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THE KAUFFMAN BRACKET SKEIN MODULE OF THE QUATERNIONIC MANIFOLD

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
John Michael Harris
B.A. in Math., Millsaps College, 1996
M.S., Louisiana State University, 1999
August 2003

Acknowledgments

This dissertation would not be possible without several contributions. It is a pleasure to thank Richard Litherland and Paul van Wamelen for their help in understanding and applying D. H. Lehmer's estimate in Chapter 5.

This work was motivated by previous computations of skein modules by Doug Bullock, Jim Hoste, Gregor Masbaum, and Jozef Przytycki, brought to my attention by my thesis advisor, Patrick Gilmer.

It is a pleasure also to thank Louisiana State University for providing me with a pleasant working environment. A special thanks to Dr. Gilmer for his ideas, guidance, and support.

This dissertation is dedicated to my family, to Christine, and to R.D.B. for their support and encouragement.

Table of Contents

Abstract

In this work, we study the structure of the Kauffman bracket skein module of the quaternionic manifold over the field of rational functions.

We begin with a brief survey of manifolds whose Kauffman bracket skein modules are known, and proceed in Chapter 2 by recalling the facts from Temperley-Lieb recoupling theory that we use in the proofs.

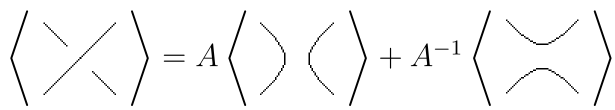
In Chapter 3, using recoupling theory and with Mathematica's assistance, we index an infinite presentation of the skein module, and conjecture that it is five-dimensional.

In Chapter 4, using a new set of relations, we prove that the skein module is indeed spanned by five elements, again using Mathematica for the difficult computations. Using the quantum invariants of these skein elements and the $\mathbb{Z}_2$-homology of the manifold, we determine that they are linearly independent in Chapter 5.

In Chapter 6, we conclude with a few brief remarks about future uses and extensions of this work. In the appendices, we present the Mathematica code referenced in Chapters 3 and 4, and we give a proof, due to Paul van Wamelen, of a lemma needed in Chapter 5 concerning Gauss sums.

1. Introduction

In [K], Kauffman presents an elegant construction of the Jones polynomial, an invariant of oriented links in S^3, by constructing a new invariant, the Kauffman bracket polynomial. The Kauffman bracket is an invariant of unoriented framed links in S^3, defined by the following skein relations:

1. 
2. $< L \cup \text{unknot} >= (-A^{-2} - A^2) < L >$

For the invariant to be well-defined, one also must normalize it by choosing a value for the empty link. $<$ empty link $>= 1$, for instance.

Alternatively, we can use the skein relations to construct a module of equivalence classes of links in S^3, or, for that matter, in any oriented 3-manifold. See Przytycki ([P1]) and Turaev ([T]).

Definition 1.1. *Let M be an oriented 3-manifold, and let R be a commutative ring with identity, with a specified unit A. The Kauffman bracket skein module of M, denoted $S(M; R, A)$, or simply $S(M)$, is the free R-module generated by the framed isotopy classes of unoriented links in M, including the empty link, quotiented by the skein relations which define the Kauffman bracket.*

Since every crossing and unknot can be eliminated from a link in S^3 by the skein relations, every skein element of $S(S^3)$ is a multiple of the empty link, and so, $S(S^3)$ is a free module, generated by the empty link.

Hoste and Przytycki have, in fact, computed the skein modules of all of the closed, oriented manifolds of genus 1: $S(L(p,q))$, which is free on $\lfloor \frac{p}{2} \rfloor + 1$ generators

FIGURE 1.1. Surgery descriptions of the quaternionic manifold

([HP1]), and $S(S^1 \times S^2; R, A) \cong R \oplus (\bigoplus_{i=1}^{\infty} R/(1 - A^{2i+4}))$, where $R = \mathbb{Z}[A^{\pm 1}]$ ([HP2]). They have also computed the skein modules of I-bundles over surfaces ([HP1]).

Additionally, Bullock has found a presentation for the complement of a $(2, q)$ torus knot in [B1], and has determined whether or not the the skein module of the result of integral surgery on a trefoil is finitely generated in [B2].

All of the computations mentioned above have been carried out in the most general case: the modules are $\mathbb{Z}[A^{\pm 1}]$-modules. By restricting ourselves to the field of rational functions of A, we are able to add an irreducible genus 2 manifold to the list.

Let M be the quaternionic manifold, S^3 quotiented by the action of the quaternionic group Q_8: M can be obtained by identifying opposite faces of a cube with one-quarter twists.

M is the three-fold cover of S^3 branched over the trefoil, and is an irreducible manifold of genus 2. Rolfsen gives surgery descriptions of this manifold in [Rol]. See Figure 1.1.

Let $\mathcal{R}$ denote $\mathbb{Q}(A)$, the field of rational functions of A. $S(M) = S(M; \mathcal{R}, A)$ is a vector space, and we prove the following result:

Theorem 1.2. $\dim(S(M; \mathcal{R}, A))) = 5$.

Before we proceed, we note that the defining relations of the Kauffman bracket skein module respect $\mathbb{Z}_2$-homology. Since $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $S(M)$ is a direct sum of four submodules $S_1(M)$, $S_i(M)$, $S_j(M)$, and $S_k(M)$.

Furthermore, the permutation group S_6 acts on the manifold, so $S_i(M)$, $S_j(M)$, and $S_k(M)$ are isomorphic. We will see that $S_1(M)$ is 2-dimensional, and that $S_i(M)$, $S_j(M)$, and $S_k(M)$ are all 1-dimensional.

2. Preliminary Remarks

We draw H_2 as a ball before the handles are attached from top to bottom along the dashed lines. Equivalently, one can view the diagrams as pictures of handlebodies whose handles have been cut.

The standard basis B for the Kauffman bracket skein module of the solid handlebody H_g of genus g is the collection of all links without crossings in the standard diagram of H_g. For H_2, we use Bullock's algebraic notation $x^i y^j z^k$ ([B2]). See Figure 2.1.

Definition 2.1. *If $|a-b| \leq c \leq a+b$ and $a+b+c \equiv 0$ mod 2, then the triple $\{a,b,c\}$ is said to be admissible.*

An arc labelled n represents the nth Jones-Wenzl idempotent f_n: f_0 is an empty tangle, f_1 is a single arc, and f_n is a linear combination of (n,n)-tangles defined by the recursive relation in Figure 2.2.

Note that, for f_n to be defined, $\Delta_{k-1} = (-1)^{k-1}[k]$ must be invertible in R for all $k \leq n$. Here the quantum integer $[n] = A^{2-2n} + A^{6-2n} + \cdots + A^{2n-6} + A^{2n-2}$.

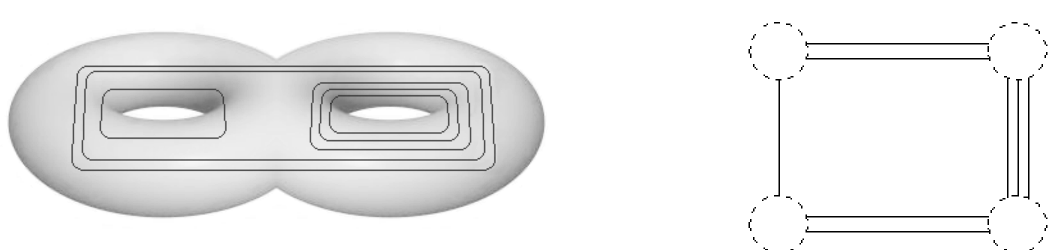


FIGURE 2.1. The link xy^2z^3 in the standard diagram and the cut-open diagram of H_2

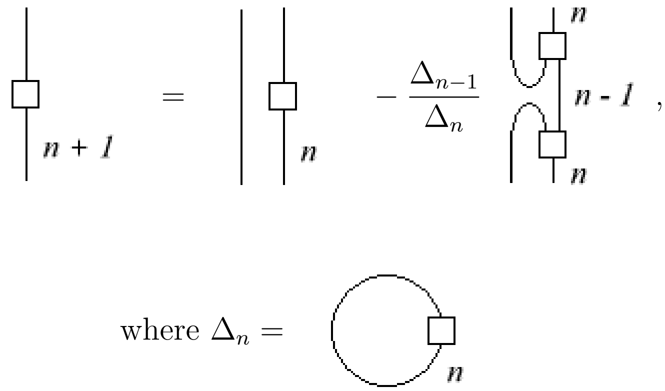


FIGURE 2.2. Definition of Jones-Wenzl idempotents

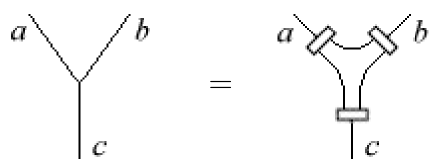
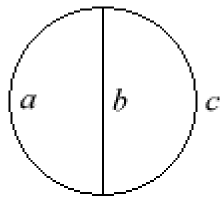


FIGURE 2.3. Trivalent graphs represent linear combinations of links

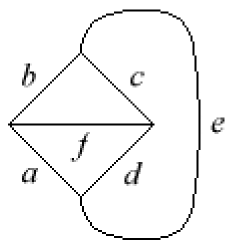
In Figure 2.2, the presence of a idempotent is indicated by a small rectangle. Hereafter, the rectangles will be dropped. Moreover, edges representing f_1 will be left unlabelled.

It is convenient to extend our view of skein modules to include banded trivalent graphs, whose edges are labelled so that at each vertex, the labels form an admissible triple. See [BHMV2, 4.5] for the precise definition of a banded trivalent graph: all trivalent graphs we discuss will be assumed to have such a banding, coming from a regular neighborhood in the surface in which they are drawn. These graphs represent linear combinations of links, as expressed in Figure 2.3.

In [MV], Masbaum and Vogel describe an algorithm for reducing every trivalent graph in a 3-ball to a scalar multiple of the empty link, and trivalent graphs are often used to denote this scalar. In addition to the labelled unknot in Figure 2.2, two of these scalars appear frequently enough to merit symbols:

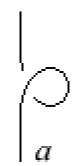


$= \theta(a,b,c)$ and

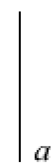


$$= Tet\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}.$$

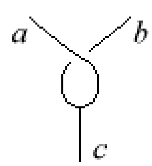
Additionally, we have the following useful local moves:



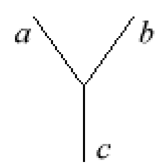
$= (-A)^{a(a+2)}$



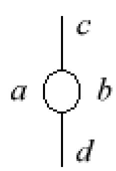
,



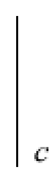
$= \lambda^{a\,b}_{c}$



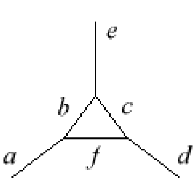
,



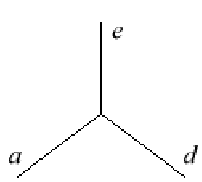
$= \delta^c_d \, \dfrac{\theta(a,b,c)}{\Delta_c}$



, and



$$= \begin{cases} \dfrac{Tet\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}}{\theta(a,d,e)} & \text{, if } (a,d,e) \text{ is admissible} \\ 0 & \text{, otherwise} \end{cases}$$

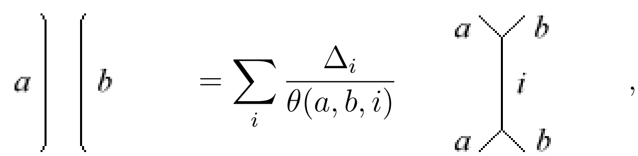


.

Explicit formulas for these scalars in terms of A are given in [MV] and [KL]. The only skein-theoretic derivation of the formula for the tet is given in [MV]. Our notation is the same as in [KL].

Working with trivalent graphs is often much easier than working directly with links, thanks mainly to the following identity:

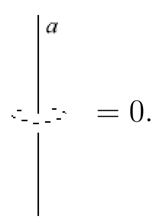
Theorem 2.2. *(Fusion Formula)*



where the sum is over all admissible labellings.

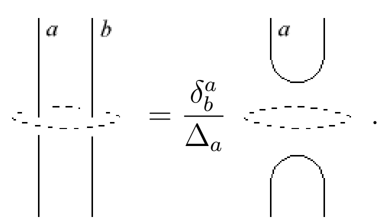
We also have the following well-known theorem. See, for example, [BFK].

Theorem 2.3. *(Sphere Lemma) If $A^k - 1$ is invertible in R for all k and a sphere intersects a skein element in exactly 1 labelled arc, then*

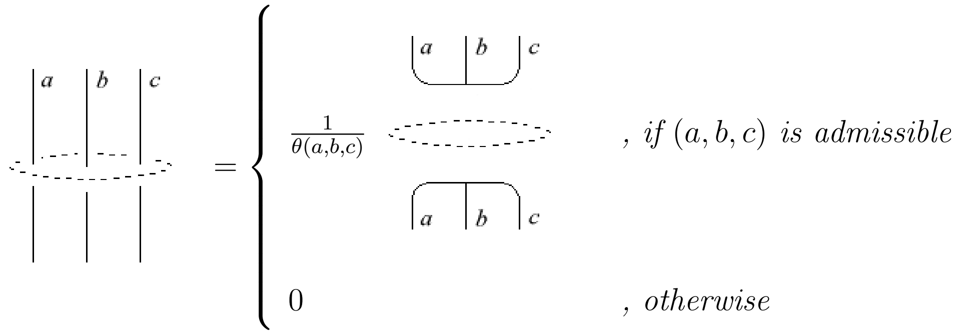


The idea of using the fusion formula in conjunction with the sphere lemma is also well-known. See, for example, [Rob].

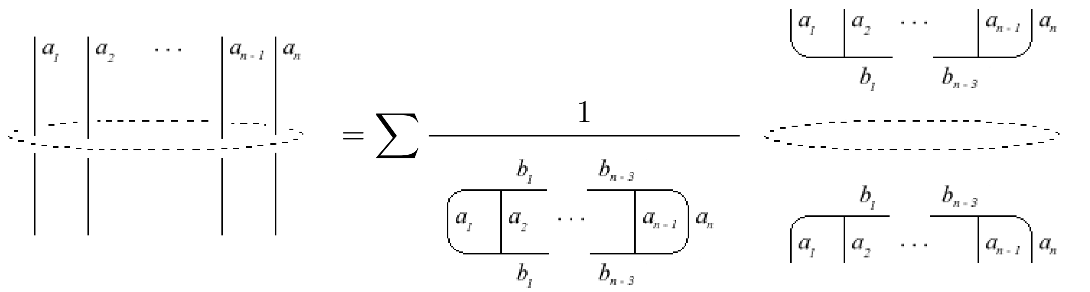
Theorem 2.4. *Suppose $A^k - 1$ is invertible in R for all k. If a sphere intersects a skein element in exactly 2 labelled arcs, then*



If a sphere intersects a skein element in exactly 3 labelled arcs, then



If a sphere intersects a skein element in exactly $n > 3$ arcs, then



where the sum is over all admissible labellings.

Proof. The proof proceeds by induction on n. The case $n = 1$ is the sphere lemma. Using the fusion formula, we can reduce the number of strands by 1 and then apply the inductve hypothesis. The factor resulting from fusion in each term can be absorbed into its coefficient, and for $n > 2$, the coefficient can be rewritten in the desired form with a bigon move. The result follows. $\square$

Hereafter, we will omit the cumbersome piecewise notation: when $\theta(a, b, c)$ appears in the denominator of an fraction, the fraction is to be taken as zero if (a, b, c) is not admissible. We also extend the Dirac delta notation as follows: $\delta(a, b, c)$ is 1 when (a, b, c) is admissible and 0 otherwise.

For our work, we will use an alternate basis for $S(H_2)$: the admissible labellings of a trivalent graph which is itself a deformation retract of H_2. We use the notation (a, b, c): see Figure 2.4. Bullock, Frohman, and Kania-Bartoszynska describe this

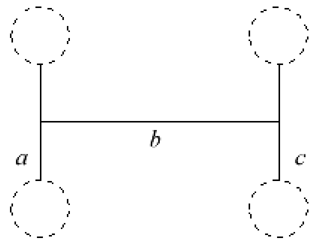


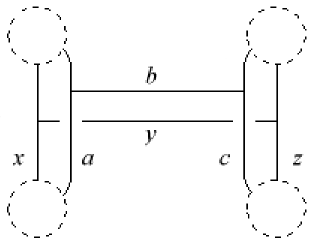
FIGURE 2.4. The trivalent graph (a, b, c) in H_2

basis in [BFK]. We remark that this basis is closely related to the basis for TQFT modules given in [G].

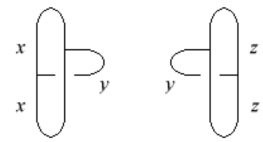
Note that, for (a, b, c) to be admissibly labelled, $b \leq 2a$, $b \leq 2c$, and b must be even.

Note that $S(S^1 \times S^2; \mathcal{R}, A) \cong \mathcal{R}$. Przytycki has shown in [P2] that over $\mathcal{R}$, the skein module of the connected sum of two manifolds is the tensor product of their skein modules. Hence, $S((S^1 \times S^2)\#(S^1 \times S^2); \mathcal{R}, A) \cong \mathcal{R}$. In [G], Gilmer defined an $\mathcal{R}$-valued bracket evaluation of admissibly labelled trivalent graphs in a connected sum of copies of $S^1 \times S^2$ which essentially coincides with this isomorphism, using fusion as above for the evaluation.

