Compact Inverses of Differential Operators.

Robert Mackenzie Kauffman

Louisiana State University and Agricultural & Mechanical College

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Robert MacKenzie Kauffman
B.A., University of the South, 1963
M.S., Louisiana State University, 1965
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Abstract

The property that the essential spectrum of an ordinary differential operator on an Lp space be null is analyzed in this paper in terms of the compactness of the inverse of the minimal operator on intervals of the type \([a, \infty)\). We show that if the associated differential expression is self adjoint, and the space is \(L^2[a, \infty)\), the essential spectrum is null if and only if the minimal operator has compact inverse. (The inverse is defined on a closed subspace of \(L^2[a, \infty)\).)

Sufficiency holds in the Lp case, whether or not the expression is self adjoint.

To show that the property occurs widely, we then develop a general example of a type of differential operator which has the compact inverse property on \(L^2[a, \infty)\). Our main example is as follows: Let \(\mathcal{T}\) be a classically self adjoint differential expression with first coefficient 1. Let \(\mathcal{T}^i\) be the formal product of \(\mathcal{T}\) with itself \(i\) times, where \(i\) is a positive integer. Let \(f_i\) be bounded for \(i > 0\), and let

\[
\lim_{x \to \infty} \text{Re}(f_0(x)) = \infty.
\]

Then \(\mathcal{T}^{2k} + \sum_{i=1}^{2k-1} f_i \mathcal{T}^i + f_0\) is a differential expression on \([a, \infty)\) such that the associated minimal operator has compact inverse defined on a closed subspace of \(L^2[a, \infty)\).

In addition, we show that the compactness property is stable under certain types of perturbations and formal products, even in the Lp case. We also discuss the case when
only a restriction of the minimal operator is compact, and finally extend our results about the essential spectrum to \((-\infty, \infty)\).
Introduction

In this paper we study the following question: Given a differential expression \( T \), what properties of the coefficients of \( T \) will assure that operators generated by \( T \) on appropriate Hilbert or Banach space have null essential spectrum? We show that for this to happen on the Hilbert space \( L_2[a,\infty) \) it is sufficient, and if \( T \) is classically self adjoint, necessary that the minimal operator generated by \( T \) have compact inverse. (We assume that we are working in an interval \([a,b)\) to assure \( L\)-ness of the minimal operator.) In the Banach space case, the sufficiency still holds. These results relate to the study by Hartman and Wintner of the essential spectrum of differential operators. (See, for example, Hartman and Wintner (1).)

We thus wish to know general conditions on the coefficients of an operator to assure compactness of the inverse of the minimal operator. If we are on an interval \([a,b]\) or \([a,b)\), with \( b \) finite, general theorems about integral operators usually ensure compactness, but if the interval is \([a,\infty)\) the infinite measure makes things much harder. However, a rather wide class of operators have this property.

Since most inverses of differential operators on finite intervals are compact, a limiting process using uniform operator convergence is suggested for infinite intervals. This is our basic tool for showing that particular examples of minimal operators have compact inverse. We find in this manner that \( T^{2n} + f(x) \), with \( T \) classically self adjoint,
generates a minimal operator with compact inverse on \( L_2[a,\infty) \) if \( \lim_{x \to \infty} \text{Re}(f(x)) = \infty \). Then we use perturbation methods to show that \( \tau^{2n} + \sum_{i=1}^{2n-1} f_i \tau^i + f_0 \) has the same property, if in addition to the above hypotheses about \( \tau \) and \( f_0 \), we assume all \( f_i \) are bounded functions. \( \tau^i \) is the ith power of \( \tau \) under the formal product. We assume all necessary differentiability here.

In studying the stability of the property under changes of coefficients, we show that the property is preserved under formal products of differential expressions provided one generates a minimal operator with closed range and the other generates a minimal operator with compact inverse. Also the preservation of the property under certain types of perturbations holds even in the \( Lp - Lq \) situation. Thus given one example we can find many more.

In existing literature the compactness property seems to only have been observed for the case when all solutions of \( \tau f = 0 \) are in \( Lp[a,\infty) \) and all solutions of \( \tau^+ f = 0 \) are in \( Lq[a,\infty) \), where we consider the differential operator from \( Lp \) to \( Lq \). Several authors, notably Weyl in the second order and Glazman in higher orders, have observed this case, which can be found in a modified form in section XIII.4 of Dunford and Schwartz (2) and also in Naimark (1). Bellman (1) studies the case when \( P \neq 2 \).

The property that the essential spectrum is void has received wider attention. Theorem 9 on page 1440 of Dunford and Schwartz (2) seems to be the most general theorem on this
subject, and it is new in Dunford and Schwartz. Theorem 2.18 generalizes the theorem in Dunford and Schwartz. There is a great deal of literature on the essential spectrum, and an excellent survey of the results can be found in Dunford and Schwartz (2). The theorem—which characterizes the essential spectrum of a self adjoint operator is apparently new in Dunford and Schwartz, but was used as the defining property by Hartman and Wintner.

Besides considering the whole minimal operator, we consider the general question of when an operator from a reflexive Banach space to an $L^p$ space is compact. This could perhaps be useful in the following situation: Let $H$ be self adjoint on $L^2[a,\infty)$. Suppose $H = \int_{-\infty}^{\infty} \lambda \, dp$, and let $0 \notin [c,b]$. Let $P = P_b - P_c$. PHP is compact $\iff$ there are no cluster points $\lambda$ of the spectrum of $H$ with $\lambda \in [c,b]$. Therefore it seems useful to discuss compactness of a restriction of a differential operator, even when the whole operator is not compact.

Along this line we use a criterion for weak compactness in $L_1$ to deduce the fact that if an operator from a reflexive Banach space to a space of bounded, continuous, $L_1$ functions on a locally compact space is continuous with the $L^p$ topology on the space of functions ($p \geq 1$), it is compact with the $L^q$ topology on the space of functions for every $q \in [1,\infty)$.

(Certain hypotheses are needed about the measure here.)

The last part of the paper is concerned with showing that $\lambda$ is in the essential spectrum of the minimal operator
on $(-\infty,\infty)$ if and only if it is in the essential spectrum as $(-\infty,0]$ or $[0,\infty)$. This extends our results to $(-\infty,\infty)$.

Many of the theorems about differential operators on $L^p$ spaces used in this paper are due to Rota (1). The book by Goldberg (1) also contains much of the background material used in this paper.
0. Background Material in Functional Analysis

0.0 Definition: Let $X$ be a normed linear space. Let $X'$ be the set of continuous linear functional on $X$. A sequence \( \{x_n\} \) in $X$ is said to **converge weakly** to $x \in X$ if for any $x' \in X'$, \( x'x_n \to x'x \). This is written \( x_n \xrightarrow{w} x \).

0.1 Definition: The **natural map**, denoted by $J_X$, of a normed linear space $X$ into its second conjugate space $X''$ (the Banach space of bounded linear functionals on $X'$, where $X'$ has the norm topology \( |x'| = \sup x'x \)) is defined by \( (J_Xx)x' = x'x \) for every $x' \in X'$. If the range of $J_X$ is all of $X''$, then $X$ is called reflexive.

0.2 Theorem: Every bounded sequence in a reflexive space $X$ contains a weakly convergent subsequence to some element $x \in X$.

Proof: This is theorem I.6.15, Goldberg (1), page 30.

0.3 Theorem: Let $(S, \Sigma, \mu)$ be a positive measure space, which is the union of countably many sets of finite $\mu$-measure. Then \( [L^p(S, \Sigma, \mu)]' = L^{p'}(S, \Sigma, \mu) \), where $\frac{1}{p} + \frac{1}{p'} = 1$ if $\infty > p > 1$, and $p' = \infty$ if $p = 1$.

Proof: The proof may be found in Dunford and Schwartz (1), pages 286-290.

0.4 Corollary: If $\infty > p > 1$, $L^p(S, \Sigma, \mu)$ is reflexive. Note that the dual of $L^\infty$ is not $L_1$, but in general a set which properly contains $L_1$. The containment is proper for lebesgue measure on the real line, which is the case we deal with in the sequel.
We now consider differential expressions for the form
\[ T = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0, \]
where \( D = \frac{d}{dt} \) and the coefficients \( a_k \) are complex valued functions of a real variable. We further assume each \( a_k \) to be infinitely differentiable, for the sake of simplicity, and assume \( a_n(x) \neq 0 \) on the interval being considered.

A differential expression \( T \) may give use to many operators \( T \) which have their domains in \( L^p(I) \) and ranges in \( L^q(I) \), where \( 1 \leq p \leq \infty, \ 1 \leq q \leq \infty \) and \( I \) is an interval of real numbers. We define the maximal operator \( T_{p,q} \) corresponding to \( T_{p,q} \) and \( I \) as follows:

0.5 Definition: Let \( X \) and \( Y \) be normed linear spaces. Let \( K \) be a linear operator with domain in \( X \) and range in \( Y \). Let \( S = \{ x \in X \mid \|x\| \leq 1 \} \). If \( KS \) is compact in \( Y \), \( K \) is said to be compact.

0.6 Definition: \( D(T_{p,q}) = \{ f \mid f \in A_n(I) \cap L^p(I), \ Tf \in L_q(I) \} \) where \( A_n(I) \) is the set of complex valued functions for which \( f^{(n-1)} = D^{n-1}f \) exists and is absolutely continuous on every compact subinterval of \( I \). For \( f \in D(T_{p,q}) \) we define \( T_{p,q} f = T^f = \sum_{k=0}^{n} a_k D^k f. \)

0.7 Definition: The operator \( T_{p,q}^R \) is defined to be the restriction of \( T_{p,q} \) to those \( f \in D(T_{p,q}) \) which have compact support in the interior of \( I \).

0.8 Definition: The minimal operator \( T_{0,q} \) corresponding to \( (T_{p,q}) \) is defined to be the minimal closed extension of \( T_{p,q}^R \) where \( 1 \leq p < \infty, \ 1 \leq q < \infty \). (When \( p \) or \( q = \infty \) we use another definition, but this case is not
important to us.)

Note: We know such an extension exists because of the following theorem.

0.9 Theorem: \( T_{p,q} \) is closed when \( 1 \leq p \leq \infty, \ l \leq q \leq \infty. \)

Proof: This is corollary VI.3.2, Goldberg (1), page 145.
(Also may be found in Rota (1.).)

0.10 Theorem: Let \( I = [a,b] \) be compact and let \( T \) be as above. Suppose \( T \) is \( 1-1 \) closed operator which is a restriction of the maximal operator corresponding to \( (T_{p,q}); \ 1 \leq p \leq \infty, \ l < q \leq \infty. \) Then \( T^{-1} \) is compact.

Proof: The above theorem is theorem VI.3.3, Goldberg, page 145.

0.11 Theorem: Let \( I \) contain one of its endpoints and let \( (p,q) \) be admissible. (i.e. \( 1 \leq p < \infty, \ l \leq q < \infty \) or \( 1 < p \leq \infty, \ l < q \leq \infty. \)) Then

i) The minimal operator corresponding to \( (T_{p,q}) \) is \( 1-1. \)

ii) The maximal operator corresponding to \( (T_{p,q}) \) has range dense in \( L_q(I), \) where \( 1 \leq p < \infty, \ l \leq q < \infty. \)

Proof: This is theorem VI.2.10, Goldberg, page 139.
(Also found in Rota (1.).)

0.12 Lemma: If \( \dim \frac{X}{M} = n, \) where \( X \) is a vector space and \( M \) is a vector subspace of \( X, \) and if \( H \supseteq M, \) then \( \dim \frac{X}{H} \leq n. \)
[Note: in other words, if \( M \) has finite deficiency in \( X, \) and \( H \supseteq M, \) then \( H \) has finite deficiency in \( X. \)]

0.13 Theorem: If \( \dim \frac{X}{M} = N_1, \) and \( \dim \frac{X}{H} = N_2, \) where \( H \supseteq M, \) then \( \dim \frac{H}{M} = N_1 - N_2. \)

Proof: First \( \dim \frac{H}{M} \leq N_1 - N_2, \) for suppose we have
\( n_1 \cdots n_{N_1-N_2+1} \) being members of \( \frac{H}{M} \). Then let \( x_1 \in \mathcal{N}_1 \),

\[ \cdots x_{N_1-N_2+1} \in \mathcal{N}_{N_1-N_2+1}, \]

where the \( x_i \) are members of \( X \).

We show that the \( x_i \) are linearly dependent modulo \( M \). For if not, we could select \( N_2 \) vectors in \( X \) which are linearly independent modulo \( H \), and combine them with the \( x_i \), and get a set of \( N_1+1 \) vectors in \( X \), linearly independent modulo \( M \).

This is a contradiction.

Reversing the process, we see that \( \dim \frac{H}{M} \geq N_1 - N_2 \).

Note: \( \dim \frac{X}{M} = N \) if and only if there is an \( N \) dimensional space \( Q \) such that \( M \oplus Q = N \).

**Theorem:** If \( M \) is a closed subspace of a Banach space, and \( Q \) is finite dimensional, then \( M \oplus Q \) is closed.

**Proof:** Suppose \( x_n \in M \oplus Q \), \( x_n \rightarrow x \). Show \( x \in M \oplus Q \).

We do this by extracting a subsequence of \( \{x_n\} \) which converges to a member of \( M \oplus Q \).

If \( x_n = y_n + z_n \) with \( y_n \in M \), \( z_n \in Q \); and if \( \{z_n\} \) have a bounded subsequence, then the theorem is proved. First, the \( z_n \) would have a convergent subsequence, \( \{z_{nk}\} \). Then if \( z_{nk} \rightarrow z \), then \( z \in Q \) and \( x_{nk} - z_{nk} \rightarrow x - z \). Therefore \( x - z \in M \), so \( x \in M \oplus Q \).

Suppose then that \( \{z_n\} \) diverge in norm to \( \infty \), then

\[
\left\{ \frac{x_n}{\|z_n\|} \right\} \text{ converges to 0. Therefore } \frac{y_n}{|z_n|} - \frac{z_n}{|z_n|} \text{ converges to 0. Now } \left\{ \frac{z_n}{|z_n|} \right\} \text{ have a convergent}
\]
subsequence to $z_0 \in \mathbb{Q}$. Let \( \frac{z_{nk}}{|z_{nk}|} \rightarrow z_0 \). Then

\[
\frac{y_n}{|z_n|} \rightarrow -z_0.
\]

So $z_0 \in M \cap \mathbb{Q}$, a contradiction of the fact that the sum is direct.
I. Compact Inverses of Differential Operators

We now consider differential expressions $\mathcal{T}$ of the type discussed in section 0, on the interval $[a, \infty)$ where $a \neq -\infty$, and where $[a, \infty)$ is given Lebesgue measure. Given $y \in [a, \infty)$, we let $f_1 \cdots f_n$ be a set of solutions of $\mathcal{T}f = 0$ defined in $[a, \infty)$, (with no assumptions as to whether the $f_i$ are in any $L^p$ space), such that

\[
\begin{bmatrix}
f_1(y) \\
f_1'(y) \\
f_1''(y) \\
\vdots \\
f_1^{(n-1)}(y)
\end{bmatrix}
= \begin{bmatrix} 1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
f_2(y) \\
f_2'(y) \\
f_2''(y) \\
\vdots \\
f_2^{(n-1)}(y)
\end{bmatrix}
= \begin{bmatrix} 0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

and so on. These are a basis for the solution space of $\mathcal{T}$, though it is to be noted that none of these need be in the domain of a maximal operator $\mathcal{T}, p, q$. If however there is some $g$ such that $\mathcal{T}, p, q(g) = 0$, then $g$ is a linear combination of the $f_i$. Also, since all the coefficients of $\mathcal{T}$ are infinitely differentiable, each $f_i$ is infinitely differentiable on $[a, \infty)$.

