Some Remarks on Integral Dependence and Noetherian Rings.

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ABSTRACT

In Chapter I of this paper, we consider the problem: "Given a ring $R$ and a family $\Gamma$ of noetherian overrings of $R$, what conditions on the family $\Gamma$ will assure us that $R$ is noetherian"? The main result of this investigation is that $R$ is noetherian if $\Gamma$ contains a ring which is finitely generated and integral over $R$. Some of the examples in Chapter II will show that a number of tempting conditions on $\Gamma$ (e.g. $\Gamma$ contains all rings of the form $R[x^{-1}]$, $x$ a non-unit of $R$) are not sufficient to imply that $R$ is noetherian.

Chapter II is devoted to examples. Together with those examples pertinent to Chapter I, there are examples of non-noetherian, finite dimensional Krull rings. The two dimensional example provides a finite dimensional Krull ring which is not the derived normal ring of a noetherian domain.
CHAPTER I
SOME RESULTS ON NOETHERIAN RINGS

In what follows the notation will be, for the most part, consistent with [14]. In addition, all rings are assumed to be commutative and to possess an identity. When we say that $S$ is an overring of $R$ or that $R$ is a subring of $S$ (denoted $R \subseteq S$) it is always understood that $S$ is a unitary module over $R$. If we wish to indicate that $S \nless R \neq \emptyset$, we write $R < S$. Finally, by the derived normal ring of a ring $R$, we mean the integral closure of $R$ in its total quotient ring.

Let $R$ and $S$ be rings such that $R \subseteq S$. The general problem of relating the ideal theory of $R$ and $S$ is a fundamental goal of ring theory. In the most general situation, virtually nothing can be said. However, with additional hypothesis on $R$ and $S$, theorems can be proved which relate the two ideal structures. A result of fundamental importance to the present investigation is the following:

**Hilbert Basis Theorem:** If $R$ is a noetherian ring and $X$ is transcendental over $R$, then $R[X]$ is noetherian.

For a proof, the reader might consult [14, p.201].
One can quickly see that if \( X \) is transcendental over \( R \) and \( R[X] \) is noetherian, then \( R \) must also be noetherian. This provides the simplest situation of the form:

\[
R < S: S \text{ noetherian} : \iff R \text{ noetherian}.
\]

With this introduction we ask the following general question.

"Let \( R \) be a commutative ring with identity and \( \Gamma \) a family of noetherian overrings of \( R \): What conditions on \( \Gamma \) will assure us that \( R \) is noetherian?"

This problem has been investigated by Davis [3] and Gilmer and Mott [7]. In case \( D \) is a domain with quotient field \( K \), they have shown that \( D \) is noetherian if every ring \( S \) such that \( D < S \subseteq K \) is noetherian. We will show that a much less restrictive hypothesis on the family of overrings will allow us to conclude that \( R \) is noetherian. We begin with several definitions.

(1.0) Definition: If \( R \) and \( S \) are rings, \( S \) is a finite integral overring of \( R \) if \( S \) is an overring of \( R \) and a finitely generated, unitary \( R \) module.

(1.1) Definition: A is a proper ideal of \( R \) if \( (0) < A < R \).

(1.2) Definition: A ring \( R \) is said to satisfy the restricted maximum condition (or be an RMX ring) if \( R/P \) is noetherian for every proper prime ideal \( P \) of \( R \).

The following are then immediately equivalent:
(1) R is an RMX ring

(2) If A is an ideal of R and P is a proper prime ideal of R, then P + A is finitely generated modulo P.

(3) If \( A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots \) is an ascending chain of ideals of R, one of which contains a proper prime ideal of R, then there exists an integer \( k \) such that \( A_i = A_j \) if \( i, j \geq k \).

(4) If S is a non-vacuous set of ideals of R and there exists \( A \in S \) and P a proper prime of R such that \( P \subseteq A \), then S has a maximal element with respect to containment.

For the present purposes, the interest in the RMX condition is that it is a tool for investigating noetherian rings. The utility of the condition is that it lends itself easily to inductive arguments.

(1.3) LEMMA. Let R be an RMX ring with S a finite integral overring of R. If P is a prime ideal of S such that \( P \cap R \neq (0) \), then S/P is noetherian.

PROOF: Under the usual isomorphism \( R/R \cap P \cong S/P \).

By hypothesis \( R \cap P \) is a proper prime of R. Thus S/P is a finite module over a noetherian ring and is thus noetherian.

(1.4) COROLLARY. If D and J are integral domains, D is RMX and J is a finite integral overring of D, then J
is RMX.

The converse of 1.4 is, in fact, true and this will follow from 1.14. If R and S are rings with \( R \subseteq S \), then A is said to be a contracted ideal of R if there exists an ideal B of S such that \( A = B \cap R \).

(1.5) LEMMA. Let S be a finite integral overring of R. Suppose

1. A is a proper contracted ideal of R
2. S is noetherian
3. No proper prime of S contracts to the zero ideal of R.

Then there exist proper prime ideals \( P_1, \ldots, P_k \) of R such \( \langle P_1 \ldots P_k \rangle^e \subseteq A \).

PROOF: Since S is noetherian there exist \( Q_1, Q_2, \ldots, Q_k \), proper prime ideals of S such that \( Q_1 \ldots Q_k \subseteq A^e \). By hypotheses \( Q_i \cap R = P_i \neq (0) \). Thus \( \langle P_1 \ldots P_k \rangle \subseteq \langle P_1 \ldots P_k \rangle^e \subseteq \langle Q_1 \ldots Q_k \rangle^c \subseteq A^e = A \).

(1.6) LEMMA. Suppose R is an RMX ring and S is a noetherian finite integral overring of R. Then if \( P_1, P_2, \ldots, P_n \) are proper primes of R, \( R/(P_1 \ldots P_n)^e \) is noetherian.

PROOF: We proceed by induction on n. The case \( n = 1 \) is included in the hypothesis; for if \( P \) is a prime of R, \( p^e = p [14; p.257] \). Suppose then that the conclusion
of the lemma holds for the product of less than \( n \) primes. Let \( Z \) be a prime ideal of \( R/(P_1 \ldots P_n)^{ec} \). Denote by \( Q \) the inverse image of \( Z \) under the canonical homomorphism. It is sufficient to show that \( Q \) is finitely generated modulo \( (P_1 \ldots P_n)^{ec} \). We can then conclude that the primes of \( R/(P_1 \ldots P_n)^{ec} \) are finitely generated. \( R/(P_1 \ldots P_n)^{ec} \) will then be noetherian by Cohen's Theorem [2; p. 29]. Since \( (P_1 \ldots P_n)^{ec} \subseteq (P_1 \ldots P_n)^{ec} \subseteq Q \), one of the \( P_i \), say \( P_n \), is contained in \( Q \). By hypothesis, \( Q \) is finitely generated modulo \( P_n \). It is therefore sufficient to show that \( P_n \) is finitely generated modulo \( (P_1 \ldots P_n)^{ec} \).

