1968

Rings Satisfying the Three Noether Axioms.

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GILBERT, Jr., John Randolph, 1941-
RINGS SATISFYING THE THREE NOETHER AXIOMS.

Louisiana State University and Agricultural and Mechanical College, Ph.D., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
RINGS SATISFYING THE THREE NOETHER AXIOMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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May, 1968
ACKNOWLEDGMENT

This dissertation was written under the supervision of Dr. Hubert S. Butts, Professor of Mathematics. I am grateful to Dr. Butts for his guidance and assistance in the writing of this dissertation and in the preparation of the manuscript.
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ABSTRACT

This dissertation is concerned with the ideal theory of a commutative ring \( R \) (which may not have an identity). A ring \( R \) is said to have property (N) provided the following three conditions are satisfied:

(1) The ascending chain condition on ideals of \( R \).

(2) Proper prime ideals of \( R \) are maximal.

(3) The ring \( R \) is integrally closed; and \( R \) has property (v) provided (1), (3) and

(2') dimension \( R \leq 1 \)

hold in \( R \). They are called N-rings and v-rings respectively. If \( R \) is a domain with an identity with either property (N) or (v) then \( R \) is a Dedekind domain and conversely.

If \( R \) is a ring and \( S \) is a ring with identity \( e \) containing \( R \) as a subring, then we denote \( \{r+ne|r \in R, n \text{ an integer} \} \) by \( R^*(S) \). In case \( D \) is a domain \( D^* \) will mean \( D^*(K) \) where \( K \) is the quotient field of \( D \).

We show that a domain \( D \) is an N-domain if and only if \( D \) is a product of distinct prime ideals in a Dedekind domain \( \tilde{D} \) which is a finite \( D^* \)-module. We also give a characterization of v-rings which contain a regular element and N-rings with zero divisors.
In Chapter IV we study rings $R$ of these three kinds: (i) each proper homomorphic image of $R$ is a general Z.P.I. ring, (ii) each proper homomorphic image of $R$ is a principal ideal ring (P.I.R.), (iii) $R$ is a domain in which each nonzero ideal is invertible. We note that any of the above properties is a generalization of the concept of a Dedekind domain.

Another generalization of the concept of a Dedekind domain is a domain $D$ with the following property: for each ideal $A$ of $D$ there exists a non-negative integer $m = m(A)$ such that $AD^m = \prod_{i=1}^{n} P_i$ where each $P_i$ is a prime ideal of $D$ (such a domain is called a $d$-domain). The domain of even integers is an example of a $d$-domain which is not a Dedekind domain.

We prove that a domain $D$ is an invertible $d$-domain if and only if $D$ is a prime ideal in the Dedekind domain $D^*$. We then give an example to show that the hypothesis that $D$ be invertible is necessary.
CHAPTER I
INTRODUCTION

This dissertation is concerned with the ideal theory of a commutative ring $R$ (which may not have an identity). We say that $R$ is \textit{integrally closed in its total quotient ring} $T$ (or, simply, \textit{integrally closed}) provided $R$ contains every element $a \in T$ such that $a$ is integral over $R$ (i.e. $a^n + r_1a^{n-1} + \ldots + r_n = 0$ for some $r_1, \ldots, r_n$ in $R$).

A ring is \textit{n-dimensional} ($n$ a non-negative integer), or has dimension $n$ ($\dim R = n$), provided there exists a chain $P_0 \subset P_1 \subset \ldots < P_n < R$ of prime ideals in $R$ and there is no such chain of prime ideals with greater length. If $R$ has no prime ideals except $R$, then we say that $\dim R = -1$.

A ring is said to have property (N) provided the following three conditions are satisfied:

(1) The ascending chain condition on ideals of $R$ (a.c.c.)

(2) Proper prime ideals (i.e. $\neq R,(0)$) of $R$ are maximal.

(3) The ring $R$ is integrally closed; and $R$ has property ($\nu$) provided (1), (3) and (2') $\dim R \leq 1$. 

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hold in R. Properties (N) and (v) are not equivalent even in a domain, but (N) always implies (v). We say that R has property (Pi) provided every ideal of R is a product of prime ideals of R (rings with this property are called general Z.P.I. rings). It is well known that if R is a domain with an identity then R has property (N) if and only if R has property (Pi). For a brief history see [3;32], and in addition see [16;53], [17;275], [9;80], [11], [15] and [13]. Rings having property (Pi) have been studied extensively - for example, see [13], [6;579], [8] and [2]. In [6] Gilmer studied domains without an identity which have property (Pi). In general (N) and (Pi) are not equivalent in a commutative ring - in fact the ring of even integers has property (N) and does not have property (Pi).

In Chapters II and III of this dissertation we investigate commutative rings having property (N), (or property (v)) and such rings will be called N-rings (v-rings). In the case when R is a domain they will be called N-domains (v-domains).

If R is a ring and S is a ring with an identity containing R as a subring, then we denote \( \{R+ne \mid r \in R, n \text{ an integer} \} \) by \( R^*(S) \). In case D is a domain \( D^* \) will mean \( D^*(K) \) where K is the quotient field of D unless stated otherwise.

We show that a domain D is an N-domain if and only
if $D$ is a product of distinct prime ideals in a Dedekind domain $D$ which is a finite $D^*$-module. In order to prove the above theorem, we first obtain a generalization of a theorem of Akizuki [14;25] which states that an integral domain $D$ with an identity has the restricted minimum (RM) condition if and only if $D$ satisfies axioms (1) and (2) above. See Theorem 3, its corollaries and Theorem 22 for this result. A ring $R$ is said to have the (RM) condition, (or be an RM-ring), provided $R/A$ has the descending chain condition (d.c.c.) on ideals, for all ideals $A \neq (0)$. In addition some results are obtained concerning $N$-rings ($\nu$-rings) with zero divisors. In particular if (1) and (2) hold in a ring $R$ which is not a domain then (3) is valid in $R$. Finally we investigate rings with the property that every proper residue class ring is an $N$-ring.

In Chapter IV we consider some alternatives to our definition of $N$-domain. For example, condition (3) is replaced by:

(3') The ring $R$ is integrally closed as an ideal (i.e. $R$ contains all elements $\alpha$ of $T$ for which there exists elements $r_1 \in R^1$ for $i = 1, \ldots, n$ such that $\alpha^n + r_1 \alpha^{n-1} + \ldots + r_n = 0$).

A ring has property $(N')$ provided (1), (2) and (3') hold in $R$. We show that a domain $D$ has property $(N')$ if and only if $D$ is an ideal in a Dedekind domain $\bar{D}$ such that $\bar{D}$ is
a finite $D^*$-module.

In addition to domains with property $(N')$, we study rings $R$ of these three kinds: (i) each proper homomorphic image of $R$ is a general Z.P.I. ring, (ii) each proper homomorphic image of $R$ is a principal ideal ring (P.I.R.), (iii) $R$ is a domain in which each non-zero ideal is invertible. We recall that a domain with identity in which either of the second two conditions hold is a Dedekind domain and we show that a domain with an identity satisfying the first property is a Dedekind domain.

Another generalization of the concept of a Dedekind domain is a domain $D$ with the following property: for each ideal $A$ of $D$ there exists a non-negative integer $m = m(A)$ such that $AD^m = \sum_{i=1}^{n} P_i$ where each $P_i$ is a prime ideal of $D$ (such a domain is called a d-domain). Clearly a d-domain with an identity is a Dedekind domain and there are d-domains without an identity - for example, the domain of even integers is a d-domain.

The notation and terminology are those of Zariski and Samuel, *Commutative Algebra* with the following exceptions - we do not require that a Noetherian ring have an identity element and we do not require that a domain have an identity element. In particular we use $\subseteq$ to denote containment and $<$ to denote proper containment. An ideal $A$ in a ring $R$ is proper provided $(0) < A < R$. The ring of integers will
be denoted by \( \mathbb{Z} \) and all rings considered are assumed to be commutative and have more than one element.

In addition we use the term **semi-prime ideal** \( A \) to mean \( A = \sqrt{A} \). Also we use the term **special primary ring** to mean a ring \( R \) with identity in which the only ideals are \( R, M, \) and powers of \( M \), where \( M \) is the unique maximal ideal of \( R \) and \( M^i = (0) \) for some \( i \in \mathbb{Z} \). An ideal \( A \) is called **regular** provided it contains a regular element of the ring.

Before considering \( N \)-rings (\( \nu \)-rings) we will first study the restricted minimum condition in domains without identity. In particular we will study the relationship between the a.c.c. and the (RM) condition in domains without identity. We first prove two lemmas which will be used in the main theorem.

**Lemma 1:** Let \( S \) be a ring with an identity containing \( R \) as a subring and let \( R^* = R^*(S) \). If \( P \) is a proper prime ideal in \( R^* \) such that \( P > R \), then \( P \) is maximal.

**Proof:** There exists a non-negative integer \( n \) such that \( R^*/R \cong \mathbb{Z}/(n) \). Since proper prime ideals are maximal in \( \mathbb{Z}/(n) \) and \( R^* > P > R \), it follows that \( P \) is maximal in \( R^* \).