The function $x \rightarrow \begin{bmatrix} f_1(x) & \cdots & f_n(x) \\
f_1'(x) & \cdots & f_n'(x) \\
\vdots & \vdots & \vdots \\
f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x)
\end{bmatrix}$ is a
function from \([a,\infty)\) to the set of \(n \times n\) complex valued matrices. This function of course depends on \(y\), in that for a different \(y\) we get a different function. If we consider the function from \([a,\infty)\) to the \(n \times n\) matrices constructed as above with respect to the point \(y\), for each \(y \in [a,\infty)\), we obtain a function in \([a,\infty) \times [a,\infty)\) defined by \((x,y) \mapsto \phi(x)\)

where \(\phi(x) = \begin{pmatrix} f_1(x) & \cdots & f_n(x) \\ f'_1(x) & \cdots & f'_n(x) \\ \vdots & \ddots & \vdots \\ f_{n-1}(x) & \cdots & f_{n-1}(x) \end{pmatrix}\) and the \(f_i\) have

the property that \(\begin{pmatrix} f_1(y) & \cdots & f_n(y) \\ f'_1(y) & \cdots & f'_n(y) \\ \vdots & \ddots & \vdots \\ f_{n-1}(y) & \cdots & f_{n-1}(y) \end{pmatrix} = I\) and are solutions of \(\mathcal{T}f = 0\). (\(I\) is the identity matrix.) We write this function \((x,y) \mapsto M(x,y)\). It is a fact that \(M(x,x) = I, M(x,y) M(y,z) = M(x,z)\).

Let \(C^n[a,\infty)\) be the set of all \(n\)-times continuously differentiable functions on \([a,\infty)\). If \(g\) is continuous on \([a,\infty)\), there is a unique \(f \in C^n[a,\infty)\) such that \(\mathcal{T}f = g\).
and
\[
\begin{pmatrix}
  f(a) \\
f'(a) \\
  \vdots \\
f^{n-1}(a)
\end{pmatrix} = 
\begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
\]

Further it follows from variation of constants that
\[
f(x) = \int_a^x M_{\ln}(x,y) a_n^{-1}(y)g(y)dy.
\]

1.0 Theorem: Suppose \( f \in A_n([a,\infty)) \) (see definition 0.6), and \( Tf = 0 \) a.e. with \( 0 = f(a) = f'(a) \cdots f^{n-1}(a) \). Then \( f \) is identically 0.

Proof: This is standard.

A corollary of the above theorem is that for a finite interval \( I \), the solution space of \( T_{τ,p,q} \) is \( n \)-dimensional and thus all solutions are infinitely differentiable, so we get no new solutions besides the classical ones.

1.1 Theorem: Let \( g \in L^q[a,x_1] \) where \( x_1 \) is finite, \( a \) is finite as always and where \( 1 \leq q < \infty \). Then there is a unique \( f \) in \( L^p[a,x_1] \) such that \( T_{τ,p,q} f = g \), and \( f(a) = f'(a) = \cdots = f^{n-1}(a) = 0 \). For this \( f \),
\[
f(x) = \int_a^x M_{\ln}(x,y) a_n^{-1}(y)g(y)dy.
\]

Proof: Uniqueness follows from theorem 1.0. Existence could be proved by quoting theorems, but we will prove existence and establish the formula at the same time.

We note that the maximal operator \( T_{τ,p,q} \) on \( L^p[a,x_1] \) is closed. Let \( g_k \to g \) in \( L^q[a,x_1] \) with \( g_k \) continuous on \([a,x_1]\). Then \( f_k(x) = \int_a^x M_{\ln}(x,y) a_n^{-1}(y)g_k(y)dy \) converges uniformly on \([a,x_1]\) to \( f(x) = \int_a^x M_{\ln}(x,y) a_n^{-1}(y)g(y)dy \). The
reason for this is that $M_{ln}(x,y)$ is continuous and therefore uniformly bounded by $B$ on $[a,x_1] \times [a,x_1]$. Therefore for each $x,M_{ln}(x,\cdot)a_n^{-1}(\cdot) \in Lp'[a,x_1]$, and

$$||M_{ln}(x,\cdot)|| p' \leq B |x_1 - a|^\frac{1}{p} \cdot \sup_{x \in [a,n]} |a_n^{-1}(x)|,$$

where $p' = 1 - \frac{1}{q}$. Since the dual of $Lq[a,x_1] = Lp'[a,x_1]$, it follows that $f_k$ converges uniformly to $f$ and incidentally that $f = \int_a^x M_{ln}(x,y)a_n^{-1}(y)g(y)dy$ is uniformly bounded and continuous, and hence is in $Lp[a,N]$. Since $\{f_k\}$ converges uniformly to $f$, it also converges in $Lp[a,N]$. Also each $f_k$ is in the domain of the maximal operator, and $Tf_k = g_k$. Now we use the fact that the maximal operator is closed to say that $f$ is in the domain of the maximal operator, and

$$T \tau, p, q f = g.$$

Now we note that $f_k'(x) = \int_a^x M_{2n}(x,y)a_n^{-1}(y)g(y)dy$, so the $f_k'$ converge uniformly on $[a,x_1]$. If we knew that $\{f_k'\} \xrightarrow{\text{unif}} f'$, we would know that $f'(a) = 0$. But $f(x) = \int_a^x f'(y)dy + f(a)$, and $f(a) = 0$, so $f(x)$

$$= \int_a^x f'(y)dy.$$  

$f$ is the uniform limit of the $f_k'$, so $f(x)$

$$= \lim_{k \to \infty} \int_a^x f_k'(y)dy \quad \text{uniformly on } [a,x_1] \text{ and so } f(x)$$

$$= \int_a^x \lim_{k \to \infty} f_k'(y)dy.$$  

But this implies that $f'(y) = \lim_{k \to \infty} f_k(y)$ for every $y$, and thus $f'(a) = 0$. Repeating the argument, we see that $f^j = 0$ for every $j \leq n - 1$. The theorem is proved.

***

We now consider the minimal operator $T_0, \tau, p, q$ corresponding to $\tau$, noting that if $T_0, \tau, p, q f = g$, we have
\( f(x) = \int_\mathcal{X} M(x,y) a_n^{-1}(y) g(y) dy. \) When it is clear which \( T, p, \) and \( q \) we mean, we shall abbreviate the minimal operator as \( T_0, \) and the maximal operator as \( T. \) Unless it is stated otherwise, we assume that \( 1 \leq p < \infty, \) and \( 1 \leq q < \infty. \)

Since the minimal operator on a finite interval is an integral operator with continuous kernel, theorem 0.10 is obviously true for it. Our job is to analyze what happens on the finite interval.

The following theorem is an important special case. It is exercise 52, page 518, Dunford and Schwartz (1).

1.2 Theorem: Let \( 1 < p < \infty, \) \( \frac{1}{p} + \frac{1}{p'} = 1. \) Let \( S \) be an interval of real numbers, \( \mu \) be Lebesgue measure and suppose \( K \) is a complex valued measurable function on \( S \times S \) such that

\[
\left( \int_S \int_S |K(s,t)|^p \mu(ds)^p \mu(dt) \right)^{\frac{1}{p'}} = M < \infty.
\]
Let \( g = Tf \) be defined by \( g(s) = \int_S K(s,t)f(t)dt. \) Then \( T \) is a compact operator in \( L^p(S,S,\mu) \) whose norm is at most \( M. \)

If \( p = 2, \) this condition reduces to \( K \) being in \( L^2(S \times S). \) When \( S = [a, \infty) \) with Lebesgue measure, and \( K(s,t) = M_{\ln}(x,y) a_n^{-1}(y), \) we may use theorem 1.2 to get some information about the minimal operator. In order to do this, we first examine more closely the make-up of \( M_{\ln}(x,y). \)

Consider a formal differential expression \( \mathcal{T} \) with \( a_n \equiv 1, \) i.e., \( \mathcal{T} = D^n + a_{n-1} D^{n-1} \cdots + a_0. \) Let \( F \) be the matrix
A fundamental matrix $\tilde{\Phi}$ for $F$ is a solution of $\tilde{\Phi}' = F\tilde{\Phi}$ with the property that $|\tilde{\Phi}(x)| = \det \tilde{\Phi}(x)$ is never 0 for any $x$ in $[a, \infty)$. Let $\Phi$ be a fundamental matrix for $F$. Then (see chapter 3 of Coddington and Levinson (1) for a discussion of these points) $M(x,y) = \tilde{\Phi}(x)\tilde{\Phi}^{-1}(y)$, where $M(x,y)$ is related to $T$ as previously. Now if $\tilde{\Phi}$ is a fundamental matrix for $F$, $\tilde{\Phi}^{-1}$ is a fundamental matrix for $-F^\ast$, where $\ast$ denotes taking the conjugate transpose. But if we have a fundamental matrix for $-F^\ast$, its bottom row is a set of $n$ linearly independent solutions of the adjoint equation $T^\ast f = 0$. ($T^\ast$ is the classical adjoint of $T$, its definition is in Coddington and Levinson (1), chapter 3.)

Now $M(x,y) = \tilde{\Phi}(x)\tilde{\Phi}^{-1}(y) = \tilde{\Phi}(x) \left[ \tilde{\Phi}^{\ast -1}(y) \right]^\ast$. The bottom row of $\tilde{\Phi}^{\ast -1}(y)$ is moved up to the last column under the operation of transposition, so $\tilde{\Phi}^{-1}(y)$ has the property that its $n$th column is a set of conjugate of linearly independent solutions of $T^\ast f = 0$. Now $M_{ln}(x,y)$ is the inner product of the first row of $\tilde{\Phi}(x)$ with the $n$th column of
Therefore $M_{1n}(x,y) = \sum_{i=1}^{n} f_i(x) \overline{\Psi}_i(y)$, there $\{f_i\}$ are $n$ linearly independent solutions of $Tf = 0$, and the $\Psi_i$ are $n$ linearly independent solutions of $T^+ \psi = 0$.

We now use these results to prove a theorem. This theorem, for $T = T^+$, is equivalent to theorem XIII.4.1, page 1330, Dunford and Schwartz (2).

1.3 Theorem: If $T = D^n + a_{n-1}D^{n-1} + \cdots + a_0$, and every solution $f$ of $Tf = 0$ is in $L^2[a,\infty)$, and further every solution $\psi$ of $T^+ \psi = 0$ is in $L^2[a,\infty)$, then the minimal operator $T_0, T, 2, 2$ is compact.

Proof: Define $K(x,y) = M_{1n}(x,y)$ for $x \geq y$ and $= 0$ for $y > x$.

$K$ is obviously in $L_2[a,\infty) \times [a,\infty)$, using the equation for $M_{1n}(x,y)$. We now use theorems 1.1 and 1.2, and the result follows.

*** ***

Remark: We can use theorems 1.1 and 1.2 to get the following stronger theorem.

1.3a Theorem: If $T$ is as above, every solution of $Tf = 0$ is in $L^p[a,\infty)$, $1 < p < \infty$, and every solution of $T^+ f = 0$ is in $L^{p'}[a,\infty)$, where $\frac{1}{p'} = 1 - \frac{1}{p}$, then the minimal operator $T_0, T, p, p$ is compact.

1.4 Definition: We define a linear operator $C^Y_x$ on the set of functions defined on $[a,\infty)$ as follows:

\[
\begin{cases}
[C^Y_x f](z) = f(z) & z \in [x,y] \\
[C^Y_x f](z) = 0 & z \notin [x,y]
\end{cases}
\]

*** ***
Let us consider the minimal operator $T_0, r, p, q$ on $[a, \infty)$, with $1 \leq p < \infty$, $1 \leq q < \infty$, and abbreviate this operator by $T_0$.

1.5 Lemma: If $x$ and $y$ are finite, $C_{x}^{y} T_{0}^{-1}$ is a compact operator from $L^q[a, \infty)$ into $L^p[a, \infty)$.

Proof: $C_{x}^{y} T_{0}^{-1}(g) = C_{x}^{y} f$, where $f = T_{0}^{-1}(g)$. $[C_{x}^{y} f](z) = \int_{a}^{z} \min(z, y) a_{n}^{-1}(T) g(T) dT$ if $z \in [x, y]$. Otherwise $[C_{x}^{y} f](z) = 0$.

Thus $[C_{x}^{y} f](z) = \int_{a}^{\infty} K(z, t) g(t) dt$ for $z \in [x, \infty]$ where

\[
\begin{align*}
K(z, t) &= 0 \quad \text{when } z \notin [a, y] \\
K(z, t) &= 0 \quad \text{when } z < t \\
K(z, t) &= \min(z, t) a_{n}^{-1}(t) \quad \text{elsewhere}
\end{align*}
\]

and $[C_{x}^{y} f]z = 0$ for $z \in [a, x]$.

Consider the operator $T'$ defined by $[T'g](z) = \int_{a}^{\infty} K(z, t) g(t) dt$ where $g \in L^q[a, \infty)$, $T' g \in L^p[a, \infty)$.

Using theorem 1.2, we see that $T'$ is compact from $L^q$ into $L^p$.

1.6 Theorem: $T_{0}^{-1}$ is compact (defined on range $T_{0}$) if and only if the sequence $T_n = C_{a}^{n} T_{0}^{-1}$ converges to $T_{0}^{-1}$ in the normed operator topology ( $||T|| = \sup ||Tf||$).

Proof: First, we note that for each $g$ in range $T_0$, $T_n g \rightarrow g$ in $L^q[a, \infty)$. So $T_n \rightarrow T_{0}^{-1}$ pointwise always, whether the convergence holds in operator norm or not.

If $T_n \rightarrow T_{0}^{-1}$ in operator norm, it is clear that $T_{0}^{-1}$ is
compact since the compact operators are a closed set under this norm. (See chapter III, Goldberg for all necessary information on compact operators.)

Conversely, suppose $T_0^{-1}$ is compact. Then, if $B$ is the unit ball in $L^q[a,\infty)$, $T_0^{-1}B$ is compact in $L^p[a,\infty)$. Therefore given $\epsilon > 0$, there is a finite set $\{f_1, \ldots, f_n\}$ in $T_0^{-1}(B)$ such that for every $f$ in $T_0^{-1}B$, there is a $j$ such that $\|f_j - f\|_p < \epsilon$. Pick $N_1$ so large that for each $i$,

(i = 1 \cdots n), $\|c_a^n f_i - f_i\| \leq \epsilon$ for all $n > N_1$. We may do this since $\lim_{n \to \infty} c_a^n f_i = f_i$ in $p$ norm for each $i$. (Note that this would not be true if $p = \infty$.)