By gluing a copy of H_2 to itself with orientation reversed along the identity map on the boundaries, we obtain a Hermitian form $<,>$: $S(H_2) \times S(H_2) \longrightarrow \mathcal{R}$, and $\{(a, b, c)\}$ is orthogonal with respect to this product. (Here $\mathcal{R}$ is equipped with the involution which sends A to A^{-1}.) See Figure 2.5. In the figure, the graph in the outer, undrawn handlebody has been pushed into the inner handlebody. This form is closely related to the Yang-Mills measure on the skein algebra of a surface discussed by Bullock, Frohman, and Kania-Bartoszynska in [BFK]. Gilmer discussed the form $<,>$ and the orthogonality of our basis of trivalent graphs in a course on quantum topology in the fall of 2001.



$$= \frac{\delta_x^a \; \delta_y^b \; \delta_z^c}{\Delta_x \; \Delta_y \; \Delta_z}$$



$$= \frac{\lambda_x^{y\,x}}{\lambda_z^{y\,z}} \; \frac{\delta_x^a \; \delta_y^b \; \delta_z^c}{\Delta_x \; \Delta_y \; \Delta_z} \; \theta(x,x,y) \; \theta(y,z,z)$$

$$= \frac{\delta_x^a \; \delta_y^b \; \delta_z^c \; \theta(x,x,y) \; \theta(y,z,z)}{\Delta_x \; \Delta_y \; \Delta_z}$$

FIGURE 2.5. Computation of $< (x,y,z), (a,b,c) >$

Note that, because $\lambda_a^{b\,a} = \lambda_c^{b\,c}$, we can twist (a,b,c) out of the way, simplifying the computation of $< (x,y,z), (a,b,c) >$. See Figure 2.6.

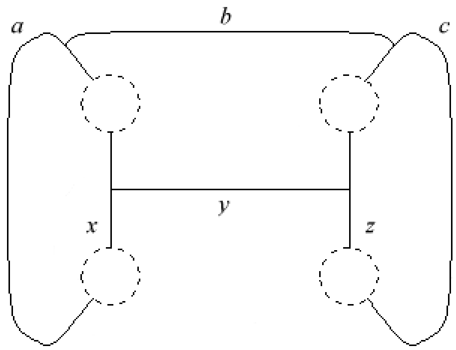
We now prove that the alternate basis described above is indeed a basis, as there does not seem to be a proof in the literature. In the proof, we use the principle of well-founded induction:

A well-founded order on a set X is a partial order such that every nonempty subset of X has a minimal element. The principle of well-founded induction states that given a property p defined on a well-founded ordered set X, if p holds for every minimal element of X, and if, for every $y \in X$, p holds for y if p holds for every $x < y$ in X, then p holds for every element of X. See, for example, [Mo].

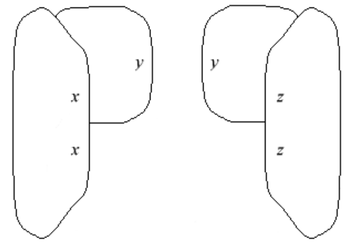
Proposition 2.5. *$\{(a,b,c)\}$ is a basis of $S(H_2; \mathcal{R}, A)$.*

Proof. Let $\leq_B$ denote the partial ordering on the standard basis B defined by

$$x^{i_1} y^{j_1} z^{k_1} \leq_B x^{i_2} y^{j_2} z^{k_2} \Leftrightarrow i_1 \leq i_2, j_1 \leq j_2, \text{ and } k_1 \leq k_2.$$



$$= \frac{\delta_x^a \, \delta_y^b \, \delta_z^c}{\Delta_x \, \Delta_y \, \Delta_z}$$



$$= \frac{\delta_x^a \, \delta_y^b \, \delta_z^c \, \theta(x,x,y) \, \theta(y,z,z)}{\Delta_x \, \Delta_y \, \Delta_z}$$

FIGURE 2.6. Alternate computation of $<(x,y,z),(a,b,c)>$

Since $\leq_B$ is well-founded, we can show that $\{(a,b,c)\}$ spans $S(H_2;\mathcal{R},A)$ by well-founded induction:

The empty link belongs to $\{(a,b,c)\}$, and so it can certainly be written as a linear combination of elements of $\{(a,b,c)\}$.

Suppose that every $x^{i_1}y^{j_1}z^{k_1} <_B x^i y^j z^k$ can be written as a linear combination of elements of $\{(a,b,c)\}$. $(i+j, 2j, k+j) = x^i y^j z^k +$ a linear combination of lesser terms, so $x^i y^j z^k$ can be expressed as a linear combination of elements in $\{(a,b,c)\}$. Hence, $\{(a,b,c)\}$ spans $S(H_2;\mathcal{R},A)$.

Now suppose that a linear combination $\sum C_{x,y,z}(a,b,c) = 0$. Then, for each (x,y,z),

$$C_{x,y,z} <(x,y,z),(x,y,z)> = <\sum C_{x,y,z}(a,b,c),(x,y,z)> = 0.$$

Hence, $C_{x,y,z} = 0$ for each (x,y,z), and so, $\{(a,b,c)\}$ is linearly independent. $\square$

3. Experimental Evidence

By viewing a closed 3-manifold in terms of its handle decomposition, we can hope to analyze the structure of its skein module. A genus 2 closed oriented manifold is built by adding 2 solid cylinders, or 1-handles, to a ball, or 0-handle, and then by attaching 2 thickened disks, or 2-handles, and closed up by adding a 3-handle. Before the 2-handles are attached, the manifold is a solid handlebody. As each 2-handle is added, a set of relations is introduced among the skein elements, namely those obtained from sliding arcs over the newly attached thickened disk. The final 3-handle has no effect on the skein module.

In this way, we obtain a presentation of the skein module. The generators are the basis elements of the solid handlebody, and the relations are given by the ways in which arcs may slide across the 2-handles. When applied to links in the manifolds, this is the most common method for generating a presentation of the module. See, for example, Hoste and Przytycki ([HP1],[HP2]), and Bullock ([B1], [B2]). In [M], Masbaum points out that one may use relative skein modules to produce a complete set of relations that emerge from sliding. Gilmer suggested that we might try something similar, using admissibly labeled trivalent graphs, and that we could use orthogonality to write the relations in terms of the basis $\{(a, b, c)\}$ of the solid handlebody.

Following the method Rolfsen applies to the Poincare homology sphere in [Rol], we can construct a Heegaard splitting of the quaternionic manifold from its surgery description. See Figure 3.1. The curves mark the boundaries of the attached 2-handles.

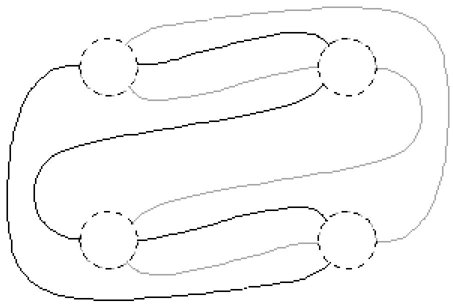


FIGURE 3.1. A Heegaard splitting of the quaternionic manifold

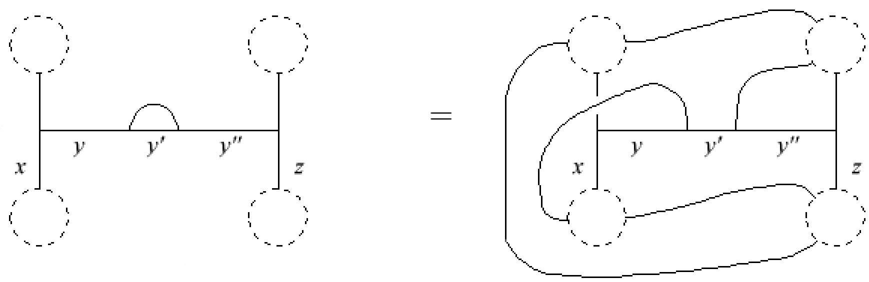


FIGURE 3.2. Sliding across an attached handle

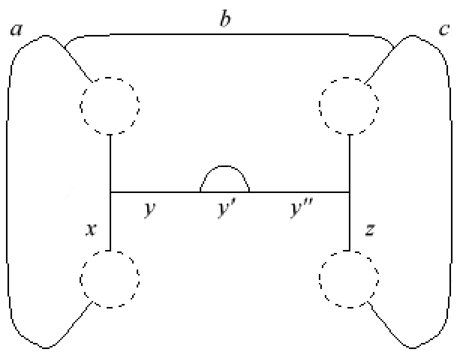
For the quaternionic manifold, a slide is depicted in Figure 3.2.

We now rewrite each side of the relation in Figure 3.2 in terms of our basis: the coefficients are computed in Figures 3.3, 3.4, 3.5, and 3.6.

Due to the symmetry of the Heegard splitting, the coefficients computed from sliding across one handle yields the coefficients obtained from sliding across the other handle. The coefficients are quite complicated, but using a symbolic algebra software package like Mathematica, we can compute them.

Order $\{(a, b, c)\}$ such that $(a', b', c') > (a, b, c)$

- if $m' = \max(a', c') > m = \max(a', c')$, or
- if $m' = m$ and $a' > a$, or



$$= \delta^{y}_{y"} \, \frac{\theta(y, y', y")}{\Delta_y} \; < (x, y, z), (a, b, c) >$$

FIGURE 3.3. Rewriting the left side of the relation

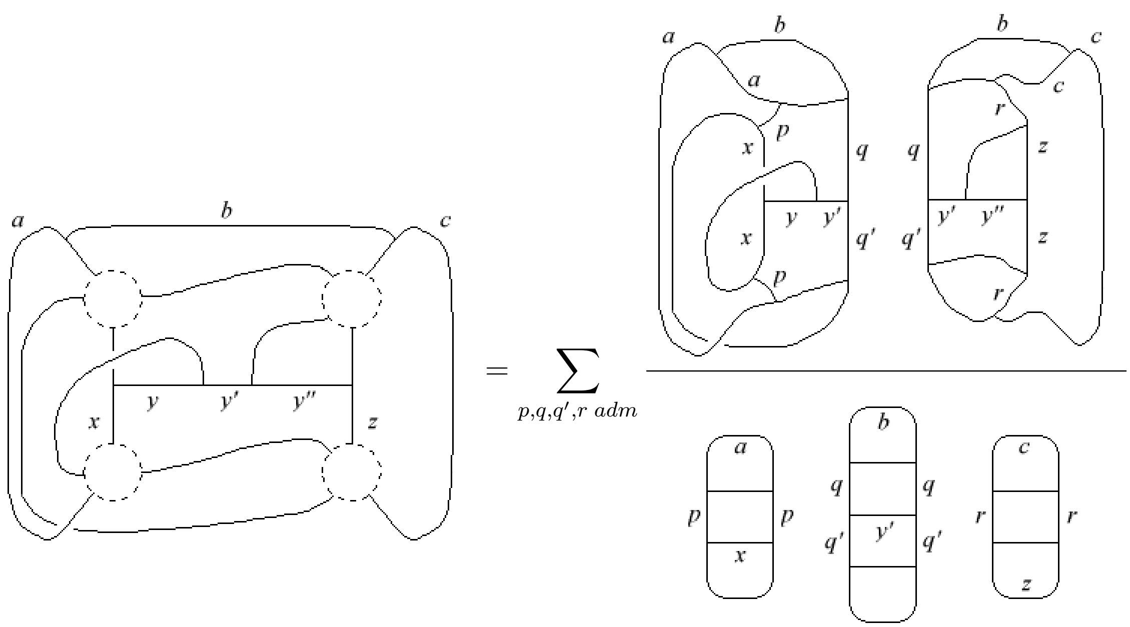
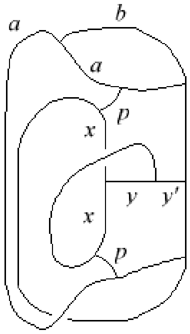
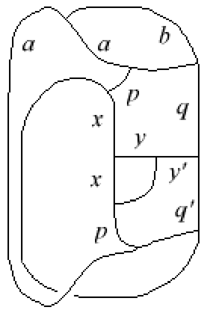


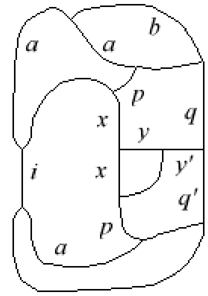
FIGURE 3.4. Rewriting the right side of the relation



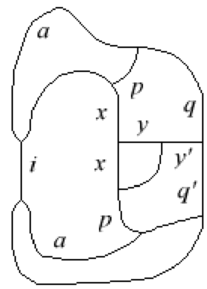
$$= \frac{\lambda^{y\,1}_{y'}}{\lambda^{x\,1}_{p}}$$



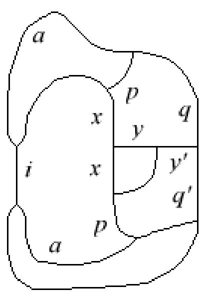
$$= \sum_{i\ adm} \frac{\lambda^{y\,1}_{y'}\ \lambda^{a\,1}_{i}\ \Delta_i}{\lambda^{x\,1}_{p}\ \theta(a,i,1)}$$



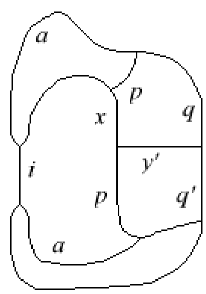
$$= \sum_{i\ as\ above} \frac{\lambda^{y\,1}_{y'}\ \lambda^{a\,1}_{i}\ \Delta_i\ \ Tet\begin{bmatrix} a & p & a \\ q & b & 1 \end{bmatrix}}{\lambda^{x\,1}_{p}\ \theta(a,i,1)\ \theta(q,a,p)}$$



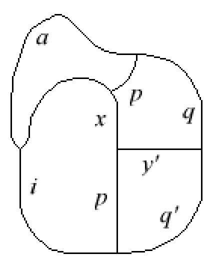
where



$$= \frac{Tet\begin{bmatrix} 1 & y' & p \\ x & x & y \end{bmatrix}}{\theta(x,p,y')}$$

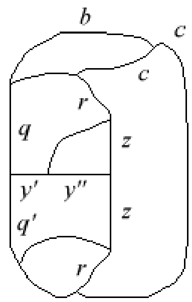


$$= \frac{Tet\begin{bmatrix} 1 & y' & p \\ x & x & y \end{bmatrix}\ \ Tet\begin{bmatrix} 1 & p & q' \\ i & 1 & a \end{bmatrix}}{\theta(x,p,y')\ \theta(p,q',i)}$$

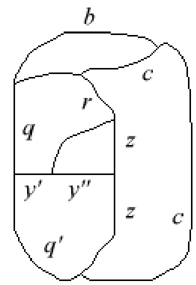


$$= \frac{Tet\begin{bmatrix} 1 & y' & p \\ x & x & y \end{bmatrix}\ \ Tet\begin{bmatrix} 1 & p & q' \\ i & 1 & a \end{bmatrix}\ \ Tet\begin{bmatrix} p & q & x \\ i & 1 & a \end{bmatrix}\ \ Tet\begin{bmatrix} q & q' & i \\ p & x & y' \end{bmatrix}}{\theta(x,p,y')\ \theta(p,q',i)\ \theta(x,q,i)}$$

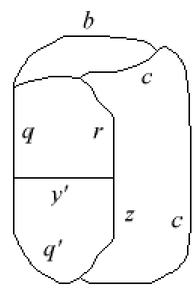
FIGURE 3.5. Rewriting the right side of the relation, continued



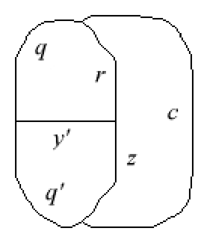
$$= \frac{Tet\begin{bmatrix} r & z & c \\ q' & 1 & 1 \end{bmatrix}}{\theta(q',c,z)}$$



$$= \frac{Tet\begin{bmatrix} r & z & c \\ q' & 1 & 1 \end{bmatrix} \; Tet\begin{bmatrix} y" & z & y' \\ r & 1 & z \end{bmatrix}}{\theta(q',c,z)\;\theta(y',r,z)}$$



$$= \frac{Tet\begin{bmatrix} r & z & c \\ q' & 1 & 1 \end{bmatrix} \; Tet\begin{bmatrix} y" & z & y' \\ r & 1 & z \end{bmatrix} \; Tet\begin{bmatrix} b & q & c \\ r & c & 1 \end{bmatrix}}{\theta(q',c,z)\;\theta(y',r,z)\;\theta(q,c,r)}$$



$$= \frac{Tet\begin{bmatrix} r & z & c \\ q' & 1 & 1 \end{bmatrix} \; Tet\begin{bmatrix} y" & z & y' \\ r & 1 & z \end{bmatrix} \; Tet\begin{bmatrix} b & q & c \\ r & c & 1 \end{bmatrix} \; Tet\begin{bmatrix} q & c & q' \\ z & y' & r \end{bmatrix}}{\theta(q',c,z)\;\theta(y',r,z)\;\theta(q,c,r)}$$

FIGURE 3.6. Rewriting the right side of the relation, completed

$$\begin{pmatrix} 1 & \frac{A^6 (-1+A^4)}{1+A^8} & -\frac{A^{12}}{(1+A^4)(1+A^4+A^8)} & 0 & 0 & 0 & \frac{A^4 (-1-A^8+A^{12})}{(1+A^4)(1+A^8)} & 0 & \frac{1+A^4+2A^8+A^{12}+4A^{16}+5A^{20}+3A^{24}-A^{32}}{A^4 (1+A^4)(1+A^8)(1+A^4+A^8)} & \frac{A^4 (2+A^4+A^8-A^{12})}{(1+A^4)(1+A^8)} \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & \frac{1}{A^8} - \frac{1}{A^4} + A^4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{A^8}{(1+A^8)(1+A^4+A^8)} & 0 & -\frac{A^4 (-1+A^4)^2 (1+A^4)}{1+A^8} & \frac{1+A^4+A^8}{1+A^8} \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{A^2}{1+A^4} & 0 & \frac{A^2 (-1+2A^4+A^8)}{1+A^4} & \frac{1+A^4+A^8}{A^2+A^6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

FIGURE 3.7. Relators for $S_1(M)$

- if $m' = m$, $a' = a$, and $c' > c$, or
- if $m' = m$, $a' = a$, $c' = c$, and $b' > b$.

