Finitely Generated Flat Modules and Determinantal Rank.

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ABSTRACT

The object of this paper is to determine sufficient conditions for a finitely generated flat R-module to be projective. All rings are commutative with identity. In Chapter I, it is shown that a finitely generated R-module M is projective if and only if M is flat and there is an exact sequence $0 \rightarrow M \rightarrow F \rightarrow E$ with F and E projective R-modules. A corollary is that a finitely generated R-module M is projective if and only if M is flat, reflexive and $M^* = \text{Hom}_R(M,R)$ is finitely presented. We give an example of a cyclic flat reflexive ideal M in a ring R such that $M^*$ is cyclic but M is not projective.

In Chapter II, we study the following condition on a ring R.

(FP) Each finitely generated flat R-module is projective.

It is shown that a subring of a ring satisfying (FP) must satisfy (FP). We also show that if the Jacobson radical of a ring R contains a finite intersection of prime ideals then R satisfies (FP).

In Chapter III, we list some questions which are still outstanding.
INTRODUCTION

By ring we mean commutative ring with identity. Let $R$ be a ring. $R$-module means unitary $R$-module. $\mathcal{C}_R$ denotes the category of $R$-modules and $R$-homomorphisms. Morphism means $R$-homomorphism, f.g. means finitely generated, and f.p. means finitely presented. A mono (epi) is a morphism which is one-to-one (onto). An iso is a mono which is also an epi. We use $(\cdot)^*$ to denote the functor $\text{Hom}_R(\cdot, R)$. $J(R)$ denotes the Jacobson radical of $R$.

**Definition.** $M \in \mathcal{C}_R$ is flat if the functor $\cdot \otimes M$ is exact.

**Definition.** $M \in \mathcal{C}_R$ is projective if the functor $\text{Hom}_R(M, \cdot)$ is exact.

It is well known that projective $R$-modules are flat. We are interested in determining when a f.g. flat $R$-module $M$ is projective. There are two ways of approaching this problem. First, one can seek a necessary and sufficient condition on $M$ for $M$ to be projective with $R$ being arbitrary. Second, one can try to characterize the class of rings $R$ which satisfy

(FP) each f.g. flat $R$-module is projective.

In the first way, one naturally searches for such a condition among the properties of a f.g. projective $R$-module.
For example, each f.g. projective R-module is f.p., and it is well known that a f.p. flat R-module is projective. Other properties of a f.g. projective R-module M are

(1) M is a submodule of a free R-module.
(2) M is a submodule of a free R-module F and \( F/M \) is a submodule of a free R-module.
(3) \( M^* \) is f.p.
(4) \( M^* \) is f.g.
(5) M is reflexive, i.e., the natural morphism \( M \to M^{**} \) is an iso.

The problem of determining if a f.g. flat R-module M satisfying (1) is projective was posed by M. Auslander in oral conversation. W. Smith considers the special case of f.g. flat ideals in [6]. For Chapter I we give an example of a f.g. flat ideal M of a ring R such that \( M^* \) is f.g., M is reflexive, but M is not projective. In the same chapter we prove that a f.g. flat R-module M satisfying (2) is projective. As a corollary we get that a f.g. flat R-module satisfying (3) and (5) is projective.

Well known conditions on a ring R sufficient to imply R satisfies (FP) are

(a) R is an integral domain.
(b) R is noetherian.
(c) R is semi-local, i.e., R has finitely many maximal ideals.
In [3], S. Endo proves that the following condition, which is a common generalization of (a), (b), and (c), also is sufficient for $R$ to satisfy (FP).

(d) $R$ has a multiplicative system $S$ of regular elements such that $R_S$ is semi-local.

Endo conjectures in [3] that (d) is also necessary and proves that it is in some special cases. The condition

(e) $R$ has primary ideals $Q_1, \ldots, Q_n$ such that $\cap Q_i = 0$

is shown to be sufficient for $R$ to satisfy (FP) by K. Mount in [5]. In Chapter II, we will show that each of the following is sufficient for $R$ to satisfy (FP).

(f) $R$ is a subring of a ring $R'$ which satisfies (FP).

(g) $R$ has prime ideals $P_1, \ldots, P_n$ such that $\cap P_i \subseteq J(R)$.

(h) $R$ is a subring of a ring $R'$ which satisfies condition (g).

Condition (h) generalizes (d) and (e). We will give an example of a ring satisfying (h) but not (d). Thus Endo's conjecture is false.
Let $R$ be a ring and $M$ an $R$-module. Let

(E) $0 \to K \to F \to M \to 0$ be an exact sequence with $F$ free.

These are equivalent:

(F1) $M$ is flat.

(F2) $x_1, \ldots, x_n \in K \to \exists F \xrightarrow{f} K \cdot 1 \cdot f(x_i) = x_i$
for all $i = 1, \ldots, n$.

(F3) $x \in K \to \exists F \xrightarrow{f} K \cdot 1 \cdot f(x) = x$.

If $M$ is f.g., then (F1) is equivalent to

(F4) $\forall$ maximal ideals $p$ of $R$, $M_p$ is a free $R_p$-module.

These are equivalent:

(P1) $M$ is projective.

(P2) The sequence (E) splits.

If $M$ is f.g., then each of (P3) and (P4) is equivalent to (P1).

(P3) $\forall$ maximal ideals $p$ of $R$, $\exists f \in R \setminus p \cdot 1$ $M_f$ is a free $R_f$-module.

(P4) $M$ is flat and f.p.

(L) If $M \xrightarrow{u} N$ is a morphism of $R$-modules, then $u$ is mono (resp. epi, iso) if and only if $\forall$ maximal ideals $p$ of $R$, $u_p$ is mono (resp. epi, iso).
If $(S)$ denotes any of the above statements and $(S')$ is the statement obtained from $(S)$ by replacing "maximal" by "prime", then $(S)$ and $(S')$ are equivalent.

For proofs see [1]. We use the above statements without reference.
In the first part of this chapter we prove that a f.g. flat submodule $M$ of a free $R$-module $F$ such that $F/M$ is a submodule of a free module is projective. For the second part we give an example of a f.g. flat submodule $M$ of a free $R$-module such that $M$ is reflexive, $M^*$ is f.g., but $M$ is not projective.