Now, if $f \in T_0^{-1}(B)$, $\|c_a^n f - f\|_p \leq \|c_a^n f_i - f_i\|_p + \|f_i - f\|_p \leq 2\epsilon$ for all $n > N_1$. For $g \in B \cap R(T_0)$, we have $\|T_0^{-1}(g) - T_n^{-1}(g)\|_p = \|c_a^n T_0^{-1}(g) - T_0^{-1}(g)\|_p \leq 2\epsilon$. So $T_n$ converges to $T_0^{-1}$ in operator norm.

***

Remark: The following theorem is really a special case of the well known theorem: "Finite dimensional extensions of compact operators are compact."

1.7 Theorem: If $T_0^{-1}$, $T, p, q$ is compact, where as usual $1 \leq p < \infty$, $1 \leq q < \infty$, then every extension $T$ of this operator to an operator from $L^q[a,\infty)$ into $L^p[a,\infty)$ is compact. [We do not necessarily assume $D(T)$ is all of $L^q[a,\infty)$. When we say $T$ is compact from $L^q[a,\infty)$, we mean $T$ is a compact operator from its domain $D(T)$ (which is
necessarily closed) into \( L^p[a, \infty) \), where \( D(T) \) has the \( L^q[a, \infty) \) topology.]

Proof: Since \( T_0 = T \), \( \tau, p, q \) has the property that \( T_0^{-1} \)
is compact, then \( T_0^{-1} \) is continuous and thus range \( T_0 = D(T_0^{-1}) \) is closed, due to the fact that \( T_0 \) is closed.
But \( T_0 = T_0^{-1} \tau, q', p' \), where \( T_0 \) means the adjoint of \( T_0 \). (This is theorem VI.1.9, Goldberg, page 130.) Therefore the kernel of \( T_0 \) is finite dimensional. But \( \overline{\text{range } T_0} = 1 \{ \ker T_0 \} \) (Goldberg (1), page 59). Also if \( N \) is finite dimensional and \( X \) is any linear space containing \( N \), \( \dim \frac{X}{\perp N} \leq \dim N \).
(Goldberg (1), page 59, equation 1.) Therefore the deficiency of the range \( T_0 \) in \( L^q[a, \infty) \) is finite, since \( \overline{\text{range } T_0} = \text{range } T_0 \). Therefore domain \( T \) is equal to \( [\text{domain } T_0^{-1}] \oplus Q \), where \( Q \) is a finite dimensional subspace of \( L^q[a, \infty) \). By theorem II.1.114, page 48, Goldberg, we see that \( Q \), being finite dimensional, admits a projection, so there is a positive real number \( C \) such that \( \| x + y \| \geq C \| x \| \) for \( x \in \text{range } T_0 \), \( y \in Q \). Therefore if \( B \) is the unit ball in \( L^q[a, \infty) \), the set of all \( x \) such that \( x \in \text{range } T_0 \) and \( x + y \in B \) has the property that \( \| x \|_p \leq \frac{1}{C} \).

Now let \( \{z_n\} = \{x_n + y_n\} \) be a sequence in \( B \cap D(T) \), where \( x_n \in \text{range } T_0 \), \( y_n \in Q \). \( T(x_n + y_n) = T(x_n) + T(y_n) \) \( = T_0^{-1}x_n + T_0 y_n \). By compactness of \( T_0^{-1} \), \( \{T_0^{-1}x_n\} \) has a convergent subsequence, since \( \{x_n\} \) is bounded (by the preceding paragraph). Call this subsequence \( \{T_0^{-1}(x_{n_k})\} \). Now
similarly \( \{y_{n_k}\} \) are bounded, and since any operator on a
finite dimensional space is compact, \( \{Ty_{n_k}\} \) have a
convergent subsequence, \( \{Ty_{n_{k_j}}\} \). Then \( \{Tz_{n_{k_j}}\} \) converge.

***

1.8 Lemma: Let \( T_1 \) and \( T_2 \) be operators from a Banach space
\( X \) to a Banach space \( Y \) such that \( T_2 \) is continuous and defined
on all of \( X \), and \( T_1 \) is closed. Then \( T_1 + T_2 \) is closed.

Proof: We show \( \text{graph} \ (T_1 + T_2) = \text{graph} \ (T_1 + T_2) \). Let
\((x_n, y_n) \in \text{graph} \ (T_1 + T_2), (x_n, y_n) \rightarrow (x, y). \) (i.e.
\( x_n \rightarrow x, y_n \rightarrow y \)). Then \( y_n = T_1x_n + T_2x_n \). So
\( y_n - T_2x_n = T_1x_n \). Now since \( T_2 \) is continuous, \( T_2x_n \) converges
to \( T_2x \). Therefore, since \( y_n \) converges, \( T_1x_n \) converges to
\( y - T_2x \). Therefore \((x_n, T_1x_n) \in \text{graph} \ T_1 \) and \( x_n \) and \( T_1x_n \)
converge. Since \( T_1 \) is closed, \((x, T_1x) \in \text{graph} \ T_1 \). Therefore
\( x \in \text{domain} \ T_1 + T_2 \), and \( (T_1 + T_2)x = y - T_2x + T_2x = y \).

***

1.9 Theorem: If \( T_0 \), \( T_1, p, q \) is compact, then for any bounded
operator \( B \), and any extension \( T \) of \( T_0, T_1, p, q \) such that
\( T \subseteq T_1, T, p, q \), \( T + B \) has closed range.

Proof: Any finite dimensional extension of a closed
operator is closed, so \( T \) is closed. Therefore \( T + B \) is
closed, by lemma 1.8. We show that the range of \( T + B \) is
closed.

Suppose it is not closed. Then by corollary III.1.10,
page 81, Goldberg (1), there is an infinite dimensional
closed subspace $S$ in $D(T + B)$ such that $(T + B)$ restricted to $S$ is compact. Let $\{y_n\}$ be a sequence in $S$ such that 

$$||y_n - y_j|| \geq \varepsilon$$

for $j \neq n$ and some fixed $\varepsilon > 0$, and 

$$||y_i|| = 1$$

for every $i$. $\{(T + B)y_n\}$ is a bounded set, since $T + B$ is bounded (because compact) on $S$.

Now $D(T) = D(T_0) \oplus \ker T \oplus Q$, where $\ker T = \{f | Tf = 0\}$. (This is true because $\ker T$ is linearly independent modulo $D(T_0)$, and $T$ is a finite dimensional extension of $T_0$.)

Now let $y_n = r_n + s_n + v_n$, $r_n \in D(T_0)$, $s_n \in \ker T$, $v_n \in Q$. Since $(T + B)y_n$ is a bounded set, and $B y_n$ is a bounded set, we have $T y_n$ is a bounded set. Therefore $\{T_0 r_n\}$ is a bounded set, because range $T_0$ admits a projection, since it is closed and of finite deficiency. Therefore $\{r_n\}$ have a convergent subsequence, because $T_0^{-1}$ is compact. But $T$ restricted to $Q$ is a bounded 1-1 linear operator onto $T(Q)$, and since every operator from a finite dimensional space to another is compact, and since $T_0 v_n$ is bounded because $T(Q)$ admits a projection, we have in particular that $\{v_{n_k}\}$ has a convergent subsequence. Call it $\{v_{n_k}\}$. $\{r_{n_k}\}$ is a subsequence of $\{r_n\}$ and so is convergent. Now

$$y_w = r_w + v_w + s_w.$$ But $\{r_w\}$ and $\{v_w\}$ are bounded, so $\{s_w\}$ is bounded. Therefore it has a convergent subsequence, $\{s_{w_i}\}$.

Now $\{y_{w_i}\} = \{r_{w_i} + s_{w_i} + v_{w_i}\}$ is a convergent subsequence of $\{y_n\}$. This contradicts the assumption that $||y_n - y_j|| \geq \varepsilon$ for $n \neq j$.

***

***
1.10 Definition: Let $T$ be a linear operator with domain and range contained in a normed space $X$. The essential spectrum of $T$, written $\sigma_0(T)$, is defined by $\sigma_0(T) = \{ \lambda | R(\lambda I - T) \text{ is not closed} \}$.

Note: If $T - \lambda I$ is 1-1 and closed and $X$ is complete, and $\lambda \notin \sigma_0(T)$, the Banach open mapping theorem says $(T - \lambda I)^{-1}$ is continuous.

1.11 Corollary: If $T_0, \mathcal{T}, p, p'$ is compact, where $1 \leq p < \infty$, and $T \supseteq T_0, \mathcal{T}, p, p'$, and $B$ is bounded linear operator from $L^p[a, \infty)$ to $L^p[a, \infty)$, then $\sigma_0(T + B)$ is empty.

Proof: Let $B_1 = B - \lambda I$, and use theorem 1.8.

***

1.12 Theorem: Let $T$ be a self adjoint operator on a Hilbert space. Then the essential spectrum $\sigma_0(T)$ is the set of non isolated points of the spectrum of $T$.

Proof: This is theorem XIII.6.5, Dunford and Schwartz (2), page 1395. (It should be noted that when $\mathcal{T}$ is classically self adjoin, and we are working in $L^2[a, \infty)$, most of the previous theorems can be extracted from section XIII.6, and XIII.7 of Dunford and Schwartz (2).)

1.13 Corollary: If $T$ is as in theorem 1.12, and $\sigma_0(T)$ is empty, then every point of the spectrum of $T$ is an eigenvalue of $T$ corresponding to some eigenvector.

***

Theorem 1.9 has a partial converse. First, we need a few lemmas.

Note: This lemma is deducible from the proof of XIII.6.28,
1.14 Lemma: Let $\mathcal{T}$ be a formal differential operator, given by $\mathcal{T} = \sum_{i=0}^{n} a_i D^i$. Define $\mathcal{T} + f$ to be the differential expression $\sum_{i=1}^{n} a_i D^i + (a_0 + f)$. Let $f$ be continuous and bounded. Then $\mathcal{T}_0, \mathcal{T} + f, p, p = T_0, \mathcal{T}, p, p + B_f$, where $B_f$ is the bounded linear transformation defined by $B_f(g) = f \cdot g$. Also $\mathcal{T}_0, \mathcal{T} + f, p, p = T, \mathcal{T}, p, p + B_f$.

Proof: To prove $\mathcal{T}_0, \mathcal{T} + f, p, p = T_0, \mathcal{T}, p, p + B_f$, we need only to show the domains are equal. The same is true for $\mathcal{T}_0, \mathcal{T} + f, p, p = T, \mathcal{T}, p, p + B_f$.

In the case of $\mathcal{T}_0, \mathcal{T} + f, p, p = T_0, \mathcal{T}, p, p + B_f$, $g \in D(\mathcal{T}_0, p, p)$ if $g$ is $n - 1$ times differentiable and $D^{n-1}$ is absolutely continuous, $g \in L^p[a, \infty)$, $\mathcal{T} g \in L^p[a, \infty)$. But $\mathcal{T} g \in L^p[a, \infty), \mathcal{T} g + fg \in L^p[a, \infty)$, since $fg \in L^p[a, \infty)$. Thus the domains are equal.

In the case of $\mathcal{T}_0, \mathcal{T} + f, p, p = T, \mathcal{T}, p, p + B_f$, $g \in D(\mathcal{T}_0, p, p)$ and there is a sequence $\{g_n\}$ of compact support functions in $(a, \infty)$ which are in $D(\mathcal{T}, p, p)$ and have the property that $g_n \rightarrow g$ and $\mathcal{T} g_n$ converges, both in $L^p[a, \infty)$. In this case, $(\mathcal{T} + f)g_n$ also converges, so $g \in D(\mathcal{T}_0, \mathcal{T} + f, p, p)$. The converse is proved the same way.

1.15 Lemma: Let $\mathcal{T}, p, p \supset T, p \supset T_0, \mathcal{T}, p, p$ (where $1 \leq p < \infty$). Then $\delta_\epsilon(T) = \delta_\epsilon(T_0)$.

Proof: If $\lambda \in \delta_\epsilon(T - \lambda I)$, $\lambda \in \delta_\epsilon(T_0)$. For if not, $T_0 - \lambda I$ has closed range. Therefore $T_0, \mathcal{T} - \lambda, p, p$ has closed range. But $T, \mathcal{T}, p, p - \lambda I = T - \lambda I \supset T - \lambda I \supset T_0 - \lambda I$.
= \text{T_0, T - \lambda, p, p}. Therefore, since range T - \lambda, p, p is a finite dimensional extension of range T_0, T - \lambda, p, p', we have by theorem 0.13 that range T - \lambda I is a finite dimensional extension of range T_0 - \lambda I and is therefore closed by theorem 0.14.

Conversely, if \lambda \in \sigma_e(T_0 - \lambda I), suppose
\lambda \notin \sigma_e(T - \lambda I). Then, since range T_0 - \lambda I has finite deficiency in Lp[a, \infty), (see theorem IV.2.3, Goldberg, coupled with the fact that (T_0)' = T^{-1, p, p}), we have that range T - \lambda I has finite deficiency in Lp[a, \infty). Therefore range T - \lambda, p, p has finite deficiency in Lp[a, \infty), and so is closed by theorem IV.1.12, Goldberg, page 100. This contradicts the fact that range T_0, T - \lambda, p, p is not closed, using theorem VI.2.7, Goldberg, page 137. (Also found in Rota(1).)

Note: A minimal operator has a continuous inverse iff its range is closed, by the Banach open mapping theorem. Thus \lambda \notin \sigma_e(T_0, T, p, p) iff T_0, T, p, p - \lambda I has continuous inverse on its range.

The following is a partial converse of 1.8

1.16 Theorem: Suppose \sigma_e(T_0, T, 2, 2) is empty, where T is classically self adjoint. Then T^{-1}_0, T, 2, 2 is compact.

Proof: Let T be a self adjoint extension of T_0, T, 2, 2' such that T \subseteq T, T, 2, 2' (There is one by theorem XIII.6.10, Dunford and Schwartz, page 1400.) By 1.15, \sigma_e(T) is empty. By 1.12, the spectrum of T consists only of isolated points.
Therefore the spectral decomposition of $T$ is $T = \sum \lambda_k p_k$, where $\{\lambda_k\}$ do not cluster and $p_k$ is the projection on the space of eigenvectors corresponding to the eigenvalue $\lambda_k$.

If $f \in D(T)$, $\|Tf\| = \sqrt{\sum \lambda_k^2 (p_k f, f)}$.

Now since $T - \lambda I \subseteq T, 2, 2 = \lambda I = T - \lambda, 2, 2$ by $l, l+$, we have $\ker (T - \lambda I)$ is finite dimensional for every $\lambda$.

Therefore $p_k$ is a projection on a finite dimensional subspace for every $k$.

Suppose $T^{-1}, 2, 2$ is not compact. Then there is a set

$\{f_n\} \subset D(T, 2, 2) \text{ with } \|T_0 f_n\| \leq 1$, and with $\|f_n - f_m\| \geq \varepsilon$ for $n \neq m$. ($\varepsilon$ is fixed.) This is true because $T_0^{-1} B$ has no finite $\varepsilon$ net for some $\varepsilon$, where $B$ is the unit ball.