Using the code in Appendix A, we can generate increasingly large matrices whose rows are the coefficients of the terms, ordered from highest to lowest, appearing in the relators obtained from sliding across one attached handle or the other. The rows themselves are similarly ordered, using the labels of the graph on the left side of the relation in 3.2. We then row-reduce the matrices, obtaining new relators for the skein module.

In Figures 3.7, 3.8, and 3.9 we give early matrices yielding relators for $S_1(M)$, $S_i(M)$, and $S_k(M)$, respectively. (Due to the symmetry of the Heegaard splitting, the matrices we obtain for $S_j(M)$ are identical to those for $S_i(M)$.)

In 3.7, the columns list, from left to right, coefficients for the terms $(2, 4, 4)$, $(2, 2, 4)$, $(2, 0, 4)$, $(0, 0, 4)$, $(2, 4, 2)$, $(2, 2, 2)$, $(2, 0, 2)$, $(2, 0, 0)$, $(0, 0, 2)$, and $(0, 0, 0)$.

From these relators alone, we see that $(0, 0, 0)$, $(0, 0, 2)$, $(2, 0, 2)$, $(2, 2, 4)$, and $(2, 0, 4)$ span the submodule generated by these ten triples.

Hereafter, we use $(R, A) = (\mathbb{Q}, 2)$, since we suspect that the module over this ring is similar to the module over $\mathcal{R}$, and because the coefficients obtained in the indeterminate case become quite complex.

For $S_i(M)$, $(1, 0, 0)$ and three other triples span the submodule generated by the triples involved.. For $S_j(M)$, $(0, 0, 1)$ and three others suffice.

$$
\begin{pmatrix}
1. & 0. & -14.938 & 3.51482 & 0. & 0. & 0. & 0. & -52.283 & 0. & 0. & 0. & 2791.96 \\
0. & 1. & 3.73541 & -0.882568 & 0. & 0. & 0. & 0. & 14.0039 & 0. & 0. & 0. & -1167.81 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & -0.235294 & 0. & 0. & 0. & 99.9626 \\
0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & -1. & 0. & 0. & 0. & 63.9385 \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -0.00364875 & 0. & 0. & 0. & -1248.97 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -0.235294 & 0. & 0. & 0. & 340.56 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & -1.29688 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 18.5294 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -5.25 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.
\end{pmatrix}
$$

FIGURE 3.8. Relators for $S_i(M)$ (and $S_j(M)$)

$$
\begin{pmatrix}
1. & 0. & -14.938 & 3.51482 & 0. & 0. & 0. & 0. & 0. & -52.2822 & 0. & 0. & 0. & 0. & 0. & 13123.1 \\
0. & 1. & 3.73541 & -0.882568 & 0. & 0. & 0. & 0. & 0. & 14.0037 & 0. & 0. & 0. & 0. & 0. & -3805.78 \\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & -0.235294 & 0. & 0. & 0. & 0. & 0. & 128.404 \\
0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & -1. & 0. & 0. & 0. & 0. & 0. & 255.938 \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & -3.5615\times 10^{-6} & 0. & 0. & 0. & 0. & 0. & 15841.1 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -0.00364875 & 0. & 0. & 0. & 0. & 0. & -4070.01 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -0.235294 & 0. & 0. & 0. & 0. & 0. & 1091.86 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 15.7385 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & -4.25 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 15.7385 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -4.25 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 2.96471 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.
\end{pmatrix}
$$

FIGURE 3.9. Relators for $S_k(M)$

$$
\left(\begin{array}{cccccccccccccccccccccccccccc}
1. & 0. & 0. & 59.7502 & -14.0589 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 209.129 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.34361\times10^{7} & -839757. \\
0. & 1. & 0. & -14.938 & 3.51482 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -52.2693 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 3.63535\times10^{6} & 227209. \\
0. & 0. & 1. & 3.73541 & -0.882568 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 14.0002 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -978399. & -61149.9 \\
0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -0.00364875 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -65023.8 & -4063.99 \\
0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & -0.235294 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 32895.7 & 2055.98 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & -1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 65535. & 4095.94 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.06615 & -0.0666342 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & -2.1737\times10^{-10} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.65725\times10^{7} & -1.03988\times10^{6} \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & -3.5615\times10^{-6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4.17468\times10^{6} & 260921. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -0.00364875 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.04445\times10^{6} & -65277.8 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -0.235294 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 279615. & 17476. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -4062.15 & -253.946 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1027.49 & 64.2187 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -273. & -17.0625 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.05859 & -0.0664063 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -4062.15 & -253.946 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 1027.49 & 64.2187 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & -273. & -17.0625 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & -1.05859 & -0.0664063 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & -238.194 & 1.05837 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 63.5294 & 3.76379 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -17. & -1.06641 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -1. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
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\end{array}\right)
$$

FIGURE 3.10. More relators for $S_1(M)$

For $S_k(M)$, $(1, 0, 1)$ and three other triples span the submodule generated by the involved triples.

In Figure 3.10, we see that of the five needed to span the submodule of $S_1(M)$ mentioned above, only $(0, 0, 0)$ and $(0, 0, 2)$ remain, joined by three new terms.

This pattern persists for all of the submodules. Hence, we conjectured that $S_1(M)$ is 2-dimensional, and each of the other three subspaces are 1-dimensional.

4. Spanning

By using simpler relations and, for the moment, giving up on working with a complete set of relations, we can show that, as the evidence presented in the previous chapter suggests, $S(M)$ is indeed generated by five elements.

To this end, we present in Figures 4.1, 4.5, 4.6, 4.9, 4.13, and 4.14 six sets of simpler relations obtained from sliding over the attached 2-handles, and rewrite them in terms of our basis.

Note that the diagrams of the second and fifth slides are just the diagrams of the first and fourth, respectively, rotated 180 degrees. Hence, the computations of the coefficients for the first and fourth relations yield the coefficients for the second and fifth as well.

Let $r_i =$ (right-hand side of relation i) - (left-hand-side of relation i).

Note that, due to admissibility conditions, (a, b, c) can only appear in r_1 if $a = \alpha, b \in \{\beta - 2, \beta, \beta + 2\}$, and $c = \gamma \pm 1$. See Figures 4.2 and 4.3.

(a, b, c) can only appear in r_2 if $a = \alpha \pm 1, b \in \{\beta - 2, \beta, \beta + 2\}$, and $c = \gamma$.

(a, b, c) can only appear in r_3 if $a = \alpha \pm 1, b \in \{0, 2\}$, and $c = \gamma \pm 1$. See Figures 4.7 and 4.8.

(a, b, c) can only appear in r_4 if $a = \alpha \pm 1, b \in \{0, 2, 4\}$, and $c \in \{0, 2\}$. See Figures 4.10 and 4.11.

(a, b, c) can only appear in r_5 if $a \in \{0, 2\}, b \in \{0, 2, 4\}$, and $c = \gamma \pm 1$.

(a, b, c) can only appear in r_6 if $a \in \{\alpha - 2, \alpha, \alpha + 2\}, b \in \{0, 2, 4\}$, and $c \in \{\alpha - 2, \alpha, \alpha + 2\}$. See Figures 4.14 and 4.15.

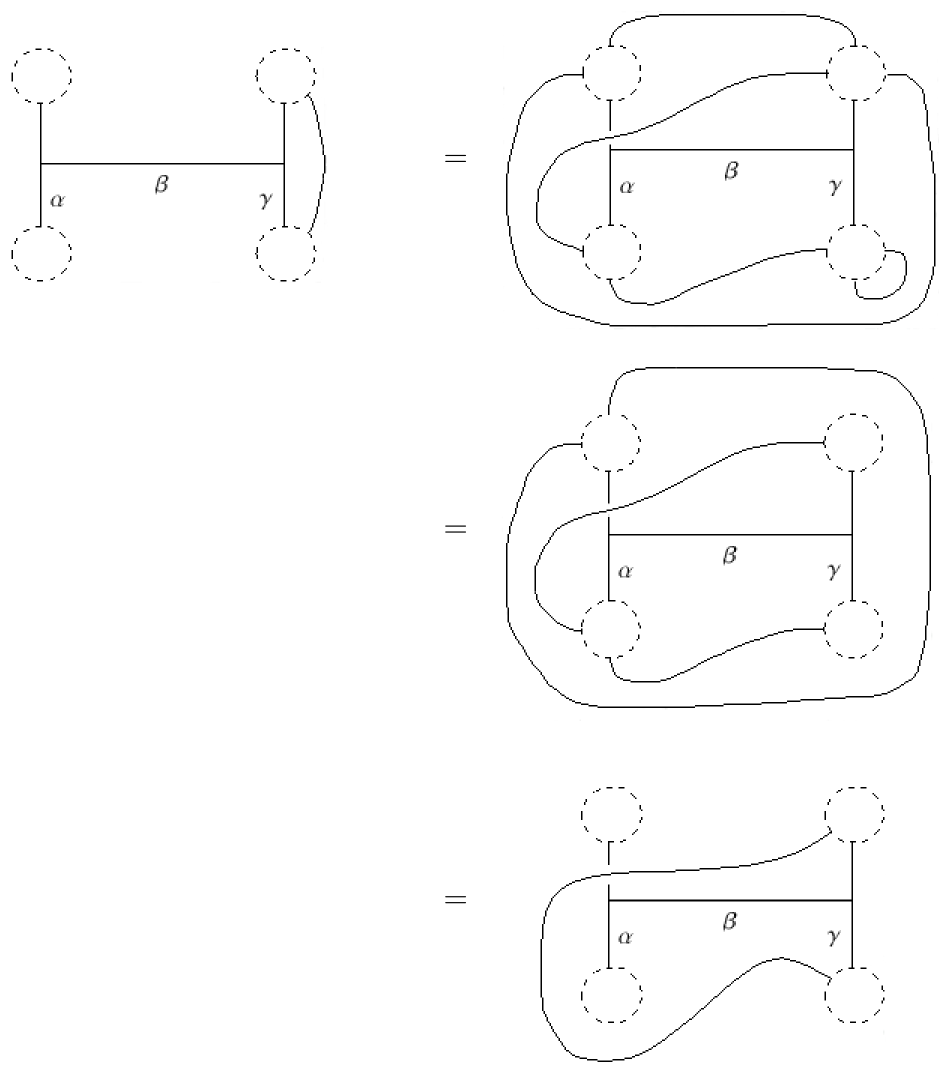
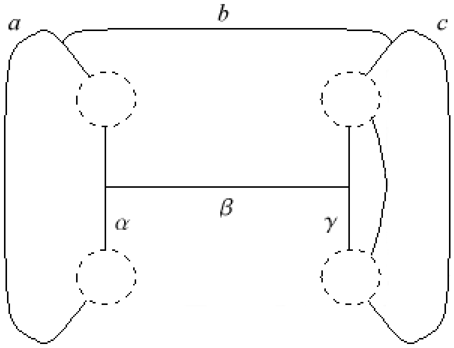
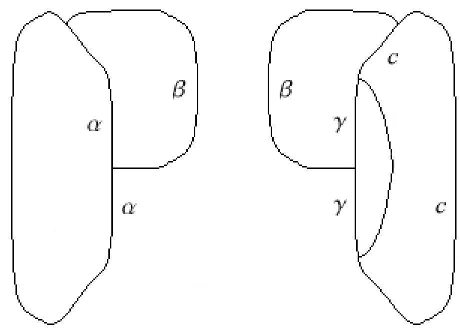


FIGURE 4.1. Relation 1



$=$ $\delta_\alpha^a \, \delta_\beta^b$



$$\overline{\qquad\qquad \Delta_\alpha \, \Delta_\beta \, \theta(c, \gamma, 1) \qquad\qquad}$$

$$= \frac{\delta_\alpha^a \, \delta_\beta^b \, \theta(\alpha, \alpha, \beta) \;\; Tet \begin{bmatrix} c & \gamma & \beta \\ \gamma & c & 1 \end{bmatrix}}{\Delta_\alpha \, \Delta_\beta \, \theta(c, \gamma, 1)}$$

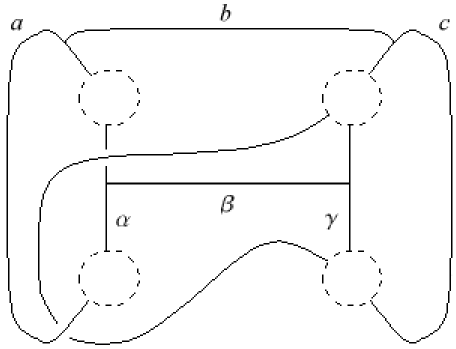
FIGURE 4.2. Rewriting the left side of relation 1

Proposition 4.1. *$S(M)$ is spanned by* $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, $(1,0,1)$, *and* $(0,0,2)$.

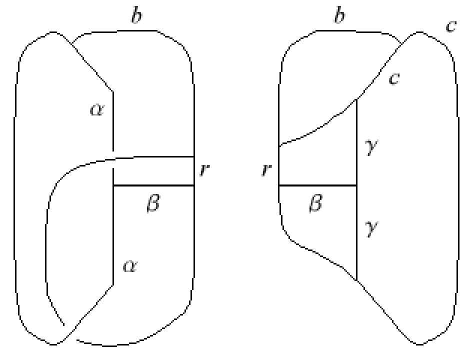
Proof. We proceed by induction on (a, b, c). Using the six sets of relations, we can rewrite (x, y, z) as a linear combination of terms appearing earlier in the ordering introduced in the previous chapter.

The proof splits into five cases:

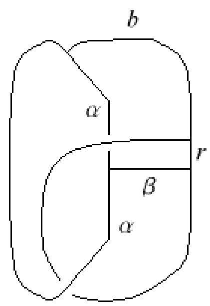
1. $x \geq 1$, $y \geq 2$, and $z \geq 1$,
2. $x \geq 1$, $y = 0$, $z \geq 1$, and $z \neq x$,
3. $x \geq 2$ and $y = z = 0$,
4. $x = y = 0$ and $z > 2$, and
5. $x \geq 2$, $y = 0$, and $z = x$.



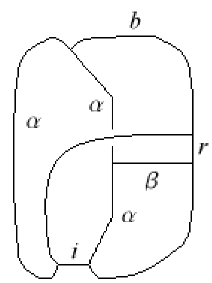
$$= \sum_{r\ adm} \frac{\delta^a_\alpha \Delta_r \quad [\text{diagram}]}{\Delta_\alpha\ \theta(\beta, r, 1)\ \theta(b, r, 1)\ \theta(c, \gamma, 1)}$$



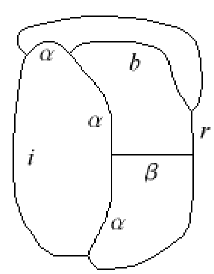
where



$$= \sum_{i\ adm} \frac{\Delta_i\ \lambda^{\alpha\,1}_i}{\theta(\alpha, 1, i)}$$

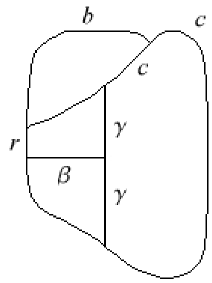


$$= \sum_{i\ adm} \frac{\Delta_i\ (\lambda^{\alpha\,1}_i)^2}{\lambda^{b\,1}_r\ \theta(\alpha, i, 1)}$$



$$= \sum_{i\ as\ above} \frac{\Delta_i\ (\lambda^{\alpha\,1}_i)^2\ \ Tet\begin{bmatrix} r & i & 1 \\ \alpha & b & \alpha \end{bmatrix}\ \ Tet\begin{bmatrix} 1 & i & r \\ \alpha & \beta & \alpha \end{bmatrix}}{\lambda^{b\,1}_r\ \theta(\alpha, i, 1)\ \theta(\alpha, r, i)}$$

FIGURE 4.3. Rewriting the right side of relation 1



$$= \frac{Tet\begin{bmatrix} c & r & b \\ 1 & c & \gamma \end{bmatrix} \; Tet\begin{bmatrix} 1 & r & c \\ \gamma & \gamma & \beta \end{bmatrix}}{\theta(c, r, \gamma)}$$

FIGURE 4.4. Rewriting the right side of relation 1, completed

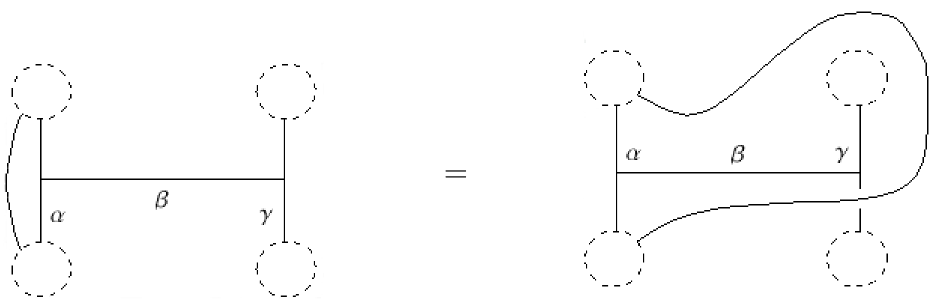


FIGURE 4.5. Relation 2

After these have been rewritten, only $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, $(1,0,1)$, and $(0,0,2)$ remain.

Case 1: $x \geq 1$, $y \geq 2$, and $z \geq 1$.

If we let $(\alpha, \beta, \gamma) = (x, y-2, z-1)$, then (x, y, z) is the highest term appearing in relator r_1, when r_1 is rewritten in our basis.

So, to show that (x, y, z) is a linear combination of lesser terms, it suffices to show that its coefficient

$$\frac{\langle r_1, (x, y, z)\rangle}{\langle (x, y, z), (x, y, z)\rangle}$$

is nonzero.

