconnected, let (A, B) be a 1-separation of $\mathcal{Q}/x$. Then

$$f(A \cup x) + f(B \cup x) = f(E) + f(x).$$

If $f(A \cup x) = f(A) + f(x)$, then the above becomes $f(A) + f(B \cup x) = f(E)$, contradicting the connectivity of $\mathcal{Q}$. Thus $f(A \cup x) < f(A) + f(x)$. Similarly, $f(B \cup x) < f(B) + f(x)$. Assume, without loss of generality, that $|A| \leq |B|$. Then $|B| \geq kn \geq n$. Since $f(A \cup x) < f(A) + f(x)$, it follows that $\mathcal{Q}/A$ is an $(f(x)-1)$-polymatroid, and we have only to show that $\mathcal{Q}/A$ is connected.

Suppose not and let $(U \cup x, V)$ be a 1-separation of $\mathcal{Q}/A$. Then

$$f(A \cup U \cup x) + f(A \cup V) = f(E) + f(A).$$

Since $A \cup V \subseteq E - x$, and $E - x$ is a matching, it follows that $A \cup V$ is a matching. Thus $f(A \cup V) = f(A) + f(V)$. From this, we have that $f(A \cup U \cup x) + f(V) = f(E)$, which contradicts the connectivity of $\mathcal{Q}$. Therefore $\mathcal{Q}/A$ is connected.

Next, we assume $f(x) \leq j$. In this case, we note that all elements have rank at most j and view $\mathcal{Q}$ as a j-polymatroid. Then

$$f^*(E) = j|E| - f(E) = j|E| - j|E - x| = j; \tag{3.1}$$

$$f^*(x) = j|x| + f(E - x) - f(E) = j; \text{ and} \tag{3.2}$$

$$|E| \geq kn. \tag{3.3}$$

Using Lemma 3.2.6 and (3.1)–(3.3), it follows that $\mathcal{Q}^*$ has $pU_{1,n}$ as a minor for some $p \in \{1, \ldots, j\}$. Thus $\mathcal{Q}$ has $pU_{n-1,n} + (j-p)U_{n,n}$ as a minor, as desired.

We finally consider the case when $f(E - x) < f(E)$ and proceed by induction on $f(x)$. If $f(x) = 1$, then, by the connectivity of $\mathcal{Q}$, $f(E - x) = f(E)$, a contradiction. We thus assume $f(x) > 1$ and that the conclusion holds for elements of rank less than $f(x)$. Observe that the k-polymatroid $T_x(\mathcal{Q}) = (E, f_x)$ is connected by Proposition 3.2.7, has $f_x(E - x) = f(E - x) = j|E - x|$, and has $f_x(x) = f(x) - 1$. We may thus use the induction hypothesis to conclude that $T_x(\mathcal{Q})$ has disjoint sets $C, D \subseteq E$ for which $T_x(\mathcal{Q})\backslash D/C \cong aU_{n-1,n} + bU_{n,n}$ for some integers $a \geq 1$ and $b \geq 0$ such that $a + b \leq k$. If $x \in C$, then $T_x(\mathcal{Q})/C = \mathcal{Q}/C$, and we are done. If $x \notin C$, then

$$f_x/C(x) = f_x(C \cup x) - f_x(C) = f(C \cup x) - f(C) - 1 = f/C(x) - 1.$$

However, since $T_e(\mathcal{Q}) \cong aU_{n-1,n} + bU_{n,n}$, we must have that $f_x/C(x) = k$ and thus, from the above, that $f/C(x) - 1 = k$, a contradiction since $f(x) \leq k$. $\square$

Corollary 3.2.9. *Choose a positive integer n and let $\mathcal{Q}$ be a connected k-polymatroid which has, for some $x \in E$, a matching $E - x$. If $|E| \geq 2k^2n$, then $\mathcal{Q}$ has either an n-element connected $(k-1)$-polymatroid minor or a minor isomorphic to $aU_{n-1,n} + bU_{n,n}$ for some integers $a \geq 1$ and $b \geq 0$ such that $a + b \leq k$.*

Proof. Let $E_i = \{a \in E - x : f(a) = i\}$ and choose $j \in \{1, \ldots, k\}$ such that $|E_j|$ is maximum. Set $S = E - (E_j \cup x)$. We show that $\mathcal{Q}/S$ satisfies the hypotheses of Lemma 3.2.8. Observe

that, since $|E-x| \geq 2k^2n$, we have $|E_j| \geq 2kn$. Also note that, since $E-x$ is a matching of $\mathcal{Q}$, all subsets of E_j have the same rank in $\mathcal{Q}/S$ as in $\mathcal{Q}$. We thus have only to show that $\mathcal{Q}/S$ is connected. Suppose not and let $(A \cup x, B)$ be a 1-separation of $\mathcal{Q}/S$. Then

$$f/S(A \cup x) + f/s(B) = f/S(E-S),$$

and thus

$$f(S \cup A \cup x) + f(B \cup S) = f(E) + f(S).$$

Since $B \cup S \subseteq E - x$ is a matching, it follows that

$$f(S \cup A \cup x) + f(B) = f(E),$$

contradicting the connectivity of $\mathcal{Q}$. Therefore, from Lemma 3.2.8, $\mathcal{Q}/S$ has either a connected $(k-1)$-polymatroid on at least n elements, or has $aU_{n-1,n} + bU_{n,n}$ as a minor for some integers $a \geq 1$ and $b \geq 0$ such that $a + b \leq k$. □

Finally, if our graph has a large path as an induced minor, this translates to there being a sequence $s_0, X_1, s_1, X_2, s_2, \ldots, X_n, s_n$ in which $s_i \in X_j$ if and only if $i \in \{j, j+1\}$. Using Lemma 3.2.10, we show that, in this case, either $\mathcal{Q}$ has one of the desired minors, or $\mathcal{Q}$ has a large connected $(k-1)$-polymatroid minor.

Lemma 3.2.10. *Choose $n \geq 1$ and suppose $\mathcal{Q} = (E, f)$ is a connected k-polymatroid with matching $S = \{s_0, s_1, \ldots, s_m\}$ such that $\mathcal{Q}/S$ has $X_1, X_2, \ldots, X_m$ as its connected components with $C(X_i) = \{s_{i-1}, s_i\}$ for each i. If $m \geq k^2n^4$, then $\mathcal{Q}$ has either a minor with at least n elements and exactly two non-essential elements or a minor whose elements have rank at most $k-1$.*

Proof. Choose $C, D \subseteq E - S$ so that $|C \cup D|$ is maximum with the property that $\mathcal{Q}\backslash D/C$ is connected. Set $\mathcal{R} = \mathcal{Q}\backslash D/C$ and let $P \subseteq S - \{s_0, s_m\}$ be maximal such that $\mathcal{R}\backslash P$ is connected. We wish to show that $\mathcal{R}\backslash P$ has exactly two non-essential elements s_0 and s_m.