Let $R$ be a ring. If $E \rightarrow F$ is a morphism of f.g. free $R$-modules, then $\text{rk}(u)$, the rank of $u$, is defined by $\text{rk}(u) = \max\{n|Au \neq 0\}$ where $\Lambda^n$ denotes $n$th exterior power. If $X$ is the matrix of $u$ relative to bases of $E$ and $F$, then $\text{rk}(u)$ is also the size of a maximal non-zero minor of $X$. If $S$ is a multiplicative system in $R$, then $\text{rk}(u_S)$ denotes the rank of $u_S$ as an $R_S$-morphism.

1.1. Lemma. Let $R$ be a ring and $S$ a multiplicative system in $R$. Then the diagram

\[
\begin{array}{ccc}
C_R & \rightarrow & C_R \\
\downarrow & & \uparrow \\
C_{RS} & \rightarrow & C_{RS}
\end{array}
\]

\[
\begin{array}{ccc}
& \Lambda & \\
\downarrow & & \uparrow \\
& \Lambda & \\
\end{array}
\]

\[
\begin{array}{ccc}
& \Lambda & \\
\downarrow & & \uparrow \\
& \Lambda & \\
\end{array}
\]

commutes in the sense that the two composite functors are
naturally isomorphic.

**Proof.** We omit the proof: it follows directly from the definition of $\mathfrak{n}^\lambda$ and the fact that $\mathfrak{n}[\lambda(\cdot)]\otimes_R S$ and $\mathfrak{n}[\lambda(\cdot)]\otimes_R S$ are naturally isomorphic.

1.2. Corollary. If $E \xrightarrow{u} F$ is a morphism of f.g. free $R$-modules and if $S$ and $T$ are multiplicative systems in $R$ such that $S \subseteq T$, then $\text{rk}(u_T) \leq \text{rk}(u_S)$.

**Proof.** If $\mathfrak{n}(u_S) = 0$, then by 1.1 $(\mathfrak{n}u)_S = 0$. As $S \subseteq T$, $(\mathfrak{n}u)_T = 0$. By 1.1 $\mathfrak{n}(u_T) = 0$. Therefore $\text{rk}(u_T) \leq \text{rk}(u_S)$.

If the condition "$S \subseteq T$" in 1.2 is replaced by "$R \rightarrow R_T$ factors through $R \rightarrow R_S$", then the resulting statement is true.

1.3. Corollary. If $E \xrightarrow{u} F$ is a morphism of f.g. free $R$-modules and if $T$ is a multiplicative system in $R$, then $\forall f \in T \cdot \text{rk}(u_f) = \text{rk}(u_T)$.

**Proof.** Let $n = \text{rk}(u_T)$. Since $\mathfrak{n}^{n+1}(u_T) = 0$, $(\mathfrak{n}^\lambda u)_T = 0$ by 1.1. Since $\mathfrak{n}^{n+1}E$ is f.g., $\forall f \in T \cdot (\mathfrak{n}^{n+1}u)_f = 0$. By 1.1, $\mathfrak{n}^{n+1}(u_f) = 0$. Hence, $\text{rk}(u_f) < n + 1 = \text{rk}(u_T) + 1$. By 1.2, $\text{rk}(u_T) \leq \text{rk}(u_f)$. Hence, $\text{rk}(u_f) = \text{rk}(u_T)$.

1.4. Theorem. If $0 \rightarrow E \xrightarrow{u} F \xrightarrow{v} G$ is an exact sequence of f.g. free $R$-modules, then $\dim(E) + \text{rk}(u) = \dim(F)$. 
The proof is somewhat technical and will be given at the end of Chapter I.

**1.5. Theorem.** If \( E \xrightarrow{v} F \xrightarrow{u} G \) is an exact sequence of f.g. free \( R \)-modules such that \( \text{Image}(v) \) is a flat \( R \)-module and \( \dim(E) + \text{rk}(u) = \dim(F) \), then \( v \) is a mono.

**Proof.** Let \( p \) be a prime ideal of \( R \) and let \( M = \text{Image}(v) \). The sequence \( E_p \xrightarrow{v_p} F_p \xrightarrow{u_p} G_p \) is exact. Since \( M \) is flat and f.g., \( M_p \) is a free \( R_p \)-module. Since \( M_p = \text{Image}(v_p) \), \( \dim(M_p) \leq \dim(E_p) \). By 1.2, \( \text{rk}(u_p) \leq \text{rk}(u) \). Hence, \( \dim(E_p) = \dim(E) = \dim(F) - \text{rk}(u) \leq \dim(F) - \text{rk}(u_p) = \dim(F_p) - \text{rk}(u_p) \). Applying 1.4 to the exact sequence \( 0 \to M_p \subseteq F_p \xrightarrow{u_p} G_p \) of f.g. free \( R_p \)-modules, we get \( \dim(M_p) = \dim(F_p) - \text{rk}(u_p) \). Therefore, \( \dim(E_p) \leq \dim(M_p) \).

Thus \( E_p \to M_p \) is an epimorphism of f.g. free \( R_p \)-modules of the same dimension. It follows from [1, p.113, Corollary 5] that \( E_p \to M_p \) is an iso. Hence, for all prime ideals \( p \) of \( R \), \( v_p \) is a mono. Therefore, \( v \) is a mono.

**1.6. Theorem.** Let \( M \) be a f.g. \( R \)-module. Then \( M \) is projective if and only if \( M \) is flat and there is an exact sequence \( 0 \to M \to F \xrightarrow{u} G \) with \( F \) and \( G \) free \( R \)-modules.

**Proof.** If \( M \) is projective, then \( M \) is a direct summand of a free \( R \)-module, so clearly one can construct the required sequence. For the converse, we first reduce to the case when
F and G are f.g. Since M is f.g., it is contained in a f.g. free submodule F' of F. Then F'/M is contained in a f.g. free submodule G' of G. So we can construct an exact sequence 0 → M → F' → G' with F' and G' f.g. free. Hence, we may assume F and G are f.g. Let p be a prime ideal of R, \( \mathfrak{m} = \text{rk}(u_p) \), and \( m = \dim(M_p) \). By 1.3, \( \mathfrak{m} \in R \setminus p \cdot \mathfrak{m} \). Since \( M_p \) is a free \( R_p \)-module of dimension \( m \), there is an \( R \)-morphism \( R^m \xrightarrow{w_p} M \cdot \mathfrak{m} \) is an epimorphism (in fact \( w_p \) is an iso). Since M is f.g., \( \mathfrak{m} \in R \setminus p \cdot \mathfrak{m} \) is an epimorphism [1,p. 136, Prop. 2]. Let \( g = f_h \).