Using Mathematica, we determine that, since $y \leq 2x$,

$$\frac{\langle r_1, (x, y, z)\rangle}{\langle (x, y, z), (x, y, z)\rangle} = A^{-2-2x-y}(A^{2+2x} - A^y))(A^{2+2x} + A^y)) \neq 0.$$

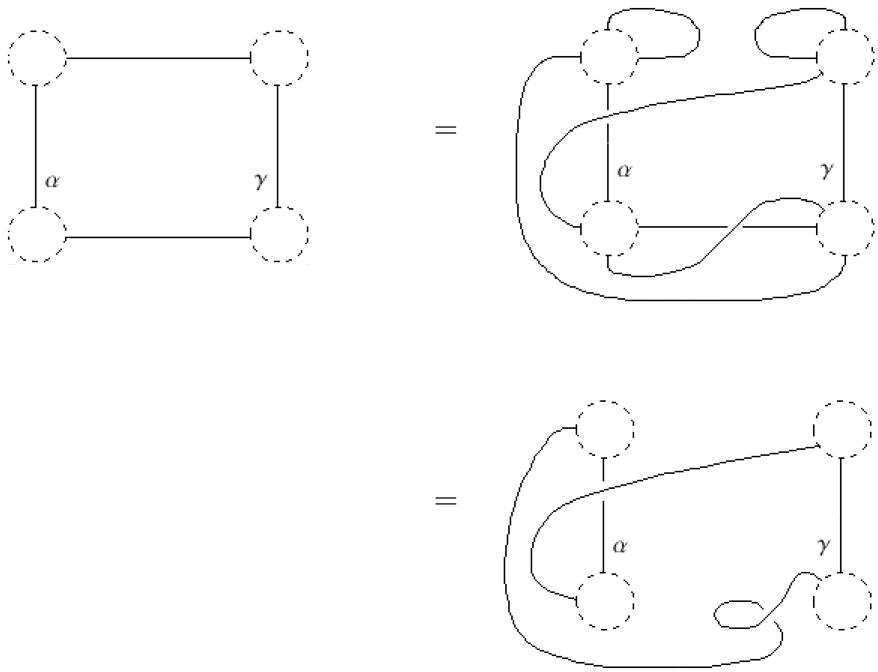
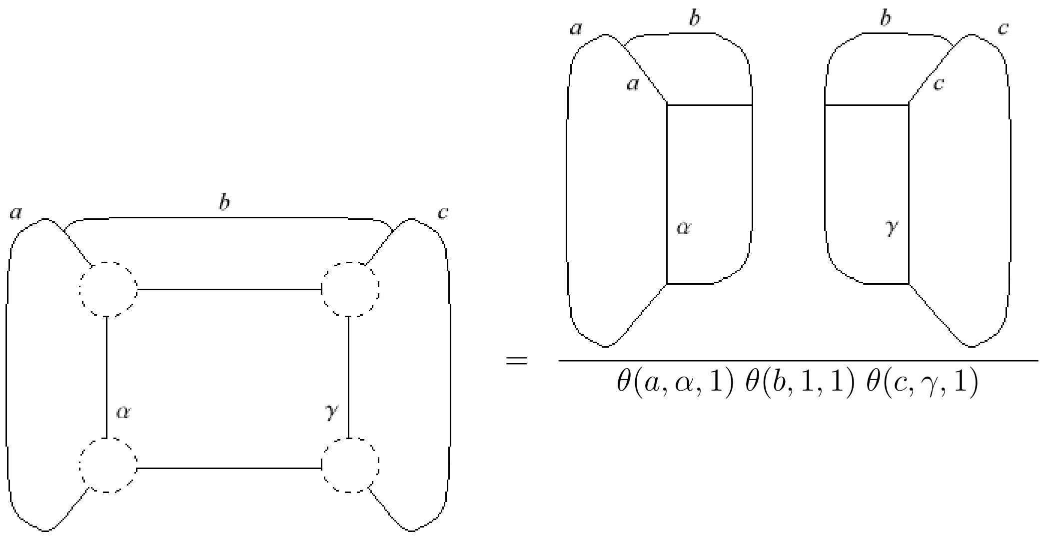
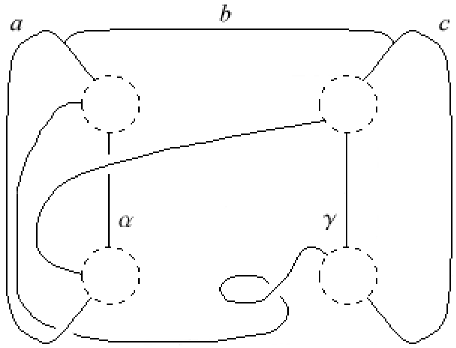


FIGURE 4.6. Relation 3

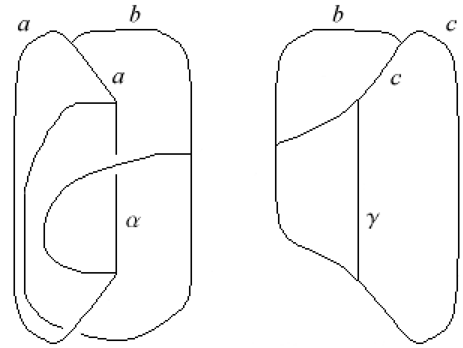


$$= \frac{Tet\begin{bmatrix} 1 & a & b \\ a & 1 & \alpha \end{bmatrix} \; Tet\begin{bmatrix} c & 1 & b \\ 1 & c & \gamma \end{bmatrix}}{\theta(a,\alpha,1)\,\theta(b,1,1)\,\theta(c,\gamma,1)}$$

FIGURE 4.7. Rewriting the left side of relation 3

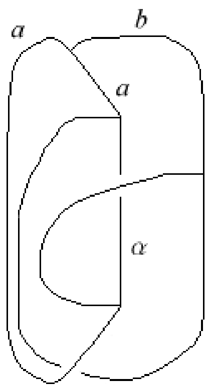


$= -A^3$

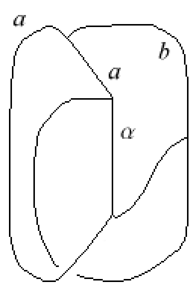


$$\overline{\theta(a,\alpha,1)\;\theta(b,1,1)\;\theta(c,\gamma,1)}$$

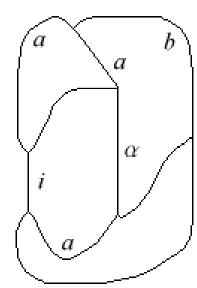
where



$= (\lambda_a^{\alpha\,1})^{-1}$

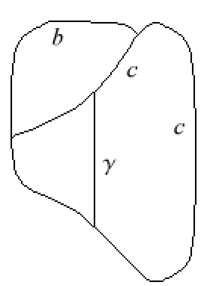


$$= (\lambda_a^{\alpha\,1})^{-1} \sum_{i\ adm} \frac{\Delta_i\ \lambda_i^{a\,1}}{\theta(a,i,1)}$$



$$= (\lambda_a^{\alpha\,1})^{-1} \sum_{i\ as\ above} \frac{\Delta_i\ \lambda_i^{a\,1}\ \ Tet\begin{bmatrix} i & a & b \\ a & \alpha & 1 \end{bmatrix}\ \ Tet\begin{bmatrix} \alpha & 1 & b \\ 1 & i & a \end{bmatrix}}{\theta(a,i,1)\;\theta(b,\alpha,i)}$$

and



$$= \quad Tet\begin{bmatrix} c & 1 & b \\ 1 & c & \gamma \end{bmatrix}$$

FIGURE 4.8. Rewriting the right side of relation 3

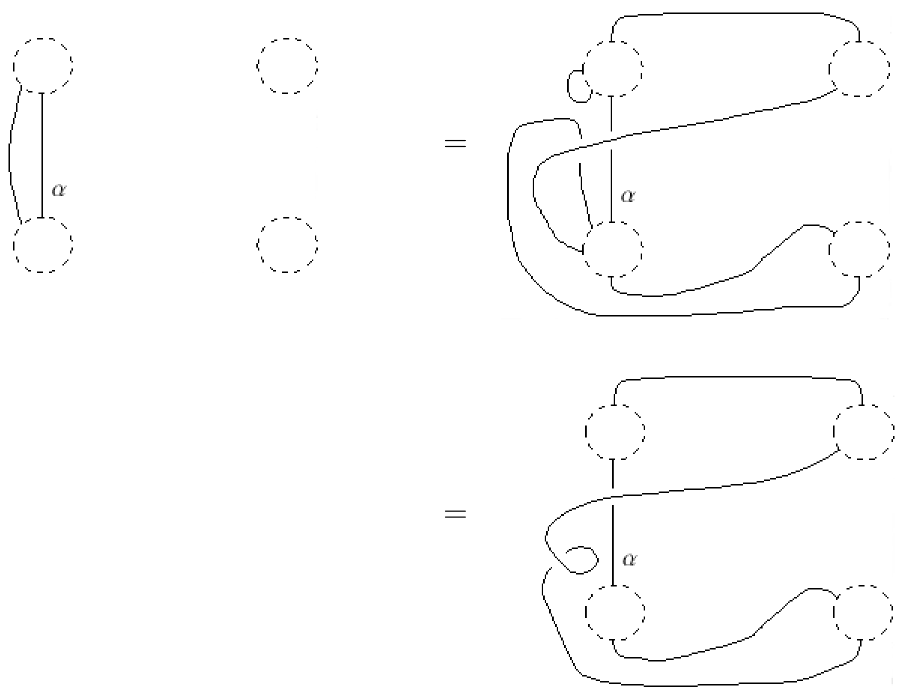
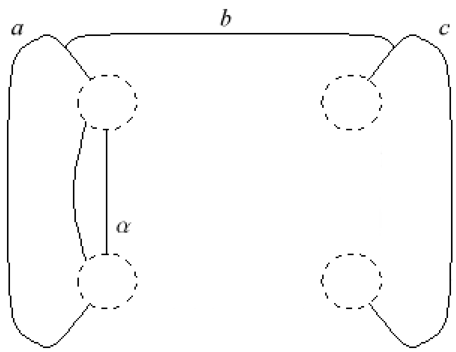
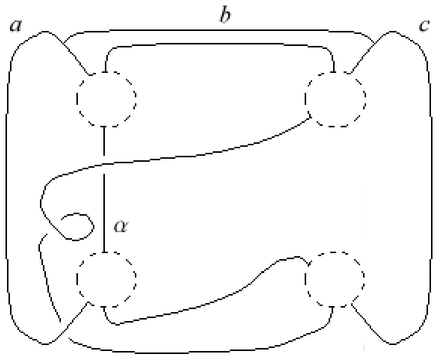


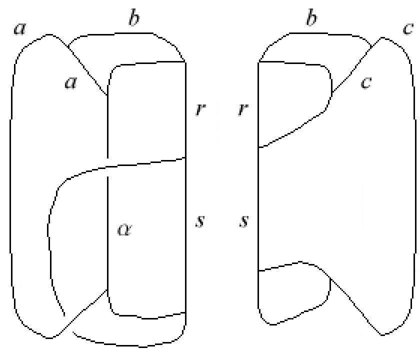
FIGURE 4.9. Relation 4



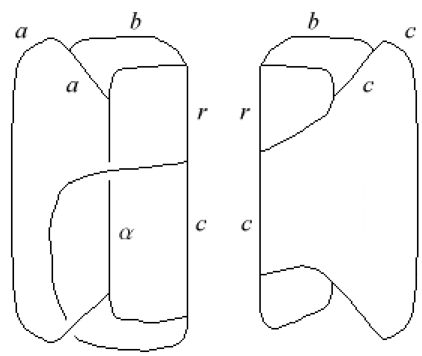
$$= \frac{\delta_0^b\,\delta_0^c\,\theta(a,\alpha,1)}{\theta(a,\alpha,1)} = \delta_0^b\,\delta_0^c\,\delta(a,\alpha,1)$$

FIGURE 4.10. Rewriting the left side of relation 4



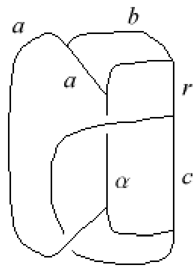
$-A^{-3}\ \Delta_s\ \Delta_r$ 

$$= \sum_{r,s\ adm} \frac{}{\theta(a,\alpha,1)\ \theta(s,1,1)\ \theta(b,r,1)\ \theta(r,s,1)\ \theta(c,1,1)}$$

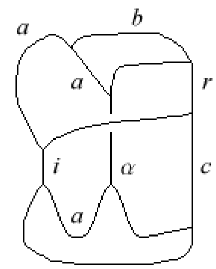
$-A^{-3}\ \Delta_c\ \Delta_r$ 

$$= \sum_{r\ adm} \frac{}{\theta(a,\alpha,1)\ \theta(c,1,1)^2\ \theta(b,r,1)\ \theta(r,c,1)}$$

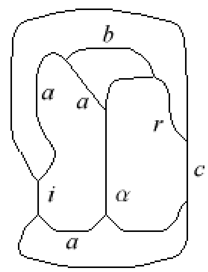
FIGURE 4.11. Rewriting the right side of relation 4



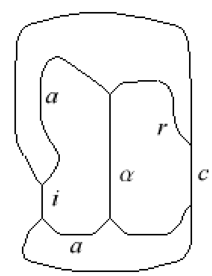
$$= \sum_{i\ adm} \frac{\Delta_i\ \lambda_i^{a\ 1}}{\theta(a,i,1)}$$



$$= \sum_{i\ adm} \frac{\Delta_i\ (\lambda_i^{a\ 1})^2}{\lambda_c^{r\ 1}\ \theta(a,i,1)}$$

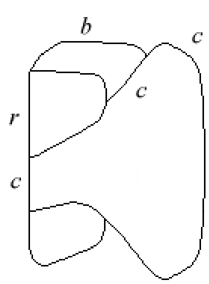


$$= \sum_{i\ adm} \frac{\Delta_i\ (\lambda_i^{a\ 1})^2\ \ Tet\begin{bmatrix} a & a & \alpha \\ r & 1 & b \end{bmatrix}}{\lambda_c^{r\ 1}\ \theta(a,i,1)\ \theta(a,r,\alpha)}$$

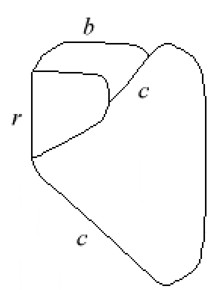


$$= \sum_{i\ as\ above} \frac{\Delta_i\ (\lambda_i^{a\ 1})^2\ \ Tet\begin{bmatrix} a & a & \alpha \\ r & 1 & b \end{bmatrix}\ \ Tet\begin{bmatrix} 1 & i & c \\ \alpha & 1 & a \end{bmatrix}\ \ Tet\begin{bmatrix} c & i & 1 \\ a & r & \alpha \end{bmatrix}}{\lambda_c^{r\ 1}\ \theta(a,i,1)\ \theta(a,r,\alpha)\ \theta(c,\alpha,i)}$$

and



$$= \frac{\theta(c,1,1)}{\Delta_c}$$



$$= \frac{\theta(c,1,1)\ \ Tet\begin{bmatrix} c & r & b \\ 1 & c & 1 \end{bmatrix}}{\Delta_c}$$