Choose $s_i \in S - P \cup \{s_0, s_m\}$. Then, since P was chosen to be maximal, it follows that $\mathcal{R}\backslash(P \cup s_i)$ is disconnected and we need only show that $\mathcal{R}\backslash P/s_i$ is disconnected.

Let $S_1 = \{s_0, \ldots, s_{i-1}\}$, $S_2 = \{s_{i+1}, \ldots, s_m\}$, $\mathcal{X}_1 = X_1 \cup X_2 \cup \cdots \cup X_{i-1}$, and $\mathcal{X}_2 = X_i \cup X_{i+1} \cup \cdots \cup X_m$. From Proposition 3.1.5, $f(S_1 \cup \mathcal{X}_1 \cup s_i) = f(S \cup \mathcal{X}_1) - f(S_2)$ and $f(S_2 \cup \mathcal{X}_2 \cup s_i) = f(S \cup \mathcal{X}_2) - f(S_1)$. Thus

$$\begin{aligned}
f/s_i(S_1 \cup \mathcal{X}_1) + f/s_i(S_2 \cup \mathcal{X}_2) &= f(S_1 \cup \mathcal{X}_1 \cup s_i) + f(S_2 \cup \mathcal{X}_2 \cup s_i) - 2f(s_i) \\
&= f(S \cup \mathcal{X}_1) + f(S \cup \mathcal{X}_2) - f(S_1) - f(S_2) - 2f(s_i) \\
&= f/S(\mathcal{X}_1) + f/S(\mathcal{X}_2) + f(S) - f(s_i) \\
&= f/S(E - S) + f(S - s_i) \\
&= f/s_i(E - s_i).
\end{aligned}$$

Therefore, $\mathcal{Q}/s_i$ is disconnected with s_0 and s_m contained in distinct components. By Corollary 3.1.4, then, $\mathcal{R}\backslash P/s_i$ is disconnected with s_0 and s_m in distinct components.

Next, let $x \in E - (S \cup C \cup D)$. We wish to show that both $\mathcal{R}\backslash(P \cup x)$ and $\mathcal{R}\backslash P/x$ are disconnected. To do this, we again use Proposition 3.1.4 and need only show that both $\mathcal{R}\backslash x$ and $\mathcal{R}/x$ have s_0 and s_m contained in distinct components. Since $|C \cup D|$ is maximum, it follows that all proper separators of $\mathcal{R}\backslash x$ and $\mathcal{R}/x$ contain elements of S or else we could delete the separator and obtain a larger $|C \cup D|$. Choose a proper separator U of $\mathcal{R}/x$ and set $V = E - (C \cup D \cup U \cup x)$. Since each of U and V must contain elements of S, we can, up to a relabeling of U and V, find $k \in (0, \ldots, m)$ such that $s_k \in U$ and $s_{k+1} \in V$. Then

$$\lambda_{\mathcal{R}/x}(U) = \lambda_{\mathcal{R}/x}(V) = 0, \tag{3.4}$$

and, by a previous argument,

$$\lambda_{\mathcal{Q}/s_i}(s_0 \cup X_1 \cup \cdots \cup s_{i-1} \cup X_i) = \lambda_{\mathcal{Q}/s_i}(X_{i+1} \cup s_{i+1} \cup \cdots \cup X_m \cup s_m) = 0 \tag{3.5}$$

for each $i \in (1, \ldots, m)$. By applying Corollary 3.1.4 to (3.5) when $i = k$ and $i = k+1$, it follows that

$$\lambda_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(X_{k+1}) = 0. \tag{3.6}$$

An application of Corollary 3.1.4 to (3.4) and (3.6) provides

$$\lambda_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(X_{k+1} \cap U) = \lambda_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(X_{k+1} \cap V) = 0. \tag{3.7}$$

A final application of Corollary 3.1.4 to (3.5) for $i = k$ and $i = k+1$ shows that

$$\lambda_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(s_0 \cup X_1 \cup \cdots s_{k-1} \cup X_k) = 0 \tag{3.8}$$

and

$$\lambda_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(X_{k+2} \cup s_{k+2} \cup \cdots \cup X_m \cup s_m) = 0. \tag{3.9}$$

Since the arguments in equations (3.7) – (3.9) partition the ground set of $\mathcal{R}/\{x, s_k, s_{k+1}\}$, it is not difficult to show that

$$\begin{aligned}
f_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(s_0 \cup & X_1 \cup \cdots s_{k-1} \cup X_k \cup (X_{k+1} \cap U)) \\
& + f_{\mathcal{R}/\{x,s_k,s_{k+1}\}}((X_{k+1} \cap V) \cup X_{k+2} \cup s_{k+2} \cup \cdots \cup X_m \cup s_m) \\
& = f_{\mathcal{R}/\{x,s_k,s_{k+1}\}}(E - (C \cup D \cup \{x, s_k, s_{k+1}\})).
\end{aligned}$$

Writing this last equality in terms of $f_{\mathcal{R}/x}$, we have

$$\begin{aligned}
f_{\mathcal{R}/x}(s_0 \cup & X_1 \cup \cdots s_{k-1} \cup X_k \cup s_k \cup (X_{k+1} \cap U) \cup s_{k+1}) \\
& + f_{\mathcal{R}/x}(s_k \cup (X_{k+1} \cap V) \cup s_{k+1} \cup X_{k+2} \cup s_{k+2} \cup \cdots \cup X_m \cup s_m) \\
& = f_{\mathcal{R}/x}(E - (C \cup D)) + f_{\mathcal{R}/x}(\{s_k, s_{k+1}\}).
\end{aligned}$$

However, from (3.4),

$$\begin{aligned}
f_{\mathcal{R}/x}(s_0 \cup & X_1 \cup \cdots s_{k-1} \cup X_k \cup s_k \cup (X_{k+1} \cap U)) \\
& + f_{\mathcal{R}/x}((X_{k+1} \cap V) \cup s_{k+1} \cup X_{k+2} \cup s_{k+2} \cup \cdots \cup X_m \cup s_m) \\
& = f_{\mathcal{R}/x}(E - (C \cup D)),
\end{aligned}$$

and thus $\mathcal{R}/x$ has s_0 and s_m in distinct connected components. The argument for $\mathcal{R}\backslash x$ is similar.