Consider the \( R \) modules \( P_n/(P_1 \ldots P_n)^{ec} \) and \( P_n/(P_1 \ldots P_n)^{ec} \). Since \( P_n^{ec} = P_n \) [14; p. 257], we have \( P_n/(P_1 \ldots P_n)^{ec} \subseteq P_n/(P_1 \ldots P_n)^{ec} \) under the canonical isomorphism. \( S \) is noetherian, thus \( P_n^{ec}/(P_1 \ldots P_n)^{ec} \) is a finitely generated \( S \) module. Further, since \( (P_1 \ldots P_{n-1})^{ec} \cdot P_n^{ec} = (P_1 \ldots P_{n-1})^{ec} \), \( P_n/(P_1 \ldots P_n)^{ec} \) is a finite \( S/(P_1 \ldots P_{n-1})^{ec} \) module. Now since \( S \) is a finite \( R \) module, \( S/(P_1 \ldots P_{n-1})^{ec} \) is a finite \( R \) module. We can thus conclude from \( (P_1 \ldots P_{n-1})^{ec} \cdot S = (P_1 \ldots P_{n-1})^{ec} \) that \( S/(P_1 \ldots P_{n-1})^{ec} \) is a finite \( R/(P_1 \ldots P_{n-1})^{ec} \) module. Thus \( P_n/(P_1 \ldots P_n)^{ec} \) is a finite \( R/(P_1 \ldots P_{n-1})^{ec} \) module. But \( (P_1 \ldots P_{n-1})^{ec} \cdot P_n \subseteq (P_1 \ldots P_n)^{ec} \) implies that \( P_n/(P_1 \ldots P_n)^{ec} \) is also an \( R/(P_1 \ldots P_{n-1})^{ec} \) module and therefore \( P_n/(P_1 \ldots P_n)^{ec} \) is a sub-module of a finitely generated
unitary module over a noetherian ring. \( P_n/(P_1\ldots P_n)^{\text{ec}} \) is therefore finitely generated over \( R/(P_1\ldots P_{n-1})^{\text{ec}} \) [14;p.158]. It then follows that \( P_n \) is a finitely generated \( R \) module modulo \( (P_1\ldots P_n)^{\text{ec}} \) and the lemma is established.

(1.7) LEMMA. If \( R \) is a ring, then \( R \) is noetherian if and only if \( R/A \) is noetherian for every proper ideal \( A \) of \( R \).

(1.8) COROLLARY. Let \( R \) be an \( \text{RMX} \) domain with quotient field \( K \). Suppose \( S \) is a finite integral overring of \( R \) such that \( R \subseteq S \subseteq K \). Then \( S \) is noetherian if and only if \( R \) is noetherian.

PROOF: The sufficiency is well known [14;p.158]. To prove the necessity, we first observe that there exists a non-zero conductor \( f \) of \( S \) over \( R \). If \( A \) is a proper ideal of \( R \), we claim that \( R/A \) is noetherian. Since \( AS = A^e \) is a proper ideal of \( S \) and \( S \) is a domain, \( A^ef = ASf = Af \) is a proper ideal of \( S \). Now \( Af \subseteq A \), thus \( Af \) is a contracted ideal. By 1.5, \( A \) contains an ideal of the form \( (P_1\ldots P_n)^{\text{ec}} \), where the \( P_i \) are proper primes of \( R \). By 1.6, \( R/(P_1\ldots P_n)^{\text{ec}} \) is noetherian. Now \( [R/(P_1\ldots P_n)^{\text{ec}}]/[A/(P_1\ldots P_n)^{\text{ec}}] \cong R/A \) and thus \( R/A \) is a homomorphic image of a noetherian ring. \( R/A \) is therefore noetherian and \( R \) is noetherian by 1.7.

In 1.8 the hypothesis that \( S \) be integral over \( R \) is necessary even if \( S = R[x], \ x \in K \). To see this, let \( R \)
be a rank 2 discrete valuation ring with maximal ideal $M$ and minimal prime $P$. Then if $1/x \in M \setminus P$, $R[x]$ is a rank 1 discrete valuation ring but $R$ is not noetherian. We note that finiteness is a necessary part of the hypothesis even if $S$ is assumed to be integral over $R$. For example, let $K$ be an infinite algebraic extension of the rational field $k$. Let $X$ be transcendental over $K$ and form the formal power series ring $K[[X]] = R$. Let $B$ denote the quotient field of $R$. We observe that $R = K + M$, where $M$ is the maximal ideal of $R$. Let $D = k + M$. Then $D < R < B$. $B$ is the common quotient field of $R$ and $D$ since they have a common proper ideal. What is more, $R$ is the integral closure of $D$ in $B$. Although $R$ is a rank 1 discrete valuation ring, $D$ is not noetherian, for $R$ is contained in the finite $D$ module $D \cdot X^{-1}$. Were $D$ noetherian, $R$ would be a finite $D$-module and it would follow that $K$ is finite over $k$.

(1.9) LEMMA. Let $D$ be a domain with quotient field $K$, $J$ a unitary overring of $D$ such that $J \cap K = D$. Then every proper ideal of $D$ contains a proper contracted ideal.

PROOF: If $a$ and $c$ are non-zero elements of $D$ such that $c \in (a)^{ec}$, then $c = ab$, $b \in J$. Setting $b = c/a$, we have $b \in J \cap K$ and thus $(a) = (a)^{ec}$. 
(1.10) LEMMA. Let $D$ be an RMX domain with quotient field $K$, $J$ a finite integral overring of $D$ and let $D^* = J \cap K$. If $D^*$ is RMX, $J$ is noetherian if and only if $D^*$ is noetherian. Moreover, if $D^*$ is a finite $D$ module, $J$ is noetherian if and only if $D$ is noetherian.

PROOF: By 1.9, every ideal of $D^*$ contains a proper contracted ideal. Thus by applying 1.5 and 1.6, we conclude that $D^*$ is noetherian if and only if $J$ is noetherian. If $D^*$ is a finite $D$ module, $D^*$ is an RMX domain by 1.4. Hence by 1.8, $D^*$ is noetherian if and only if $D$ is noetherian and therefore $D$ is noetherian if and only if $J$ is noetherian.

(1.11) LEMMA. Suppose $D$ is a domain with quotient field $K$. Let $L$ be a field properly containing $K$ and $q$ an element of $L$ integral over $D$. Then there exist domains $D^* \subset K$ and $J \subset L$ such that:

1. $D^*$ is a finite integral overring of $D$.
2. $J$ is a finite integral overring of $D[q]$.
3. $J \cap K = D^*$.

PROOF: Let $t_1, t_2, \ldots, t_{k+2}$ be the coefficients of the monic irreducible polynomial satisfied by $q$ over $K$. The $t_i$ are integral over $D$ and $1, q, \ldots, q^k$ is a basis for the field $K(q)$ over $K$. Now let $D^* = D[t_1, \ldots, t_{k+2}]$ and $J = D[q, t_1, \ldots, t_{k+2}] = D^*[q]$. The representation for $a \in J$
as $a = a_0 + a_1 q + \ldots + a_k q^k$ is unique. Thus $a \in K$ implies $a = a_0 \in D^*$ so that $J \cap K = D^*$. It is obvious that $J$ is a finite integral overring of $D[q]$ and that $D^*$ is a finite integral overring of $D$.

(1.12) THEOREM. Let $D$ be an RMX domain with quotient field $K$. Suppose that $S$ is a domain and a finite integral overring of $D$. Then $S$ is noetherian only if $D$ is noetherian.