FIGURE 4.12. Rewriting the right side of relation 4, completed

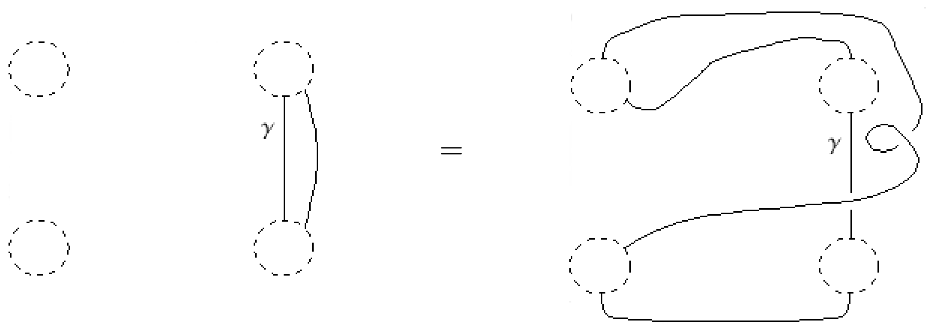


FIGURE 4.13. Relation 5

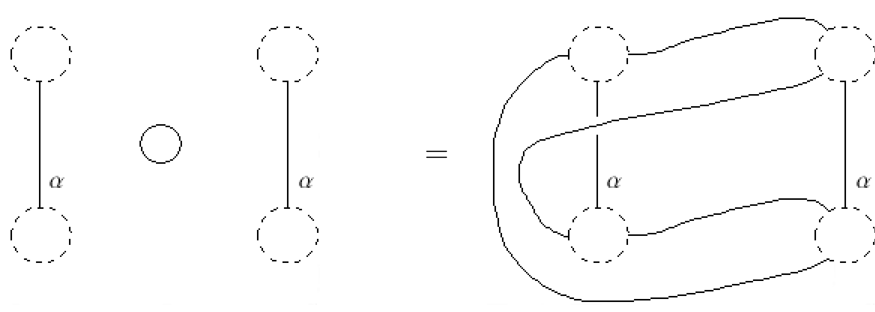


FIGURE 4.14. Relation 6

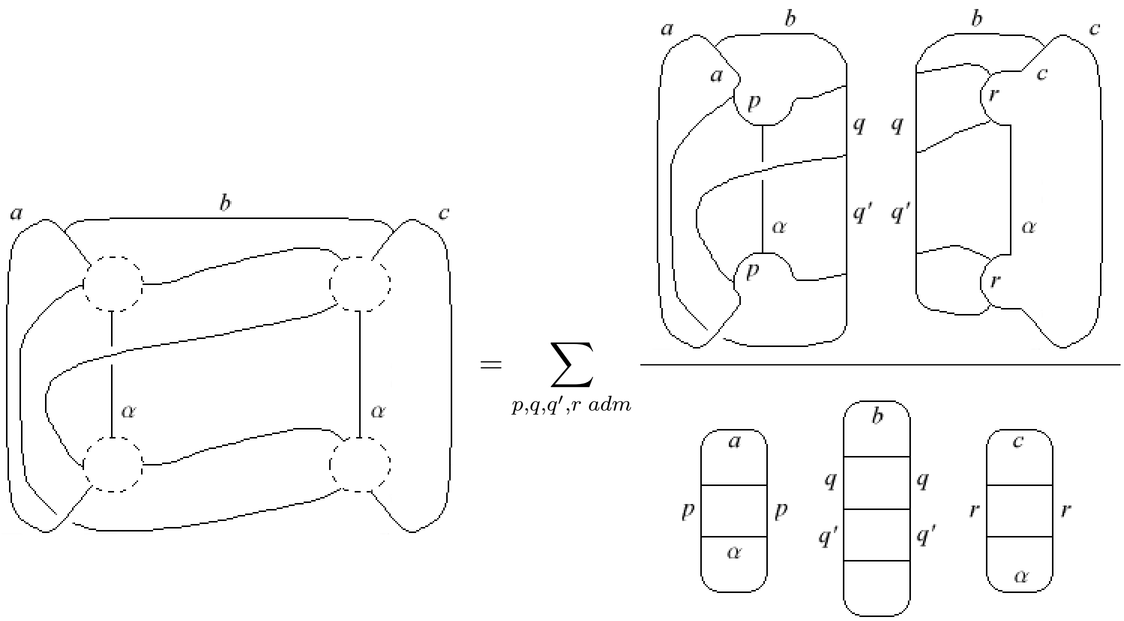


FIGURE 4.15. Rewriting the right side of relation 6

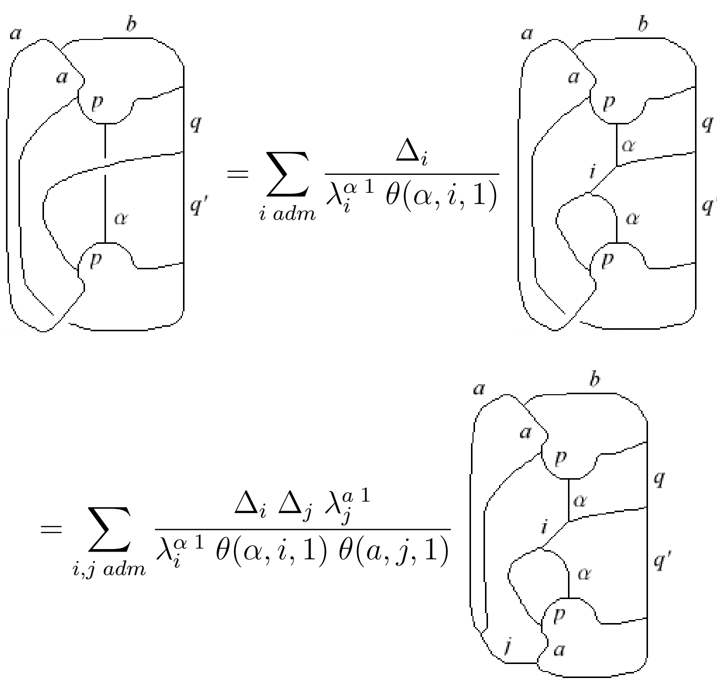
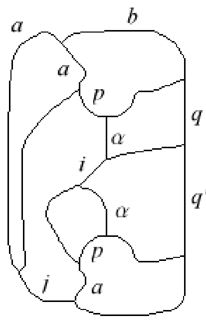
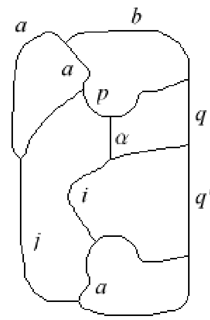


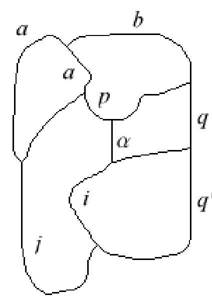
FIGURE 4.16. Rewriting the right side of relation 6, continued



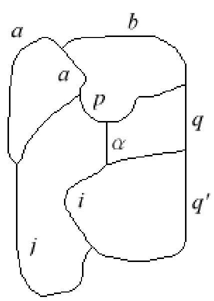
$$= \frac{Tet\begin{bmatrix} p & 1 & a \\ i & 1 & \alpha \end{bmatrix}}{\theta(a,i,1)}$$



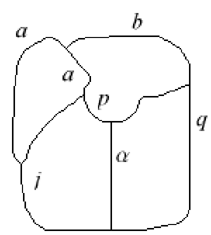
$$= \frac{Tet\begin{bmatrix} p & 1 & a \\ i & 1 & \alpha \end{bmatrix} \; Tet\begin{bmatrix} 1 & q' & j \\ i & a & 1 \end{bmatrix}}{\theta(a,i,1)\;\theta(i,j,q')}$$



where

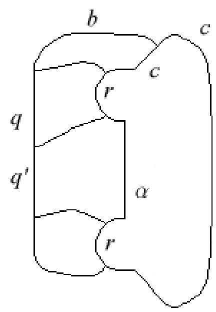


$$= \frac{Tet\begin{bmatrix} q' & q & j \\ \alpha & i & 1 \end{bmatrix}}{\theta(\alpha,j,q)}$$

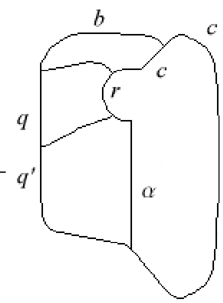


$$= \frac{Tet\begin{bmatrix} q' & q & j \\ \alpha & i & 1 \end{bmatrix} \; Tet\begin{bmatrix} q & b & j \\ p & \alpha & 1 \end{bmatrix} \; Tet\begin{bmatrix} j & a & b \\ a & p & 1 \end{bmatrix}}{\theta(\alpha,j,q)\;\theta(p,b,j)}$$

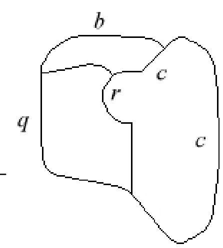
FIGURE 4.17. Rewriting the right side of relation 6, continued



$$= \frac{Tet\begin{bmatrix} 1 & c & q' \\ \alpha & 1 & r \end{bmatrix}}{\theta(c,\alpha,q')}$$



$$= \frac{Tet\begin{bmatrix} 1 & c & q' \\ \alpha & 1 & r \end{bmatrix} \; Tet\begin{bmatrix} q' & c & q \\ r & 1 & \alpha \end{bmatrix}}{\theta(c,\alpha,q')\;\theta(c,q,r)}$$



$$= \frac{Tet\begin{bmatrix} 1 & c & q' \\ \alpha & 1 & r \end{bmatrix} \; Tet\begin{bmatrix} q' & c & q \\ r & 1 & \alpha \end{bmatrix} \; Tet\begin{bmatrix} q & b & c \\ c & r & 1 \end{bmatrix}}{\theta(c,\alpha,q')\;\theta(c,q,r)}$$

FIGURE 4.18. Rewriting the right side of relation 6, completed

Case 2: $x \geq 1$, $y = 0$, $z \geq 1$, and $z \neq x$.

Let $(\alpha, \beta, \gamma) = (x, 0, z-1)$ in r_1, and let $(\alpha, \beta, \gamma) = (x-1, 0, z)$ in r_2. Then $(x, 2, z)$ and $(x, 0, z)$ are the two highest terms appearing in the relators.

Rearranging terms, we have

$$a_1(x, 2, z) + a_2(x, 0, z) = \text{lesser terms}$$

and

$$b_1(x, 2, z) + b_2(x, 0, z) = \text{lesser terms},$$

where a_i is the coefficient of the ith-highest term appearing in r_1 and b_i is the coefficient of the ith-highest term appearing in r_2.

So, we can rewrite $(x, 0, z)$ if

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

is nonzero.

Using Mathematica, we can see that this determinant is

$$-A^{-2-2x-2z}(-1+A^x)(1+A^x)(A^x - A^z)(-1+A^z)(1+A^z)(A^x + A^z),$$

which is nonzero when $x \geq 1$, $z \geq 1$, and $x \neq z$.

Case 3: $x \geq 2$, $y = z = 0$.

Using r_1 with $(\alpha, \beta, \gamma) = (x, 2, 1)$, r_2 with $(\alpha, \beta, \gamma) = (x-1, 0, 2)$, r_3 with $(\alpha, 0, \gamma) = (x-1, 0, 1)$, and r_4 with $\alpha = x-1$, we obtain four relations with $(x, 4, 2), (x, 2, 2), (x, 0, 2)$, and $(x, 0, 0)$ appearing as the four highest terms.

Thus, we can rewrite $(x, 0, 0)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is nonzero, where a_i, b_i, c_i, and d_i are the coefficients of the ith-highest terms appearing in r_1, r_2, r_3, and r_4, respectively.

This determinant is

$$-A^{-10-2x}(-1+A)(1+A)(1+A^2)(-A+A^x)^2(A+A^x)^2(A^2+A^{2x})^2,$$

which is nonzero when $x \geq 2$.

Case 4: $x = y = 0$, $z > 2$.

Using r_1 with $(\alpha, \beta, \gamma) = (2, 2, z-1)$, r_2 with $(\alpha, \beta, \gamma) = (1, 0, z)$, r_3 with $(\alpha, 0, \gamma) = (1, 0, z-1)$, and r_5 with $\gamma = z-1$, we obtain four relations with $(2, 4, z), (2, 2, z), (2, 0, z)$, and $(0, 0, z)$ appearing as the four highest terms. Note that this only holds for $z > 2$: for $z = 2$, $(0, 0, z)$ is no longer the fourth-highest term.

Thus, we can rewrite $(0, 0, z)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is nonzero, where a_i, b_i, c_i, and d_i are the coefficients of the ith-highest terms appearing in r_1, r_2, r_3, and r_5, respectively.

This determinant is

$$-A^{-6-2z}(-1+A)(1+A)(1+A^2)(-1+A^z)(1+A^z)(-A+A^z)(A+A^z)(A^2+A^{2z}),$$

which is nonzero when $z > 2$.

Case 5: $x \geq 2$, $y = 0$, and $z = x$.

Using r_2 with $(\alpha, \beta, \gamma) = (x-1, 2, x)$, r_3 with $(\alpha, 0, \gamma) = (x-1, 0, x-1)$, and r_6 with $\alpha = x-2$, we obtain three relations with $(x, 4, x), (x, 2, x)$, and $(x, 0, x)$ appearing as the three highest terms.

Thus, we can rewrite $(x, 0, x)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is nonzero, where a_i, b_i, and c_i are the coefficients of the ith-highest terms appearing in r_2, r_3, and r_6, respectively.

This determinant is $A^{-4+2x}(-1+A^x)(1+A^x)$, which is nonzero for $x \geq 2$. However, the calculations for $x = 2$ and $x > 2$ must be done separately, due to admissibility conditions.

For all five cases, the determinants are indeed nonzero, though their complexity compels us to use Mathematica for their evaluations. See Appendix B for the code and the output. $\square$

5. Linear Independence

This chapter is the result of joint work with Patrick Gilmer.

Proposition 5.1. $(0,0,0)$*,* $(0,0,1)$*,* $(1,0,0)$*,* $(1,0,1)$*, and* $(0,0,2)$ *are linearly independent in* $S(M)$*.*

Proof. First, we recall that $S(M)$ is a direct sum of four submodules $S_1(M)$, $S_i(M)$, $S_j(M)$, and $S_k(M)$, and that the latter three are isomorphic.

Hence, our task is greatly simplified. We only have to show that $(0,0,0)$ and $(0,0,2)$ are linearly independent, and that $(0,0,1)$ is nonzero.

Definition 5.2. *The triple* a,b,c *is said to be* r*-admissible if it is admissible and* $a+b+c \leq 2r-4$*.*

The recoupling theory we have used in previous chapters works when A is a primitive $2r$th root of unity, for odd $r > 1$. But we must replace "admissible" with "r-admissible" and restrict our labels to the range 0, ..., r-2: our insistence on r-admissibility ensures that the fusion formula still makes sense.

For the remainder of this chapter, let $\mathcal{A}$ denote the indeterminate we have previously denoted A.

Let A be the $2r$th root of unity $e^{\pi i/r}$ with $r > 1$ odd, let e_i denote the core of the solid torus, labelled i. Let $\Omega_r = \sum_{i=0}^{\lfloor (r-3)/2 \rfloor} \Delta_i e_i$, choose η such that $\eta^2 < \Omega_r >= 1$, and let κ be a root of unity such that $\kappa^6 = A^{-6-r(r+1)/2}$.

Definition 5.3. *For a framed link K in a closed, connected, oriented 3-manifold M described by surgery on a framed link $L \subset S^3$, we define the quantum invariant*

$$I_r(M,K) = \kappa^{3(b_-(L)-b_+(L))} < L(\eta\Omega_r) \cup K >,$$

where $b_+(L)$ and $b_-(L)$ are the numbers of the positive and negative eigenvalues of the linking matrix of L.

We follow the notation of Masbaum and Roberts in [MR]. See [BHMV1], [Lic], [RT], and [W] for for the origins of this formula.

Note that for the quaternionic manifold M, with the surgery description L presented in the introduction, $b_+(L) = b_-(L)$, and so $I_r(M,K) = \eta^2 < L(\Omega_r) \cup K >$.

Proposition 5.4. *For odd $r > 1$,*

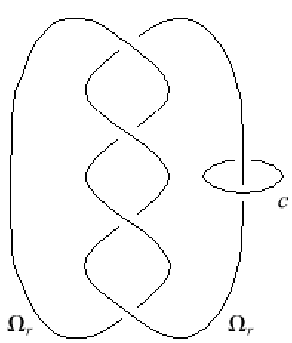
$$(1-A^4)I_r(M) = \sum_{k=1}^{r-1}(-1)^k A^{2k^2+2k},$$

$$I_r(M,(0,0,1)) = (-1)^{\frac{r-1}{2}}\frac{A^{-2}}{A^2+1},$$

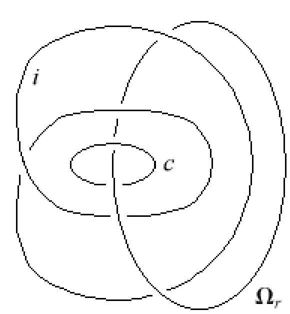
and

$$A^4 I_r(M,(0,0,2)) = I_r(M) - 1.$$

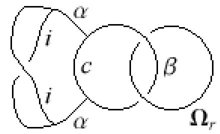
Proof.



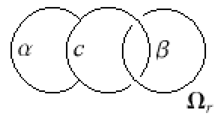
$$= \sum_{i=0}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i$$



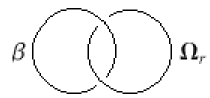
$$= \sum_{i=0}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i \sum_{\alpha,\beta\ adm} \frac{\Delta_\alpha \Delta_\beta}{\theta(i,i,\alpha)\theta(\alpha,\beta,c)}$$



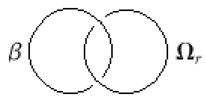
$$= \sum_{i=0}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i \sum_{\alpha,\beta\ as\ above} \frac{\Delta_\beta \lambda_\alpha^{i\,i}}{\theta(\alpha,\beta,c)}$$



$$= \sum_{i=0}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i \sum_{\alpha,\beta\ as\ above} \lambda_\alpha^{i\,i}$$



By the encirclement lemma,



$= \eta^{-2}\Delta_\beta$ if $\beta = 0$ or $\beta = r-2$, and 0 otherwise. See, for example, [Lic].

Furthermore, the sums are restricted to r-admissible labellings, and so, $0 \le \alpha \le 2i$, $|\alpha - c| \le \beta \le \alpha + c$, and $\beta \equiv \alpha + c \mod 2$.

Hence, a contribution to the sum can only occur when $\alpha = 0, \beta = 0$ for $c = 0$, $\alpha = r-3, \beta = r-2, i = \frac{r-3}{2}$ for $c = 1$, and $\alpha = 2, \beta = 0$ for $c = 2$.

Hence,

$$I_r(M) = \sum_{i=0}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i \lambda_0^{i\,i},$$

$$I_r(M,(0,0,2)) = \sum_{i=1}^{(r-3)/2} (-A)^{i(i+2)} \Delta_i \lambda_2^{i\,i},$$

and

$$I_r(M,(0,0,1)) = (-A)^{\frac{r-3}{2}(\frac{r-3}{2}+2)} \Delta_{\frac{r-3}{2}} \Delta_{r-2} \lambda_{r-3}^{\frac{r-3}{2}\ \frac{r-3}{2}} = (-1)^{\frac{r-1}{2}} \frac{A^{-2}}{A^2+1}.$$

Hence, $A^4 I_r(M,(0,0,2)) = I_r(M) - 1$, and letting $q = A^2$ and $k = i+1$, we obtain

$$-q(q-q^{-1})I_r(M) = \sum_{k=1}^{(r-1)/2} (-1)^k (q^{k^2+k} - q^{k^2-k}).$$

Since $q^r = 1$, $(-1)^{r-k} q^{(r-k)^2+(r-k)} = -(-1)^k q^{k^2-k}$. Hence,

$$(1-q^2)I_r(M) = \sum_{k=1}^{r-1}(-1)^k q^{k^2+k}.$$

Rewritten in terms of A,

$$(1-A^4)I_r(M) = \sum_{k=1}^{r-1}(-1)^k A^{2k^2+2k}.$$

$\square$

Before proceeding, we need more notation and a useful lemma:

Notation 5.5. *Let $\zeta_N = e^{2\pi i/N}$. (Hence, $A = \zeta_{2r}$.)*

Lemma 5.6. *If F is a rational complex-valued function, then the imaginary part of $F(\zeta_{2r})$ cannot change sign infinitely often as $r \to \infty$.*

Proof. If $F(z)$ is a rational function, then $F(z^{-1})$ is also rational, as is $G(z) = F(z) - F(z^{-1})$, and so, G must also be rational.