Now \( \text{rk}(u_g) = \mathfrak{m} \), \( g \in R \setminus p \), the sequence \( R^m \xrightarrow{w_g} F_g \xrightarrow{u_g} G_g \) of f.g. free \( R_g \)-modules is exact, and \( M_g = \text{Image}(w_g) \) is a flat \( R_g \)-module. By 1.4, \( \dim(M_p) + \text{rk}(u_p) = \dim(F_p) \). So \( m + \mathfrak{m} = \dim(F) \), i.e., \( \dim(R^m) + \text{rk}(u_g) = \dim(F_g) \). Therefore, by 1.5, \( w_g \) is a monomorphism. Hence \( M_g = \text{Image}(w_g) \) is a free \( R_g \)-module. We have shown that for each prime ideal \( p \) of \( R \), \( M_g \) is a free \( R_g \)-module. Hence, \( M \) is projective.

1.7. Corollary. Let \( M \) be a f.g. \( R \)-module. Then \( M \) is projective if and only if \( M \) is flat and there is an exact sequence \( 0 \rightarrow M \rightarrow F \rightarrow G \) with \( F \) and \( G \) projective \( R \)-modules.

Proof. If \( M \) is projective, we can let \( v = 1_M \) and \( u = 0 \). Conversely, since \( F \) is projective, \( \mathfrak{m} \cdot \mathfrak{m} \cdot F \oplus N \) is a free
R-module. Let \( F^\mathcal{C} \to F \oplus N \) be the canonical injection. The sequence \( 0 \to M \xrightarrow{\alpha} F \oplus N \xrightarrow{\iota \oplus N} G \oplus N \) is exact. Since \( G \) and \( N \) are projective, \( G \oplus N \) is also projective. Hence, \( G \oplus N \) is contained in a free \( R \)-module. Thus, by 1.6, \( M \) is projective.

1.8. Corollary. Let \( M \) be a f.g. \( R \)-module. Then \( M \) is projective if and only if \( M \) is flat, reflexive, and \( M^* \) is finitely presented.

Proof. If \( M \) is projective, then \( M \) is a direct summand of a f.g. free \( R \)-module. The additivity of the functor \((\cdot)^*\) and the fact that f.g. free modules are reflexive imply that \( M \) is reflexive and \( M^* \) is f.p. Of course, \( M \) is flat. Conversely, since \( M^* \) is f.p., there is an exact sequence \( F^* \to E^* \to M^* \to 0 \) with \( F \) and \( E \) f.g. free \( R \)-modules. Let \( M \xrightarrow{\alpha} M^{**} \) be the natural morphism. \( M \) reflexive means \( \alpha \) is an iso. So the sequence \( 0 \to M \xrightarrow{\epsilon^*} E^* \xrightarrow{f^*} F^* \) is exact with \( F^* \) and \( E^* \) f.g. free \( R \)-modules. By 1.6, \( M \) is projective.

1.9. Example. We give a construction which yields a cyclic flat non-projective ideal. Let \( S \) be a ring which admits a commutative \( S \)-algebra \( A \neq 0 \) satisfying

(1) there is a regular element \( t \) of \( S \) such that \( tA = 0 \).

(2) \( A \) has no multiplicative identity.
Let \( R = S \times A \) with the usual coordinate addition and multiplication defined by
\[(r,a)(s,b) = (rs, rb + sa + ab).\]

\( R \) is a ring. Let \( M \) be the ideal of \( R \) generated by \( r = (t,0) \) where \( t \) is a regular element of \( S \) such that \( tA = 0 \). The sequence
\[(E) \quad 0 \rightarrow \text{Ann}(M) \rightarrow R \rightarrow M \rightarrow 0\]
is exact. The morphism into \( R \) is the inclusion and the morphism out of \( R \) is defined by \( 1 \rightarrow r \). By (F3), \( M \) is flat if and only if \( x \in \text{Ann}(M) \) implies \( \exists y \in \text{Ann}(M) : xy = x \).

Let \( x \in \text{Ann}(M) \), say \( x = (s,a) \). Since \( xr = 0 \), \( ts = 0 \). As \( t \) is regular, \( s = 0 \). By (3), \( \exists b \in A : 3 \cdot ba = a \). Let \( y = (0,b) \).

Then \( yr = (0,b)(t,0) = (0, tb) = 0 \) since \( tA = 0 \). So \( y \in \text{Ann}(M) \). Also, \( yx = (0,b)(0,a) = (0, ba) = (0, a) = x \).

Therefore, \( M \) is flat. \( M \) is not projective: If \( M \) is projective, then \( (E) \) splits. So \( \text{Ann}(M) \) is generated by an idempotent \( g \) of \( R \). Since \( gr = 0 \) and \( t \) is regular \( g = (0,u) \) for some \( u \in A \). Since \( A \nmid 0 \), \( \text{ann}(M) \nmid 0 \). So \( u \nmid 0 \). Let \( c \in A \). Since \( tA = 0 \), we have \( (0,c)r = 0 \). So \( (0,c) \in \text{Ann}(M) \). Therefore, \( (0,c) = (0,c)g = (0,c)(0,u) = (0, cu) \).

Hence, \( c = uc \). Since \( c \) is arbitrary, \( u \) is a multiplicative identity of \( A \). This contradicts (2) so \( M \) is not projective.

We will specialize the construction given in 1.9 in
such a way that $M$ is also reflexive and $M^*$ is cyclic. We will need the following lemma.

1.10. Lemma. If $M$ is a cyclic ideal of a ring $R$ such that $\text{Ann} \text{Ann} (M) = M$, then $M^*$ is cyclic and $M$ is reflexive.