PROOF: $S$ has the form $D[q_1, \ldots, q_n]$. By induction we can assume $n = 1$, for $D[q_1, \ldots, q_{n-1}]$ is an RMX ring by 1.4. We are then reduced to the situation $D < D[q]$. By 1.11 there exists $D^*$, a finite integral overring of $D$ with quotient field $K$ such that $D^* = D^*[q] \cap K$. If $D[q]$ is noetherian, then $D^*[q]$ is noetherian. It then follows from 1.10 that $D$ is noetherian.

(1.13) LEMMA. Suppose $R$ is a ring and $S$ is an integral overring of $R$. Then if $S$ satisfies the ascending chain condition (a.c.c.) on prime ideals, so does $R$.

PROOF: 1.13 is an immediate consequence of the "lying over theorem" [14;p.257].

(1.14) THEOREM. If $R$ is a ring and $S$ is a finite integral overring of $R$, then $S$ is noetherian if and only if $R$ is noetherian.
PROOF: The sufficiency is well known. To prove the necessity we first note that $R$ satisfies the a.c.c. on primes by 1.13. $R$ must also be an RMX ring. If not, then $E = \{P \mid P$ is a prime ideal of $R$ and $R/P$ is not noetherian$\}$ is not vacuous. The a.c.c. on primes is equivalent to the maximum condition on primes. Thus let $P$ be a maximal element of $E$. By the lying over theorem, there is a prime $Q$ of $S$ lying over $P$. Passing over to residues modulo $Q$ we have $R/P \subset S/Q$ with $R/P$ an RMX domain and $S/Q$ a finite integral over ring. Thus by 1.12, $R/P$ is noetherian. This contradicts the existence of $P$ and therefore $R$ is RMX.

If $S$ is a domain, then by 1.12 $R$ is noetherian. If $S$ is not a domain there are two cases to consider. In the first case, suppose a prime $N$ of $S$ contracts to the zero ideal of $R$. Then passing over to residues modulo $N$ we have $R \subset S/N$ and we are reduced to the domain case. If every prime of $S$ contracts to a proper prime of $R$, then $(0)$ has the form $(P_1\ldots P_n)^{ec}$, where the $P_i$ are proper primes of $R$. Then by 1.6, $R/(P_1\ldots P_n)^{ec} = R/(0) = R$ is noetherian and the theorem is proven.

Theorem 1.14 has numerous obvious corollaries. We include one simple application as a computational device.

Let $k$ be a field and $\{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over $k$. Let $(X_1, X_2, \ldots, X_n)$ be the ideal
of $k[X_1, X_2, \ldots, X_n]$ generated by the $X_1$. Then if $n$ is any positive integer, the ring $D = k + (X_1, X_2, \ldots, X_n)^n$ is noetherian since each $X_1$ is integral over $D$ and $D[X_1, X_2, \ldots, X_n] = k[X_1, X_2, \ldots, X_n]$.

As a special case of our general problem we consider the following question: if $R$ is a domain with quotient field $K$ such that $R[\alpha]$ is noetherian for every $\alpha \in K \setminus R$, need $R$ be noetherian? In general the answer is no. However if $R$ is not a valuation ring we will see that the answer is yes.

Let $Q$ be a linearly ordered group and $Q \oplus \mathbb{Z}$ the lexicographically ordered direct sum of $Q$ and the integers. Suppose $R$ is a valuation ring with the resulting ordered group as its value group [9,p.78]. If $K$ denotes the quotient field of $R$ and $\alpha \in k \setminus R$, then $R[\alpha]$ is either a rank one discrete valuation ring on $K$. If $J$ is a rank one, non-discrete valuation ring with quotient field $F$ and $\alpha \in F \setminus J$, then $J[\alpha] = F$. We thus have that every ring strictly between one of these rings and its quotient field is noetherian, but neither of them is noetherian. It turns out that these are essentially the only exceptions.

(1.15) THEOREM. Suppose $R$ is an integral domain which is not a valuation ring and $K$ is its quotient field. Then $R[\alpha]$ is noetherian for every $\alpha \in K \setminus R$ if and only if $R$ is noetherian.
PROOF: The sufficiency is a consequence of the Hilbert Basis Theorem. To prove the necessity of $R$ being noetherian, we first observe that by 1.14 we can assume that $R$ is integrally closed. By hypothesis $R$ is not a valuation ring, hence there exist $a$ and $b$ in $R$ such that neither $a/b$ nor $b/a$ is in $R$. Thus by hypothesis, $R[a/b]$ and $R[b/a]$ are noetherian. Let $\{V_a\}_{a \in \Delta}$ be the family of valuation overrings of $R$. Then given $a \in \Delta$, either $a/b$ or $b/a$ is in $V_a$ and consequently either $R[a/b]$ or $R[b/a]$ (the respective derived normal rings) is contained in $V_a$. Since $R$ is integrally closed, we have

$$R \subseteq R[a/b] \cap R[b/a] \subseteq \bigcap_{a \in \Delta} V_a = R.$$ 

Since $R[a/b]$ and $R[b/a]$ are noetherian, $R[a/b]$ and $R[b/a]$ are Krull \cite[12,p.118]{12} and therefore $R$ is Krull \cite[13,p.10]{13}. Now one dimensional Krull rings are Dedekind domains \cite[13,p.13]{13} and are therefore noetherian. We then assume that dimension $R \geq 2$. We now observe that such Krull rings have an interesting number theoretic property which we state in the form of a lemma.

(1.16) LEMMA. Let $R$ be a Krull ring with an infinite number of minimal primes and $a \in R$ with $a \neq 0$. Then there exists $b \in R$ such that:

(1) $a/b \notin R$

(2) If $c \in R$ and $ac/b \in R$, then $c/b \in R$. 


Let us assume this for the moment and complete the proof of 1.15. Since we can assume $D$ Krull and of dimension $\geq 2$, for $x \in D$, let $y$ be an element of $D$ which satisfies the conclusions of the lemma. We claim 

$$(\frac{x}{y})^c = (\frac{x}{y})D[\frac{x}{y}] \cap D = xD.$$ 

If $x_0 \in (\frac{x}{y})^c$, then let 

$$(*) \quad 0 = x_0 + \lambda_1(\frac{x}{y}) + \ldots + \lambda_n(\frac{x}{y})^n$$

with $\lambda_i \in D$ and $n$ minimal. If $n > 1$, then multiplying both sides of the expression $(*)$ by $\frac{\lambda_n}{y}$, and collecting terms, we arrive at:

$$0 = \lambda_{n-1}x_0 + \lambda_{n-2} \frac{\lambda_n}{y} + \ldots + \left(\frac{\lambda_n}{y}\right)^n,$$

which is an equation of integral dependence for $\frac{\lambda_n}{y}$ over $D$. Thus $\frac{\lambda_n}{y} \in D$ which allows us to write:

$$0 = x_0 + \lambda_1(\frac{x}{y}) + \ldots + \lambda_{n-1}(\frac{x}{y})^{n-1} + \left(\frac{\lambda_n}{y}\right)(\frac{x}{y})^{n-1},$$

which contradicts the minimality of $n$. Hence $(*)$ has the form $x_0 = qx$ with $q \in D$. It then follows that 

$D/xD \subseteq D[\frac{x}{y}]/(\frac{x}{y})D[\frac{x}{y}].$ However, modulo $(\frac{x}{y})$ every element of $D[\frac{x}{y}]$ is congruent to an element of $D$ and the inclusion thus becomes equality. Since $D[\frac{x}{y}]/(\frac{x}{y})D[\frac{x}{y}]$ is noetherian, $D/xD$ is noetherian for any $x \in D$. Therefore $D$ is noetherian by 1.7.