**Lemma 2:** Let \( R, S \) and \( R^* \) be as in Lemma 1. If \( P \) is a
proper prime ideal in $R^*$ such that $P \cap R$ is a maximal ideal of $R$, then $P$ is maximal in $R^*$.

Proof: Clearly $P \cap R$ is a prime ideal in $R$ so $R/(P \cap R)$ is a field since $P \cap R$ is maximal and prime. Hence $R/(P \cap R) \cong (P + R)/P$ [17,144] and consequently $(P + R)/P$ is a field. Let $\theta + P$ be the identity of $(P + R)/P$ and $e + P$ the identity in $R^*/P$; then $(e + P)(\theta + P) = (\theta + P) = (\theta + P)^2$, which implies $e + P = \theta + P$ since $R^*/P$ is a domain. Since $(P + R)/P$ is an ideal in $R^*/P$ which contains the identity, we have $(P + R)/P = R^*/P$ and $R^*/P$ is a field.

Theorem 3: A domain $D$ has properties (1) and (2) if and only if $D^*$ has properties (1) and (2).

Proof: If $D$ has an identity then $D = D^*$ and the theorem is valid. Suppose $D$ does not have an identity and that properties (1) and (2) hold in $D$; then $D^*$ is Noetherian [5,184]. If $P$ is a proper prime ideal in $D^*$ such that $P > D$ then $P$ is maximal by Lemma 1. If $P$ is a proper prime in $D^*$ such that $P \not\in D$, then $D \neq P \cap D \supseteq PD \neq (0)$ and by Lemma 2 we see that $P$ is maximal. Finally we will show that if $D$ is prime in $D^*$ then $D$ is maximal. If $D$ is prime in $D^*$ and not maximal then there exists a maximal ideal $M$ of $D^*$ such that $D^* > M > D > (0)$. By [17,240] there exists a chain $D^* > M > P > (0)$ of prime ideals in $D^*$ such that $P \not\in D$. But we have just shown
that all prime ideals of $D^*$ different from $D$ are maximal and we have a contradiction. Therefore all proper prime ideals in $D^*$ are maximal. Conversely, if $D^*$ has properties (1) and (2) then clearly $D$ has property (1) since ideals of $D$ are ideals of $D^*$. By the theorem of Akizuki [14;25] $D^*$ has the (RM) condition, and consequently $D$ has the (RM) condition since ideals of $D$ are ideals of $D^*$. Let $P$ be a proper prime ideal of $D$; then $D/P$ is a domain with the d.c.c. (and hence is a field) so $P$ is maximal.

**Corollary 4:** The (RM) condition holds in a domain $D$ if and only if conditions (1) and (2) hold in $D$.

**Proof:** In [1;342] Akizuki proved that a regular RM-ring has the a.c.c. In any ring with the (RM) condition proper prime ideals are maximal, so conditions (1) and (2) hold. Conversely, if conditions (1) and (2) hold in $D$, then they hold in $D^*$ by Theorem 3. By [3;29] $D^*$ is therefore an RM-domain and $D$ is an RM-domain.

**Corollary 5:** A domain $D$ is an RM-domain if and only if $D^*$ is an RM-domain.
CHAPTER II
CHARACTERIZATION OF REGULAR $v$-RINGS AND N-DOMAINS

Theorem 6: If $R$ is a ring with an identity and $A$ is a regular ideal of $R$, then $A$ is a Noetherian ring if and only if $R$ is Noetherian and $R$ is a finite $A^* = A^*(R)$-module.

Proof: If $A$ is Noetherian, then $A^*$ is Noetherian by [5;184]. Since $A$ is an ideal in $R$ and in $A^*$, $A$ is contained in the conductor of $R$ over $A^*$. Let $a \in R$ and let $r$ be an element of $A$ regular in $R$; then $a \in A \subseteq A^*$, which implies that $a \in r^{-1}A^*$. Since $A^*$ is Noetherian and $r^{-1}A^*$ is finite over $A^*$, we see that $r^{-1}A^*$ is a Noetherian $A^*$-module. But $R \subseteq r^{-1}A^*$, so $R$ is a Noetherian $A^*$-module and hence $R$ is a Noetherian ring. Conversely, suppose $R$ is Noetherian and $R$ is a finite $A^*$-module; then by [4] $A^*$ is Noetherian and by [5;184] $A$ is Noetherian. Note that we did not use the hypothesis that $A$ is a regular ideal in the proof of the converse.

Lemma 7: Let $R$ be a subring of a ring $S$ with identity and let $R^* = R^*(S)$. If $P$ and $Q$ are prime ideals in $R^*$ such that $R^* > P > Q$ and $P \not
subseteq R$, then $R^* > P \cap R > Q \cap R$.

Proof: It is clear that $R > R \cap P \supseteq R \cap Q$.
Now suppose that $P \cap R = Q \cap R$, and choose $r \in R -(P \cap R) = R -(Q \cap R)$ and $p \in P - Q$; then $rp \in P \cap R = Q \cap R$ which implies $rp \in Q$ and hence $r \in Q$. But this contradicts our choice of $r \in R -(Q \cap R)$, so $P \cap R > Q \cap R$.

Lemma 8: Let $S$ be a ring with identity containing $R$ as a subring and let $R^* = R^*(S)$. If $P_1 < P_2 < \ldots < P_n < R$ is a chain of prime ideals in $R$, then there exists a chain of prime ideals $P_1^* < P_2^* < \ldots < P_n^*$ in $R^*$ such that $P_i^* \cap R = P_i$.

Proof: We first prove the lemma in the case that $S$ is a domain. Set $P_i^* = P_i R_{n-1} \cap R^*$, where $R_{n-1}$ is the quotient ring of $R$ with respect to $P_n$. Since $R_{n-1} \supset R^* \supset R$ and $P_1 R_{n-1} \cap R = P_1$, we have $P_i^* \cap R = P_i$ for $i = 1, \ldots, n$.

We now consider the case in which $S$ is a ring. There exists a prime ideal $P_i^*$ of $R^*$ such that $P_i^* \cap R = P_i$, and the proof is completed by applying the domain case to the domains $R/P_1 \subset R^*/P_i^*$. 

Lemma 9: If $A$ is a regular ideal of a ring $R$, then the total quotient ring of $A$ is equal to the total quotient ring of $R$.

Proof: Let $r$ be an element of $A$ which is regular in $R$, and let $a$ be a regular element of the ring $A$. If $ax = 0$ for $x \in R$, then $a(rx) = 0$ implies that
rx = 0 and x = 0. Hence a is regular in R.

**Theorem 10:** If A is a regular ideal of an integrally closed ring R, then A is integrally closed if and only if $A = \sqrt{A}$ in R.

**Proof:** Suppose A is integrally closed. If $x \in \sqrt{A}$ then $x^n \in A$ which implies $x \in A$ since A is integrally closed and therefore $A = \sqrt{A}$ in R. Conversely, suppose $A = \sqrt{A}$ in R and let x be an element of the quotient ring of A which is integral over A. Since R is integrally closed, it follows from Lemma 9 that $x \in R$. Furthermore, we have $x^{n+1} + a_n x^n + \ldots + a_0 = 0$ with $a_i \in A$ for $i = 0, \ldots, n$. This implies that $x^{n+1} \in A$ since $x \in R$ and A is an ideal of R. Hence $x \in \sqrt{A} = A$ and A is integrally closed.

**Theorem 11:** If R is a regular ring with total quotient ring T then R is a regular $\omega$-ring if and only if all of the following hold:

(a) R is a semi-prime ideal in a Noetherian, integrally closed ring S with identity;

(b) $R^*(T) = R^* \subseteq S \subseteq T$, S is a finite $R^*$-module, and dim $S \leq 2$;

(c) if $P$ is a prime ideal of S such that $P \not\subseteq R$, then height $P \leq 1$ [17,240].
Proof: Suppose that $R$ is a regular $\nu$-ring and let $S$ be the integral closure of $R^*$ in $T$. If $\alpha \in S$ and $d \in R$, then $d\alpha$ is integral over $R$ and hence $d\alpha \in R$; so $R$ is an ideal of $S$. Since $R$ is Noetherian, it follows that $S$ is Noetherian and $S$ is a finite $R^*$-module by Theorem 6. Theorem 10 gives us $\sqrt{R} = R$ in $S$ and $R$ is a semi-prime ideal of $S$.

To establish that $\dim S \leq 2$, let $R^* \supset P_1^* \supset P_2^* \supset P_3^* \supset P_4^*$ be a chain of prime ideals in $R^*$. If $P_1^* \supset R$ and $P_2^* \nsubseteq R$, it follows from Lemma 1 that $P_2^* \nsubseteq R$ and applying Lemma 7 we have $R \cap P_2^* > R \cap P_3^* > R \cap P_4^*$, contradicting $\dim R \leq 1$. If $P_1^* \nsubseteq R$ and $P_2^* = R$, then there exists a prime ideal $P_2^*$ in $R^*$ such that $P_1^* > P_2^* > P_3^*$ and $P_2^* \nsubseteq P_2^*$ since $R^*$ is Noetherian [17;240], and Lemma 1 yields $P_2^* \nsubseteq R$; again we contradict $\dim R \leq 1$. If $P_1^* \nsubseteq R$, it is clear that we have a contradiction by Lemma 7; hence $\dim R^* \leq 2$. Since $S$ is integral over $R^*$, it follows from the lying over theorem [17;259] that $\dim R^* = \dim S \leq 2$.