For $|z| = 1$, $\mathrm{Im}F(z) = \frac{1}{2i}(F(z) - \overline{F(z)}) = \frac{1}{2i}(F(z) - F(\overline{z})) = \frac{1}{2i}G(z)$.

Hence, $\mathrm{Im}F(z)$ is also rational when restricted to the unit circle, and so, has finitely many zeros and poles.

Choose N such that $F(z)$ has no poles for $|z| = 1$ with $\mathrm{Arg}z \in (0, \frac{2\pi}{N})$.

Let $h(x) = \mathrm{Im}F(e^{2\pi i x})$, with domain $(0, \frac{1}{N})$. Then h is a real-valued continuous function defined on a subset of the reals, and since $\mathrm{Im}F$ has finitely many zeros on the unit circle, $h(x)$ can change signs only finitely often. Hence, $\mathrm{Im}F(\zeta_{2r})$ itself can change signs only a finite number of times.

Thus, if F is rational, then $\mathrm{Im}F(\zeta_{2r})$ cannot change sign infinitely often as $r \to \infty$. $\square$

Proposition 5.7. $(0,0,0)$ *and* $(0,0,2)$ *are linearly independent.*

Proof. Suppose that $(0,0,0)$ and $(0,0,2)$ are linearly dependent. Then there are rational functions f_0/g_0 and f_2/g_2 such that

$$\frac{f_0(\mathcal{A})}{g_0(\mathcal{A})}(0,0,0) + \frac{f_2(\mathcal{A})}{g_2(\mathcal{A})}(0,0,2) = 0.$$

Hence, there is a linear combination of Kauffman bracket skein relators such that

$$\frac{f_0(\mathcal{A})}{g_0(\mathcal{A})}(0,0,0) + \frac{f_2(\mathcal{A})}{g_2(\mathcal{A})}(0,0,2) = \sum_i \frac{a_i(\mathcal{A})}{b_i(\mathcal{A})} R_i$$

in the free $\mathcal{R}$-module generated by all isotopy classes of framed links in M.

Clearing denominators on both sides, we obtain

$$h_0(\mathcal{A})(0,0,0) + h_2(\mathcal{A})(0,0,2) = \sum_i c_i(\mathcal{A}) R_i,$$

which must hold over $\mathbb{Z}[\mathcal{A}^{\pm 1}, [2]^{-1}]$, and hence, over $\mathbb{C}$.

Taking quantum invariants on both sides,

$$h_0(A) I_r(M) + h_2(A) I_r(M,(0,0,2)) = 0.$$

Since $A^4 I_r(M,(0,0,2)) = I_r(M) - 1$ for odd $r > 1$,

$$h_0(A) I_r(M) + A^{-4} h_2(A)(I_r(M) - 1) = 0,$$

for odd $r > 1$.

Hence,

$$I_r(M) = \frac{h_2(A)}{A^4 h_0(A) + h_2(A)},$$

for all but finitely many odd r.

Thus, it suffices to show that there is no rational function f such that $f(\zeta_{2r}) = I_r(M)$ for all but finitely many odd r.

By Proposition 5.4, $(1 - A^4) I_r(M) = \sum_{k=1}^{r-1} (-1)^k A^{2k^2+2k}$.

Notation 5.8. $g_N(m) = \frac{2}{\sqrt{N}} \sum_{k=0}^{m} \zeta_N^{k^2}$.

Lemma 5.9. *(van Wamelen)*

$$\sum_{k=1}^{r-1} \zeta_{2r}^{rk} \zeta_{2r}^{2k^2+2k} + 1 = \zeta_{16r}^{-(r+2)^2} (\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))).$$

Proof. See Appendix C. ☐

Hence, $(1 - A^4)I_r(M) + 1 = \zeta_{16r}^{-(r+2)^2} (\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})))$.

Now, we can apply a result in [Le] by D.H. Lehmer to estimate each of the new sums.

Theorem 5.10. *(Lehmer) For $N \geq 100$ and $\sqrt{\frac{N}{2}} \leq m \leq \frac{N}{4}$, $g_N(m)$ lies within the circle with center $(h, k) = (C(\sqrt{(2)}, S(\sqrt{2}) - \frac{1}{\sqrt{2\pi}}) \approx (0.529, 0.489)$ and radius $\frac{1}{\sqrt{2\pi}} + \frac{101}{40\sqrt{N}}$, where $C(u) = \int_0^u \cos(\frac{1}{2}\pi x^2)dx$ and $S(u) = \int_0^u \sin(\frac{1}{2}\pi x^2)dx$ are the Fresnel integrals.*

The proof of this result in Lehmer's paper is only sketched, and moreover, there are some minor errors. However, Litherland has given a detailed proof [Lit] along the lines indicated by Lehmer.

Let $R = \frac{1}{\sqrt{2\pi}} + 0.0001$. Then, for sufficiently large r, $g_{4r}(\frac{r-1}{2})$ lies inside a circle with center (h, k) and radius R, and $2g_{16r}(r-1)$ lies inside a circle with center $(2h, 2k)$ and radius $2R$.

Let D be the distance from (h, k) to the intersection of the tangent lines between the circles depicted in Figure 5.1, let D' be the distance from $(2h, 2k)$ to the point of intersection, let L be the distance between the two centers, let θ be the angle between one of the tangent lines and the line joining the centers, and let ϕ be the angle between the line joining the centers and the x-axis.

$$\phi - \theta < \arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) < \phi + \theta,$$

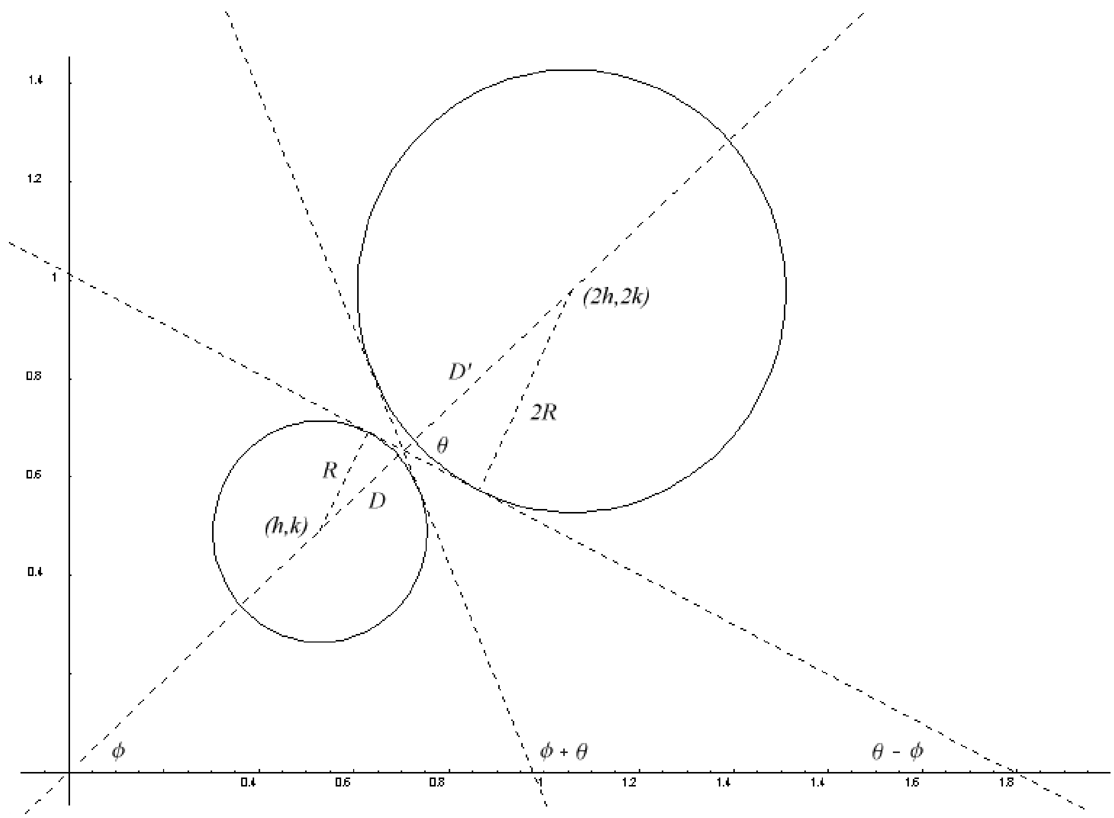


FIGURE 5.1. Estimating $g_{4r}(\frac{r-1}{2})$ and $2g_{16r}(r-1)$

where arg takes values in $[-90^{\circ}, 270^{\circ})$.

We will now show that multiplication by $\zeta_{16r}^{-(r+2)^2}$ rotates the difference into the upper half-plane for certain values of r and into the lower half-plane for other values of r.

Since $\frac{R}{D} = \sin(\theta) = \frac{2R}{D'}$, $D' = 2D$. $L = D + D' = 3D$, so $\sin(\theta) = \frac{3R}{3D} = \frac{3R}{L} = \frac{3R}{\sqrt{h^2+k^2}}$, and so $\theta = \sin^{-1}(\frac{3R}{\sqrt{h^2+k^2}}) \approx 69.7078^{\circ}$.

Also, $\tan(\phi) = \frac{k}{h}$, so $\phi \approx 42.7495^{\circ}$.

As $r \to \infty$, $\zeta_{16r}^{-(r+2)^2} \to \zeta_{16}^{3}$, for $r \equiv 9 \bmod 16$ and $\zeta_{16r}^{-(r+2)^2} \to \zeta_{16}^{11}$, for $r \equiv 1 \bmod 16$.

$$0^{\circ} < 40.541^{\circ} < \phi - \theta + 67.5^{\circ} < \arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) + 67.5^{\circ}$$

and

$$\arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) + 67.5^{\circ} < \phi + \theta + 67.5^{\circ} < 179.96^{\circ} < 180^{\circ}.$$

Hence, for sufficiently large r, multiplication by $\zeta_{16r}^{-(r+2)^2}$ rotates $(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))$ into the upper half-plane for $r \equiv 9 \bmod 16$ and into the lower half-plane for $r \equiv 1 \bmod 16$.

Hence, for sufficiently large r, the imaginary part of

$$\frac{(1-A^4)I_r(M)+1}{2\sqrt{r}} = \zeta_{16r}^{-(r+2)^2}(\frac{\zeta_{16r}^{9r^2}}{2\sqrt{r}} - (2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})))$$

is positive for $r \equiv 1 \bmod 16$ and is negative for $r \equiv 9 \bmod 16$, and the same must hold for $(1-A^4)I_r(M)+1$.

Hence, the imaginary part of a function F such that $F(\zeta_{2r}) = (1-\zeta_{2r}^4)I_r(M)+1$ for all but finitely many odd r changes sign infinitely often, and so, by Lemma 5.6, F cannot be rational. Hence, there can be no rational function f such that $f(\zeta_{2r}) = I_r(M)$ for all but finitely many r, as required. $\square$

Proposition 5.11. $(0, 0, 1) \neq 0$

Proof. If $(0, 0, 1) = 0$, then there exists a linear combination of Kauffman bracket skein relators R_i such that $(0, 0, 1) = \sum_i \frac{a_i(\mathcal{A})}{b_i(\mathcal{A})} R_i$ in the free module over $\mathcal{R}$, and hence as in the previous proof, there is a polynomial k such that $k(A)I_r(M, (0, 0, 1)) = 0$ for all r.

As shown in Proposition 5.4, $I_r(M, (0, 0, 1)) = (-1)^{\frac{r-1}{2}} \frac{A^{-2}}{A^2+1} \neq 0$ for all odd $r > 1$, and so no such polynomial can exist.

Hence, $(0, 0, 1) \neq 0$. $\square$

Thus, $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, and $(0, 0, 2)$ are all linearly independent in $S(M; \mathcal{R}, \mathcal{A})$, and so, $\dim(S(M; \mathcal{R}, \mathcal{A})) = 5$. $\square$

6. Concluding Remarks

We plan to apply the methods presented here to analyze the skein modules of other 3-manifolds. The method presented in Chapter 2 is immediately applicable to any 3-manifold, given its Heegaard splitting. In particular, we hope to obtain similar results for the Poincare homology sphere.

Another avenue of further research lies in the choice of the underlying ring R and unit A. The proof given here extends immediately to the case when A is transcendental, but not to the case where A is not a root of unity.

$S(M; \mathcal{R}, A)$ is a vector space, and so there can be no torsion. We wonder about the possibility of having torsion when R is chosen to be, say, $\mathbb{Z}[A^{\pm 1}]$, $\mathbb{Q}[A^{\pm 1}]$, or one of these with the inverses of the quantum integers adjoined.

Finally, we are also interested in investigating the spin-refined skein module of M as described by Masbaum in [M].

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Appendix A. Mathematica Notebook for Chapter 3

qi[n] = $[n]$, qif[n] = $[n]!$, and lambdas, thetas, and tets are evaluated as in [KL].

Here, lambda[a, b, c] = $\lambda_c^{a\,b}$.

```
qi[n_] := Sum[A^i, {i, 2 - 2 n, 2 n - 2, 4}];

delta[n_] := (-1)^n qi[n + 1] ;

adm[a_, b_, c_] := Mod[a + b + c, 2] == 0 && Abs[a - b] ≤ c && c ≤ a + b;

qif[n_] := Product[qi[k], {k, 1, n}];

lambda[a_, b_, c_] := (-1)^((a + b - c) / 2) A^((a (a + 2) + b (b + 2) - c (c + 2)) / 2);

theta[a_, b_, c_] := Module[{
    m = (a + b - c) / 2,
    n = (b + c - a) / 2,
    p = (a + c - b) / 2
   },

   If[adm[a, b, c],
    (-1)^(m + n + p) qif[m + n + p + 1] qif[m] qif[n] qif[p] / qif[m + n] / qif[n + p] / qif[m + p],
    0]
  ];
```

$$\text{tet[a, b, c, d, e, f]} = \quad Tet\left[\begin{array}{ccc} a & b & e \\ c & d & f \end{array}\right].$$

```
admtet[a_, b_, c_, d_, e_, f_] := adm[a, d, e] && adm[b, c, e] && adm[a, b, f] && adm[c, d, f];
```

```
tet[a_, b_, c_, d_, e_, f_] := Module[{
   a1 = (a + d + e) / 2 // Simplify,
   a2 = (b + c + e) / 2 // Simplify,
   a3 = (a + b + f) / 2 // Simplify,
   a4 = (c + d + f) / 2 // Simplify,
   av,

   b1 = (b + d + e + f) / 2 // Simplify,
   b2 = (a + c + e + f) / 2 // Simplify,
   b3 = (a + b + c + d) / 2 // Simplify,
   bv,

   m, M,
  },

  av = {a1, a2, a3, a4}; bv = {b1, b2, b3};
  m = Max[a1, a2, a3, a4]; M = Min[b1, b2, b3];

  If[admtet[a, b, c, d, e, f],
   intfac = Product[qif[bv[[j]] - av[[i]]], {i, 1, 4}, {j, 1, 3}];
   extfac = qif[a] qif[b] qif[c] qif[d] qif[e] qif[f];

   (intfac / extfac)
    Sum[(-1)^s qif[s + 1] / Product[qif[s - av[[i]]], {i, 1, 4}] / Product[qif[bv[[j]] - s], {j, 1, 3}],
     {s, m, M}],
   0]
 ];
```

quatcoef1[] and quatcoef2[] compute the coefficients of the terms in the relations coming from the slides across the attached 2-handles.

```
fusion[a_, b_, c_, x_, y1_, z_, p_, q_, q1_, r_] :=
  If[adm[p, a, 1] && adm[p, x, 1] && adm[b, 1, q] && adm[q, q1, y1] && adm[q1, 1, 1] && adm[c, 1, r] &&
    adm[z, 1, r], delta[p] delta[q] delta[q1]
    delta[r] / theta[p, a, 1] / theta[p, x, 1] / theta[b, 1, q] / theta[q, q1, y1] / theta[q1, 1, 1] /
      theta[c, 1, r] / theta[z, 1, r], 0];


leftquat[a_, b_, x_, y_, y1_, p_, q_, q1_] := If[adm[q, a, p] && adm[x, p, y1],
  (lambda[y, 1, y1] / lambda[x, 1, p]) tet[a, p, q, b, a, 1] tet[1, y1, x, x, p, y]
   Sum[If[adm[a, i, 1] && adm[i, q1, p] && adm[i, x, q],
       (lambda[a, 1, i] delta[i] / theta[a, i, 1]) tet[1, p, i, 1, q1, a] tet[p, q, i, 1, x, a]
        tet[q, q1, p, x, i, y1] / theta[i, q1, p] / theta[i, x, q],
       0],
      {i, Abs[a - 1], a + 1, 2}] / theta[q, a, p] / theta[x, p, y1],
  0];


rightquat[b_, c_, y1_, y2_, z_, q_, q1_, r_] :=
 If[adm[q1, c, z] && adm[r, y1, z] && adm[r, c, q],
  tet[r, z, q1, 1, c, 1] tet[y2, z, r, 1, y1, z] tet[b, q, r, c, c, 1]
   tet[q, c, z, y1, q1, r] / theta[q1, c, z] / theta[r, y1, z] / theta[r, c, q], 0];


quatcoef1[a_, b_, c_, x_, y_, y1_, y2_, z_] :=
 Module[{ans}, ans = (delta[a] delta[b] delta[c] / theta[a, a, b] / theta[b, c, c])
    Sum[If[adm[a, a, b] && adm[b, c, c],
      fusion[a, b, c, x, y1, z, p, q, q1, r] leftquat[a, b, x, y, y1, p, q, q1]
       rightquat[b, c, y1, y2, z, q, q1, r],
      0],
     {p, Max[Abs[a - 1], Abs[x - 1]], Min[a + 1, x + 1], 2}, {q1, 0, 2, 2},
     {q, Max[Abs[b - 1], Abs[q1 - y1]], Min[b + 1, q1 + y1], 2},
     {r, Max[Abs[c - 1], Abs[z - 1]], Min[c + 1, z + 1], 2}] ;
  If[a == x && b == y && c == z && y2 == y, ans - theta[y, 1, y1] / delta[y], ans]];


quatcoef2[a_, b_, c_, x_, y_, y1_, y2_, z_] := quatcoef1[c, b, a, z, y2, y1, y, x];
```