We now prove 1.16. Let $b$ be a non-unit of $R$ with the property that no minimal prime of $b$ contains $a$. There must exist such an element for by assumption $R$ has
infinitely many minimal primes and a is in only a finite number of them. Suppose \( bx = ac \) with \( x, c \in D \). In order to show that \( c/b \in R \), it is necessary and sufficient that \( v_\alpha(c) \geq v_\alpha(b) \) for each essential valuation \( v_\alpha \) of \( R \). Since both \( c \) and \( b \in R \), \( v_\alpha(c) \geq v_\alpha(b) \) whenever \( v_\alpha(b) = 0 \).

Let \( \{v_1\}_{i=1}^n \) be the essential valuations of \( b \). Then from \( bx = ac \) we have \( v_1(b) + v_1(x) = v_1(a) + v_1(c) \) for each \( i \).

By choice of \( b \), \( v_1(a) = 0 \) all \( i \). Thus \( v_1(b) + v_1(x) = v_1(c) \)

and since \( v_1(x) \geq 0 \) we have \( v_1(b) \leq v_1(c) \). We thus conclude \( (c/b) \in R \).

(1.17) COROLLARY. If \( D \) is a domain with quotient field \( K \), then every ring \( S \) such that \( D < S \subseteq K \) is noetherian if and only if:

1. \( D \) is a rank one, non-discrete valuation ring.
2. \( D \) is a rank two valuation ring such that \( D_P \) is rank one, discrete where \( P \) is the prime of height one.
3. \( D \) is a noetherian, one dimensional domain.

PROOF: It is easily seen that rings of the type (1) and (2) have the property that every ring between one of them and its quotient field is noetherian. If \( D \) is not a valuation ring, by the previous corollary, \( D \) is noetherian. What is more, for every finite chain \( C \) of primes of \( D \), there exists a valuation overring \( U \) of \( D \) whose chain of primes is centered on \( C \) [13;p.257]. But in order that \( U \)
be noetherian, it must be rank one. Thus the chain $C$ would have only a unit length and $D$ would have to be one-dimensional. That every ring between a one-dimensional noetherian domain and its quotient field is noetherian is well known. [2; p.30].
(2.0) **Rings.**

Although some ring theoretic properties are preserved under intersection, even under very favorable circumstances, the noetherian condition is not.

(2.1) **An example of a non-noetherian domain which is the intersection of two one dimensional, local subrings of its quotient field.**

Let $k$ be a field of characteristic zero and $X$ a transcendental element over $k$. Then $k(X^2)$ and $k(X^2+X)$ are subfields of $k(X)$ and it is not hard to show that $k(X^2) \cap k(X^2+X) = k$ [8;p.31]. Let $Y$ be transcendental over $k(X)$ and denote by $D$ the formal power series ring $k(X)[[Y]]$. $D = k(X) + M$, where $M$ is the maximal ideal of $D$. Let $R = k(X^2) + M$ and $J = k(X^2+X) + M$. By 1.14 each of $R$ and $J$ is noetherian. However, their intersection, $k + M$, is not noetherian (the argument is similar to that used in the example after 1.8).

We might remark that Example 2.13 will sound another note on this theme.

**Note:** The above constructions and the example which
follows Corollary 1.8 are special cases of a quite useful method which Gilmer [6,A-2] refers to as "The D+M Construction".

The next collection of examples requires a bit of introduction:

(2.2) Non-noetherian, Finite Dimensional Krull Rings.

Let k be a field and \( \{X_\alpha \}_{\alpha \in \Gamma} \) an infinite collection of algebraically independent elements over k. Then \( k[\{X_\alpha \}_{\alpha \in \Gamma}] \) is a non-noetherian Krull ring. Finite dimensional Krull rings which are not noetherian are not so easy to find. One-dimensional Krull rings are Dedekind domains and are therefore all noetherian. The best known example is due to Nagata [12,p.207]. This is his example of a three-dimensional noetherian domain whose integral closure is not noetherian. Since Nagata has shown that the derived normal ring of a noetherian domain is a Krull ring [12,p.118], his example also yields a three-dimensional, non-noetherian Krull ring. Since the derived normal ring of a two-dimensional noetherian domain must again be noetherian [11,p.120], a non-noetherian, two-dimensional Krull ring cannot be found in this manner. In [1,p.83], Bourbaki indicated a method for constructing a two-dimensional Krull ring with a non-finitely generated minimal prime. However, in [4, p.2], Eakin and Heinzer demonstrate that the Bourbaki example, or any obvious modification of it, must be
noetherian. This apparently left open the question of the existence of a non-noetherian, two-dimensional Krull ring. Using methods similar to those of Nagata, we will provide such an example. We also give a somewhat different analysis of Nagata's example, as well as an explicit formulation of the derived normal ring in question. We feel that this is a more elementary (though admittedly more computational) analysis of the example. We begin with a general construction due to Nagata [12,p.206], which gives quasi-local rings whose dimension and integral closure can be controlled. We will be contented with the cases giving dimension two and three, but the general case is obvious.

If K is a field of characteristic $p \neq 0$ and $K^*$ is a subfield of K, then elements $\{ z_{a} \}_{a \in \Gamma}$ of K are said to be $p$-independent over $K^*$ if $[K^*(z_{a_1}, \ldots, z_{a_n}) : K^*] = p^n$ for any finite collection $z_{a_1}, \ldots, z_{a_n}$. The collection $\{ z_{a} \}_{a \in \Gamma}$ is said to be a $p$-basis for K over $K^*$ if

(a) $K^p \subseteq K^*$, (b) $K^*([z_{a} \}_{a \in \Gamma}) = K$ and (c) $\{ z_{a} \}_{a \in \Gamma}$ is $p$-independent over $K^*$.

(2.3) Let $K$ denote a field of characteristic 2 which has an infinite 2-basis over a subfield $k$. (For example; if $\{ X_{i} \}_{i=0}^{\infty}$ is a collection of transcendental elements over $GF(2)$, set $K = GF(2) (X_1, X_2, \ldots, X_n, \ldots)$ and $k = GF(2) (X_1^2, X_2^2, \ldots, X_n^2, \ldots)$.) Let $\{ b_{i} \}_{i=1}^{\infty}$ and $\{ c_{i} \}_{i=1}^{\infty}$
denote disjoint collections of distinct elements of the 2-basis. For simplicity, we assume \( K = \bigcup K_i \) where \( K_0 = k \) and \( K_{i+1} = K_i(b_k,c_i) \). Let \( R_i = K_i[[x,y,z]] \) (the formal power series ring in three indeterminates over \( K_i \)), \( R = \bigcup R_i \) (notice \( R_i \subset R_{i+1} \)) and \( R^* = K[[x,y,z]] \). Let \( S \) be a ring such that \( R \subset S \subset R^* \) and let \( L \) be the quotient field of \( S \). The following are true:

1. \( R \) is a regular local ring of dimension 3 and \((x,y,z)\) is a regular system of parameters of \( R \)
2. If \( \xi \in S \) then \( \xi^2 \in R_0 \)
3. \( S \) is 3-dimensional, quasi local, and the derived normal ring of \( S \) is \( L \cap R^* \).