We have shown that $\mathcal{R}\backslash P$ has exactly two non-essential elements s_0 and s_m. If $|E - (C \cup D \cup P)| \geq n$, we are done. We thus assume $|E - (C \cup D \cup P)| < n$ and thus find $k' \in \{1, \ldots, m\}$ for which $X_{k'} \cup X_{k'+1} \cup \cdots \cup X_{k'+p'} \subseteq C \cup D$, where $p' \geq k^2 n^3$. Further, since $|E - (C \cup D \cup P)| < n$, we can find $k \in \{k', \ldots, k'+p'\}$ for which $\{s_k, \ldots, s_{k+p}\} \subseteq P$ for some $p \geq k^2 n^2$. Suppose, for some $a \in \{1, \ldots, p\}$, that $\sqcap_{\mathcal{R}}(s_k, s_{k+a}) = 0$ and set $Y_i = X_i - (C \cup D)$ for each i. With an application of Corollary 3.1.4 to (3.5),

$$\lambda_{\mathcal{Q}/\{s_k,s_{k+a}\}}(s_0 \cup X_1 \cup \cdots \cup s_{k-1} \cup X_k) = 0;$$

$$\lambda_{\mathcal{Q}/\{s_k,s_{k+a}\}}(X_{k+1} \cup s_{k+1} \cup \cdots \cup X_{k+a-1} \cup s_{k+a-1} \cup X_{k+a}) = 0; \text{ and}$$

$$\lambda_{\mathcal{Q}/\{s_k,s_{k+a}\}}(X_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup X_m \cup s_m) = 0.$$

With a second application of Corollary 3.1.4, the above equalities become

$$\lambda_{\mathcal{R}/\{s_k,s_{k+a}\}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k) = 0; \tag{3.10}$$

$$\lambda_{\mathcal{R}/\{s_k,s_{k+a}\}}(\{s_{k+1}, \ldots, s_{k+a-1}\}) = 0; \text{ and} \tag{3.11}$$

$$\lambda_{\mathcal{R}/\{s_k,s_{k+a}\}}(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m) = 0. \tag{3.12}$$

As the arguments of (3.10)–(3.12) partition the ground set of $\mathcal{R}/\{s_k, s_{k+a}\}$, it follows that

$$\begin{aligned}
&f_{\mathcal{R}}/\{s_k, s_{k+a}\}[E - (C \cup D \cup \{s_k, s_{k+a}\})] \\
&\qquad = f_{\mathcal{R}}/\{s_k, s_{k+a}\}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k) \\
&\qquad + f_{\mathcal{R}}/\{s_k, s_{k+a}\}(\{s_{k+1}, \ldots, s_{k+a-1}\}) \\
&\qquad + f_{\mathcal{R}}/\{s_k, s_{k+a}\}(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m).
\end{aligned}$$

With a deletion of $\{s_{k+1}, \ldots, s_{k+a-1}\}$, the above becomes

$$\begin{aligned}
f_{\mathcal{R}}/\{s_k, s_{k+a}\}&[E - (C \cup D \cup \{s_k, \ldots, s_{k+a}\})] \\
&= f_{\mathcal{R}}/\{s_k, s_{k+a}\}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k) \\
&+ f_{\mathcal{R}}/\{s_k, s_{k+a}\}(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m).
\end{aligned}$$

Rewriting this equality in terms of the rank function of $\mathcal{R}$, we have

$$f_{\mathcal{R}}[E - (C \cup D \cup \{s_{k+1}, \ldots, s_{k+a-1}\})] + f_{\mathcal{R}}(\{s_k, s_{k+a}\}) \tag{3.13}$$

$$= f_{\mathcal{R}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup \{s_k, s_{k+a}\}) \tag{3.14}$$

$$+ f_{\mathcal{R}}(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m \cup \{s_k, s_{k+a}\}). \tag{3.15}$$

Finally, observe that, from (3.14), we have

$$\begin{aligned}
f_{\mathcal{R}}&(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup \{s_k, s_{k+a}\}) \\
&= f_{\mathcal{R}}/s_k(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup \{s_{k+a}\}) + f_{\mathcal{R}}(s_k) \\
&= f_{\mathcal{R}}/s_k(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k) + f_{\mathcal{R}}/s_k(s_{k+a}) + f_{\mathcal{R}}(s_k) \\
&= f_{\mathcal{R}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup s_k) + f_{\mathcal{R}}(\{s_k, s_{k+a}\}) - f_{\mathcal{R}}(s_k) \\
&= f_{\mathcal{R}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup s_k) - \sqcap_{\mathcal{R}}(s_k, s_{k+a}) + f_{\mathcal{R}}(s_{k+a}) \\
&= f_{\mathcal{R}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup s_k) + f_{\mathcal{R}}(s_{k+a}).
\end{aligned}$$

Similarly, from (3.15),

$$\begin{aligned}
f_{\mathcal{R}}&(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m \cup \{s_k, s_{k+a}\}) \\
&= f_{\mathcal{R}}(s_k) + f_{\mathcal{R}}(Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m \cup s_{k+a}).
\end{aligned}$$

Thus (3.13)–(3.15) becomes

$$\begin{aligned}
f_{\mathcal{R}}&[E - (C \cup D \cup \{s_{k+1}, \ldots, s_{k+a-1}\})] \\
&= f_{\mathcal{R}}(s_0 \cup Y_1 \cup \cdots \cup s_{k-1} \cup Y_k \cup s_k) \\
&+ f_{\mathcal{R}}(s_{k+a} \cup Y_{k+a+1} \cup s_{k+a+1} \cup \cdots \cup Y_m \cup s_m).
\end{aligned}$$

Therefore, $\mathcal{R} - \{s_{k+1}, \ldots, s_{k+a-1}\}$ is disconnected with s_0 and s_m in distinct components and thus $\mathcal{R} - P$ is disconnected, a contradiction. From this, we have that $\sqcap_{\mathcal{R}}(s_k, s_{k+a}) > 0$ for each $a \in \{1, \ldots, p\}$. That is, each of $\{s_{k+1}, \ldots, s_{k+p}\}$ intersects s_k in $\mathcal{R}$. Observe, then, that $\mathcal{R}|\{s_k, \ldots, s_{k+p}\}$ is a connected k-polymatroid with maximal matching of size one and ground set on at least k^2n^2 elements. By Lemma 3.2.6, then, we have that $\mathcal{R}|\{s_k, \ldots, s_{k+p}\}$ either contains a minor isomorphic to $aU_{1,n} + bU_{n,n}$, where $a \geq 1$ and $k \geq a + b \geq 1$, or $(\mathcal{R}|\{s_k, \ldots, s_{k+p}\})/s_k$ has a connected component of size greater than n. In the former case, we are done. In the latter case, observe that, since each element of $\mathcal{R}|\{s_k, \ldots, s_{k+p}\}$ intersects s_k, when s_k is contracted we are left with a connected $(k-1)$-polymatroid with at least n elements, as desired. $\square$

