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OPTIMAL ACTUATION IN ACTIVE VIBRATION CONTROL USING POLE-PLACEMENT

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mechanical Engineering

by
Carla Ann Guzzardo
B.S., Embry-Riddle Aeronautical University
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NOMENCLATURE

$\mathbf{a}$	=	vector of coefficients
$\mathbf{A}$	=	system matrix in state-space form
$\mathbf{b}$	=	force selection vector, also called input vector
$\mathbf{C}$	=	damping matrix
$\mathbf{e}_i$	=	i^{th} unit vector
$\mathbf{F}$	=	net force applied to the system
$\mathbf{f}, \mathbf{g}$	=	velocity and position gain vectors, respectively
$\mathbf{I}$	=	Identity matrix
$\mathbf{K}$	=	stiffness matrix
$\mathbf{M}$	=	mass matrix
m	=	number of eigenvalues to be assigned
n	=	system order, degrees-of-freedom
r	=	number of controllable degrees-of-freedom
$\mathbf{S}$	=	matrix of open-loop eigenvalues
s	=	complex Laplace frequency
t	=	time
$\mathbf{u}$	=	control input

$\mathbf{V}$ = matrix of open-loop eigenvectors

v_i = i^{th} eigenvector

$\mathbf{x}$ = state vector

$\mathbf{y}$ = control output

$\mathbf{y}_1$ = vector of controllable eigenvectors

$\boldsymbol{\beta}$ = input matrix in state-space form (controllability)

$\boldsymbol{\Lambda}$ = matrix of open-loop eigenvalues

λ = open-loop pole or eigenvalue

μ = closed-loop (assigned) pole or eigenvalue

η = cost function for optimization

ω = open-loop natural frequency

ϑ, γ, q = partial pole, partial eigenvalue, partial natural frequency placement factors

ϕ = Lagrangian constraint

ρ_I = internal solution

ρ_B = boundary solution

$\Im$ = controllability matrix

τ, ξ = Lagrange multiplier

w = weighting parameter

ABSTRACT

The purpose of this study was to find and demonstrate a method of optimal actuation in a mechanical system to control its vibration response. The overall aim is to develop an active vibration control method with a minimum control effort, allowing the smallest actuators and lowest control input.

Mechanical systems were approximated by discrete masses connected with springs and dampers. Both numerical and analytical methods were used to determine the optimum force selection vector, or input vector, to accomplish the pole placement, finding the optimal location of actuators and their relative gain so that the control effort is minimized. The problem was of finding the optimal input vector of unit norm that minimizes the norm of the control gain vector.

The methods of pole placement and partial pole placement were introduced, and used to solve various problems, including the active natural frequency modification problem associated with resonance avoidance in undamped systems, and the single-input-multiple-output pole assignment problem for second order systems. Both full and limited controllability were addressed.

During the numerical analysis, it was discovered that the system is uncontrollable if a control input vector is chosen that is mathematically orthogonal to an eigenvector associated with a reassigned eigenvalue. Conversely, the optimal input vector was discovered to be mathematically parallel to an eigenvector. This was proven analytically through mathematical proofs and demonstrated with various examples. Simulations were performed in MATLAB and Maple to verify the results numerically.

An example using realistic units was developed to show the order of magnitude improvement expected by using this method of optimization. All initial conditions and system parameters were held the same, but the input vector was changed. The optimal input vector provided an order of magnitude improvement over an evenly distributed input vector.

The principal conclusion was that by choosing a state feedback input vector that is mathematically parallel to the eigenvector associated with the open-loop eigenvalue to be reassigned, or in the case of multiple assignments, in the subspace of the eigenvectors, the control effort to accomplish pole placement can be reduced to its minimal value.

CHAPTER 1: INTRODUCTION AND LITERATURE REVIEW

Vibration is defined by Meirovitch (2001, pg. xvi) as "a subset of dynamics in which a system subjected to restoring forces swings back and forth about an equilibrium position." There are beneficial vibrations, such as in musical instruments to create sounds or in electrical massage units that offer comfort to tired muscles. However, in many mechanical systems, vibration can cause damage and shortening of service life. Constant vibration of a motor can cause fatigue and fracture of supports, earthquake-induced vibration can damage buildings, and vibration in a spacecraft can jeopardize a mission.

Free vibration occurs when a system is moved from equilibrium and then released, with no further input. A system with damping will eventually dissipate energy and return to equilibrium, while a system without damping will vibrate indefinitely. *Forced vibration* occurs when an outside force continually adds energy to the system. If this energy is not dissipated quickly enough, the vibration will become larger and larger until a breakdown occurs.

Vibration control is used to eliminate or at least attenuate vibration so that it does not affect the performance or design life of a mechanical system. Vibration can be controlled through passive or active means. There is also the possibility of combinations of passive and active technologies, known as hybrid or semi-active methods as mentioned by Song (1996).

Passive vibration control consists of parameter modification, including modifying or adding components by changing the geometry of the system, changing materials for different elasticity, or adding mass or damping material to the system. This all works to alter the response of the system to outside forces. If these outside, exciting forces are known in advance, this can be an efficient means of shifting the response of the system so that vibration does not occur due to those forces.

Resonance is a common problem in undamped or lightly damped systems. Vibration is subject to superposition, so when the forcing excitation that a system experiences is close to the natural frequency of the system, the vibration will constructively interfere and increase in amplitude over time. This can eventually lead to failure in the system if the amplitude increases beyond design limits. Often, dynamic absorbers, a type of passive control, are used to alter the response of the system. An absorber consists of a spring and mass added to the original system and chosen so that the frequency response of the original system goes to zero at the operating frequency. This can be a very powerful approach if the problem frequencies are known in advance. Using a dynamic absorber eliminates resonance at the original frequency, but it creates two more resonant frequencies around it at different values. In many systems, the excitation may be unknown or random, meaning that there is still the danger of resonance and making this type of passive control unsuitable.

Active vibration control can adjust to varying excitation forces as they occur, so that any vibration is removed from the system. Much work has been done on this topic and many different methods have been developed, but they all have one unifying trait: the use of sensors to measure the vibration and actuators to apply forces to the system components to destructively interfere with the vibration until it is cancelled out. Alkhatib and Golnaraghi (2003) provides a comprehensive review of active vibration control topics and methods, including their typical use.

All active control methods are implemented by either feedback, feed-forward control, or a combination of the two. Figure 1.1 illustrates a general block diagram of a control system. The input $\mathbf{u}$ is on the left. It is altered by various matrices and computations in the control law, then the output is displayed as $\mathbf{y}$. In feedback control, as shown by matrix $\mathbf{A}$ in Figure 1.1, the state of each degree of freedom is measured and sent as input to the control system. The control system then calculates the output signal which is sent to the actuators, so that they can apply the force needed to bring the system back to equilibrium. Feed-forward control, as shown by matrix $\mathbf{D}$ in Figure 1.1, requires anticipated values of state variables. This can be accomplished in systems where the excitation forces are known in advance and the response of the system is well understood, but for a system with random excitation, feedback control is better suited.

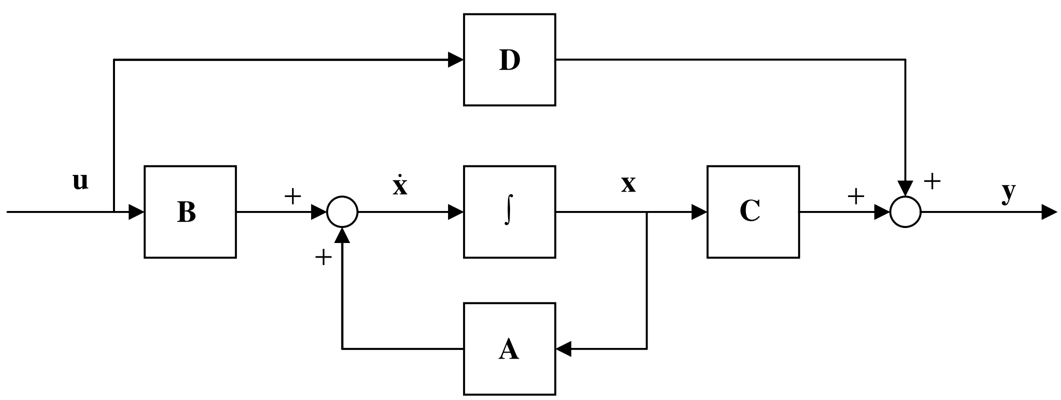

Figure 1.1: General matrix block diagram of the state and output equations from D'Azzo and Houpis (1995, pg. 148).

Feedback control can be implemented in various ways. Single-input, single-output (SISO) control is used in a single degree-of-freedom system. The state of the system is measured with sensors and those measurements are sent to the control system as input. From this input, the output signal is calculated and sent to the actuators. In multiple-input, multiple-output (MIMO) control, measurements for each degree of freedom are used to generate separate control signals to each actuator and all actuators work separately. This requires computation of a separate control law for each degree of freedom. In single-input, multiple-output (SIMO) control, the sensor measurements are used to generate one control signal that is modified by a separate gain for each actuator. SIMO has the advantage of working for multiple degrees-of-freedom without needing the much larger computational power or having the added complexity of a MIMO system.

This thesis investigates the use of SIMO feedback control to accomplish *eigenvalue assignment* (also known as *pole placement*). This is a powerful method of active vibration control that relies on modifying the response of a system by modifying its eigenvalues to lie in the left-half of the complex plane, resulting in a stable system that returns to equilibrium quickly.

1.1 Eigenvalues/Natural Frequencies and Stability

The vibration response of a system can be described by its eigenvalues and eigenvectors. The eigenvalues may also be called *mode frequencies,* or *poles* if the system is completely controllable and observable. The eigenvectors may also be referred to as *mode shapes*.

In a system with damping, the eigenvalues are complex conjugate pairs that describe the frequency of vibration and the rate at which the vibration decreases or increases. A system of n degrees-of-freedom will have n complex-conjugate pairs for a total of $2n$ eigenvalues. The complex part of each eigenvalue describes the frequency of the vibration response. A higher magnitude complex part indicates a higher frequency of vibration. The real part of each eigenvalue describes how quickly the vibration response will decrease or increase. A negative real part indicates that the vibration response will decrease in amplitude over time, and conversely, a positive real part indicates that the response will increase in amplitude over time. A higher magnitude real part indicates a faster increase or decrease.

A system with eigenvalues that all lie in the left-half plane (LHP) of the complex *s*-plane, in other words with negative real part, is called a *stable* system. Stability means that vibration of all degrees of freedom will decrease over time and the system will converge to, or at least oscillate about, equilibrium without added control. If any eigenvalues lie in the right-half plane (RHP), the system is said to be unstable because one or more degrees of freedom will have increased vibration over time and will require added control to bring the entire system back to equilibrium. Figure 1.2 shows the vibration response of systems with eigenvalues in various locations on the complex plane.

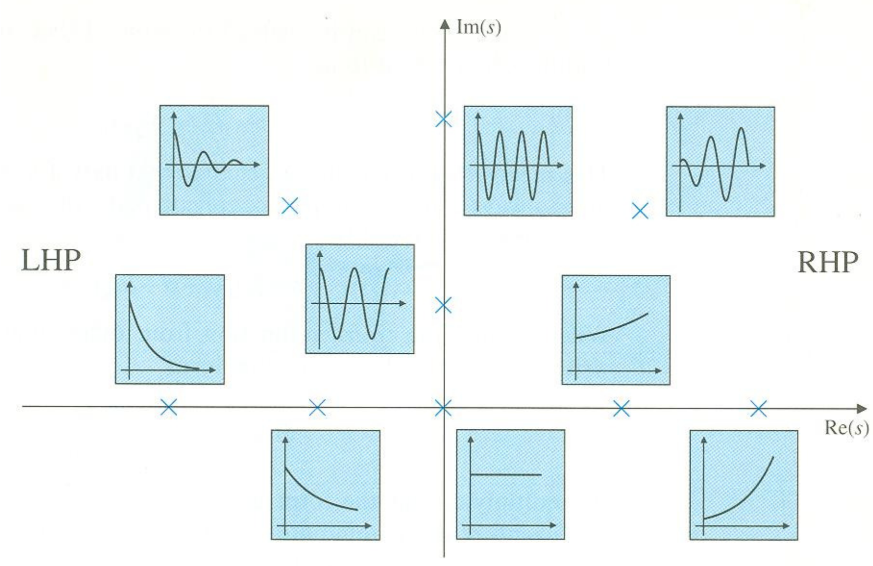

Figure 1.2: Graph of complex plane (s-plane) showing response and stability of various systems by position of eigenvalues, from Franklin et al (1994, pg. 121).

Each eigenvector gives the relative multiplication of displacement between each degree of freedom for vibration at its associated eigenvalue. A system of n degrees-of-freedom can have up to $2n$ eigenvectors and each eigenvector is associated with an eigenvalue.

In a system with no damping, the eigenvalues are given by the square of the natural frequencies, as shown in (4). In this case, each eigenvalue can be thought of as a pair of pure imaginary complex roots,

$$\lambda_k = (i\omega_k)(-i\omega_k) = \omega_k^2, \; k = 1, 2, \ldots, n. \quad (1)$$

If no outside forces exist, the system will oscillate about equilibrium indefinitely due to its conservative nature; there are no dissipative forces to release energy from the system and allow it to return to equilibrium. It is inherently stable if there is no input of energy to increase the amplitude of vibration. However, even a very small outside force acting on the system at or near a natural frequency can cause resonance to occur, where the amplitude of vibration gradually grows beyond the physical limits of the system. It may be desirable to shift the natural frequency of the system to prevent resonance. It also may be desirable to use active control to add damping to the system, so that instead of oscillating indefinitely, the system will come back to equilibrium.

Many algorithms have been developed to accomplish eigenvalue assignment. Ackermann's formula is the classical method, developed in 1972, but it is limited in its applicability. Miminis and Paige (1988) give an algorithm for pole placement by state feedback and also review many other pole placement algorithms, each concerned with the condition number of the resulting gain matrix and the numerical accuracy of the resulting eigenvalues rather than optimizing the actuation used. Balas (1978) investigates vibration suppression in large space structures by Direct Velocity Feedback, where the velocity output from sensors is multiplied by a gain in the control system, then applied by force actuators at the same location. Kimura (1975) shows that eigenvalue assignment by gain feedback control is possible on systems that are not completely observable. Mottershead and Ram (2006) offer a good background on full and partial pole assignment. Datta and Sarkissian (2002) establish the uniqueness and completeness of solution for the partial eigenvalue assignment problem with single or multiple inputs, and also discuss controllability.

The classical method of control design involves transforming the equation of motion of the system to a frequency domain equation. This allows the use of simple algebraic equations to solve for the gain necessary for pole assignment. However, transforming the system destroys the symmetry in the second-order nature of the equations and can lead to computational errors in the final design.

1.2 Spillover

With any active control used, if care is not taken, the system may actually be made unstable. Any time outside forces, such as from the actuators, are applied, energy is being added to the system. If applied improperly, the control could result in more vibration in the system even when less was intended. This can sometimes happen in partial pole assignment.

In physical cases, there may be a large number of eigenvalues but only a few that are undesirable. In this case, partial pole assignment can be used to shift these undesirable eigenvalues to a more favorable position on the complex plane, while ignoring the originally favorable eigenvalues. However, in some instances of partial pole assignment, the originally favorable eigenvalues may be inadvertently altered and moved to an unstable position on the complex plane. This is called *spillover*. In order to avoid spillover, Datta, Elhay, and Ram (1997) developed a method for partial pole assignment that does not reduce the model to a first-order transformation first, as is often done in control design. This allows the second-order nature of the problem to be maintained and allows a mathematical way of assuring that only the unfavorable eigenvalues chosen by the designer are reassigned, eliminating spillover.

1.3 Optimization

In most current methods of pole assignment, the input vector, $\mathbf{b}$, is selected by the designer and the feedback gain vectors, $\mathbf{f}$ and $\mathbf{g}$, are unknown. This thesis proposes solving for optimal actuation, thus letting $\mathbf{b}$, $\mathbf{f}$, and $\mathbf{g}$ be unknown and finding the combination that allows for the minimal control effort.

In practice, excessive control force from the actuators can lead to damage of the structure or saturation and improper functioning of the actuators. Optimization can prevent this. Also by optimizing the actuation for minimal control force, the system designer can select smaller actuators. In aerospace missions where mass is a strong system constraint this can be vital to mission success.

Various methods have been researched and used to implement pole assignment optimally. Optimization in this case refers to effectively applying the control to place the eigenvalues while minimizing some cost function, typically related to the amount of control force necessary. Chang and Yu (1996) attempt to have minimal control force to place the eigenvalues, but instead of choosing new values in advance, a technique is used to find the optimal eigenvalues within a given region which require the minimum control force to assign, and the gain is found from that. This thesis does not put a limitation on the eigenvalues, but allows the designer to determine what eigenvalues they would like to assign and optimizes the gain needed to accomplish that. Gao et al (2003) are concerned with the placement of actuators in the optimal control of a building with random parameters. Feedback gain optimization is only done after placement optimization. Hong, Park, and Park (2006) use H^2 and $H\infty$ controls for robustness in the control of a composite beam with an embedded piezoelectric layer. Jiang and Moore (1996) use least squares feedback to assign optimal eigenvalues, but the authors admit that this is only a means of finding a local minimum to the cost function. Karbassi (2001) establishes an algorithm to minimize the control force during eigenvalue assignment, but uses a different, more computatively-involved method than is used in this thesis. Lam and Yan (1997) use robustness, as measured by the spectral condition number, as the cost function for complete pole placement optimization. Qian and Xu (2005) offer a method of optimal partial eigenvalue assignment with the condition number of the matrix of eigenvectors as the cost function, but use an already assigned force selection vector. Ram and Inman (1999) also offer a method of optimal control while maintaining the second order nature of the vibration equations, instead of relying on first-order realization. An optimization solution with a cost function weighted on both control force and response of the system is offered, but it does not use the same method of pole assignment as this thesis.