lessthan[] and greaterthan[] order the trivalent graphs (a,b,c) as described in Chapter 2.

```
lessthan[x1_, y1_, z1_, x2_, y2_, z2_] := Module[{m1 = Max[x1, z1], m2 = Max[x2, z2]},
  m1 < m2 ||
   (m1 == m2 && x1 < x2) ||
   (m1 == m2 && x1 == x2 && z1 < z2) ||
   (m1 == m2 && x1 == x2 && z1 == z2 && y1 < y2)];

greaterthan[x1_, y1_, z1_, x2_, y2_, z2_] := Module[{m1 = Max[x1, z1], m2 = Max[x2, z2]},
  m1 > m2 ||
   (m1 == m2 && x1 > x2) ||
   (m1 == m2 && x1 == x2 && z1 > z2) ||
   (m1 == m2 && x1 == x2 && z1 == z2 && y1 > y2)];
```

fullcoefmatrix[] constructs the matrices presented in Chapter 2, before row-reduction.

```
colorderedlist[x_, y_, z_] := Module[{ol = {}, temp},
  Do[If[j ≤ 2 Min[i, k], ol = Append[ol, {i, j, k}]], {i, Mod[x, 2], x + 2, 2}, {j, 0, 2 Min[x, z] + 4, 2},
   {k, Mod[z, 2], Max[x + 2, z + 2], 2}];
  Do[If[lessthan[ol[[m, 1]], ol[[m, 2]], ol[[m, 3]], ol[[n, 1]], ol[[n, 2]], ol[[n, 3]]],
    temp = ol[[m]]; ol[[m]] = ol[[n]]; ol[[n]] = temp], {m, 1, Length[ol] - 1}, {n, m + 1, Length[ol]}]; ol];


roworderedlist[x_, z_] := Module[{ol = {}, temp},
  Do[If[j ≤ 2 Min[i, k], ol = Append[ol, {i, j, k}]], {i, Mod[x, 2], x, 2}, {j, 0, 2 Min[x, z + 1], 2},
   {k, Mod[z, 2], z, 2}];
  Do[If[lessthan[ol[[m, 1]], ol[[m, 2]], ol[[m, 3]], ol[[n, 1]], ol[[n, 2]], ol[[n, 3]]],
    temp = ol[[m]]; ol[[m]] = ol[[n]]; ol[[n]] = temp], {m, 1, Length[ol] - 1}, {n, m + 1, Length[ol]}]; ol];


fullcoefmatrix[xf_, zf_] :=
 Module[{col = colorderedlist[xf, 2 Min[xf, zf], zf], rol = roworderedlist[xf, zf], mat = {}, x, y, z},

  Do[x = rol[[i, 1]]; y = rol[[i, 2]]; z = rol[[i, 3]];

   h1rel1[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef1[a, b, c, x, y, y - 1, y - 2, z]];
   h1rel2[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef1[a, b, c, x, y, y - 1, y, z]];
   h1rel3[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef1[a, b, c, x, y, y + 1, y, z]];
   h1rel4[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef1[a, b, c, x, y, y + 1, y + 2, z]];

   h2rel1[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef2[a, b, c, x, y, y - 1, y - 2, z]];
   h2rel2[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef2[a, b, c, x, y, y - 1, y, z]];
   h2rel3[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef2[a, b, c, x, y, y + 1, y, z]];
   h2rel4[t_] := Module[{a = t[[1]], b = t[[2]], c = t[[3]]}, quatcoef2[a, b, c, x, y, y + 1, y + 2, z]];

   If[y ≥ 2 && y ≤ 2 x && y - 2 ≤ 2 z,
    mat = Append[mat, Map[h1rel1, col]];
    mat = Append[mat, Map[h2rel1, col]]
   ];
   If[y ≥ 1 && y ≤ 2 x && y ≤ 2 z,
    mat = Append[mat, Map[h1rel2, col]];
    mat = Append[mat, Map[h2rel2, col]]
   ];
   If[y ≥ 0 && y ≤ 2 x && y ≤ 2 z,
    mat = Append[mat, Map[h1rel3, col]];
    mat = Append[mat, Map[h2rel3, col]]
   ];
   If[y ≥ 0 && y ≤ 2 x && y + 2 ≤ 2 z,
    mat = Append[mat, Map[h1rel4, col]];
    mat = Append[mat, Map[h2rel4, col]]
   ],

   {i, 1, Length[rol]}];
  mat];
```

Appendix B. Mathematica Notebook for Chapter 4

oddq[] and evenq[] extend Oddq[] and Evenq[] to variables.

```
oddq[a_ b_ /; oddq[a] && oddq[b]] := True;
oddq[a_ + b_ /; (oddq[a] && evenq[b]) || (evenq[a] && oddq[b])] := True;
oddq[a_] := OddQ[a];

evenq[a_ b_ /; (evenq[a] && IntegerQ[b]) || (evenq[b] && IntegerQ[a])] := True;
evenq[a_ + b_ /; (evenq[a] && evenq[b]) || (oddq[a] && oddq[b])] := True;
evenq[a_] := EvenQ[a];
```

Quantum integers and their factorials are left unevaluated. Lambdas, thetas, and tets are evaluated as in [KL].

Here, lambda[a, b, c] $= \lambda_c^{a\,b}$.

```
qi[0] = 0; qi[1] = 1;

qif[0] = 1; qif[n_ /; n ≥ 1] := qif[n - 1] qi[n];
qif[n_ + x_ /; n ≥ 1] := qif[n + x - 1] qi[n + x];

delta[n_] := (-1)^n qi[n + 1] ;


adm[a1_, b1_, c1_] := Module[{a = Simplify[a1], b = Simplify[b1], c = Simplify[c1]},
  Simplify[a ≥ 0 && b ≥ 0 && c ≥ 0 && Abs[a - b] ≤ c && c ≤ a + b, given] && evenq[a + b + c]];


lambda[a_, b_, c_] := (-1)^((a + b - c) / 2) A^((a (a + 2) + b (b + 2) - c (c + 2)) / 2);


theta[a_, b_, c_] := Module[{
   m = (a + b - c) / 2 // Simplify,
   n = (b + c - a) / 2 // Simplify,
   p = (a + c - b) / 2 // Simplify
  },

  If[adm[a, b, c],
   (-1)^(m + n + p) qif[m + n + p + 1] qif[m] qif[n] qif[p] / qif[m + n] / qif[n + p] / qif[m + p] // Simplify,
   0]
 ];
```

$$\text{tet[a, b, c, d, e, f]} = \quad Tet\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}.$$

If the tet is a one-term sum (this is the case in each tet we evaluate in this notebook), then it is evaluated without Sum[], preventing Mathematica from leaving Sum[]'s in the output.

```
admtet[a_, b_, c_, d_, e_, f_] := adm[a, d, e] && adm[b, c, e] && adm[a, b, f] && adm[c, d, f];


 tet[a_, b_, c_, d_, e_, f_] := Module[{
    a1 = (a + d + e) / 2 // Simplify,
    a2 = (b + c + e) / 2 // Simplify,
    a3 = (a + b + f) / 2 // Simplify,
    a4 = (c + d + f) / 2 // Simplify,
    av,

    b1 = (b + d + e + f) / 2 // Simplify,
    b2 = (a + c + e + f) / 2 // Simplify,
    b3 = (a + b + c + d) / 2 // Simplify,
    bv,

    m, M, cv, s
   },

   av = {a1, a2, a3, a4}; bv = {b1, b2, b3};
   m = Max[a1, a2, a3, a4]; M = Min[b1, b2, b3];

   If[admtet[a, b, c, d, e, f],
    intfac = Product[qif[bv[[j]] - av[[i]]], {i, 1, 4}, {j, 1, 3}];
    extfac = qif[a] qif[b] qif[c] qif[d] qif[e] qif[f];
    cv = Intersection[av, bv];
    (intfac / extfac) If[Length[cv] > 0,  s = cv[[1]];
       (-1)^s qif[s + 1] / Product[qif[s - av[[i]]], {i, 1, 4}] / Product[qif[bv[[j]] - s], {j, 1, 3}],
       Sum[(-1)^s qif[s + 1] / Product[qif[s - av[[i]]], {i, 1, 4}] / Product[qif[bv[[j]] - s], {j, 1, 3}],
        {s, m, M}]] // Simplify,
    0]
  ];
```

norm[a, b, c] $= <(a,b,c),(a,b,c)>$,

r1[x, y, z, a, b, c] $= \dfrac{<\text{relator } r_1 \text{ with } _{(\alpha,\beta,\gamma)=(x,y,z),(a,b,c)>}}{\text{norm[a, b, c]}}$, and

r2[x, y, z, a, b, c] $= \dfrac{<\text{relator } r_2 \text{ with } _{(\alpha,\beta,\gamma)=(x,y,z),(a,b,c)>}}{\text{norm[a, b, c]}}$.

Here we begin computing coefficients. In this computation and those which follow, the sums are expanded to prevent Mathematica from leaving Sum[]'s in the output.

```
norm[a_, b_, c_] := theta[a, a, b] theta[b, c, c] / delta[a] / delta[b] / delta[c];


lhsr1[x_, y_, z_, a_, b_, c_] := If[adm[c, z, 1] && a == x && b == y,
  theta[x, x, y] tet[c, z, z, c, y, 1] / delta[x] / delta[y] / theta[c, z, 1],
  0];

rhsr1[x_, y_, z_, a_, b_, c_] :=
 If[adm[b, y - 1, 1] && adm[c, z, 1] && adm[c, z, y - 1],
   delta[y - 1] tet[c, y - 1, 1, c, b, z] tet[1, y - 1, z, z, c, y] (If[adm[x, x - 1, y - 1],
          delta[x - 1] (lambda[x, 1, x - 1]^2) tet[y - 1, x - 1, x, b, 1, x]
           tet[1, x - 1, x, y, y - 1, x] / theta[x, x - 1, 1] / theta[x, x - 1, y - 1],
          0] + If[adm[x, x + 1, y - 1],
          delta[x + 1] (lambda[x, 1, x + 1]^2) tet[y - 1, x + 1, x, b, 1, x]
           tet[1, x + 1, x, y, y - 1, x] / theta[x, x + 1, 1] / theta[x, x + 1, y - 1],
          0]) / delta[x] / lambda[b, 1, y - 1] / theta[y, y - 1, 1] / theta[b, y - 1, 1] / theta[c, z, 1] /
    theta[c, z, y - 1],
   0] + If[adm[b, y + 1, 1] && adm[c, z, 1] && adm[c, z, y + 1],
   delta[y + 1] tet[c, y + 1, 1, c, b, z] tet[1, y + 1, z, z, c, y] (If[adm[x, x - 1, y + 1],
          delta[x - 1] (lambda[x, 1, x - 1]^2) tet[y + 1, x - 1, x, b, 1, x]
           tet[1, x - 1, x, y, y + 1, x] / theta[x, x - 1, 1] / theta[x, x - 1, y + 1],
          0] + If[adm[x, x + 1, y + 1],
          delta[x + 1] (lambda[x, 1, x + 1]^2) tet[y + 1, x + 1, x, b, 1, x]
           tet[1, x + 1, x, y, y + 1, x] / theta[x, x + 1, 1] / theta[x, x + 1, y + 1],
          0]) / delta[x] / lambda[b, 1, y + 1] / theta[y, y + 1, 1] / theta[b, y + 1, 1] / theta[c, z, 1] /
    theta[c, z, y + 1],
   0];

r1[x_, y_, z_, a_, b_, c_] :=
 If[adm[x, x, y] && adm[z, z, y] && adm[a, a, b] && adm[c, c, b],
   (rhsr1[x, y, z, a, b, c] - lhsr1[x, y, z, a, b, c]) / norm[a, b, c], 0] // Simplify;

r2[x_, y_, z_, a_, b_, c_] := r1[z, y, x, c, b, a];
```

$$\text{r3[x, z, a, b, c]} = \frac{<\text{relator } r_3 \text{ with } (\alpha,0,\gamma)=(x,0,z),(a,b,c)>}{\text{norm[a, b, c]}}.$$

```
lhsr3[x_, z_, a_, b_, c_] := If[adm[a, x, 1] && adm[b, 1, 1] && adm[c, z, 1],
   tet[1, a, a, 1, b, x] tet[c, 1, 1, c, b, z] / theta[a, x, 1] / theta[b, 1, 1] / theta[c, z, 1],
   0];

 rhsr3[x_, z_, a_, b_, c_] := If[adm[a, x, 1] && adm[b, 1, 1] && adm[c, z, 1],
   -A^3 tet[c, 1, 1, c, b, z] (If[adm[b, x, a - 1],
          delta[a - 1] tet[a - 1, a, a, x, b, 1] tet[x, 1, 1, a - 1, b, a]
           lambda[a, 1, a - 1] / theta[a, a - 1, 1] / theta[b, x, a - 1],
          0] + If[adm[b, x, a + 1],
          delta[a + 1] tet[a + 1, a, a, x, b, 1] tet[x, 1, 1, a + 1, b, a]
           lambda[a, 1, a + 1] / theta[a, a + 1, 1] / theta[b, x, a + 1],
          0]) / lambda[x, 1, a] / theta[a, x, 1] / theta[b, 1, 1] / theta[c, z, 1],
   0];

 r3[x_, z_, a_, b_, c_] := If[adm[a, a, b] && adm[c, c, b],
    (rhsr3[x, z, a, b, c] - lhsr3[x, z, a, b, c]) / norm[a, b, c], 0] // Simplify;
```

$$\text{r4[x, a, b, c]} = \frac{<\text{relator } r_4 \text{ with } \alpha=x,(a,b,c)>}{\text{norm[a, b, c]}}, \text{ and}$$

$$\text{r5[z, a, b, c]} = \frac{<\text{relator } r_5 \text{ with } \gamma=z,(a,b,c)>}{\text{norm[a, b, c]}}.$$

```
rhsr4[x_, a_, b_, c_] := -A^(-3) If[adm[a, x, 1] && adm[c, 1, 1],
    (If[adm[b, c - 1, 1] && adm[c, c - 1, 1] && adm[a, c - 1, x],
        delta[c - 1] tet[c, c - 1, 1, c, b, 1] tet[a, a, c - 1, 1, x, b]
         (If[adm[a, a - 1, 1] && adm[x, c, a - 1], delta[a - 1] lambda[a, 1, a - 1]^2
               tet[1, a - 1, x, 1, c, a] tet[c, a - 1, a, c - 1, 1, x] / lambda[c - 1, 1, c] / theta[a, a - 1, 1] /
                theta[x, c, a - 1], 0] + If[adm[a, a + 1, 1] && adm[x, c, a + 1],
              delta[a + 1] lambda[a, 1, a + 1]^2 tet[1, a + 1, x, 1, c, a]
               tet[c, a + 1, a, c - 1, 1, x] / lambda[c - 1, 1, c] / theta[a, a + 1, 1] / theta[x, c, a + 1], 0]) /
           theta[b, c - 1, 1] / theta[c, c - 1, 1] / theta[a, c - 1, x], 0] +
       If[adm[b, c + 1, 1] && adm[c, c + 1, 1] && adm[a, c + 1, x],
        delta[c + 1] tet[c, c + 1, 1, c, b, 1] tet[a, a, c + 1, 1, x, b]
         (If[adm[a, a - 1, 1] && adm[x, c, a - 1], delta[a - 1] lambda[a, 1, a - 1]^2
               tet[1, a - 1, x, 1, c, a] tet[c, a - 1, a, c + 1, 1, x] / lambda[c + 1, 1, c] / theta[a, a - 1, 1] /
                theta[x, c, a - 1], 0] + If[adm[a, a + 1, 1] && adm[x, c, a + 1],
              delta[a + 1] lambda[a, 1, a + 1]^2 tet[1, a + 1, x, 1, c, a]
               tet[c, a + 1, a, c + 1, 1, x] / lambda[c + 1, 1, c] / theta[a, a + 1, 1] / theta[x, c, a + 1], 0]) /
           theta[b, c + 1, 1] / theta[c, c + 1, 1] / theta[a, c + 1, x], 0]) / theta[a, x, 1] / theta[c, 1, 1],
    0];

 r4[x_, a_, b_, c_] := If[adm[a, a, b] && adm[c, c, b],
    (rhsr4[x, a, b, c] - If[adm[a, x, 1] && b == 0 && c == 0, 1, 0]) / norm[a, b, c], 0] // Simplify;


 r5[z_, a_, b_, c_] := r4[z, c, b, a];
```