3.3 Main Results

In this section, we state and prove the main results of the chapter, Theorems 3.3.2 and 3.3.4. The first of these finds the unavoidable minors of large connected k-polymatroids. In general, an explicit description of one of the classes of unavoidable minors seems difficult. But, when $k = 2$, we are able to give such a description and this appears in Theorem 3.3.4. We begin the section by identifying the unavoidable minors of large k-polymatroids when no connectivity condition is imposed.

Theorem 3.3.1. *For all positive integers n, there is an integer r such that every k-polymatroid with at least r elements has a minor isomorphic to one of $U_{0,n}$; $U_{n,n}$; $2U_{n,n}$; $\ldots$; $kU_{n,n}$.*

Proof. Choose $n \in \mathbb{N}$ and let $\mathcal{Q} = (E, f)$ be a k-polymatroid. We show that if $r = \frac{2}{k}[(2^{k+1} - k - 2)n - 2^k + 2]$, then the conclusion is satisfied. Assume first that $f(E) \geq f^*(E)$. Then

$$k|E| = f(E) + f^*(E) \leq 2f(E)$$

and thus

$$f(E) \geq \frac{k}{2}|E| \geq (2^{k+1} - k - 2)n - 2^k + 2.$$

From Lemma 3.2.1, it follows that $\mathcal{Q}$ contains at least one of $U_{n,n}$; $2U_{n,n}$; $\ldots$; $kU_{n,n}$ as a minor. If, on the other hand, $f^*(E) \geq f(E)$, then $\mathcal{Q}^*$ has one of $U_{n,n}$; $2U_{n,n}$; $\ldots$; $kU_{n,n}$ as a minor. It is easy to check that, for any positive integer t, the k-dual of $tU_{n,n}$ is $(k-t)U_{n,n}$, where we assume that $0U_{n,n} = U_{0,n}$. The theorem follows. □

Theorem 3.3.2. *For every positive integer n, there is an integer m such that every connected k-polymatroid with at least m elements either has a minor isomorphic to $aU_{1,n} + bU_{n,n}$ or $aU_{n-1,n} + bU_{n,n}$, for some $a \geq 1$ and $1 \leq a + b \leq k$; or has a connected minor with at least n elements and exactly two non-essential elements.*

Proof. We fix n and observe that if $n = 1$, then the statement is trivial and thus assume $n > 1$. If $k = 1$, then the theorem reduces to Theorem 3.1.3 and we are done. We thus assume $k > 1$ and that, for any $j < k$, there is an integer m_j such that if $\mathcal{R}$ is a connected j-polymatroid on at least m_j elements, then $\mathcal{R}$ has a minor isomorphic to $aU_{1,n} + bU_{n,n}$ or $aU_{n-1,n} + bU_{n,n}$, for some $a \geq 1$ and $1 \leq a + b \leq j$; or has a connected minor with at least n elements and exactly two non-essential elements.

It is a well-known Ramsey result (see, for example, [3]) that, for every $p \in \mathbb{N}$, there is an $R(p) \in \mathbb{N}$ such that every connected graph of order at least $R(p)$ contains K_p, $K_{1,p}$, or P_p as an induced subgraph. We show that if $m = m_{k-1}R((k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}})$, then $\mathcal{Q}$ has one of the desired minors. Let S be a maximal matching of $\mathcal{Q}$ and observe that, for any $x \in E - S$, $f/S(x) < k$ or else $S \cup x$ would be a matching properly containing S. Thus $\mathcal{Q}/S$ is a $(k-1)$-polymatroid. If there is a connected component C of $\mathcal{Q}/S$ which has at least m_{n-1} elements, then, by induction, $(\mathcal{Q}/S)|C$ has a minor which satisfies the theorem. We thus assume every component of $\mathcal{Q}/S$ has fewer than m_{n-1} elements and label these components $X_1, X_2, \ldots, X_t$.

We now construct a graph G with vertices labeled by the members of S and the components $X_1, \ldots, X_m$ of $\mathcal{Q}/S$. Two vertices of G are adjacent if and only if one is labeled by a

member s of S, one is labeled by a component X_i, and $s \in C(X_i, S)$. We show that G is connected.

Choose $X_A \subseteq \{X_1, \ldots, X_m\}$ and $S_A \subseteq S$ such that $X_A \cup S_A$ labels the vertices of a connected component of G. Set $X_B = \{X_1, \ldots, X_m\} - X_A$ and $S_B = S - S_A$. We denote, by $\cup X_A$ and $\cup X_B$, the union of all elements in X_A and X_B, respectively. Observe that, since $X_A \cup S_A$ labels a connected component of G, if $X_i \in X_A$, then $C(X_i, S) \subseteq S_A$. It follows from Proposition 3.1.5, then, that

$$\begin{aligned} f((\cup X_A) \cup S) &= f(\bigcup_{X_i \in X_A} C(X_i, S)) + f(S - \bigcup_{X_i \in X_A} C(X_i, S)) \\ &= f((\cup X_A) \cup S_A) + f(S_B), \end{aligned}$$

and, similarly,

$$f((\cup X_B) \cup S) = f((\cup X_B) \cup S_B) + f(S_A).$$

However, this means

$$\begin{aligned} &f((\cup X_A) \cup S_A) + f((\cup X_B) \cup S_B) \\ &\qquad = f((\cup X_A) \cup S) + f((\cup X_B) \cup S) - f(S_A) - f(S_B) \\ &\qquad = f/S(\cup X_A) + f/S(\cup X_B) + f(S) \\ &\qquad = f/S(X_A \cup X_B) + f(S) \\ &\qquad = f(E). \end{aligned}$$

As $\mathcal{Q}$ is connected, it follows that $X_B = S_B = \varnothing$ and, therefore, G is connected.