1.4 Collocated Sensor/Actuator Pairs

Both Schulz and Heimbold (1983) and Yang and Lee (1993) study the optimization of feedback control on a system with non-collocated sensors and actuators. This non-collocation can lead to an unstable system if not implemented properly. However, Dosch, Inman, and Garcia (1992) show that a collocated sensor/actuator pair is possible with a self-sensing piezoelectric actuator; therefore, the assumption of collocated sensor/actuator pairs is used in this thesis.

1.5 Technology

Vasques and Rodrigues (2006) compare various control schemes, both classical and modern, and show the response of a piezo-electrically controlled beam to those controls. It shows that application of the technology is possible at this time, though work is needed to implement it on a large scale. Matsuzaki, Ikeda, and Boller (2005) introduce a new Smart Metal Alloy which is partially magnetized and actuated by electromagnetic field excitation. This is the sort of material needed for devices that are fast enough to perform pole placement on a large scale in structural systems. The material was not available at the time of publication, so numerical results are given in place of lab experimentation results. Zhang et al (2004) also present a future actuator material that could be used for active vibration control. An experimental setup is presented and results show good response by the closed-loop system, again showing that the technology exists to implement this type of control.

1.6 Organization of Thesis

The thesis is organized as follows. Chapter 2 introduces the mathematical formulae necessary to discuss vibration of an open-loop system, both with and without damping. The equation of motion is developed and state space analysis is used to determine the eigenvalues and eigenvectors of a system. The vibration response is found from initial conditions. An example problem is offered to demonstrate.

Chapter 3 continues with the development of a vibration control method using single-input-multiple-output feedback control. The formula for partial pole assignment developed by Datta, Ram, and Elhay (1997) is introduced. The system of the first example is now used in an example of this pole placement technique to increase stability.

Chapter 4 uses the results of Chapters 2 and 3 to develop an optimization method, through variation of the input vector and gain vectors, to minimize control effort as defined by a cost function. Observations on the controllability of the system and a theory developed from this observation are extended through various examples. The work of much of Chapter 4 is to be published in a future issue of Mechanical Systems and Signal Processing and is presented here with permission.

Chapter 5 introduces units to the equations and solves Example 9 again to find the magnitude of control force used in actuation of the system, both optimal and arbitrary.

Chapter 6 involves a brief discussion of the results of Chapter 5 and draws conclusions on the effectiveness of the theory.

Finally, the Appendices provide additional resources, including a proof of the solution of Problem 1 from Chapter 4 being the minimum on the domain, and all computer files used to generate the solutions seen in the thesis.

1.7 Scope and Limitations

This thesis serves as an initial investigation into the topic of optimal actuation. Very general and simplified linear models and examples are used to demonstrate the theory and to verify the mathematical proofs. Only feedback control is used to implement pole placement and partial pole placement.

Due to the constraints of time and equipment, no physical experimentation has been done, only computer simulations. There is no consideration in this work for non-linear systems or systems where time delay in the controls is a factor. There is also no statement made to what value eigenvalues should be assigned, and no control of eigenvectors. Those decisions are left up to the reader.

CHAPTER 2: OPEN-LOOP ANALYSIS

2.1 Equation of Motion

A mechanical system can be modeled as a simplified combination of lumped masses joined by springs and dampers. The masses model the inertia of the system; the springs model the resistance to motion or stiffness of the system; and the dampers model the energy dissipation of the system. Each mass-spring-damper combination represents one degree of freedom of the physical system. A two degrees-of-freedom system is modeled in Figure 2.1.

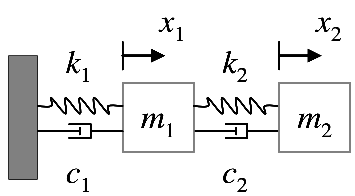

Figure 2.1: Simplified model of a two degrees-of-freedom system.

Each block has mass m; each spring has a coefficient of stiffness k; and each damper has a coefficient of damping c. The position of each block from equilibrium is measured as x. Each block is constrained to move only in the x direction.

The equation of motion of a system with n degrees-of-freedom is derived from Newton's second law of motion. If there are no outside forces on the system, the summation of forces on the i^{th} block is

$$\sum F_i = m_i \ddot{x}_i + c_i \dot{x}_i + k_i x_i = 0 \qquad (2)$$

The equation of motion of the entire system is represented in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \qquad (3)$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are n x n real matrices. Dots denote derivatives with respect to time.

Separation of variables is used to find a solution for Equation (3). This allows identification of the eigenvalues and eigenvectors of the system.

Consider a solution of the form

$$\mathbf{x} = \mathbf{v}e^{st} \tag{4}$$

and substitute into Equation (3). The equation of motion becomes

$$s^2\mathbf{M}\mathbf{v}e^{st} + s\mathbf{C}\mathbf{v}e^{st} + \mathbf{K}\mathbf{v}e^{st} = \mathbf{0} \tag{5}$$

The exponential function is nonzero for all positive values of time t; therefore it can be cancelled out of the equation. The quadratic eigenvalue problem remains.

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0}. \tag{6}$$

The solution $\mathbf{v} = \mathbf{0}$ exists for all values of s. This trivial solution, when $\mathbf{x} = \mathbf{0}$ for all time, does not tell us anything about the vibration response of the system, so we concentrate only on solutions where $\mathbf{v} \neq \mathbf{0}$, or when

$$\left|s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right| = 0. \tag{7}$$

The determinant equation is a polynomial of order $2n$ with roots s_i, $i = 1, 2, ..., 2n$. Each root s_i is an eigenvalue of the system. Once the eigenvalues are known, each eigenvector $\mathbf{v}_i$ is found by solving Equation (6), where s_i is the associated eigenvalue.

2.2 State Space Analysis

Two state space variables are defined for the system in Figure 2.1. Both position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$ are vectors of n length. The state space equations of motion are

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{M} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}. \tag{8}$$

By defining

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{M} \end{bmatrix}, \tag{9}$$

and rearranging, Equation (8) becomes

$$\mathbf{Az} - \mathbf{B}\dot{\mathbf{z}} = \mathbf{0}. \tag{10}$$

Try a solution of the form

$$\mathbf{z}(t) = \mathbf{U}e^{st} \tag{11}$$

where $\mathbf{U}$ is the constant vector

$$\mathbf{U} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{2n} \\ s_1\mathbf{v}_1 & s_2\mathbf{v}_2 & \cdots & s_{2n}\mathbf{v}_{2n} \end{bmatrix}. \tag{12}$$

This results in the problem

$$\mathbf{AU}e^{st} - s\mathbf{BU}e^{st} = \mathbf{0} \tag{13}$$

which can be simplified to the generalized eigenvalue problem,

$$\left(\mathbf{A} - s\mathbf{B}\right)\mathbf{U} = \mathbf{0}. \tag{14}$$

Equation (14) can be easily solved for the eigenvalues and eigenvectors of the system using an off-the-shelf commercial software program such as MATLAB.

2.3 Vibration Response

The output of the system is the position and velocity of each degree-of-freedom,

$$\mathbf{y}(t) = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \tag{15}$$

assuming all degrees-of-freedom to be observable.

The general solution of position for each degree of freedom is the linear combination of the $2n$ solutions for each eigenvalue and eigenvector pair, from Equation (4),

$$\mathbf{x} = \sum_{i=1}^{2n} a_i \mathbf{v}_i e^{s_i t} \ , \tag{16}$$

where a_i are constant coefficients determined by initial conditions of the system. Similarly, the solution of velocity is the time derivative of Equation (16),

$$\dot{\mathbf{x}} = \sum_{i=1}^{2n} a_i s_i \mathbf{v}_i e^{s_i t} \ . \tag{17}$$

To impose initial conditions, solve Equations (16) and (17) at time $t = 0$. This gives initial position

$$\mathbf{x}(0) = \sum_{i=1}^{2n} a_i \mathbf{v}_i \ , \tag{18}$$

and initial velocity

$$\dot{\mathbf{x}}(0) = \sum_{i=1}^{2n} a_i s_i \mathbf{v}_i \ . \tag{19}$$

In matrix form, this is written

$$\mathbf{Ua} = \begin{pmatrix} \mathbf{x}_0 \\ \dot{\mathbf{x}}_0 \end{pmatrix} \tag{20}$$

where, $\mathbf{U}$ is from Equation (12) and $\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{2n} \end{pmatrix}^T$. To determine the coefficients a_i in Equation (20), $\mathbf{U}$ must be invertible.

Once the eigenpairs $(s_i, \mathbf{v}_i), i = 1, 2, ..., 2n$ and coefficients a_i are known, the value of $\mathbf{y}(t)$ is known from Equations (15)-(17).

Example 1: Eigenvalues of a Two Degree-of-Freedom System

Consider a simple uncontrolled, vibrating system of two degrees of freedom, as shown in Figure 2.2.

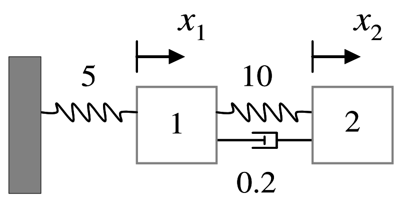

Figure 2.2: Example system with two degrees of freedom.

The system can be modeled as in Equation (3), with equation of motion

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 15 & -10 \\ -10 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{21}$$

State space analysis leads to Equation (10) as

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & 0 & 0 \\ 10 & -10 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.2 & -0.2 & 1 & 0 \\ -0.2 & 0.2 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{22}$$

Solving the generalized eigenvalue problem of Equation (14) using MATLAB, the eigenvalues and eigenvectors of the system in Figure 2.2 are found to be

$$\mathbf{S} = \begin{bmatrix} -0.1472 - 4.3170i & 0 & 0 & 0 \\ 0 & -0.1472 + 4.3170i & 0 & 0 \\ 0 & 0 & -0.0028 - 1.1575i & 0 \\ 0 & 0 & 0 & -0.0028 + 1.1575i \end{bmatrix}, \tag{23}$$

$$\mathbf{U} = \begin{bmatrix} 0.0263 - 0.1995i & 0.0263 + 0.1995i & 0.0082 - 0.6269i & 0.0082 + 0.6269i \\ -0.0114 + 0.0728i & -0.0114 - 0.0728i & 0.0055 - 0.8563i & 0.0055 + 0.8563i \\ 0.8573 + 0.1427i & 0.8573 - 0.1427i & 0.7257 + 0.0113i & 0.7257 - 0.0113i \\ -0.3127 - 0.0601i & -0.3127 + 0.0601i & 0.9912 + 0.0088i & 0.9912 - 0.0088i \end{bmatrix}. \tag{24}$$

The first two rows of the matrix **U** are the eigenvectors of the system, also known as matrix $\mathbf{v}$. Note there are $2n = 4$ eigenvalues and eigenvectors, $n = 2$ pairs of complex conjugates.

Also note that the eigenvalues of the system all have negative real part, therefore the system is stable.

Assume a given initial position and velocity of

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \dot{\mathbf{x}}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{25}$$

The coefficients for Equation (16) are

$$\mathbf{a} = \begin{bmatrix} 0.0025 + 1.9711\mathrm{i} \\ 0.0025 - 1.9711\mathrm{i} \\ 0.3872 + 0.1652\mathrm{i} \\ 0.3872 - 0.1652\mathrm{i} \end{bmatrix}. \tag{26}$$

The output $\mathbf{y}$ can be found as a function of time following Equations (15)-(17) and using the calculated values. A simulation of this example system is shown in Figure 2.3. Each figure shows the high and low frequency modes in the initial response of position and velocity. The high frequency response stabilizes more quickly than the low frequency, due to the eigenvalue being further in the left-half plane on the complex plane. The lower frequency response gradually decreases amplitude over time.

This example shows that the system is indeed stable and will converge to equilibrium; however, that convergence may take much longer than desired. There may be constraints on the performance characteristics of the design to minimize vibration or to more quickly damp out such oscillations to below a threshold of amplitude.

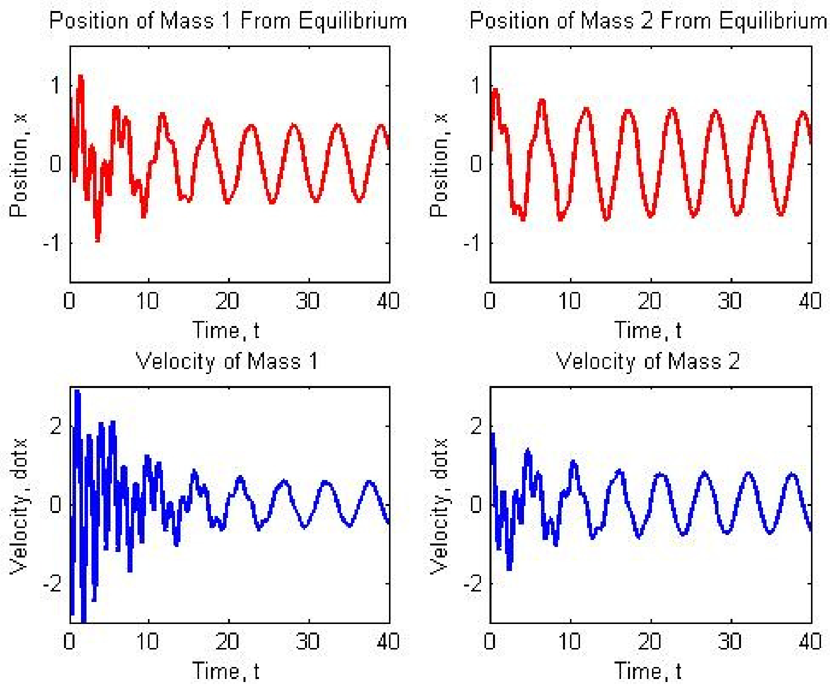

Figure 2.3: Simulation of example system displacement (top) and velocity (bottom), showing high frequency response stabilization from *t=0-30* and gradual decrease in amplitude of vibration for all *t*.

CHAPTER 3: POLE PLACEMENT

3.1 Pole Placement

Pole placement or *pole assignment*, also called *eigenvalue assignment* in various papers, involves reassigning the eigenvalues of the system to reduce its dynamic response. This can include moving eigenvalues to the LHP for stability, or moving further to the left if they are already stable, to reduce the time to convergence at equilibrium. This assignment is achieved through active damping and active stiffness, modifying the damping and stiffness of the closed loop system through applied forces.

A *control force* $\mathbf{b}u(t)$ is applied to the system, as in Figure 3.1. The *control input*, $u(t)$, includes the *velocity and position gain vectors*, $\mathbf{f}$ and $\mathbf{g}$, where

$$u(t) = \mathbf{f}^T \dot{\mathbf{x}} + \mathbf{g}^T \mathbf{x}. \quad (27)$$

This results in a new equation of motion,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u(t) \quad (28)$$

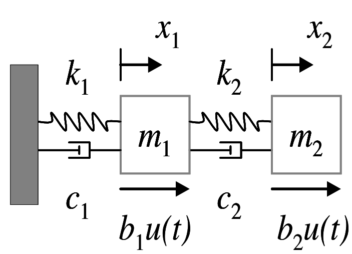

Figure 3.1: Discrete mass-spring-damper system of two degrees of freedom with applied control forces.

Only those eigenvalues which lie outside of the performance constraints in the design require reassignment. The *force selection vector*, $\mathbf{b}$, determines on which masses the control input is applied and with what gain, where

$$\mathbf{b} = \begin{pmatrix} b_1 & b_2 & \dots & b_m \end{pmatrix}^T . \tag{29}$$

This is an example of *single-input, multiple output (SIMO)* control. The only input is from u, but the force selection vector applies this input to multiple degrees-of-freedom, resulting in multiple outputs.

MATLAB's `place` command can be used to assign all new eigenvalues to the system. Begin by assigning a matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{M}^{-1}\mathbf{C} \end{bmatrix}. \tag{30}$$

The command `gf=place(A,-[zeros(n,1);inv(M)*b],s)` assigns the vector

$$\mathbf{gf} = \begin{bmatrix} \mathbf{g} \\ \mathbf{f} \end{bmatrix}, \tag{31}$$

which contains both position and velocity gain vectors necessary to reassign the system eigenvalues to the set s.

There is a related problem associated with the avoidance of resonance and near resonance phenomena in harmonically excited undamped systems. In this problem it is desired to shift a few natural frequencies from the spectral neighborhood of the exciting forces. There is a wealth of literature associated with this problem where the spectral modification is achieved by passive means, i.e., by physical structural modification altering the rigidity and density of the system, see e.g., Elishakoff (2000), Lawther (2007), McMillan and Keane (1996), Mottershead and Ram (2006), Ram (1994), and Ram and Blech (1991). Here we address the associated problem where the spectral modification is done by active vibration control implemented by state feedback. The problem may be regarded as a reduced form of the pole placement problem where $\mathbf{C} = \mathbf{0}$ and $\mathbf{f} = \mathbf{0}$. We name this problem the *active natural frequency modification problem*.