Now for every integer $N$, $\sum_{k=1}^{N} p_k$ is a projection on a finite dimensional subspace, since, if $k \neq m$, $p_k \perp p_m$.

For every $N$, there is an $n_1$ such that $n > n_1$

$\Rightarrow \| \sum_{k=1}^{N} p_k f_n - f_n \| > \frac{\varepsilon}{4}$. This is true because $\{f_n\}$ is a bounded set since $T_0^{-1}, 2, 2$ is continuous, and our statement were false, then there would be a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $\| \sum_{k=1}^{N} p_k f_{n_j} - f_{n_j} \| \leq \frac{\varepsilon}{4}$. However $\sum_{k=1}^{N} p_k f_{n_j} = s_j$ is a bounded sequence in a finite dimensional space. Therefore there are two terms, $s_\ell$ and $s_m$, such that $\|s_\ell - s_m\| < \frac{\varepsilon}{4}$.

We then have $\|f_{n_\ell} - f_{n_m}\| \leq \|f_{n_\ell} - s_\ell\| + \|s_\ell - s_m\| + \|s_m - f_{n_m}\| \leq \frac{3\varepsilon}{4}$. This is a contradiction of the choice
of \{f_n\}, and establishes the statement.

Pick \( N \) so large that \( k > N \implies \left| \lambda_k \right| > \frac{8}{\varepsilon} \). (This is possible since \( T \) is unbounded.) Pick \( n_1 \) such that
\[ n > n_1 \implies \left\| \sum_{k=1}^{N} p_k f_n - f_n \right\| > \frac{\varepsilon}{4} \cdot \left\| T f_n \right\|^2 \]
\[ \leq \sum_{k=N+1}^{\infty} \left( p_k f_n, p_k f_n \right) \geq \frac{64}{\varepsilon^2} \sum_{k=N+1}^{\infty} (p_k f_n, p_k f_n) \]
\[ = \frac{64}{\varepsilon^2} \left\| \sum_{k=1}^{N} p_k f_n - f_n \right\|^2 \cdot (\sum_{k=1}^{\infty} p_k f_n = f_n). \]

Therefore \( \left\| T f_n \right\|^2 \geq \frac{64}{\varepsilon^2} \cdot \frac{\varepsilon^2}{16} = 4 \). Therefore \( \left\| T f_n \right\| > 2 \).

This contradicts the choice of \( \{f_n\} \).

---

Remark: There is a good deal of literature about when the essential spectrum of \( T_0, \tau, 2, 2 \) is empty. (See chapter XIII.7, Dunford and Schwartz (2), for example.) Thus we have a large class of non-trivial examples where theorem 1.16 holds.

1.17 Theorem: If \( T_0, \tau, p, p \) has compact inverse, then \( T_0, \tau, p, p + \lambda I \) has compact inverse for each \( \lambda \in \mathbb{C} \).

Proof: First, \( T_0, \tau, p, p + \lambda I = T_0, \tau + \lambda, p, p \) and is therefore 1-1. Also, we know that range \( T_0, \tau, p, p - \lambda I \) is closed, so that \( (T_0 - \lambda I)^{-1} \) is continuous. (See theorem 1.9.)

If \( T_0^{-1} \) is not compact, then there is a sequence \( x_n \) in range \( T_0 - \lambda I \) such that \( \left\| x_n \right\| \leq 1 \), \( x_n = (T_0 - \lambda I)y_n \), and \( \left\| y_n - y_k \right\| \geq \varepsilon \) for \( n \neq k \). But \( \{y_n\} \subseteq D(T_0) \), so that by compactness of \( T_0^{-1} \) we have \( \left\| T_0 y_n \right\| \not\to \infty \). However, since
\{x_n\} are in the unit ball of \(L^p[a,\infty)\), we see by continuity of \(T_0^{-1}\) that \(\{y_n\}\) are uniformly bounded. i.e. There is a \(k > 0\) such that \(\|y_n\| \leq k\) for every \(n\). Therefore \(\|\lambda y_n\| \leq k|\lambda|\) for every \(n\). So that since \(\|(T_0 - \lambda I)y_n\| \geq |\|T_0y_n\| - \|y_n\||\), we have \(\|(T_0 - \lambda I)y_n\| \to 0\) as \(n \to \infty\).

However we assumed that \(\|(T_0 - \lambda I)y_n\| = 1\) for each \(n\).

This contradiction establishes the theorem.

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1.16 Theorem: Let \(T_0, T, p, q\) be a minimal operator from \(L^p[a,\infty)\) to \(L^q[a,\infty)\), with \(1 < p < \infty\), \(1 < q < \infty\). Let \(T\) be a restriction of \(T_0, T, p, q\) such that range \(T\) is closed, and \(D(T) \subseteq L^1[a,\infty) \cap L^\infty[a,\infty)\). Then \(T^{-1}\) is a compact operator from its range into \(L^p[a,\infty)\), where range \(T\) has the topology of \(L^q[a,\infty)\).

Proof: Let \(T_\infty\) be the operator from domain \(T\) with \(L^\infty\) topology to range \(T\) with the \(L^q\) topology, such that \(T_\infty f = T f\). In other words, \(T_\infty\) is algebraically \(T\), but not topologically. We claim that \(T_\infty\) is closed.

Proof of claim: The proof is in several steps. First, range \(T\) is closed, and \(T_\infty\) is 1-1 since \(T_\infty \subseteq T_0, T, p, q\), so \(T = T_\infty\) and thus \(T\) is closed from \(L^p[a,\infty)\) to \(L^q[a,\infty)\).

Second, if \(\{x_n\} \subseteq D(T_\infty)\) and \(\|x_n - x\|_\infty \to 0\), \(\|T_\infty x_n - y\|_q \to 0\), then by the closed graph theorem \(\|T^{-1}(T_\infty x_n) - T^{-1}(y)\| \to 0\). We need to show that \(x = T^{-1}(y)\). But \(T^{-1}(T_\infty x_n) = x_n\), and since \(\|T^{-1}(T_\infty x_n) - T^{-1}(y)\| \to 0\), we have \(x_n \to T^{-1}(y)\) pointwise a.e. on \([a,\infty)\). But by assumption, \(\{x_n\}\) converge uniformly to \(f\) on
\[ f = T^{-1}(y) \text{ a.e. But both functions are continuous, so the equality holds everywhere. Therefore } T_\infty \text{ is closed.} \]

Now we define \( T_1 \) to be an operator from \( D(T) \) with the \( L_1 \) topology to range \( T \) with the \( L_q \) topology by \( T_1(f) = T(f) \). We claim \( T_1 \) is closed.

In fact, suppose \( \{x_n\} \subset D(T_1) \), and \( \|x_n - x\|_1 \to 0 \), and suppose \( \|T_1 x_n - y\|_q = 0 \). Then, since \( T_\infty \) is a closed operator with closed range, \( T^{-1}_\infty \) is continuous and so \( \|T^{-1}_\infty (x_n - y)\|_\infty \to 0 \). Now there is a subsequence \( x_{nk} \) such that \( x_{nk} \to x \text{ a.e.} \) But \( \{x_{nk}\} \to T^{-1}_\infty (y) \) uniformly, so \( x = T^{-1}_\infty y \) and \( x \in D(T_\infty) = D(T_1) \). So \( T_1 \) is closed. By the closed graph theorem, \( T^{-1}_1 \) is bounded.

We now come to the main part of the proof. It can best be motivated by a few pictures:

![Graphs of functions](image)

Consider the sequence \( f_1, f_2, f_3, \text{ etc.} \) illustrated above. It will be weakly convergent to \( 0 \) in \( L_p[a, \infty) \) for \( p > 1 \), because everything in the dual of \( L_p \) "gets small" outside compact sets. However, since the dual of \( L_1 \) is \( L_\infty \), a
sequence in $L_1$ which looks like the above will not be weakly convergent to 0, intuitively because $L_\infty$ functions do not have to get small outside compact sets. By theorem 1.6, if we have a sequence $\{x_n\}$ such that $\{x_n\}$ has no cluster points in $L_p$ and $\|Tx_n\|_Q = 1$, the $x_n$ look somewhat like the $f_n$ in the picture. (At least they do not get small outside compact sets.) However we shall show that the $x_n$ have an $L_1$-weakly convergent subsequence, so that it has to get small outside compact sets. This will be a contradiction. We use the hypothesis about $L_\infty$ to show that if the subsequence gets small in $L_1$ norm outside compact sets, then it gets small in $L_p$ norm outside compact sets. With this somewhat shaky navigational guide, we attempt the proof.

Suppose $T^{-1}$ is not compact. Since $C_0^{N_1}T^{-1}$ is compact, for each $N$ (since it is a restriction of $[C_0^{N_1}T^{-1}]$), this means that $C_0^{N_1}T^{-1}$ does not approach $T^{-1}$ in operator norm as $N \to \infty$. ($\{N\}$ contained in the positive integers.) Therefore, there is a set $\{y_n\}$ such that $\|y_n\|_q = 1$ for each $n$, and $\|C_0^{N_1}T^{-1}y_n - T^{-1}y_n\|_p > \varepsilon$. For some fixed $\varepsilon$.

Now range $T$ is a reflexive Banach space since it is a closed subspace of the reflexive Banach space $L_q[a,\infty)$. (See theorem I.6.12, Goldberg (1), page 29.) Therefore $\{y_n\}$ has a subsequence $\{y_{nk}\} \to y$ weakly, $y \in$ range $T$. Note that $T^{-1}_1$ is continuous from $L_q[a,\infty)$ to $L_1[a,\infty)$, and therefore is weakly continuous and thus takes weakly convergent sequences onto weakly convergent sequences. Therefore $T^{-1}_1(y_{nk}) \to T^{-1}_1(y)$ weakly in $L_1[a,\infty)$. Therefore
\{ |T^{-1}_1(y_{nk})| \} \cup |T^{-1}_1(y)| is weakly sequentially compact in 
$L_1[a,\infty)$. ( | | indicates absolute value, and the statement 
is deduced from corollary IV.8.10, page 293, Dunford and 
Schwartz (1).) Therefore, given \( \varepsilon_1 > 0 \), there is a positive 
integer \( N \) such that \( \int_0^\infty |[T^{-1}_1 y_{nk}](x)| dx < \varepsilon_1 \), by theorem 
IV.8.9, Dunford and Schwartz (1), page 292. (Taking the sets 
\( E_n \) at the bottom of page 292 to be \([n,\infty)\).) Now 
\[ |[T^{-1}_\infty (y_{nk})](x)| < B \] for every \( x \in [a,\infty) \) and every \( y_{nk} \), due 
to the fact that \( T^{-1}_\infty \) is continuous from \( L_q \) to \( L_\infty \). (Note that 
since \( y_{nk} \) is continuous, \( \| y_{nk} \| = \sup_{x \in [a,\infty)} |y_{nk}(x)| \).) 
Therefore \( \int_0^\infty |T^{-1}_n (y_{nk})|^p dx \leq B^{p-1/\infty} \int_0^\infty \| T^{-1}_1 (y_{nk}) \| dx \). Let 
\( \varepsilon_1 = \frac{\varepsilon p^{1/p}}{4^{1/p}} \), where \( \varepsilon \) is the number such that 
\( \| C_{\infty} T^{-1}_n (y_{nk}) - T^{-1}_n (y_n) \|_p > \varepsilon \). Then let \( N \) correspond to \( \varepsilon_1 \) as 
above. Let \( n_k > N \). Then \( \| C_{\infty} T^{-1}_n (y_{nk}) - T^{-1}_n (y_{nk}) \|_p \)
\[ = \int_0^\infty |[T^{-1}_n (y_{nk})](x)|^p dx \leq \frac{\varepsilon}{4^p} \]. This is a contradiction.

***

We note that with \( p = 1 \), the hypothesis about \( L_\infty \) is
unnecessary for the above argument.

Note also that for \( f \in D(T_0, \tau, p, q) \), \( f' \in L_1[a,\infty) \)
implies \( f \in L_\infty[a,\infty) \).

1.19 Corollary \( T^{-1}_0, \tau, p, q \) is compact if and only if there is 
a sequence of restrictions \( T_n \) of \( T_0, \tau, p, q \) (\( 1 \leq p < \infty \), 
\( 1 < q < \infty \)) such that \( \| \) for each \( n \),
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D(T_n) \subset L_1(a,\omega) \cap L_\infty(a,\omega), \quad \text{for each } n, \text{ range } T_n \text{ is closed.} \quad \text{(1)}

If \epsilon > 0, then for some \(N\), there is a bounded set \(S_N\) in range \(T_N\) such that if \(f \in D(T_0, T, p, q)\),

\[ ||Tf||_q = 1, \text{ then } d(f, T_N^{-1}S_N) \leq \epsilon. \text{ (i.e. } \text{glb}\{||f - g||\} \leq \epsilon, \text{ where } g \text{ ranges over } T_N^{-1}S_N.\)

Proof: Suppose \(T_0^{-1}, T, p, q\) is not compact. Then there is a set \(\{x_j\} \in D(T_0, T, p, q)\) such that \(||T_0, T, p, q(x_j)|| = 1\) for each \(n\), and \(||x_j - x_k|| \geq \epsilon_1\) for some \(\epsilon_1 > 0\). Let \(\epsilon = \frac{\epsilon_1}{3}\), and pick \(n\) as guaranteed in \(\text{(3)}\). Pick \(y_j\) such that

\[ y_j \in T_n^{-1}S_n, \quad ||y_j - x_j|| < \frac{\epsilon_1}{4}. \]

Now \(\{x_0\}\) have a cluster point. So we can find \(j_1\) and \(j_2\) such that \(||y_{j_1} - y_{j_2}|| \leq \frac{\epsilon_1}{4} \).

Then \(||x_{j_1} - x_{j_2}|| \leq ||x_{j_1} - y_{j_1}|| + ||y_{j_1} - y_{j_2}|| \leq \frac{\epsilon_1}{4} + \frac{\epsilon_1}{4}.\)

This is a contradiction of the choice of \(x_n\).