Since G is a connected graph on at least $R((k^2 m_{k-1}+1)^2 (m_{k-1})^{4m_{k-1}})$ vertices, it has one of K_γ, $K_{1,\gamma}$, and P_γ as an induced minor, for some $\gamma \geq (k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}}$. Since G is bipartite, however, we will have at least one of $K_{1,\gamma}$ and P_γ as an induced minor. Suppose first that G has $K_{1,\gamma}$ as an induced minor. The vertex of high degree in this minor will be either an element of S or a component of $\mathcal{Q}/S$. We begin by assuming the latter.

Let X_i be this component of $\mathcal{Q}/S$ and observe that its high degree in G translates to $|C(X_i, S)| \geq (k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}}$. Since $n \geq 2$ and $|X_i| < m_{k-1}$, we have

$$\begin{aligned}|C(X_i, S)| &\geq (k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}} \\ &\geq (k^2 m_{k-1} + 1)^2 (|X_i|)^{4|X_i|} \\ &> (k^2 m_{k-1} + 1)(|X_i|)^{2|X_i|}.\end{aligned}$$

Let $\mathcal{R} = \mathcal{Q}|_{X_i \cup C(X_i,S)}$. We have, from Lemma 3.2.2, that $\mathcal{R}$ is connected and, from Lemma 3.2.3, that $\mathcal{R}$ has an element $x \in X_i$ and matching $T \subseteq S$ for which $|C_{\mathcal{R}}(x, T)| > k^2 m_{k-1}$. It is immediate, then, by Corollary 3.2.9, that $\mathcal{R}$ has either an m_{k-1}-element connected $(k-1)$-polymatroid minor, or a minor isomorphic to either $aU_{m_{k-1}-1, m_{k-1}} + bU_{m_{k-1}, m_{k-1}}$ or $aU_{1, m_{k-1}}$, for integers $a \geq 1$ and $b \geq 0$ such that $a + b \leq k$. Since $m_{k-1} \geq n$, we are, in either case, done.

Next, we consider the case when the vertex of high degree is labeled by an element $s \in S$. Let $\{X_{a_1}, X_{a_2}, \ldots, X_{a_m}\}$ be the maximal collection of components of $\mathcal{Q}/S$ for which $s \in \bigcap_{i=1}^{m} C(X_{a_i}, S)$ and recall that

$$m \geq (k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}} \geq k^2 (m_{k-1})^2.$$

It follows, from Lemma 3.2.5, that we can find a connected minor $\mathcal{R}$ of $\mathcal{Q}$ which has $\{s\}$ as a matching and greater than $k^2(m_{k-1})^2$ elements. From Lemma 3.2.6, then, either $\mathcal{R}$ has $aU_{1,m_{k-1}} + bU_{m_{k-1}, m_{k-1}}$, where $a \geq 1$ and $k \geq a + b \geq 1$ as a minor, or $\mathcal{R}/s$ has a connected component of size greater than m_{k-1}. Since $\mathcal{R}/s$ is a $(k-1)$-polymatroid, we are, in either case, done.

Finally, we suppose G has P_γ as an induced minor and let $s_0, X_1, s_1, \ldots, s_{m-1}, X_m, s_m$ be a maximal path of G where

$$m \geq \gamma \geq (k^2 m_{k-1} + 1)^2 (m_{k-1})^{4m_{k-1}} \geq k^2 (m_{k-1})^4.$$

It is not difficult to check that if $\mathcal{R} = (\mathcal{Q}/(S - \{s_0, s_1, \ldots, s_m\}))|(\{s_0, s_1, \ldots, s_m\} \cup X_1 \cup X_2 \cup \cdots \cup X_m)$, then $\mathcal{R}$ satisfies the hypotheses of Lemma 3.2.10 and thus has either a minor with

at least n elements and exactly two non-essential elements or a connected $(k-1)$-polymatroid minor with at least m_{k-1} elements. $\square$

As discussed earlier, the above theorem does not satisfactorily characterize the unavoidable minors for connected k-polymatroids as it does not fully described the class of k-polymatroids with exactly two non-essential elements. In the case of 2-polymatroids, however, the following description of this class is provided in [9].

Let E be a non-empty finite set and choose distinct elements $a, b \notin E$. Take M_1 to be a matroid with ground set $E \cup \{a, b\}$ isomorphic to $U_{1,|E|+1} \oplus U_{0,1}$, where b is the loop, and M_2 to be a matroid on the same ground set isomorphic to $U_{|E|,|E|+1} \oplus U_{0,1}$, where a is the loop. Then the 2-polymatroid $\mathcal{S}_{|E|} = M_1 + M_2$ has a and b as its only non-essential elements. For convenience, we set $\mathcal{S}_0 = U_{1,2} + U_{1,1}$.

Informally, the following theorem says that 2-polymatroids with exactly two non-essential elements are 2-sums of $\mathcal{S}_i$.

Theorem 3.3.3. *Let $\mathcal{Q}$ be a connected 2-polymatroid with at least three elements. Then $\mathcal{Q}$ has exactly two non-essential elements if and only if, for some $n \geq 1$, there is a sequence $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ of 2-polymatroids such that*

(i) each $\mathcal{Q}_i$ is isomorphic to some member of $\{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots\}$;

(ii) if either $n = 1$ or $2 \leq i \leq n-1$, then $\mathcal{Q}_i$ is isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$;

(iii) the ground sets of $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_n$ are disjoint except that, for each i in $\{1, 2, \ldots, n-1\}$, the sets $E(\mathcal{Q}_i)$ and $E(\mathcal{Q}_{i+1})$ meet in a single rank-1 element; and

(iv) $\mathcal{Q} \cong \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 \oplus_2 ... \oplus_2 \mathcal{Q}_n$.

We use Theorem 2.3.9, together with Theorem 3.3.2, to obtain the following characterization of the unavoidable minors for connected 2-polymatroids.

Theorem 3.3.4. *For every positive integer n, there is an integer r such that every connected 2-polymatroid with at least r elements has a minor isomorphic to $U_{1,n}$; $U_{n,n}+U_{n-1,n}$; $U_{n-1,n}$; $U_{n,n}+U_{1,n}$; $2U_{1,n}$; $2U_{n-1,n}$; or*
$\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 \oplus_2 \cdots \oplus_2 \mathcal{Q}_n$, where each $\mathcal{Q}_i$ is isomorphic to one of $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots,$ or $\mathcal{S}_{n-1}$.