If $P$ is a prime ideal of $S$ such that $P \nsubseteq R$, then $P^* = P \cap R^*$ is a prime ideal of $R^*$ such that $P^* \nsubseteq R$; applying the lying over theorem and Lemma 7, it follows from $\dim R \leq 1$ that height $P \leq 1$.

Conversely, suppose (a), (b) and (c) hold. Then $R$ is Noetherian and integrally closed by Theorems 6 and 10.
Since $S$ is a finite $R^*$-module, then $S$ is integral over $R^*$ [17,254] and $\dim R^* = \dim S \leq 2$ by the lying over theorem [17;259]. Now we wish to show that $\dim R \leq 1$.

Suppose $P_1 < P_2 < P_3 < R$ is a chain of prime ideals of $R$, then by Lemma 8 there exists a chain $P_1^* < P_2^* < P_3^*$ of prime ideals of $R^*$ such that $P_1^* \cap R = P_1$. Now $P_3^* \nsubseteq R$ since height $P_3^* = 2$ so $P_3^* \nsubseteq R$ which also yields a contradiction so $\dim R \leq 1$.

**Theorem 12:** A domain $D$ is an $N$-domain if and only if $D$ is a product of distinct prime ideals in a Dedekind domain $\overline{D}$ such that $\overline{D}$ is a finite $D^*$-module.

**Proof:** Let $D$ be an $N$-domain with quotient field $K$ and let $\overline{D}$ be the integral closure of $D^*$ in $K$.

Conditions (1) and (2) hold in $D^*$ by Theorem 6, and $\dim \overline{D} = 1$ by the lying over theorem. As in the proof of Theorem 11, $D$ is an ideal in $\overline{D}$, $\overline{D}$ is Noetherian, integrally closed and a finite $D^*$-module. Hence $\overline{D}$ is a product of distinct prime ideals since $\sqrt{D} = D$ in $\overline{D}$.

The converse follows from Theorem 11 and Theorem 3.

**Theorem 13:** If $A$ is a product of distinct prime ideals in a general Z.P.I. ring $R$ with an identity and $R$ is a finite $A^* = A^*(R)$ module, then $A$ is a $\nu$-ring.
Proof: Since $R$ is a general Z.P.I. ring, we have $R = R_1 \oplus \ldots \oplus R_n$ where $R_i$ is either a Dedekind domain or a special primary ring for $i = 1, \ldots, n$ [2,89]. Set $A_i = AR_i$, $A_i^* = A_i^*(R_i)$, and note that $A_i$ is a product of distinct prime ideals in $R_i$ (including $R_i$) for $i = 1, \ldots, n$. Since $R$ is a finite $A_i^*$-module, we have $R = \sum_{i=1}^t s_i A_i^*$ where $s_i \in R$ for $i = 1, \ldots, t$. Now, $s_i = \sum_{j=1}^n r_{ij}$ with $r_{ij} \in R_j$ for $i = 1, \ldots, t$ and it follows readily that $R_j = \sum_{i=1}^t r_{ij} A_i^* j$ and $R_j$ is a finite $A_i^*$ module for $j = 1, \ldots, n$. If $R_j$ is a Dedekind domain, then $A_j = (0)$ or $A_j$ is a $\nu$-ring by Theorem 12. If $R_j$ is a special primary ring then $A_j$ is the maximal ideal in $R_j$ (or $A_j = R_j$ and $A_j$ is a $\nu$-ring). Since $A_j$ is a nilpotent ring, we have $\dim A_j = -1$ or $A_j = (0)$. Furthermore, $R_j$ is Noetherian, which implies that $A_j^*$ is Noetherian [4] hence $A_j$ is Noetherian [5;184]. Since $A_j$ is integrally closed (trivially) then $A_j$ is a $\nu$-ring. Finally, $A$ is a $\nu$-ring since a finite direct sum of $\nu$-rings is a $\nu$-ring.

The converse to Theorem 13 is false; in fact, if $A$ is a ring with an identity then $A$ is an ideal in a general Z.P.I. ring if and only if $A$ is a general Z.P.I. ring (as we will presently show), and in example 16 we exhibit a $\nu$-ring with an identity which is not a general Z.P.I. ring.
Proposition 14: If \( R \) is a ring and \( A \) is a finitely generated ideal of \( R \) such that \( A = A^2 \), then \( R = A \oplus R_1 \).

Proof: If \( R \) does not have an identity, let \( S \) be a ring with identity containing \( R \) as a subring \([12; 87]\) and set \( R^* = R^*(S) \). If \( R \) has an identity, set \( R = R^* \). In either case \( A \) is an ideal of \( R^* \). Since \( A = A^2 \) there exists an \( e \in A \) such that \( ea = a \) for all \( a \in A \) \([5; 185]\). If \( e^* \) is the identity of \( R^* \), then \( e \) and \( e^* - e \) are orthogonal idempotents and \( R^* = eR^* \oplus (e^* - e)R^* \). It follows that \( R = eR \oplus (e^* - e)R \), \( eR = A \), and \( R = A \oplus R_1 \).

Corollary 15: If \((0) \neq A = A^2\) is an ideal in a general Z.P.I. ring \( R \), then \( A \) is a general Z.P.I. ring.

Proof: Since \( R \) is Noetherian \([13; 125]\), it follows by proposition 14 that \( R = A \oplus R_1 \) and \( A \cong R/R_1 \) is a general Z.P.R. ring.

Example 16: Let \( x \) and \( y \) be indeterminants over a field \( F \) and set \( R = F[x, y]/(x, y)^2 \). The ring \( R \) has exactly one proper prime ideal \( P = (x, y)/(x, y)^2 \) and consequently \( R \) is its own total quotient ring and is integrally closed. It is clear that \( R \) is Noetherian and \( \dim R = 0 \), hence \( R \) is an N-ring. Obviously \( R \) is not a general Z.P.I. ring since \( P^2 = (0) \).

It follows from Theorem 12 that a v-domain (N-domain) can be imbedded as an ideal in a Dedekind domain (i.e. Z.P.I.,
domain with identity) in a special way. However, Corollary 15 and example 16 show that in general a \(\nu\)-ring can not be imbedded as an ideal in a general Z.P.I. ring.

We complete this section with a sufficient condition that \(D^*\) be a Dedekind domain when \(D\) is an \(N\)-domain, and give two examples.

**Theorem 17:** If there exists \(d \in D\) such that \(D = dD + dZ\) and \(D\) is an \(N\)-domain (\(\nu\)-domain), then \(D^*\) is a Dedekind domain.

**Proof:** It suffices to prove that \(D^*\) is integrally closed since \(D^*\) has properties (1) and (2) by Theorem 3. Let \(\alpha\) be an element of the quotient field of \(D^*\) which is integral over \(D^*\); then \(\alpha = a/b\) with \(a\) and \(b \in D\) and there exist \(d_i^* \in D^*, i = 0, \ldots, n-1\), such that

\[
\alpha^n + d_{n-1}^* \alpha^{n-1} + \ldots + d_0^* = 0. 
\]

Hence \((d \alpha)^n + \sum_{i=0}^{n-1} (d \alpha)^{n-i} = 0\) and \(d \alpha\) is integral over \(D\), which implies \(d \alpha \in D\) since \(D\) is integrally closed. Therefore \(d \alpha = d(a/b) = kd + nd\) where \(k \in D\) and \(n \in Z\) and consequently \(\alpha = a/b = (kd + nd)/d = k + n \in D^*\) and \(D^*\) is integrally closed.

**Example 18:** This example shows that the domain \(D^*\) of Theorem 12 may not be a Dedekind domain (i.e. \(\bar{D} > D^*\)). Let \(w = (1 + \sqrt{5})/2\), \(S = \{a + bw | a, b \in Z\}, 2S = (2)\), and \((2)^* = \{n + 2a + 2bw | a, b, n \in Z\}\). Then \((2)\) is a prime ideal in the Dedekind domain \(S\) \([10;33,66]\), \(S = (2)^* + w(2)^*\) is a finite \((2)^*\)-module, and \(S \nmid (2)^*\) since \(w \notin (2)^*\). It
follows from Theorem 12 that (2) is an N-domain, but \((2)^*\) is not a Dedekind domain since the integral closure of \((2)^*\) is \(S\) (however, \((2)^*\) is an RM-domain).

