Laguerre functions associated to Euclidean Jordan algebras

Michael Aristidou

Louisiana State University and Agricultural and Mechanical College

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LAGUERRE FUNCTIONS ASSOCIATED TO EUCLIDEAN JORDAN ALGEBRAS

A Dissertation
Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy
in
The Department of Mathematics

by
Michael Aristidou
B.S. in Math., Aristotle University of Thessaloniki, 1999
M.S. in Math., Louisiana State University, 2001
M.A. in Phil., Louisiana State University, 2004
August 2005
"Μηδενὶ ἀλλὰ πείθεσθαι ἢ τῷ λόγῳ"

ΣΩΚΡΑΤΗΣ
(Πλάτων, 'Κρίτων' 46β)

[Nothing can convince me but reason]

SOCRATES
(Plato, 'Crito' 46 b)
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Abstract

Certain differential recursion relations for the Laguerre functions, defined on a symmetric cone $\Omega$, can be derived from the representations of a specific Lie algebra on $L^2(\Omega, d\mu_\nu)$. This Lie algebra is the corresponding Lie algebra of the Lie group $G$ that acts on the tube domain $T(\Omega) = \Omega + iV$, where $V$ is the associated Euclidean Jordan algebra of $\Omega$. The representations involved are the highest weight representations of $G$ on $L^2(\Omega, d\mu_\nu)$. To obtain these representations, we start from the highest weight representations of $G$ on $H_\nu(T(\Omega))$, the Hilbert space of holomorphic functions on $T(\Omega)$, and we transfer the representations to $L^2(\Omega, d\mu_\nu)$ via the Laplace transform. The Laguerre functions correspond to an orthogonal set of functions in $H_\nu(T(\Omega))$ and they form an orthogonal basis in $L^2(\Omega, d\mu_\nu)^L$, where $L$ is a specific subgroup of $G$. The recursion relations result by restricting the representation to a distinguished 3-dimensional subalgebra which is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. First, we construct the differential recursion relations for Laguerre functions defined on $\Omega = Sym^+(n, \mathbb{R})$, the cone of positive definite real symmetric matrices, from the highest weight representations of $Sp(2n, \mathbb{R})$. These relations generalize the 'classical' relations for Laguerre functions on $\mathbb{R}^+$. Then, we consider highest weight representations of any simple Lie group $G$ to construct general differential recursion relations, for Laguerre functions defined on any symmetric cone, that generalize both the 'classical' recursion relations for Laguerre functions on $\Omega = \mathbb{R}^+$ and the ones for Laguerre functions on $\Omega = Sym^+(n, \mathbb{R})$. 
Introduction

In the late 18th and early 19th century, mathematicians and other scientists realized that the ordinary functions of the time (algebraic, exponential and trigonometric functions) had many limitations. For example, solutions to differential equations of significant problems in physics, such as the motion of the planets, could not be expressed by ordinary functions. What these mathematicians discovered was that `certain’ new functions were showing up in the context of, even unrelated sometimes, physical problems. These functions, by the latter half of 19th century, were known as ‘Special Functions’ and naturally some of them bore the name of the people who discovered them. Hence, we have special functions that came to be known as Bessel, Jacobi, Legendre, Hermite and Laguerre functions, but also special functions with names such as Beta, Gamma and Hypergeometric functions. Many of their functional properties, such as orthogonality relations, recursion relations and functional relations among them, were also known by the end of the 19th century.

The first to discover the close connection between special functions and representations of Lie groups was É. Cartan in the beginning of the 20th century. Basically, special functions appear as matrix entries of representations of Lie groups on certain spaces. This is well documented in classic texts by N. Vilenkin, N. Vilenkin-A. Klimyk and W. Miller in the second half of the 20th century [24, 25, 17]. The representation theory of Lie groups was proved to be a decisive tool in establishing several properties of the special functions, especially deriving several of their differential recursion relations.
All the above were done for special functions defined on $\mathbb{R}$, more specifically on $\mathbb{R}^+$. In the second half of the 20th century, we also have the first generalizations. C. Herz in 1955 [10], considered matrix valued Bessel functions and S. Gindikin in an important paper in 1964 [9], influenced by earlier work from Siegel [23], defined special functions on homogeneous convex cones. In particular, the Beta and Gamma functions were generalized on homogeneous convex cones. M. Koecher, also in the 60’s, developed his analysis on symmetric cones and Jordan algebras obtaining many important results. Finally, in the excellent book by J. Faraut and A. Koranyi [8] the generalized Laguerre functions are defined and a complete treatment of the harmonic analysis associated with symmetric cones and Jordan algebras is given.

The first decisive use of representation theory, though, in relation to recursion relations of special functions, comes from M. Davidson, G. Ólafsson and G. Zhang in 2002 [3]. The showed that representations of $\text{SL}_2(\mathbb{R})$ on $\mathcal{H}_\nu(\mathbb{H}, x^{\nu-1}dz)$, where $\mathbb{H} = \mathbb{R} + i\mathbb{R}^+$, when transferred on $L^2(\mathbb{R}^+, x^\nu dx)$ give rise to the ‘classical’ recursion relations for Laguerre functions, the latter defined on the cone $\mathbb{R}^+$. In 2003, the first two authors above extended their results when they showed that representations of $\text{SU}(n,n)$ give rise to generalized recursion relations of Laguerre functions defined on the cone $\text{Herm}^+(n, \mathbb{C})$ [5]. In this work, we use representations of $\text{Sp}(2n, \mathbb{R})$ to build the recursion relations of Laguerre functions defined on the cone $\text{Sym}^+(n, \mathbb{R})$, and then we construct general recursion relations for Laguerre functions defined on any symmetric cone $\Omega$ from representations of any simple Lie group $G$.

We summarize now some of the basic facts regarding Laguerre functions: The ‘classical’ Laguerre functions are defined through the Laguerre polynomials that can be defined, from the Rondriguez formula, by:

$$L^\nu_m(x) = \frac{e^x x^{-\nu}}{m!} \frac{d^m}{dx^m} e^{-x} x^{\nu+m}, \quad x \in \mathbb{R}^+, \ m, \nu \in \mathbb{N}.$$
It is not hard to see that the set \( \{ L^\nu_m(x) \} \) is an orthonormal basis for \( L^2(\mathbb{R}^+, e^{-x^\nu} dx) \).

The Laguerre functions are defined by:

\[
\ell^\nu_m(x) = e^{-x} L^\nu_m(2x)
\]

and the set \( \{ \ell^\nu_m(x) \} \) forms an orthogonal basis for \( L^2(\mathbb{R}^+, e^{-x}x^\nu dx) \). Furthermore, \( \ell^\nu_m \) satisfy certain recursion relations (see [3],[16]), such as:

\[
ex^2 \frac{d^2}{dx^2} \ell^\nu_m(x) + (\nu + 1) \frac{d}{dx} \ell^\nu_m(x) + (2m + \nu + 1 - x) \ell^\nu_m(x) = 0 \tag{1}
\]

\[
ex^2 \frac{d^2}{dx^2} \ell^\nu_m(x) + (2x + \nu + 1) \frac{d}{dx} \ell^\nu_m(x) + (x + \nu + 1) \ell^\nu_m(x) = -2(m + \nu) \ell^\nu_{m-1}(x) \tag{2}
\]

\[
ex^2 \frac{d^2}{dx^2} \ell^\nu_m(x) - (2x - \nu - 1) \frac{d}{dx} \ell^\nu_m(x) + (x - \nu - 1) \ell^\nu_m(x) = -2(m + 1) \ell^\nu_{m+1}(x) \tag{3}
\]

The Laplace Transform of \( \ell^\nu_m(x) \) gives:

\[
\mathcal{L}\nu (\ell^\nu_m)(z) = \int_{\mathbb{R}^+} e^{-zx} \ell^\nu_m(x) \, d\mu_\nu(x) = c_{\nu,m} (z - 1)^m (z + 1)^{-(m+\nu+1)}
\]

Denote the polynomials on the right-hand side of the equation above by \( q^\nu_m(z) \). Then, \( \{ q^\nu_m(z) \} \) is an orthogonal basis of the space \( \mathcal{H}_\nu(\mathbb{H}, x^{\nu-1} dz) \), where \( \mathbb{H} = \mathbb{R} + i\mathbb{R}^+ \).

Observe that \( \mathbb{R}^+ \) is a symmetric cone, \( \mathbb{R} \) is a Euclidean Jordan algebra, and highest weight representations of \( SL_2(\mathbb{R}) \) on \( \mathcal{H}_\nu(\mathbb{H}, x^{\nu-1} dz) \) are derived through the action of \( SL_2(\mathbb{R}) \) on \( \mathbb{H} \). Hence, highest weight representations of \( SL_2(\mathbb{R}) \) can also be constructed on \( L^2(\mathbb{R}^+, x^\nu dx) \) via \( \mathcal{L}_\nu^{-1} \), since \( \mathcal{L}_\nu \) is a unitary isomorphism from \( L^2(\mathbb{R}^+, x^\nu dx) \) to \( \mathcal{H}_\nu(\mathbb{H}, x^{\nu-1} dz) \). This idea, enabled the authors in [3] to generate relations (1), (2) and (3) above, using representation theory.

The first time, though, that \( L^\nu_m(x) \) were related to representation theory was when
$L_n(x)$ showed up as matrix coefficients of group representations. Following [24], we show briefly the main idea. Consider the following matrix group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, c \neq 0 \right\}$$

and the Hilbert space

$$\mathcal{H} = \{ f : \mathbb{C} \rightarrow \mathbb{C}, f \text{ analytic} \}.$$

The representations of $G$ on $\mathcal{H}$ are given by:

$$\pi(g)f(z) = e^{dz+b}(cz+a)$$

The set $\{ z^k \}$, where $0 \leq k < \infty$, is a basis for $\mathcal{H}$. Hence, the representation on that basis is given by:

$$\pi(g)z^n = e^{dz+b}(cz+a)^n$$

Considering now the fact that

$$e^{-xz}(1+z)^\nu = \sum_{m=0}^{\infty} L^\nu_{m-m}(x)z^m$$

we have that:

$$\pi(g)z^n = \sum_{m=0}^{\infty} e^b a^{n-m} L^\nu_{m-m}(-\frac{ad}{c})z^m$$

Recall that the formula that gives the matrix entries of a matrix $A$ is given by $A_{ij} = (Ae_i|e_j)$, where $\{e_i\}$ is an orthonormal basis. Similarly, in our case we have:

$$\pi^{\nu}_{n,m}(g) = (\pi(g)z^n|z^m) = e^b a^{n-m} L^\nu_{m-m}(-\frac{ad}{c}),$$
which shows that the Laguerre polynomials are expressed as matrix coefficients.

The generalized Laguerre polynomials are defined in [8] in terms of certain $L$-invariant polynomials $\psi_n(x)$, where $n = (n_1, ..., n_n) \in \mathbb{C}^n$ and $L$ is the compact group that fixes the cone $\Omega$. They are defined as follows:

$$L_\nu^\mu_m(x) = (\nu)_m \sum_{|n| \leq |m|} \left( \begin{array}{c} m \\ n \end{array} \right) \frac{1}{(\nu)_n} \psi_n(-x), \quad x \in \Omega$$

where $(\nu)_n$ and $\left( \begin{array}{c} m \\ n \end{array} \right)$ are both given in terms of the generalized Gamma function

$$\Gamma_\Omega(m) = \int_\Omega e^{-\text{tr}x} \Delta_m(x) \Delta(x)^{-\frac{d}{2}} dx,$$

with $\Delta_m(x) = \Delta_{m_1 - m_2}(x) \Delta_{m_2 - m_3}(x) \cdots \Delta_{m_n}(x)$ and $\Delta(x) := \Delta_n(x) = \det(x)$.

Finally, the generalized Laguerre functions are defined, in terms of $L_\nu^\mu_m(x)$, by:

$$\ell_\nu^\mu_m(x) = e^{-\text{tr}(x)} L_\nu^\mu_m(2x)$$

and in [8] is shown that the set $\{\ell_\nu^\mu_m\}_{m \geq 0}$ forms an orthogonal basis in $L_\nu^2(\Omega)^L$, the Hilbert space of $L$-invariant functions in $L_\nu^2(\Omega)$.

This dissertation is organized as follows: In Chapter 1, we introduce the basic concepts on symmetric cones and Jordan algebras. In Chapter 2, we construct generalized recursion relations for Laguerre functions defined on $\text{Sym}^+(n, \mathbb{R})$ from the representations of $\text{Sp}(2n, \mathbb{R})$. These new relations generalize the recursion relations of the 'classical' Laguerre functions. In Chapter 3, we use representations of any simple Lie group $G$ to construct general differential recursion relations, for Laguerre functions defined on any symmetric cone, that generalize both the 'classical' recursion relations and the ones for Laguerre functions defined on $\text{Sym}^+(n, \mathbb{R})$. 
Chapter 1

Symmetric Cones and Jordan Algebras

In this chapter, we present some basic definitions and facts regarding symmetric cones and Jordan algebras. We keep the proofs to the minimum, and give more examples instead, since a detailed exposition of the subject can be found in [8].

1.1 Symmetric Cones

Let $V$ be a finite-dimensional real Euclidean space. We denote the inner product in $V$ by $(\cdot | \cdot)$.

Definition 1.1.1. A non-empty open subset $\Omega$ of $V$ is called an open cone if for every $x \in \Omega$ and every $\lambda > 0$, we have $\lambda x \in \Omega$. If, in addition, for every $x, y \in \Omega$ and $\lambda, \mu > 0$ we have $\lambda x + \mu y \in \Omega$, then $\Omega$ is said to be an open convex cone.

Example 1.1.2. A simple example of an open convex cone is $\mathbb{R}^+$. Another, is the set $\text{Sym}^+(n, \mathbb{R})$, the space of positive definite symmetric matrices, defined by $\text{Sym}^+(n, \mathbb{R}) = \{ A \in \text{Sym}(n, \mathbb{R}) \mid A > 0 \}$. Recall that $A > 0$ means that $A$ is positive definite (i.e. $(Ax|x) > 0, \forall x \in V, x \neq 0$). Finally, the set $Q = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid q(t, x) > 0, \ t > 0\}$, where $q(t, x)$ is a bilinear form defined on $\mathbb{R} \times \mathbb{R}^n$ by $q(t, x) = t^2 - \|x\|^2$, is also an open convex cone. The cone $Q$ is usually called the light cone.
From now on, by cone we will mean a non-empty open convex cone.

**Definition 1.1.3.** Let $\Omega$ be a cone, and $\bar{\Omega}$ be its closure. Then, the cone $\Omega$ is said to be **proper** if $\bar{\Omega} \cap (-\bar{\Omega}) = \{0\}$.

**Definition 1.1.4.** The set $\Omega^* = \{y \in V \mid (x|y) > 0, \forall x \in \bar{\Omega} - \{0\}\}$ is called the (open) **dual cone** of the cone $\Omega$. The cone $\Omega$ is called **self-dual** if $\Omega = \Omega^*$.