Proof. Let $c$ be the inclusion of $M \subseteq R$. Let $x$ generate $M$. The sequence $0 \rightarrow M \xrightarrow{c} R \xrightarrow{\phi} R/M \rightarrow 0$ is exact and yields an exact sequence $0 \rightarrow (R/M)^* \xrightarrow{c^*} R^* \rightarrow M^*$. In fact, $c^*$ is an epi: Let $f \in M^*$, i.e. $M \xrightarrow{f} R$. Then $f(x) \in \text{Ann} \text{Ann} (M)$ since $(r \in R, rx = 0) \Rightarrow rf(x) = f(rx) = 0$. Since $\text{Ann} \text{Ann} (M) = M = Rx$, $g \in R \cdot 3 \cdot f(x) = gx$. Let $g$ also denote the morphism $R \rightarrow R$ which is multiplication by $g$. Then $gc = f$ or, equivalently, $c^*(g) = f$. Thus $c^*$ is an epi. It follows that $M^*$ is cyclic and that $M^{**} \xrightarrow{c^{**}} R^{**}$ is a mono. The diagram

\[
\begin{array}{ccc}
M & \xrightarrow{c} & R \\
\downarrow{\alpha_M} & & \downarrow{\alpha_R} \\
M^{**} & \xrightarrow{c^{**}} & R^{**}
\end{array}
\]

commutes where the $\alpha$'s are the natural morphisms. Also $\alpha_R$ is an iso. Since the diagram commutes, $\alpha_M$ is a mono. Now $\text{Image}(c) = M$ and $\text{Image}(\alpha_R^{-1}c^{**}) = \text{Ann} \text{Ann}(M) = M$. Therefore, because $c^{**}$ is a mono, $\alpha_M$ is an epi. Hence, $M$
is reflexive.

1.11. Example. Consider the following special case of the construction in 1.9: Let $S$ be the ring of integers. Let $A$ be the set of functions $f$ from an infinite set $I$ into $S/(2)$ which are zero on the complement of a finite subset of $I$. With pointwise operations, $A$ is an $S$-algebra satisfying (1) - (3) in 1.9 where $t = 2$ in (1). Hence, the ideal $M$ of $R = S \times A$ generated by $r = (2,0)$ is flat but not projective.

In this case, $M^*$ is cyclic and $M$ is reflexive: By 1.10 we need only show $\text{Ann} \text{Ann}(M) = M$. Let $x \in \text{Ann} \text{Ann}(M)$. Write $x = (n,a)$. It is clear that $\text{Ann}(M) = (0) \times A$, so $x \in \text{Ann}((0) \times A)$. Hence $\forall g \in A, 0 = x(0,g) = (n,a)(0,g) = (0,ng+ag)$. Therefore, $\forall g \in A$, $ng + ag = 0$. Choose $g \in A$ for some $i \in I$ $g(i) = 1$ and $a(i) = 0$. Then $n1 = 0$, so $2k = n$ for some $k \in S$. Since $a \in A$, $0 = na + aa = 2ka + aa = 0 + aa = aa = a$. Therefore, $x = (n,a) = (n,0) = (2k,0) = (2,0)(k,0) \in R(2,0) = M$. Hence $\text{Ann} \text{Ann}(M) = M$. Therefore, $M$ is cyclic, flat, reflexive, not projective and $M^*$ is cyclic. This shows that the condition "$M^*$ is f.p." in 1.8 cannot be weakened to "$M^*$ is f.g."

Now we start the proof of 1.4. It is divided into three parts. Before starting Part A, we list some notation and results from linear algebra.

If $p$ and $q$ are integers, then $[p,q]$ denotes
the set of integers \( x \) such that \( p \leq x \leq q \). (If \( p > q \), then \([p,q] = \emptyset \)). If \( I \) is a set, \( |I| \) denotes the cardinality of \( I \). If \( k \) is an integer and \( I \subseteq [1,k] \), then \( I'(k) \) denotes the complement of \( I \) in \([1,k] \). If the role of \( k \) is clear from context, we will write \( I' \) rather than \( I'(k) \). Throughout the rest of this discussion \( R \) is a fixed ring and all modules mentioned are free. If \( E \rightarrow F \) is a morphism of \( R \)-modules and if \((e_j)\) and \((f_i)\) are bases of \( E \) and \( F \) respectively, we agree to calculate the matrix \( U = (u_{ij}) \) of \( u \) relative to these bases by writing \( u(e_j) = \sum_{i \in I} u_{ij}f_i \) for each \( j \in J \). If \( p \) is a positive integer and if \( H \subseteq I, K \subseteq J \) such that \( |H| = |K| = p \), then \( U_{H,K} \) denotes the corresponding \( p \)-minor of \( U \), i.e., \( U_{H,K} \) is the determinant of the \( p \times p \)-submatrix of \( U \) determined by the rows in \( H \) and the columns in \( K \). If \( p \) is a positive integer, then \( D(u,p) \) is the ideal of \( R \) generated by the \( p \)-minors of \( U \). \( D(u,p) \) is independent of the choice of bases [4]. If \( p \) is a positive integer and \( H \subseteq [1,p] \), then \( \rho_H = (-1)^\ell \) where \( \ell \) is the number of pairs \((i,j)\) in \( H \times H' \) such that \( j < i \).

Laplace's Expansion. Let \( X \) be an \( n \times n \)-matrix over \( R \) and let \( H \subseteq [1,n] \). Then

\[
\det(X) = \rho_H \sum_K \rho_K X_{H,K} X_{H',K'},
\]

where the summation is over all \( K \subseteq [1,n] \) such that \( |K| = |H| \).
1.12. Lemma. Let \( E \) and \( F \) be \( R \)-modules of finite dimensions \( m \) and \( n \) respectively. Then a morphism \( E \overset{u}{\rightarrow} F \) is a mono if and only if \( m \leq n \) and \( \text{Ann}(D(u,m)) = 0 \).

Proof of Laplace's Expansion appears in [2]. 1.12 is stated in [2,p. 98, Exercise 3].