**Example 19:** In this example we show that a prime ideal in a Dedekind domain need not be an N-domain (in fact, need not be Noetherian). Denote by \(Q\) the field of rational numbers, let \(x\) be an indeterminant over \(Q\), and set \(D = Q[x](x)\) (i.e. the quotient ring of \(Q[x]\) with respect to the prime ideal \((x))\). The ideal \(D = xD\) is a prime in \(D\), and we will show that \(D\) is not Noetherian by showing that \(D^*\) is not Noetherian. If \(p_n\) denotes the \(n\)th prime number and \(A_1 = (x/2)D^*\), then define \(A_n\) for \(n > 1\) by \(A_n = A_{n-1} + (x/p_n)D^*\). It follows easily that \(x/p_{n+1}\) does not belong to \(A_n\) for \(n \geq 1\), and therefore the sequence \(A_1 < A_2 < \ldots\) is strictly increasing - which implies that \(D^*\) (and hence \(D\)) is not Noetherian.
CHAPTER III

A CHARACTERIZATION OF N-RINGS WITH PROPER DIVISORS OF ZERO
AND ALMOST N-RINGS

We state without proof the following theorem, which is an easy consequence of Theorem 4 of [1,339].

**Theorem 20:** Let $R$ be a ring and let $P_1, \ldots, P_r$ be ideals of $R$ such that $R/P_i$ is a field for $i = 1, \ldots, r$ and such that $(0) = \prod_{i=1}^{r} P_i^{m_i}$. Then there exists a positive integer $n$ such that $R = R^n \oplus N$ where $R^n = R^{n+1}$ has an identity, $N$ is nilpotent, and in $R^n$, $(0) = \prod_{i=1}^{r} P_i^{m_i}$ where $P_i = P_i \cap R^n$ and $R^n/P_i$ is a field for $i = 1, \ldots, r$.

**Corollary 21:** Let $R$ be a regular ring in which $(0)$ is not a prime ideal. If conditions (1) and (2) hold in $R$, then $R$ has an identity.

**Proof:** Since $R$ is Noetherian every ideal of $R$ contains a product of prime ideals, hence $(0) = \prod_{i=1}^{k} P_i$. The $P_i$ are maximal by (2) and we apply Theorem 20 to $R$ and see that $N = (0)$ since $R$ is regular.

**Theorem 22:** If $R$ is a ring with a regular element, then
R is an RM-ring if and only if conditions (1) and (2) hold in R.

**Proof:** The result follows from Corollary 4 in case R is a domain, so we may assume that (0) is not prime in R. If conditions (1) and (2) hold, then Corollary 21 applies and R has an identity, and hence R is an RM-ring [3;29]. Conversely, if R is an RM-ring with a regular element then the a.c.c. is valid in R [1;342] and since property (2) holds in any RM-ring, the proof is complete.

**Remark 23:** We note that it follows from the proof of Theorem 22 that a regular RM-ring, in which (0) is not a prime ideal, has an identity. However, an RM-domain need not have an identity (e.g. the even integers).

**Lemma 24:** If R has the d.c.c., then R is equal to its total quotient ring T (and R is integrally closed).

**Proof:** If there are no regular elements in R, then R = T. If r is regular in R, then \( (r)^n = (r)^{n+1} \) for some integer n. Hence \( r^n = sr^{n+1} + mr^{n+1} \) with \( s \in R, m \in Z \) so that \( r = r(sr+mr) \) and \( e = sr+mr \) is an identity for R. It follows easily that every regular element of R has an inverse in R and R = T.

**Proposition 25:** Let R be a ring with a regular element
which is not a domain. Then \( R \) is an N-ring if and only if \( R \) is a ring with identity in which the d.c.c. holds.

**Proof:** Suppose \( R \) is an N-ring. It follows from Theorem 22 and remark 23 that \( R \) has an identity. Now since \( R \) is a Noetherian ring with an identity and every prime ideal different from \( R \) is maximal, \( R \) has the d.c.c. [3,28].

Conversely, by [3;28] \( R \) is Noetherian and every prime ideal \( \neq R \) is maximal. By Lemma 24, \( R \) is integrally closed and therefore \( R \) is an N-ring.

**Corollary 26:** If \( R \) is a ring in which \((0)\) is not prime, then \( R \) is an N-ring if and only if conditions (1) and (2) hold in \( R \).

**Proof:** Suppose (1) and (2) hold in \( R \). If \( R \) has a regular element then Corollary 21 implies that \( R \) has an identity, and therefore \( R \) has the d.c.c. by [3;28]. It follows from Lemma 24 that \( R \) is an N-ring. If \( R \) has no regular elements then \( R = T\), its total quotient ring, and \( R \) is an N-ring.

**Theorem 27:** Let \( R \) be a ring in which \((0)\) is not prime. Then \( R \) is an N-ring if and only if \( R \cong R_1 \oplus \ldots \oplus R_k \oplus N \) where each \( R_i \) is a Noetherian primary ring with identity and \( N \) is a Noetherian nilpotent ring.
Proof: Suppose $R$ is an $N$-ring. If $R$ has a proper prime ideal $P$, then $(0) = \prod_{i=1}^{k} P_{i}^{e_{i}}$ where $R/P_{i}$ is a field for $i = 1, \ldots, k$ because every ideal in a Noetherian ring contains a product of prime ideals. (If $(0) = \prod_{i=1}^{k} P_{i}^{e_{i}}$ then $(0) = \prod_{i=1}^{k} P_{i}^{e_{i}}$ where $P$ is a proper prime, hence $(0) = \prod_{i=1}^{k} P_{i}^{e_{i}}$.) By Theorem 20, $R = R^{n} \oplus N$ where $R^{n}$ has an identity, and $(0) = \prod_{i=1}^{k} P_{i}^{e_{i}}$ in $R^{n}$ where the $R^{n}/P_{i}$ are fields. Therefore $R^{n} \cong \prod_{i=1}^{k} R^{n}/P_{i}^{e_{i}}$ by [17;176] and each $R^{n}/P_{i}^{e_{i}} = R_{i}$ is a Noetherian primary ring with identity. If $R$ has no proper prime ideals, then $\sqrt{(0)} = R$ and $R^{n} = (0)$ so $R = N$.

Conversely if $R \cong R_{1} \oplus \cdots \oplus R_{k} \oplus N$, where each $R_{i}$ is a Noetherian primary ring with identity, then it is clear that properties (1) and (2) hold in $R$. We consider two cases. If $N = (0)$, then $R$ satisfies the d.c.c. and $R$ is an $N$-ring by proposition 25. Second, if $N \nmid (0)$, then there are no regular elements and $R$ is an $N$-ring since $R$ is integrally closed.

Theorem 28: Let $R$ be a general Z.P.I. ring with an identity and let $A \nmid (0)$ be an ideal in $R$ such that $A = \prod_{i=1}^{k} P_{i}$, where the $P_{i}$ are distinct prime ideals in $R$. Further suppose $R$ is a finite $A^{*} = A^{*}(R)$ module and that proper prime ideals are maximal in $A$. Then $A$ is an $N$-ring and one of the following holds:
i) Either $A$ is a Dedekind domain $D$ or a product of distinct prime ideals in a Dedekind domain $D$ such that $D$ is a finite $A^* \cap D$ module.

ii) The ideal $A$ is a product of prime ideals in a general Z.P.I. ring $R_1$ such that primes different from $R_1$ are maximal in $R_1$, and $R = R_1 \oplus D_2 \oplus \ldots \oplus D_k$ where the $D_i$ are Dedekind domains.

iii) $A$ is a general Z.P.I. ring $R_1$ with identity in which prime ideals are maximal, and $R = R_1 \oplus D_2 \oplus \ldots \oplus D_k$ where the $D_i$ are Dedekind domains.

Proof: Since $R$ is Noetherian [13;125] and $R$ is a finite $A^*$-module, $A^*$ is Noetherian by [4] and hence $A$ is Noetherian. By [2;83], $R = D_1 \oplus \ldots \oplus D_k \oplus \Pi_{k+1} \oplus \ldots \oplus \Pi_t$ where the $D_i$ are Dedekind domains and the $\Pi_i$ are special primary rings.

Case 1: $A$ is a domain. Since $A$ is a domain, then $A \subseteq D_i$ for some $i$ (say $i = 1$) which is a Dedekind domain or a field. Hence either $A = D_1$ or $A$ is a product of distinct prime ideals in $D_1$, which implies that $A$ is an N-domain by Theorem 12.

Case 2: $A$ is not a domain. By the proof of Theorem 8 the only ideal of $A^*$ which could be prime but not maximal is $A$. If $A$ is not prime in $A^*$, then all primes of $A^*$ different from $A^*$ are maximal in $A^*$ - which implies that
all primes different from $R$ in $R$ are maximal by the
lying over theorem [17;259]. Thus we have conclusion (ii).
with $k = 1$ and $A$ is a product of maximal ideals in a
direct sum of special primary rings. In this case
$R \cong \Pi_1 \oplus \ldots \oplus \Pi_s$ and $A = A_1 \oplus \ldots \oplus A_s$ where each $A_i$ is
either $\Pi_1$ or $M_1$ the maximal idea of $\Pi_1$. Each $\Pi_1$ is
integrally closed since each $\Pi_1$ has the d.c.c. and each $M_1$
is integrally closed since it has no regular elements. There­
fore $A$ is integrally closed since it is a direct sum of
integrally closed rings. Hence $A$ is an N-ring.