Proof. Set $p = n^n$. From Theorem 3.3.2, we have that there is an integer r' such that every connected 2-polymatroid with at least r' elements either has a minor isomorphic to one of $U_{1,p}$; $U_{p,p}+U_{p-1,p}$; $U_{p-1,p}$; $U_{p,p}+U_{1,p}$; $2U_{1,p}$; and $2U_{p-1,p}$; or has a connected minor with at least p elements and exactly two non-essential elements. In the former case, each of these has a desired minor and we are done.

We thus assume $\mathcal{Q}$ has a minor $\mathcal{R}$ which has at least p elements and exactly two non-essential elements. We can thus find a sequence $\mathcal{Q}_1, \mathcal{Q}_2, ..., \mathcal{Q}_j$ of 2-polymatroids satisfying the conditions of Theorem 2.3.9. In particular, $\mathcal{R} \cong \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 \oplus_2 ... \oplus_2 \mathcal{Q}_j$. If $j \geq n$, then it is easy to check that $\mathcal{Q}$ has a desired minor.

On the other hand, suppose $j < n$. Since $\mathcal{R}$ has at least n^n elements, it follows that there is some $\mathcal{Q}_i$ with $|E(\mathcal{Q}_i) \cap E(\mathcal{Q})| = q \geq n$. Let $\{a, b\} = E(\mathcal{Q}_i) - E(\mathcal{Q})$. Since $\mathcal{Q}_i \cong \mathcal{S}_q$, $\mathcal{Q}_i \backslash \{a, b\} \cong U_{1,q} + U_{q,q}$, which has $U_{1,n} + U_{n,n}$ as a minor. Observe that $\mathcal{R}$ and thus $\mathcal{Q}$ has this $U_{1,n} + U_{n,n}$ as a minor as well. $\square$

Chapter 4
A Characterization of Tangle Matroids

4.1 Introduction

Tangles are a way of identifying highly connected regions in matroids of low connectivity. As these highly-connected regions are the obstructions to large branchwidth (see [4]), tangles play a fundamental role in Geelen, Gerards, and Whittle's structure theorem for $GF(q)$-representable matroids in [5]. The reader is also referred to [5] for a history and enlightening discussion of tangles.

The connectivity function of a matroid M is defined, for $X \subseteq E(M)$, to be

$$\lambda_M(X) = r(X) + r(E(M) - X) - r(M).$$

Our definitions of tangles and of tangle matroids follow [8]. Let M be a matroid, and $\mathcal{T}$ a collection of subsets of $E(M)$. Then $\mathcal{T}$ is a *tangle of order θ of M* if

(T1) For all $X \in \mathcal{T}$, $\lambda_M(X) < \theta$;

(T2) For all $X \subseteq E(M)$ with $\lambda_M(X) < \theta$, either $X \in \mathcal{T}$ or $E(M) - X \in \mathcal{T}$;

(T3) If $X, Y, Z \in \mathcal{T}$, then $X \cup Y \cup Z \neq E(M)$;

(T4) For each $e \in E(M)$, $E(M) - \{e\} \notin \mathcal{T}$.

A useful tool for studying tangles is the *tangle matroid*. Let M be a matroid and $\mathcal{T}$ a tangle of M of order θ. Let $\rho : 2^{E(M)} \to \mathbb{N}$ be defined by

$$\rho(X) := \begin{cases} \min\{\lambda_M(Y) : X \subseteq Y \in \mathcal{T}\} & \text{if } X \subseteq Y \in \mathcal{T} \\ \theta & \text{otherwise.} \end{cases}$$

In [4], it is shown that ρ is the rank function of a matroid on $E(M)$. We call this matroid $M(\mathcal{T})$. If $\mathcal{T}$ is a tangle of a matroid M, we call $M(\mathcal{T})$ a *tangle matroid of* M.

In this chapter, we characterize those matroids that are tangle matroids. For example, $PG(2,3)$ is a tangle matroid while $PG(2,2)$ is not. The reason lies in the arrangement of the hyperplanes in these matroids. In Section 4.3, we prove the following.

Theorem 4.1.1. *A matroid M other than $U_{1,1}$ is a tangle matroid if and only if M has no three hyperplanes whose union is $E(M)$.*

In addition, we show some consequences of this theorem. In particular, a tangle matroid of M is a quotient of M, and all binary tangle matroids have rank less than two. Much of the work here appears in [11].

4.2 High-order Tangles

An immediate consequence of the following lemma [6] is that a matroid cannot have a tangle whose order exceeds its rank.

Lemma 4.2.1. *Let $\mathcal{T}$ be a tangle of order θ at least 2 in a matroid M. Then each subset of $E(M)$ with rank less than θ is in $\mathcal{T}$.*

In view of this, it is natural to ask when a matroid has a tangle whose order equals the rank of the matroid.

Proposition 4.2.2. *A matroid M other than $U_{1,1}$ has an order-$r(M)$ tangle if and only if M has no three hyperplanes whose union is $E(M)$.*

Proof. If a matroid has three such hyperplanes, then, from Lemma 4.2.1, each of these hyperplanes is a member of the tangle, a contradiction of (T3). If no three hyperplanes cover the entire ground set of M, we show that the following is an order-$r(M)$ tangle of M.

$$\mathcal{T} = \{X : r(X) < r(M)\}$$

For (T1), we choose $X \in \mathcal{T}$ and observe that $\lambda_M(X) = r(X) + r(E(M) - X) - r(M)$. If $r(E(M)-X) < r(M)$, then X and $E(M)-X$ are two non-spanning sets that cover $E(M)$. As such, there are two hyperplanes H_1 and H_2 containing X and $E(M) - X$, respectively, such that $H_1 \cup H_2 = E(M)$. Thus $r(E(M) - X) = r(M)$, and therefore $\lambda_M(X) = r(X) < r(M)$, as desired.

For (T2), choose a partition (X, Y) of $E(M)$ such that $\lambda_M(X) < r(M)$. Expanding out $\lambda_M(X)$, this becomes $r(X) + r(Y) < 2r(M)$. It follows, then, that either $r(X) < r(M)$ or $r(Y) < r(M)$, and thus either $X \in \mathcal{T}$ or $Y \in \mathcal{T}$.

The remaining conditions, (T3) and (T4), follow immediately from the hypothesis. We conclude that $\mathcal{T}$ is indeed a tangle of order $r(M)$. □

We now know which matroids have a tangle with order equal to the rank of the matroid. A natural next challenge is to find all such tangles. It turns out that, if a matroid M has an order-$r(M)$ tangle, then that tangle is exactly the one used in the proof of Proposition 4.2.2. The following proposition shows this.