Part A. Let \( E,F,G \) be \( R \)-modules with \( F \) of finite dimension \( n \). Let \( E \overset{V}{\rightarrow} F \overset{U}{\rightarrow} G \) be morphisms such that \( \text{Image}(V) \subseteq \text{Kernel}(U) \). If \( p \) and \( q \) are integers such that \( 0 \leq p,q \leq n \) and \( p + q > n \), then \( D(u,p)D(v,q) = 0 \).

Proof. We first reduce to the case when \( \dim(E) \geq n \) and \( \dim(G) \geq n \). Let \( H \) be an \( R \)-module. Let \( E \oplus H \overset{V}{\rightarrow} F \) be the morphism induced by \( V \) and \( H \overset{0}{\rightarrow} F \). Clearly, \( uV = 0 \) and \( D(\tilde{v},k) = D(v,k) \) for all \( k = 1,2,\ldots \). So we may assume \( \dim(E) \geq n \). Let \( G \overset{G}{\rightarrow} G \oplus H \) be the canonical injection. Then \( (cu)v = 0 \) and \( D(u,k) = D(cu,k) \) for all \( k = 1,2,\ldots \). So we may assume \( \dim(G) \geq n \). Now we reduce to the case when \( \dim(E) = \dim(F) = \dim(G) \). Suppose that Part A is true under these conditions. Let \( \{e_\ell\}_{\ell \in I} \), \( \{f_\ell\}_{\ell \in [1,n]} \) and \( \{g_j\}_{j \in J} \) be bases of \( E,F \) and \( G \) respectively. Let \( V \) and \( U \) be the matrices of \( V \) and \( U \), respectively, relative to these bases. Let \( K,T \subseteq [1,n] \), \( H \subseteq J \), \( S \subseteq I \cdot 3 \cdot |H| = |K| = p \) and \( |S| = |T| = q \). We must show \( V_S \cdot T_H \cdot K = 0 \). Choose \( N \subseteq I \cdot 3 \cdot |L| = n \) and \( L \supseteq H \). Choose \( L \subseteq J \cdot 3 \cdot |N| = n \) and \( N \supseteq S \). Let \( E_N \overset{C}{\subseteq} E \) be the inclusion of the submodule
generated by \( \{e_i\} \) into \( E \). Let \( G \twoheadrightarrow G_L \) be the projection of \( G \) onto the submodule \( G_L \) of \( G \) generated by \( \{g_j\} \).

Consider \( E_N \xrightarrow{\nu} F \xrightarrow{\mu} G_L \). We have \((\mu\nu)(vc) = 0\). It is clear that if \( V^# \) and \( U^# \) are the matrices of \( vc \) and \( \muu \), respectively, relative to \( (e_i)_{i \in N} \), \( (f_e)_{i \in [1,n]} \), and \( (g_j)_{j \in L} \), then \( V_{S,T}^v = V_{S,T}^u \) and \( U_{H,K}^v = U_{H,K}^u \). Since \( \dim(E_N) = \dim(F) = \dim(G_L) \), \( D(\muu,p)D(vc,q) = 0 \). Hence, \( U_{H,K}^v V_{S,T}^u = U_{H,K}^v V_{S,T}^u = 0 \). Thus we may assume \( \dim(E) = \dim(F) = \dim(G) \). Under this assumption, we may assume that, in fact, \( E = F = G \). Thus we have the following situation: \( u \) and \( v \) are endomorphisms of an \( n \)-dimensional \( R \)-module \( E \).

\[ uv = 0, \quad 0 \leq p, q \leq n \quad \text{and} \quad p + q > n. \]

We must show \( D(u,p)D(v,q) = 0 \). Let \( e_1, \ldots, e_n \) be a basis of \( E \) and let \( U = (u_{ij}) \) and \( V = (v_{ij}) \) be the matrices of \( u \) and \( v \), respectively, relative to \( e_1, \ldots, e_n \). Let \( H, K, S, T \subseteq [1, n] \).

\[ |H| = |K| = p \quad \text{and} \quad |S| = |T| = q. \]

We must show that \( V_{S,T} U_{H,K} = 0 \). To do this we construct an endomorphism \( w \) of \( E \) such that \( \det(w) = t V_{S,T} U_{H,K} \) for some regular element \( t \) of \( R \). Then to finish the proof we show that \( \det(w) = 0 \). Before constructing \( w \) we name some special endomorphisms of \( E \). For \( L \subseteq [1, n] \), \( E \twoheadrightarrow E \) is the projection defined by \( \pi_L(e_i) = 0 \) if \( i \in L \) and \( \pi_L(e_i) = e_i \) if \( i \in L \). Select one-to-one correspondences \( K' \xrightarrow{\sigma} H' \) and \( T' \xrightarrow{\tau} S' \). Let \( \alpha = \sigma^{-1} \) and \( \beta = \tau^{-1} \). Define \( E \xrightarrow{f} E \) by \( f(e_i) \) if \( i \in K \) and \( f(e_i) = e_{\sigma(i)} \) if \( i \).
Define \( f_\tau, f_\alpha, f_\beta \) similarly. We have the following properties.

(i) \( \pi_S + \pi_S' = \pi_K + \pi_{K'} = 1_E \)

(ii) \( f_\tau f_\beta = \pi_S', f_\alpha f_\sigma = \pi_K' \)

(iii) \( \pi_T f_\beta = f_\alpha \pi_H = \pi_H f_\sigma = \pi_S f_\tau = 0 = f_\tau \pi_T \)

(iv) \( f_\beta v \pi_T + 1_E \) and \( \pi_H u f_\alpha + 1_E \) are monos.

(i) - (iii) are routine calculations. For (iv), observe that \( (f_\beta v \pi_T + 1_E)(x) = 0 \Rightarrow -x = f_\beta v \pi_T(x) \Rightarrow \pi_T(x) = -f_\beta v \pi_T(x) = 0 \Rightarrow \pi_T'(x) = x = x = -f_\beta v \pi_T(\pi_T'(x)) = 0 \).