3.2 Partial Pole Placement

In full pole placement, all modes of the open loop system are reassigned to new eigenvalues. In practice, this can be an impossible and unnecessary task. A flexible structure may have a very large number of modes, but only a selection of those may be unstable or outside of performance requirements. Higher frequency modes will typically damp out much faster than low frequency modes. It is necessary only to reassign those modes which will cause problems in operation. Partial pole placement reassigns only those eigenvalues chosen while leaving all other open loop eigenvalues unchanged.

The original set of open loop eigenvalues, Λ, consists of those to be replaced and the remaining eigenvalues,

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}, \tag{32}$$

where

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} \tag{33}$$

is the set to be replaced.

Similarly, the open-loop eigenvectors consist of those to be replaced and those remaining unchanged,

$$\mathbf{V} = \left[\mathbf{V}_1 \mid \mathbf{V}_2 \right], \tag{34}$$

where

$$\mathbf{V}_1 = \left[\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\right] \tag{35}$$

is the set to be replaced.

If the number of reassigned eigenvalues is m, then the new eigenvalues of the system will be assigned to the set $\{\mu_1, \mu_2, ..., \mu_m\}$ and the remaining unchanged eigenvalues will be assigned to the set $\{s_{m+1}, s_{m+2}, ...s_{2n}\}$.

The velocity gain vector is chosen as

$$\mathbf{f = MV_1\Lambda_1q} \tag{36}$$

and the position gain vector is chosen as

$$\mathbf{g = -KV_1q} \tag{37}$$

where

$$\mathbf{q}_j = \frac{1}{\mathbf{b}^T\mathbf{v}_j}\frac{\mu_j - s_j}{s_j}\prod_{\substack{i=1 \\ i\neq j}}^{m}\frac{\mu_i - s_j}{s_i - s_j},\ j = 1,2,...,m \tag{38}$$

The result is a modified eigenvalue matrix with the new assigned eigenvalues but retaining the initial eigenvalues not meant to be changed, as shown in Datta, Elhay, and Ram (1997).

Once the force control vectors are known, the equation of motion can be solved for the new eigenvectors by including the control forces, giving the equation

$$\mathbf{M\ddot{x}} + \mathbf{C\dot{x}} + \mathbf{Kx} = \mathbf{b}(\mathbf{f}^T\mathbf{\dot{x}} + \mathbf{g}^T\mathbf{x})\,. \tag{39}$$

This can be solved by grouping terms of **x** and solving by separation of variables, as in the previous section.

$$\left[\mathbf{M}s^2 + \left(\mathbf{C} - \mathbf{bf}^T\right)s + \left(\mathbf{K} - \mathbf{bg}^T\right)\right]\mathbf{v} = \mathbf{0} \tag{40}$$

The state space equation of motion, similar to Equation (8), becomes

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \left(\mathbf{C}-\mathbf{b}\mathbf{f}^T\right) & \mathbf{M} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\left(\mathbf{K}-\mathbf{b}\mathbf{g}^T\right) & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}. \tag{41}$$

Following the procedure of Equations (9)-(14), the eigenvectors of the new eigenvalues can be found. The response of the system can also be found by following the procedure of Equations (15)-(20).

Example 2: Partial Pole Placement

Consider the system from Example 1, having eigenvalues given by Equation (23). We can determine the gain vectors needed to replace the second set of eigenvalues to have a larger real value and slightly higher frequency, i.e. $-1 \pm 1i$. Let the force selection vector be

$$\mathbf{b} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T. \tag{42}$$

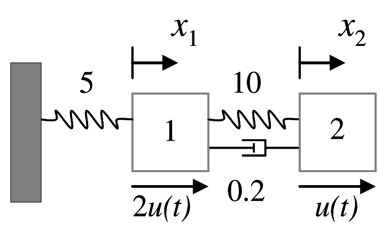

Figure 3.2: Example system with applied control forces.

The system, shown in Figure 3.2, has new eigenvalues

$$\mathbf{\Lambda} = \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & s_4 \end{bmatrix}, \tag{43}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \text{-1 - 1i} & 0 & 0 & 0 \\ 0 & \text{-1 + 1i} & 0 & 0 \\ 0 & 0 & \text{-0.1472 + 4.3170i} & 0 \\ 0 & 0 & 0 & \text{-0.1472 - 4.3170i} \end{bmatrix}. \quad (44)$$

Using the procedure from Datta, Elhay, and Ram (1997),

$$\begin{aligned} \mathbf{q}_1 &= \frac{1}{\mathbf{b}^T\mathbf{v}_1}\frac{\mu_1 - s_1}{s_1}\frac{\mu_2 - s_1}{s_2 - s_1} \\ \mathbf{q}_2 &= \frac{1}{\mathbf{b}^T\mathbf{v}_2}\frac{\mu_2 - s_2}{s_2}\frac{\mu_1 - s_2}{s_1 - s_2} \end{aligned}. \quad (45)$$

This leads to position gain vector

$$\mathbf{f} = \begin{pmatrix} -0.5930 \\ -1.6168 \end{pmatrix} \quad (46)$$

and velocity gain vector

$$\mathbf{g} = \begin{pmatrix} -0.1478 \\ -0.5771 \end{pmatrix}. \quad (47)$$

These are the gain vectors necessary to move only the undesired eigenvalues to a new value while retaining the other eigenvalues of the system.

The response of the closed loop system is shown in Figures 3.3 and 3.4. Note that the response decreases in amplitude much more quickly than that of the open loop system, shown in Figures 2.3 and 2.4. By $t = 30$, the amplitude of vibration has decreased two orders of magnitude.

Note that this example is limited by the choice of force selection vector, $\mathbf{b}$. The value chosen does not allow assigning the eigenvalue to any higher frequencies using this method. If the attempt is made, a solution cannot be found that correctly assigns the eigenvalues. The resulting performance is that of an unstable system.

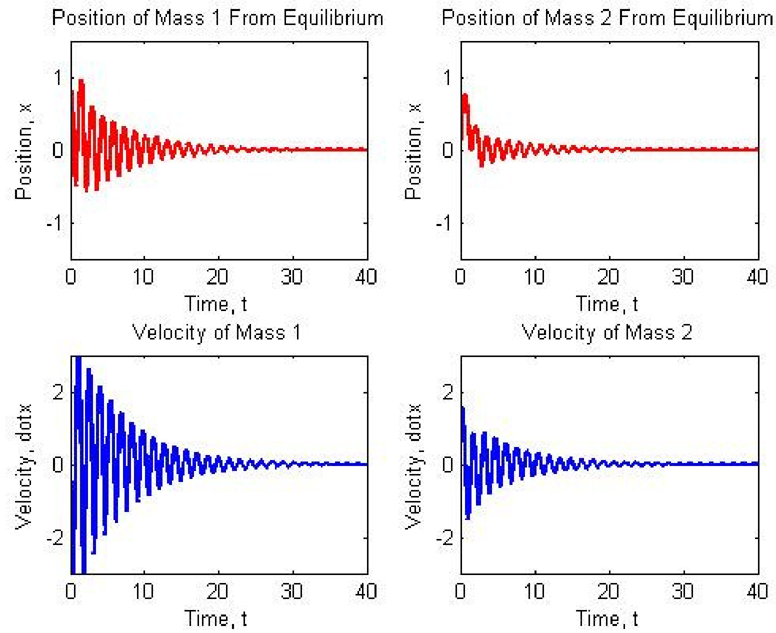

Figure 3.3: Simulation of controlled example system displacement and velocity.

CHAPTER 4: OPTIMIZATION

4.1 Definition of Cost Function

The choice of force selection vector, $\mathbf{b}$, affects the values of the position and velocity gain vectors, $\mathbf{f}$ and $\mathbf{g}$. These values, in turn, determine how much control force must be exerted by the actuators in the physical system. Minimizing the control force allows use of the smallest possible actuators and the minimum applied voltage during actuation.

We leave the definition of control force up to the designer of the system and show that any definition can be achieved through this method. We will use the cost function

$$\eta = \|\mathbf{f}\|^2 + w\|\mathbf{g}\|^2 \tag{48}$$

to demonstrate the method.

In addition, the force selection vector is constrained to

$$\|\mathbf{b}\| = 1 . \tag{49}$$

Without this constraint, the force selection vector could theoretically be made very large to allow the position and velocity gain vectors to be very small, with the same control effect. However, physically this would not minimize the actuation needed.

4.2 Controllability

A system is said to be completely controllable if each output state is constrained by the input control vector **[D'Azzo]**. To determine if this is true, a controllability matrix can be assembled from the state equations of the system. The optimization criterion decided upon in Section 4.1 can also show where the system becomes uncontrollable. There is a peak in the graph of optimization criterion versus force selection vector, shown in Figure 4.1, which represents the choice of force selection control vector that is not able to control the given system of Example 2. This is demonstrated through both state space formulation and vibration formulation.

4.2.1 Controllability Matrix

Using the equation of motion of the closed-loop system (28), the state-space formulation is

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \boldsymbol{\beta}u(t), \tag{50}$$

where

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \boldsymbol{\beta} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{b} \end{pmatrix}, \tag{51}$$

$$u(t) = \boldsymbol{\varphi}^T\mathbf{z}, \text{ and } \boldsymbol{\varphi} = \begin{pmatrix} \mathbf{g} \\ \mathbf{f} \end{pmatrix}. \tag{52}$$

The controllability matrix is defined as

$$\Im = \begin{bmatrix} \boldsymbol{\beta} & \mathbf{A}\boldsymbol{\beta} & \cdots & \mathbf{A}^{2n-1}\boldsymbol{\beta} \end{bmatrix}. \tag{53}$$

The system is only controllable if the controllability matrix has full rank, when

$$\text{rank}(\Im) = 2n. \tag{54}$$

Otherwise, it is uncontrollable. In the case where $\mathbf{b} = \Re^{n\times 1}$, the system is uncontrollable if $\det(\Im) = 0$.

Example 3: Demonstration of Controllability - State Space Formulation

Graphical evidence of controllability can be shown by repeating Example 2 and analyzing a plot of force selection vector versus the optimization criterion, as in Figure 4.1. To create the graph, the first component of the force selection vector is varied from -1 to 1 with a step size of 0.001. The force selection vector is constrained to having a norm of 1, so the second element is calculated as

$$b_2 = \sqrt{1 - b_1^{\,2}}\ , \tag{55}$$

and the force selection vector is

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{56}$$

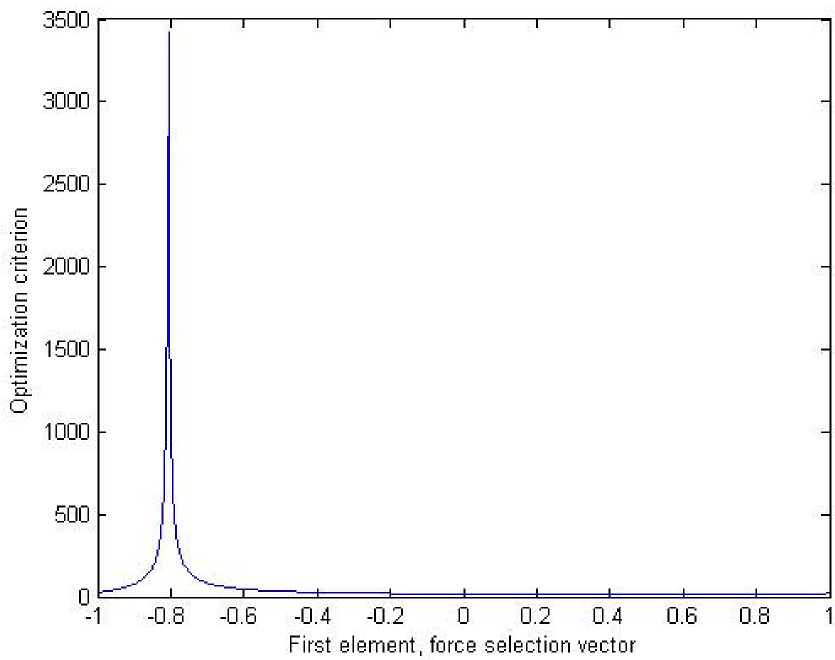

Figure 4.1: Plot of force selection vector versus optimization criteria, showing lack of controllability at peak of $b_1 = -0.807$.

The graph peaks when the first element of the force selection vector is -0.806. The resulting value of the second element is 0.592, therefore

$$\mathbf{b} = \begin{bmatrix} -0.806 \\ 0.592 \end{bmatrix}. \tag{57}$$

Using this value of $\mathbf{b}$, the state space controllability matrix is calculated as

$$\Im = \begin{bmatrix} 0 & \text{-0.8060} & 0.2204 & 14.9835 \\ 0 & 0.2960 & \text{-0.1102} & \text{-5.4767} \\ \text{-0.8060} & 0.2204 & 14.9835 & \text{-8.4999} \\ 0.2960 & \text{-0.1102} & \text{-5.4767} & 3.6990 \end{bmatrix} \tag{58}$$

Care must be taken when analyzing the controllability matrix. Checking the rank of the matrix in MATLAB, gives the result that $rank(\Im) = 4 = 2n$. However, the determinant is $\det(\Im) = -9.2028 \times 10^{-4}$. Because of the discretization error involved with a step size of 0.001, the determinant is not exactly zero, but is approaching zero meaning that the system is still controllable in the mathematical sense, but for practical applications, the forces necessary to achieve the control will exceed anything the actuators are capable of supplying. Therefore the system is, in a practical sense, uncontrollable.

4.2.2 Vibration Formulation

The vibration formulation helps to give a better understanding of areas where the system will become uncontrollable. The state space formulation is equivalent to the vibration formulation used in Chapter 3. This equivalency is proven by the state space formulation of (50) which expands to give back the closed-loop equation of motion (28) used in the vibration formulation and the identity equation, $\dot{\mathbf{x}} = \dot{\mathbf{x}}$.

From (38), we see that the control forces $\mathbf{f}$ and $\mathbf{g}$ require the calculation of

$$\frac{1}{\mathbf{b}^T \mathbf{v}_j}, \ j = 1, 2, ..., m. \tag{59}$$

where $\mathbf{v}_j$ are those eigenvectors associated with eigenvalues to be reassigned.

If the force selection vector, $\mathbf{b}$, is orthogonal to any of these eigenvectors of the open-loop system, this calculation results in a division by zero. This would require an infinite control force to completely control the system using that force selection vector. Physically this is impossible, making the system uncontrollable for chosen force selection vectors that are orthogonal to any eigenvector associated with a reassigned eigenvalue of the open-loop system.

Example 4: Demonstration of Controllability - Vibration Formulation

Using the vibration formulation, the system is analyzed for the same force selection vector, $\mathbf{b}$, as in Example 3. This vector, $\mathbf{b}$, is checked for orthogonality with the open-loop eigenvectors, $\mathbf{v}_j$, $j = 1, 2, ..., 2n$. Orthogonality is proven if

$$\mathbf{b}^T \mathbf{v}_j = 0. \tag{60}$$

Since each eigenvector is part of a complex conjugate pair, only one of each pair needs to be checked.

For this example,

$$\mathbf{b}^T \mathbf{v}_1 = \begin{bmatrix} -0.806 & 0.592 \end{bmatrix} \begin{bmatrix} -0.0133 - 0.2256i \\ 0.0028 + 0.0827i \end{bmatrix} = 0.0124 + 0.2308i, \tag{61}$$

$$\mathbf{b}^T \mathbf{v}_1 = \begin{bmatrix} -0.806 & 0.592 \end{bmatrix} \begin{bmatrix} -0.0040 - 0.6259i \\ 0.0112 + 0.8548i \end{bmatrix} = -0.0034 - 0.0015i. \tag{62}$$

Again, discretization errors keep (62) from equaling zero exactly, but the value approaches zero. Thus, the force selection vector of (57) is very close to orthogonal to the eigenvector $\mathbf{v}_3$. By the vibration formulation, the system is not controllable at this force selection vector.

4.3 Statement of Hypothesis

This uncontrollability associated with a mutually orthogonal eigenvector and force selection vector leads to the hypothesis that, conversely, the optimal force selection vector exists parallel to the reassigned eigenvector. In the case where multiple eigenvectors exist, the optimal force selection vector exists in the subspace of those eigenvectors. This hypothesis is proven mathematically in a journal article, written with co-authors Su-Seng Pang and Yitshak M. Ram, accepted for publication in Mechanical Systems and Signal Processing, the body of which is reprinted by permission in sections 4.4 through 4.7.

4.4 Introduction of Equations Used

We use the notation

$$\frac{\partial \gamma}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial \gamma}{\partial x_1} & \frac{\partial \gamma}{\partial x_2} & \cdots & \frac{\partial \gamma}{\partial x_n} \end{pmatrix}^T \tag{63}$$

to define the partial derivatives of a scalar function $\gamma(\mathbf{x})$ with respect to the elements of $\mathbf{x}$. We also use the following basic relations,

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}, \tag{64}$$

which holds for any constant symmetric matrix $\mathbf{A}$, and

$$\frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}\,, \tag{65}$$

which holds for any constant vector $\mathbf{a}$. By norm we mean the Euclidian norm.