Conversely, if \(T_0^{-1}, T, p, q\) is compact, then consider \(\{x\}\) such that \(||T_0^{-1}, T, p, q(x)|| = 1\) and \(x\) has compact support in \((a,\omega)\). This set has compact closure, so there is a finite \(\epsilon\) net in the set for every \(\epsilon > 0\). Given \(\epsilon\), let the \(\epsilon\) net be \(\{x_i\}_{i=1}^n\). Pick an interval \(J\) such that support \(x_i \subset J\) for every \(i\). Define our sequence \(B_n = T_0, T, p, q\) restricted to \(\{f | \text{support } f \subset (a,\omega)\}\). Let \(T_n = \overline{B_n}\). Clearly range \(T_n\) is closed. Now suppose \((f_1, T_{f_1}) \in \text{graph } \overline{B_n}\). Then support \(f_1 \subset (a,\omega)\). This is proved as follows. If \((f_1, T_{f_1})\) is in \text{graph } \overline{B_n}, there is a sequence \(\{g_i, Tg_i\}\) such that \(g_i \to f_1, \quad Tg_i \to T_{f_1}\), and \(g_i \in D(B_n)\). Thus support \(f_1\) is
contained in \((a,n]\), since \(f^1\) is continuous and vanishes almost everywhere outside \((a,n]\). (If it didn't, 
\[\|f^1 - f^i\| \to 0.\]) So since \(D(T_n)\) consists of a set of continuous functions on \([a,N]\), \(D(T_n) \subseteq L_1 \cap L_\infty\). Now pick \(N\) such that \((a,N] \supseteq J\). Then each \(\{x_i\}^n_{i=1}\) is in \(D(T_n)\). Thus if \(x\) is of compact support, and \(\|T_0, \tau, p, q x\| = 1, \exists x_i \in D(T_n)\) such that \(\|x_i - x\| < \varepsilon, \) and \(\|T x_i\|_p = 1.\)

Thus, if \(f\) is any element of \(T_0, \tau, p, q\), with 
\[\|T_0, \tau, p, q f\|_q = 1,\] then there is an \(i\) such that 
\[\|x_i - f\|_p < 2\varepsilon.\]

***

**1.20 Corollary:** Suppose \(\delta_e(T_0, \tau, p, p)\) is not the complete set of complex numbers and suppose \(D(T_0, \tau, p, p) \subset L_1(a,\infty) \cap L_\infty(a,\infty)\), where \(1 < p < \infty\). Then \(T_0, \tau, p, p\) is compact.

Proof: \(D(T_0, \tau, p, p - \lambda I) = D(T_0, \tau, p, p)\). However, if \(\lambda \notin \delta_p(T_0, \tau, p, p)\), then \(T_0, \tau, p, p - \lambda I = T_0, \tau - \lambda, p, p\) has closed range and thus we use theorem 1.18 to get \(T_0, \tau, p, p - \lambda I,\) and Thus \(T_0, \tau, p, p\), compact.

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Note: From the construction of the inverse of \(T_0, \tau, p, p\), the conditions of corollary 1.20 will hold if \(\frac{1}{a_n} \in L_\infty,\) where 
\[\tau = \sum_{i=1}^n a_i D^i,\] and also (1) each solution of \(\tau f = 0\) is in \(L_1(a,\infty) \cap L_\infty(a,\infty),\) and (2) each solution of \(\tau^* f = 0\) is in \(L^{1/p'}(a,\infty),\) where \(\frac{1}{p} + \frac{1}{p'} = 1.\) Compactness has already been noticed (theorem 1.6) in this case, however.
The next theorem follows from the proofs above. The proof is relatively easy, using the Dunford-Pettis theorem. In our context, we need only the special cases above. In the theorem, let $S$ be a locally compact Hausdorff space and let $\mu$ be a measure generated by a positive linear functional on the set of continuous compact support functions in $S$.

**1.22 Theorem:** Suppose $T$ is a map from a reflexive Banach space $B$ to a space $R$ of bounded continuous functions on $S$. Suppose $R \subseteq L^1(S, \mu)$. Suppose finally that $T$ is continuous from $B$ to $R_p$, where $R_p$ is $R$ with the $L^p$ topology, for some $1 \leq p \leq \infty$. Then $T$ is compact from $B$ to $R_q$ where $q$ is any number $1 \leq q < \infty$.

**Proof:** Let us pick any compact set $k \subseteq S$, and show $C_k T$ is compact, where $C_k f = f|k$. (i.e. $[C_k f](x) = f(x)$ if $x \in k$, $[C_k f](x) = 0$ if $x \notin k$.)

Let $\{y_n\} \subseteq B$ with $\|y_n\| = 1$ for each $n$. We show that $\{C_k T(y_n)\}$ has a cluster point in $L^q(S, \mu)$. To do this we will need to develop a few facts.

First, letting $R_\infty$ be $R$ with the $L_\infty$ topology, we note that $T$ is a closed map from $B$ to $R_\infty$. In fact, if $g_n \to g$ in $B$ and $\|T(g_n) - h\|_\infty \to 0$, where $h$ is a continuous bounded function on $S$, then $\|T(g_n) - T(g)\|_q \to 0$ by hypothesis, so as above $T(g) = h$. Thus $T$ is closed from $B$ to $R_\infty$ and therefore continuous.

The same argument as before shows $T$ is closed from $B$ to $R_1$ and therefore continuous from $B$ to $R_1$.

Now $C_k T$ is a continuous map from $B$ into $R|k$ where $R|k$
has the sup norm. It is thus compact to \( R^{|k([R|k]} \) with
\( L^q \) norm) by several theorems (ex. 56, page 519, Dunford and
Schwartz (1)), but we include a proof.

Since \( C_k \) is weakly continuous, then it takes weakly
convergent sequences to weakly convergent sequences. Now
\( \{y_n\} \) chosen above has a weakly convergent subsequence \( \{y_{nk}\} \).
Thus \( \{Ty_{nk}\} \) has a weakly convergent subsequence in \( R^\infty \).
Define \( F_x(f) = f(x) \), for \( x \in S \) and \( f \in R \). \( F_x \) is a
continuous linear functional on \( R^\infty \). Thus \( [Ty_{nk}]_x(x) \to [Ty]_x(x) \)
for every \( x \), if \( y_{nk} \xrightarrow{w} y \) in \( B \). Also, since \( T \) is bounded
from \( B \) to \( R^\infty \), \( \{y_{nk}\} \) are a uniformly bounded set of functions
in \( R \). Thus \( \{y_{nk}|k\} \) converge in \( L^q(S,\mu) \) by the Lebesgue
dominated convergence theorem. So \( C_kT \) is compact from \( B \) to
\( L^q(S,\mu) \).

However, \( \{Ty_{nk}\} \) is also weakly convergent in \( L^1(S,\mu) \).
Suppose that \( \{Ty_{nk}\} \) has no convergent subsequence in
\( L^q(S,\mu) \). Then there is an \( \varepsilon_1 > 0 \) such that \( \|Ty_{nk_1}\ |
- Ty_{nk_2} \|_q > \varepsilon_1 \) for \( k_1 \neq k_2 \). Pick \( k \) so large that
\[
\int_{S-k} \left| [Ty_{nk}](s) \right|^q d\mu(s) < \frac{\varepsilon_1^q}{4Q^{q-1}}
\]
where \( Q = \text{l.u.b.} \{\|Ty_{nk}\|_\infty \} \).
We can do this, as above in the proof of 1.18, by theorem
IV.8.11, Dunford and Schwartz (1), page 294. Now \( \{Ty_{nk}\} \)
restricted to \( k \) has a cauchy subsequence in \( L^q(S,\mu) \), and
\[
\int_{S-k} \left| [Ty_{nk}](s) \right|^q d\mu(s) \frac{1}{q} < \frac{\varepsilon}{4}. \]
This is a contraction, as above.

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II. Formal Differential Expressions

Unless we say otherwise, in this section we work with differential expressions $\tau$. We first discover several algebraic operations which preserve the compact inverse property, and use these to give large classes of examples.

2.0 Definition: Let $T_0, \tau, p, q$ be a minimal operator, $1 \leq p < \infty$, $1 \leq q < \infty$. We say $T_0, \tau, p, q$ is $n$-bounded if $f \in D(T_0, \tau, p, q)$ implies $\|D^N f\|_q < \infty$. (For this definition, $\tau$ is not necessarily of order $n$.)

2.1 Theorem: If $T_0, \tau, p, q$ is $N$ bounded and has closed range, then there is a $c > 0$ such that $\|T_0, \tau, p, q f\|_q \geq c \|D^N f\|_q$ for each $f \in D(T_0, \tau, p, q)$.

Proof: Consider the linear operator $Q$ which maps $T_0 f$ to $D^N f$. It is closed, for if $T_0 f_n \to g$ and $D^N f_n \to h$, then since $T_0^{-1}$ is continuous, $\{f_n\} \to f$ in $L^p[a, \infty)$, and thus $f \in D(T_0)$ and $T_0 f = g$. But since $D^N, f, q$ is closed, we have $D^N f = h$. Therefore $Q(g) = h$.

Furthermore, $D(Q) = \text{range } T_0$ and is closed in $L^q$. Therefore $Q$ is a continuous linear mapping by the closed graph theorem, and the theorem is proved.

***

The following theorem can be extracted from the proof of theorem VI.8.1, Goldberg (1), page 166.

2.2 Theorem: Let $T_0, \tau, p, p$ be $N$-bounded with closed range. Let $v = \sum b_i D^i$, where each $b_i$ is infinitely differentiable.
and \( \sup_{s \leq s < \infty} \int_{s}^{s+1} \left| b_k(t) \right|^p dt < \infty, \ 0 \leq k \leq N - 1 \). Let

\[ D(B) = \{ f \in L^p[a, \infty) \text{ and } \nu f \in L^p[a, \infty) \}. \]

If \( f \in D(B) \), let \( Bf = \nu f \). Then \( T_0, T + \nu, p, p = T_0, T, p, p + B \), and

\[ D(B) \supseteq D(T_0, T, p, p). \]

**Proof:** We note that by theorem VI.8.11, Goldberg, page 166, \( \nu \) is \( T \) bounded, and for any \( \varepsilon > 0 \) there is a \( k \) such that \( \| Bf \| \leq k \| f \| + \varepsilon \| T D^N, p, p \| f \| \). Thus

\[ D(B) \supseteq D(T_0, T, p, p). \]

Now if \( f \in D(T_0, T, p, p) \) then by theorem 2.1, \( \| T_0, T, p, p f \| \leq c \| D^N f \| \). Therefore if \( f \in D(T_0, T, p, p) \), \( \| Bf \| \leq k \| f \| + \frac{\varepsilon}{c} \| T_0, T, p, p f \| \). Therefore \( \| Bf \| \leq k \| f \| + \frac{\varepsilon}{c} (\| (T_0, T, p, p + B)f \| + \| Bf \|) \).

Pick \( \varepsilon \) so small that \( \frac{\varepsilon}{c} < 1 \). Then \( \| Bf \| (1 - \frac{\varepsilon}{c}) \leq k \| f \| + \frac{\varepsilon}{c} \| (T_0, T, p, p + B)f \| \), or

\[ \| Bf \| \leq \frac{k}{1 - \frac{\varepsilon}{c}} \| f \| + \frac{\varepsilon}{1 - \frac{\varepsilon}{c}} \| (T_0, T, p, p + B)f \|. \tag{1} \]

Now suppose \( \{f_n\} \) are of compact support, and \( \{f_n\} \) are Cauchy in \( L^p[a, \infty) \), and also \( (T + \nu)f_n \) are Cauchy. Then by inequality \( \tag{1} \), \( \{\nu f_n\} \) are a Cauchy sequence. Therefore \( \{Tf_n\} \) are Cauchy. Thus if \( f_n \rightarrow h \), then \( h \in D(T_0, T, p, p) \), and if \( \{\nu f_n\} \rightarrow g_1 \) and \( \{Tf_n\} \rightarrow g_2, T_0, T + \nu, p, p (h) = g_1 + g_2. \) Now since \( T_0, T, p, p \) is closed, \( T h = g_2. \) But also \( \nu h \) is \( g_1 \), because of the following two facts. First, \( h \in D(T^N, p, p) \), and since \( \{T_0, T, p, p f_n\} \) is Cauchy then
\{D^N f_n\} is Cauchy, so \(D^N f_n \rightarrow D^N h\) because \(D^N, p, p\) is closed.

(We don't know that \(B\) is closed.) Second, this fact implies by theorem VI.8.1 in Goldberg (1) that \(h \in D(B)\), and
\[
\|B(f_n - h)\| \leq k_1 \|f_n - h\| + \varepsilon \|D^N f - h\|,
\]
so \(Bf_n \rightarrow B(h)\).

So \(h \in D(T_0, \tau, p, p + B)\), and \(T_0, \tau, p, p + B\) is closed. Thus we have showed that \(T_0, \tau + v, p, p\)
\[
\subseteq T_0, \tau, p, p + B.\] However, suppose \(f \in D(T_0, \tau, p, p + B)\).

Then \(f \in D(T_0, \tau, p, p)\). Thus there is a set \{\(f_n\)\} each of compact support in \([a, \infty)\) with \(\{f_n\} \rightarrow f\) and \(\{Tf_n\} \rightarrow Tf\).
Thus \(D^N f_n \rightarrow D^N f\), and so \(vf_n \rightarrow vf\). Therefore
\[f \in D(T_0, \tau + v, p, p)\).

***

2.3 Theorem: Let \(\tau\) and \(v\) be as in theorem 2.2 with \(T_n, \tau, p, p\) bounded. Suppose \(T_n^{-1}, \tau, p, p\) is compact. Then
\(T_n^{-1}, \tau + v, p, p\) is compact.

Proof: Suppose not. Then there is a set \{\(x_n\)\}
\[
\subseteq D(T_n, \tau + v, p, p)\) such that \(\|T_n, \tau + v, p, p x_n\|\) is 1 for each \(n\), and \(\|x_n - x_m\| > \varepsilon\) for \(n \neq m\). Now by theorem 2.2, \{\(x_n\)\}
\[
\subseteq D(T_0, \tau, p, p)\). Since \(T_n^{-1}, \tau, p, p\) is compact, we have
\[\|\tau x_n\| \rightarrow \infty.\] We now temporarily assume \(T_n^{-1}, \tau + v, p, p\) is continuous. Then \{\(x_n\)\} is a bounded set.

Suppose \{\(D^N x_n\)\} is a bounded set. Since \(\|vf\| p \leq k\|f\| p + \varepsilon\|D^N f\| p\) for each \(f \in D(T_n, \tau, p, p)\), we have
\{\(vx_n\)\} is a bounded set. However, since \(\|\tau x_n\| \rightarrow \infty\), \{\(D^N x_n\)\} is a bounded set, and \(\|\tau v x_n\| \geq \|\tau x_n\| - \|vx_n\|\),
therefore \( \{ \| (T + v)x_n \| \} \to \infty \). But we assumed
\( \| (T + v)x_n \| = 1 \) for each \( n \), a contradiction.

Suppose, however that \( \| D^N x_n \| \) is not a bounded set.

Then there is a set \( \{x_{nk}\} \) with \( \| D^N x_{nk} \| \to \infty \). Let \( \varepsilon_1 < \frac{c}{2} \)
where \( c \) is as in 2.1 and \( \| vf \| \leq k \| f \| + \| D^N f \| \) for each
\( f \) such that \( \| f \| \) and \( \| D^N f \| \) are both finite. Then

\[ \| vx_n \| \leq k \| x_n \| + \varepsilon_1 \| D^N x_n \| . \]

Since \( \| (T + v)x_{nk} \| = 1 \), we have \( (T + v) \frac{x_{nk}}{\| D^N x_{nk} \|} \to 0 \).

By (1) \( \exists \ N_1 \exists \left\| \frac{vx_{nk}}{\| D^N x_{nk} \|} \right\| \leq \varepsilon_1 + \frac{\varepsilon_1}{2} \) for \( n_k > N_1 \).