Consider now the **automorphism group** of $\Omega$, defined by:

$$G(\Omega) = \{g \in GL(V) \mid g\Omega = \Omega\},$$

where $GL(V)$ is the general linear group of invertible linear transformations on $V$. Notice that $G(\Omega)$ is a closed subgroup of the Lie group $GL(V)$, since $g \in G(\Omega)$ if and only if $g\bar{\Omega} = \bar{\Omega}$, hence $G(\Omega)$ is a Lie group. The adjoint of $g$ is defined to be the element $g^*$ such that $(gx|y) = (x|g^*y)$, and consequently we have $G(\Omega^*) = G(\Omega)^*$ if and only if $\Omega = \Omega^*$ (see [8], p.4). Obviously, $G(\Omega)$ acts on $\Omega$, so let us denote by $g \cdot x$ the action of $G(\Omega)$ on $\Omega$.

**Definition 1.1.5.** The cone $\Omega$ is said to be **homogeneous** if $G(\Omega)$ acts transitively on $\Omega$. Furthermore, if the cone $\Omega$ is homogeneous and self-dual, then the cone is said to be **symmetric**.

**Example 1.1.6.** Let $V = Sym(n, \mathbb{R})$ be the space of real symmetric matrices, and $\Omega = Sym^+(n, \mathbb{R})$ the space of positive definite symmetric matrices. For symmetric matrices we can define an inner product by $(x|y) = \text{Tr}(xy)$. It is not hard to show that $\Omega \subset V$ is a symmetric cone:

(i) First, let us show that $\Omega$ is self-dual: Let $y \in \Omega^*$. Then, $(x|y) > 0, \forall x \in \bar{\Omega} - \{0\}$. Take $x = uu^t, \forall u \in \mathbb{R}^n - \{0\}$. This $x$ belongs to $\bar{\Omega} - \{0\}$. Indeed, $(xv|v) = (uu^tv|v) = (u^tv|u^tv) = \|u^tv\|^2 > 0, \forall v \in \mathbb{R}^n - \{0\}$. Therefore, $(uu^ty) > 0 \Rightarrow (u|uy) > 0, \forall u \in \mathbb{R}^n$. Hence, $\Omega$ is self-dual.
$\mathbb{R}^n - \{0\}$, which says that $y \in \Omega$. For the reverse inclusion, let $y \in \Omega$. Since $y$ is symmetric, then $y$ is diagonalizable. Hence by the Spectral Theorem, there exists $a \in O(n)$ such that $y = ada^t$ where $d$ is the diagonal matrix:

$$d = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}, \quad \lambda_i > 0.$$ 

Now, $\forall x \in \tilde{\Omega} - \{0\}$, we have:

$$(x|y) = \text{Tr}(xy) = \text{Tr}(xada^t) = \text{Tr}(x'a'd) = \text{Tr}(xa^td)$$

$$= \text{Tr}(x'd)$$

$$= \sum_{i=1}^{n} (x'de_i|e_i)$$

$$= \sum_{i=1}^{n} (x'e_i|de_i)$$

$$= \sum_{i=1}^{n} (x'e_i|\lambda_ie_i)$$

$$= \sum_{i=1}^{n} \lambda_i(x'e_i|e_i) > 0,$$

because $\lambda_i > 0$ and $x'$ is positive definite. Since $(x|y) > 0$, $\forall x \in \tilde{\Omega} - \{0\}$, we have that $y \in \Omega^*$.

(ii) Finally, we show that $\Omega$ is homogeneous: Clearly, $G(\Omega) = GL_+(n, \mathbb{R})/\{\pm I_n\}$. Now let $GL_+(n, \mathbb{R})$ act on $\Omega$ by $g \cdot x = gxg^t$. This action is transitive. Indeed, since $x \in Sym^+(n, \mathbb{R})$, by the Spectral Theorem, $x$ is diagonalizable with positive eigenvalues. In other words, there exists $a \in O(n) \subset GL_+(n, \mathbb{R})$ such that $axa^t = d_1$, where $d_1$ is the
diagonal matrix:

\[
d_1 = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}, \quad \lambda_i > 0.
\]

For another \( y \in Sym^+(n, \mathbb{R}) \), we also have that \( byb^t = d_2 \) for some \( b \), where \( d_2 \) is the diagonal matrix:

\[
d_2 = \begin{pmatrix}
s_1 & 0 & \ldots & 0 \\
0 & s_2 & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \ldots & s_n
\end{pmatrix}, \quad s_i > 0.
\]

That is, \( y = b^t d_2 b \). Notice now that \( d_2 = cd_1 c \), where \( c \) is given by:

\[
c = \begin{pmatrix}
\sqrt{\frac{s_1}{\lambda_1}} & 0 & \ldots & 0 \\
0 & \sqrt{\frac{s_2}{\lambda_2}} & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \sqrt{\frac{s_n}{\lambda_n}}
\end{pmatrix}.
\]

Hence, \( y = b^t d_2 b = b^t c d_1 c b = b^t c c^t r^t x a^t c b = r^t x r \). Since, \( \forall x, y \in \Omega \) there exists \( r \in GL_+(n, \mathbb{R}) \) such that \( y = r \cdot x \), that means the action is transitive. Therefore, \( \Omega \) is homogeneous.

**Example 1.1.7.** Let \( \Omega = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid q(t, x) > 0, \ t > 0\} \) be the light cone, with \( q(t, x) \) as defined in Example 1.1.2. Using the inner product \((t, x)|(s, y)) = ts + (x|y)\) on \( V = \mathbb{R} \times \mathbb{R}^n \), one can show that \( \Omega \) is a symmetric cone:
(i) First, let us show that $\Omega$ is self-dual: Let $(t, x) \in \Omega$. Then, for all $(s, y) \in \bar{\Omega} - \{0\}$, we have by Schwarz’s Inequality

$$((t, x)(s, y)) = ts + (x|y) \geq ts - \|x\||y\| > 0.$$ 

That implies that $(t, x) \in \Omega^*$. Notice that the last part of the inequality is clearly positive, because since $(t, x) \in \Omega \Rightarrow t^2 - \|x\|^2 > 0 \Rightarrow (t - \|x\|)(t + \|x\|) \Rightarrow t - \|x\| > 0$, as $t + \|x\|$ is always positive. Since $t - \|x\| > 0$ and $s - \|y\| > 0$ we have that $ts - \|x\||y\| > 0$.

For the reverse inclusion, let $(t, x) \in \Omega^*$. We have $t > 0$, because otherwise $((t, x)(s, y))$ would not be positive for all $(s, y) \in \bar{\Omega} - \{0\}$. Take, for example, $(s, y) = (1, 0)$. We want to show that $(t, x) \in \Omega$, i.e. $t^2 - \|x\|^2 > 0$. Notice, though, that if $x = 0$, then obviously $(t, x) \in \Omega$. If $x \neq 0$, then some of the coordinates of $x$ are not zero. In other words, $x_i \neq 0$ for some $i$. Since $((t, x)(s, y)) > 0, \forall (s, y) \in \bar{\Omega} - \{0\}$ we can take $(s, y)$ as follows: $s = t$ and $y = -x$. Then, $((t, x)(s, y)) > 0 \Rightarrow t^2 - \|x\|^2 > 0$, which says that $(t, x) \in \Omega$.

(ii) Finally, we show that $\Omega$ is homogeneous: Let $(t, x) \in \Omega$. For every such $(t, x)$ we can take $\lambda = \sqrt{\frac{1}{t^2 - \|x\|^2}}$ which is positive. Notice that $(\lambda t, \lambda x) \in \Omega$ and $q(\lambda t, \lambda x) = 1$. Hence, we can always assume, without loss of generality, that $q(t, x) = 1$. Take $b \in \mathbb{R}$ such that $t = \cosh b$ and $\|x\|^2 = \sinh b$. Clearly, $q(t, x) = 1$. There exists an $h$ in $SO(n) \subset SO(1, n)$ such that $h \cdot (t, x) = (t, (0, 0, ..., \sinh b))$. Consider the element:

$$a_b^{-1} = \begin{pmatrix} \cosh b & 0 & \sinh b \\ 0 & I_{n-1} & 0 \\ \sinh b & 0 & \cosh b \end{pmatrix} \in SO(1, n) = G(\Omega).$$

Take $g = h^{-1}a_b^{-1} \in SO(1, n)$. Then, $g \cdot e = (t, x)$ which says that the action is transitive.

Let $\Omega$ be symmetric. Let $H := G(\Omega)^0$ denote the connected component of $G(\Omega)$ containing the identity and $O(V)$ the orthogonal group of $V$ defined, as usual, by $O(V) = \{g \in GL(V)| g^*g = id\}$. Let $L = H \cap O(V)$. Then $L$ is a maximal compact subgroup in $H$ and $L = \text{Stab}(a)$, for some $a \in \Omega$. In other words, $L = \{h \in H| h \cdot a = a\}$. Notice
now that $\Omega \simeq H/L$ via the mapping $\omega = h \cdot a \mapsto hL$, which means that we can think of $\Omega$ also as a Riemannian symmetric space.

**Example 1.1.8.** We have seen in Example 1.1.6 that by the action of $GL_+(n, \mathbb{R})$, namely $g \cdot x = gx^t$, $\Omega = Sym^+(n, \mathbb{R})$ becomes a symmetric cone. Now, for $L$ we have:

$L = \text{Stab}(I_n) = \{g \in GL_+(n, \mathbb{R}) \mid gI_ng^t = I_n\} = \{g \in GL_+(\mathbb{R}) \mid gg^t = I_n\} = O_+(n)$.

Hence, $Sym^+(n, \mathbb{R}) \simeq GL_+(n, \mathbb{R})/O_+(n)$.

**Example 1.1.9.** In Example 1.1.7, the action of $SO(1,n)$, namely $g \cdot x = gx$, makes $\Omega = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid q(t,x) > 0, t > 0\}$ into a symmetric cone. We will calculate $L = \text{Stab}(e)$, where $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. First note that if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then we have the following relations among $A, B, C$ and $D$:

$A^tA - C^tC = 1$

$A^tB - C^tD = 0$

$B^tA - D^tC = 0$

$B^tB - D^tD = -1$

Now, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow A = 1, C = 0$. Using this in the relations above, we also get $B = 0$ and $D^tD = 1$. Therefore, $\text{Stab}(e) = \{\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \mid D^tD = 1\} \cong SO(n)$, and hence, $\Omega \cong SO(1,n)/SO(n)$.

By *rank* of $\Omega$ we mean the rank of the space $H/L$, where the second is defined as follows: Let $\theta$ be a Cartan involution on $\mathfrak{h}$, the Lie algebra of $H$. Then $\theta$ decomposes $\mathfrak{h}$ as $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{s}$, where $\mathfrak{l}$ is the Lie algebra of $L$. The *rank* of $H/L$ is the dim of $\mathfrak{a}$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{s}$. Let $r = \text{rank}(\Omega)$, $d = \text{dim}V$ and $p = \frac{2d}{r}$. Then, with the help of $p$, we can actually define an $H$-invariant measure on $\Omega$ as follows:
**Definition 1.1.10.** The function \( \phi(x) = \int_{\Omega} e^{-(x|y)} \, dy, \ x \in \Omega \) is called the **characteristic function** of \( \Omega \), and the function \( \Delta(x) = c\phi(x)^{-\frac{2}{p}} \) is called the **Koecher norm function**, where \( c \) is such that \( \Delta(a) = 1 \), for some \( a \in \Omega \).

The function \( \Delta \) is an \( L \)-invariant polynomial on \( V \), and extends uniquely to an \( L \)-invariant polynomial on \( V_{\mathbb{C}} \) (see [7], p.12).

**Proposition 1.1.11.** For all \( h \in H, x \in \Omega \), we have:

\[
\begin{align*}
(a) \quad \Delta(h \cdot x) &= (\text{det} \, h)^{\frac{2}{p}} \Delta(x) \\
(b) \quad d_{p}x := \Delta(x)^{-\frac{2}{p}} \, dx \text{ is an } H \text{-invariant measure on } \Omega.
\end{align*}
\]

**Proof.** Let \( h \) and \( x \) be as in the Proposition. Then we get:

\[
\begin{align*}
(a) \quad \Delta(h \cdot x) &= c\phi(h \cdot x)^{-\frac{2}{p}} = c \left( \int_{\Omega} e^{-(h \cdot x|y)} \, dy \right)^{-\frac{2}{p}} \\
&= c \left( \int_{\Omega} e^{-(x|h^{*}y)} \, dy \right)^{-\frac{2}{p}} \\
&= c \left( \int_{\Omega} e^{-(x|u)} (\text{det} \, h^{*})^{-1} \, du \right)^{-\frac{2}{p}}, \text{ by letting } h^{*} \cdot y = u \\
&= c (\text{det} \, h)^{\frac{2}{p}} \phi(x)^{-\frac{2}{p}} \\
&= (\text{det} \, h)^{\frac{2}{p}} \Delta(x)
\end{align*}
\]

\[
\begin{align*}
(b) \quad d_{p}(h \cdot x) = \Delta(h \cdot x)^{-\frac{2}{p}} d(h \cdot x) &= \left[ (\text{det} \, h)^{\frac{2}{p}} \Delta(x) \right]^{-\frac{2}{p}} \text{det} \, h \, dx \\
&= \text{det} \, h^{-1} \text{det} \, h \Delta(x)^{-\frac{2}{p}} \, dx \\
&= \Delta(x)^{-\frac{2}{p}} \, dx \\
&= d_{p}x
\end{align*}
\]

\[\square\]
Remark 1.1.12. Consider the Cartan involution on $H$, given by $\theta(h) = (h^*)^{-1}$. Then, $L = \{h \in H | \theta(h) = h\}$. It is also clear that the mapping, $h \mapsto \Delta(h \cdot a)^p = (\det h)^2$, is a 1-dimensional representation of $H$ into $\mathbb{R}$.

1.2 Jordan Algebras

In this section, we assume that all Jordan algebras have finite dimension.

Definition 1.2.1. A vector space $V$ over $\mathbb{R}$ is called a Jordan algebra, if there is a bilinear map $(x, y) \mapsto x \circ y$, from $V \times V$ to $V$, that satisfies the following two conditions:

(i) \hspace{1cm} x \circ y = y \circ x

(ii) \hspace{1cm} x \circ (x^2 \circ y) = x^2 \circ (x \circ y), \ \forall \ x, y \in V.

Clearly, $x^2$ denotes $x \circ x$, and notice that a Jordan algebra is not, in general, associative but is always commutative.

For any element $x \in V$ consider the linear endomorphism on $V$, defined by:

\[ L(x)y = x \circ y. \]

In terms of $L(x)$, condition (ii) in the definition above is equivalent to:

\[ L(x)L(x^2) = L(x^2)L(x). \]

Notice, also, that the condition above is equivalent to $[L(x), L(x^2)] = 0$, where $[,]$ is the Lie bracket in the Lie algebra $\text{Lie}(\text{End}(V))$. It is also clear that if $V$ is associative, then $L^2(x) = L(x^2)$.