$$\mathrm{r6[x,\ a,\ b,\ c]} = \frac{<\mathrm{relator}\ r_6\ \mathrm{with}\ \alpha{=}x,(a,b,c)>}{\mathrm{norm[a,\ b,\ c]}}.$$

```
lhsr6[x_, a_, b_, c_] := If[a == x && c == x && b == 0, delta[1], 0];

 rhsr6f[x_, a_, b_, c_, p_, q_, q1_, r_] :=
  theta[x, 1, p] theta[a, 1, p] theta[q1, 1, 1] theta[q, q1, 1] theta[b, q, 1] theta[x, r, 1]
   theta[c, r, 1] / delta[p] / delta[q] / delta[q1] / delta[r];

 rhsr6r[x_, a_, b_, c_, q_, q1_, r_] :=
  If[adm[x, q1, c] && adm[r, q, c], tet[1, c, x, 1, q1, r] tet[q1, c, r, 1, q, x]
    tet[q, b, c, r, c, 1] / theta[x, q1, c] / theta[r, q, c], 0];

 rhsr6l[x_, a_, b_, c_, p_, q_, q1_] :=
  If[adm[a, x - 1, 1] && adm[a - 1, x - 1, q1] && adm[x, q, a - 1] && adm[p, b, a - 1],
    delta[x - 1] delta[a - 1] lambda[a, 1, a - 1] tet[p, 1, x - 1, 1, a, x] tet[1, q1, x - 1, a, a - 1, 1]
     tet[q1, q, x, x - 1, a - 1, 1] tet [q, b, p, x, a - 1, 1]
     tet[a - 1, a, a, p, b, 1] / lambda[x, 1, x - 1] / theta[x, x - 1, 1] / theta[a, a - 1, 1] / theta[a, x - 1, 1] /
        theta[a - 1, x - 1, q1] / theta[x, q, a - 1] / theta[p, b, a - 1], 0] +
   If[adm[a, x - 1, 1] && adm[a + 1, x - 1, q1] && adm[x, q, a + 1] && adm[p, b, a + 1],
    delta[x - 1] delta[a + 1] lambda[a, 1, a + 1] tet[p, 1, x - 1, 1, a, x] tet[1, q1, x - 1, a, a + 1, 1]
     tet[q1, q, x, x - 1, a + 1, 1] tet [q, b, p, x, a + 1, 1]
     tet[a + 1, a, a, p, b, 1] / lambda[x, 1, x - 1] / theta[x, x - 1, 1] / theta[a, a + 1, 1] / theta[a, x - 1, 1] /
        theta[a + 1, x - 1, q1] / theta[x, q, a + 1] / theta[p, b, a + 1], 0] +
   If[adm[a, x + 1, 1] && adm[a - 1, x + 1, q1] && adm[x, q, a - 1] && adm[p, b, a - 1],
    delta[x + 1] delta[a - 1] lambda[a, 1, a - 1] tet[p, 1, x + 1, 1, a, x] tet[1, q1, x + 1, a, a - 1, 1]
     tet[q1, q, x, x + 1, a - 1, 1] tet [q, b, p, x, a - 1, 1]
     tet[a - 1, a, a, p, b, 1] / lambda[x, 1, x + 1] / theta[x, x + 1, 1] / theta[a, a - 1, 1] / theta[a, x + 1, 1] /
        theta[a - 1, x + 1, q1] / theta[x, q, a - 1] / theta[p, b, a - 1], 0] +
   If[adm[a, x + 1, 1] && adm[a + 1, x + 1, q1] && adm[x, q, a + 1] && adm[p, b, a + 1],
    delta[x + 1] delta[a + 1] lambda[a, 1, a + 1] tet[p, 1, x + 1, 1, a, x] tet[1, q1, x + 1, a, a + 1, 1]
     tet[q1, q, x, x + 1, a + 1, 1] tet [q, b, p, x, a + 1, 1]
     tet[a + 1, a, a, p, b, 1] / lambda[x, 1, x + 1] / theta[x, x + 1, 1] / theta[a, a + 1, 1] / theta[a, x + 1, 1] /
        theta[a + 1, x + 1, q1] / theta[x, q, a + 1] / theta[p, b, a + 1], 0];

 rhsr6term[x_, a_, b_, c_, p_, q_, q1_, r_] :=
  If[adm[a, p, 1] && adm[b, q, 1] && adm[c, r, 1],
   rhsr6l[x, a, b, c, p, q, q1] rhsr6r[x, a, b, c, q, q1, r] / rhsr6f[x, a, b, c, p, q, q1, r], 0];

 rhsr6[x_, a_, b_, c_] := rhsr6term[x, a, b, c, x - 1, 1, 0, x - 1] + rhsr6term[x, a, b, c, x - 1, 1, 0, x + 1] +
   rhsr6term[x, a, b, c, x - 1, 1, 2, x - 1] + rhsr6term[x, a, b, c, x - 1, 1, 2, x + 1] +
   rhsr6term[x, a, b, c, x - 1, 3, 2, x - 1] + rhsr6term[x, a, b, c, x - 1, 3, 2, x + 1] +
   rhsr6term[x, a, b, c, x + 1, 1, 0, x - 1] + rhsr6term[x, a, b, c, x + 1, 1, 0, x + 1] +
   rhsr6term[x, a, b, c, x + 1, 1, 2, x - 1] + rhsr6term[x, a, b, c, x + 1, 1, 2, x + 1] +
   rhsr6term[x, a, b, c, x + 1, 3, 2, x - 1] + rhsr6term[x, a, b, c, x + 1, 3, 2, x + 1];

 r6[x_, a_, b_, c_] := If[adm[a, a, b] && adm[c, c, b],
    (rhsr6[x, a, b, c] - lhsr6[x, a, b, c]) / norm[a, b, c], 0] // Simplify;
```

Case 1

```
eref1[e_] :=
 ReplaceRepeated[ e // Expand, {qif[x + y/2] → qi[x + y/2] qif[x + y/2 - 1],
     qif[y/2 + z] → qi[y/2 + z] qif[y/2 + z - 1], qif[x] → qi[x] qif[x - 1], qif[y] → qi[y] qif[y - 1],
     qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2))}] // Expand // Factor;

IntegerQ[x] ^= True; EvenQ[y] ^= True; IntegerQ[z] ^= True;
given = x ≥ 1&& y ≥ 2 && z ≥ 1&& y ≤ 2 x&& y ≤ 2 z;
r1[x, y - 2, z - 1, x, y, z] // eref1
```

$A^{-2-2x-2y}(A^{2+2x}-A^{y})(A^{2+2x}+A^{y})$

This is nonzero, as required.

Case 2

```
eref2[e_] :=
 ReplaceRepeated[ e // Expand, {qif[x] → qi[x] qif[x - 1], qif[z] → qi[z] qif[z - 1],
     qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2))}] // Expand // Factor;

IntegerQ[x] ^= True; IntegerQ[z] ^= True;
given = x ≥ 1&& z ≥ 1;
r1[x, 0, z - 1, x, 2, z] r2[x - 1, 0, z, x, 0, z] - r1[x, 0, z - 1, x, 0, z] r2[x - 1, 0, z, x, 2, z] //
 eref2
```

$-A^{-2-2x-2z}(-1+A^{x})(1+A^{x})(A^{x}-A^{z})(-1+A^{z})(1+A^{z})(A^{x}+A^{z})$

For $x \neq z$, nonzero.

Case 3

```
eref3[e_] :=
 ReplaceRepeated[ e // Expand, {qif[x] → qi[x] qif[x - 1], qif[x - 1] → qi[x - 1] qif[x - 2],
     qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2))}] // Expand // Factor;

IntegerQ[x] ^= True; given = x ≥ 2;
mat[1, 1] = r1[x, 2, 1, x, 4, 2];
mat[1, 2] = r1[x, 2, 1, x, 2, 2];
mat[1, 3] = r1[x, 2, 1, x, 0, 2];
mat[1, 4] = r1[x, 2, 1, x, 0, 0];
mat[2, 1] = 0;
mat[2, 2] = r2[x - 1, 0, 2, x, 2, 2];
mat[2, 3] = r2[x - 1, 0, 2, x, 0, 2];
mat[2, 4] = 0;
mat[3, 1] = 0;
mat[3, 2] = r3[x - 1, 1, x, 2, 2];
mat[3, 3] = r3[x - 1, 1, x, 0, 2];
mat[3, 4] = r3[x - 1, 1, x, 0, 0];
mat[4, 1] = r4[x - 1, x, 4, 2];
mat[4, 2] = r4[x - 1, x, 2, 2];
mat[4, 3] = r4[x - 1, x, 0, 2];
mat[4, 4] = r4[x - 1, x, 0, 0];
Det[Table[mat[i, j], {i, 1, 4}, {j, 1, 4}]] // eref3
```

$$(-1 + A)\, A^{-10-2x}\, (1 + A)\, (1 + A^2)\, (-A + A^x)^2\, (A + A^x)^2\, (A^2 + A^{2x})^2$$

For $x >= 2$, nonzero.

Case 4

```
eref4[e_] :=
 ReplaceRepeated[ e // Expand, {qif[z] → qi[z] qif[z - 1], qif[z - 1] → qi[z - 1] qif[z - 2],
     qif[z - 2] → qi[z - 2] qif[z - 3], qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2))}] // Expand // Factor;

IntegerQ[z] ^= True; given = z > 2;
mat[1, 1] = r1[2, 2, z - 1, 2, 4, z];
mat[1, 2] = r1[2, 2, z - 1, 2, 2, z];
mat[1, 3] = r1[2, 2, z - 1, 2, 0, z];
mat[1, 4] = 0;
mat[2, 1] = 0;
mat[2, 2] = r2[1, 0, z, 2, 2, z];
mat[2, 3] = r2[1, 0, z, 2, 0, z];
mat[2, 4] = r2[1, 0, z, 0, 0, z];
mat[3, 1] = 0;
mat[3, 2] = r3[1, z - 1, 2, 2, z];
mat[3, 3] = r3[1, z - 1, 2, 0, z];
mat[3, 4] = r3[1, z - 1, 0, 0, z];
mat[4, 1] = r5[z - 1, 2, 4, z];
mat[4, 2] = r5[z - 1, 2, 2, z];
mat[4, 3] = r5[z - 1, 2, 0, z];
mat[4, 4] = r5[z - 1, 0, 0, z];
Det[Table[mat[i, j], {i, 1, 4}, {j, 1, 4}]] // eref4
```

$$-(-1 + A)\, A^{-6-2z}\, (1 + A)\, (1 + A^2)\, (-1 + A^z)\, (1 + A^z)\, (-A + A^z)\, (A + A^z)\, (A^2 + A^{2z})$$

Since $z > 2$, nonzero, as required.

Case 5

Due to admissibility conditions, Case 5 breaks into two subcases.

Case 5a: $x = 2$.

```
eref5a[e_] := e /. qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2)) // Factor;

mat[1, 1] = r2[1, 2, 2, 2, 4, 2];
mat[1, 2] = r2[1, 2, 2, 2, 2, 2];
mat[1, 3] = r2[1, 2, 2, 2, 0, 2];
mat[2, 1] = 0;
mat[2, 2] = r3[1, 1, 2, 2, 2];
mat[2, 3] = r3[1, 1, 2, 0, 2];
mat[3, 1] = r6[0, 2, 4, 2];
mat[3, 2] = r6[0, 2, 2, 2];
mat[3, 3] = r6[0, 2, 0, 2];
Det[Table[mat[i, j], {i, 1, 3}, {j, 1, 3}]] // eref5a
```

$(-1 + A)\ (1 + A)\ (1 + A^2)$

Since $x = 2$, nonzero.

Case 5b: $x > 2$.

```
eref5b[e_] :=
 ReplaceRepeated[ e // Expand, {qif[x] → qi[x] qif[x - 1], qif[x - 1] → qi[x - 1] qif[x - 2],
     qif[x - 2] → qi[x - 2] qif[x - 3], qi[n_] → (A^(2 n) - A^(-2 n)) / (A^2 - A^(-2))}] // Expand // Factor;

IntegerQ[x] ^= True; given = x ≥ 3;
mat[1, 1] = r2[x - 1, 2, x, x, 4, x];
mat[1, 2] = r2[x - 1, 2, x, x, 2, x];
mat[1, 3] = r2[x - 1, 2, x, x, 0, x];
mat[2, 1] = 0;
mat[2, 2] = r3[x - 1, x - 1, x, 2, x];
mat[2, 3] = r3[x - 1, x - 1, x, 0, x];
mat[3, 1] = r6[x - 2, x, 4, x];
mat[3, 2] = r6[x - 2, x, 2, x];
mat[3, 3] = r6[x - 2, x, 0, x];
Det[Table[mat[i, j], {i, 1, 3}, {j, 1, 3}]] // eref5b
```

$A^{-4+2x}\ (-1 + A^x)\ (1 + A^x)$

Since $x > 2$, nonzero, as required.

Appendix C. Gauss Sum Lemma

The results presented in this appendix are due to Paul van Wamelen.

Let r be an odd integer. Let $\zeta_N = e^{2\pi i/N}$ and let $\zeta = \zeta_{16r}$.

Lemma C.1.

$$\sum_{k=1}^{r-1} \zeta_{2r}^{rk}\zeta_{2r}^{2k^2+2k} + 1 = \zeta_{16r}^{-(r+2)^2}(\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))).$$

Proof. Note that

$$\begin{aligned} \sum_{k=1}^{r-1} \zeta_{2r}^{rk}\zeta_{2r}^{2k^2+2k} &= \sum_{k=1}^{r-1} \zeta_{2r}^{2k^2+(r+2)k} \\ &= -1 + \sum_{k=0}^{r-1} \zeta_{16r}^{16k^2+8k(r+2)} \\ &= -1 + \zeta_{16r}^{-(r+2)^2} \sum_{k=0}^{r-1} \zeta_{16r}^{(4k^2+r+2)^2}. \end{aligned}$$

Lemma C.2.

$$\sum_{k=0}^{r-1} \zeta^{(4k+2+r)^2} = \zeta^{9r^2} - 2\sum_{k=0}^{r-1} \zeta^{k^2} + 2\sum_{k=0}^{\frac{r-1}{2}} \zeta_{4r}^{k^2}.$$

Proof. Let $\varphi : \mathbb{Z} \to \mathbb{Z}_{16r}$. For n such that $0 \leq n < 16$ let

$$I_n = \{\varphi(i)^2 | i \equiv 1 \mod 4 \text{ and } nr < i < (n+1)r\},$$

and

$$J_n = \{\varphi(i)^2 | i \equiv 3 \mod 4 \text{ and } nr < i < (n+1)r\}.$$

For $n < 0$ or $n \geq 16$ replace n by its remainder on division by 16.

As $i^2 = (-i)^2$ we see that $I_n = J_{15-n}$ for all n. As $(i \pm 8r)^2 \equiv i^2 \mod 16r$ we see that $I_n = I_{n\pm 8}$ and $J_n = J_{n\pm 8}$ for all n.

If $r \equiv 1 \mod 4$ then

$$\begin{aligned}(4k+3+2r)^2 - (4k+3)^2 &= 4r(4k+r+3)\\ &\equiv 0 \mod 16r,\end{aligned}$$

so that $J_n = I_{n+2}$ for all n.

If $r \equiv 3 \mod 4$ then

$$\begin{aligned}(4k+1+2r)^2 - (4k+1)^2 &= 4r(4k+r+1)\\ &\equiv 0 \mod 16r,\end{aligned}$$

so that $I_n = J_{n+2}$ for all n.

Using the above rules we see that in each of the sets below all elements are equal. For $r \equiv 1 \mod 4$

$$\{I_0, I_1, I_8, I_9, J_6, J_7, J_{14}, J_{15}\},$$

$$\{I_2, I_7, I_{10}, I_{15}, J_0, J_5, J_8, J_{13}\},$$

$$\{I_3, I_6, I_{11}, I_{14}, J_1, J_4, J_9, J_{12}\},$$

$$\{I_4, I_5, I_{12}, I_{13}, J_2, J_3, J_{10}, J_{11}\},$$

For $r \equiv 3 \mod 4$

$$\{I_0, I_5, I_8, I_{13}, J_2, J_7, J_{10}, J_{15}\},$$

$$\{I_6, I_7, I_{14}, I_{15}, J_0, J_1, J_8, J_9\},$$

$$\{I_1, I_4, I_9, I_{12}, J_3, J_6, J_{11}, J_{14}\},$$

$$\{I_2, I_3, I_{10}, I_{11}, J_4, J_5, J_{12}, J_{13}\}.$$

Using this we have, for $r \equiv 1 \mod 4$

$$\begin{aligned}
&\sum_{k=0}^{r-1} \zeta^{(4k+2+r)^2} \\
=&\zeta^{9r^2} + \sum_{i \in J_1 \cup J_2 \cup J_3 \cup J_4} \zeta^i \\
=&\zeta^{9r^2} + 2 \sum_{i \in I_4 \cup J_4} \zeta^i,
\end{aligned}$$

and for $r \equiv 3 \mod 4$

$$\begin{aligned}
&\sum_{k=0}^{r-1} \zeta^{(4k+2+r)^2} \\
=&\zeta^{9r^2} + \sum_{i \in I_1 \cup I_2 \cup I_3 \cup I_4} \zeta^i \\
=&\zeta^{9r^2} + 2 \sum_{i \in I_4 \cup J_4} \zeta^i.
\end{aligned}$$

Now as $\zeta^{8r} = -1$ we have, for k odd, that $\zeta^{(k+4r)^2} = \zeta^{k^2}(-1)^{rk} = -\zeta^{k^2}$. Combining this with the above we see that

$$\sum_{i \in I_4 \cup J_4} \zeta^i = - \sum_{i \in I_0 \cup J_0} \zeta^i,$$

and we conclude that

$$\begin{aligned}
\sum_{k=0}^{r-1} \zeta^{(4k+2+r)^2} =& \zeta^{9r^2} - 2 \sum_{k=0}^{\frac{r-1}{2}-1} \zeta^{(2k+1)^2} \\
=&\zeta^{9r^2} - 2 \sum_{k=0}^{r-1} \zeta^{k^2} + 2 \sum_{k=0}^{\frac{r-1}{2}} \zeta^{(2k)^2} \\
=&\zeta^{9r^2} - 2 \sum_{k=0}^{r-1} \zeta^{k^2} + 2 \sum_{k=0}^{\frac{r-1}{2}} \zeta_{4r}^{k^2}.
\end{aligned}$$

□

Hence,

$$\sum_{k=1}^{r-1} \zeta_{2r}^{rk} \zeta_{2r}^{2k^2+2k} + 1 = \zeta_{16r}^{-(r+2)^2} (\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))).$$

□

Vita

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