So there is an \( N_2 \) such that

\[ \left\| \left( T + v \right) \frac{x_{nk}}{\| D^N (x_{nk}) \|} - \frac{v(x_{nk})}{\| D^N x_{nk} \|} \right\| \leq \varepsilon_1 + \frac{\varepsilon_1}{2} \]

for \( n_k > N_2 \). But \( \left\| T \frac{x_{nk}}{\| D^N x_{nk} \|} \right\| \geq \frac{c}{\| D^N x_{nk} \|} = c \) for
each \( n_k \). Since \( \varepsilon_1 < \frac{c}{2} \), this is a contradiction. So under
the hypothesis that \( T_{0, T + v, p, p} \) is a continuous, we have
completed the proof.

Now suppose \( T_{0, T + v, p, p} \) is not continuous. Then its
range is not closed, so by corollary III.1.10, page 61,
Goldberg (1), there is an infinite dimensional closed
subspace \( s \) contained in \( D(T_{0, T + v, p, p}) \) such that \( T_{0, T + v, p, p} \)
is compact on \( s \). Now the unit ball in \( s \) is not compact.
Pick \( \{x_n\} \) such that \( \|x_n\| = 1 \), and \( \|x_n - x_m\| > \varepsilon \). Since 
\( T_0, \mathcal{L} + \nu, p, p \) restricted to \( s \) is compact, then it is 
continuous. So \( \{T_0, \mathcal{L} + \nu, p, p^*x_n\} \) is a bounded set. Let 
\[ \|T_0, \mathcal{L} + \nu, p, p^*x_n\| \leq M \text{ for each } n. \] We now repeat the 
argument above.

Pick \( \varepsilon_1 < \frac{\varepsilon}{2} \), and let \( \|vf\| \leq k \|f\|p + \varepsilon_1 \|D^Nf\|p \) for 
each \( f \) such that both \( \|f\| \) and \( \|D^Nf\| \) are finite. Then 
take two cases. First, if \( \{D^N x_n\} \) is a bounded set, then 
\( \{\nu x_n\} \) is a bounded set. But \( \{\|T x_n\|\} \not\to \infty \), since 
\( T_0^{-1}, \mathcal{L}, p, p \) is compact. This \( \| (\mathcal{L} + \nu)x_n \| \not\to \infty \), and this 
contradicts the fact that \( \|T_0, \mathcal{L} + \nu, p, p^*x_n\| \leq M \text{ for each } n. \)

However, if \( \{D^N x_n\} \) is not a bounded set, then there is

a subsequence \( \{x_{nk}\} \) with \( \|D^N x_{nk}\| \not\to \infty \). Now

\[ \left\| (\mathcal{L} + \nu) \frac{x_{nk}}{\|D^N x_{nk}\|} \right\| \leq \frac{M}{\|D^N x_{nk}\|} \quad \to 0 \text{ as } n_k \to \infty. \]

Therefore as above

\[ \left\| \mathcal{L} \frac{x_{nk}}{\|D^N x_{nk}\|} \right\| = \left\| (\mathcal{L} + \nu) \frac{x_{nk}}{\|D^N x_{nk}\|} \right\| - \frac{\nu(x_{nk})}{\|D^N x_{nk}\|} \leq \varepsilon_1 + \frac{\varepsilon_1}{2} \text{ for } n_k > N. \]

But \( \frac{T x_{nk}}{\|D^N x_{nk}\|} \geq \varepsilon. \)

This is a contradiction.

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We note that as a result of theorem 2.2, the domain of an 
\( N \)-bounded operator is preserved under adding of the operator 
\( \nu \), so \( T_0, \mathcal{L} + \nu, p, p \) is \( N \)-bounded.
We now prove a few more theorems before generating examples.

2.4 Theorem: Let $T_1$ be a formal differential expression such that $T_0^{-1}T_1$, $p,q$ is continuous. Let $T_2$ be a differential expression such that $T_0^{-1}T_2$, $q,r$ is compact. Let $T_1T_2(f) = \tilde{T}_1(T_2(f))$ for $f$ sufficiently differentiable. Then $T_0^{-1}T_1T_2$, $p,r$ is compact, where $1 \leq p < \infty$, $1 \leq q < \infty$, and $1 \leq r < \infty$.

Proof: We first note that $T_1T_2$ is a differential expression with nowhere vanishing leading coefficient. Now suppose $T_0^{-1}T_1T_2$, $p,r$ is not compact. Then there is a set \{f_n\} $\subset$ $D(T_0, T_1T_2, p,r)$ such that $\|f_n - f_m\|_p > \varepsilon$ for $r \neq m$, and $\|T_1T_2f_n\| = 1$.

We show first that $D(T_0, T_1T_2, p,r)$ $\subset$ \{f|$f \in D(T_0, T_2, p,q)$ and $T_2f \in D(T_0, T_1, q,r)$\}. In fact, if $h \in D(T_0, T_1T_2, p,r)$ then there is a sequence $h_n \rightarrow h$ and $T_2T_2(h_n) \rightarrow T_1T_2(h)$. Therefore since $T_2h_n$ is of compact support for each $n$, and $T_1(T_2h_n)$ is in $L_r[a,\infty)$, $T_2h_n \in D(T_0, T_1, q,r)$. But since \{T_1T_2(h_n)\} converge in $L_r[a,\infty)$, and $T_0^{-1}T_1, q,r$ is continuous, then \{T_2h_n\} converge in $L_q[a,\infty)$. Thus $h \in D(T_0, T_2, p,q)$ and $T_2h \in D(T_0, T_1, q,r)$.

Now consider \{f_n\} chosen above. $\|T_2f_n\|_q \rightarrow \infty$ since $T_0^{-1}T_2, p,q$ is compact. Therefore $\|T_1T_2f_n\|_r \rightarrow \infty$ since
$T_0, \tau_1, q, r$ is continuous. This contradicts the possibility that $\| \tau_1 \tau_2 f_n \|_r = 1$.

2.5 Theorem: If $T_{-1}^0, \tau_2, p, q$ is continuous, and $T_{-1}^0, \tau_1, q, r$ is compact, then $T_{-1}^0, \tau_1 \tau_2, p, r$ is compact.

Proof: As above, if $f \in D(T_0, \tau_1 \tau_2, p, r)$ then $f \in D(T_0, \tau_2, p, q)$ and $T_2 f \in D(T_0, \tau_1, q, r)$. Now if $\| f_n - f_m \|_p > \varepsilon$ for $r \neq m$, and $\| \tau_1 \tau_2 f_n \|_r = 1$, then there is an $\varepsilon$ such that $\| \tau_2 f_n - \tau_2 f_m \|_q \geq \varepsilon_1$ for $r \neq m$, since $T_{-1}^0, \tau_2, p, q$ is continuous. Thus, since $T_{-1}^0, \tau_1, q, r$ is compact, $\| \tau_1(\tau_2 f_n) \|_r \to \infty$.

2.6 Theorem: If $T_0, \tau_1, p, p$ is n bounded and $T_{-1}^0, \tau_1, p, p$ is continuous, and $\tau_2$ is a constant coefficient operator of order $m$, then $T_0, \tau_1 \tau_2, p, p$ is $n + m$ bounded.

Proof: First, we show that $D(T_0, \tau_1 \tau_2, p, p)$

$c \{ f | f \in D(T_0, \tau_2, p, p) \text{ and } \tau_2 f \in D(T_0, \tau_1, p, p) \}$. For if $\| f_n \|_p \to 0$ and $f_n$ is of compact support for each $n$ and $\| \tau_1 \tau_2 f_n - g \|_p \to 0$, then $\tau_2 f_n$ is of compact support for each $n$, $\tau_1(\tau_2 f_n) \in L^p[a, \infty)$, so $\tau_2 f_n \in D(T_0, \tau_1, p, p)$. But since $T_{-1}^0, \tau_1, p, p$ is continuous, then $\tau_2 f_n \to T_{-1}^0 \tau_1, p, p(g)$, so $f \in D(T_0, \tau_2, p, p)$ and
\[ T_2 f \in D(T_0, \mathcal{T}_1, p, p) . \]

Now since \( f \in D(T_0, \mathcal{T}_1, \mathcal{T}_2, p, p) \) implies that \( T_2 f \in D(T_0, \mathcal{T}_1, p, p) \), and since \( T_0, \mathcal{T}_1, p, p \) is bounded, we have \( (D^n T_2)f \in L^p[a, \infty) \). However \( D^n T_2 \) is itself a constant coefficient differential expression of order \( n + m \).

Therefore by theorem VI.6.2, Goldberg (1), page 160, \( f^{n+m} \in L^p[a, \infty) \) and so \( T_0, \mathcal{T}_1, \mathcal{T}_2, p, p \) is \( h + m \) bounded.

2.7 Theorem: If \( T_0^{-1}, \mathcal{T}_1, p, q \) is continuous, and \( \mathcal{T}_2 \) has \( r \)th coefficient 1, (i.e. \( \mathcal{T}_2 = D^n + a_{n-1}D^{n-1} + \cdots + a_0 \)) then

\[ D(T_0, \mathcal{T}_1, \mathcal{T}_2, r, q) = \{f | f \in D(T_0, \mathcal{T}_2, r, p) \} \]

\[ \mathcal{T}_2 f \in D(T_0, \mathcal{T}_1, p, q) \] where \( 1 \leq p < \infty, 1 \leq q < \infty, 1 \leq r < \infty \).

Proof: Let \( f \) have compact support. If \( f \in D(T_0, \mathcal{T}_1, \mathcal{T}_2, r, q) \) then \( f^{n+m-1} \) is absolutely continuous and \( \mathcal{T}_1 \mathcal{T}_2(f) \in L^q[a, \infty) \). (\( n = \text{degree } \mathcal{T}_2, m = \text{degree } \mathcal{T}_1 \))

Thus \( f^n \in L^p[a, \infty) \) since it is continuous and of compact support.

Now if \( f \in D(T_0, \mathcal{T}_1, \mathcal{T}_2, r, q) \) then there is a sequence

\[ \| f_j - f \|_r \to 0, \| (\mathcal{T}_1 \mathcal{T}_2)f_j - \mathcal{T}_1 \mathcal{T}_2 f \|_q \to 0, \] where \( f_j \)

is of compact support in \([a, \infty)\) for each \( j \). Since the \( \mathcal{T}_2 f_n \)

are also of compact support, then \( \mathcal{T}_2 f_n \in D(T_0, \mathcal{T}_1, p, q) \) for each \( n \). But \( \| \mathcal{T}_1 \mathcal{T}_2 f_n \|_r \to 0 \) implies that

\[ \{ \mathcal{T}_2 f_n \} \] is a Cauchy sequence in \( L^p[a, \infty) \), since \( T_0^{-1}, \mathcal{T}_1, p, q \) is continuous. Now since \( T_0, \mathcal{T}_1, r, p \) is closed, this means that
\[ f \in D(T_0, \tau^r_{l,p}, p) \text{ and } \| T_2 f^n - T_2 f \|_p \to 0. \]

2.8 Corollary: If \( T_0, \tau^r_{1,p,p} \) is \( N \)-bounded, for some \( N \), then if \( T_0, \tau^r_{1,p,p} \) is continuous, \( T_0, \tau^r_{1,2,p,p} \) is \( N \)-bounded.

Remark: Theorem VI.6.2, page 160, Goldberg tells us that if \( \tau = D^n + a_{n-1}D^{n-1} + \cdots + a_0 \), where all \( a_i \) are in \( L^\infty \), then \( T_0, \tau, p, p \) is \( n \)-bounded.

2.9 Theorem: If \( \tau = D^n + a_{n-1}D^{n-1} + \cdots + a_0 \), where all \( a_i \) are in \( L^\infty \), then \( T^{-1}_0, \tau, p, p \) is not compact on \( L^p[a, \infty) \).

Proof: Let \( \phi \) be any function which is \( n \)-times continuously differentiable, and has compact support in \( [a + 1, \infty) \). Suppose support \( \text{support} \phi(x) \subseteq [a, N] \). Let \( \phi(x) = \phi_j(x + 3jN) \) define a sequence of functions \( \phi_j \). Domain \( \phi_j = [a + 3jN, \infty) \).

Let \( Q_j(x) = \phi_j(x) \) for \( x \in [a + 3jN, \infty] \) and \( Q_j(x) = 0 \) for \( x \in [a, a + 3jN] \).

Now \( \| Q_i - Q_j \|_p = 2 \| \phi \|_p \) if \( i \neq j \), but \( Q_j \) is \( n \) times continuously differentiable and \( \| \tau(Q_j) \|_p \leq L_0 \| Q_j \|_p \) + \( L_1 \| DQ_j \|_p + L_2 \| D^2Q_j \|_p + \cdots + \| D^nQ_j \|_p \), where \( L_i = \| a_i \|_{\infty} \). But \( \| Q_j \|_p = \| \phi \|_p \), \( \| D(Q_j) \|_p = \| D\phi \|_p \), and \( \| D^n\phi \|_p = \| D^n\phi \|_p \). Therefore \( \{ \| \tau Q_j \|_p \} \) is a bounded set of real numbers, so \( T^{-1}_0, \tau, p, p \) is not compact.

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2.10 Corollary: If \( \mathcal{T} = D^N + \sum_{0}^{N-1} a_i D^i \), then there is at least one non \( L_2 \) solution of \( \mathcal{T}f = 0 \) or \( \mathcal{T}^+f = 0 \) on \([a, \infty)\).

Proof: If all solutions were \( L_2 \), the minimal operator would have compact inverse.

***

We now have the theorems to generate examples. Given one \( \mathcal{T} \) such that \( \mathcal{T}_{0,-1}^{-1} \) is compact, we can multiply on the right or left by differential expressions whose minimal operators have closed range in the appropriate topologies, and get more compact inverse minimal operators. Further, if \( \mathcal{T}_{0,-1}^{-1} \) is \( n \)-bounded and has compact inverses, we can add coefficients of form \( \sum_{0}^{n-1} a_i D^i \) where

\[
\sup_{s \leq t < \infty} |b_{k}(t)|^{p} dt < \infty
\]

and preserve the compact inverse property. If \( \mathcal{T}_{0,-1}^{-1} \) is compact, we can multiply on the right by any \( \mathcal{T}_{1}^{-1} \) with \( \mathcal{T}_{0,-1}^{-1} \) continuous and where

\[
\mathcal{T}_{1} = D^N + a_{n-1}D^{n-1} \ldots + a_0 \text{ with } a_i \text{ bounded, and get}
\]

\( \mathcal{T}_{0, \mathcal{T}_{1}}^{-1} \) \( \mathcal{T}_{2}^{-1} \) \( n \)-bounded with compact inverse.

To generate an example to use as a starting point is easy when \( p = 2 \). We shall give a general example below. We have several lemmas below which will allow us to get compact inverse minimal operators \( \mathcal{T}_{0,-1}^{-1} \) from an \( L_2 \) example, where \( p \geq 2, q \leq 2 \).

The essential spectrum of constant coefficient operators from \( L^p(0, \infty) \) into \( L^p(0, \infty) \) is completely known (see Goldberg (1), theorem VI.7.2, page 163, and also Balslev and
Thus we have no difficulty in constructing continuous inverse minimal operators of this type. Also perturbation theorems such as VI.8.1, page 167, Goldberg, allow us to proceed to non constant coefficient minimal operators with continuous inverse.