A similar argument establishes that \( \pi_H u f_\alpha + 1_E \) is also a mono. Now let \( w = (\pi_H u + f_\sigma)(v \pi_T + f_\tau) \). We calculate \( \det(\pi_H u + f_\sigma) \) as follows: Using (i) - (iii) we get that

\[
(\pi_H u f_\alpha + 1_E)(\pi_H u f_{K'} + f_\sigma) = \pi_H u + f_\sigma.
\]

By (iv) and 1.12, \( t_1 = \det(\pi_H u f_\alpha + 1_E) \) is a regular element of \( R \). Hence, \( \det(\pi_H u + f_\sigma) = t_1 \det(\pi_H u f_{K'} = f_\sigma) \), where \( t_1 \) is regular. It is easy to see that the matrix of \( \pi_H u \pi_K + f_\sigma \) relative to \( e_1, \ldots, e_n \) has the following form:

\[
\begin{bmatrix}
K' & K \\
H & H' \\
\end{bmatrix}
\begin{bmatrix}
u_{ij} & 0 \\
- & 0 \\
- & 0 \\
0 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

Hence, \( \det(\pi_H u \pi_K + f_\sigma) = \pm U_{H,K} \) by Laplace's Expansion. Thus, we have \( \det(\pi_H u + f_\sigma) = \pm t_1 U_{H,K} \). Similar arguments
show that \( \det(v\pi_T + f_T) = \pm t_2 V_S, T \) where \( t_2 = \det(f_E v\pi_T + l_E) \) is a regular element of \( R \). Hence, \( \det(w) = \det(\pi_H u + f_0) \det(v\pi_T + f_T) = t U_H, K V_S, T \) where \( t = \pm t_1 t_2 \) is regular. Now we show that \( \det(w) = 0 \). Let \( W \) be the matrix of \( w \) relative to \( e_1, \ldots, e_n \). We have

\[
\det(W) = \rho_T \Sigma_{L,L',T} W_L, T^W L', T \] 

by Laplace's Expansion. Let \( L \subseteq [1,n] \cdot 3 \cdot |L| = |T| = q \). Then \( L \cap H \neq \emptyset \) since \( |H| = p \) and \( p + q > n \). Choose \( j \in L \cap H \). Then \( \pi_{[j]} \pi_T \pi_{[j]} = (\pi_{[j]} \pi_T) (\pi_{[j]} \pi_T) = (\pi_{[j]} \pi_T) = 0 \). Hence, the \( j \)th row of the \( q \times q \) submatrix of \( W \) determined by rows in \( L \) and columns in \( T \) is zero. Therefore \( W_{L,T} = 0 \), and \( \det(W) = 0 \). Finally, since \( t \) is regular, \( U_H, K V_S, T = 0 \).

**Part B.** If \( E \rightarrow F \rightarrow G \) are morphisms of \( f.g. \) free \( \mathbb{R} \)-modules such that \( \text{Image}(v) \supset \text{Kernel}(u) \), then either \( \dim(F) \leq \text{rk}(u) + \dim(E) \) or \( D(v, \dim(E)) \subseteq \mathbb{V}_0 \).

**Proof.** Let \( m = \dim(E), n = \dim(F) \). Suppose \( D(v,m) \notin \mathbb{V}_0 \). Since we wish to show that \( m + \text{rk}(u) \geq n \), we may suppose \( m \leq n \). Since \( D(v,m) \notin \mathbb{V}_0 \), \( p \), a prime ideal of \( \mathbb{R}, 3 \cdot D(v,m) \notin p \). Clearly, \( [D(v,m)]_p = D(v_p, \mathbb{R}) = R_p \) and \( D((v_p)^*, m^*) = D(v_p, m) \). Hence, by \([2, p. 98, \text{Exercise 5}],[v_p]^*: (F_p)^* \rightarrow (E_p)^* \) is an epi. Thus, by \([1, p. 108, \text{Proposition 6}],[v_p] \) is a mono and \( v_p(E_p) \) is a direct summand of \( F_p \). Write \( v_p(E_p) \oplus H = F_p \) and let \( H \leq F_p \) be
the canonical injection. Evidently, $H$ is a free $R_p$-module of dimension $n-m$. Since $\text{Image}(v_p) \supseteq \text{Kernel}(u_p)$, $u_p c$ is a mono. Since $H$ is free of dimension $n-m$, $\text{rk}(u_p c) = n-m$. Since $\text{rk}(u_p c) \leq \text{rk}(u_p)$, we have $n-m \leq \text{rk}(u_p)$. By 1.2, $\text{rk}(u_p) \leq \text{rk}(u)$. Therefore $n-m \leq \text{rk}(u)$.

**Part C.** If $D$ is a f.g. ideal of $R$ such that $\text{Ann}(D) = 0$, then $D \not\subseteq V_0$.

**Proof.** Suppose $D \subseteq V_0$. Since $D$ is f.g., there exists an integer $3 \cdot D^n = 0$. Let $m + 1$ ($m \geq 0$) be the least such integer. Then, $0 = D^{m+1} = DD^m = D^m \subseteq \text{Ann}(D) = 0$ $\Rightarrow D^m = 0$. This contradiction proves $D \not\subseteq V_0$.

**Proof of 1.4.** Recall that we are given an exact sequence $0 \rightarrow E \rightarrow F \rightarrow G$ of f.g. free $R$-modules and must prove that $\text{rk}(u) + \text{dim}(E) = \text{dim}(F)$. Let $m = \text{dim}(E), n = \text{dim}(F)$. By 1.12, $\text{Ann}(D(v,m)) = 0$. Hence, by Part B and Part C, $\text{rk}(u) + m \geq n$. By Part A, $D(u, n-m+1)D(v,m) = 0$. Hence, $D(u, n-m+1) \subseteq \text{Ann}(D(v,m)) = 0$. Therefore $\bigwedge^n u = 0$, i.e., $\text{rk}(u) \leq n-m$. Therefore, $\text{rk}(u) + m = n$. 
We call a ring $R$ an FP-ring if each f.g. flat $R$-module is projective. In [3, Theorem 2] S. Endo proves that if $R$ is a ring which admits a multiplicative system $S$ of regular elements such that $R_S$ is semi-local, then $R$ is an FP-ring. In this chapter we obtain a strict generalization.

2.1. Theorem. Let $R$ be a subring of a ring $S$ (having the same identity). Let $M$ be a f.g. $R$-module. Then $M$ is projective if and only if $M$ is flat and $M_S$ is a projective $S$-module.