**Proof:** (2) is easily verified. By (2), \( R^* \) is integral over \( R_0 \) and hence over both \( R \) and \( S \). Since \( R^* \) is a power series ring in three indeterminates over a field, \( R^* \) is a three-dimensional, regular local ring [15,p.301]. Therefore \( S \) is a three-dimensional ring since \( R^* \) is integral over \( S \). As any ring which has a quasi-local integral over it must be quasi-local, both \( R \) and \( S \) are quasi-local. Since \( R^* \) is integrally closed [15,p.302], the derived normal ring of \( S \) must be \( L \cap R^* \). We have thus established (3). To see that \( R \) is a regular local ring we first note that \( x,y \) and \( z \) obviously generate the non-units of \( R \). Thus the maximal ideal of \( R \) has the
Let $a_1, \ldots, a_m$ generate an ideal $A$ of $R$. We claim that $AR^* \cap R = A$. Suppose $b \in AR^* \cap R$. Then

$$b = \sum_{i=1}^{m} a_i f_i^*$$

where $f_i^* \in R^*$. The elements $b, a_1, \ldots, a_m$ are in $K_n[[x,y,z]]$ for some $n$. We write $f_i^* = f_i + f'_i$ where the coefficients of $f_i$ are in $K_n$ and no coefficient of $f'_i$ is in $K_n$. Then let $\{l_i\}_{i=1}^{n}$ be a linear basis for $K$ over $K_n$ with $l_1 = 1$. Each coefficient of $f'_i$ thus has the form $\sum_{i=1}^{n} s_i l_i$ with $s_i \in K_n$ and not all $s_i = 0$. Hence each coefficient of $\Sigma a_i f'_i$ must also have this form. But $\Sigma a_i f'_i \in K_n[[x,y,z]]$ which implies every coefficient of $\Sigma a_i f'_i$ must be zero. We then have $b = \Sigma a_i f'_i$ and conclude that $AR^* \cap R = A$.

We have shown that finitely generated ideals of $R$ extend and contract to themselves through $R^*$. Since $R^*$ is noetherian [15,p.138] it follows that $R$ has the a.c.c. on finitely generated ideals. But this implies that $R$ has the a.c.c. on all ideals and we have established that $R$ is noetherian.

\[(2.4) \text{Example (Nagata).} \text{ Set } d = \Sigma_{i=0}^{\infty} b_i x^i + \Sigma_{i=0}^{\infty} c_i x^i.\]

Then $R[\hat{d}]$ is a three-dimensional local domain whose derived normal ring $T$ is not noetherian.

**PROOF:** From (1) and (3) of 2.3 we have that $R[\hat{d}]$
is a three-dimensional local ring. Let $F$ denote the quotient field of $R$. By 2.3, $T = F(d) \cap R^*$. We will now compute $T$.

Note first that there is an obvious family of elements of $T$ that are not in $R[d]$. This is the family $\{d_i\}^\infty_{i=0}$ which is defined inductively as follows:

\[ d_0 = d \]

\[ d_{i+1} = d - \left[ y_\sum_{j=0}^{i} b_j x^j + z_\sum_{j=0}^{i} c_j x^j \right] \]

Notice: \[ d_1 = y_\sum_{j=0}^{\infty} b_{1+j} x^j + z_\sum_{j=0}^{\infty} c_{1+j} x^j \]

Let \[ T' = R[d_0, d_1, d_2, \ldots, d_n, \ldots] \].

We claim: $T = T'$

To see this we first observe that $T' \subseteq R^*$ and $R[d] \subseteq T' \subseteq F(d)$. Thus $R[d] \subseteq T' \subseteq R^* \cap F(d) = T$. Hence to prove $T = T'$, it is sufficient to show that $T'$ is integrally closed. To this end we first note:

\[ T' = R[\{d_i\}^\infty_{i=0}] = \bigcup_{i=1} R_i[\{d_i\}^\infty_{i=0}] = \bigcup_{i=1} R_i[d_i] \].

To see this last equality, we see that the expression on the right is obviously contained in the middle expression. To see the other containment, observe the identity:

\[ d_i = b_i y + c_i z + x d_{i+1} \]

Thus $d_i \in R_{i+1}[d_{i+1}]$. It follows that if $0 \leq t \leq n$ and
0 \leq r \leq n, then

\begin{equation}
\tag{**}
R[d_r] \subseteq R_n[d_n].
\end{equation}

Any element \( \xi \) of \([\bigcup R_i][[d_i]]_{i=0}^{\infty} \) must be in \( R_{\ell}[d_{i=0}^{\ell}] \) for

some \( \ell \) and \( q \). By (**) \( \xi \in R_{\ell+q}[d_{\ell+q}] \) so \( T' = \bigcup R_i[d_i] \).

Since \( R_i[d_i] \subseteq R_{i+1}[d_{i+1}] \), to show \( T' \) integrally closed

it is sufficient to show that \( R_i[d_i] \) is integrally closed

for each \( i \). Let \( T_i \) be the integral closure of \( R_i[d_i] \).

Since \( R_i \) is integrally closed and \( T_i \) is an integral

overring, principal ideals of \( R_i \) extend and contract to

themselves through both \( T_i \) and \( R^* \) [5]. We thus have

\begin{align*}
XR^* \cap R_i &= xR_i \\
xT_i \cap R_i &= xR_i.
\end{align*}

Similarly we have

\begin{align*}
XR^* \cap T_i &= xT_i.
\end{align*}

Thus under the usual isomorphism:

\begin{align*}
R_i/xR_i \subseteq (R_i[d_i]/XR^*) \cap (R_i[d_i]) &\subseteq T_i/xT_i \subseteq R^*/xR^*
\end{align*}

Under this isomorphism, we have \( T_i/xT_i \) is integral over

\( R_i/xR_i \). We now gain some information about \( T_i/xT_i \).

Let \( \xi = \frac{f_1 + f_2d_i}{f_3} \in T_i \) with \( f_1, f_2, f_3 \in R_i \). Recall

\( \xi \in R^* \) and consider \( \xi f_3 = f_1 + f_2d_i \). In \( R^*/xR^* \) we have:

\begin{align*}
\xi f_3 &= \bar{f}_1 + \bar{f}_2 \bar{d}_i = \bar{f}_1 + \bar{f}_2(b_1y + c_1z) = \bar{f}_1 + b_1 \bar{f}_2y + c_1 \bar{f}_2z
\end{align*}

where \( \bar{f}_1 \) denotes the residue of \( f_1 \) modulo x. If \( \bar{f}_3 = 0 \),

then by the linear independence of \( 1, b_1 \) and \( c_1 \) over \( K_i \)

we have \( \bar{f}_1 = \bar{f}_2 = 0 \). Thus if \( x \) divides \( f_3 \), \( x \) divides

both \( f_1 \) and \( f_2 \) and by factoring, \( f_3 \) can be chosen so
that \( f_3 \neq 0(x) \). Hence, under the homomorphism \( R^* \to R^*/xR^* \), we have:

\[
\tilde{g} = \frac{\bar{f}_1 + \bar{f}_2 d_1}{\bar{f}_3} = \frac{f_1 + f_2(b_1y + c_1z)}{f_3}.
\]

Now \( \tilde{g} \) integral over \( R_1/xR_1 \) implies \( \tilde{g} \in \) integral closure \( (R_1/xR_1)[d_1] \cong (R_1/xR_1)[b_1y + c_1] \cong K_1[[y,z]] [b_1y + c_1z] \). We now apply the following.