4.5 Optimal Actuation in the Single Natural Frequency Modification Problem

The equations of motion for an open-loop undamped system are a simplified version of those presented in Chapter 2 for a full system, and can be modeled as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}\,. \tag{66}$$

The solution to (66) takes the form

$$\mathbf{x}(t) = \mathbf{v}\sin \omega t \tag{67}$$

where $\mathbf{v}$ is a constant vector. Substituting (67) in (66) gives the generalized eigenvalue problem

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{v} = \mathbf{0}\,, \qquad \lambda = \omega^2\,, \tag{68}$$

where $\{\lambda_k\}_{k=1}^n$ and $\{\mathbf{v}_k\}_{k=1}^n$ are the eigenvalues and eigenvectors of the open-loop system. In the natural frequency assignment problem, where the eigenvalues of (68) are assigned to be real, the closed-loop system (27)-(28) is reduced to

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u(t)\,, \tag{69}$$

where

$$u(t) = \mathbf{g}^T \mathbf{x}\,. \tag{70}$$

This leads to the eigenvalue problem

$$\left(\mathbf{K} - \mathbf{b}\mathbf{g}^T - \mu \mathbf{M}\right)\mathbf{w} = \mathbf{0}\,, \tag{71}$$

where $\{\mu_k\}_{k=1}^n$ and $\{\mathbf{w}_k\}_{k=1}^n$ are the eigenvalues and eigenvectors of the closed-loop system. In the partial natural frequency assignment problem we wish to change by the control some m eigenvalues, $m < n$, of the open-loop system $\{\lambda_k\}_{k=1}^m$ to a given real set $\{\mu_k\}_{k=1}^m$ while keeping the rest of the eigenvalues unchanged.

Note that control here is accomplished only through induced stiffness, in other words, only by using the position gain vector g. Although adding induced damping by using a velocity gain vector f may help control the system more efficiently, it is easier to introduce the concepts and proofs by using this simpler form of only one gain vector. Section 4.7 uses both gain vectors to find the solution. The procedure does not change, however the Lagrange multiplier complexity and number of equations increases with use of both gain vectors.

Lemma 1

With

$$\mathbf{g} = \sum_{k=1}^{m} \vartheta_k \mathbf{M}\mathbf{v}_k \tag{72}$$

where

$$\vartheta_k = \frac{\lambda_k - \mu_k}{\mathbf{b}^T \mathbf{v}_k} \prod_{\substack{i=1 \\ i \neq k}}^{m} \frac{\lambda_k - \mu_i}{\lambda_k - \lambda_i}, \tag{73}$$

the eigenvalues of (71) are

$$\{\mu_k\} = \{\mu_1 \quad \cdots \quad \mu_m \quad \lambda_{m+1} \quad \cdots \quad \lambda_n\}. \tag{74}$$

The lemma is a straightforward reduction of Theorem 3.2 in Datta, Elhay and Ram [2].

Consider the partial natural frequency assignment problem where $m=1$. That is an undamped system where only one natural frequency is to be reassigned. The problem of optimal actuation in this case may be formulated as follows:

Problem 1

Given: $\mathbf{M}$, $\mathbf{K}$, μ_1

Find: $\mathbf{b}$, and $\mathbf{g}$ such that

$$\{\mu_k\}_{k=1}^{n} = \{\mu_1 \quad \lambda_2 \quad \cdots \quad \lambda_n\} \tag{75}$$

and where $\left|\mathbf{b}\mathbf{g}^T\right|$ attains its minimum.

Solution

The solution is

$$\mathbf{b} = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \qquad\qquad \mathbf{g} = \gamma \mathbf{M}\mathbf{v}_1 \tag{76}$$

where

$$\gamma = \frac{\lambda_1 - \mu_1}{|\mathbf{v}_1|} \tag{77}$$

Proof (partial)**:**

We first note that by physical reasoning Problem 1 has a minimal norm solution. The solution is either internal to the domain of the physical parameters, $b_k \in \Re$, or on the boundary of the domain, where for some, but not all, $b_k = 0$.

Let $\mathbf{b}$ and $\mathbf{g}$ be one solution of Problem 1. Then $\beta\mathbf{b}$ and $\beta^{-1}\mathbf{g}$ is another solution for any real scalar $\beta \neq 0$. Hence, without loss of generality, we may look for a solution where $|\mathbf{b}|=1$. This prerequisite is satisfied by the first equation in (76). Since $|\mathbf{b}\mathbf{g}^T|=|\mathbf{b}||\mathbf{g}|$, the solution to Problem 1 is obtained by minimizing $\mathbf{g}^T\mathbf{g}$ subject to $\mathbf{b}^T\mathbf{b}=1$. By Lemma 1

$$\mathbf{g}=\frac{\lambda_1-\mu_1}{\mathbf{b}^T\mathbf{v}_1}\mathbf{M}\mathbf{v}_1\,, \tag{78}$$

hence if there exists a local minimum within the domain it could be located by finding the stationary values of the Lagrangian

$$L(\mathbf{b})=\frac{(\lambda_1-\mu_1)^2}{(\mathbf{b}^T\mathbf{v}_1)^2}\mathbf{v}_1^T\mathbf{M}^2\mathbf{v}_1+\xi\mathbf{b}^T\mathbf{b}\,, \tag{79}$$

where ξ is a Lagrange multiplier imposing the unit norm constraint on $\mathbf{b}$. Differentiating (79) with respect to $\mathbf{b}$ gives

$$\frac{\partial L}{\partial \mathbf{b}}=-2(\lambda_1-\mu_1)^2\mathbf{v}_1^T\mathbf{M}^2\mathbf{v}_1\frac{\mathbf{b}^T\mathbf{v}_1}{(\mathbf{b}^T\mathbf{v}_1)^4}\mathbf{v}_1+2\xi\mathbf{b}=\mathbf{0}\,. \tag{80}$$

Note that Equation (80) has a unique solution, up to a sign change,

$$\mathbf{b}=\frac{(\lambda_1-\mu_1)^2(\mathbf{v}_1^T\mathbf{M}^2\mathbf{v}_1)}{\xi|\mathbf{v}_1|^3}\mathbf{v}_1\,, \tag{81}$$

and

$$\xi=\frac{(\lambda_1-\mu_1)^2(\mathbf{v}_1^T\mathbf{M}^2\mathbf{v}_1)}{\mathbf{v}_1^T\mathbf{v}_1}\,. \tag{82}$$

It is shown in Appendix A that this solution is a local minimum.

To show that the internal solution is in fact the global minimum we need to prove that $|\mathbf{b}\mathbf{g}^T|$ of the internal solution is smaller than the minimal norm solution on the boundary of the domain. A formal proof is given at the end of this section.

Meanwhile we would be satisfied with the heuristic argument that the minimal norm solution on the boundary of the domain is equivalent to the optimal solution where some degrees of freedom are not subject to actuation. Such a system is less flexible to control and requires larger control effort.

Example 5: Single Natural Frequency Modification

Consider the two-degree-of-freedom system shown in Figure 4.2, where $k=1$ and $m=1$. The mass and stiffness matrices of the open-loop system are

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad \mathbf{K}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}.$$

The eigenvalues and normalized eigenvectors of (68) are

$$\left\{\lambda_1=0.3820 \quad \mathbf{v}_1=\begin{pmatrix}0.5257\\ 0.8507\end{pmatrix}\right\}, \qquad \left\{\lambda_2=2.6180 \quad \mathbf{v}_2=\begin{pmatrix}-0.8507\\ 0.5257\end{pmatrix}\right\}.$$

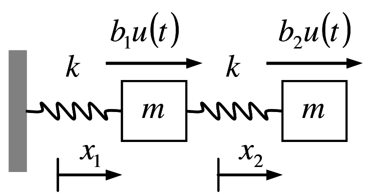

Figure 4.2: Two degree-of-freedom controlled system.

We wish to find the input vector $\mathbf{b}=(b_1 \quad b_2)^T$, where $b_1^2+b_2^2=1$, and the minimal norm vector $\mathbf{g}$ such that the eigenvalues μ_k of the closed-loop system (71), discarding the damping term, are

$$\mu_1=2 \quad \mu_2=\lambda_2=2.6180.$$

We have changed the parameter b_1 in the range $-1\le b_1\le 1$ and evaluated $\mathbf{g}$ that assigns the eigenvalues of the closed-loop system as required. The graph of $|\mathbf{g}|$ as a function of b_1 is shown in Figure 4.3a. The singularity at $b_1 = -0.8507$ corresponds to the maximal control effort where $\mathbf{b}$ is orthogonal to $\mathbf{v}_1$ and the system is not controllable as shown in section 4.2.

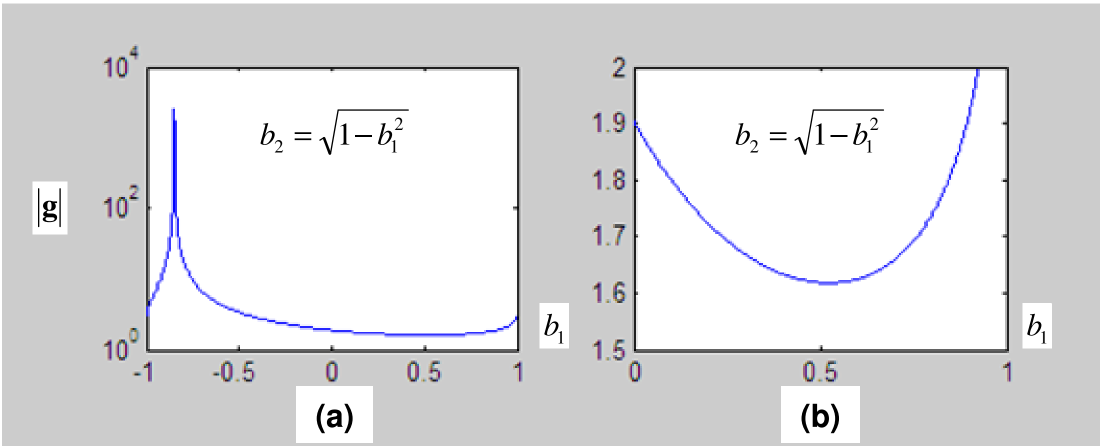

Figure 4.3: The norm of $\mathbf{g}$ as a function of b_1.

Figure 4.3b zooms on the graph in the interval $0 \le b_1 \le 1$. It shows that the minimum of $|\mathbf{g}|$ attains at $b_1 = 0.5257$, with corresponding $b_2 = 0.8507$, where as predicted by (76) $\mathbf{b} = \mathbf{v}_1$ since $|\mathbf{b}| = 1$.

Generally the eigenvector $\mathbf{v}_1$ is fully populated and hence the optimal input vector $\mathbf{b}$ that solves Problem 1 should be fully populated as well. This implies that in physical applications there is an actuator at each degree of freedom to realize the control. However some degrees of freedom in a realistic system are usually not accessible to actuation and therefore the number of actuators r is smaller than the number of degrees of freedom, $r < n$. We therefore define below the problem of finding the optimal input vector for the case where some specified elements in $\mathbf{b}$ vanish by design. Since the degrees of freedom are numbered arbitrarily, without loss of generality, we may number the degrees of freedom in such a way that

$$\mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{0} \end{pmatrix}, \tag{83}$$

where $\mathbf{b}_1 \in \Re^r$. The related optimal assignment of one eigenvalue in this case is formulated as follows.

Problem 2

Given: $\mathbf{M}$, $\mathbf{K}$, μ_1, and an integer r, $r < n$.

Find: $\mathbf{b}$, and $\mathbf{g}$ such that

$$\{\mu_k\}_{k=1}^n = \{\mu_1 \quad \lambda_2 \quad \cdots \quad \lambda_n\} \tag{84}$$

subject to the constraints

$$\mathbf{e}_k^T \mathbf{b} = 0\,, \qquad k = r+1, r+2, ..., n \tag{85}$$

where $\mathbf{e}_k$ is the k^{th} unit vector and where $\left|\mathbf{b}\mathbf{g}^T\right|$ attains its minimum.

Solution:

Denote

$$\mathbf{v}_1 = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \qquad \mathbf{y}_1 \in \Re^r \tag{86}$$

Then

$$\mathbf{b} = \frac{1}{|\mathbf{y}_1|}\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{0} \end{pmatrix} \tag{87}$$

Proof:

We wish to minimize

$$\mathbf{g}^T\mathbf{g} = \vartheta_1^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1 \tag{88}$$

subject to

$$\mathbf{b}^T\mathbf{b} = 1 \tag{89}$$

and

$$\mathbf{e}_k^T\mathbf{b} = 0\,, \qquad k = r+1, r+2, ..., n\,. \tag{90}$$

Define the Lagrangian

$$L(\mathbf{b}) = \vartheta_1^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1 + \xi \mathbf{b}^T\mathbf{b} + \boldsymbol{\tau}^T \mathbf{E}^T \mathbf{b} \tag{91}$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_{r+1} & \mathbf{e}_{r+2} & \cdots & \mathbf{e}_n \end{bmatrix} \tag{92}$$

and ξ and τ_k are Lagrange multipliers. Differentiating

$$\frac{\partial L}{\partial \mathbf{b}} = -(\lambda_1 - \mu_1)^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1 \frac{2\mathbf{b}^T\mathbf{v}_1}{(\mathbf{b}^T\mathbf{v}_1)^4} \mathbf{v}_1 + 2\xi\mathbf{b} + \mathbf{E}\boldsymbol{\tau} = \mathbf{0} \tag{93}$$

gives

$$\frac{2(\lambda_1 - \mu_1)^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1}{(\mathbf{b}^T\mathbf{v}_1)^3} \mathbf{v}_1 - 2\xi\mathbf{b} = \mathbf{E}\boldsymbol{\tau}\,. \tag{94}$$

We will now show that with $\mathbf{b}$ given by (87) there exist ξ and $\boldsymbol{\tau}$ such that the equations in (94) are all satisfied.

Substituting (87) in (94) gives for the first r equations

$$\frac{(\lambda_1 - \mu_1)^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1}{(\mathbf{y}_1^T\mathbf{y}_1)^{1.5}} \mathbf{y}_1 - \frac{\xi}{|\mathbf{y}_1|} \mathbf{y}_1 = \mathbf{0} \tag{95}$$

Hence with

$$\xi = \frac{(\lambda_1 - \mu_1)^2 \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1}{\mathbf{y}_1^T\mathbf{y}_1} \tag{96}$$

the equations in (95) are satisfied. The other $n-r$ equations of (94) are obviously satisfied when the vector of Lagrange multipliers $\boldsymbol{\tau}$ is chosen as

$$\boldsymbol{\tau} = \frac{2(\lambda_1 - \mu_1)^2 \, \mathbf{v}_1^T \mathbf{M}^2 \mathbf{v}_1}{\left(\mathbf{b}^T \mathbf{v}_1\right)^3} \mathbf{y}_2 \, . \tag{97}$$

Similar to the proof in Appendix A it could be shown that this solution is a local minimum. We note that by (87) we have $|\mathbf{b}| = 1$ and hence

$$\sqrt{\mathbf{g}^T \mathbf{g}} = \left|\mathbf{b}\mathbf{g}^T\right| \tag{98}$$

i.e., by minimizing $\mathbf{g}^T \mathbf{g}$ the minimum of $\left|\mathbf{b}\mathbf{g}^T\right|$ is attained.

Example 6: Single Natural Frequency Modification with Limited Actuation

Consider the five degree-of-freedom mass-spring system shown in Figure 4.4 where $m = 1$ and $k = 5$. The mass and stiffness matrices for this system are

$$\mathbf{M} = \mathbf{I} \qquad \mathbf{K} = \begin{bmatrix} 10 & -5 & & & \\ -5 & 10 & -5 & & \\ & -5 & 10 & -5 & \\ & & -5 & 10 & -5 \\ & & & -5 & 5 \end{bmatrix}$$

where elements not shown are zeros.

We wish to assign the eigenvalue $\lambda = 8.5769$ to $\mu_1 = 2$ while keeping the other eigenvalues unchanged.

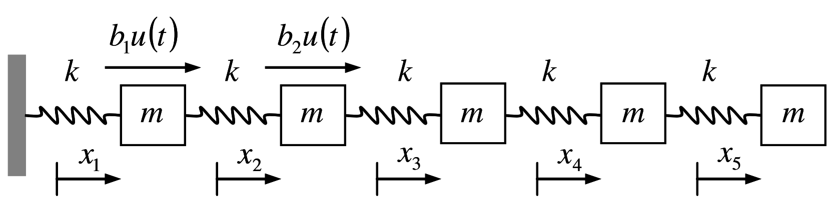

Figure 4.4: Five degree-of-freedom system.

The eigenvector of the open-loop system corresponding to the assigned eigenvalue is

$$\mathbf{v} = (0.5969 \quad 0.1699 \quad -0.5485 \quad -0.3260 \quad 0.4557)^T .$$

The graph shown in Figure 4.5 indicates that the minimum of $|\mathbf{g}|$ corresponds to $b_1 = -0.9618$. The associated $b_2 = -0.2738$ satisfies the unit norm constraint (89).

We note that

$$\frac{b_1}{b_2} = \frac{v_1}{v_2} = 3.513 ,$$

as expected from definition (86) and equation (87).

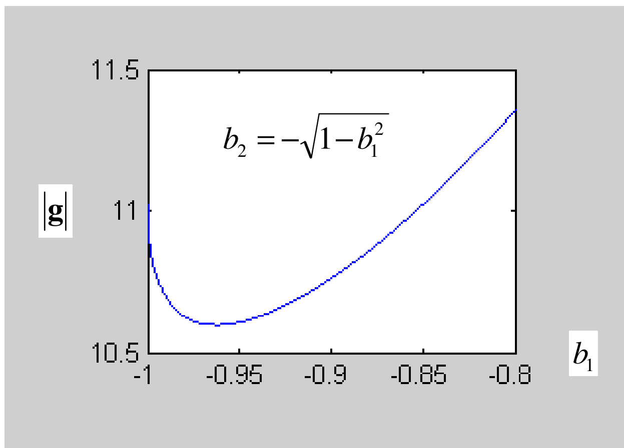

Figure 4.5: The norm of $\mathbf{g}$ as a function of b_1.