If $M$ is projective, it is immediate that $M$ is flat and $M_S$ is a projective $S$-module. Two proofs of the converse are given. The first is due to M. Auslander. S. Endo proves 2.1 for the case when $S$ is the total quotient ring of $R$ in [3]. If $I$ is a set and $X$ an $R$-module, then $X^I$ denotes the $R$-module of functions from $I$ into $X$ with pointwise operations. $R^I \otimes X \rightarrow X^I$ is the natural transformation defined by $\varphi (f \otimes x)(i) = f(i)x \ \forall i \in I \ \forall f \in R^I \ \forall x \in X$.

2.2. Lemma. An $R$-module $X$ is f.p. if and only if for each set $I$ $R^I \otimes X \cong X^I$ is an isomorphism.
M. Auslander has proved this result using the theory of coherent functors. We supply a direct proof as follows:

2.3. Lemma. An R-module $X$ is f.g. if and only if for each set $I$, $R^I \otimes X \rightarrow X^I$ is an epimorphism.

Proof of 2.3. If $X$ is f.g., then there is a f.g. free $R$-module $F$ and an epi $F \rightarrow X$. Let $I$ be a set. The diagram

\[
\begin{array}{ccc}
R^I \otimes F & \xrightarrow{\sigma_F} & F^I \\
I \otimes \varphi \downarrow & & \downarrow \varphi^I \\
R^I \otimes X & \xrightarrow{\sigma} & X^I
\end{array}
\]

commutes. Since $F$ is f.g. and free, $\sigma_F$ is an iso. Clearly, $\varphi^I$ is an epi. Therefore $\sigma_X$ is an epi. Conversely since $\sigma_X$ is an epi for all sets $I$, $R^X \otimes X \rightarrow X^X$ is an epi. Hence, $\exists y \in R^X \otimes X \cdot \sigma(y) = 1_X$. Write $y = f_1 \otimes x_1 + \ldots + f_n \otimes x_n$ with $f_i \in R^X, x_i \in X$. Then, for $x \in X$, $x = l_X(x) = \sigma(y)(x) = f_1(x)x_1 + \ldots + f_n(x)x_n$. Hence, $x_1, \ldots, x_n$ generate $X$.

Proof of 2.2. If $X$ is f.p., then by [1,p. 62,Exercise 9], $R^I \otimes X \rightarrow X^I$ is an iso for each set $I$. Conversely, $X$ must be f.g. by 2.3. Hence, there is an exact sequence

$0 \rightarrow K \rightarrow F \rightarrow X \rightarrow 0$ with $F$ f.g. and free. Let $I$ be a set. The diagram

\[
\begin{array}{ccc}
R^I \otimes K & \xrightarrow{\sigma_K} & R^I \otimes F & \xrightarrow{\sigma_F} & R^I \otimes X & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
K^I & \xrightarrow{\sigma_K} & F^I & \xrightarrow{\sigma_F} & X^I & \rightarrow 0
\end{array}
\]
commutes and has exact rows. $\sigma_X$ and $\sigma_F$ are isos. Hence $\sigma_K$ is an epi. Since $I$ is arbitrary, $K$ is f.g. by 2.3. Therefore, $X$ is f.p.

**First proof of 2.1.** It is sufficient to show that $M$ is f.p. Let $I$ be a set. Denote by $c$ the inclusion $R^I \subseteq S^I$. The following diagram commutes.

$$
\begin{array}{ccc}
R^I \otimes_R M & \xrightarrow{c \otimes \text{id}_M} & S^I \otimes_R M \\
\sigma_M & \downarrow & \sigma_S \otimes M \\
M^I & \rightarrow & (S \otimes_R M)^I
\end{array}
$$

Since $S \otimes_R M$ is a f.g. projective $S$-module, it is f.p. So $\sigma_S \otimes M$ is an iso by 2.2. Since $M$ is flat, $c \otimes \text{id}_M$ is a mono. As the diagram commutes, $\sigma_M$ is a mono. Since $M$ is f.g., $\sigma_M$ is an epi. Therefore $\sigma_M$ is an iso. Hence by 2.2. $M$ is f.p.

The second proof of 2.1. uses the Fitting invariants of $M$.

**2.4. Definition.** Let $R$ be a ring and $M$ a f.g. $R$-module. Let $F \xrightarrow{u} E \rightarrow M \rightarrow 0$ be an exact sequence with $F$ and $E$ free and $E$ f.g. Say $\dim(E) = n$. We use the notation of [5]. For each non-negative integer $j$, let $f(j,M)$ be the ideal of $R$ generated by the $(n-j)$-minors of a matrix of
The ideals \( f(j,M) \) are the Fitting invariants of \( M \). They depend only on \( M \) \([4]\). Let \( F(j,M) = R/f(j,M) \).

K. Mount proves in \([5]\) that a f.g. \( R \)-module \( M \) is flat (respectively projective) if and only if each of the cyclic \( R \)-modules \( F(j,M) \) is flat (respectively projective).

**Second proof of 2.1.** By \([5, Theorem 3]\) it is sufficient to show that each of the cyclic \( R \)-modules \( F(j,M) \) is projective. As \( M \) is flat, so is \( F(j,M) \) by \([5, Theorem 2]\). Since \( F(j,S \otimes_R M) = S \otimes_R F(j,M) \) and since \( S \otimes_R M \) is a projective \( S \)-module, \( S \otimes_R F(j,M) \) is a projective \( S \)-module by \([5, Theorem 3]\). Hence, it is sufficient to prove 2.1. when \( M \) is cyclic. For \( M \) cyclic, we let \( K = \text{Ann}_R(M) \). Because \( SK = \text{Ann}_S(S \otimes_R M) \) and because \( S \otimes_R M \) is a projective \( S \)-module, \( SK \) is generated by an idempotent \( e \) of \( S \).

Thus for \( y \in SK \) \( ey = y \). Write \( e = s_1k_1 + \ldots + s_nk_n \) with \( s_i \in S \) and \( k_i \in K \). Since \( M \) is flat, \( \exists y \in K \cdot 3 \cdot yk_i = k_i \) for all \( i = 1, \ldots, n \). Hence, \( e = ey = y \). Therefore, \( K = Re \). So \( M \) is projective.