Example 1.2.2. Take $V = \text{Sym}(n, \mathbb{R})$, the space of real symmetric matrices. Define a
product on $V$ by $x \circ y = \frac{1}{2}(xy + yx)$. Then, it is easy to see that $V$, with this product (Jordan product), is a Jordan algebra. In fact, more generally, any associative algebra $V$ with the Jordan product is a Jordan algebra. Let us call such Jordan algebras as special Jordan algebras.

**Example 1.2.3.** Let $V = \mathbb{R} \times \mathbb{R}^n$. Define a product on $V$ by $(t, u) \circ (s, v) = (ts + q(u, v), tv + su)$, where $q(u, v)$ is a symmetric bilinear form on $\mathbb{R}^n$ (e.g. $q(u, v) = (u|v)$). Then, it is easy to see that $V$ with this product is a Jordan algebra.

**Definition 1.2.4.** Let $V$ be a Jordan algebra over $\mathbb{R}$. An element $e \in V$ is called the identity element if $e \circ x = x \circ e = x, \forall x \in V$.

Notice that in a Jordan algebra, the identity element is necessarily unique. Let $V$ be a Jordan algebra over $\mathbb{R}$, with identity element $e$. Let $\mathbb{R}[X]$ denote the algebra of polynomials in one-variable over $\mathbb{R}$. For an $x \in V$, consider the map:

$$\mathbb{R}[X] \longrightarrow V$$

$$p \mapsto p(x).$$

The image of this map is all linear combinations of $e, x, x^2, ..., x^k, ...$, where $x^k = \underbrace{x \circ x \circ ... \circ x}_{k \text{ times}}$. But since $\dim V < \infty$, then there exist a minimal $N > 0$ such that $e, x, x^2, ..., x^N$ are linearly dependent. Thus, $x^N - a_{N-1}(x)x^{N-1} + ... + (-1)^N a_0(x)e = 0$, where $a_j$ are unique homogeneous polynomials of degree $j$ (i.e. $a_j(\lambda x) = \lambda^j a_j(x), \lambda > 0$) from $V$ to $\mathbb{R}$ (see [8], p.38). The polynomial $p_x(X) = X^N - a_{N-1}(x)X^{N-1} + ... + (-1)^N a_0(x)e$ is called the minimal polynomial of $x$, and it is unique. Obviously, for different $x$ we might get different $N$, so the number $\sup_{x \in V} \{N \mid N = \deg p_x(X)\}$ which clearly exists is called the rank of $V$ and is denoted by $r$. The rank of $V$ is the same as the rank of $\Omega$, which is that is why they are both denoted by $r$.

**Definition 1.2.5.** An element $x \in V$ is called regular, if $N = r$. 

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In other words, \( x \) is regular if \( p_x \) has maximal degree. We write the minimal polynomial of a regular element as \( p_x(X) = X^r - a_{r-1}(x)X^{r-1} + \ldots + (-1)^ra_0(x)e \).

**Example 1.2.6.** Consider \( V = \text{Sym}(n, \mathbb{R}) \). First, notice that squaring in \( V \) is the usual squaring, since \( x^2 = x \circ x = \frac{1}{2}(xx + xx) = x^2 \), and that the identity is:

\[
e = I_n = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

By the Cayley-Hamilton Theorem we know that for any \( x \) there exist a monic polynomial of minimal degree such that \( p(x) = 0 \). It is a fact then that \( p(x) = p_x(x) \), since the two polynomials cannot be different and both minimal. Since \( x \) is symmetric, we can assume that \( x \) is diagonal, such that:

\[
x = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}, \quad \text{with } \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.
\]

Then, \( p_x(x) = (x - \lambda_{i_1})(x - \lambda_{i_2})\ldots(x - \lambda_{i_n}) \), where \( \lambda_{i_j} \) are different eigenvalues. Notice that \( r \) can go from 1 up to \( n \), which means that when \( r = n \) (i.e. when \( x \) is regular) all \( \lambda_i \)'s are different. In other words, if \( x \) is regular, with eigenvalues \( \lambda_1 > \lambda_2 > \ldots > \lambda_n \), then \( p_x(x) = (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_n) \). That is, \( p_x(x) = x^n - (\lambda_1 + \lambda_2 + \ldots + \lambda_n)x^{n-1} + \ldots + (-1)^n\lambda_1\lambda_2\ldots\lambda_n \).

**Example 1.2.7.** Let \( V = \mathbb{R} \times \mathbb{R}^n \). Set \( x = (t, u) \). Then, \( x^2 = (t^2 + q(u, u), 2tu) \). Observe
that $x^2 - 2tx + (t^2 - q(u,u))e = 0$, where $e = (1, 0)$, which means that $V$ has rank = 2. Hence, $p_x(x) = x^2 - 2tx + (t^2 - q(u,u))e$.

**Definition 1.2.8.** Let $x \in V$ be regular. Then, the coefficients $a_{r-1}$ and $a_0$ of $p_x$ are called, respectively, the trace and the determinant and are denoted by: $\text{tr}(x)$ and $\text{det}(x)$.

**Example 1.2.9.** In Example 1.2.6, we have $p_x(x) = x^n - (\lambda_1 + \lambda_2 + ... + \lambda_n)x^{n-1} + ... + (-1)^n\lambda_1\lambda_2...\lambda_n$. Hence, $\text{tr}(x) = \lambda_1 + \lambda_2 + ... + \lambda_n$ and $\text{det}(x) = \lambda_1\lambda_2...\lambda_n$.

**Example 1.2.10.** From Example 1.2.7, we have $p_x(x) = x^2 - 2tx + (t^2 - q(u,u))e$. Hence, $\text{tr}(x) = 2t$ and $\text{det}(x) = t^2 - q(u,u)$.

Notice that in Example 1.2.9 above, $\text{tr}(x) = \lambda_1 + \lambda_2 + ... + \lambda_n = \text{Tr}(x)$ and $\text{det}(x) = \lambda_1\lambda_2...\lambda_n = \text{Det}(x)$, where $\text{Tr}(x)$ and $\text{Det}(x)$ are the usual trace and determinant of the linear map (matrix) $x$. This can be generalized, as the following proposition shows.

**Proposition 1.2.11.** Let $L_0(x)$ be the restriction of $L(x)$ on $\mathbb{R}[x]$, where $x$ is regular. Then, $\text{tr}(x) = \text{Tr}(L_0(x))$ and $\text{det}(x) = \text{Det}(L_0(x))$.

**Proof.** By the definition of $L(x)$ we have that $L(x)x^n = x^{n+1}$. Now, consider the linear map $L_0(x) = L(x)|_{\mathbb{R}[x]} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$. We will calculate the corresponding matrix of $L_0(x)$, with respect to the basis $e, x, ..., x^{r-1}$. We have:

\[
L_0(x)e = x = 0 + 1x + 0x^2 + ... + 0x^{r-1}
\]

\[
L_0(x)x = x^2 = 0 + 0x + 1x^2 + ... + 0x^{r-1}
\]

\[
\vdots
\]

\[
L_0(x)x^{r-1} = x^r = (-1)^{r-1}a_0(x)e + a_1(x)x + ... + a_{r-1}(x)x^{r-1},
\]
as $x$ is regular. So, the corresponding matrix of $L_0(x)$ is the $r \times r$ matrix given by:

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & (-1)^{r-1}a_0(x) \\
1 & 0 & \ldots & 0 & (-1)^{r-2}a_1(x) \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & a_r(x)
\end{pmatrix}
$$

Now clearly, $\text{Tr}(L_0(x)) = a_r(x) = \text{tr}(x)$ and $\text{Det}(L_0(x)) = a_0(x) = \text{det}(x)$ which is what we wanted to show.

\[\square\]

**Proposition 1.2.12.** Let $x \in V$ and $u, v \in \mathbb{R}[x]$. Then:

(a) $\det(u \circ v) = \det(u) \det(v)$

(b) $\det(e) = 1$, $\text{tr}(e) = r$.

**Proof.** See [8], p.30.

\[\square\]

**Definition 1.2.13.** An element $x \in V$ is said to be invertible if there exists an element $y \in \mathbb{R}[x]$ such that $x \circ y = y \circ x = e$.

The element $y$ is called the inverse of $x$ and it is unique. It is denoted by $x^{-1}$.

**Remark 1.2.14.** The condition $u, v \in \mathbb{R}[x]$ in the proposition above is necessary, as the following examples suggest:

(a) Take, for instance, $V = \text{Sym}(2, \mathbb{R})$ and the elements $x = u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $V$. Then, $u \circ v = \frac{1}{2} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = 0$, which means $\det(u \circ v) = 0$. But on the other hand, $\det(u) = -1$ and $\det(v) = -1$, which gives $\det(u) \det(v) = 1$. Therefore, $\det(u \circ v) \neq \det(u) \det(v)$. That is because $v \notin \mathbb{R}[x]$. 

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(b) Take, again, $V = \text{Sym}(2, \mathbb{R})$ and the elements $x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ in $V$.

Notice that $x^{-1} = x$, and although $x \circ y = e$, $y \neq x^{-1}$. That is because $y \notin \mathbb{R}[x]$.

**Proposition 1.2.15.** An element $x$ is invertible if and only if $\det(x) \neq 0$. Furthermore, $x^{-1} = \frac{g(x)}{(-1)^{r-1}\det(x)}$, where $g(x) = x^{r-1} - a_{r-1}(x)x^{r-2} + \ldots + (-1)^{r-1}a_1(x)e$.

**Proof.** See [8], p.31.

**Definition 1.2.16.** Let $x \in V$, where $V$ is a Jordan algebra with identity. The map $P(x) = 2L^2(x) - L(x^2)$ is called the **quadratic representation** of $V$.

We denote the directional derivative of $P(x)$ in $y$ by $P(x, y)$. In particular, $P(x, y) = \frac{1}{2}D_yP(x)$. The factor $\frac{1}{2}$ is just for reasons of simplification, as the following formulas will show.

**Lemma 1.2.17.** Let $P(x)$ be the quadratic representation of a Jordan algebra $V$. Then:

(a) $P(x + y) - P(x) - P(y) = 2(L(x)L(y) + L(y)L(x) - L(xy))$

(b) $P(x, y) = \frac{1}{2}[P(x + y) - P(x) - P(y)], \ \forall \ x, y \in V$

(c) $P(x, x) = P(x)$

(d) $P(x, y) = P(y, x)$

(e) $P(x, y)y = x(y^2)$.

**Proof.** (a) On the one hand, we have:

$$[P(x + y) - P(x) - P(y)]z = P(x + y)z - P(x)z - P(y)z$$

$$= 2L^2(x + y)z - L(x + y)^2z - 2L^2(x)z$$

$$+ L(x)^2z - 2L^2(y)z + L(y)^2z$$
\[
2(x + y)[(x + y)z] - (x + y)^2z - 2x(xz) + x^2z
- 2y(yz) + y^2z
= 2x(xz) + 2y(yz) + 2y(xz) + 2x(yz) - x^2z - 2(xy)z
- y^2z - 2x(xz) + x^2z - 2y(yz) + y^2z
= 2y(xz) + 2x(yz) - 2(xy)z.
\]

On the other hand, we have:

\[
2 \left( L(x)L(y) + L(y)L(x) - L(xy) \right) z = 2L(x)L(y)z + L(y)L(x)z - L(xy)z
= 2x(yz) + 2y(xz) - 2(xy)z.
\]

Since the left hand side equals the right hand side, part (a) is established.

(b) \( P(x, y) = \frac{1}{2}D_yP(x) = \frac{1}{2} \frac{d}{dt}P(x + ty)|_{t=0} \)
    \[= \frac{1}{2} \frac{d}{dt} \left[ 2L^2(x + ty) - L((x + ty)^2) \right]|_{t=0} \]
    \[= \frac{1}{2} \left[ 2L(x)L(y) + 2L(y)L(x) - L(xy + yx) \right] \]
    \[= \frac{1}{2} \left[ 2 \left( L(x)L(y) + L(y)L(x) - L(xy) \right) \right] \]
    \[= \frac{1}{2} \left[ P(x + y) - P(x) - P(y) \right], \text{ by (a)}. \]

(c) By (b), \( P(x, x) = \frac{1}{2}[P(x + x) - P(x) - P(x)] = \frac{1}{2}P(2x) = P(x). \)

(d) Clear by (b).

(e) From (a) and (b), \( P(x, y) = L(x)L(y) + L(y)L(x) - L(xy) \). Hence,

\[
P(x, y)y = [L(x)L(y) + L(y)L(x) - L(xy)]y
= x(y^2) + y(xy) - (xy)y
\]
\[ \begin{align*}
= & \quad x(y^2) + (xy)y - (xy)y \\
= & \quad x(y^2).
\end{align*} \]

**Proposition 1.2.18.** Let \( V \) be a special Jordan algebra \( V \). Then:

(a) \( P(x)y = xyx \)

(b) \( D_yP(x)z = xzy + yzx \)

(c) \( P(xyx) = P(x)P(y)P(x) \).

**Proof.**

(a) \( P(x)y = 2L^2(x)y - L(x^2)y = 2L(x)(x \circ y) - x^2 \circ y \)

\[ \begin{align*}
= & \quad 2x \circ (\frac{xy + yx}{2}) - \frac{x^2y + yx^2}{2} \\
= & \quad \frac{2x(xy + yx)}{2} + \frac{(xy + yx)x}{2} - \frac{x^2y + yx^2}{2} \\
= & \quad \frac{x^2y + xyx + yyx + yx^2}{2} - \frac{x^2y + yx^2}{2} \\
= & \quad \frac{2xyx}{2} \\
= & \quad xyx.
\end{align*} \]

(b) \( D_yP(x)z = P(x + y)z - P(x)z - P(y)z \), by Lemma 1.2.17 (b)

\[ \begin{align*}
= & \quad (x + y)z(x + y) - xzx - yzy, \quad \text{by (a)} \\
= & \quad xzx + xzy + yzx + yzy - xzx - yzy \\
= & \quad xzy + yzx.
\end{align*} \]
(c) \( P(xy)z = xyxzxyx = P(x)yxzxy = P(x)P(y)xzx = P(x)P(y)P(x)z, \)

which shows that \( P(xy) = P(x)P(y)P(x). \)

\[ \]

**Proposition 1.2.19.** Let \( x \in V. \) Then \( x \) is invertible if and only if \( P(x) \) is invertible. Furthermore:

(a) \( x^{-1} = P(x)^{-1}x \)

(b) \( P(x)^{-1} = P(x^{-1}) \)

(c) \( P(x)L(x^{-1}) = L(x). \)

**Proof.** See [8], p. 32.

**Proposition 1.2.20.** Let \( x, y \in V, \) where \( V \) is a Jordan algebra. Then:

(a) \( \) The derivative of the map \( x \mapsto x^{-1} \) is \( -P(x)^{-1}, \) i.e. \( D_y(x^{-1}) = -P(x)^{-1} \)

(b) \( \) If \( x, y \) are invertible, then \( P(x)y \) invertible and \( (P(x)y)^{-1} = P(x^{-1})y^{-1} \)

(c) \( \) For any \( x, y \in V, \) we have \( P(P(x)y) = P(x)P(y)P(x). \)

**Proof.** See [8], p. 33.

1.3 Euclidean Jordan Algebras

In this section, all Jordan algebras will have identity and finite dimension.