(2.5) LEMMA. \( K_1[[y,z]] [b_1y + c_1z] \) is integrally closed.

Assuming the lemma for the moment, we have

\( \tilde{g} \in (R_1/xR_1)[d_1] \). Thus by uniqueness of representation, \( \frac{\bar{f}_1}{\bar{f}_3} \) and \( \frac{\bar{f}_2}{\bar{f}_3} \in R_1/xR_1 \). Thus there exist \( f_1', f_2' \in R_1 \) and \( f_1^*, f_2^* \in R^* \) such that

\[
f_3 f_2' + xf_2^* = f_2
\]

and

\[
f_3 f_1' + xf_1^* = f_1.
\]

Thus:

\[
\tilde{g} = \frac{(f_1'f_3 + xf_1^*) + (f_2'f_3 + xf_2^*)d_1}{f_3}
\]

We then have

\[
\tilde{g} = f_1 + f_2 d_1 + x[f_1^* + f_2^* d_1] \frac{d_1}{f_3}
\]

Since \( f_3 \neq 0(x) \) and \( x[f_1^* + f_2^* d_1] \in R^* \), we conclude

that \( \tilde{g}_1 = [\frac{f_1^* + f_2^* d_1}{f_3}] \in R^* \) [15, p. 148]. Now \( \tilde{g}, f_1', f_2' d_1 \in T_1 \),

so \( x[\tilde{g}_1] \in xR^* \cap T_1 = xT_1 \). Therefore \( \tilde{g}_1 \in T_1 \) and we have
\( \mathfrak{g} = f'_1 + f'_2 d_1 + xg_1 \). Let \( M_1 \) denote the maximal ideal of \( R_1 \). Since \( x \in M_1 \) we then have \( T_1 = R_1[d_1] + M_1 T_1 \). Since \( R_1 \) is complete, we can conclude that \( T_1 = R_1[d_1] \). [12,p.105]. Therefore, modulo 2.5 we have \( T = \bigcup R_1[d_1] \).

**Proof of 2.5.** Set \( S = K_1[[y,z]] \). We claim that \( S' = S[\mathfrak{b}_1 y + c_1 z] \) is integrally closed. First note \( S' \subseteq K_{i+1}[[y,z]] \) which is integrally closed. Thus \( K_{i+1}[[y,z]] \) contains \( J \), the derived normal ring of \( S' \). Notice that \( \{1, b_1, b_1 c_1, c_1\} \) is an independent module basis for \( K_{i+1}[[y,z]] \) over \( S \). Thus if \( \tau = (g_1 + g_2 (b_1 y + c_1 z))/g_3 \in J^2 \) (with \( g_1, g_2, g_3 \in S \)) we must have

\[
\frac{g_1}{g_3} + \frac{g_2}{g_3} (b_1 y + c_1 z) = g_4 + b_1 g_5 + c_1 b_1 g_6 + c_1 g_7 \text{ with each } g_i \in S. \]

By uniqueness of representation we must have:

\[
\begin{align*}
g_1 &= g_3 g_4 \\
g_2 y &= g_3 g_5 \\
g_2 z &= g_3 g_7.
\end{align*}
\]

The first equation says \( g_1/g_3 \in S \). The second and third yield \( g_2/g_3 \in S \). This is because \( S \) is a UFD \([15,148]\): the second equation says that any irreducible factor of \( g_3 \) which is not equal to \( y \), must divide \( g_2 \) and the third says all irreducible factors of \( g_3 \) other than \( z \) are factors of \( g_2 \). Thus \( \tau \in S' \) and 2.5 is established.
We now show that $T$ is not noetherian. Since $T = \bigcup_{i=1}^{\infty} K_i[[x,y,z]][d_i]$, the non-units of $T$ are generated by $x,y,z$ and the collection $\{d_i\}_{i=0}^{\infty}$. But since $x d_{i+1} + y b_i + z c_i = d_i$, it follows that $x,y,$ and $z$ generate the non-units of $T$. Thus if $T$ were noetherian, it would have to be a regular local ring. Hence $T/zT$ would also be a regular local ring [15, p.303]. Now:

$$T/zT \cong \bigcup_{i=0}^{\infty} K_i[[x,y]][e_i]$$

where $e_i = y \sum_{j=0}^{\infty} b_j x^j$. A regular local ring must be integrally closed [15, p.302]. Thus $e_0/y = \sum_{j=0}^{\infty} b_j x^j$ must be in $T/zT$.

Hence $\sum_{j=0}^{\infty} b_j x^j \in K_t[[x,y]][c_t]$ for some $t$. We must then have: $\sum_{j=0}^{\infty} b_j x^j = f_t + g_t y \sum_{j=0}^{\infty} b_j x^j$ with $f_t, g_t \in K_t[[x,y]]$.

Reducing this expression modulo $y$, we get $\sum_{j=0}^{\infty} b_j x^j \in K_t[[x]]$.

This implies $b_j \in K_t$ for every $j$, however $K_t$ is a finite extension of $K_0$. The contradiction thus implies that $T$ is not noetherian.

This completes our analysis of Nagata's example. We might also note the following: Our formulation of $T$ shows that $T/zT = \bigcup_{i=0}^{\infty} K_i[[x,y]][e_i] = K_0[[x,y]][K][\{e_i\}_{i=0}^{\infty}]$. This identity shows that $T/zT$ is, in fact, another example of Nagata [12, p.207]. This is his example of a non-noetherian, quasi-local ring $T'$ which is between a two dimensional local ring $(\bigcup_{i=0}^{\infty} K_i[[x,y]][e_0])$ and its derived normal ring.
The ring \( T \), thus provides the following:

(2.6) Example. A three dimensional, quasi-local, non-
noetherian Krull ring \( T \) with a principal minimal prime \( zT \)
such that the derived normal ring of \( T/zT \) is not a finite
module over \( T/zT \).

PROOF: We have observed that \( T/zT \) is between a
two dimensional noetherian domain and its derived normal
ring. Thus the derived normal ring of \( T/zT \) is noetherian
for the derived normal ring of a two-dimensional noetherian
domain is noetherian [12, p.120]. It then follows from 1.14
that the derived normal ring could not be a finite module
lest \( T/zT \) be noetherian.

The following lemma is useful for constructing non-
noetherian rings.

(2.7) LEMMA. Suppose that \( R \) and \( S \) are domains, \( R \subseteq S \),
that \( S \) is integral over \( R \), and that \( R \) is integrally
closed with quotient field \( L \). Let \( \{y_i\}_{i=1}^{\infty} \) be a
collection such that:

1. \( y_i \in S \) for each \( i \).
2. \( [L(\{y_i\}_i):L(\{y_i\}_{i \neq j})] = [L(y_j):L] > 0 \)

Then if \( T = R[\{y_i\}_{i=1}^{\infty}] \) and \( A = \Sigma y_i T \), then \( A \) is finitely
generated if and only if \( A = T \).