If $A$ is prime in $A^*$ and $A = \prod_{i=1}^{k} P_i$ in $R$, with
all of the $P_i$ maximal, then we will show that all primes
of $R$ except $R$ are maximal. If $P$ is a prime in $R$,
then $P \cap A^*$ is prime in $A^*$. Since $A$ is the only possible
non-maximal prime in $A^*$, if $P \cap A^* \neq A$ then it follows that
$P \cap A^*$ is maximal in $A^*$ and $P$ is maximal in $R$. If $P \cap A^* = A$
then $P \supset P_i$ for some $i$, which implies that $P$ is maximal.

Therefore all primes of $R$ except $R$ are maximal and $R = \Pi_1 \oplus \ldots \oplus \Pi_s$
as in the previous case. If some $P_i$, say $P_1$, is not maximal,
them $P_1 = P_1^2$ [13;129] and by proposition 14, $R \cong P_1 \oplus R/P_1$ and
$P_1 \supset A$. If $P_1 = A$ we have case (iii) since $R/P_1$ is a
Dedekind domain. If $P_1 > A$ then $A = \Pi P_{i1}$ a product of
distinct primes in $P_1$, and either all $P_{i1}$ are maximal or
one $P_{i1}$ is not maximal and we can apply the above argument
to the ring $P_1$. Continuing in this manner we get either
case (ii) or (iii) in the statement. In either case, A is an N-ring by the argument in the first part of case 2.

Next we investigate rings with the property that every proper homomorphic image is an N-ring.

Theorem 29: A ring $R$ has the property that $R/A$ is an N-ring for every ideal $A \neq (0)$ if and only if $R$ is one of the following types of rings.

(a) $R$ is a one dimensional Noetherian ring with a non-maximal prime ideal $P \neq (0)$ such that $P^2 = (0)$, there are no ideals between $P$ and $(0)$, and $R/P$ is an N-domain.

(b) $R = D \oplus K$ where $D$ is an N-domain and $K$ is a field.

(c) $R = R_1 \oplus \ldots \oplus R_k \oplus N$ where each $R_i$ is a Noetherian primary ring with identity and $N$ is a nilpotent ring with the a.c.c.

(d) $R$ is an RM-domain.

Proof: The ring $R$ is Noetherian since $R/A$ is Noetherian for all $A \neq (0)$.

Case 1: $R$ is a domain. If $R$ has a proper prime, then let $P$ denote one such prime. If $0 \neq x \in P$, then $R/(x^2)$ is an N-ring. Let $\phi: R \to R/(x^2)$ be the natural map. Then $\phi(P)$ is maximal and $P$ is maximal since $P \supseteq (x^2)$. Therefore $R$ is an RM-domain by Corollary 4. If $R$ has no proper prime ideals then $R$ is again an RM-domain by Corollary 4.
Case 2: \( R \) is not a domain and \( R \) has at least one proper prime ideal \( P \) which is not maximal. Let \( P_1 \notdiv R \) be a prime of \( R \). Then \( R/P_1 \) is an N-domain and \( \dim R/P_1 \leq 1 \). There are no ideals between \( P \) and \((0)\) because \( P > A > (0) \) implies \( P/A \) is maximal in \( R/A \) which is a contradiction. Therefore either \( P = P^2 \) or \( P^2 = (0) \). If \( P^2 = (0) \), then \( R \) is a ring of type (a). If \( P = P^2 \) then by proposition 14, \( R = P \oplus R(1-e) \). Since there are no ideals between \( P \) and \((0)\) and \( P \) has an identity, \( P \) must be a field since any ideal of \( P \) is an ideal of \( R \). Therefore \( R \cong K \oplus D \), where \( K \) is a field and \( D \cong R/P \) is an N-domain, and \( R \) is of type (b).

Case 3: \( R \) is not a domain and every prime ideal of \( R \) except \( R \) is maximal. If \( R \) has no proper primes then \( \sqrt{(0)} = R \) which implies that \( R^k = (0) \) since \( R \) is Noetherian, and \( R \) is a ring of type (c). If \( R \) has at least one proper prime ideal then \((0) = P_1^{e_1} \cdots P_k^{e_k} \) where the \( P_i \) are maximal and prime. By Theorem 20, \( R = R^n \oplus N \) where \( R^n \) has an identity and \( N \) is nilpotent. In \( R^n = \bar{R} \), 

\( (0) = \bar{P}_1^{e_1} \cdots \bar{P}_k^{e_k} \) such that \( \bar{R}/\bar{P}_i \) is a field. Therefore \( \bar{R} = \bar{R}/\bar{P}_1^{e_1} \oplus \cdots \oplus \bar{R}/\bar{P}_k^{e_k} \) by [17;178] and \( R \cong \bar{R}_1 \oplus \cdots \oplus R_k \oplus N \) where \( R_i = \bar{R}/\bar{P}_i^{e_i} \) is a Noetherian primary ring with identity for each \( i \) and \( N \) is a Noetherian nilpotent ring.

Conversely, suppose \( R \) is a ring of type (a) and let \( B \notdiv (0) \) be an ideal of \( R \). If \( B = P \), then \( R/B \) is
an $N$-ring by hypothesis. If $B \not\subseteq P$, then all proper primes of $R/B$ are maximal and $R/B$ is Noetherian. Hence $R/B$ is an $N$-ring by Corollary 26, or $R/B$ is a field which is an $N$-ring. Suppose $R \simeq D \oplus K$, where $D$ is an $N$-domain and $K$ is a field. If $B \not\subseteq (0)$ is an ideal of $R$ then $B = B_1 \oplus B_2$ and $R/B \simeq D/B_1 \oplus K/B_2$. By considering the cases $B_2 = (0)$ and $B_2 = K$, it follows easily (see Corollary 26) that $R/B$ is an $N$-ring. Similarly, if $R$ is of type (c) or (d) then it follows readily that $R/B$ is an $N$-ring for each $B \not\subseteq (0)$ in $R$.
CHAPTER IV

COMMUTATIVE RINGS SATISFYING SOME PROPERTIES OF
DEDEKIND DOMAINS

In Chapter I we studied N-domains and we recall that an N-domain with identity is a Dedekind domain. In this chapter we will consider some properties (or characterizations) of Dedekind domains and study rings having these properties. First we consider a variation of the concept of N-domains obtained by replacing (1), (2), and (3) by (1), (2) and (3'). Since D is an ideal in D*, (3') simply states that D is integrally closed as an ideal of D* in the sense of [18;349].

We recall that a Dedekind domain is a domain with identity having property (Π), and notice that every homomorphic image of a ring with property (Π) has property (Π). In the case of domains the converse is also valid (see Corollary 32).

Before proving the next theorem we make the following observation. An ideal D of R has property (3') if and only if D is complete. A domain D is complete if D = \( \cap_{v \in S} dR_v \), where S is the set of all valuations v of K non-negative on R and R_v is the valuation ring.
corresponding to the valuation $v$ [18;350].

**Theorem 30:** A domain $D$ with quotient field $K$ is complete (integrally closed as an ideal in $D^*$) and has properties (1) and (2) if and only if $D$ is an ideal in a Dedekind domain $\tilde{D}$ such that $\tilde{D}$ is a finite $D^*$-module.

**Proof:** Suppose $D$ satisfies conditions (1), (2) and (3'). If $D$ has an identity then $D$ is a Dedekind domain. If $D$ does not have an identity then let $\tilde{D}$ be the integral closure of $D^*$ in $K$. We will show that $D$ is an ideal in $\tilde{D}$; $D = D' = (D\tilde{D})' \supset D\tilde{D}$ where $A'$ denotes the completion of $A$ [18;347,348], therefore $D$ is an ideal of $\tilde{D}$. Since $D$ is an ideal of both $D^*$ and $\tilde{D}$, $D$ is contained in the conductor of $\tilde{D}$ over $D^*$. Fix $0 \neq d \in D$ and let $\tilde{d} \in \tilde{D}$; then $\tilde{d}d \in D$, which implies that $\tilde{d} \in Dd^{-1} \subseteq D^*d^{-1}$ and $\tilde{D} \subseteq D^*d^{-1}$. Now $D$ is Noetherian so $D^*$ is Noetherian [5;184] and $D^*d^{-1}$ is a Noetherian $D^*$-module since it is finite over $D^*$ [17;158]. Hence $\tilde{D}$ is a Noetherian $D^*$-module [17;156], so $\tilde{D}$ is a Noetherian ring since any ideal of $\tilde{D}$ is a $D^*$-submodule. By Theorem 3, $D^*$ has properties (1) and (2) so $\tilde{D}$ is an RM-domain [3;29], and consequently $\tilde{D}$ is a Dedekind domain.