We now complete the proof that the solution given by (76)-(77) is the global minimum of Problem 1. We use the notation $\mathbf{v} = \mathbf{v}_1$. By (76) and (77) the minimum norm associated with the internal solution is

$$\rho_I = |\mathbf{g}| = \frac{|\lambda_1 - \mu_1|}{|\mathbf{v}|} \left(\mathbf{v}^T \mathbf{M}^2 \mathbf{v} \right)^{0.5} . \qquad (99)$$

We now look at the minimal norm solution on the boundary of the domain. Without loss of generality we may number the degrees of freedom such that an arbitrary solution on the boundary of the domain is defined by Problem 2 with some $r < n$. The boundary minimal norm solution ρ_B is then given by (87), (72) and (73)

$$\rho_B = |\mathbf{g}| = \frac{|\lambda_1 - \mu_1|}{|\mathbf{y}_1|}\left(\mathbf{v}^T \mathbf{M}^2 \mathbf{v}\right)^{0.5}. \tag{100}$$

From (86) we have $|\mathbf{y}_1| \le |\mathbf{v}|$ and hence $\rho_I \le \rho_B$. It thus follows that the solution (76)-(77) to Problem 1 is the global minimum. Similar reasoning applies to the proof of the solution of Problem 2.

Example 7: Checking the Solution on the Physical Domain

Equations (99) and (100) applied to Example 5 give the norm for the interior minimum

$$\mathbf{b} = \mathbf{v}/|\mathbf{v}| \qquad \rho_I = 1.6180,$$

and the norms for the boundaries of the physical domain

$$\mathbf{b} = (\pm 1 \quad 0)^T \qquad \mathbf{y}_1 = v_1 \qquad \rho_{B1} = 3.0777,$$

$$\mathbf{b} = (0 \quad \pm 1)^T \qquad \mathbf{y}_1 = v_2 \qquad \rho_{B2} = 1.9021,$$

as indicated in Figure 4.3a. The interior minimum is the global one as predicted by the solution to Problem 1.

Now that it was shown with due mathematical rigor that the solutions given to Problems 1 and 2 are the minimal norm solutions it is instructive to examine the strength of the physical argument. The physical domain of parameters is characterize by $\mathbf{b} \in \Re$ and $|\mathbf{b}| = 1$. When $\mathbf{b}$ is orthogonal to $\mathbf{v}_1$, the problem is not controllable and $|\mathbf{g}| \to +\infty$. We have also the inequality constraint $|\mathbf{g}| > 0$. It thus follows that there is a minimum somewhere inside the domain or on the boundary of the domain. On the boundary of the domain some of the elements of $\mathbf{b}$ vanish, which means that no actuation is applied to some of the degrees of freedom. From a physical point of view it is unlikely that the optimal actuation is achieved without a complete set of actuators, unless by chance where some of the elements of $\mathbf{v}_1$ vanish. The conclusion is that in general the minimum norm solution is internal to the domain and that it is necessarily defined by the stationary values of the Lagrangian. Since in Problems 1 and 2 the stationary values are unique up to a sign change there is no ambiguity in determining the solution.

4.6 Optimal Actuation in the Multiple Natural Frequency Modification Problem

We now consider the case of optimal actuation where several natural frequencies are intended to be changed while keeping the rest of the spectrum unaltered. For simplicity and clarity of exposition we will address the case where $m = 2$. The extension to higher dimensions $m > 2$ is straightforward.

Problem 3

Given: $\mathbf{M}$, $\mathbf{K}$, μ_1, μ_2

Find: $\mathbf{b}$, and $\mathbf{g}$ such that

$$\{\mu_k\}_{k=1}^{n} = \{\mu_1 \quad \mu_2 \quad \lambda_3 \quad \cdots \quad \lambda_n\}$$

and where $\left|\mathbf{b}\mathbf{g}^T\right|$ attains its minimum.

Here we wish to minimize

$$\mathbf{g}^T\mathbf{g} = \left(\vartheta_1\mathbf{v}_1^T\mathbf{M} + \vartheta_2\mathbf{v}_2^T\mathbf{M}\right)\left(\vartheta_1\mathbf{M}\mathbf{v}_1 + \vartheta_2\mathbf{M}\mathbf{v}_2\right) \tag{101}$$

where ϑ_k, $k = 1,2\ldots,m$ are given by (73), subject to $\mathbf{b}^T\mathbf{b} = 1$. Note that

$$\mathbf{g} = \frac{1}{\mathbf{b}^T \mathbf{v}_1} \boldsymbol{\tau}_1 + \frac{1}{\mathbf{b}^T \mathbf{v}_2} \boldsymbol{\tau}_2 \tag{102}$$

where

$$\boldsymbol{\tau}_1 = \frac{(\lambda_1 - \mu_1)(\lambda_1 - \mu_2)}{(\lambda_1 - \lambda_2)} \mathbf{M}\mathbf{v}_1 \qquad \boldsymbol{\tau}_2 = \frac{(\lambda_2 - \mu_2)(\lambda_2 - \mu_1)}{(\lambda_2 - \lambda_1)} \mathbf{M}\mathbf{v}_2 \tag{103}$$

It thus follows from (102) and (103) that

$$\mathbf{g}^T \mathbf{g} = \frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_1}{(\mathbf{b}^T \mathbf{v}_1)^2} + \frac{2\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_1)(\mathbf{b}^T \mathbf{v}_2)} + \frac{\boldsymbol{\tau}_2^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_2)^2}. \tag{104}$$

We define the Lagrangian

$$L(\mathbf{b}) = \frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_1}{(\mathbf{b}^T \mathbf{v}_1)^2} + \frac{2\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_1)(\mathbf{b}^T \mathbf{v}_2)} + \frac{\boldsymbol{\tau}_2^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_2)^2} + \xi \mathbf{b}^T \mathbf{b} \tag{105}$$

where ξ is a Lagrange multiplier. The stationary principle gives

$$\frac{\partial L}{\partial \mathbf{b}} = -\frac{2\boldsymbol{\tau}_1^T \boldsymbol{\tau}_1}{(\mathbf{b}^T \mathbf{v}_1)^3} \mathbf{v}_1 - \frac{2\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_1)^2 (\mathbf{b}^T \mathbf{v}_2)} \mathbf{v}_1 - \frac{2\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_1)(\mathbf{b}^T \mathbf{v}_2)^2} \mathbf{v}_2 - \frac{2\boldsymbol{\tau}_2^T \boldsymbol{\tau}_2}{(\mathbf{b}^T \mathbf{v}_2)^3} \mathbf{v}_2 + 2\xi \mathbf{b} = \mathbf{0}. \tag{106}$$

We define

$$\hat{\mathbf{b}} = \xi^{1/4} \mathbf{b}, \tag{107}$$

and obtain from (106)

$$\left(\frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_1}{(\hat{\mathbf{b}}^T \mathbf{v}_1)^3} + \frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\hat{\mathbf{b}}^T \mathbf{v}_1)^2 (\hat{\mathbf{b}}^T \mathbf{v}_2)} \right) \mathbf{v}_1 + \left(\frac{\boldsymbol{\tau}_2^T \boldsymbol{\tau}_2}{(\hat{\mathbf{b}}^T \mathbf{v}_2)^3} + \frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2}{(\hat{\mathbf{b}}^T \mathbf{v}_1)(\hat{\mathbf{b}}^T \mathbf{v}_2)^2} \right) \mathbf{v}_2 = \hat{\mathbf{b}}. \tag{108}$$

We may thus solve (108) for $\hat{\mathbf{b}}$ and obtain the optimal input vector $\mathbf{b}$ via the normalization

$$\mathbf{b} = \frac{\hat{\mathbf{b}}}{|\hat{\mathbf{b}}|} \qquad \xi = |\hat{\mathbf{b}}|^4. \tag{109}$$

Note that (108) may be written in the form

$$\chi_1 \mathbf{v}_1 + \chi_2 \mathbf{v}_2 = \mathbf{b} \tag{110}$$

with the obvious definition of χ_k, $k = 1,2$. Note that (110) indicates that $\mathbf{b}$ lies in the subspace spanned by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

By the physical insight gained in Section 4.5 it is clear that the minimal norm solution is generally internal to the domain and that it is therefore one of the solutions of (110) by necessity.

Example 8: Multiple Natural Frequency Modification

We consider the mass-spring system shown in Figure 4.2. We wish to assign the eigenvalues of the system to $\mu_1 = 1$ and $\mu_2 = 2$ with minimal control effort. Since $\mathbf{M} = \mathbf{I}$ the bi-orthogonal condition $\mathbf{v}_1^T \mathbf{M} \mathbf{v}_2 = 0$ implies via (103) that $\boldsymbol{\tau}_1^T \boldsymbol{\tau}_2 = 0$ and the system of equations (108) reduces to

$$\frac{\boldsymbol{\tau}_1^T \boldsymbol{\tau}_1}{\left(\hat{\mathbf{b}}^T \mathbf{v}_1\right)^3} \mathbf{v}_1 + \frac{\boldsymbol{\tau}_2^T \boldsymbol{\tau}_2}{\left(\hat{\mathbf{b}}^T \mathbf{v}_2\right)^3} \mathbf{v}_2 = \hat{\mathbf{b}}. \tag{111}$$

With

$$\boldsymbol{\tau}_1 = \begin{pmatrix} 0.2351 \\ 0.3804 \end{pmatrix} \qquad \boldsymbol{\tau}_2 = \begin{pmatrix} -0.3804 \\ 0.2351 \end{pmatrix}$$

and $\mathbf{v}_1$, $\mathbf{v}_2$ as given in Example 1, we obtain two solutions to (111)

$$\hat{\mathbf{b}}_1 = \begin{pmatrix} 0.2584 \\ -1.0946 \end{pmatrix}, \qquad \hat{\mathbf{b}}_2 = \begin{pmatrix} -1.0946 \\ -0.2584 \end{pmatrix}.$$

By (109) the optimal input vector is

$$\mathbf{b}_1 = \begin{pmatrix} 0.2298 \\ -0.9732 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} -0.9732 \\ -0.2298 \end{pmatrix}.$$

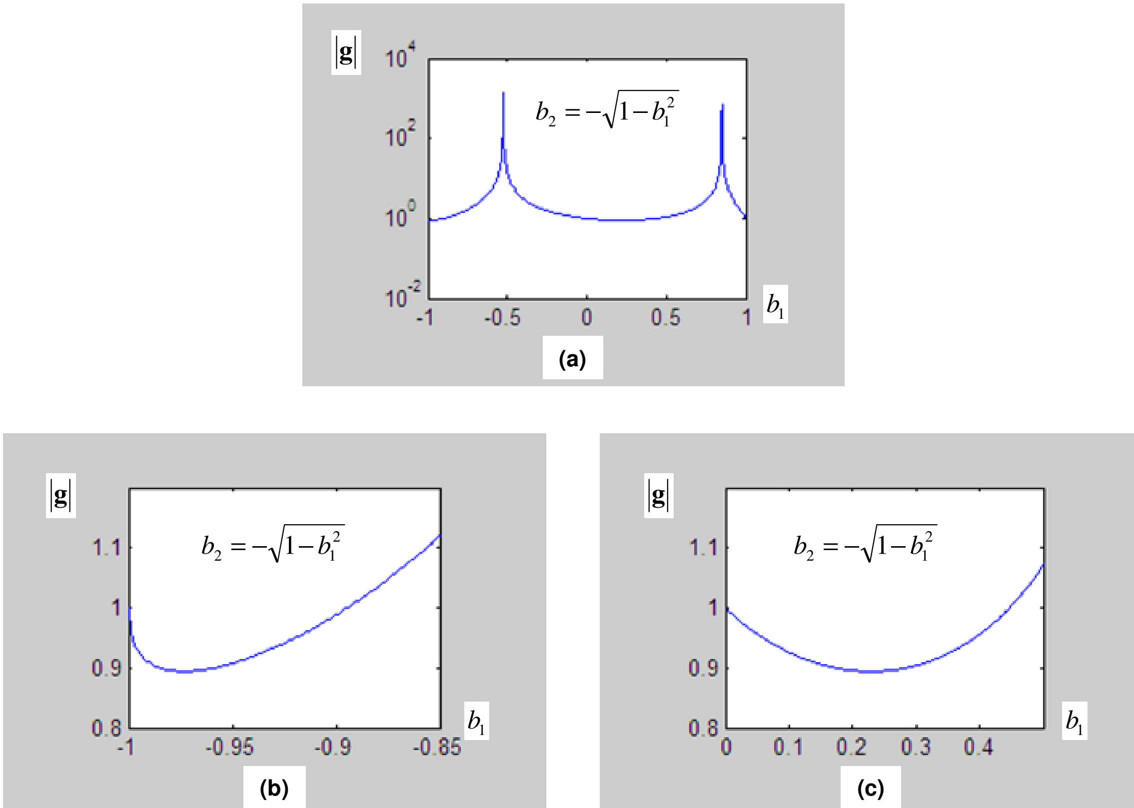

Figure 4.6: The norm of $\mathbf{g}$ as a function of b_1, (a) in the complete physical range $-1 \le b_1 \le 1$, (b) zoom on the left minimum, and (c) zoom on the right minimum.

Figure 4.6a shows the norm of $\mathbf{g}$ as a function of b_1 in the range of $-1 \le b_1 \le 1$. It is apparent from the graph that there are two local minima in this complete range. Figures 4.6b and 4.6c zoom on these minima and show the two solutions for the optimal input vector $\mathbf{b}$ obtained above.

4.7 Pole Placement by Optimal Actuation

We now consider the problem of finding the optimal actuation for complete pole assignment for damped systems. The pole placement problem is formulated in equations (27)-(28) and (40). Here there are two feedback vectors $\mathbf{f}$ and $\mathbf{g}$ hence the cost function η to be minimized involves a given weighting parameter w,

$$\eta \equiv |\mathbf{f}|^2 + w|\mathbf{g}|^2 . \tag{112}$$

The characteristic equation of the closed-loop system is defined as follows

$$\phi(s) \equiv \det\left(s^2\mathbf{M} + s\left(\mathbf{C} - \mathbf{b}\mathbf{f}^T\right) + \left(\mathbf{K} - \mathbf{b}\mathbf{g}^T\right)\right) \tag{113}$$

and the problem to be solved is:

Problem 4

Given $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$, w, and the eigenvalues of the closed-loop system $s_1, s_2, ..., s_{2n}$

Find $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$

which minimize η

subject to the constraints

$$\phi(s_k) = 0 \qquad\qquad k = 1,2,..,2n \tag{114}$$

and

$$|\mathbf{b}| = 1 . \tag{115}$$

To solve the problem we define the Lagrangian

$$L(\mathbf{b},\mathbf{f},\mathbf{g}) = \mathbf{f}^T\mathbf{f} + w\mathbf{g}^T\mathbf{g} - \xi_0\mathbf{b}^T\mathbf{b} - \sum_{k=1}^{2n}\xi_k\phi(s_k) \tag{116}$$

where ξ_k, $k = 0,1,...,2n$ are Lagrange multipliers. The solution to the problem is given by the set of equations

$$\frac{\partial L(\mathbf{b},\mathbf{f},\mathbf{g})}{\partial \mathbf{b}} = \mathbf{0} \qquad \frac{\partial L(\mathbf{b},\mathbf{f},\mathbf{g})}{\partial \mathbf{f}} = \mathbf{0} \qquad \frac{\partial L(\mathbf{b},\mathbf{f},\mathbf{g})}{\partial \mathbf{g}} = \mathbf{0} \tag{117}$$

together with (114) and (115). Note that this set of equations gives $5n+1$ equations for the $5n+1$ unknowns in $\mathbf{b}$, $\mathbf{f}$, $\mathbf{g}$ and ξ_k, $k = 0,1,...,2n$. The explicit forms of the equations in (117) are

$$\sum_{k=1}^{2n} \xi_k \frac{\partial \phi(s_k)}{\partial \mathbf{b}} = -2\xi_0 \mathbf{b}\,, \qquad \sum_{k=1}^{2n} \xi_k \frac{\partial \phi(s_k)}{\partial \mathbf{f}} = 2\mathbf{f}\,, \qquad \sum_{k=1}^{2n} \xi_k \frac{\partial \phi(s_k)}{\partial \mathbf{g}} = 2w\mathbf{g}\,. \quad (118)$$

A numerically viable method for finding the derivative of a matrix determinant with respect to a parameter without expanding the determinant by its fundamental definition is given in Ram [12].

Example 9: Full Eigenvalue Assignment

Consider the open-loop system shown in Figure 4.7 with $k = 5$, $m = 1$ and $c = 0.2$. The mass damping and stiffness matrix for this system are:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.2 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} 15 & -10 & 0 \\ -10 & 15 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

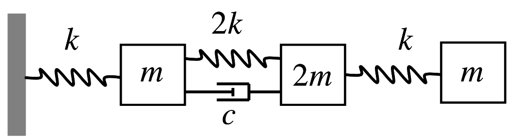

Figure 4.7: The open-loop system of Example 9.

We wish to assign the eigenvalues of the system to the set

$$s_1 = -1\,, \qquad s_2 = -3\,, \qquad s_{3,4} = -0.5 \pm 2i\,, \qquad s_{5,6} = -0.75 \pm 5i$$

by using optimal actuation with a norm weighting of $w = 1$.