PROOF: If \( A \) is finitely generated, we can assume
A = \sum_{i=1}^{n} y_i T. Hence

\[ (*) \quad y_{n+1} = \sum_{i=1}^{n} f_i y_i \]

where \( f_i \in R \{ y_i \}_{i=1}^{t} \) for some integer \( t \) (which can surely be greater than \( n \)). We then write this sum as

\[ (**) \quad y_{n+1} = h(y_1, \ldots, y_n, y_{n+2}, \ldots, y_t) + g(y_1, \ldots, y_t) y_{n+1} \]

where \( h \) and \( g \) have coefficients in \( R \) and no term in \( h \) has \( y_{n+1} \) as a factor. If \( y_{n+1} \) is of degree \( d \) over \( L \), it can be assumed that no power of \( y_{n+1} \) greater than \( d - 2 \) occurs in the expression \( g \). This is because, by the integral closure of \( R \), the minimal monic polynomial of \( y_{n+1} \) over \( L \) is an expression with coefficients in \( R \) [14, p. 260]. Hence by assumption (2) of the hypothesis and uniqueness of representation we conclude that

\[ h(y_1, \ldots, y_n, y_{n+2}, \ldots, y_t) = 0. \]

Thus it must be true that

\[ g(y_1, \ldots, y_t) = 1. \]

However, since \( y_{n+1} \) can occur on the right side of (*) only in the form \( y_i y_{n+1} \) for some \( i \) such that \( 1 \leq i \leq n \), it must follow that \( g(y_1, \ldots, y_t) \in A \). Hence \( A = T \).

(2.8) Example: With the notation of 2.3, let

\[ D = K_0[[x, y]][(b_1 x + c_1 y)]_{i=0}^{\infty}. \]

Then \( D \) is a non-noetherian, quasi-local, two dimensional Krull ring.

**PROOF:** In 2.7 set \( R = K_0[[x, y]], S = K[[x, y]] \) and \[ (**) \quad y_{n+1} = h(y_1, \ldots, y_n, y_{n+2}, \ldots, y_t) + g(y_1, \ldots, y_t) y_{n+1} \]

where \( h \) and \( g \) have coefficients in \( R \) and no term in \( h \) has \( y_{n+1} \) as a factor. If \( y_{n+1} \) is of degree \( d \) over \( L \), it can be assumed that no power of \( y_{n+1} \) greater than \( d - 2 \) occurs in the expression \( g \). This is because, by the integral closure of \( R \), the minimal monic polynomial of \( y_{n+1} \) over \( L \) is an expression with coefficients in \( R \) [14, p. 260]. Hence by assumption (2) of the hypothesis and uniqueness of representation we conclude that

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where \( h \) and \( g \) have coefficients in \( R \) and no term in \( h \) has \( y_{n+1} \) as a factor. If \( y_{n+1} \) is of degree \( d \) over \( L \), it can be assumed that no power of \( y_{n+1} \) greater than \( d - 2 \) occurs in the expression \( g \). This is because, by the integral closure of \( R \), the minimal monic polynomial of \( y_{n+1} \) over \( L \) is an expression with coefficients in \( R \) [14, p. 260]. Hence by assumption (2) of the hypothesis and uniqueness of representation we conclude that

\[ h(y_1, \ldots, y_n, y_{n+2}, \ldots, y_t) = 0. \]

Thus it must be true that

\[ g(y_1, \ldots, y_t) = 1. \]

However, since \( y_{n+1} \) can occur on the right side of (*) only in the form \( y_i y_{n+1} \) for some \( i \) such that \( 1 \leq i \leq n \), it must follow that \( g(y_1, \ldots, y_t) \in A \). Hence \( A = T \).
follows from 2.7 that the ideal of $D$ generated by the collection $\{(b_1x+c_1y)\}_{i=0}^\infty$ is not finitely generated. Hence $D$ is not noetherian. Since $K[[x,y]]$ is integral over $D$, $D$ is quasi-local and two dimensional. We now need only establish that $D$ is Krull.

Since $K[[x,y]]$ is Krull [13,p.12] and (integral closure $D)$ = $K[[x,y]] \cap$ (quotient field $D$), the integral closure of $D$ is therefore Krull [13,p.10]. We then need only show that $D$ is integrally closed. Write $\mathcal{D} = \bigcup_{i=0}^\infty D_i$ where $D_0 = K_0[[x,y]][b_0x+c_0y]$ and $D_{i+1} = D_i[b_i x+c_i y]$. It is sufficient to prove $D_n$ integrally closed for each $n$. We proceed by induction. The case $n=0$ is Lemma 2.5. If $D_n$ is integrally closed, we now show that $D_{n+1}$ is integrally closed.

Let $f_1, f_2, f_3 \in D_n$ and $g = f_1/f_3 + f_2/f_3 (b_{n+1}x+c_{n+1}y)$ be an element of the quotient field of $D_{n+1}$ which is integral over $D_{n+1}$. Now $D_{n+1} \subseteq K_{n+1}[[x,y]][b_{n+1}x+c_{n+1}y]$ which is integrally closed by 2.5. Hence $g \in K_n[[x,y]][b_{n+1}x+c_{n+1}y]$. By uniqueness of representation, we thus have $f_1/f_3$ and $f_2/f_3 \in K_n[[x,y]]$ and conclude that $f_1/f_3$ and $f_2/f_3$ are integral over $D_n$. By the induction hypothesis, $D_n$ is integrally closed. Thus $f_1/f_3, f_2/f_3 \in D_n$ and $g \in D_n[b_{n+1}x+c_{n+1}y]$. Therefore $D_{n+1}$ is integrally closed.
(2.9) Definition: A domain, $D$, is said to be prefactorial if every minimal prime of $D$ is the radical of a principal ideal.

(2.10) Lemma: The domain, $D$, of (2.8) is prefactorial.

Proof: Set $K[[x,y]] = D_0$ and $K[[x,y]] = D^*$. We obviously have $D_0 < D < D^*$. Each of $D_0$ and $D^*$ is a UFD [15,p.148] and therefore minimal primes of these domains are principal [13,p.16]. Let $P$ be a minimal prime of $D$. By the "lying over" theorem [14,p.257], there exists $P^*$, a minimal prime of $D^*$ such that $P^* \cap D = P$. Let $P_0 = P \cap D_0$. Since $D^*$ and $D_0$ are UFD's there are irreducible elements $\Pi$ and $\xi$ of $D_0$ and $D^*$ respectively such that $P_0 = \Pi D_0$ and $P^* = \xi D^*$. We have $\Pi = \xi^k f$ where $\xi$ does not divide $f$ since $\Pi \in P^*$. It follows that $k$ must be either 1 or 2 and $f$ must be a unit of $D^*$. To see this, recall that the square of any element of $D^*$ is in $D_0$. Since $\Pi^2 = \xi^{2k} f^2$. Since $D_0$ is a UFD and $f^2 \notin \Pi D_0$ we have $\Pi^2$ divides $\xi^{2k}$. Thus $\xi^{2k} = \Pi^2 j$ and it follows that $1 = j f$. Hence both $j$ and $f$ are units and we have $j^{-1}(\xi^2)^k = \Pi^2$. Since $\Pi$ is irreducible and $\xi^2 \in D_0$, $k$ is either 1 or 2. We then have that either $\xi^2 = y \Pi$ or $\xi = y \Pi$ for some unit $y$ of $D^*$. Each of these cases implies that $P$ has a principal primary. If $\xi = y \Pi$, we have $P = \Pi D$ and the assertion is obvious.
If \( \mathfrak{g}^2 = \mathfrak{N}D \), we first recall that \( D \) is a Krull ring and hence \( D_p \) is a rank one discrete valuation ring. We then have

\[
[(P^*)^2D_p^*] \cap D_p = PD_p \text{ or } P^2D_p
\]

Intersect both sides with \( D \) and we have

\[
[(P^*)^2 \cap D_p^*] \cap D = (PD_p \cap D) \text{ or } (P^2D_p \cap D)
\]

Since \( D \subseteq D^* \) we have

\[
[((P^*)^2D_p^*) \cap D^*] \cap D = P \text{ or } P^{(2)}.
\]

Hence:

(1) \((P^*)^{(2)} \cap D = P \text{ or } P^{(2)}\).