Conversely, if $D$ is an ideal of a Dedekind domain $\tilde{D}$ such that $\tilde{D}$ is a finite $D^*$-module, then $D^*$ is Noetherian by [4]. By the lying over theorem [17;259] $\dim D^* = \dim \tilde{D} \leq 1$ so $D^*$ is an RM-domain [17;203] and
consequently D has properties (1) and (2). Any ideal A in a Dedekind domain is complete because \( \bigcap_{v \in S} AR_v = 0 \) \( A \bigcap_{v \in S} \bar{A} \bar{P}_v = A \) \([18; 84]\), so D is complete.

Next we consider rings with the property that every homomorphic image is a general Z.P.I. ring (i.e. a ring with property (π)).

**Theorem 31:** If R is a ring, then \( R/A \) is a general Z.P.I. ring for every proper ideal A if and only if one of the following is valid for R:
(a) \( R^2 = (0) \) and the additive group of R is a finite abelian group whose order is a prime or the product of two primes.
(b) R is a general Z.P.I. ring.
(c) R is a ring with identity having a composition series of ideals \( R > M > A > (0) \) of length three, where \( M^2 = (0) \).
Note that M is maximal, generated by two elements, and the only prime ideal of R different from R.
(d) \( R = R^2 \oplus N \) where \( R^2 \nmid (0) \) is a general Z.P.I. ring with identity with precisely one proper prime ideal \( \bar{P} \), \( \bar{P}^2 = (0) \) and N is a ring with no proper ideals. (Note that \( N^2 = (0) \) since \( R^2 = R^1 \)).
(e) \( R = K_1 \oplus K_2 \oplus N \) where the \( K_i \) are fields, \( N^2 = (0) \), and N has no proper ideals.

**Proof:** Suppose \( R/A \) is a general Z.P.I. ring for
every ideal $A$ in $R$. We first consider the case when $R^2 = (0)$. If $R = (a)$ for all $a \neq 0$ in $R$, then $R>(0)$ is a composition series of ideals for $R$. If $a \neq 0$ is any element of $R$ such that $R \nmid (a)$, then $R=(a,b)$ for any $b$ in $R-(a)$ since $R/(a)$ is a general Z.P.I. ring with no proper prime ideals. It follows that $R>(a)>(0)$ is a composition series of ideals of $R$ for any $a \neq 0$ such that $R \nmid (a)$. Since each additive subgroup of $R$ is an ideal it follows easily from elementary group theory that $R$ is finite with order either a prime or the product of two primes.

From this point on we suppose that $R^2 \nmid (0)$.

**Case 1.** If $P$, $P_1$ are any two ideals which are maximal and prime, then $PP_1 \nmid (0)$.

In this case we show that $R$ is a general Z.P.I. ring. First the a.c.c. for ideals holds in $R$ since it holds in the general Z.P.I. ring $R/A$ for every proper ideal $A$ in $R$ [13;125]. Second, if $P$ and $P_1$ are two ideals which are maximal and prime in $R$, then $R/(PP_1)$ is a general Z.P.I. ring, implying that there are no ideals between $\bar{P}$ and $\bar{PP_1}$ and this implies there are no ideals between $P$ and $PP_1$. Finally, $R/R^2$ is a general Z.P.I. ring, so there are no ideals between $\bar{R}$ and $\bar{R}^2$ and this implies that there are no ideals between $R$ and $R^2$. These three conditions characterize general Z.P.I. rings by [13;131].
Case 2. There exist two distinct ideals $P$ and $P_1$ which are maximal and prime such that $PP_1 = (0)$.

We show in this case that $R$ satisfies either (b) or (c). Choose $x \in P_1 - P$. Then $\bar{x} = x + P$ is a unit in $\bar{R} = R/P$ and there exists $x'$ in $R$ such that $\bar{xx'} = 1$, the identity in $\bar{R}$, and $xx' \in P_1 - P$. Set $r = xx'$. Then $r - r^3 \in P$ since $r^3 = r$, and $r(r - r^3) \in PP_1 = (0)$, which implies that $r^2 = r^4$ and $r^2 \notin P$. Letting $e = r^2$, we have

$(0) < (e) < R$ since $e \in P_1 - P$. If $R$ has an identity, let $R = R^*$; if $R$ does not have an identity, then let $R^* = R^*(R)$. If $1$ is the identity of $R^*$, then $e$ and $1 - e$ are mutually orthogonal idempotent elements in $R^*$ and we have

$R^* = R^* e \oplus R^* (1 - e)$

If $A$ is any ideal in $R$, it follows from (i) that $A = Ae \oplus A (1 - e)$ and in particular

$(ii) \quad R = Re \oplus R (1 - e)$ and $P = Pe \oplus P (1 - e)$.

Since $e \in P_1$, then $Pe \subseteq PP_1 = (0)$; $P \supseteq (0) = R \cdot R (1 - e)$ implies that $P \supset R (1 - e)$, $P \subseteq R$ yields $P (1 - e) \subseteq R (1 - e)$, and therefore $P (1 - e) = R (1 - e)$. It now follows from $(ii)$ that $P = R (1 - e)$ and

$(iii) \quad R = Re \oplus P$.

Since $P \cong R/Re$ and $Re \cong R/P$, it follows that $Re$ is a field and $P$ is a general Z.P.I. ring. From $PP_1 = (0)$ in $R$ and $(iii)$ we conclude that $P$ has exactly one prime ideal $P^* \notin P$ and $P^*$ is maximal - in fact $Re \oplus P^* = P_1$ (it can
be shown that \( P^* = P_{1}(1-e) \), and therefore \( PRe \oplus P_{1}P^* = PP_{1} = (0) \) and \( PP^* = (0) \). Hence the only ideals in \( P \) are the powers of \( P, P^* \) and \( (0) \) (note that \( P^k \neq (0) \) since \( (0) \subset P_{1} \neq P \)). If \( P \) has an identity, then it follows from (iii) that \( R \) is a general Z.P.I. ring (also, \( P^* = (0) \) in this case and \( P \) is a field). Suppose \( P \) does not have an identity. Then \( P > P^2 \) by [5;185]. Since \( P^2 \neq P^* \), then \( P = P^2 + P^* \). Furthermore \( P^2 \cap P^* = (0) \) since the only ideals in \( P \) are \( (0), P^* \), and the powers of \( P \). Hence \( P = P^2 \oplus P^* \) and \( P/P^* \cong P^2 \) is a field. Since \( P \) is a general Z.P.I. ring, there are no ideals properly between \( P \) and \( P^2 \) and hence \( P^* \) has no proper ideals. Consequently \( R = Re \oplus P^2 \oplus P^* \) is of type \((e)\), since (iii) implies that \( R^2 = Re \oplus P^2 \).

Case 3. There exists an ideal \( P \), which is maximal and prime, such that \( P^2 = (0) \). (Note that \( P \) is the only prime in \( R \) different from \( R \)).

Since a field is a general Z.P.I. ring, we may assume that \( P > P^2 = (0) \). Suppose that \( R \) has an identity. If there are no ideals properly between \( P \) and \( P^2 = (0) \) then \( R \) is a general Z.P.I. ring since the a.c.c. for ideals holds in \( R \) and there are no ideals properly between \( R \) and \( R^2 \) [13;131]. Suppose there exists an ideal properly between \( P \) and \( P^2 = (0) \). If \( A \) and \( B \) are any ideals of \( R \) such that \( (0) < A < B < P \), then \( B = P \) since \( \bar{R} = R/A \) is a general Z.P.I. ring with
identity with exactly one prime ideal $\mathfrak{p} \subseteq \mathfrak{r}$ and $\mathfrak{p}^2 = (0)$ in $\mathfrak{r}$. It follows that if $A$ is any proper ideal of $R$ such that $A \nsubseteq P$, then $(0) < A < P < R$ is a composition series of ideals of $R$, and $R$ satisfies (c).

We consider now the case in which $R$ does not have an identity (and therefore $R > R^2$ by [5;185]). Since $R^2 \nsubseteq P$, then $R = R^2 + P$ and $R^2 = (R^2 + P)^2 = R^4 + PR^2 = R^2(R^2 + P) = R^3$. Consequently $R^2 = R^4$ and $R^2$ has an identity $e$ since the a.c.c. holds in $R$ [5;185]. Since $(0) < (e) < R$ and $e = e^2$, it follows as in case 2 that there exists an ideal $N$ in $R$ such that

(iv) $R = Re \Theta N$.

We have $R^2 = R^2 e \subseteq Re \subseteq R^2$, so that $Re = R^2$ and

(v) $R = R^2 \Theta N$.

Since $\mathfrak{r} = R/R^2$ is a general Z.P.I. ring, there are no ideals properly between $\mathfrak{r}$ and $\mathfrak{r}^2$, and therefore $N$ is a ring with no proper ideals (also, since $R^2 \cdot R^2 = R^2$, it follows from (v) that $N^2 = (0)$ and $N$ is the only prime ideal in the ring $N$). Since $P$ is the only prime ideal of $R$ different from $R$ and $P$ is maximal in $R$, it follows from (v) that $R^2$ has exactly one prime ideal $\mathfrak{p} \nsubseteq R^2$ and $\mathfrak{p}$ is maximal. Moreover, $P = \mathfrak{p} \Theta N$ implies that $\mathfrak{p}^2 = (0)$, and $R$ satisfies (d).