The sixteen equations defined by (114), (115) and (118) yield two physical solutions. One solution

$$\mathbf{b} = \begin{pmatrix} -0.9864 \\ -0.1637 \\ 0.0149 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 5.0492 \\ 14.5344 \\ -1.9836 \end{pmatrix} \quad \mathbf{g} = \begin{pmatrix} 15.2817 \\ -6.5870 \\ -1.1243 \end{pmatrix}$$

with seven Lagrange multipliers

$$\xi_0 = -518.8605\,, \qquad \xi_1 = 0.2791\,, \quad \xi_2 = -0.1196\,,$$

$$\xi_{3,4} = -0.1209 \pm 0.0959i\,, \qquad \xi_{5,6} = 0.0333 \pm 0.0178i\,,$$

corresponds to the local minimum $\eta_{\min} = 518.8605$ of the cost function. The graph of η in the neighborhood of $\mathbf{b}$ associated with this solution is shown in Figure 4.8.

The second solution

$$\mathbf{b} = \begin{pmatrix} 0.5828 \\ -0.3998 \\ 0.7074 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 2.3184 \\ -2.7946 \\ -11.4637 \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} -10.8479 \\ 14.0012 \\ -7.4205 \end{pmatrix}$$

with

$$\xi_0 = -513.3758\,, \qquad \xi_1 = 0.3736\,, \quad \xi_2 = -0.1403\,,$$

$$\xi_{3,4} = -0.2999 \pm 0.1314i\,, \qquad \xi_{5,6} = 0.0351 \pm 0.0191i\,,$$

corresponds to the global minimum $\eta_{\min} = 513.3758$ of the cost function. The graph of η in the neighborhood of $\mathbf{b}$ associated with the global optimization is shown in Figure 4.9.

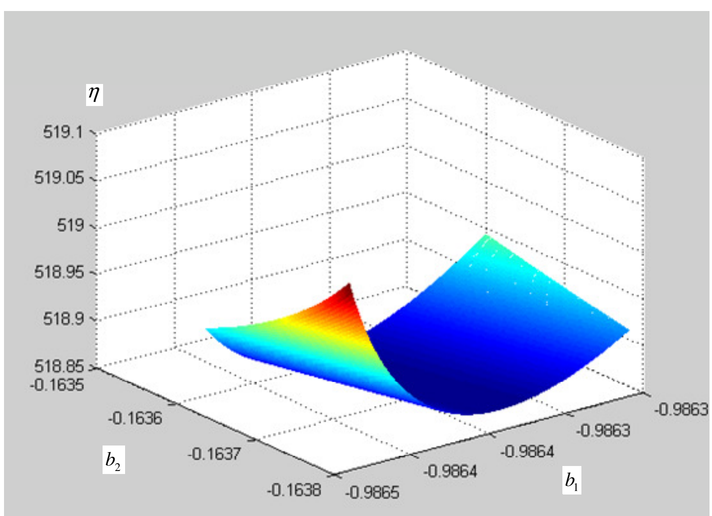

Figure 4.8: Local minimum of η, $b_3 = \sqrt{1 - b_1^2 - b_2^2}$.

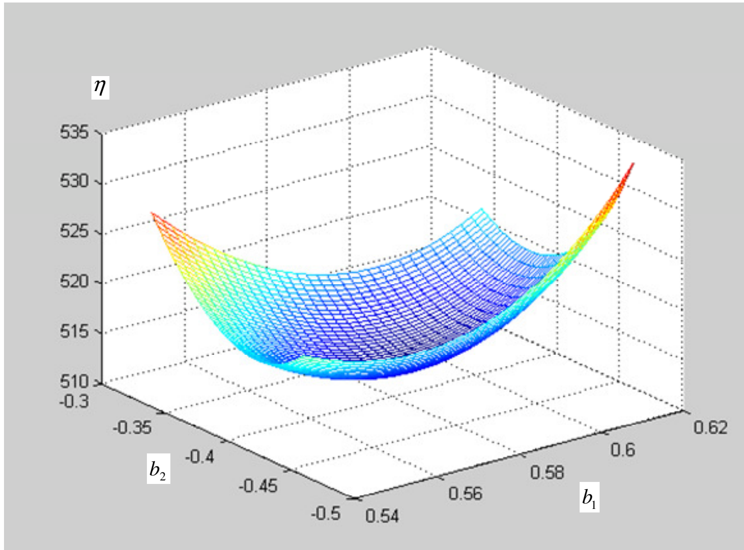

Figure 4.9: Global minimum of η, $b_3 = \sqrt{1 - b_1^2 - b_2^2}$.

CHAPTER 5: DEMONSTRATION WITH UNITS

Previous examples all demonstrated minimization of a cost function, but the true measure of this method is in the magnitude reduction of control effort. A physical system can be modeled with units to demonstrate this.

The force selection vector, b, has no units and is merely the multiplication of control force applied to each degree of freedom. In a physical system using SI units, mass can be given in units of kilograms, stiffness in units of Newtons per millimeter, and damping in units of Newtons per millimeter per second. The eigenvalues are frequencies in units of radians per second, and the eigenvectors are a normalized unit length as millimeters. The position and velocity are given as millimeters and millimeters per second, respectively. This means the calculation of q from (38) results in units of millimeters^{-1}. Thus, the velocity gain vector, f, from (36) has units of kilograms per second, and the position gain vector, g, from (37) has units of 10^3 kilograms per second squared. The control force, u, would then, from (27), have units of Newtons that are multiplied by the force selection vector and applied to each degree of freedom. Note that a 10^{-3} correction must be applied to the velocity gain multiplication in (27) to account for the use of millimeters instead of meters. The optimization laid out in this thesis works to minimize the magnitude of this control force input, u, as well as the magnitude of the total control effort, b*u.

If Example 9 is run with SI units considered for each of the parameters, the system can be rewritten as

$$\mathbf{M}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\text{kg} \qquad \mathbf{C}=\begin{bmatrix}0.2 & -0.2 & 0\\ -0.2 & 0.2 & 0\\ 0 & 0 & 0\end{bmatrix}\text{N/mm/s} \qquad \mathbf{K}=\begin{bmatrix}15 & -10 & 0\\ -10 & 15 & -5\\ 0 & -5 & 5\end{bmatrix}\text{N/mm.}$$

The optimum force selection vector and feedback gain vectors would then be

$$\mathbf{b}=\begin{pmatrix}0.5828\\ -0.3998\\ 0.7074\end{pmatrix}, \qquad \mathbf{f}=\begin{pmatrix}2.3184\\ -2.7946\\ -11.4637\end{pmatrix}\text{kg/s}, \qquad \mathbf{g}=\begin{pmatrix}-10.8479\\ 14.0012\\ -7.4205\end{pmatrix}\text{kg/s}^2.$$

Given an initial position and velocity of

$$\mathbf{x}_0=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\text{mm}, \quad \dot{\mathbf{x}}_0=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\text{mm/s},$$

the system will respond with a specific dynamic response and the control system will bring all masses back to equilibrium by assigning the eigenvalues to the set

$$s_1=-1, \qquad s_2=-3, \qquad s_{3,4}=-0.5\pm 2i, \qquad s_{5,6}=-0.75\pm 5i$$

as demonstrated in Example 9.

Using the optimal force selection vector and associated gain vectors, the control input necessary over time is shown in Figure 5.1.

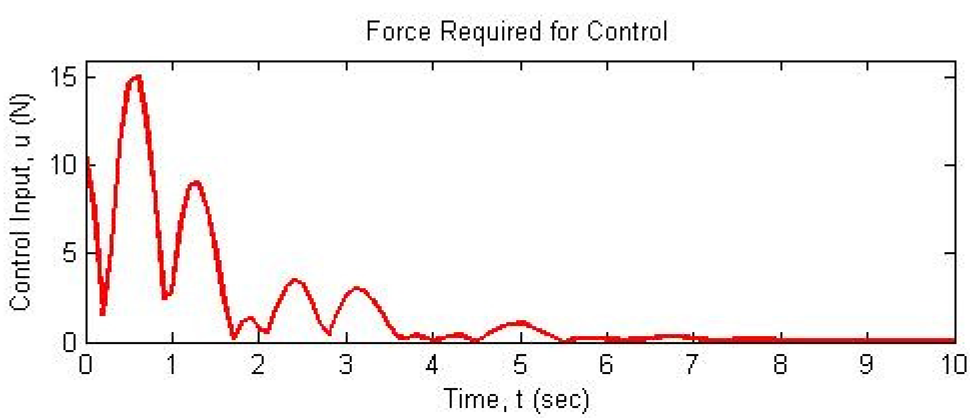

Figure 5.1: Control force input needed to control the system when using the optimal force selection vector.

The peak control input, u, is 15.11 N. The sum of all control inputs for each 0.1-sec time step from 0 to 20 seconds is 180.65 N. When multiplied by the force selection vector and distributed across all masses, the magnitude of the total control effort, b*u, applied over the 20 seconds is 305.31 N.

We can compare these values to an arbitrary force selection vector. For this example, a force selection vector of equal components is chosen,

$$\mathbf{b} = \begin{pmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{pmatrix} = \begin{pmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{pmatrix}.$$

This results in gain vectors

$$\mathbf{f} = \begin{pmatrix} 77.66 \\ 115.08 \\ -145.94 \end{pmatrix} \text{kg/s}, \qquad \mathbf{g} = \begin{pmatrix} 50.96 \\ 138.28 \\ -143.99 \end{pmatrix} \text{kg/s}^2.$$

This arbitrary control accomplishes the same pole placement. Under the same initial conditions as the previous example, the control input over time is shown in Figure 5.2. Note that the y-axis has been increased to accommodate the higher values.

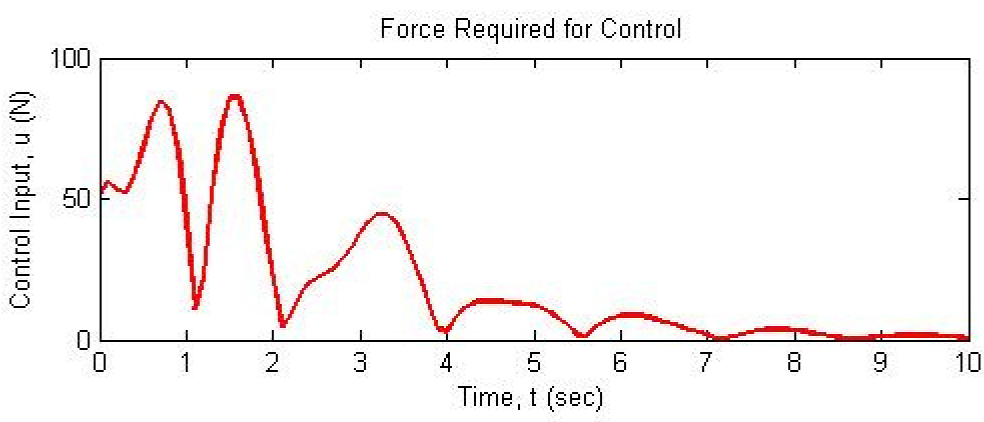

Figure 5.2: Control force input needed to control the system when using the optimal force selection vector.

The peak control input using equally distributed actuation is 86.00 N. The sum of all control inputs for each 0.1-sec time step from 0 to 20 seconds is 2022.96 N. When multiplied by the force selection vector and distributed across all masses, the magnitude of the total control effort, b*u, applied over the 20 seconds is 3503.9 N.

The peak control input of the optimized actuation is 17.6% of the equally distributed actuation. Also, the total control input of the optimized actuation over the 20 seconds of simulation is also 8.9% of that of the equally distributed actuation. The total control effort from actuation across all degrees of freedom over the 20 second interval by the optimized actuation is only 8.7% of the equally distributed actuation.

In the system studied, holding all other parameters and initial conditions to be the same, using optimal actuation instead of an equally-distributed actuation decreases the amount of control force necessary to assign the eigenvalues as desired by an order of magnitude. In this example, there was an 1100 times reduction in control effort needed.

CHAPTER 6: CONCLUSIONS

The results show that at least an order of magnitude improvement can be achieved by paying attention to the open-loop properties of the system and choosing the control input that is parallel to the eigenvector associated with the eigenvalue to be reassigned. The term “parallel” in this case can be confusing. Consider a string that is vibrating. In each mode of vibration, there will be points in the string which experience no deflection, and there will be other points which experience maximum deflection. If a sensor/actuator pair is placed at a zero-point, the point with no deflection, there will be no measurable change in state and thus no control applied. This would be analogous to using a force selection vector, b, which is orthogonal to an open-loop eigenvector, resulting in an uncontrollable system. If the sensor/actuator pair is moved to a maximum-point, where there is maximum deflection, the control is able to act directly at the point where it is needed most. This is analogous to using the optimal force selection vector which is parallel to an open-loop eigenvector. In the case where more than one eigenvalue is to be reassigned, the optimal force selection vector will be in the subspace of the associated eigenvectors and as near to parallel to all of them as possible. The results have shown this to be true for damped and undamped systems and even systems with limited actuation. Use of Lagrange multipliers has been developed to find the optimal control input in the case of a damped system with full pole placement. All examples result in minimal control effort to accomplish the desired pole assignment.

Due to the nature of the mathematical equations, this theory could be explored for use in more than just mechanical systems. Simple electrical systems use a similar differential equation, thus this theory may apply to the problem of controlling fluctuations in current in such systems. Future research could use a simple electrical circuit to prove the theory through experimentation.

There is also possible application of this theory in vibration enhancement instead of vibration control. It is usually desirable to shift the eigenvalues of a system to the left-half of the complex plane, in order to reduce vibration. However, in the field of energy harvesting, it may be desirable to increase vibration by moving the poles of a system further towards the right. The theory presented here works to minimize the control effort needed, no matter what value is chosen for the new eigenvalues.

It should be noted that this work is still theoretical and has not been demonstrated outside of computer simulation. Future work is needed to test the theory with a physical demonstration.

REFERENCES

Alkhatib, Rabih, and M. F. Golnaraghi. "Active Structural Vibration Control: A Review." *The Shock and Vibration Digest* 35, no. 5 (2003): 367-83.

Bai, Z.J., B.N. Datta and J.W. Wang. "Robust and minimum norm partial quadratic eigenvalue assignment in vibrating systems: A new optimization approach." *Mechanical Systems and Signal Processing* 24 (2010): 766-783.

Balas, Mark J. "Direct Velocity Feedback Control of Large Space Structures." *Journal of guidance and control* 2, no. 3 (1978): 252-53.

Chang, Chih-Chen, and Lung-On Yu. "The Application of Optimal Pole Placement Technique in Structural Control." *Active control of vibration and noise* 93 (1996): 161-71.

D'Azzo, John J., and Constantine H. Houpis. *Linear Control System Analysis and Design: Conventional and Modern*. 4th ed. New York, NY: McGraw-Hill, 1995.

Datta, Biswa N., Sylvan Elhay, and Yitshak M. Ram. "Orthogonality and Partial Pole Assignment for the Symmetric Definite Quadratic Pencil." *Linear algebra and its applications* 257 (1997): 29-48.

Datta, B.N. and V. Sokolov. "Quadratic inverse eigenvalue problems, active vibration control and model updating." *Applied and Computational Mathematics* 8 (2009): 170-191.

Datta, Biswa N., and Daniil R. Sarkissian. "Partial Eigenvalue Assignment in Linear Systems: Existence, Uniqueness and Numerical Solution." Paper presented at the 15th International symposium on the mathematical theory of networks and systems, University of Notre Dame, South Bend, IN, August 14, 2002 2002.

Dosch, Jeffrey J., Daniel J. Inman, and Ephrahim Garcia. "A Self-Sensing Piezoelectric Actuator for Collocated Control." *Journal of intelligent material systems and structures* 3, no. 1 (1992): 166-85.

Elishakoff, I.. "A selective review of direct, semi-inverse and inverse eigenvalue problems for structures described by differential equations with variable coefficients." *Archives of Computational Methods in Engineering* 7 (2000): 451-526.

Franklin, Gene F., J. David Powell, and Abbas Emami-Naeini. *Feedback Control of Dynamic Systems*. 3rd ed. Reading, MA: Addison-Wesley, 1994.

Gao, W., J. J. Chen, H. B. Ma, and X. S. Ma. "Optimal Placement of Active Bars in Active Vibration Control for Piezoelectric Intelligent Truss Structures with Random Parameters." *Computers and structures* 81 (2003): 53-60.

Golub, G. H. and C. F. Van Loan. *Matrix Computations*. 2nd ed. Baltimore, MD: The Johns Hopkins University Press, 1989.

Hong, Seongil, Chul H. Park, and Hyun C. Park. "Vibration Control of Beams Using Multiobjective State-Feedback Control." *Smart materials and structures* 15 (2006): 157-63.

Jiang, Dan-chi, and J. B. Moore. "Least Squares Pole Assignment by Memoryless Output Feedback." *Systems and control letters* 29 (1996): 31-42.

Karbassi, S. M. "An Algorithm for Minimizing the Norm of State Feedback Controllers in Eigenvalue Assignment." *Computers and mathematics with applications* 41 (2001): 1317-26.

Kimura, Hidenori. "Pole Assignment by Gain Output Feedback." *IEEE transactions on automatic control* 20, no. 4 (1975): 509-16.

Korn, G.A. and T.M. Korn. *Mathematical Handbook for Scientists and Engineers*. 2nd ed. New York, NY: McGraw Hill Book Company, 1968.

Lam, James, and Wei-Yong Yan. "Pole Assignment with Optimal Spectral Conditioning." *Systems and control letters* 29 (1997): 241-53.