**Definition 1.3.1.** A Jordan algebra over \( \mathbb{R} \) is called **Euclidean** if there exists an inner product \( (\cdot | \cdot) \) on \( V \) such that \( (L(x)y|z) = (y|L(x)z), \forall x, y, z \in V. \)
**Remark 1.3.2.** It is clear by the definition above that $P(x)$ and $P(x,y)$ are also symmetric with respect to inner product, since both are expressed in terms of $L(x)$.

**Example 1.3.3.** The Jordan algebra $V = Sym(n, \mathbb{R})$, equipped with the inner product $(x|y) = \text{tr}(xy)$, is a Euclidean Jordan algebra. Indeed,

\[
(L(x)y|z) = \text{tr}(L(x)y z) = \text{tr}((x \circ y) z)
\]

\[
= \text{tr}(\frac{xy + yx}{2} z)
\]

\[
= \text{tr}(\frac{xyz + yxz}{2})
\]

\[
= \text{tr}(\frac{xyz}{2}) + \text{tr}(\frac{yxz}{2})
\]

\[
= \text{tr}(\frac{yxz}{2}) + \text{tr}(\frac{yxz}{2})
\]

\[
= \text{tr}(y(x \circ z))
\]

\[
= \text{tr}(yL(x)z)
\]

\[
= (y|L(x)z).
\]

**Example 1.3.4.** Let $V = \mathbb{R} \times \mathbb{R}^n$, equipped with the inner product $((t, u)|(s, v)) = ts + q(u, v)$. Then, $V$ is a Euclidean Jordan algebra. Indeed,

\[
(L((r, w)(t, u))(s, v)) = ((r, w)(t, u)|(s, v)) = ((rt + q(w, u), ru + tw)|(s, v))
\]

\[
= (rt + sq(w, u)) + q(ru + tw, v)
\]

\[
= rts + sq(w, u) + rq(u, v) + tq(w, v).
\]

On the other hand, $((t, u)|L((r, w))(s, v))$ also equals $rts + sq(w, u) + rq(u, v) + tq(w, v)$, which makes $(L((r, w)(t, u))(s, v)) = ((t, u)|L((r, w))(s, v))$, since their right-hand sides are equal. Hence, $V = \mathbb{R} \times \mathbb{R}^n$ is a Euclidean Jordan algebra.
Definition 1.3.5. An element $c \in V$ is called idempotent if $c^2 = c$. Two idempotents $c, d \in V$ are said to be orthogonal if $c \circ d = 0$.

Notice that, in a Euclidean Jordan algebra, orthogonality of idempotents as defined above implies usual orthogonality, since $(c|d) = (c^2|d) = (c \circ c|d) = (L(c)c|d) = (c|L(c)d) = (c|c \circ d) = (c|0) = 0$.

Definition 1.3.6. The set of elements $c_1, c_2, ..., c_k$ in $V$ is called a complete system of idempotents, if the following conditions are satisfied:

(i) \[ c_i^2 = c_i, \ i = 1, 2, ..., k \]
(ii) \[ c_i \circ c_j = 0, \ i \neq j \]
(iii) \[ c_1 + c_2 + ... + c_k = e. \]

Example 1.3.7. Consider $V = \text{Sym}(2, \mathbb{R})$, with the Jordan product. First, notice that squaring in $V$ is the usual squaring, since $x^2 = x \circ x = \frac{1}{2}(xx + xx) = x^2$. Take $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $V$. Clearly, $c, d$ are idempotents, since $c^2 = c$ and $d^2 = d$. They are also orthogonal, since $c \circ d = 0$. Finally, $c$ and $d$ form a complete system of idempotents, since they further satisfy that $c + d = I_2$, where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example 1.3.8. Let $V = \mathbb{R} \times \mathbb{R}^n$, with product $(t, u) \circ (s, v) = (ts + q(u, v), tv + su)$. Notice that $(t, u)^2 = (t^2 + q(u, u), 2tu)$. Take $c = \left( \frac{1}{2}, w \right)$ and $d = \left( \frac{1}{2}, -w \right)$ in $V$, such that $q(w, w) = \frac{1}{4}$ (e.g. take $q(w, w) = (w|w) = ||w||^2$, and $w = (\frac{1}{\sqrt{4n}}, \frac{1}{\sqrt{4n}}, ..., \frac{1}{\sqrt{4n}})$). Then, $c, d$ are idempotents, since $c^2 = c$ and $d^2 = d$. They are also orthogonal, since $c \circ d = 0$, and furthermore $c + d = e$, where $e = (1, 0)$.

Definition 1.3.9. Let $c$ be a non-zero idempotent in $V$. Then $c$ is called primitive if it cannot be written as a sum of two non-zero idempotents.
Example 1.3.10. Let $V = Sym(4, \mathbb{R})$. Consider the element:

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in V.$$  

Then, $c$ is an idempotent which is not primitive since:

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_{11} + E_{22}.$$  

Notice, though, that each $E_{ii}$ for $i = 1, 2, 3, 4$ is primitive.

Example 1.3.11. Let $V = \mathbb{R} \times \mathbb{R}^n$. Then $c = (\frac{1}{2}, w)$, with $q(w, w) = \frac{1}{4}$, is a primitive idempotent. Notice that the identity $e = (1, 0)$ is an idempotent which is not primitive.

Definition 1.3.12. A complete system of idempotents, in which all idempotents are primitive, is called a Jordan frame.

Example 1.3.13. Let $V = Sym(4, \mathbb{R})$. Then, from example 1.3.10, $\{E_{ii}\}_{i=1,2,3,4}$ is a Jordan frame, but the following set:

$$\left\{ c, d \in V \mid c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

is not.
Example 1.3.14. Let $V = \mathbb{R} \times \mathbb{R}^n$. Then $\{c, d \in V \mid c = (\frac{1}{2}, w), d = (\frac{1}{2}, -w)\}$, with $q(w, w) = \frac{1}{4}$, is a Jordan frame.

Theorem 1.3.15. (Spectral Theorem) Let $V$ be a Euclidean Jordan algebra with rank $r$. Then, for every $x \in V$ there exists a Jordan frame $c_1, c_2, ..., c_r$ and $\lambda_1, \lambda_2, ..., \lambda_r \in \mathbb{R}$, such that $x = \sum_{i=1}^r \lambda_i c_i$. Furthermore, $\det(x) = \prod_{i=1}^r \lambda_i$ and $\text{tr}(x) = \sum_{i=1}^r \lambda_i$.

Proof. See [8], p. 44. □

Let $V$ be a Euclidean Jordan algebra, and let $P$ denote the set of all squares of elements in $V$, i.e. $P = \{x^2 \mid x \in V\}$. Let $\Omega_V$ be the interior of of $P$, i.e. $\Omega_V := \text{Int}P$.

Theorem 1.3.16. $\Omega_V$ is a symmetric cone. Furthermore, the following are equivalent descriptions of $\Omega_V$:

(a) $\Omega_V = \{x \in V \mid L(x) > 0\}$
(b) $\Omega_V = \exp V$
(c) $\Omega_V = \mathcal{J}_o$,

where $\mathcal{J}_o$ denotes the connected component of the identity of the set of invertible elements $\mathcal{J}$ in $V$.

Proof. See [8], p. 46. □

The cone $\Omega_V$, as defined above, is called the **symmetric cone associated** to the Jordan algebra $V$.

Conversely, we can define the Jordan algebra associated with a symmetric cone $\Omega$. This can be done as follows:

Let $\Omega \subset V$ be a symmetric cone in $V$, where $V$ is a Euclidean vector space. Recall, from Section 1.1, the groups $H = G(\Omega)^o$ and $L = H \cap O(V)$. Denote by $\mathfrak{h}$ and $\mathfrak{l}$ the Lie algebras of $H$ and $L$ respectively. Consider the Cartan involution $\theta$ on $\mathfrak{h}$ given by
\( \theta(X) = -X^t. \) Let \( \mathfrak{l} = \{X \in \mathfrak{h} \mid \theta(X) = X\} \) and \( \mathfrak{s} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}. \) Then, \( \mathfrak{h} = \mathfrak{l} \oplus \mathfrak{s}. \)

**Lemma 1.3.17.** The spaces \( V \) and \( \mathfrak{s} \) are isomorphic.

**Proof.** Consider the map:

\[
\phi : \mathfrak{s} \longrightarrow V \\
X \mapsto X \cdot a,
\]

where \( X \cdot a \) indicates the action of the Lie algebra \( \mathfrak{s} \) on \( V \), given by \( X \cdot a = \frac{d}{dt} \exp(tX) \cdot a|_{t=0}. \) First, notice that if \( X \in \mathfrak{l} \), then \( X \cdot a = 0. \) Indeed, if

\[
X \in \mathfrak{l} \Rightarrow \exp(tX) \in L \Rightarrow \exp(tX) \cdot a = a.
\]

Hence,

\[
X \cdot a = \frac{d}{dt} \exp(tX) \cdot a|_{t=0} = \frac{d}{dt} a|_{t=0} = 0.
\]

Now, \( \phi \) is a homomorphism since:

\[
\phi(X_1 + X_2) = (X_1 + X_2) \cdot a = X_1 \cdot a + X_2 \cdot a = \phi(X_1) + \phi(X_2).
\]

Also, \( \phi \) is a one-to-one map because \( \forall X_1, X_2 \in \mathfrak{s}, \) we have:

\[
\phi(X_1) = \phi(X_2) \Rightarrow X_1 \cdot a = X_2 \cdot a \Rightarrow (X_1 - X_2) \cdot a = 0.
\]

The last implication says that \( (X_1 - X_2) \in \mathfrak{l} \cap \mathfrak{s}. \) But \( \mathfrak{l} \cap \mathfrak{s} = \{0\}, \) hence \( X_1 = X_2. \)

Finally, \( \phi \) is clearly onto. Therefore, \( \phi \) is an isomorphism. \( \square \)

**Remark 1.3.18.** Let \( L = \phi^{-1} \) and \( x \in V. \) Then, \( L(x) \) is the unique element in \( \mathfrak{s} \) such that \( L(x)a = x. \)
Theorem 1.3.19. Let $\Omega \subset V$ be a symmetric cone in $V$, where $V$ is a Euclidean vector space. Then $V$, equipped with the product $x \circ y = L(x)y$, is a Euclidean Jordan algebra with identity element $e = a$. Furthermore, $\Omega_V = \Omega$.

Proof. See [8], p. 49.

The Jordan algebra defined in the Theorem above is called the Jordan algebra associated to the symmetric cone $\Omega$.

1.4 Symmetric Tube Domains

Let $V$ be a simple Euclidean Jordan algebra and $\Omega$ its associated symmetric cone. By $V_C = V + iV$ we denote the complexification of $V$, which is a complex Jordan algebra, and equipped with the inner product $(x | y) = \text{tr}(xy)$ is a Hermitian vector space.

Definition 1.4.1. The domain $T(\Omega) = \Omega + iV \subset V_C$ is called a symmetric tube domain.

Notice that $\Omega + iV \cong V + i\Omega$. In particularly, $T(\Omega) = V + i\Omega$ is called the Siegel upper-half plane. We will also refer to a symmetric domain of the form $T(\Omega) = \Omega + iV$ as Siegel right-half plane.

Example 1.4.2. The following are all examples of symmetric tube domains:

(a) $\mathbb{H} = \mathbb{R} + i\mathbb{R}^+$ (Classical upper-half plane)
(b) $\mathbb{P} = \text{Sym}(n, \mathbb{R}) + i\text{Sym}^+(n, \mathbb{R})$ (Siegel upper-half plane)
(c) $\mathbb{S} = \text{Sym}^+(n, \mathbb{R}) + i\text{Sym}(n, \mathbb{R})$ (Siegel right-half plane)
(d) $\mathbb{L} = Q + iV$ (Lorentz right-half plane), where $Q$ denotes the light cone and $V$ the Jordan algebra $V = \mathbb{R} \times \mathbb{R}^n$.

Let $\text{Aut}(T(\Omega))$ be the group of biholomorphic automorphisms of $T(\Omega)$. We define

$$G(T(\Omega)) := \text{Aut}(T(\Omega))^o$$
to be the connected component of the identity in $Aut(T(\Omega))$. It is a well known fact, that the group $G(T(\Omega))$ is a finite dimensional Lie group that acts transitively on $T(\Omega)$. The group $K = \text{Stab}(a)$, for some $a \in T(\Omega)$, is a maximal compact subgroup in $G(T(\Omega))$. As the action is transitive, we can always take $a = e$, where $e$ is the identity in $V$. Hence, $K = \{k \in G(T(\Omega)) | k \cdot e = e\}$. Furthermore, $T(\Omega) \cong G(T(\Omega))/K$ via the mapping $z = g \cdot e \mapsto gK$, which says that $T(\Omega)$ is a Hermitian symmetric space.

**Example 1.4.3.** Consider $S = \text{Sym}^+(n, \mathbb{R}) + i\text{Sym}(n, \mathbb{R})$. Then, $G(S) \cong \text{Sp}(2n, \mathbb{R})$. Recall that $\text{Sp}(2n, \mathbb{R})$ is usually defined as:

$$\text{Sp}(2n, \mathbb{R}) = \{g \in \text{SL}(2n, \mathbb{R}) | g^t J g = J\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The actual $\text{Sp}(2n, \mathbb{R})$ acts on the Siegel upper-half plane $\mathbb{P}$, but not on the Siegel right half-plane $S$. For $S$ we use the group $G(S)$, an isomorphic copy of $\text{Sp}(2n, \mathbb{R})$, which is defined as follows:

$$G(S) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}(2n, \mathbb{C}) | \begin{pmatrix} A & -iB \\ iC & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}) \right\}.$$  

The action of $G(S)$ on $S$ is given by linear transformations as follows:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z = (Az + B)(Cz + D)^{-1}.$$  

Note that this action gives a surjective homomorphism onto $G(T(\Omega))$ with kernel equal to $Z$, the center of $G(S)$. Hence, $G(T(\Omega)) \cong G(S)/Z$, but we actually use $G(S)$ instead of $G(T(\Omega))$.