Conversely, suppose $R$ satisfies (a). As a consequence of $R^2 = (0)$, it follows that $(0)$ is not prime
in $R$; also, if $A$ is a proper ideal of $R$ and $\bar{R} = R/A$, then $R^2 = (0)$. If $|R|$ is prime, then $R$ has no proper ideals and is a general Z.P.I. ring. Suppose $|R| = pq$ where $p$ and $q$ are prime integers, and let $A$ be a proper ideal of $R$. Then $|A| = p$ or $q$, $|R/A| = q$ or $p$, and $R/A$ is a general Z.P.I. ring.

It is clear that if $R$ satisfies (b), then $R/A$ is a general Z.P.I. ring for every proper ideal $A$ in $R$.

If $R$ is of type (c) and $B$ is a proper ideal of $R$, then $\bar{R} = R/B$ is a field in case $B = M$, and when $B < M$ then $(0) < M < R$ is a composition series and the only ideals in $\bar{R}$ are $\bar{R}$, $\bar{M}$, and $\bar{M}^2 = (0)$. Since $M$ is the unique maximal ideal of $R$, this establishes the converse in case (c) holds.

If $R$ is either of type (d) or (e), then the ideals of $R$ are easily computed and the converse follows directly.

**Corollary 32:** If $D$ is a domain with an identity, then $D$ is a Dedekind domain if and only if $D/A$ is a general Z.P.I. ring for all proper ideals $A$ of $D$.

**Definition 33:** Let $D$ be a domain and $A \not= (0)$ an ideal of $D$; then $A$ is said to be invertible if there exists a fractionary ideal $F$ of $D^*$ such that $AF = D^*$.

**Theorem 34:** If $D$ is a domain in which every prime ideal $P \not= (0)$ is invertible, then $D^*$ is an RM-domain.
Proof: Let $P \not= (0)$ be a prime ideal; then there exists an $F$ such that $PF = D^*$, which implies $P$ is finitely generated [6;580] and that $D$ is Noetherian [3;29] so $D^*$ is Noetherian [5;184]. If $P$ is a proper prime ideal of $D$, then $D_P$ is a domain with identity in which every proper prime ideal is invertible, which implies that $D_P$ is a Dedekind domain [15;233]. Hence proper prime ideals of $D$ are maximal and $D^*$ is an RM-domain by Corollary 4.

Although we are not able to give a complete characterization of domains in which prime ideals are invertible we offer the following theorem about domains in which all non-zero ideals are invertible (we call such a domain an \underline{\textit{L-domain}}).

\textbf{Theorem 35:} If $D$ is an \underline{\textit{L-domain}}, then $D^*$ is a Dedekind domain; and conversely, if $A \not= (0)$ is an ideal in a Dedekind domain $\bar{D}$ such that $A^* = A^*(\bar{D}) = \bar{D}$, then $A$ is an \underline{\textit{L-domain}}.

\textbf{Proof:} We first assume that $D$ is an \underline{\textit{L-domain}}. By Theorem 34 $D^*$ is an RM-domain; since every ideal is invertible, we see that the cancellation law holds in $D$ and consequently in $D^*$ also [7;284]. The cancellation law in $D^*$ implies $D^*$ is integrally closed [7;283]. Therefore $D^*$ is a domain with properties (1), (2), and (3) so $D^*$ is a Dedekind domain [17;275].

Conversely, let $A \not= (0)$ be an ideal of $\bar{D}$ such
that \( A^* = \bar{D} \) and \( \bar{D} \) is a Dedekind domain. If \( B \) is an ideal of \( A \), then \( B \) is an ideal of \( A^* = \bar{D} \), implying that \( B \) is invertible and \( A \) is an \( \mathcal{L} \)-domain.

**Corollary 36:** If \( D \) is a domain such that every ideal is principal, then \( D^* \) is a Dedekind domain.

**Proof:** This corollary follows from the fact that principal ideals of the form \( aD + az \) are invertible and Theorem 35.

We note that neither an \( \mathcal{L} \)-domain nor a domain in which all ideals are principal need be integrally closed, for example every ideal of \( D = 4Z \) is principal and \( D \) is not integrally closed since \( 2^2 \in D \) and \( 2 \notin D \).

**Definition 37:** A domain \( D \) is called a \( \mathcal{d} \)-domain if for each ideal \( A \) of \( D \) there exists a non-negative integer \( k = k(A) \) such that \( \bar{A}D^k = \prod_{i=1}^{n} P_i^{e_i} \) where the \( P_i \) are prime ideals of \( D \) (possibly \( P_i = D \), for some \( i \)). We observe here that the ring of even integers is a \( \mathcal{d} \)-domain but not a Dedekind domain.

**Lemma 38:** For each prime ideal \( P \) in \( D \), there exists a prime ideal \( P^* \) in \( D^* \) such that \( P^* \cap D = P \). If \( D^* \) is one dimensional, then the proper prime ideals of \( D \) are precisely the ideals \( P^*D \) where \( P^* \) is a proper prime ideal of \( D^* \) such that \( P^* + D = D^* \).
**Proof:** For a proper prime $P$ in $D$, we have $PD_p \cap D^* = P^*$ is a proper prime ideal in $D^*$ and $P^* \cap D = PD_P \cap D = P$. Furthermore, $P = P^* \cap D = P^*D$ in case $P^*$ is maximal in $D^*$. In addition, for a proper prime $P^*$ in $D^*$ such that $D + P^* = D^*$, we have $P^* \cap D = P^*D$ is a proper prime of $D$.

**Theorem 39:** A domain $D$ is an invertible d-domain if and only if $D^*$ is a Dedekind domain and $D$ is a prime ideal in $D^*$.

**Proof:** We show first that if $P$ is a proper prime ideal of $D$, then $P$ is invertible. Since $D$ is a d-domain, it follows that every ideal in the quotient ring $D_P$ is a product of prime ideals and $D_P$ is a Dedekind domain; hence $P$ is maximal. For $0 \neq a \in P^-$, we have $P \triangleright (a)D^k = P_1 \cdots P_n$ ($P_i$ prime in $D$), $P \triangleright P_i$ for some $i$, and $P = P_i$. It follows that $P$ is invertible since $(a)D^k$ is invertible. We conclude that every non-zero prime ideal in $D$ is invertible. Consequently, if $A^*$ is a nonzero ideal in $D^*$, then $A^*D^k$ is a product of prime ideals in $D$ and $A^*$ is invertible since $D$ is invertible. Therefore $D^*$ is a Dedekind domain [17;275]. Let $A^*$ be an ideal of $D$ such that $D^* \triangleright A^* > D$. Now, $A^*D$ is an ideal of $D$ which is not contained in any proper prime ideal $P = P^*D$ (Lemma 38) of $D$ since $P^* \not \supset D$; hence $A^*D \cdot D^u = D^v$ and $A^* = D^*$ since $A^* > D$ and $D$ is invertible. Therefore $D$ is...
maximal and prime in $D^*$.

Conversely, if $D^*$ is a Dedekind domain and $D$ is prime in $D^*$, then $D$ is invertible. If $A \neq (0)$ is an ideal of $D$, then $A$ is an ideal of $D^*$ and $A = P^* \cdots P_n^*D^n$, where $P_i^*$ is a proper prime ideal of $D^*$ such that $P_i^* \not\subseteq D$. It follows that $AD^n = P_1 \cdots P_n D^n$ where $P_1 = P_i^*D$ is a proper prime of $D$ for $i = 1, \ldots, n$.

Example 40: We now give an example of a $d$-domain $D$ such that $D^*$ is not a Dedekind domain.

Let $A$ be a nonzero ideal in the domain $J = \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\}$, let $a$ be the smallest positive integer in $A$, and let $b$ be the smallest positive value of $y$ among all numbers $x + y\sqrt{5}$ in $A$. Using the division algorithm, the "minimal" property of $b$, and the fact that $A$ is an ideal in $J$, it follows directly that $b | a$ and $b | x$, $b | y$ for all $x + y\sqrt{5}$ in $A$. Let $t + b\sqrt{5} \in A$, where $0 < t < a$, and set $a = bm$, $t = br$ with $m, r \in \mathbb{Z}$. Then $bm$ and $b(r + \sqrt{5})$ form an independent $\mathbb{Z}$-basis for $A$ and $r^2 \equiv 5 \pmod{m}$, i.e.

$$A = bm\mathbb{Z} + b(r + \sqrt{5})\mathbb{Z}, \quad 0 \leq r < m, r^2 \equiv 5 \pmod{m}.$$  

Conversely, any $\mathbb{Z}$-module of type (1), such that $r^2 \equiv 5 \pmod{m}$, is an ideal in $J$.

Denote by $S$ the set of odd integers in $\mathbb{Z}$, and set $R = J_S = \{a/s \mid a \in J, s \in S\}, i.e. R$ is the quotient ring of $J$ formed with respect to the multiplicative system $S$. 