Lawther, R. "Assessing how changes to a structure can create gaps in the natural frequency spectrum." *International Journal of Solids and Structures* 44 (2007): 614-635.

Matsuzaki, Yuji, Tadashige Ikeda, and Christian Boller. "New Technological Development of Passive and Active Vibration Control: Analysis and Test." *Smart materials and structures* 14 (2005): 343-48.

McMillan, A.J. and A.J. Keane. "Shifting resonances from a frequency band by applying concentrated masses to a thin rectangular plate." *Journal of Sound and Vibration* 192 (1996): 549–562.

Mierovitch, Leonard. *Fundamentals of Vibrations*. New York, NY: McGraw-Hill, 2001.

Miminis, George S., and Chris C. Paige. "A Direct Algorithm for Pole Assignment of Time-Invariant Multi-Input Linear Systems Using State Feedback." *Automatica* 24, no. 3 (1988): 343-56.

Mottershead, John E., and Yitshak M. Ram. "Inverse Eigenvalue Problems in Vibration Absorption: Passive Modification and Active Control." *Mechanical systems and signal processing* 20, no. 1 (2006): 5-44.

Qian, Jiang, and Shufang Xu. "Robust Partial Eigenvalue Assignment Problem for the Second-Order System." *Journal of sound and vibration* 282 (2005): 937-48.

Ram, Y. M. “Enlarging a spectral gap by structural modification.” *Journal of Sound and Vibration* 176 (1994): 225-234.

Ram, Y. M. “A method for finding repeated roots in transcendental eigenvalue problems.” *Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science* 222 (2008): 1665-1671.

Ram, Y. M. and J. J. Blech. “The dynamic behaviour of a vibratory system after modification.” *Journal of Sound and Vibration* 150 (1991): 357-370.

Ram, Y. M. and S. Elhay. “Pole assignment in vibratory systems by multi-input control.” *Journal of Sound and Vibration* 230 (2000): 309-321.

Ram, Y. M., and D. J. Inman. "Optimal Control for Vibrating Systems." *Mechanical systems and signal processing* 13, no. 6 (1999): 879-92.

Schulz, G., and G. Heimbold. "Dislocated Actuator/Sensor Positioning and Feedback Design for Flexible Structures." *Journal of guidance, control, and dynamics* 6 (1983): 361-67.

Song, Y. D., and T. L. Mitchell. "Active Damping Control of Vibrational Systems." *IEEE transactions on aerospace and electronic systems* 32, no. 2 (1996): 569-77.

Vasques, C. M. A., and J. Dias Rodrigues. "Active Vibration Control of Smart Piezoelectric Beams: Comparison of Classical and Optimal Feedback Control Strategies." *Computers and structures* 84 (2006): 1402-14.

Xu, S.F. and J. Qian. “Orthogonal basis selection method for robust partial eigenvalue assignment problem in second-order control systems.” *Journal of Sound and Vibration* 317 (2008): 1-19.

Yang, S. M., and Y. J. Lee. "Optimization of Non-Collocated Sensor/Actuator Location and Feedback Gain in Control Systems." *Smart materials and structures* 2 (1993): 96-102.

Zhang, Tianli, Chengbao Jiang, Hu Zhang, and Huibin Xu. "Giant Magnetostrictive Actuators for Active Vibration Control." *Smart materials and structures* 13 (2004): 473-77.

APPENDIX A: PROOF OF MINIMUM IN PROBLEM 1

The necessary and sufficient conditions for $\mathbf{p}$ to be a local minimum of a multi-variable scalar function $L(\mathbf{b})$ is that

$$\left.\frac{\partial L}{\partial b_k}\right|_{\mathbf{b}=\mathbf{p}} = 0 \qquad k = 1,2,...,n \tag{A.1}$$

and that for all non-zero variations $\Delta b_k \in \Re$, $k = 1,2,...,n$,

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left.\frac{\partial^2 L}{\partial b_i \partial b_j}\right|_{\mathbf{b}=\mathbf{p}} \Delta b_i \Delta b_j > 0, \tag{A.2}$$

see e.g., [6], pp. 333-334. The condition (A.2) may be written equivalently in matrix notations as

$$\left.\Delta\mathbf{b}^T \mathbf{J}(\mathbf{b})\Delta\mathbf{b}\right|_{\mathbf{b}=\mathbf{p}} > 0 \qquad , \tag{A.3}$$

for all $\Delta\mathbf{b} \neq \mathbf{0}$, where

$$\Delta\mathbf{b} = (\Delta b_1 \quad \Delta b_2 \quad \cdots \quad \Delta b_n)^T, \tag{A.4}$$

and where $\mathbf{J}$ is the matrix of partial derivatives

$$\mathbf{J} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial b_i \partial b_j}\right). \tag{A.5}$$

We denote $\lambda = \lambda_1$, $\mu = \mu_1$, $\mathbf{v} = \mathbf{v}_1$ and define

$$\psi = (\lambda - \mu)^2 \mathbf{v}^T \mathbf{M}^2 \mathbf{v} > 0. \tag{A.6}$$

Then (21) may be written in the form

$$L(\mathbf{b}) = \frac{\psi}{(\mathbf{b}^T\mathbf{v})^2} + \xi \mathbf{b}^T \mathbf{b}. \tag{A.7}$$

We have shown in Section 3 that

$$\mathbf{b} = \frac{\mathbf{v}}{|\mathbf{v}|} \,, \tag{A.8}$$

satisfies the n equations in (A.1). To show that the condition in (A.2) is also satisfied we differentiate

$$\frac{\partial L}{\partial b_j} = \frac{-2\psi v_j}{(\mathbf{b}^T \mathbf{v})^3} + 2\xi b_j \,, \tag{A.9}$$

and obtain by second differentiation

$$\frac{\partial^2 L}{\partial b_i \partial b_j} = \begin{cases} 2\gamma v_i v_j + 2\xi & i = j \\ 2\gamma v_i v_j & i \neq j \end{cases} \,, \tag{A.10}$$

where

$$\gamma = \frac{3\psi}{(\mathbf{b}^T \mathbf{v})^4} \,. \tag{A.11}$$

From (A.5) we have

$$\mathbf{J} = \xi \mathbf{I} + \gamma \mathbf{v}\mathbf{v}^T \,. \tag{A.12}$$

With $\mathbf{b}$ chosen as in (A.8) Equation (A.11) gives

$$\gamma = \frac{3\psi}{(\mathbf{v}^T \mathbf{v})^2} > 0 \,, \tag{A.13}$$

by virtue of (A.6). From (24)

$$\xi = \frac{(\lambda - \mu)^2 (\mathbf{v}^T \mathbf{M}^2 \mathbf{v})}{\mathbf{v}^T \mathbf{v}} > 0 \,. \tag{A.14}$$

It thus follows from the monotonicity property of the eigenvalues of symmetric matrices that $\mathbf{J}$ is a positive definite matrix, see e.g., [5], p. 462. The condition (A.3) is therefore satisfied for all $\Delta \mathbf{b} \neq \mathbf{0}$, and the solution (A.8) is a local minimum.

APPENDIX B: COMPUTER FILES USED IN EXAMPLES

MATLAB file for Example 1

```
clear all;
% input system parameters
n=2;  % two-dimensional system
m1=1; m2=2;  % define mass of each dimension
k1=5; k2=10;  % define spring constants
% k1 is spring from ground to mass 1,
% k2 is from mass 1 to mass 2
c1=0; c2=0.2;  % define damping constants
% c1 is dashpot from ground to mass 1,
% c2 is from mass 1 to mass 2
%
% begin calculations
I=eye(n);  % identity matrix
O=zeros(n,n);  % zero matrix
M=[m1 0;   % mass matrix
   0 m2];
C=[c1+c2 -c2;  % damping matrix
   -c2 c2];
K=[k1+k2 -k2;  % spring matrix
   -k2 k2];
A=[O I;   % first-order realization
   -K O];
B=[I O;
   C M];
[U,S]=eig(A,B);   % eigenvalues and eigenvectors
%
% For verification
% s=-0.1472 -+ 4.3170i,-0.0028 -+ 1.1575i
% v=0.0263 -+ 0.1995i, 0.0082 -+ 0.6269i
%  -0.0114 +- 0.0728i, 0.0055 -+ 0.8563i
%
% Initial conditions
x0=[1 0]';  % define initial position
xdot0=[0 1]';   % define initial velocity
a=U\[x0;xdot0];  % calculate coefficients of solution
%
% For verification
% a=[0.0025 +- 1.9711i, 0.3872 +- 0.1652i]'
%
% Calculate solution for each time-step
k=0;
for j=0:0.1:100  % define time range and step
    k=k+1;
    t(k)=j;  % time
    x1(k)=0;  % initialize positions and velocities of masses
    x2(k)=0;
    xdot1(k)=0;
    xdot2(k)=0;
    for i=1:2*n  % begin calculations for time step
       x1(k)=x1(k)+a(i)*U(1,i)*exp(S(i,i)*j);  % position of mass 1
       x2(k)=x2(k)+a(i)*U(2,i)*exp(S(i,i)*j);  % position of mass 2
```

```
        xdot1(k)=xdot1(k)+a(i)*S(i,i)*U(1,i)*exp(S(i,i)*j);   % velocity
of mass 1
        xdot2(k)=xdot2(k)+a(i)*S(i,i)*U(2,i)*exp(S(i,i)*j);   % velocity
of mass 2
    end
end
% remove discretization errors by rounding off any imaginary parts less
% than tolerance
tol=1e-10;  % define tolerance setting
if imag(x1)<tol
    x1=real(x1);
end
if imag(x2)<tol
    x2=real(x2);
end
if imag(xdot1)<tol
    xdot1=real(xdot1);
end
if imag(xdot2)<tol
    xdot2=real(xdot2);
end

% RESULTS - POSITION AND VELOCITY OF BOTH MASSES
frame=401; % plot up to t=40
    subplot(2,2,1)  % top left box shows plot of mass 1 position
    plot(t(1:frame),x1(1:frame),'-r','LineWidth',2)
    axis([0 40 -1.5 1.5])
    xlabel('Time, t')
    ylabel('Position, x')
    title('Position of Mass 1 From Equilibrium')

    subplot(2,2,2)  % top right box shows plot of mass 2 position
    plot(t(1:frame),x2(1:frame),'-r','LineWidth',2)
    axis([0 40 -1.5 1.5])
    xlabel('Time, t')
    ylabel('Position, x')
    title('Position of Mass 2 From Equilibrium')

    subplot(2,2,3)  % bottom left box shows plot of mass 1 velocity
    plot(t(1:frame),xdot1(1:frame),'-b','LineWidth',2)
    axis([0 40 -3 3])
    xlabel('Time, t')
    ylabel('Velocity, dot{x}')
    title('Velocity of Mass 1')

    subplot(2,2,4)  % bottom right box shows plot of mass 2 velocity
    plot(t(1:frame),xdot2(1:frame),'-b','LineWidth',2)
    axis([0 40 -3 3])
    xlabel('Time, t')
    ylabel('Velocity, dot{x}')
    title('Velocity of Mass 2')
```

MATLAB file for Example 2

```
clear all;
% input system parameters
```

```
n=2;  % two-dimensional system
m1=1; m2=2;  % define mass of each dimension
k1=5; k2=10;  % define spring constants
% k1 is spring from ground to mass 1,
% k2 is from mass 1 to mass 2
c1=0; c2=0.2;  % define damping constants
% c1 is dashpot from ground to mass 1,
% c2 is from mass 1 to mass 2
%
% calculate open loop eigenvalues
I=eye(n);  % identity matrix
O=zeros(n,n);  % zero matrix
M=[m1 0;   % mass matrix
   0 m2];
C=[c1+c2 -c2;  % damping matrix
   -c2 c2];
K=[k1+k2 -k2;  % spring matrix
   -k2 k2];
Ao=[O I;
    -K O];
Bo=[I O;
    C M];
[Uo,So]=eig(Ao,Bo);
b=[2 1]'; % arbitrary control selection vector
i=sqrt(-1);
for k=1:2*n
    vo(:,k)=Uo(1:n,k);  % Uo is given as normalized set
end;
so=[So(1,1); So(2,2); So(3,3); So(4,4)]; % open loop eigenvectors
mu=[so(1); so(2); -1-i; -1+i]; % define new eigenvalues to be assigned
Num1=((mu(3)-so(3))/so(3))*((mu(4)-so(3))/(so(4)-so(3)));
Num2=((mu(4)-so(4))/so(4))*((mu(3)-so(4))/(so(3)-so(4)));
Den1=b'*vo(:,3);
Den2=b'*vo(:,4);
q(1,1)=Num1./Den1;
q(2,1)=Num2./Den2;
f=M*vo(:,3:4)*diag(so(3:4))*q;
g=-K*vo(:,3:4)*q;% solve for new eigenvalues of the system
A=[O I;   % first-order realization including control
   -(K-b*g') O];
B=[I O;
   (C-b*f') M];
[U,S]=eig(A,B);   % eigenvalues and eigenvectors
%
% Initial conditions
x0=[1 0]';  % define initial position
xdot0=[0 1]';   % define initial velocity
a=U\[x0;xdot0];  % calculate coefficients of solution
%
% Calculate solution for each time-step
k=0;
for j=0:0.1:100  % define time range and step
    k=k+1;
    t(k)=j;  % time
    x1(k)=0;  % initialize positions and velocities of masses
    x2(k)=0;
    xdot1(k)=0;
```

```
    xdot2(k)=0;
    for i=1:2*n  % begin calculations for time step
       x1(k)=x1(k)+a(i)*U(1,i)*exp(S(i,i)*j);  % position of mass 1
       x2(k)=x2(k)+a(i)*U(2,i)*exp(S(i,i)*j);  % position of mass 2
       xdot1(k)=xdot1(k)+a(i)*S(i,i)*U(1,i)*exp(S(i,i)*j);  % velocity
of mass 1
       xdot2(k)=xdot2(k)+a(i)*S(i,i)*U(2,i)*exp(S(i,i)*j);  % velocity
of mass 2
    end
end
% remove discretization errors by rounding off any imaginary parts less
% than tolerance
tol=1e-10;  % define tolerance setting
if imag(x1)<tol
    x1=real(x1);
end
if imag(x2)<tol
    x2=real(x2);
end
if imag(xdot1)<tol
    xdot1=real(xdot1);
end
if imag(xdot2)<tol
    xdot2=real(xdot2);
end

% RESULTS - POSITION AND VELOCITY OF BOTH MASSES
frame=401; % plot to t=40
    subplot(2,2,1)  % top left box shows plot of mass 1 position
    plot(t(1:frame),x1(1:frame),'-r','LineWidth',2)
    axis([0 40 -1.5 1.5])
    xlabel('Time, t')
    ylabel('Position, x')
    title('Position of Mass 1 From Equilibrium')

    subplot(2,2,2)  % top right box shows plot of mass 2 position
    plot(t(1:frame),x2(1:frame),'-r','LineWidth',2)
    axis([0 40 -1.5 1.5])
    xlabel('Time, t')
    ylabel('Position, x')
    title('Position of Mass 2 From Equilibrium')

    subplot(2,2,3)  % bottom left box shows plot of mass 1 velocity
    plot(t(1:frame),xdot1(1:frame),'-b','LineWidth',2)
    axis([0 40 -3 3])
    xlabel('Time, t')
    ylabel('Velocity, dot{x}')
    title('Velocity of Mass 1')

    subplot(2,2,4)  % bottom right box shows plot of mass 2 velocity
    plot(t(1:frame),xdot2(1:frame),'-b','LineWidth',2)
    axis([0 40 -3 3])
    xlabel('Time, t')
    ylabel('Velocity, dot{x}')
    title('Velocity of Mass 2')
```

MATLAB file for Examples 3 and 4

```
clear all
n=2;
i=sqrt(-1);
mu1=-1+i;
mu2=-1-i;
M=[1 0;
    0 2];
C=[0.2 -0.2;
    -0.2 0.2];
K=[15 -10;
    -10 10];
I=eye(n);
O=zeros(n,n);
OO=zeros(n,1);
A=[O I;
    -K -C];
B=[I O;
    O M];
[U,S]=eig(A,B);
s=[S(1,1);S(2,2);S(3,3);S(4,4)];
for k=1:2*n
    v(:,k)=U(1:n,k);  % Uo is given as normalized set
end;
V1=[v(:,3) v(:,4)];
S1=[s(1) 0;
    0 s(2)];
k=0;
for b1=-1:0.001:1,
    k=k+1;
    b2=sqrt(1-b1^2);
    b=[b1;b2];
    Num(1)=(mu1-s(3))/s(3)*(mu2-s(3))/(s(4)-s(3));
    Num(2)=(mu2-s(4))/s(4)*(mu1-s(4))/(s(3)-s(4));
    Den(1)=b'*v(:,3);
    Den(2)=b'*v(:,4);
    q=Num./Den;
    f=M*V1*S1*q';
    g=-K*V1*q';
    B1(k,1)=b1;
    p1(k,1)=norm(f)+norm(g);
end
plot(B1,p1)
grid on
% use b from peak to analyze controllability
bb1=-0.806;
bb2=sqrt(1-bb1^2);
AA=[O I; -(M^(-1))*K -(M^(-1))*C];
BB=[OO;(M^(-1))*[bb1;bb2]];
FF=[BB AA*BB AA^2*BB AA^3*BB];
```