We will show now that the action is well defined, but first notice that from the definition
of $G(\mathbb{S})$ one has the following relations among $A, B, C,$ and $D$:

\[
A'C - C'A = 0 \quad ABt - BA' = 0 \\
A'D - C'B = I \quad ADt - BCt = I \\
B'D - D'B = 0 \quad CDt - DCt = 0 \\
B'C - D'A = -I \quad ADt - BCt = -I.
\]

Notice also that $z \in \mathbb{S}$ if and only if $\frac{1}{2}(z + z^*) > 0$. Hence, $(Az + B)(Cz + D)^{-1} \in \mathbb{S}$ if and only if $\frac{1}{2}[(Az + B)(Cz + D)^{-1} + ((Az + B)(Cz + D)^{-1})^*] > 0$. Denote the left-hand side of the last inequality by $M$. We prove that $M$ is positive definite:

\[
M = \frac{1}{2}[(Az + B)(Cz + D)^{-1} + ((Az + B)(Cz + D)^{-1})^*] \\
= \frac{1}{2}[(Az + B)(Cz + D)^{-1} + ((Cz + D)^{-1})^*(Az + B)^*] \\
= \frac{1}{2}((Cz + D)^{-1})^*[ (Cz + D)^*(Az + B) + (Az + B)^*(Cz + D) ](Cz + D)^{-1} \\
= \frac{1}{2}((Cz + D)^{-1})^*[ (z^*C^* + D^*)(Az + B) + (z^*A^* + B^*)(Cz + D) ](Cz + D)^{-1} \\
= \frac{1}{2}((Cz + D)^{-1})^*[ (z^*(-C^t) + D^t)(Az + B) + (z^*A^t - B^t)(Cz + D) ](Cz + D)^{-1} \\
= \frac{1}{2}((Cz + D)^{-1})^*[ (-z^*C^tAz - z^*C^tB + D^tAz + D^tB + z^*A^tCz + z^*A^tD - B^tCz \\
- B^tD)](Cz + D)^{-1} \\
= \frac{1}{2}((Cz + D)^{-1})^*[ z^*(-C^tA + A^tC)z + z^*(-C^tB + A^tD) + (D^tA - B^tC)z + D^tB \\
- B^tD)](Cz + D)^{-1} \\
= \frac{1}{2}((Cz + D)^{-1})^*[ z^*(0)z + z^*(I)z + 0 ](Cz + D)^{-1}, \text{ by the relations from } G(\mathbb{S}) \\
= \frac{1}{2}((Cz + D)^{-1})^*[ z^* + z ](Cz + D)^{-1} > 0, \text{ since it is congruent to the matrix } \frac{1}{2}(z^* + z) \\
\]

which is positive definite.

Hence, $M > 0$. Therefore, the action of the group $G(\mathbb{S})$ on $\mathbb{S}$ is well defined.

**Proposition 1.4.4.** Let $p = \frac{2d}{r}$. The measure $d_z = \Delta(x)^{-p}dz$, where $z = x + iy \in T(\Omega) = \Omega + iV$, is a $G(T(\Omega))$-invariant measure on $T(\Omega)$.
Proof. We have to show that \( d_p(g \cdot z) = d_p z, \forall g \in G(T(\Omega)) \). The group \( G(T(\Omega)) \) is generated (according to [8], p.207) by the elements \( \tau_u, h \) and \( j \), with \( u \in V \) and \( h \in H \), which act on \( T(\Omega) \) as follows:

\[
\begin{align*}
\tau_u \cdot z &= \tau_u(z) = z + iu \\
h \cdot z &= h(z) = hz \\
j \cdot z &= j(z) = z^{-1},
\end{align*}
\]

Hence, it is easier to prove the invariance of the measure case by case. First, notice though that any \( x \) in \( \Omega \) can be written as:

\[
x = \frac{z + \bar{z}}{2}.
\]

Now consider the following cases for each generator:

Case (1). Let \( g = \tau_u \). Then, we have:

\[
d_p(g \cdot z) = d_p(\tau_u \cdot z) = \Delta \left( \frac{\tau_u \cdot z + \bar{\tau_u \cdot z}}{2} \right)^{-p} \ d(\tau_u \cdot z) = \Delta \left( \frac{z + iu + \bar{z} - iu}{2} \right)^{-p} \ d(z + iu) = \Delta \left( \frac{z + \bar{z}}{2} \right)^{-p} \ dz = \Delta(x)^{-p} \ dz = d_p z
\]

Case (2). Let \( g = h \). Similarly, we have:

\[
d_p(g \cdot z) = d_p(h \cdot z) = \Delta \left( \frac{h \cdot z + \bar{h \cdot z}}{2} \right)^{-p} \ d(h \cdot z)
\]
\[= \Delta \left( \frac{hz + h\bar{z}}{2} \right)^{-p} \, d(hz)\]
\[= \Delta \left( \frac{h(z + \bar{z})}{2} \right)^{-p} \, d(hz)\]
\[= \Delta \left( \frac{h(z + \bar{z})}{2} \right)^{-p} \, \det(h)^2 \, dz\]
\[= \Delta (hx)^{-p} \det(h)^2 \, dz\]
\[= \Delta (h \cdot x)^{-p} \det(h)^2 \, dz\]
\[= \left( (\det h)^{\frac{p}{2}} \Delta(x) \right)^{-p} \det(h)^2 \, dz, \quad \text{by Prop.1.1.11}\]
\[= \Delta(x)^{-p} \, dz\]
\[= d_p z\]

Case (3). Let \( g = j \). Similar calculations give:

\[d_p (g \cdot z) = d_p (j \cdot z) = \Delta \left( \frac{j \cdot z + j \cdot \bar{z}}{2} \right)^{-p} \, d(j \cdot z)\]
\[= \Delta \left( \frac{z^{-1} + z^{-1}}{2} \right)^{-p} \, d(z^{-1})\]
\[= \Delta \left( \frac{z^{-1} + \bar{z}^{-1}}{2} \right)^{-p} \, d(z^{-1})\]
\[= \Delta \left( \frac{z + \bar{z}}{2} \right)^{-p} \det(P(z)^{-1})^2 \, dz, \quad \text{by Prop.1.2.20(a)}\]
\[= \Delta \left( \frac{1}{z\bar{z}} \right)^{-p} \Delta \left( \frac{z + \bar{z}}{2} \right)^{-p} \det(P(z))^{-2} \, dz\]
\[= \Delta(z\bar{z})^p \Delta(x)^{-p} \det(P(z))^{-2} \, dz\]
\[= \det(z)^{2p} \Delta(x)^{-p} \det(z)^{-2p} \, dz, \quad \text{by Prop.III.4.2, p.52 in [8]}\]
\[= \Delta(x)^{-p} \, dz\]
\[= d_p z\]

Remark 1.4.5. Notice that through the Cayley transform \( c(z) = (e + z)(e - z)^{-1} \),
one can identify the symmetric domain $T(\Omega)$ with the **bounded symmetric domain** $D = \{ z \in V_C \mid e - z^*z > 0 \}$, i.e. $c(D) = T(\Omega)$. In that case, we have that $G(D) = e^{-1}(T(\Omega)c$ and that $K_D = \{ k \in G(D) \mid k \cdot 0 = 0 \}$ is a maximal compact subgroup of $G(D)$. Furthermore, we can identify $D$ with the Hermitian symmetric space $G(D)/K_D$ via the the mapping $z = g \cdot e \mapsto gK_D$. 


Chapter 2

Laguerre Functions and Representations

In this chapter, we present some basic definitions and facts regarding Laguerre functions over symmetric cones, and their relation to representations of Lie groups and Lie algebras. We are particularly interested in Laguerre functions defined over $\Omega = \text{Sym}^+(n, \mathbb{R})$ for which we derive their recursion relations through the action of the Lie algebra of the group $\text{Sp}(2n, \mathbb{R})$ on certain Hilbert spaces of holomorphic functions over $T(\Omega)$.

2.1 Laguerre Functions

The 'classical' Laguerre functions are defined through the Laguerre polynomials that can be defined in many ways. One way is by the Rondriquez formula:

\[
L^\nu_m(x) = \frac{e^{x} e^{-\nu}}{m!} \frac{d^m}{dx^m} e^{-x} x^{\nu+m}, \quad x \in \mathbb{R}^+, \quad m, \nu \in \mathbb{N},
\]

Definition 2.1.1. The polynomials defined by $L^\nu_m(x) = \frac{e^{x} e^{-\nu}}{m!} \frac{d^m}{dx^m} e^{-x} x^{\nu+m}, \quad x \in \mathbb{R}^+, \quad m, \nu \in \mathbb{N},$ are called Laguerre polynomials.

In terms of the hypergeometric function $_1F_1$, the Laguerre polynomials $L^\nu_m(x)$ can also be defined as follows:

\[
L^\nu_m(x) = \frac{\Gamma(\nu + m + 1)}{\Gamma(m + 1)} \, _1F_1(-m; \nu + 1; x),
\]
\[ \Gamma(z) = \int_{\mathbb{R}^+} e^{-x} x^{z-1} \, dx \text{ with } \Re(z) > 0, \quad \text{and} \quad \pFq{p}{q}{a; \gamma_s; z} = \sum_{k=0}^{\infty} \frac{\prod_{r=1}^{p}(a_{\nu})_k}{\prod_{s=1}^{q} \Gamma(\gamma_s)_k} \frac{z^k}{k!}. \]

Recall that \((a_{\nu})_k = \frac{\Gamma(a_{\nu}+k)}{\Gamma(a_{\nu})}\) and \(\Gamma(n+1) = n!\). It is clear to see that the set \(\{\sqrt{\frac{\Gamma(m+1)}{\Gamma(\nu+m+1)}} L^\nu_m(x)\}\) forms an orthonormal basis for \(L^2(\mathbb{R}^+, e^{-x} x^\nu \, dx)\).

One can define now the Laguerre functions as follows:

**Definition 2.1.2.** The functions defined by \(\ell^\nu_m(x) = e^{-x} L^\nu_m(2x), \ x \in \mathbb{R}^+, \ m, \nu \in \mathbb{N}\), are called Laguerre functions.

It is not hard to show now that \(\{\ell^\nu_m(x)\}\) forms an orthonormal basis for \(L^2(\mathbb{R}^+, x^\nu \, dx)\).

It is also known that \(\ell^\nu_m\) satisfy certain recursion relations (see [3] p.273), like the following:

\[ x \frac{d^2}{dx^2} \ell^\nu_m(x) + (\nu + 1) \frac{d}{dx} \ell^\nu_m(x) + (2m + \nu + 1 - x) \ell^\nu_m(x) = 0 \quad (2.1) \]

\[ x \frac{d^2}{dx^2} \ell^\nu_m(x) + (2x + \nu + 1) \frac{d}{dx} \ell^\nu_m(x) + (x + \nu + 1) \ell^\nu_m(x) = -2(m + \nu) \ell^\nu_{m-1}(x) \quad (2.2) \]

\[ x \frac{d^2}{dx^2} \ell^\nu_m(x) - (2x - \nu - 1) \frac{d}{dx} \ell^\nu_m(x) + (x - \nu - 1) \ell^\nu_m(x) = -2(m + 1) \ell^\nu_{m+1}(x) \quad (2.3) \]

The Laplace Transform of \(\ell^\nu_m(x)\) is:

\[ \mathcal{L}_\nu(\ell^\nu_m)(z) = \int_{\mathbb{R}^+} e^{-z|x|} \ell^\nu_m(x) \, d\mu_\nu(x) \]

\[ = \frac{\Gamma(\nu + m + 1)}{\Gamma(m+1)} (z - 1)^m (z + 1)^{-(m+\nu+1)} \]

Denote the polynomials on the right-hand side of the equation above by \(q^\nu_m(z)\). Then, \(\{q^\nu_m(z)\}\) is an orthogonal basis of the space of \(\mathcal{H}_\nu(\mathbb{H}, x^{-1} \, dz)\), where \(\mathbb{H} = \mathbb{R} + i \mathbb{R}^+\).

Observe that \(\mathbb{R}^+\) is a symmetric cone, \(\mathbb{R}\) is a Euclidean Jordan algebra, and highest weight representations of \(\text{SL}_2(\mathbb{R})\) on \(\mathcal{H}_\nu(\mathbb{H})\) are derived through the action of \(\text{SL}_2(\mathbb{R})\) on the tube domain \(\mathbb{H}\) (the classical upper half-plane). In their paper, Davidson, Ólafsson...
and Zhang (see [3]), also show that one can generate the classical recursion relations (2.1),(2.2) and (2.3), by transferring the representations mentioned above on the space $L^2(\mathbb{R}^+, x^\nu dx)$.

One wants to check now whether this can be done for other cones and Euclidean Jordan algebras. That is, given a symmetric cone $\Omega \subset V$, where $V$ is a Jordan algebra, and a connected semisimple Lie group $G$, we want to build highest weight representations of $G$ on $\mathcal{H}_\nu(T(\Omega))$. Then, we want to transfer the representations on $L^2(\Omega, d\mu_\nu)$ to establish recursion relations for the generalized Laguerre functions, for which we give the basic concepts below. The following cases have been settled:

1. $\Omega = \mathbb{R}^+, V = \mathbb{R}, G = SL_2(\mathbb{R})$ (see [3])
2. $\Omega = Herm^+(n,n), V = Herm(n,n), G = SU(n,n)$ (see [4]).

In this chapter, we will present the results for the following case:

3. $\Omega = Sym^+(n,\mathbb{R}), V = Sym(n,\mathbb{R}), G = Sp(2n,\mathbb{R})$ (see also [1]).

Note that case (1) was done for $T(\Omega) = V + i\Omega$ (the classical upper half-plane $\mathbb{H}$), and gives the classical recursion relations including (2.1),(2.2) and (2.3) (see [3]). Case (2) was done for $T(\Omega) = \Omega + iV$ (right half-plane) and gives a generalization of the classical relations (see [4]). Case (3) is treated here also for the right half-plane, and gives a generalization of the classical relations as well.

### 2.1.1 $L$-invariant Polynomials

Let $E_{ii}$ be the diagonal $n \times n$ matrix with 1 in the $ii$-position and zeros elsewhere. Then \{$E_{11}, \ldots, E_{nn}$\} is a Jordan frame for $V = Sym(n,\mathbb{R})$. Let $V^{(k)}$ be the $+1$-eigenspace of the idempotent $E_{11} + \cdots + E_{kk}$ acting on $V$ by multiplication. Each $V^{(k)}$ is a Jordan subalgebra and we have:

$$V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(n)} = V.$$
If $\det_k$ is the determinant function for $V(k)$ and $P_k$ is orthogonal projection of $V$ onto $V(k)$ then the function $\Delta_k(x) = \det_k P_k(x)$ is the usual $k$th principal minor for an $n \times n$ symmetric matrix; it is homogeneous of degree $k$. In particular $\Delta(x) := \Delta_n(x) = \det(x)$.

Let $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$. We say that $m \geq 0$, if each $m_i$ is a nonnegative integer and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Let $\Lambda = \{m \mid m \geq 0\}$.