For each ideal $B$ of $R$, we have $B \cap J$ is an ideal of $J$ and $(B \cap J)R = B$. Since every nonzero ideal of $J$ is a $Z$-module of type (1), it follows that every nonzero ideal $B$ in $R$ is a two-dimensional $Z_S$-module where $Z_S = \{a/s \mid a \in Z, s \in S\}$, i.e.

$$B = bmZ_S + b(r + \sqrt{5})Z_S, \quad 0 \leq r < m, \quad r^2 \equiv 5 \pmod{m} \quad \text{where} \quad b, m, r \in Z \quad \text{and} \quad b, m \quad \text{are positive.}$$

We may assume that $b$ and $m$ are powers of 2, say $b = 2^k$ and $m = 2^e$, and reduce $r \mod 2^e$, so that

$$B = 2^k \cdot 2^e Z_S + 2^k (r_0 + \sqrt{5})Z_S, \quad 0 \leq r_0 < 2^e,$$

$r_0^2 \equiv 5 \pmod{2^e}$. However, $x^2 \equiv 5 \pmod{8}$ has no solutions, and therefore $0 \leq e \leq 2$. If we denote by $2^k [2^e, r_0 + \sqrt{5}]$ the $Z_S$-module in (3), then it follows that the only nonzero ideals in $R$ are given by the following $Z_S$-modules: $2^kR$, $2^kM$, $2^kB$, and $2^k\bar{B}$ where $R = [1, \sqrt{5}]$, $M = [2, 1 + \sqrt{5}]$, $B = [4, 1 + \sqrt{5}]$, $\bar{B} = [4, 1 - \sqrt{5}] = [4, 3 + \sqrt{5}]$, and $k = 0, 1, 2, \ldots$.

Since every proper ideal can be generated by a basis of two elements, then $R$ is an RM-domain [3;37] and therefore $R$ is Noetherian and one-dimensional. It is clear that each proper ideal of $R$ is contained in $M$, and consequently $M$ is the unique proper prime ideal of $R$. We note that $R$ is not integrally closed (hence, not a Dedekind domain) since $(1 + \sqrt{5})/2$ is a root of $x^2 - x - 1 = 0$ and $(1 + \sqrt{5})/2$ is not in $R$ (in fact, the integral closure $\bar{R}$ of $R$ in its quotient field $F$ is a Noetherian valuation ring, and $\bar{R}$ is
the only proper overring of $R$ in $F$). A direct computation shows that $M^{k+1} = 2^k M$, $2^k B \cdot M = M^{k+2}$, $2^k B \cdot M = M^{k+2}$, and $2^k R \cdot M = M^{k+1}$ for $k = 0, 1, 2, \ldots$. Hence $AM$ is a power of $M$ for each nonzero ideal $A$ of $R$. We observe that $M$ is not invertible since $R$ is not a Dedekind domain; however, $B = (1 + \sqrt{5})R$ and $\bar{B} = (1 - \sqrt{5})R$, so that every ideal of $R$ is principal except for the powers of $M$.

If we set $D = M$, then it follows easily that $D^* = R$. Consequently, the ideals of $D$ are precisely the ideals of $D^*$ which are different from $D^*$; furthermore, $D$ has no proper prime ideals. However, $D$ is a d-domain such that $D^*$ is not a Dedekind domain (note that $D$ is also an RM-domain).

A Dedekind domain has the property that every proper homomorphic image is a principal ideal ring (P.I.R.) and the converse is also valid. In the next theorem we characterize rings with an identity which have the property that every proper homomorphic image is a principal ideal ring. Before proving this theorem, we will need the following definition and lemma.

**Definition 41:** A principal ideal ring is called a **special P.I.R.** if it has only one prime ideal $P \neq R$ and if $P^n = (0)$ for some positive integer $n$.

We will use the following theorem from [17; 245]:
"a ring R with an identity is a P.I.R. if and only if R is a direct sum of P.I.D.'s and special P.I.R.'s".

Lemma 42: If R is a P.I.R. with identity, then R is a general Z.P.I. ring.

Proof: Due to the above theorem we need only show that a special P.I.R. is a general Z.P.I. ring. Let P be the unique prime ideal in a special P.I.R., then P = (p) and $p^n = 0$. For $0 \neq x \in R$, then either $R = (x)$ or $x \in (p)$, and $x \in (p^i) - (p^{i+1})$; hence $x = p^i e$ where $e \notin (p)$, so e is a unit and $(x) = (p^i) = P^i$. Consequently, R is a general Z.P.I. ring since a finite direct sum of general Z.P.I. rings with identity is a general Z.P.I. ring.

Theorem 43: Let R be a ring with an identity. Then R/A is a principal ideal ring (P.I.R.) for every proper ideal A if and only if one of the following holds:

1) R is a Dedekind domain.
2) R is a P.I.R. with an identity.
3) R is a ring with an identity with a unique prime ideal M ⊆ R, such that $M = (a,b)$, $M^2 = (0)$ and $R \succ M \succ (a) \succ (0)$ is a composition series.

Proof: If R/A is a P.I.R. for all proper ideals A of R, then by Lemma 42 R/A is a general Z.P.I. ring for all proper ideals A. Therefore by Theorem 31, R is
either a general Z.P.I. ring with identity or a ring of type (iii). First consider the case when \( R \) is a general Z.P.I. ring; then by [2;89], \( R = R_1 \oplus \ldots \oplus R_n \) where \( R_i \) is either a Dedekind domain or a special primary ring. If \( n = 1 \), then \( R \) is either a Dedekind domain or a special primary ring. If \( R \) is a special primary ring then by [2;89] \( R \) is a P.I.R. because \( R \) is a general Z.P.I. ring with the d.c.c. \( (R \) has the d.c.c. since prime ideals are maximal). If \( n > 1 \), then \( R/(\sum_{j=1}^n R_j) \cong R_i \) is a P.I.R. for \( i = 1, \ldots, n \) and \( R \) is a P.I.R.

The converse is well known in case (i) or (ii) hold. Now suppose \( R \) has property (iii) and we will show that every ideal \( B \subset M \) is principal. Since \( R \triangleright M \triangleright (a) \triangleright (0) \) is a composition series, then \( R \triangleright M \triangleright B \triangleright (b) \triangleright (0) \) can be refined to a composition series of the same length and \( B = (b) \). It is clear that \( \bar{M} = M/B \) is principal in \( R/B = \bar{R} \), since \( \bar{M} = (\bar{x}) \) for any \( x \in M-B \) \((R \triangleright (x) + B \triangleright B \triangleright (0) \) is a composition series).

In a large part of this paper we are concerned with the relationship between the a.c.c. and the d.c.c. The following theorem yields some information about this relationship, especially in light of Theorem 20.

**Theorem 44:** If \( N \) is a nilpotent ring with a composition series, then \( N \) is finite.

**Proof:** Suppose \( N^{k+1} = (0) \), \( N^k \nmid (0) \), and refine
Consider \( N^j \supset A_i > A_{i+1} \supset N^{j+1} \), with no ideals between \( A_i \) and \( A_{i+1} \); then \( A_i^2 \subseteq A_i N^j \subseteq N^{j+1} \), so \( A_i^2 + A_{i+1} \subseteq N^{j+1} + A_{i+1} = A_{i+1} \) and \( A_i^2 \subseteq A_{i+1} \). Hence \( [A_i/A_{i+1}]^2 = (A_i^2 + A_{i+1})/A_{i+1} = (0) \), and therefore \( A_i/A_{i+1} \) has no proper additive subgroups since subgroups are ideals in a ring whose square is zero; consequently \( A_i/A_{i+1} \) is finite \([8;9]\). Using the fact \( A_i/A_{i+1} \) is finite we see \( N \) is finite since

\[
|N| = |N/A_1| \cdot |A_1/A_2| \cdots |A_{e-1}/A_e| \cdot |A_e/(0)|.
\]

**Theorem 45:** Let \( R \) be a ring with the d.c.c. Then \( R \) has the a.c.c. if and only if \( R \) is either a nilpotent ring \( N \) or \( R = R_1 \oplus N \) where \( R_1 \) is a ring with identity with the d.c.c. and \( N \) is a finite nilpotent ring.

**Proof:** If \( R \) has the a.c.c. and d.c.c., then \( R \) has a composition series \([17;161]\) and \( (0) = P_1 \cdots P_n \) where \( R/P_i \) is a field for each \( i = 1, \ldots, n \). By Theorem 27 \( R = R_1 \oplus N \), where \( R_1 \) has an identity and \( N \) is nilpotent, and the previous theorem shows that \( N \) is finite. The converse is clear.
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BIOGRAPHY

John Randolph Gilbert, Jr. was born on April 28, 1941, in Stonewall, Mississippi. He attended the public school system in Prichard, Alabama, and graduated from Vigor High School in 1959. He received his Bachelor of Arts degree in Mathematics from the University of Alabama in 1962, and his Master of Arts degree in 1964. At present he is a student at Louisiana State University in Baton Rouge, where he is a candidate for the Doctor of Philosophy degree in the Department of Mathematics.
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