MATLAB file for Example 5

```
clear all
n=2;
M=eye(n);
K=[2 -1;-1 1];
```

```
[U,S]=eig(K,M);
ss=diag(S);
mu1=2;
mu2=ss(2);
mu=[mu1;mu2];
k=0;
for b1=-1:0.001:1,
    k=k+1;
    b2=sqrt(1-b1^2);
    b=[b1;b2];
    g=place(K,b,mu);
    s=eig(K-b*g,M);
    B1(k,1)=b1;
    B2(k,1)=b2;
    G1(k,1)=g(1);
    G2(k,1)=g(2);
    N(k,1)=norm(g);
end
subplot(2,2,1)
semilogy(B1,N)
    text(-0.5,10^3,'$b_{2}=\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
    title('(a)')
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    text(-1.5,10^2,'$\left | \mathbf{g} \right
|$','Interpreter','latex','FontSize',12)
subplot(2,2,2)
plot(B1,N)
    text(0.25,1.88,'$b_{2}=\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
    title('(b)')
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    axis([0 1 1.5 2])
```

MATLAB file for Example 6

```
clear all
n=5;m=2;i=1;j=2;
mu=2;
M=eye(n);
E=eye(n)-diag(ones(n-1,1),1);
K=5*E*E';
[U,S]=eig(K,M);
s1=S(3,3);
u1=U(:,3);
k=0;
for b1=-1:0.0001:1,
    k=k+1;
    b2=-sqrt(1-b1^2);
    b=[zeros(i-1,1);b1;zeros(j-i-1,1);b2;zeros(n-j,1)];
    g=(s1-mu)/(b'*u1);
    f=g*M*u1;
    B1(k,1)=b1;
    B2(k,1)=b2;
    Nf(k,1)=norm(f);
end
subplot(2,2,1)
```

```
plot(B1,Nf)
    axis([-1 -0.8 10.5 11.5])
    title(' ')
    text(-0.965,11.25,'$b_{2}=-\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    text(-1.05,11,'$\left | \mathbf{g} \right
|$','Interpreter','latex','FontSize',12)
    text(-0.78,10.5,' ')
```

MATLAB file for Example 8

```
clear all
n=2;
M=eye(n);
K=[2 -1;-1 1];
[U,S]=eig(K,M);
ss=diag(S);
mu1=1;
mu2=2;
mu=[mu1;mu2];
k=0;
for b1=-1:0.001:1,
    k=k+1;
    b2=-sqrt(1-b1^2);
    b=[b1;b2];
    g=place(K,b,mu);
    s=eig(K-b*g,M);
    B1(k,1)=b1;
    B2(k,1)=b2;
    G1(k,1)=g(1);
    G2(k,1)=g(2);
    N(k,1)=norm(g);
end
subplot(2,2,1.5)
    semilogy(B1,N)
    text(-0.38,9.9^3,'$b_{2}=\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
    title('(a)')
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    text(-1.55,10^1,'$\left | \mathbf{g} \right
|$','Interpreter','latex','FontSize',12)
    text(1.4,10^-2,' ')
subplot(2,2,3)
    plot(B1,N)
    axis([-1 -0.85 0.8 1.2])
    text(-0.97,1.1,'$b_{2}=\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
    title('(b)')
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    text(-1.04,0.95,'$\left | \mathbf{g} \right
|$','Interpreter','latex','FontSize',12)
subplot(2,2,4)
    plot(B1,N)
    axis([0 0.5 0.8 1.2])
    text(0.11,1.1,'$b_{2}=\sqrt{1-
b_{1}^{2}}$','Interpreter','latex','FontSize',12)
```

```
    title('(c)')
    xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
    text(-0.12,0.95,'$\left | \mathbf{g} \right
|$','Interpreter','latex','FontSize',12)
    text(0.6,0.8,' ')
```

MAPLE file for Example 9

```
> restart;
> with(LinearAlgebra):
> Digits:=16:
> M:=<<1 | 0 | 0> , <0 | 2 | 0> , <0 | 0 | 1>>:
> K:=<<15 | -10 | 0> , <-10 | 15 | -5> , <0 | -5 | 5>>:
> C:=<<0.2 | -0.2 | 0> , <-0.2 | 0.2 | 0> , <0 | 0 | 0>>:
> bf:=<<b1*f1 | b1*f2 | b1*f3> , <b2*f1 | b2*f2 | b2*f3> , <b3*f1 |
b3*f2 | b3*f3>>:
> bg:=<<b1*g1 | b1*g2 | b1*g3> , <b2*g1 | b2*g2 | b2*g3> , <b3*g1 |
b3*g2 | b3*g3>>:
> eta:=(f1^2+f2^2+f3^2)+a*(g1^2+g2^2+g3^2):
> QP1:=s1^2*M+s1*(C-bf)+K-bg:
> QP2:=s2^2*M+s2*(C-bf)+K-bg:
> QP3:=s3^2*M+s3*(C-bf)+K-bg:
> QP4:=s4^2*M+s4*(C-bf)+K-bg:
> QP5:=s5^2*M+s5*(C-bf)+K-bg:
> QP6:=s6^2*M+s6*(C-bf)+K-bg:
> D_QP1:=Determinant(QP1):
> D_QP2:=Determinant(QP2):
> D_QP3:=Determinant(QP3):
> D_QP4:=Determinant(QP4):
> D_QP5:=Determinant(QP5):
> D_QP6:=Determinant(QP6):
> L:=eta-p0*(b1^2+b2^2+b3^2)-p1*D_QP1-p2*D_QP2-p3*D_QP3-p4*D_QP4-
p5*D_QP5-p6*D_QP6:
> s1:=-1:s2:=-3:s3:=-0.5+2*I:s4:=-0.5-2*I:s5:=-0.75+5*I:s6:=-0.75-
5*I:a:=1:
> Eq1:=diff(L,b1):
> Eq2:=diff(L,b2):
> Eq3:=diff(L,b3):
> Eq4:=diff(L,f1):
> Eq5:=diff(L,f2):
> Eq6:=diff(L,f3):
> Eq7:=diff(L,g1):
> Eq8:=diff(L,g2):
> Eq9:=diff(L,g3):
> Eq10:=D_QP1:
> Eq11:=D_QP2:
> Eq12:=D_QP3:
> Eq13:=D_QP4:
> Eq14:=D_QP5:
> Eq15:=D_QP6:
> Eq16:=b1^2+b2^2+b3^2-1:
>
fsolve({Eq1,Eq2,Eq3,Eq4,Eq5,Eq6,Eq7,Eq8,Eq9,Eq10,Eq11,Eq12,Eq13,Eq14,Eq
15,Eq16},{b1=0.5795,b2=-
0.4053,b3=0.7070,f1,f2,f3,g1,g2,g3,p0,p1,p2,p3,p4,p5,p6})
```

MATLAB file for Example 9 – first graphic

```
clear all;
n=3;
i=sqrt(-1);
M=diag([1 2 1]);
C=[0.2 -0.2 0;
    -0.2 0.2 0;
    0 0 0];
K=[15 -10 0;
    -10 15 -5;
    0 -5 5];
s=[-1;-3;-0.5+2*i;-0.5-2*i;-0.75+5*i;-0.75-5*i];
% b=[-0.9863909563940264;-0.1637371834322245;0.01493371707780675];
% b=[-0.9845047747761751;-0.1746871909092368;0.01532102395901377];
% b=[0.5827969264810778;-0.3998426084738319;0.7074416095573197];
dd=0.000001;
k1=0;
for b1=-0.98648:dd:-0.9863,
    k1=k1+1;
    k2=0;
    for b2=-0.16378:dd:-0.1636,
        k2=k2+1;
        b3=sqrt(1-b1^2-b2^2);
        b=[b1;b2;b3];
        A=[zeros(n,n) eye(n);
            -inv(M)*K -inv(M)*C];
        gf=place(A,-[zeros(n,1);inv(M)*b],s);
        g=gf(1:n)';
        f=gf(n+1:2*n)';
        k=1;
        NN(k2,k1)=f'*f+g'*g;
        B2(k2,1)=b2;
    end
    B1(k1,1)=b1;
end
mesh(B1,B2,NN)
ylabel('$b_{2}$','Interpreter','latex','FontSize',12)
xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
zlabel('$\eta  $','Interpreter','latex','FontSize',12)
set(get(gca,'ZLabel'),'Rotation',0.0)
```

MATLAB file for Example 9 – second graphic

```
clear all;
n=3;
i=sqrt(-1);
M=diag([1 2 1]);
C=[0.2 -0.2 0;
    -0.2 0.2 0;
    0 0 0];
K=[15 -10 0;
    -10 15 -5;
    0 -5 5];
s=[-1;-3;-0.5+2*i;-0.5-2*i;-0.75+5*i;-0.75-5*i];
% b=[-0.9863909563940264;-0.1637371834322245;0.01493371707780675];
% b=[-0.9845047747761751;-0.1746871909092368;0.01532102395901377];
% b=[0.5827969264810778;-0.3998426084738319;0.7074416095573197];
dd=0.0025;
```

```
k1=0;
for b1=0.54:dd:0.62,
    k1=k1+1;
    k2=0;
    for b2=-0.46:dd:-0.34,
        k2=k2+1;
        b3=sqrt(1-b1^2-b2^2);
        b=[b1;b2;b3];
        A=[zeros(n,n) eye(n);
            -inv(M)*K -inv(M)*C];
        gf=place(A,-[zeros(n,1);inv(M)*b],s);
        g=gf(1:n)';
        f=gf(n+1:2*n)';
        k=1;
        NN(k2,k1)=f'*f+g'*g;
        B2(k2,1)=b2;
    end
    B1(k1,1)=b1;
end
mesh(B1,B2,NN)
ylabel('$b_{2}$','Interpreter','latex','FontSize',12)
xlabel('$b_{1}$','Interpreter','latex','FontSize',12)
zlabel('$\eta  $','Interpreter','latex','FontSize',12)
set(get(gca,'ZLabel'),'Rotation',0.0)
```

MATLAB file for Chapter 5

```
clear all;
% input system parameters
n=3;  % two-dimensional system
m1=1; m2=2; m3=1; % define mass of each dimension
k1=5; k2=10; k3=5;  % define spring constants
c1=0; c2=0.2; c3=0;  % define damping constants
I=eye(n);  % identity matrix
O=zeros(n,n);  % zero matrix
M=[m1 0 0;   % mass matrix
   0 m2 0;
   0 0 m3];
C=[c1+c2 -c2 0;  % damping matrix
   -c2 c2+c3 -c3;
   0 -c3 c3];
K=[k1+k2 -k2 0;  % spring matrix
   -k2 k2+k3 -k3;
   0 -k3 k3];
Ao=[O I;
    -K O];
Bo=[I O;
    C M];
[Uo,So]=eig(Ao,Bo);
i=sqrt(-1);
for k=1:2*n
    vo(:,k)=Uo(1:n,k);  % Uo is given as normalized set
end;
b=[0.5828; -0.3998; 0.7074];  % optimal from Maple solution
% b=[0.57735; 0.57735; 0.57735]; % arbitrary of equal components
b=b/norm(b); % normalize the control
```

```
s=[So(1,1); So(2,2); So(3,3); So(4,4); So(5,5); So(6,6)]; % set of
open-loop eigenvalues
mu=[-1; -3; -0.5+2*i; -0.5-2*i; -0.75+5*i; -0.75-5*i]; % define new
eigenvalues to be assigned
Num1=((mu(1)-s(1))/s(1))*((mu(2)-s(1))/(s(2)-s(1)))*((mu(3)-
s(1))/(s(3)-s(1)))*((mu(4)-s(1))/(s(4)-s(1)))*((mu(5)-s(1))/(s(5)-
s(1)))*((mu(6)-s(1))/(s(6)-s(1)));
Num2=((mu(2)-s(2))/s(2))*((mu(1)-s(2))/(s(1)-s(2)))*((mu(3)-
s(2))/(s(3)-s(2)))*((mu(4)-s(2))/(s(4)-s(2)))*((mu(5)-s(2))/(s(5)-
s(2)))*((mu(6)-s(2))/(s(6)-s(2)));
Num3=((mu(3)-s(3))/s(3))*((mu(1)-s(3))/(s(1)-s(3)))*((mu(2)-
s(3))/(s(2)-s(3)))*((mu(4)-s(3))/(s(4)-s(3)))*((mu(5)-s(3))/(s(5)-
s(3)))*((mu(6)-s(3))/(s(6)-s(3)));
Num4=((mu(4)-s(4))/s(4))*((mu(1)-s(4))/(s(1)-s(4)))*((mu(2)-
s(4))/(s(2)-s(4)))*((mu(3)-s(4))/(s(3)-s(4)))*((mu(5)-s(4))/(s(5)-
s(4)))*((mu(6)-s(4))/(s(6)-s(4)));
Num5=((mu(5)-s(5))/s(5))*((mu(1)-s(5))/(s(1)-s(5)))*((mu(2)-
s(5))/(s(2)-s(5)))*((mu(3)-s(5))/(s(3)-s(5)))*((mu(4)-s(5))/(s(4)-
s(5)))*((mu(6)-s(5))/(s(6)-s(5)));
Num6=((mu(6)-s(6))/s(6))*((mu(1)-s(6))/(s(1)-s(6)))*((mu(2)-
s(6))/(s(2)-s(6)))*((mu(3)-s(6))/(s(3)-s(6)))*((mu(4)-s(6))/(s(4)-
s(6)))*((mu(5)-s(6))/(s(5)-s(6)));
Den1=b'*vo(:,1);
Den2=b'*vo(:,2);
Den3=b'*vo(:,3);
Den4=b'*vo(:,4);
Den5=b'*vo(:,5);
Den6=b'*vo(:,6);
q(1,1)=Num1./Den1;
q(2,1)=Num2./Den2;
q(3,1)=Num3./Den3;
q(4,1)=Num4./Den4;
q(5,1)=Num5./Den5;
q(6,1)=Num6./Den6;
f=M*vo*So*q;
g=-K*vo*q;
% solve for new eigenvalues of the system
A=[O I;   % first-order realization including control
   -(K-b*g') O];
B=[I O;
   (C-b*f') M];
[U,S]=eig(A,B);   % eigenvalues and eigenvectors
% Initial conditions
x0=[1 0 0]';  % define initial position
xdot0=[0 1 0]';   % define initial velocity
a=U\[x0;xdot0];  % calculate coefficients of solution
%
% Calculate solution for each time-step
k=0;
for j=0:0.1:20  % define time range and step
    k=k+1;
    t(k)=j;  % time
    x1(k)=0;  % initialize positions and velocities of masses
    x2(k)=0;
    x3(k)=0;
    xdot1(k)=0;
    xdot2(k)=0;
```

```
    xdot3(k)=0;
    for i=1:2*n  % begin calculations for time step
       x1(k)=x1(k)+a(i)*U(1,i)*exp(S(i,i)*j);  % position of mass 1
       x2(k)=x2(k)+a(i)*U(2,i)*exp(S(i,i)*j);  % position of mass 2
       x3(k)=x3(k)+a(i)*U(3,i)*exp(S(i,i)*j);  % position of mass 3
       xdot1(k)=xdot1(k)+a(i)*S(i,i)*U(1,i)*exp(S(i,i)*j);  % velocity
of mass 1
       xdot2(k)=xdot2(k)+a(i)*S(i,i)*U(2,i)*exp(S(i,i)*j);  % velocity
of mass 2
       xdot3(k)=xdot3(k)+a(i)*S(i,i)*U(3,i)*exp(S(i,i)*j);  % velocity
of mass 3
    end
end
% remove discretization errors by rounding off any imaginary parts less
% than tolerance
tol=1e-10;  % define tolerance setting
if imag(x1)<tol
    x1=real(x1);
end
if imag(x2)<tol
    x2=real(x2);
end
if imag(x3)<tol
    x3=real(x3);
end
if imag(xdot1)<tol
    xdot1=real(xdot1);
end
if imag(xdot2)<tol
    xdot2=real(xdot2);
end
if imag(xdot3)<tol
    xdot3=real(xdot3);
end
% CALCULATE CONTROL FORCE AND EFFORT
k=0;
for j=0:0.1:20
    k=k+1;
    u(k)=abs(10^(-3)*f'*[xdot1(k); xdot2(k); xdot3(k)]+g'*[x1(k);
x2(k); x3(k)]); % control force at each time step, u (N), 10^-3
correction for millimeters instead of meters
    ce1(k)=abs(b(1)*u(k));  % control force applied on mass 1 at each
time step (N)
    ce2(k)=abs(b(2)*u(k)); % control force applied on mass 2 at each
time step (N)
    ce3(k)=abs(b(3)*u(k)); % control force applied on mass 3 at each
time step (N)
end;
tce=sum(ce1(:))+sum(ce2(:))+sum(ce3(:));  % total control effort
% % RESULTS - CONTROL EFFORT OVER TIME
frame=201;
    subplot(2,1,1) % plot of control input
    plot(t(1:frame),u(1:frame),'-r','LineWidth',2)
    axis([0 10 0 16])
    xlabel('Time, t (sec)')
    ylabel('Control Input, u (N)')
    title('Force Required for Control')
```

VITA

Carla was born in 1977, in Baton Rouge, Louisiana. She graduated from Scotlandville Magnet High School's High School for the Engineering Professions in Baton Rouge. She then went on to graduate from Embry-Riddle Aeronautical University with a Bachelor of Science in Engineering Physics in December 2000.

After completing her undergraduate degree, she remained at Embry-Riddle as a payload technician in the Atmospheric Physics Research Laboratory where she worked on sounding rocket scientific payloads.

Wanting to further her education in engineering, Carla returned to her home state and began a direct Ph.D. program in Mechanical Engineering in 2006. She will graduate in December 2012. She plans to pursue a career in aerospace after graduation.