In the case of $L^2$, we can improve the $n$-boundness theorems in certain situations. We shall do this in the last theorem to get an extremely general class of examples in $L^2$.

Part b of the following theorem is generalized in theorem 2.18, but the proof is included because it motivates 2.18.

2.11 Theorem: Let \( \lim_{x \to \infty} \Re(a_0(x)) = \infty \). Let \( \mathcal{T} = (-1)^n D^{2n} + a_0 \). Then

- (a) \( T_0, \mathcal{T}, 2, 2 \) is $n$ bounded;
- (b) \( T_0^{-1}, \mathcal{T}, 2, 2 \) is compact.

Proof: We select a number \( c \) such that \( \Re(a_0(x) + c) > 1 \) for every \( x \in [a, \infty) \). By 1.8, the domain of \( T_0, \mathcal{T}, 2, 2 + cI \) is the same as the domain of \( T_0, \mathcal{T}, 2, 2 \). By 1.13,

\[
T_0, \mathcal{T} + c, 2, 2 = T_0, \mathcal{T}, 2, 2 + cI.
\]

So that if we define \( \mathcal{T}_1 = (-1)^n D^{2n} + (a_0 + c) \), we are finished if we can show that \( T_0, \mathcal{T}_1, 2, 2 \) is $n$ bounded.

Let \( \| f \| = 1 \), where \( f \) has compact support in the interior of \([a, \infty)\) and is in \( D(T_0, \mathcal{T}, 2, 2) \). Then

\[
\| \mathcal{T}_1 f \| \geq |(\mathcal{T}_1 f, f)| = |(-1)^n (D^{2n} f, f) + ((a_0 + c)f, f)|.
\]

Integrating by parts, we see that
\[(T_1 f, f) = ||D^nf||^2 + \int_a^\infty (a_0 + c)f(t)\]. Now \(|(T_1 f, f)| \geq \text{Re}(T_1 f, f) = ||D^nf||^2 + \int_a^\infty \text{Re}(a_0 + c)f(t)\]. So

\[||T_1 f|| \geq ||D^nf||^2.\]

If \(f\) is not of compact support, then by 0.8 there is a Cauchy sequence of compact support functions \(f_j\) such that \(\{f_j\} \rightharpoonup f\), and \(\{T_1 f_j\} \rightharpoonup T_1 f\). Thus \(\{D^nf_j\}\) form a Cauchy sequence, and since \(T_0, D^nf, 2, 2\) is closed, \(D^nf \in L_2[a, \infty)\). Thus \(T_0, T_1, 2, 2\) is n-bounded.

To show that \(T^{-1}_0, T_1, 2, 2\) is compact, we note that it is sufficient, by theorem 1.17, to show that \(T^{-1}_0, T_1, 2, 2\) is compact.

We first show that \(T^{-1}_0, T_1, 2, 2\) is continuous. In fact, if \(||f|| = 1\) and \(f \in D(T_0, T_1, 2, 2)\) and also \(f\) has compact support in \((a, \infty)\), we have as above \(|(T_1 f, f)| \geq \text{Re}(T_1 f, f) = ||D^nf||^2 + \text{Re}(a_0 + c)f(t)\) \[= ||D^nf||^2 + \text{Re}((a_0 + c)f(t)) \geq ||D^nf||^2 + (f, f) \geq 1.\]

Therefore, passing to the closure, \(||T^{-1}_0, T_1, 2, 2|| \leq 1.\)

Now, if \(T^{-1}_0, T_1, 2, 2\) is not compact, there is a sequence \(\{f_j\} \subset D(T_0, T_1, 2, 2)\) such that \(||T_1 f_j|| = 1\) and \(||c^j f_j - f_j|| \geq \varepsilon_1\). Pick a subsequence \(\{f_{j_k}\}\) such that \(\text{Re}(a_0(x) + c) > k\) if \(x \geq j_k\). Then \(\text{Re}(a_0 + c)f_j k\). Thus \(\text{Re}(a_0 + c)f_j k > k \varepsilon_1^2.\)

Since, for \(f\) of compact support in \(D(T_0, T_1, 2, 2)\), we have \(\text{Re}(T_1 f, f) \geq \text{Re}((a_0 + c)f(t), f)\), we may pass to the closure and see that this holds for all \(f \in D(T_0, T_1, 2, 2)\). Thus
Re(\mathcal{T}, f_{j_k}, f_{j_k}) \to \infty. But since \| f_{j_k} \| = 1, this means that
\| \mathcal{T} f_{j_k} \| \to \infty, a contradiction of the definition of \{f_{j_k}\}.

***

The similarity of this theorem with theorem XIII.7.9, page 144, Dunford and Schwartz (2) should be observed. We
shall eventually prove that for a somewhat more general \mathcal{T}
than the one in this theorem, \mathcal{T}^{-1}_0, \mathcal{T}, 2, 2 is compact. Thus
theorem XIII.7.9 will be a corollary.

We use the example above to generate an example in the
case p \neq 2.

2.12 Lemma: Suppose Re(\mathcal{A}) > 0. Then if s > 2, \mathcal{T}^{-1}_0, \mathcal{T}, s, 2
is continuous, where \mathcal{T} = D - \mathcal{A}, and the interval
I = [0, \infty).

Proof: The solution to (D - \mathcal{A})f = 0 is f(x) = e^{-\mathcal{A}x}
= e^{-(\text{Re}\ \mathcal{A})x} \cdot e^{-i(\text{Im}\ \mathcal{A})x}. Now e^{-\mathcal{A}x} \in L^p[0, \infty)-for
1 \leq p \leq \infty. Also if g \in D(T_0, \mathcal{T}, s, 2) and \mathcal{T}g = h, then
g(x) = \int_0^\infty e^{-\mathcal{A}(x-y)}h(y)dy = \int_{-\infty}^x e^{-\mathcal{A}(x-y)}h(y)dy
= \int_{-\infty}^\infty \phi(x - y)h(y)dy, where \phi(T) = \begin{cases} 0 & T < 0 \\ e^{-\mathcal{A}T} & T > 0 \end{cases}

Note that \phi \in L^p(-\infty, \infty) for 1 \leq p \leq \infty. We now use exercise
VI.11.6, Dunford and Schwartz (1), page 528, to get our
conclusion. (Solving for p \geq 1 such that \frac{1}{s} = \frac{1}{p} + \frac{1}{2} - 1 and
noting that \phi \in L^p[-\infty, \infty].)

2.13 Lemma: Suppose Re(\mathcal{A}) > 0. Then if r \leq 2, \mathcal{T}^{-1}_0, \mathcal{T}, 2, r
is continuous, where \mathcal{T} = D - \mathcal{A}, and the interval I = [0, \infty).
Proof: Exactly the same as 2.8, except here we solve for $p > 1$ such that \( \frac{1}{2} = \frac{1}{p} + \frac{1}{r} - 1 \). We can do this since $r \leq 2$.

**2.14 Remark:** These examples may be used in conjunction with theorem 2.11 and theorem 2.5 to generate differential expressions with compact inverse in the non $L^2$ case.

We now use the following sequence of lemmas to develop a very general example in the $L^2$ case.

**2.15 Lemma:** Let $\mathcal{T}$ be classically self-adjoint. Let

$$\mathcal{T}_1 = \mathcal{T}^{2N} + a_0, \text{ where } \text{Re}(a_0(x)) \geq 1 \text{ for } x \in [a, \infty).$$

Suppose

$$\lim_{x \to \infty} \text{Re}(a_0(x)) = \infty.$$ 

Then, if $\{f_n\}$ is a sequence in $D(T_0, \mathcal{T}_1, 2, 2)$ such that \( \|C_{a_0}^{n \rightarrow 2} f_n - f_n \|_2 > \epsilon \), and such that each $f_n$ is of compact support in $(a, \infty)$, then

$$|\langle \mathcal{T}_1 f_n, f_n \rangle| \to \infty.$$ 

**Proof:** By the generalized Green's formula (VI.1.8, page 130, Goldberg) we have

$$\langle \mathcal{T}_1 f_n, f_n \rangle = \langle \mathcal{T}^{2N} f_n, \mathcal{T}^{2N} f_n \rangle + \langle a_0 f_n, f_n \rangle.$$ 

Now

$$|\langle a_0 f_n, f_n \rangle| \geq \int_{[n, \infty)} \text{Re}(a_0(x)) \, dx \geq \left( \text{glb}_{x \in [a, \infty)} \text{Re}(a_0(x)) \right) \epsilon.$$ 

But the term on the right approaches infinity.

***

**2.16 Lemma:** Let $T$ be a self-adjoint operator on a Hilbert space $H$. Suppose there is a sequence $\{f_n\}$, each of norm 1, contained in $D(T^{2k})$, such that

$$\lim_{k \to \infty} \frac{1}{k} \langle T^{2k} f_n, f_n \rangle \to 0.$$ 

Then if $k_1 < 2k$, \( \frac{1}{k} \langle T^{2k} f_n, f_n \rangle \to 0. \)
Proof: Let $T = \int_{-\infty}^{\infty} k dP(\lambda)$. Then

$$T^{2k} = \int_{-\infty}^{\infty} \lambda^{2k} dP(\lambda),$$

and $T^1 = \int_{-\infty}^{\infty} \lambda^1 dP(\lambda)$. Thus

$$(T^{2k} f, f) = \int_{-\infty}^{\infty} \lambda^{2k} d(P(\lambda)f, f),$$

and $(T^1 f, f) = \int_{-\infty}^{\infty} \lambda^1 d(P(\lambda)f, f),$ for $f \in D(T^{2k})$.

Now, if $f \in D(T^{2k})$, then

$$\int_{-\infty}^{\infty} \lambda^1 d(P(\lambda)f, f) \leq \int_{-\infty}^{\infty} |\lambda|^{k_1} d(P(\lambda)f, f).$$

Also if $\lambda_1 > 0$,

$$\int_{-\infty}^{\infty} \lambda^1 d(P(\lambda)f, f) = \int_{-\infty}^{\infty} |\lambda|^{k_1-2k} d(P(\lambda)f, f).$$

Also, $\int_{-\infty}^{\infty} \lambda^1 d(P(\lambda)f, f) \leq |\lambda_1|^{k_1}$.

Now let $\lambda_n = (T^{2k} f_n, f_n)^{\frac{1}{k_1+1}}$. Then

$$\int_{\lambda_n} \lambda^1 d(P(\lambda)f_n, f_n) \leq \int_{-\infty}^{\infty} \lambda^{k_1} d(P(\lambda)f_n, f_n) + \int_{\lambda_n} \lambda^{k_1} d(P(\lambda)f_n, f_n)$$

$$+ \int_{-\infty}^{\infty} \lambda^{k_1} d(P(\lambda)f_n, f_n)$$

$$\leq \int_{-\infty}^{\infty} \lambda^{k_1} d(P(\lambda)f_n, f_n)$$

$$+ \int_{\lambda_n} \lambda^{k_1} d(P(\lambda)f_n, f_n).$$
\[
\frac{2(\mathcal{A}_n)^{k_1}}{(\mathcal{A}_n)^{k_1+1}} + 2(\mathcal{A}_n)^{k_1-2k} = 2(\mathcal{A}_n)^{-1} + 2(\mathcal{A}_n)^{k_1} - 0.
\]

2.17 Lemma: Let \( T \) be as in lemma 2, and \( \{f_n\} \) bounded. If \(|(T^{2k}f_n,f_n)|\) is bounded, so is \(|(T^k f_n,f_n)|\), for \( k_1 \leq 2k \).

Proof: \( \int_\infty^{\infty} \lambda d(P(A)f_n,f_n) \geq \int_1^{\infty} \lambda d(P(A)f_n,f_n) \). Also \( \int_{-1}^{1} \lambda d(P(A)f_n,f_n) \geq \int_{-1}^{\infty} \lambda d(P(A)f_n,f_n) \) and \( \int_{-1}^{1} \lambda d(P(A)f_n,f_n) \leq |f_n| \). Thus, since \(|T^{2k}f_n,f_n|\) is bounded, and \( \{f_n\} \) is bounded, the conclusion follows.

2.18 Theorem: Let \( \mathcal{V} = \sum_{1}^{2k-1} b_i \mathcal{T}_i \), where \( \mathcal{T} \) is classically self adjoint, and \( b_i \) is a bounded function for every \( i \). Let \( \mathcal{T}_1 = \mathcal{T}^{2k} + a_0 \), \( \text{Re}(a_0(x)) \geq 1 \), \( \lim_{x \to \infty} \text{Re}(a_0(x)) = \infty \). Then \( T^{-1}_{0}, T^{-1}_{1} + \mathcal{V}, 2, 2 \) is compact.

Proof: Let \( T \) be a self adjoint extension of the symmetric, closed operator \( T_0, \mathcal{T}, 2, 2 \), to a larger Hilbert space if necessary. Such an extension is guaranteed by exercise 25, page 1261, Dunford and Schwartz (2).

(Originally due to Naimark.) Then if \( f \) is of compact support in \((a, \infty)\), and \( f \in D(T_0, \mathcal{T}_1 + \mathcal{V}, 2, 2) \), \( f \in D(T^{2k}) \). In fact,
since \( f \in D(T_0, \Gamma_2^2) \), then \( f^{N-1} \) is absolutely continuous on \([a, \infty)\), where \( \Gamma_1 \) is of order \( N \). Since \( f \) is of compact support in \((a, \infty)\), it follows that \( f \in D([T_0, \Gamma_2^2]) \) for any \( k_1 < 2k \). Therefore \( f \in D(T^{2k-1}) \). We now show that \( T^{k-1}f \in D(T_0, \Gamma_2^2) \). Since \( T^{k-1}f \) has compact support in \((a, \infty)\), we need only show that \( T(T^{2k-1}f) \in L^2[a, \infty) \). We know, however, that if \( K = \text{support } f, (\Gamma_1 + \mathcal{V})f \in L^2[a, \infty) \) and therefore \( L^2(K) \). Now, using theorem VI.6.2, Goldberg (1), we see that for every \( k \leq 2kN, f^{(k)} \in L^2(K) \). (Remember \( \Gamma \) is of order \( N \).) Thus \( T^{2k}f \in L^2(K) \), and therefore \( T^{2k}f \in L^2[a, \infty) \). So \( f \in D(T^{2k}) \).