For each $m \in \mathbb{C}^n$, we define the **generalized power functions** as follows:

$$
\Delta_m(x) = \Delta_1^{m_1-m_2}(x) \Delta_2^{m_2-m_3}(x) \cdots \Delta_n^{m_n}(x).
$$

The degree of $\Delta_m$ is $|m| := m_1 + \cdots + m_n$. Observe that each generalized power function extends to a holomorphic polynomial on $V_C = \text{Sym}(n, \mathbb{C})$ in a unique way.

For each $m \in \Lambda$, we define an $L$-invariant polynomial $\psi_m$ on $J_C$ by:

$$
\psi_m(z) = \int_L \Delta_m(lz) \, dl, \quad z \in V_C
$$

where $L$ is the group that fixes $e$ in $\Omega$ and $dl$ is the normalized Haar measure on $L$. Notice that for the case of $\mathbb{H}$, i.e. $n = 1$, we have $\psi_m(z) = \psi_m(z) = z^m$, as $L = \{1\}$.

**Lemma 2.1.3.** If $\mathcal{P}(V_C)$ is the space of all polynomial functions on $V_C$ and $\mathcal{P}(V_C)^L$ denotes the space of $L$-invariant polynomials, then $\{\psi_m\}_{m \geq 0}$ is a basis of $\mathcal{P}(V_C)^L$. Furthermore, if $\mathcal{P}_k(V_C)^L$ denotes the space of $L$-invariant polynomials of degree less than or equal $k$, then $\{\psi_m\}_{|m| \leq k}$ is a basis of $\mathcal{P}_k(J_C)^L$.

**Proof.** See [22], p.61-90. □

The lemma above implies that $\psi_m(e + x)$ is a linear combination of $\psi_n$, $|n| \leq |m|$. This allows us to define the **generalized binomial coefficients** $\binom{m}{n}$ from the equation:

$$
\psi_m(e + x) = \sum_{|n| \leq |m|} \binom{m}{n} \psi_n(x).
$$

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2.1.2 The Generalized Gamma Function

The **generalized Gamma function** is defined as follows:

\[
\Gamma_\Omega(m) = \int_\Omega e^{-tr(x)} \Delta_m(x) \Delta(x)^{-\frac{d}{2}} \, dx,
\]

where \( x \in \Omega \) and \( m \in \Lambda \). The numbers \( d \) and \( r \) are, respectively, the dimension and the rank of the Jordan algebra \( V = \text{Sym}(n, \mathbb{R}) \). Convergence conditions for the integral above, and other properties of \( \Gamma_\Omega(m) \), are given in the following proposition.

**Proposition 2.1.4.** Let \( m = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n \). Then the following hold:

1. The integral defining \( \Gamma_\Omega(m) \) converges if \( \text{Re}(m_j) > \frac{1}{2}(j - 1) \), where \( j = 1, \ldots, n \).

   Furthermore,

   \[
   \Gamma_\Omega(m) = (2\pi)^{\frac{n(n-1)}{4}} \prod_{i=1}^{n} \Gamma \left( m_i - \frac{1}{2}(i - 1) \right),
   \]

   where \( \Gamma \) is the classical Gamma function.

2. Take \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^t \), with 1 in the \( j \)th position. Then, \( \forall m \in \mathbb{C}^n \), we have:

   \[
   \begin{align*}
   (a) & \quad \frac{\Gamma_\Omega(m)}{\Gamma_\Omega(m - e_j)} = m_j - 1 - \frac{1}{2}(j - 1) \\
   (b) & \quad \frac{\Gamma_\Omega(m + e_j)}{\Gamma_\Omega(m)} = m_j - \frac{1}{2}(j - 1).
   \end{align*}
   \]

**Proof.** See Theorem VII.1.1 in [8], p.123, for (1). Part (2) follows easily from (1). \( \square \)

Let \( \lambda \in \mathbb{R} \). We correspond \( \lambda \) to the multi-index \( (\lambda, \cdots, \lambda) \) and denote the latter by \( \lambda \) as well. The context of use should distinguish the two. Then, we define:

\[
(\lambda)_m = \frac{\Gamma_\Omega(m + \lambda)}{\Gamma_\Omega(\lambda)}.
\]
2.1.3 The Generalized Laguerre Functions

Let \( \nu > 0 \) and \( m \in \Lambda \). Then, the \textit{generalized Laguerre polynomials} are defined (see [8], p. 343) by:

\[
L_\nu^m(x) = (\nu)_m \sum_{|n| \leq |m|} \binom{m}{n} \frac{1}{(\nu)_n} \psi_n(-x), \quad x \in \Omega
\]

The \textit{generalized Laguerre functions} are defined in terms of \( L_\nu^m(x) \) by:

\[
\ell_\nu^m(x) = e^{-\text{tr}(x)} L_\nu^m(2x).
\]

\textbf{Remark 2.1.5.} Notice that for \( \Omega = \mathbb{R}^+ \), i.e. \( n = 1 \), the generalized Laguerre polynomials and functions defined above are precisely the classical Laguerre polynomials and functions defined on \( \mathbb{R}^+ \) (see [3]).

Recall that from Prop.1.1.11(b) we know that the measure \( d_\mu x = \Delta(x)^{-\frac{\nu}{2}} dx \), where \( p = \frac{2d}{r} \), is an \( H \)-invariant measure on \( \Omega \). Define now the following measure:

\[
d_\mu(x) = \Delta(x)^{\nu - \frac{d}{2}} dx.
\]

\textbf{Theorem 2.1.6.} The set \( \{\ell_\nu^m\}_{m \geq 0} \) is an orthogonal basis of \( L_\nu^2(\Omega, d_\mu) \), the Hilbert space of \( L \)-invariant functions in \( L_\nu^2(\Omega, d_\mu) \).

\textit{Proof.} See Theorem 7.8 in [4], p.191.

Finally, observe that by Prop.1.1.11(b) it follows that \( H \) acts unitarily on \( L_\nu^2(\Omega, d_\mu) \) by the formula:

\[
\lambda_\nu(h) f(x) = \det(h)^\nu f(h^t \cdot x).
\]
2.2 \( \text{Sp}(2n, \mathbb{R}) \) and Its Lie Algebra \( \mathfrak{sp}(2n, \mathbb{R}) \)

In this section, we use a non-standard model of \( \text{Sp}(2n, \mathbb{R}) \) suitable for the action on the right half-plane. This non-standard model is a group \( G(S) \) isomorphic to \( \text{Sp}(2n, \mathbb{R}) \). We describe some important subgroups of \( G(S) \). Then, we introduce some subalgebras of \( g_C \), the complexification of the Lie algebra of \( G(S) \).

The group \( \text{Sp}(2n, \mathbb{R}) \) is called the \textit{symplectic group} and is usually defined as:

\[
\text{Sp}(2n, \mathbb{R}) = \{ g \in \text{SL}(2n, \mathbb{R}) \mid g^t J g = J \},
\]

where \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Defined this way, \( \text{Sp}(2n, \mathbb{R}) \) acts on the upper half-plane by linear transformations.

Consider now the map:

\[
\text{Sp}(2n, \mathbb{R}) \ni \begin{pmatrix} A & -B \\ iC & D \end{pmatrix} P^* \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(S) \subset \text{SL}(2n, \mathbb{C}),
\]

where \( P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \). This map is an isomorphism, in other words \( P^* \text{Sp}(2n, \mathbb{R}) P \cong G(S) \).

Hence, the group \( G(S) \) is an isomorphic copy of \( \text{Sp}(2n, \mathbb{R}) \) in \( \text{SL}(2n, \mathbb{C}) \), and it can be defined as follows:

\[
G(S) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SL}(2n, \mathbb{C}) \mid \begin{pmatrix} A & -B \\ iC & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}) \right\}.
\]

As in p.28, we note once again that \( G(T(\Omega)) \cong G(S)/\mathcal{Z} \), where \( \mathcal{Z} \) denotes the center of \( G(S) \), but we will actually use \( G(S) \) for the action on the right-half plane \( T(\Omega) \). Consequently, by the definition of \( G(S) \), we have the following relations among \( A, B, C \),
and $D$:

$$
A^t C - C^t A = 0 \quad AB^t - BA^t = 0 \\
A^t D - C^t B = I \quad AD^t - BC^t = I \\
B^t D - D^t B = 0 \quad CD^t - DC^t = 0 \\
B^t C - D^t A = -I \quad AD^t - BC^t = -I
$$

We will use precisely this copy of $\text{Sp}(2n, \mathbb{R})$, namely $G(\mathbb{S})$, to realize an action of $\text{Sp}(2n, \mathbb{R})$ on the right half-plane. This action of $G(\mathbb{S})$ on $T(\Omega)$, where $T(\Omega) = \mathbb{S} = \text{Sym}^+(n, \mathbb{R}) + i\text{Sym}(n, \mathbb{R})$, is given by linear transformations as follows:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \cdot z = (Az + B)(Cz + D)^{-1}.
$$

It is already shown in Example 1.4.3 that the action above is well defined, using the fact that $z \in T(\Omega)$ if and only if $\frac{z + z^*}{2} > 0$.

Some important subgroups of $G(\mathbb{S})$ are the following:

$$
K = \text{Stab}(I) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \in G(\mathbb{S}) \mid A \pm B \in U(n) \right\} \cong U(n)
$$

$$
H = G(\Omega) = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \in G(\mathbb{S}) \mid A \in \text{GL}(n, \mathbb{R}) \right\} \cong \text{GL}(n, \mathbb{R})
$$

and

$$
L = K \cap H = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in G(\mathbb{S}) \mid A \in \text{SU}(n) \right\} \cong \text{SU}(n),
$$

where $K$ is the stabilizer of the identity (it is maximal compact), $H$ is the group that fixes $\Omega$, and $L$ is the intersection of $K$ and $H$. We found $H$ by observing that $ix + (ix)^* = 0, \forall x \in \Omega$, as the following proposition suggests:
Proposition 2.2.1. The group $H$ that fixes the cone $\Omega$ is given by:

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \in G(S) \mid A \in GL(n, \mathbb{R}) \right\}.$$ 

Proof. First note that $ix + (ix)^* = 0, \forall x \in \Omega$. Indeed, $ix + (ix)^* = ix - ix^* = ix - ix^t = ix - ix = 0$. Since $H$ preserves the cone it means that $g \cdot x \in \Omega, \forall x \in \Omega$. As $g \cdot x \in \Omega$, we have that $ig \cdot x + (ig \cdot x)^* = 0$, i.e. $g \cdot x = (g \cdot x)^*$. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Hence, $g \cdot x = (Ax + B)(Cx + D)^{-1}$. Then, since $g \cdot x = (g \cdot x)^*, \forall x \in \Omega$, we have:

$$(Ax + B)(Cx + D)^{-1} = [(Ax + B)(Cx + D)^{-1}]^*$$

Applying $\ast$, we get:

$$(Ax + B)(Cx + D)^{-1} = ((Cx + D)^{-1})^*(Ax + B)^*$$

Multiplying both sides of the equation above, successively, by $(Cx + D)^*$ (on the left) and $(Cx + D)$ (on the right), we have:

$$(Cx + D)^*(Ax + B) = (Ax + B)^*(Cx + D)$$

Applying $\ast$ again, we get:

$$(-xC^t + D^t)(Ax + B) = (xA^t - B^t)(Cx + D)$$

After the multiplication, we have:

$$-xC^tAx - xC^tB + D^tAx + D^tB = xA^tCx + xA^tD - B^tCx - B^tD$$
Which finally gives:

\[ -x(C^tA + A^tC)x - x(C^tB + A^tD) + (D^tA + B^tC)x + D^tB + B^tD = 0, \forall x \in \Omega. \]

From the relations of \( G(\mathbb{S}) \) we have \( C^tA = A^tC \) and \( B^tD = D^tB \). Hence, the last equation above becomes:

\[ -x(2C^tA)x - x(C^tB + A^tD) + (D^tA + B^tC)x + 2D^tB = 0, \forall x \in \Omega \quad (1) \]

Since we can write every \( z \in V \) as \( z = x_1 - x_2, \ x_i \in \Omega \) (see [8], p.1), that means we can actually extend (1) on \( V \) by linearity. Since (1) is true \( \forall z \in V \) taking \( z = kI, \ k \in \mathbb{R} \), we get:

\[ -2k^2C^tA - k(C^tB + A^tD - D^tA - B^tC) + 2D^tB = 0, \forall k \in \mathbb{R} \quad (2) \]

Since (2) is zero for every \( k \), we have the following relations:

\[ C^tA = 0, \quad C^tB + A^tD - D^tA - B^tC = 0, \quad D^tB = 0 \quad (3) \]

Now, from the second relation in (3) and the from the relations of \( G(\mathbb{S}) \), we get that:

\[ A^tD = D^tA \quad (4) \]

Putting the first and the third relations from (3) and (4) back in (1), we get:

\[ x(C^tB + A^tD) = (C^tB + A^tD)x, \forall x \in \Omega \]

This last equation implies that \( C^tB + A^tD = mI \), by Shur’s Lemma. Using the relations of \( G(\mathbb{S}) \) once again, we get:
\[ C^tB + A^tD = mI \Rightarrow 2C^tB + I = mI \Rightarrow 2C^tB = (m - 1)I \]
\[ \Rightarrow C^tB = \frac{m - 1}{2}I. \]

Case (1): \( m = 1 \). When \( m = 1 \), we have that \( C^tB = 0 \). This implies that \( C = B = 0 \).

From \( C^tB + A^tD = mI \), we get \( A^tD = I \) (because \( C = B = 0 \) and \( m = 1 \)), which says that \( A, D \) are invertible and \( D = (A^t)^{-1} \). Therefore, \( g = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \).

Case (2): \( m \neq 1 \). Take for instance \( m = 3 \). Then \( C^tB = I \), which says that \( C, B \) invertible, and \( C = (B^t)^{-1} \). But from the previous relations \( C^tA = 0 \) and \( D^tB = 0 \), we get \( A = 0 \) and \( D = 0 \) since \( C \) and \( B \) are invertible and hence non-zero. Therefore, \( g = \begin{pmatrix} 0 & B \\ (B^t)^{-1} & 0 \end{pmatrix} \). We exclude elements of this form, since the group \( H \) was defined as the connected component of the group that fixes the cone.