Proposition 4.2.3. *If a matroid M has an order-$r(M)$ tangle $\mathcal{T}$, then $\mathcal{T} = \{X : r(X) < r(M)\}$.*

Proof. Set $\mathcal{T}' = \{X : r(X) < r(M)\}$ and let $\mathcal{T}$ be an order-$r(M)$ tangle of M. Assume first that $r(M) = 1$. If x is a loop of M, then, by combining (T2) and (T3), we have that $\{x\} \in \mathcal{T}$. From this and Lemma 2.7 of [8], it follows that, if $X \subseteq E(M)$ where $r(X) = 0$, then $X \in \mathcal{T}$. Thus $\mathcal{T}' \subseteq \mathcal{T}$ and we have only to show that $\mathcal{T}$ contains no rank-1 elements. Assuming the contrary, let $A \subseteq E(M)$ such that $r(A) = 1$ and $A \in \mathcal{T}$. Since, in this case, $\lambda_M(A) = 0$, it follows that $r(E(M) - A) = 0$. From the above argument, $E(M) - A \in \mathcal{T}$, contradicting (T3) since both A and its complement would be in $\mathcal{T}$. Therefore $\mathcal{T}' = \mathcal{T}$.

If $r(M) \geq 2$, then we are able to use Lemma 4.2.1 to show that $\mathcal{T}' \subseteq \mathcal{T}$. To prove equality, suppose $Y \in \mathcal{T}$ such that $r(Y) = r(M)$. Then, by (T1),

$$r(M) > \lambda_M(Y) = r(Y) + r(E(M) - Y) - r(M) = r(E(M) - Y).$$

Thus, since $r(E(M) - Y) < r(M)$, both Y and its complement are members of $\mathcal{T}$, contradicting (T3). Therefore, $\mathcal{T}' = \mathcal{T}$. □

4.3 Tangle Matroids

The first question we answer concerning tangle matroids is that of when a matroid has itself as a tangle matroid.

Proposition 4.3.1. *A matroid M other than $U_{1,1}$ has itself as a tangle matroid if and only if M has no three hyperplanes whose union is $E(M)$.*

Proof. Let $\mathcal{T}$ be a tangle of M for which $M(\mathcal{T}) = M$. Then the order of $\mathcal{T}$ is necessarily $r(M)$ and thus, from Proposition 4.2.2, M has no three hyperplanes whose union is $E(M)$. We thus assume M has no three hyperplanes whose union is $E(M)$. Then, by Proposition 4.2.2, M has an order-$r(M)$ tangle $\mathcal{T}$. Further, Proposition 4.2.3 tells us that $\mathcal{T} = \{X : r(X) < r(M)\}$. We now have only to show that $M(\mathcal{T}) = M$.

Choose $X \subseteq E(M)$. We wish to show that $\rho(X) = r(X)$. If $r(X) = r(M)$, then neither X, nor any set containing X, is a member of $\mathcal{T}$. Thus $\rho(X) = \theta = r(M)$. On the other hand, if $r(X) < r(M)$, then $X \in \mathcal{T}$ and $\lambda_M(X) = r(X) + r(E(M) - X) - r(M)$. Since $\mathcal{T}$ is, by Proposition 4.2.3, a tangle, we use (T3) to observe that $E - X \notin \mathcal{T}$. Thus $r(E - X) = r(M)$ and therefore $\lambda_M(X) = r(X)$. If $X \subseteq Y \in \mathcal{T}$, then, as above we see that $\lambda_M(Y) = r(Y) \geq r(X) = \lambda_M(X)$. Thus,

$$\rho(X) = \min\{\lambda_M(Y) : X \subseteq Y \in \mathcal{T}\} = \lambda_M(X) = r(X),$$

and therefore $M(\mathcal{T}) = M$. □

We are now ready to prove the main theorem. We show that a matroid M other than $U_{1,1}$ is a tangle matroid if and only if it has no three hyperplanes that cover $E(M)$. We omit $U_{1,1}$ in this statement as it is the only matroid without three covering hyperplanes that is not a tangle matroid.

Proof of Theorem 1.1. If M has no three hyperplanes whose union is $E(M)$, then, by Proposition 4.3.1, M has a tangle $\mathcal{T}$ such that $M = M(\mathcal{T})$. We thus assume M has three hyperplanes H_1, H_2, and H_3 such that $H_1 \cup H_2 \cup H_3 = E(M)$. Assume that $\mathcal{T}$ is a tangle for which $M(\mathcal{T}) = M$. Observe that the order of $\mathcal{T}$ is $r(M)$. Thus, from the definition of $M(\mathcal{T})$, there exist sets Y_1, Y_2, and Y_3 such that each $H_i \subseteq Y_i \in \mathcal{T}$. It follows, then, that $Y_1 \cup Y_2 \cup Y_3 = E(M)$, contradicting (T3). $\square$

We now list several immediate consequences of Theorem 4.1.1. The first gives several ways to restate the main theorem.

Corollary 4.3.2. *Let M be a matroid with ground set E. Then the following are equivalent.*

1. *M is either a tangle matroid or $U_{1,1}$;*

2. *M has no three hyperplanes whose union is E;*

3. *In every partition (X, Y, Z) of E, at least one of X, Y, and Z is a spanning set;*

4. *M has no three cocircuits whose intersection is empty.*

A matroid is *round* if it has no two disjoint cocircuits. The following is another immediate consequence of Theorem 4.1.1.

Corollary 4.3.3. *Every tangle matroid is round.*

The next consequence follows immediately from Theorem 4.1.1 and Geelen and Kabell's result [7] that finds an Erdős-Pósa property for matroid circuits. The bicircular matroid of a graph G is denoted by $B(G)$.

Corollary 4.3.4. *There exists an integer-valued function $\gamma(n)$ such that, for any $n \in \mathbb{N}$, if M is a tangle matroid with $r(M) \geq \gamma(n)$, then M has a minor isomorphic to $U_{n,2n}$; $M(K_n)$; or $B(K_n)$.*

Corollary 4.3.5. *If M is a tangle matroid, then every parallel minor of M is either a tangle matroid or $U_{1,1}$.*

Let $\mathcal{C}_\mathcal{T}$ be the class consisting of $U_{1,1}$ together with all tangle matroids. Corollary 4.3.5 establishes that $\mathcal{C}_\mathcal{T}$ is closed under taking parallel minors. One might hope, then, to obtain an excluded parallel minor-based characterization of $\mathcal{C}_\mathcal{T}$. Unfortunately, such a list of excluded minors would be infinite. For example, the class $\{U_{k,3k-3} : k \geq 2\}$ are all excluded parallel minors of $\mathcal{C}_\mathcal{T}$.