Now we prove the theorem. Suppose \( \{f_n\} \) is a norm bounded sequence in \( D(T_0, \Gamma_1^2 + \mathcal{V}, 2^2) \) such that 

\[
\| C_{a_n}^{\Gamma_2} f_n - f_n \| > \varepsilon, \text{ and } \| (\Gamma_1 + \mathcal{V})f_n \| = 1.
\]

Then we can get a norm bounded sequence of compact support functions \( \{g_n\} \) such that \( \| C_{a_n}^{\Gamma_2} g_n - g_n \| > \varepsilon \) \( \frac{\varepsilon}{2} \) and \( \| (\Gamma_1 + \mathcal{V})g_n \| < 2 \). By lemma 2.11, \( |(\Gamma_1 g_n, g_n)| \to \infty \). Let \( S = \{(T^{2k}g_n, g_n)\} \). If it is a bounded sequence, then by lemma 3 \( |(\mathcal{V} g_n, g_n)| \) is a bounded sequence. But since \( |(\Gamma_1 f_n, f_n)| \to \infty \), we then see that \( |(\Gamma_1 + \mathcal{V})g_n, g_n)| \to \infty \). Since \( \{g_n\} \) is assumed norm bounded this is a contradiction of the fact that 

\[
\| (\Gamma_1 + \mathcal{V})g_n \| < 2.
\]

If, however, \( \{(T^{2k}g_n, g_n)\} \) is unbounded, there is a sequence \( \{f_n\} \) such that \( (T^{2k}g_n, g_n) \to \infty \). Since \( \{g_n\} \) is
norm bounded, by assumption, we see that \( \frac{\langle T^{2k}_n f_n, f_n \rangle}{\|g_{n_j}\|^2} \to \infty \).

Let \( \phi_j = \frac{g_{n_j}}{\|g_{n_j}\|} \). \( \|\phi_j\| = 1 \) for each \( j \), and

\( \langle T^{2k}_j \phi_j, \phi_j \rangle \to \infty \). Now \( \{g_{n_j}\} \) does not converge to 0, because

\( \|c_{a,j} g_j - g_{n_j}\| > \epsilon \). Therefore there is a subsequence \( \{g_{n_{j_i}}\} \) bounded away from 0, i.e. \( \|g_{n_{j_i}}\| > \epsilon \). Then

\( \{T_1 + \gamma\} g_{n_{j_i}} \) is a bounded sequence, for since

\( \|T_1 + \gamma\| g_{n_{j_i}} \| < 2 \), we have \( \|T_1 + \gamma\| g_{n_{j_i}} \| < \frac{2}{\epsilon} \). We derive a contradiction from this.

By Green's formula, \(|(T^{2k}_j \phi_j, \phi_j) + (a_0 \phi_j, \phi_j)|\)

\[= |(T^{k}_j \phi_j, T^{k}_j \phi_j) + (a_0 \phi_j, \phi_j)| \geq \Re[(T^{k}_j \phi_j, T^{k}_j \phi_j) + (a_0 \phi_j, \phi_j)] \]

\[= (T^{k}_j \phi_j, T^{k}_j \phi_j) + 1 > (T^{2k}_j \phi_j, \phi_j) \]. Let

\[ \lambda_i = (T^{2k}_j \phi_j, \phi_j). \]

\[ \lambda_i \to 0, \text{ since} \]

\( \{\lambda_i\} \to \infty, \text{ and } \|\phi_j\| = 1, \text{ and } \|T_1 + \gamma\| \phi_j \| < \frac{2}{\epsilon} \). But

\[ \frac{(T_1 + \gamma) \phi_j, \phi_j)}{\lambda_i} = \frac{T_1 \phi_j, \phi_j)}{\lambda_i} + \frac{(\gamma \phi_j, \phi_j)}{\lambda_i}. \]

\[ \left| \frac{(T_1 \phi_j, \phi_j)}{\lambda_i} \right| > 1, \text{ by the above discussion. But we} \]
shall show, using lemma 2.15, that \( \frac{(\forall \phi_{j_1}, \phi_{j_1})}{\Lambda_i} \rightarrow 0 \), and thus contradict the fact that \( \frac{((\mathcal{T}_1 + \mathcal{V})\phi_{j_1}, \phi_{j_1})}{\Lambda_i} \rightarrow 0 \). Let

\[
1_u.b. \quad |a_i(x)| = M_1. \quad (\forall \phi_{j_1}, \phi_{j_1}) \leq \frac{2^{k-1}}{x \in [a, \infty)} \sum_{r=1}^{M_r} |(T_{1}^r \phi_{j_1}, \phi_{j_1})|.
\]

Therefore, by lemma 2.15, \( \frac{(\forall \phi_{j_1}, \phi_{j_1})}{\Lambda_i} \rightarrow 0 \). This is a contradiction. Therefore we have shown that there is no norm bounded sequence of functions \( \{f_n\} \subset D(T_{0}, \mathcal{T}_1 + \mathcal{V}, 2, 2) \) such that \( \|C_{\alpha_n} f_n - f_n\| > \varepsilon \), and \( \|(T_{1} + \mathcal{V})f_n\| = 1 \). If we knew that \( T_{0}^{-1}, \mathcal{T}_1 + \mathcal{V}, 2, 2 \) was continuous, we would be finished by theorem 1.6. As it is, however, we must prove continuity in order to guarantee that if \( T_{0}^{-1}, \mathcal{T}_1 + \mathcal{V}, 2, 2 \) is not compact, there is such a norm bounded sequence \( \{f_n\} \).

If \( T_{0}^{-1}, \mathcal{T}_1 + \mathcal{V}, 2, 2 \) is not continuous, then \( \exists \) is a sequence \( \{f_n\} \) such that \( \|f_n\| = 1 \), \( \|(T_{1} + \mathcal{V})f_n\| \rightarrow 0 \), and \( f_n \in D(T_{0}, \mathcal{T}_1 + \mathcal{V}, 2, 2) \). Now there is a sequence of compact support functions \( \{g_n\} \subset D \subset \|g_n - f_n\| < \frac{1}{2^n} \), \( \|(T_{1} + \mathcal{V})g_n - (T_{1} + \mathcal{V})f_n\| < \frac{1}{2^n} \). So \( \|(T_{1} + \mathcal{V})g_n\| \rightarrow 0 \), and \( \{g_n\} \) is norm bounded. In the preceding argument we derived a contradiction from the existence of a norm bounded sequence \( \{g_n\} \) of compact support functions supported in \((a, \infty)\) such that \( \|C_{\alpha_n} g_n - g_n\| > \varepsilon \),
and \( \| (\mathcal{T}_1 + \mathcal{Y})g_n \| < 2 \). We thus see that in this case we must have that \( \forall \varepsilon > 0 \exists N \in \mathbb{N} \ni n > N \implies \| c^N_a g_n - g_n \| < \varepsilon \).

For if not, then for every \( N \in \mathbb{N} \) there exists \( n > N \) such that
\[
\| c^N_a g_j - g_j \| > \varepsilon.
\]
Therefore \( \{g_j\} \) is a sequence of the type that has been shown not to exist. In particular, this must be true for \( \varepsilon = \frac{1}{16} \). So there is an \( N_1 \) such that
\[
\| c^N_a g_n - g_n \| < \frac{1}{16} \text{ for } n > N_1.
\]
But now we note that \( \{c^N_a g_n\} \) is a sequence of functions in the domain of \( W \), where \( W \) is the maximal operator on \( L^2[a,N] \) generated by \( (\mathcal{T}_1 + \mathcal{Y}) \).

Further \( [D^j(g_n)](a) = 0 \) for \( 0 \leq j \leq 2kN - 1 \). \( W \) is thus a \( 1-1 \) extension of the minimal operator generated by \( \mathcal{T}_1 + \mathcal{Y} \) on \( L^2[a,N] \), so \( W^{-1} \) is compact, and thus certainly continuous.

However, since \( \| (\mathcal{T}_1 + \mathcal{Y})g_n \| \to 0 \), then
\[
\| c^N_a (\mathcal{T}_1 + \mathcal{Y})g_n \| \to 0.
\]
Therefore, in other words,
\[
W(c^N_a g_n) \to 0. \text{ Therefore, } \| c^N_a g_n \| \to 0. \text{ But } \| g_n \| \to 1
\]
and for \( n > N_1 \), \( \| c^N_a g_n - g_n \| < \frac{1}{16} \). This is a contradiction, and at least the theorem is proved.

\[\star\star\star \quad \star\star\star\]

2.19 Corollary: Let \( \mathcal{T} \) be classically self adjoint with \( \sigma(e(T_0, \mathcal{T}, 2, 2)) \neq \mathbb{R} \). Let \( \text{Re } a_0(x) \to \infty \) as \( x \to \infty \). Let \( \{a_1\} \) be bounded functions. Let
\[
\mathcal{T}_1 = \mathcal{T}^{2k} + \sum_{i=1}^{2k-1} a_i \mathcal{T}_i^i + a_0.
\]
Then \( T^{-1}_0, \mathcal{T}_1, 2, 2 \) is compact.

Proof: Use theorem 1.17 and the above theorems.
Theorem 1.17 is used to add $\lambda$ to $a_0$ so that
$\operatorname{Re}(a_0 + \lambda) \geq 1$.

2.20 Remark: Letting $\mathcal{T} = i \frac{d}{dx}$ in the above theorem we deduce a generalization of theorem 9, page 1449, Dunford and Schwartz (2).

We now wish to extend our results to $(-\infty, \infty)$. The next theorem is our tool. We extend the results about the spectrum rather than results about the compactness of the inverse of the minimal operator, since the minimal operator may not be $(1,1)$ in $(-\infty, \infty)$.

Let $\mathcal{T}$ be a differential expression on $(-\infty, \infty)$. Let $\mathcal{T}_1$ be $\mathcal{T}$ considered as an expression on $[a, \infty)$, and let $\mathcal{T}_2$ be $\mathcal{T}$ considered as an expression on $(-\infty, 0]$.

2.21 Theorem: $\lambda \in \sigma_e(T_0, \mathcal{T}, \rho, \pi) \iff \lambda \in \sigma_e(T_0, \mathcal{T}_1, \rho, \pi)$ or $\lambda \in \sigma_e(T_0, \mathcal{T}_2, \rho, \pi)$.

Proof: $\implies$ is clear.

But if $\lambda \in \sigma_e(T_0, \mathcal{T}, \rho, \pi)$, then the range $T_0, \mathcal{T} - \lambda, \rho, \pi$ is not closed. However, consider $f \in D(T_0, \mathcal{T} - \lambda, \rho, \pi)$ with $f(0) = c_0$. Call the vector \( \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = V \). The set of all possible such $V$ is an $n$-dimensional vector space. Let $\{V_i\}$
be a basis. Select $f_i$ of compact support with

$$
\begin{pmatrix}
    f_i(0) \\
    f'_i(0) \\
    \vdots \\
    f^{n-1}_i(0)
\end{pmatrix}
= V_i.
$$

Then if $g \in D(T_0)$ is of compact support, $g = g_0 + \sum c_i f_i$, where

$$
\begin{pmatrix}
    g_0(0) \\
    g'_0(0) \\
    \vdots \\
    g^{n-1}_0(0)
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    0
\end{pmatrix},
$$

and $g_0$ is of compact support. Let $R = \text{span } f_i$, $Q = \{g| g(0) = 0\}$. Note that $\{f_i\}$ are linearly independent modulo $Q$.

By Lagrange's identity, any $n$-differentiable function $h$ of compact support in $[0, \infty)$ with $h(0) = 0$ is in

$$
\begin{pmatrix}
    h(0) \\
    h'(0) \\
    \vdots \\
    h^{n-1}(0)
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    0
\end{pmatrix}
$$
$D(T_0, T_2^- \lambda, p, q)$. Similarly, any $n$-differentiable function $h$ of compact support in $(-\infty, 0]$ with
\[
\begin{pmatrix}
h(0) \\
h'(0) \\
\vdots \\
h^{n-1}(0)
\end{pmatrix} = 0
\]
is in $D(T_0, T_2^- \lambda, p, q)$.

Now since $\lambda \in \mathcal{E}(T_0)$, then $T_0, T^- \lambda, p, q$ is not continuous, where $T_0, T^- \lambda, p, q$ is the induced operator on $L_p$, with $K$ the kernel of the minimal operator $T_0, T^- \lambda, p, q$.

Therefore $\exists \{h_n\}$ with $d(h_n, K) > 1$, such that

\[
\|(T - \lambda)h_n\| \to 0. \quad \text{We may assume each } h_n \text{ is of compact support, since the map } L_p \to \frac{L_p}{K} \text{ is continuous. As above,}
\]
decompose $h_j$ into $h_j = q_j + r_j$, $q_j \in Q$, $r_j \in R$. Decompose $R$ into $K \oplus S$, where $K = \text{kernel } T_0, T^- \lambda, p, q$, and $r_j = k_j + s_j$.

Now $\|q_j + s_j\| \geq 1$. Thus either $q_j \not\to 0$, or $s_j \not\to 0$.

Let $T = T_0, T^- \lambda, p, q$.

If $s_j \not\to 0$, then $\|T s_j\| \not\to 0$, since $T|S$ is finite dimensional with no kernel. However, if

$\lambda \in \mathcal{E}(T_0, T_1, p, q) \text{ or } \mathcal{E}(T_0, T_2, p, q)$, we see that

$T(q_j) \not\to 0$ where $q_j \not\to 0$, because $q_j = q_{1j} + q_{2j}$,

$q_{1j} \in D(T_0, T_1, p, q)$, $q_{2j} \in D(T_0, T_2, p, q)$. Therefore we must have $T s_j \not\to 0$, $T q_j \not\to 0$, but $T s_j + T q_j \not\to 0$. 

\[ \frac{\| T_{s_jk} \|}{\| T_{s_jk} \|} + \frac{\| T_{q_{jk}} \|}{\| T_{s_jk} \|} \rightarrow 0 \text{ for some subsequence} \]

Since \( \{T_{s_jk}\} \) are contained in a finite dimensional subspace, there is a subsequence \( \left\{ \frac{T_{s_{jk}}}{\| T_{s_{jk}} \|} \right\} \rightarrow f. \)

Then \( \frac{T_{q_{jk}}}{\| T_{s_{jk}} \|} \rightarrow s' \text{ for some } s'. \) But range

\( T_0, T_1, p, q \oplus \text{range } T_0, T_2, p, q \) is closed, so

\[ f \in \text{range } T_0, T_1, p, q \oplus \text{range } T_0, T_2, p, q \text{ (since each } T_{q_{jk}} \text{ is).} \]

Also \( f \in T(S), \) and also \( f \neq 0. \) Therefore, since \( S \) is linearly independent mod \( Q, \) we have \( f = T_l f_1 = T_l f_2, f_1 \in Q, f_2 \in S \) and therefore \( f_1 = f_2. \) Now \( f_1 - f_2 \in \ker T, \) but \( S \) is linearly independent mod \( \ker T \oplus Q. \) This is a contradiction.

\[ *** \quad *** \]

2.22 Corollary: If the essential spectra of \( T_0, T_1, p, p \) and \( T_0, T_2, p, p \) are both null, then the essential spectrum of \( T_0, T_1, p, p \) is null.
Bibliography


Vita

Robert M. Kauffman was born in Muskegon, Michigan on August 26, 1941. He was graduated from Crystal Lake Community High School, Crystal Lake, Illinois, in June, 1959. He then attended The University of the South, Sewanee, Tennessee, majoring in mathematics and physics and receiving his B.A. degree in June, 1963.

He received his M.S. degree from Louisiana State University in August, 1965. At present he is an assistant professor at Western Washington State College, and a candidate for the Doctor of Philosophy degree in the Department of Mathematics at Louisiana State University.
EXAMINATION AND THESIS REPORT

Candidate: Robert MacKenzie Kauffman

Major Field: Mathematics-Analysis

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Approved:

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