Since \( G(S) \cong P^*\text{Sp}(2n, \mathbb{R})P \), then for the Lie algebra we have \( \mathfrak{g} = P^*\mathfrak{sp}(2n, \mathbb{R})P \), and hence \( \mathfrak{g} \) is given by:

\[ \mathfrak{g} = \left\{ \begin{pmatrix} a & ib \\ -ic & -a^t \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \ | \ a, b, c \text{ real, } b = b^t, c = c^t \right\}. \]

There is an involution on \( \mathfrak{g} \), call it \( \theta \), defined by:

\[ \theta: \mathfrak{g} \rightarrow \mathfrak{g} \]
\[ \theta(X) = -X^*. \]

Let \( \mathfrak{t} = \{ X \in \mathfrak{g} | \theta(X) = X \} \) and \( \mathfrak{p} = \{ X \in \mathfrak{g} | \theta(X) = -X \} \). Then, we have:
\[ \mathfrak{t} = \left\{ \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = -a^t, b = b^t, \text{tr}(a) = 0 \right\} \]

and

\[ \mathfrak{p} = \left\{ \begin{pmatrix} a & ib \\ -ib & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = a^t, b = b^t \right\}. \]

Their complexification gives:

\[ \mathfrak{t}_C = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = -a^t, b = b^t, \text{tr}(a) = 0 \right\} \]

and

\[ \mathfrak{p}_C = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = a^t, b = b^t \right\}. \]

Hence, \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) and, consequently, \( \mathfrak{g}_C = \mathfrak{t}_C \oplus \mathfrak{p}_C \). Now, \( \mathfrak{p}_C \) breaks further, if we consider

\[ X_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in Z(\mathfrak{t}_C), \]

where \( Z(\mathfrak{t}_C) \) denotes the center of \( \mathfrak{t}_C \). Then, \( \text{ad}(iX_0) : \mathfrak{g}_C \to \mathfrak{g}_C \)

has eigenvalues 0, 2, −2, where 0 corresponds to the subalgebra \( \mathfrak{t}_C \), and 2 and −2 correspond to the subalgebras \( \mathfrak{p}^+ \) and \( \mathfrak{p}^- \) that are are given by:

\[ \mathfrak{p}^+ = \left\{ \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{g}_C \mid a = a^t \right\} \text{ and } \mathfrak{p}^- = \left\{ \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{g}_C \mid a = a^t \right\}. \]

Hence, \( \mathfrak{g}_C = \mathfrak{p}^+ \oplus \mathfrak{t}_C \oplus \mathfrak{p}^- \). Observe also that \( \mathfrak{p}^+ \cong V \), by the map \( \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \mapsto a \).
2.3 Representations of $\mathfrak{sp}(2n, \mathbb{R})_\mathbb{C}$ on $\mathcal{H}_\nu(T(\Omega))$

In this section we want to build representations of $G(\mathbb{S})$ and its Lie Algebra $\mathfrak{g}$ on $\mathcal{H}_\nu(\mathbb{S})$, the Hilbert space of holomorphic functions on $\mathbb{S}$. Since here $T(\Omega) = \mathbb{S} = \text{Sym}^+(n, \mathbb{R}) + i\text{Sym}(n, \mathbb{R})$, that means $d = \frac{n(n+1)}{2}$ and $r = n$.

2.3.1 The Hilbert Space $\mathcal{H}_\nu(T(\Omega))$

Consider the following Hilbert space of holomorphic functions:

$$\mathcal{H}_\nu(T(\Omega)) = \{ F \mid F : T(\Omega) \rightarrow \mathbb{C}, \|F\|^2 < \infty \}, \nu \in \mathbb{R},$$

where

$$\|F\|^2 = \alpha_\nu \int_{T(\Omega)} |F(x + iy)|^2 \Delta(x)^{\nu-(n+1)} \, dx \, dy,$$

with $\Delta(x) = \det(x)$ and $\alpha_\nu = \frac{\pi^{n\nu}}{(4\pi)^{n/2} \Gamma(\nu - \frac{n+1}{2})}$. Recall that in general:

$$\Gamma_\Omega(m) = \int_\Omega e^{-\text{tr}(x)} \Delta_m(x) \Delta(x)^{-\frac{d}{2}} \, dx$$

and

$$\Delta_m(x) = \Delta_1^{m_1-m_2}(x) \Delta_2^{m_2-m_3}(x) \cdots \Delta_n^{m_n}(x).$$

It is clear that the norm came from the inner product on $\mathcal{H}_\nu(T(\Omega))$, which is defined by:

$$(F|G) = \alpha_\nu \int_{T(\Omega)} F(x + iy) \overline{G(x + iy)} \Delta(x)^{\nu-(n+1)} \, dx \, dy.$$

Finally, notice that $\mathcal{H}_\nu(T(\Omega)) \neq \{0\}$ if and only if $\nu \geq n + 1$. For $\nu = n + 1$, $\mathcal{H}_\nu(T(\Omega))$ is called the Bergman space.

The space $\mathcal{H}_\nu(T(\Omega))$ is a reproducing kernel Hilbert space (see [4] and [20] for more details). This means that point evaluation $E_z : \mathcal{H}_\nu((T(\Omega)) \rightarrow \mathbb{C}$ given by $E_z(F) =$
$F(z)$ is continuous, $\forall z \in T(\Omega)$. This implies the existence of a kernel function $K_z \in \mathcal{H}_\nu(T(\Omega))$, such that $F(z) = (F|K_z)$ for all $F \in \mathcal{H}_\nu(T(\Omega))$ and $z \in T(\Omega)$. Set $K(z, w) = K_w(z)$. Then $K(z, w)$ is holomorphic in the first variable and antiholomorphic in the second variable. The function $K(z, w)$ is called the reproducing kernel for $\mathcal{H}_\nu(T(\Omega))$.

We note that the Hilbert space is completely determined by the function $K(z, w)$. In particular, we have the following theorem:

**Theorem 2.3.1.** Suppose that $\nu \geq n + 1$. Then for the Hilbert space $\mathcal{H}_\nu(T(\Omega))$ we have:

1. The reproducing kernel of $\mathcal{H}_\nu(T(\Omega))$ is given by $K\nu(z, w) = \Gamma(\nu)\Delta(z + \bar{w})^{-\nu}$

2. The functions $q\nu_m(z) := \Delta(z + e)^{-\nu}\psi_m(z) - \nu \psi_m(z + e)$, $m \in \Lambda$, form an orthogonal basis of $\mathcal{H}_\nu(T(\Omega))^L$, the space of $L$-invariant functions in $\mathcal{H}_\nu(T(\Omega))$.

**Proof.** See Theorem 2.9 in [4], and Prop.'s XIII.1.2 and XIII.1.3 in [8], p.261, for the proofs. \qed

**Remark 2.3.2.** We close this discussion with the following remarks:

1. $F \in \mathcal{H}_\nu(T(\Omega))^L$ if $\pi_\nu(l)F(z) = F(z)$, $\forall l \in L$, where $\pi_\nu$ is a representation of $L$.

2. $\mathcal{H}_\nu(T(\Omega))^o := \{ \sum c_j K_{w_j} | c_j \in \mathbb{C}, w_j \in T(\Omega) \}$, the space of finite linear combinations, is dense in $\mathcal{H}_\nu(T(\Omega))$.

3. The inner product in $\mathcal{H}_\nu(T(\Omega))^o$ is given by:

$$\left( \sum_j c_j K_{w_j} | \sum_k d_k K_{z_k} \right) = \sum_{j,k} c_j \bar{d}_k K(z_k, w_j).$$

**2.3.2 The Action of $\mathfrak{sp}(2n, \mathbb{R})_\mathbb{C}$ on $\mathcal{H}_\nu(T(\Omega))$**

The representation of $G(\mathbb{S})$ on $\mathcal{H}_\nu(\mathbb{S})$ is given by a multiplier representation as follows:

$$\pi_\nu(g)F(z) = J(g^{-1}, z)^{\tilde{x}} F(g^{-1} \cdot z).$$
where $J(g, z)$ is complex Jacobian of the action of $G(S)$ on $S$, i.e. $J(g, z) = \det D(g \cdot z)$, and $p = \frac{2d}{r}$. In our case, $J(g, z) = \det(Cz + D)^{-(n+1)}$, whenever $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The Lie algebra representation is given, by differentiation, as follows:

$$\pi_\nu(X) F(z) = \frac{d}{dt} \pi_\nu(\exp(tX)) F(z)|_{t=0} = \frac{d}{dt} J(\exp(-tX), z) \bar{\nu} F(\exp(-tX) \cdot z)|_{t=0}$$

**Proposition 2.3.3.** For each piece of the Lie algebra $g_C$, we have:

1. $\pi_\nu(X) F(z) = -\nu \text{tr}(az + a) F(z) + D_{v(a,z)} F(z), \quad X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in p^+$

2. $\pi_\nu(X) F(z) = -\nu \text{tr}(-az + a) F(z) + D_{v(-a,z)} F(z), \quad X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in p^-$

3. $\pi_\nu(X) F(z) = \nu \text{tr}(bz) F(z) + D_{w(a,b,z)} F(z), \quad X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in k_C$

where $v(a, z) = -az - a - zaz - za$ and $w(a, b, z) = -az - b + zbz + za$.

**Proof.** We prove the Proposition case by case:

Case (1): Let $X \in p^+$. Then, $X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$. Now, as $X^n = 0$ for $n \geq 2$, we have:

$$\exp(-tX) = \begin{pmatrix} 1 - ta & -ta \\ ta & 1 + ta \end{pmatrix}.$$
As $J(g, z) = \det(Cz + D)^{-(n+1)}$, whenever $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we also have:

$$J(\exp(-tX), z) = \det(taz + 1 + ta)^{-(n+1)}.$$ 

Hence, using also the fact that:

$$\det'(1 + tu) = \text{tr}(u)$$

we have:

$$\pi_\nu(X)F(z) = \frac{d}{dt}J(\exp(-tX), z)^{\nu} F(\exp(-tX) \cdot z)|_{t=0}$$

$$= \frac{d}{dt} \left[ \det(taz + 1 + ta)^{-\nu} F \left( \frac{(1 - ta)z + (-ta)}{taz + 1 + ta} \right) \right]_{t=0}$$

$$= -\nu \det(taz + 1 + ta)^{-\nu - 1} \det'(1 + t(az + a))F \left( \frac{(1 - ta)z - ta}{taz + 1 + ta} \right) |_{t=0}$$

$$+ \det(taz + 1 + ta)^{-\nu} F' \left( \frac{(1 - ta)z - ta}{taz + 1 + ta} \right) |_{t=0}$$

$$= -\nu \det(1)^{-\nu - 1} \text{tr}(az + a)F(z) + \det(1)^{-\nu} F'(z) \left( \frac{(1 - ta)z - ta}{taz + 1 + ta} \right)' |_{t=0}$$

$$= -\nu \text{tr}(az + a)F(z) + F'(z)(-az - a - zaz - za)$$

$$= -\nu \text{tr}(az + a)F(z) + D_{az - a - zaz - za}F(z)$$

$$= -\nu \text{tr}(az + a)F(z) + D_{e(a,z)}F(z).$$

Case (2): Let $X \in \mathfrak{p}^-$. Then, $X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}$. Again, since $X^n = 0$ for $n \geq 2$, we have:

$$\exp(-tX) = \begin{pmatrix} 1 - ta & ta \\ -ta & 1 + ta \end{pmatrix}.$$
Since \( J(g, z) = \det(Cz + D)^{-(n+1)} \), whenever \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), we have:

\[
J(\exp(-tX), z) = \det(-taz + 1 + ta)^{-(n+1)}.
\]

Therefore,

\[
\pi_\nu(X)F(z) = \frac{d}{dt} J(\exp(-tX), z) \left. \right|_{t=0}^{\nu} F(\exp(-tX) \cdot z) |_{t=0}^{\nu}.
\]

\[
\pi_\nu(X)F(z) = \frac{d}{dt} \left[ \det(-taz + 1 + ta)^{-\nu} \left( \frac{(1 - ta)z + ta}{-taz + 1 + ta} \right) \right] |_{t=0}^{\nu}.
\]

\[
\pi_\nu(X)F(z) = -\nu \det(-taz + 1 + ta)^{-\nu-1} \det'(1 + t(-az + a)) F\left( \frac{(1 - ta)z + ta}{-taz + 1 + ta} \right) |_{t=0}^{\nu}.
\]

\[
\pi_\nu(X)F(z) = -\nu \det(1)^{-\nu-1} \text{tr}(-az + a) F(z) + \det(1)^{-\nu} F'(z) \left( \frac{(1 - ta)z - ta}{taz + 1 + ta} \right) |_{t=0}^{\nu}.
\]

\[
\pi_\nu(X)F(z) = -\nu \text{tr}(-az + a) F(z) + F'(z)(-az + a + za - za).
\]

\[
\pi_\nu(X)F(z) = -\nu \text{tr}(-az + a) F(z) + D_{-az+a+za-za} F(z)
\]

\[
\pi_\nu(X)F(z) = -\nu \text{tr}(-az + a) F(z) + D_{v(-a,-z)} F(z).
\]

Case (3): Let \( X \in k_C \). Then, \( X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), and for \( \exp(-tX) \) we have:

\[
\exp(-tX) = \begin{pmatrix} 1 - ta + t^2(a^2 + b^2) - ... & -tb + \frac{t^2}{2!}(ab + ba) - ... \\ -tb + \frac{t^2}{2!}(ab + ba) - ... & 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - ... \end{pmatrix}.
\]

Similarly, \( J(g, z) = \det(Cz + D)^{-(n+1)} \), whenever \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). Therefore,
\[ J(\exp(-tX), z) = \det \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \]^{- (n+1)}.

Hence,

\[
\pi_{\nu}(X)F(z) = \left. \frac{d}{dt} J(\exp(-tX), z) \right|_{t=0} \bigg[ \pi_{\nu} F(\exp(-tX) \cdot z) \bigg] = \left. \frac{d}{dt} \det \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \right]^{- \nu} \bigg|_{t=0} \det' \left( 1 + t \left( -b + t(ab + ba) - \ldots \right) z - a + t(a^2 + b^2) - \ldots \right) \bigg|_{t=0} \frac{d}{dt} \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \bigg|_{t=0}^{- \nu} + \det \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \bigg|_{t=0}^{- \nu} F' \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \bigg|_{t=0}^{- \nu} = -\nu \det(1)^{-\nu-1} \operatorname{tr}(-bz - a) F(z) + \det(1)^{-\nu} F'(z) \bigg|_{t=0} \frac{d}{dt} \left( -tb + \frac{t^2}{2!}(ab + ba) - \ldots \right) z + 1 - ta + \frac{t^2}{2!}(a^2 + b^2) - \ldots \bigg|_{t=0}^{- \nu} \operatorname{tr}(-bz - a) F(z) + F'(z)(-az - b + zb + za) = \nu \operatorname{tr}(bz) F(z) + D_{-az+a+az-za} F(z), \quad \text{since } \operatorname{tr}(a) = 0
\]

\[ = \nu \operatorname{tr}(bz) F(z) + D_{w(a,b,z)} F(z). \]
2.3.3 Highest Weight Representations

In this subsection, we introduce the basic concepts on highest weight representations. The representation \( \pi_\nu \) we constructed in the previous subsection is a highest weight representation, and that is important for the recursion relations for Laguerre functions.