If we restrict $\mathcal{C}_\mathcal{T}$ to only its $GF(q)$-representable members, a similar problem occurs. Let M_q be the class of $GF(q)$-representable matroids. If $q \geq 3$, then the class $M_q \cap \mathcal{C}_\mathcal{T}$ has an infinite number of excluded parallel minors, as we show in the following example.

Example 4.3.6. Choose two integers $r, q \geq 3$. Let M_1 and M_2 be two matroids that are isomorphic to $PG(r-1, q)$ such that $E(M_1) \cap E(M_2) = T$ where $M_1|T = M_2|T \cong PG(r-2, q)$. Recall that $P_T(M_1, M_2)$ denotes the generalized parallel connection of M_1 and M_2 over T (see [14]). Then $M = P_T(M_1, M_2)\backslash T$ is not a tangle matroid since $E(M_1) - T$ and $E(M_2) - T$ are two hyperplanes whose union covers $E(M)$. However, for $a \in E(M)$, M/a is isomorphic to M_1, where each member of T is replaced with $q-1$ elements in parallel. Thus M/a is a $GF(q)$-representable tangle matroid and so M is an excluded minor of $M_q \cap \mathcal{C}_\mathcal{T}$.

We conclude this section by characterizing the binary tangle matroids.

Theorem 4.3.7. *A matroid M other than $U_{1,1}$ is a binary tangle matroid if and only if $r(M) < 2$.*

Proof. Let M be a binary tangle matroid. It is well known (see, for example, [14]), that a matroid is binary if and only if the matroid has no parallel minor isomorphic to $U_{2,k+2}$ for

any $k \geq 2$. Suppose $r(M) \geq 2$. Let H be an independent set of M such that $|H| = r(M)-2$. Then $si(M/H)$ is a rank-2 parallel minor of M and, as both $\mathcal{C}_{\mathcal{T}}$ and M_2 are closed under parallel minors, $si(M/H)$ is a member of $\mathcal{C}_{\mathcal{T}} \cap M_2$. However, since neither $U_{2,2}$ nor $U_{2,3}$ is a tangle matroid, $si(M/H)$ is isomorphic to one of the excluded parallel minors for binary matroids, a contradiction. Therefore, $r(M) < 2$. The converse follows immediately from Theorem 4.1.1. $\square$

4.4 Quotients

It is easy to check that, if a matroid M has a tangle matroid M', then $r(M') \leq r(M)$. Because of this, a tangle matroid M' can be thought of as a minimal matroid with M' as a tangle matroid. The following theorem is used by Corollary 4.4.2 to show this formally, where the ordering is that induced by matroid quotients.

Theorem 4.4.1. *Let M be a matroid and $X \subseteq E(M)$. If $\mathcal{T}$ is a tangle of M, then $cl_M(X) \subseteq cl_{\mathcal{T}}(X)$.*

Proof. Choose $a \in cl_M(X) - cl_{\mathcal{T}}(X)$ and let $\mathcal{T}$ be a tangle of order θ. Then $r_M(X \cup a) = r_M(X)$ while $r_{\mathcal{T}}(X \cup a) = r_{\mathcal{T}}(X) + 1$. If $\lambda_M(X \cup a) > \lambda_M(X)$, then

$$\begin{aligned} r_M(X) + r_M(E - X) - r_M(E) &= \lambda_M(X) \\ &< \lambda_M(X \cup a) \\ &= r_M(X \cup a) + r_M(E - (X \cup a)) - r_M(E) \\ &= r_M(X) + r_M(E - (X \cup a)) - r_M(E) \\ &\leq r_M(X) + r_M(E - X) - r_M(E), \end{aligned}$$

a contradiction. Thus $\lambda_M(X \cup a) \leq \lambda_M(X)$. Suppose $X \cup a \in \mathcal{T}$. As $r_{\mathcal{T}}(X) < r_{\mathcal{T}}(X \cup a)$, there is a set A contained in X such that $a \notin A$ and $A \in \mathcal{T}$. Then it follows that $\lambda_M(A) + \lambda_M(X \cup a) \geq \lambda_M(A \cup a) + \lambda_M(X) \geq \lambda_M(A \cup a) + \lambda(X \cup a)$, and so $\theta > \lambda_M(A) \geq \lambda_M(A \cup a)$.

If $A \cup a \in \mathcal{T}$, then $r_{\mathcal{T}}(X \cup a) \leq \lambda_M(A \cup a) \leq \lambda_M(A) = r_{\mathcal{T}}(A)$, a contradiction since $r_{\mathcal{T}}(X \cup a) > r_{\mathcal{T}}(A)$. Thus $A \cup a \notin \mathcal{T}$ and so $a \notin \mathcal{T}$. This may only occur when $\theta = 1$ and $r_M(a) = 1$. Observe, however, that since $0 = \lambda_M(A) = r_M(A) + r_M(E - A) - r_M(E)$, it follows that A is a connected component of M. Since $a \notin A$, we have that $r_M(A \cup a) = r_M(A) + 1$, and thus $r_M(X \cup a) = r_M(X) + 1$, a contradiction. $\square$

The following corollary follows immediately from Theorem 4.4.1 using results from [14, §7.3].

Corollary 4.4.2. *Let M be a matroid with tangle $\mathcal{T}$. Then the following statements hold.*

(i) $M(\mathcal{T})$ is a quotient of M.

(ii) Every flat of $M(\mathcal{T})$ is a flat of M.

(iii) If $X \subseteq Y \subseteq E(M)$, then $r_M(Y) - r_M(X) \geq r_{\mathcal{T}}(Y) - r_{\mathcal{T}}(X)$.

(iv) Every circuit of M is a union of circuits of $M(\mathcal{T})$.

(v) The identity map $i : M \longrightarrow M(\mathcal{T})$ is a strong map.

The result that $M(\mathcal{T})$ is a quotient of M has the following consequence.

Proposition 4.4.3. *Let M be a matroid such that both M and M^* are tangle matroids. Then $M = M^*$.*

Proof. If M is a tangle matroid, then there is a tangle $\mathcal{T}$ of M such that $M = M(\mathcal{T})$. As M and M^* share the same connectivity function λ_M, it follows that $\mathcal{T}$ is a tangle of M^* and thus $M(\mathcal{T}) = M$ is a quotient of M^*. Similarly, M^* is a quotient of M. Therefore, $M = M^*$. $\square$

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Vita

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