Since \( D^* \) is a UFD, \((P^*)^{(2)} = (P^*)^2\). But
\[(P^*)^2 = (\mathfrak{g}D^*)^2 = \mathfrak{g}^2D^* = \mathfrak{N}D^* \text{ and since } D^* \text{ is integral}
\]
over \( D \), we have \( \mathfrak{N}D^* \cap D = \mathfrak{N}D \). Equation (1) therefore yields: \( \mathfrak{N}D = P \) or \( P^{(2)} \). Thus either \( P \) or \( P^{(2)} \) is principal. In particular \( P^{(2)} \) is principal for every minimal prime \( P \) of \( D \).

We can now see that the domain, \( D \), of 2.8 provides other interesting examples.

If \( D \) is a domain, a family, \( \Delta \), of non-zero ideals of \( D \) is said to be a multiplicative system of ideals of \( D \) if \( A, B \in \Delta \) implies \( AB \in \Delta \). One can then generalize the
usual notion of quotient ring.

(2.11) Definition: (Krull-Ohm) If $D$ is a domain with quotient field $K$ and $\Delta$ is a multiplicative system of ideals of $D$, the quotient ring with respect to the multiplicative system $\Delta$, $D_\Delta = \{x \in K | \text{there exists } B \in \Delta \text{ such that } xB \subseteq D\}$. $D_\Delta$ is easily seen to be a ring and, in fact, $D_\Delta = \bigcup_{A \in \Delta} A_T$ where $A_T$ denotes the transform of $D$ with respect to $A$ [10,p.41]. These "generalized quotient rings" therefore also generalize the $A$-transform.

(2.12) Example: The domain, $D$, of 2.8 is a non-noetherian ring with the following property: If $\Delta$ is a multiplicative system of ideals of $D$, then $D < D_\Delta$ if and only if $D_\Delta$ is a Dedekind domain.

**Proof:** We always have $D \subseteq D_\Delta$; since $D$ is not noetherian, if $D_\Delta$ is Dedekind we must have $D < D_\Delta$. Conversely, if $D < D_\Delta = \bigcup_{A \in \Delta} A_T$, it must follow that $D < A_T$ for some $A \in \Delta$. Since $D$ is Krull, $A_T = \bigcap_{P \in \Gamma} D_P$ where $\Gamma$ is the collection of minimal primes of $D$ which do not contain $A$ [11,p.43]. Thus $A_T = (A^{-1})^{-1} \Gamma$ and we can therefore assume that $A$ is divisorial. We can then represent $A$ in the form $P_{\Delta}^{(e_1)} \cap \ldots \cap P_{n}^{(e_n)}$ for some finite collection of minimal primes of $D$. A trivial modification of the proof
that $D$ is prefactorial shows that $P^{(2k)}$ is principal for every $k \geq 1$. $A^2 = [P_1^{(e_1)} \cap \ldots \cap P_n^{(e_n)}]^2 \subseteq P_1^{(2e_1)} \cap \ldots \cap P_n^{(2e_n)}$.

This ideal is the intersection of principal primaries and is therefore principal. Let $q$ be the generator. Then $A^2 \subseteq qD$ and it follows that $q^{-1}A^2 \subseteq D$. Thus $q^{-1} \in A_T$ and we have $D[q^{-1}] \subseteq A_T$. Now $q$ is a non-unit of $D$ which is a two dimensional, quasi-local Krull ring. Hence $D[q^{-1}]$ is a one dimensional Krull ring and is therefore a Dedekind domain [13,p.13]. This implies that $A_T$ is an overring of a Dedekind domain and is therefore Dedekind [10,p.781].

(2.13) Example: The domain, $D_1$ of 2.8 is a non-Noetherian, two dimensional Krull ring which is, in an infinite number of ways, the intersection of a Dedekind domain and a rank one discrete valuation ring.

PROOF: We have seen that $P^{(2)}$ is principal for any minimal prime $P$ of $D$. If $P^{(2)} = \Pi D$, then $D[1/\Pi] \cap D_P = D$, as $D[1/\Pi]$ is the intersection of all essential valuations of $D$ whose centers on $D$ do not contain $\Pi$ [12,p.116]. Since $P^{(2)} = \Pi D$, no minimal prime other than $P$ contains $\Pi$.

We remark that we have shown that the ring $D$ of 2.8 has the following interesting property. Let $\Delta$ denote the family of minimal primes of $D$. Then if $\Gamma \subseteq \Delta$, $\bigcap_{P \in \Gamma} D_P$ is a Dedekind domain if and only if $\Gamma < \Delta$. 
In [5] Gilmer asks the following question: Suppose $D$ is an integrally closed domain with quotient field $K$. Let $L$ be a finite, separable extension of $K$ and let $J$ be the integral closure of $D$ in $L$. Then if $J$ is noetherian, need $D$ be noetherian? Although we cannot answer the question in full, we can make the following observations:

If $J$ is a finite $D$ module, as for instance when the discriminant is a unit of $D$, then $D$ must be noetherian by 1.14. Let $d^{-1}$ be the discriminant and consider the rings $D[d]$ and $J[d]$. $J[d]$ is noetherian and a finite $D[d]$ module. Thus by 1.14, $D[d]$ is noetherian. $J$ is an integrally closed, noetherian domain. Therefore $J$ is a Krull domain [12,p.118] and since $J \cap K = D$, $D$ is a Krull domain. $D[d]$ is then a quotient ring of a Krull ring. Thus $D[d]$ is simply the intersection of all essential valuation rings of $D$ which do not contain $d$ [12,p.116]. Since $d^{-1}$ is a non-unit in at most a finite number of the essential valuation rings of $D$, we conclude that $D$ can be represented as the intersection of a noetherian Krull ring and a finite number of rank one discrete valuation rings. In 2.13 we see that there are such rings.

(2.14) Question: Is the integral closure of the domain of Example 2.8 in some finite, separable extension of its quotient field noetherian?
We close with a few more questions.

(2.15) Question: Does either Example 2.8 or Nagata's Example (2.4) have a minimal prime that is not finitely generated? Does either ring have a minimal prime that is not principal? In general, is there a finite dimensional Krull ring with a non-finitely generated minimal prime?

(2.16) Question: With the notation of 2.3, is $k[[x,y]][b_0x+c_0y]$ a UFD? Is $k[x,y][b_0x+c_0y]$ a UFD?
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BIOGRAPHY

Paul M. Eakin, Jr. was born on August 7, 1942, in New Orleans, Louisiana. He attended the public schools of Bunkie and Alexandria, Louisiana. After graduation from high school he entered Louisiana State University in September, 1960. He received the degree of Bachelor of Science in Mathematics in June, 1964.

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