**2.5. Theorem.** If \( R \) is a ring having prime ideals \( A_1, \ldots, A_n \) such that \( \cap A_i \subseteq J(R) \), then \( R \) is an FP-ring.

**Proof.** (cf. \([5, Corollary 3.2]\)) It is sufficient to prove each cyclic flat \( R \)-module is projective by \([5, Corollary 3.1]\). Let \( M \) be a cyclic flat \( R \)-module, \( K = \text{Ann}(M) \). Let \( p \) be a prime ideal of \( R \). Then \( M_p = 0 \) or \( M_p = R_p \). Hence,
K ⊆ p or K + p = R. We may assume K ⊆ A_i if and only if i ≤ t. Let B = A_1 ∩ A_2 ∩ ... ∩ A_t (B = R if t = 0) and let C = A_{t+1} ∩ ... ∩ A_n (C = R if t = n). Then K ⊆ B and K + C = R. Let x ∈ K, y ∈ C with x + y = 1. Let p be any maximal ideal of R. Since B ∩ C ⊆ J(R) ⊆ p, B ⊆ p or C ⊆ p. If B ⊆ p, then K ⊆ p so (Rx)_p ⊆ K_p = 0. If C ⊆ p, then x | p (else 1 = x + y ∈ p). Hence K_p ⊆ (Rx)_p = R_p. Thus, (Rx)_p = K_p for all maximal ideals p of R. Therefore, Rx = K. Therefore, M is f.p. and necessarily projective.

Remark. The condition in 2.5 that each A_i is prime may be replaced by the condition that each ideal A_i has the following property: If K ⊆ R and R/K is flat, then K ⊆ A_i or K+A_i=R.

Together 2.1 and 2.5 imply the following corollary.

2.6. Corollary. If R is a ring contained in a ring R' such that J(R') contains a finite intersection of prime ideals of R', then R is an FP-ring.

2.7. Example. Let D be an integral domain with infinitely many maximal ideals. Let M = ⊕D/p where the sum is over all maximal ideals p of D. Let R = D × M with the usual coordinate addition and multiplication defined by (d,m)(d',m') = (dd', d'm + dm'). By [l,p.179,Exercise 12] R is its own total quotient ring. The projection R → D has kernel M. Since D is a domain, M is a prime ideal of R.
$M^2 = 0$ so $M \subseteq J(R)$. Hence, by 2.5, $R$ is an FP-ring. For any multiplicative system $S$ of $R$ consisting of regular elements, $R = R_S$. Clearly, $R$ is not semi-local. Thus the criterion given in 2.6 is a strict generalization of the criterion given in [2, Theorem 2].
CHAPTER III

The following are problems related to the topics of Chapter I and II which are still outstanding.

3.1: Characterize FP-rings. In particular generalize the sufficient condition of Corollary 2.6. to a necessary and sufficient condition.

3.2: Suppose $M$ is a f.g. flat submodule of a free $R$-module. Find a necessary and sufficient condition for $M$ to be projective. Generalize the results of [5] about f.g. flat ideals.

3.3: Let $F$ be a free $R$-module of dimension $n < \infty$. For $k \leq n$ characterize the projective submodules of $F$ of rank $k$ (projective $R$-module $M$ is of rank $k$ if for every prime ideal $p$ of $R$ $\dim(M_p) = k$). W. Smith gives such a condition for $n = k = 1$ in [5]. The following is a partial generalization of his result for arbitrary $n$ but $n = k$. A f.g. submodule $M$ of $F$ is projective of rank $n$ if and only if $M$ is flat and $(F/M)^* = 0$.

Proof: We have an exact sequence $0 \to M \to F \to F/M \to 0$.

Suppose $M$ is flat and $(F/M)^* = 0$. Let $p$ be a prime ideal of $R$. Clearly, $\dim(M_p) \leq \dim(F_p) = n$. Since $F/M$ is f.p., $[(F/M)_p]^* \cong [(F/M)^*]_p$. Hence, $[(F/M)_p]^* = 0$. So $(v_p)^*$ is a mono. Therefore, $\dim(M_p) = \dim((M_p)^*) \geq \dim((F_p)^*) = n$. 26
Hence, \( \dim(M_p) = n \) for every prime ideal \( P \) of \( R \). So \( M \) is projective of rank \( n \). Suppose \( M \) is projective of rank \( n \). Let \( p \) be a prime ideal of \( R \). By Lemma 1.12, \( \text{Ann}(D(v_p,n)) = 0 \). Since \( D(v_p,n) = D((v_p)^*,n) \), \((v_p)^*\) is a mono by Lemma 1.12. Therefore \( [(F/M)_p]^* = 0 \). Since \( F/M \) is f.p., \( [(F/M)_p]^* \simeq [(F/M)^*]_p \). Therefore, \( [(F/M)^*]_p = 0 \) for every prime ideal \( p \) of \( R \). Hence \( (F/M)^* = 0 \).

3.4: Characterize rings \( R \) with the property that each f.g. flat submodule of a free \( R \)-module is projective. M. Auslander has shown that a sufficient condition for \( R \) to have this property is that arbitrary products of \( R \) with itself be submodules of flat \( R \)-modules.

3.5: Let \( M \) be an \( R \)-module. Determine necessary and sufficient conditions for \( M \) to be a submodule of a flat (or free) \( R \)-module.

3.6: If \( E \xymatrix{ \ar[r] & F \ar[r] & G } \) are morphisms of f.g. free \( R \)-modules such that \( \text{Image}(v) \supseteq \text{Kernel}(u) \), then is it necessary that \( \text{rk}(u) + \text{rk}(v) \geq \dim(F) \)?
BIBLIOGRAPHY


BIOGRAPHY

Samuel Harry Cox, Jr. was born on September 8, 1942, in San Francisco, California. He received a high school diploma from R. L. Paschal High School, Fort Worth, Texas, in 1959. He received the B.A. and M.S. degrees from Texas Christian University in 1963 and 1965 respectively. He entered the graduate school of Louisiana State University in 1965.
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Title of Thesis:  FINITELY GENERATED FLAT MODULES AND DETERMINATAL RANK

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