Suppose that \( G \) is a Hermitian group. That means \( G \) is simple and the maximal compact subgroup \( K \) has \( \dim \mathcal{Z}(K) = 1 \). The Hermitian groups have been classified in terms of their Lie algebras, where the latter are \( \mathfrak{su}(p,q) \), \( \mathfrak{sp}(n,\mathbb{R}) \), \( \mathfrak{so}^*(2n) \), \( \mathfrak{so}(2,n) \) and two exceptional Lie algebras. The fact that \( \dim \mathcal{Z}(K) = 1 \) implies that \( G/K \) is isomorphic to a bounded symmetric complex domain. It also implies that \( \mathfrak{g}_C \) has a decomposition of the form \( \mathfrak{g}_C = \mathfrak{p}^+ \oplus \mathfrak{k}_C \oplus \mathfrak{p}^- \), where \( \mathfrak{k}_C \) is the Lie algebra of \( K_C \). The subspaces \( \mathfrak{p}^+ \), \( \mathfrak{k}_C \) and \( \mathfrak{p}^- \) are, respectively, the \(-2,0,2\)-eigenspaces of \( \text{ad}(z) \), for some \( z \in \mathcal{Z}(\mathfrak{k}_C) \).

**Lemma 2.3.4.** We have the following inclusions for the spaces \( \mathfrak{p}^+, \mathfrak{k}_C \) and \( \mathfrak{p}^- \):

(a) \( [\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{k}_C \)

(b) \( [\mathfrak{k}_C, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm \)

**Proof.** (a) Suppose that \( z \in \mathcal{Z}(\mathfrak{k}_C) \) and let \( X \in \mathfrak{p}^+, Y \in \mathfrak{p}^- \) and \( Z \in \mathfrak{k}_C \). Then we have

\[
\text{ad}(z)[X,Y] = [\text{ad}(z)X,Y] + [X,\text{ad}(z)Y] = -2[X,Y] + 2[X,Y] = 0,
\]

which implies that \( [X,Y] \in \mathfrak{k}_C \).

(b) Similarly, \( \text{ad}(z)[Z,X] = [\text{ad}(z)Z,X] + [Z,\text{ad}(z)X] = -2[Z,X] \), which implies that \( [Z,X] \in \mathfrak{p}^+ \). Similar calculations give that \( [Z,Y] \in \mathfrak{p}^- \). \( \square \)
Suppose $\pi$ that is an irreducible representation of $G$ on a Hilbert Space $H$. We say $\pi$ is a **highest weight representation** if there is a nonzero vector $v \in \mathbb{H}$ such that

$$\pi(X)v = 0, \ \forall X \in p^+. $$

Let $\mathbb{H}_o = \{v \in \mathbb{H} \mid \pi(X)v = 0, \ \forall X \in p^+\}$. Then, we have the following important theorem.

**Theorem 2.3.5.** Suppose $\pi$ is an irreducible unitary highest weight representation of $G$ on $H$ and $\mathbb{H}_o$ is defined as above. Then $(\pi|_K, \mathbb{H}_o)$ is irreducible. Furthermore, there is a scalar $\lambda$ such that:

$$\pi(Z)v = \lambda v, \ \forall v \in \mathbb{H}_o. $$

If $H_n = \{v \in H \mid \pi(Z)v = (\lambda + 2n)v\}$, then $H = \bigoplus_{n \geq 0} H_n$. Furthermore, we have:

$$\pi(Z) : H_n \rightarrow H_n, \quad Z \in \mathfrak{k}_C$$

$$\pi(X) : H_n \rightarrow H_{n-1}, \quad X \in p^+$$

$$\pi(Y) : H_n \rightarrow H_{n+1}, \quad Y \in p^-,$$

where, in the case $n = 0$, $H_{-1}$ is understood to be the $\{0\}$ space.

**Proof.** By Lemma 2.3.4, $\mathbb{H}_o$ is an invariant $K$-space. Suppose $V_o$ is a nonzero invariant subspace of $\mathbb{H}_o$ and $W_o$ is its orthogonal complement in $\mathbb{H}_o$. Define $V_n$ inductively as follows:

$$V_n = \text{span} \{ \pi(Y)v \mid Y \in p^-, v \in V_{n-1} \}.$$  

Let $V = \bigoplus V_n$. Define $W_n$ in the same way as $V_n$ and let $W = \bigoplus W_n$. Then, by Lemma 2.3.4, $V$ and $W$ are invariant $\mathfrak{g}_C$ subspaces of $H$. Since $\pi$ is unitary $V$ and
W are orthogonal. However, since π is irreducible and V is nonzero, it follows that V = H and hence W = 0. This implies W₀ = 0 and thus π|ₚ is irreducible. Since π(ℤ) commutes with π(𝐾) Schur’s lemma implies that π(ℤ) = λI on ᴨ for some scalar λ. Since V₀ = ᴨ, induction, Lemma 2.3.4, and irreducibility of π implies that ᴨ = ᴨ. The remaining claims follow from Lemma 2.3.4. □

Remark 2.3.6. The operators π(X), X ∈ ℓ, are called **annihilation operators** because, for v in the algebraic direct sum ⊕ ᴨ, sufficiently many applications of π(X) annihilates v. The operators π(Y), Y ∈ ℓ, are called **creation operators**.

Remark 2.3.7. For ᴨ₁(T(Ω)), a straightforward calculation gives:

\[ πν(X)qν₀ = 0, \forall X ∈ ℓ. \]

Indeed, observe that qν₀ = Δ(ℤ + ℓ)−ν from Theorem 2.3.1. Then use Prop.2.3.3(1) and (∗) from the proof of Lemma 2.4.2. Finally, we also have that ᴨ₁(T(Ω)) = ℂqν₀, which says that πν is an irreducible unitary highest weight representation of G.

2.4 Representations of **sp(2n, ℂ)** on \( L²(Ω, dμ_ν) \)

In the previous section we have seen representations of ℓ on ᴨ₁(T(Ω)). In this section we want to build representations of ℓ on \( L²(Ω, dμ_ν) \). We will actually transfer the previous representations that are on ᴨ₁(T(Ω)) to \( L²(Ω, dμ_ν) \) with the help of the Laplace transform.

2.4.1 The Laplace Transform

Consider the space \( L²(Ω, dμ_ν) \), where \( dμ_ν(x) = Δ(x)^{ν−\frac{d}{2}} \). The Laplace transform is defined by:
\[ L_\nu(f)(z) = \int_{\Omega} e^{-\langle z|x \rangle} f(x) \, d\mu_{\nu}(x) \]
\[ = \int_{\Omega} e^{-\langle z|x \rangle} f(x) \det(x)^m \, dx, \]

where \( m = \nu - \frac{d}{r} \). In our case, \( m = \frac{2\nu - n - 1}{2} \), since \( d = \frac{n(n+1)}{2} \) and \( r = n \). Notice that \( d\mu_{\nu}(x) \) is a generalization of the measure \( x^{\nu-1} \, dx \) on \( \mathbb{R}^+ \).

**Proposition 2.4.1.** Let \( f \in L^2(\Omega, d\mu_{\nu}) \). Then, \( L_\nu(f) \in \mathcal{H}_\nu(T(\Omega)) \) and the map
\[
L_\nu : L^2(\Omega, d\mu_{\nu}) \rightarrow \mathcal{H}_\nu(T(\Omega))
\]
is a unitary isomorphism.

**Proof.** See [4], p.187-190. \qed

### 2.4.2 The Derivatives of Some Important Functions

In this section we will give the differentiation formulas of the functions \( e^x \), \( \det(x) \), and \( x_{kl} \) (the function that sends the matrix \( x \) to its \( kl \)-entry). These formulas will be needed in the next section.

Let
\[
E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} + E_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
be the natural basis of \( V = Sym(n, \mathbb{R}) \). The following is also a basis of \( V \):
\[
\tilde{E}_{11} = \frac{E_{11} + E_{11}}{2}, \quad \tilde{E}_{12} = \frac{E_{12} + E_{21}}{2}, \quad \tilde{E}_{22} = \frac{E_{22} + E_{22}}{2}.
\]
In general, the vectors above are given by:

\[ \tilde{E}_{ij} = \frac{E_{ij} + E_{ji}}{2}, \quad 1 \leq i \leq j \leq 2. \]

Notice also that \( \tilde{E}_{11} = E_{11} \) and \( \tilde{E}_{22} = E_{22} \).

We define the derivative of \( f \), now, to be:

\[ D_{ij}f(x) = D_{\tilde{E}_{ij}}f(x). \]

That is, by \( D_{ij}f \) we denote the directional derivative of \( f \) in the direction of \( \tilde{E}_{ij} \). Based on the way \( D_{ij} \) was define, now, one has the folowing formulas for \( e^{-(z|x)} \), \( \det(x)^m \) and \( x_{kl} \):

**Lemma 2.4.2.** The derivatives of \( e^{-(z|x)} \), \( \det(x)^m \) and \( x_{kl} \) are given, respectively, as follows:

\[
\begin{align*}
(i) & \quad D_{ij} e^{-(z|x)} = -e^{-(z|x)} z_{ij} \\
(ii) & \quad D_{ij} \det(x)^m = m \det(x)^{m-1} \text{cof}_{ij}(x) \\
(iii) & \quad D_{ij} x_{kl} = \frac{1}{2} [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] 
\end{align*}
\]

**Proof.** (i) Notice first that if \( f \) is given by inner product, i.e. \( f(x) = (z|x) \), then

\[ D_u(z|x) = (z|u). \]

Using the Chain Rule we have:

\[ D_{ij} e^{-(z|x)} = e^{-(z|x)} D_{ij} (- (z|x)) = -e^{-(z|x)} (z|\tilde{E}_{ij}) = -e^{-(z|x)} z_{ij}. \]
(ii) First, we show the more general result that if $z, w$ are $n \times n$ matrices over $\mathbb{C}$ and $z$ is invertible, then

$$D_w \det(z)^m = m \det(z)^m \text{tr}(z^{-1}w). \quad (*)$$

This follows from the chain rule and the fact that

$$D_w \det(z) = \frac{d}{dt} \det(z + tw)|_{t=0} = \det z \frac{d}{dt} \det(1 + tz^{-1}w)|_{t=0} = \det(z) \text{tr}(z^{-1}w).$$

Taking $w = \tilde{E}_{ij}$, we obtain (ii) since

$$D_{ij} \det(z)^m = m \det(z)^m \text{tr}(z^{-1}\tilde{E}_{ij})$$

$$= m \det(z)^m \text{tr}((z^{-1})_{ij})$$

$$= m \det(z)^m \frac{1}{\det(z)} \text{cof}_{ij}(z)$$

$$= m \det(x)^m \text{cof}_{ij}(z).$$

(iii) Finally, notice that $f(x) = x_{kl} = \frac{x_{kl} + x_{lk}}{2} = (x|\tilde{E}_{kl}).$ Then, one has:

$$D_{ij}x_{kl} = D_{ij}(x|\tilde{E}_{kl}) = (\tilde{E}_{ij}|\tilde{E}_{kl}) = \left(\frac{E_{ij} + E_{ji}}{2}, \frac{E_{kl} + E_{lk}}{2}\right)$$

$$= \frac{1}{4}[ (E_{ij}|E_{kl}) + (E_{ij}|E_{lk}) + (E_{ji}|E_{kl}) + (E_{ji}|E_{lk}) ]$$

$$= \frac{1}{4}[ \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \delta_{jk}\delta_{il} + \delta_{jl}\delta_{ik} ]$$

$$= \frac{1}{2}[ \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} ].$$

\[\square\]

Remark 2.4.3. Let us just check that (ii) is true, for $n = 2, (i, j) = (1, 1)$ and $m = 1$. Indeed, let $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}$. Then, we have:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} = \begin{pmatrix} \frac{x_{11} + x_{22}}{2} & \frac{x_{12} + x_{21}}{2} \\ \frac{x_{21} + x_{12}}{2} & \frac{x_{22} + x_{11}}{2} \end{pmatrix}.$$
\[ D_{11} \det(x) = \frac{d}{dt} \det(x + t\tilde{E}_{11})|_{t=0} \]
\[ = \frac{d}{dt} \det \begin{pmatrix} x_{11} + t & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \big|_{t=0} \]
\[ = \frac{d}{dt} ((x_{11} + t)x_{22} - x^2_{12})|_{t=0} \]
\[ = x_{22} \]
\[ = \text{cof}_{11}(x), \]

which makes (ii) true.

**Proposition 2.4.4.** Suppose \( f, g \in L^2(\Omega, d\mu) \) are smooth and \( f \) vanishes on the boundary of the cone \( \Omega \). Let \( v \in V \). Then, the following hold:

1. \( \int_{\Omega} D_v f(s)g(s)ds = -\int_{\Omega} f(s)D_v g(s)ds \)
2. \( \int_{\Omega} e^{-(z|s)}(z|v)f(s)ds = \int_{\Omega} e^{-(z|s)}D_v f(s)ds \)

**Proof.** (1) is Stokes Theorem, and (2) follows from (1) and the fact that \( D_v e^{-(z|s)} = -e^{-(z|s)}(z|v), \quad \forall \ z \in V_C \) (using Lemma 2.4.2(i)). \qed

Finally, define the **gradient** of \( f \) to be:

\[ (\nabla f(x)|u) = D_u f(x). \]

Clearly, if \( f \) is given by inner product, i.e. \( f(x) = (z|x) \), then \( (\nabla f(x)|u) = (z|u) \). Note also that \( D_u f(x) = \text{tr}(u\nabla)f(x) \).
2.4.3 Transferred Representations

We transfer now the representations of \( g \) (on \( \mathcal{H}_\nu(T(\Omega)) \)) to \( L^2(\Omega, d\mu_\nu) \). This can be done via:

\[
\lambda_\nu(X)f(x) = (\mathcal{L}_\nu^* \pi_\nu(X) \mathcal{L}_\nu) f(x). \tag{2.4}
\]

The results are contained in the following Theorem.

**Theorem 2.4.5.** For each piece of the Lie algebra \( g_\mathbb{C} \), we have:

1. \( \lambda_\nu(X)f(x) = \text{tr}(\nu a + a x + (ax + xa + \nu a) \nabla + a \nabla x \nabla) f(x), \quad X = \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{p}^+ \)

2. \( \lambda_\nu(X)f(x) = \text{tr}(\nu a - ax + (ax + xa - \nu a) \nabla - a \nabla x \nabla) f(x), \quad X = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{p}^- \)

3. \( \lambda_\nu(X)f(x) = \text{tr}(bx + (ax - xa - \nu b) \nabla - b \nabla x \nabla) f(x), \quad X = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{k}_\mathbb{C} \)

**Proof.** We prove the Theorem case by case:

**Case (1):** Let \( X \in \mathfrak{p}^+ \). By Proposition 2.3.3 we know how the Lie algebra acts on functions of \( \mathcal{H}_\nu(T(\Omega)) \). So, at first, it is better if we calculate the part \( \pi_\nu(X) \mathcal{L}_\nu f(z), \quad X \in \mathfrak{p}^+ \), of equation (2.4) as \( \mathcal{L}_\nu f(z) \in \mathcal{H}_\nu(T(\Omega)) \). So,

\[
\pi_\nu(X) \mathcal{L}_\nu f(z) = -\nu \text{tr}(az + a) \mathcal{L}_\nu f(z) + D_{v(a,z)} \mathcal{L}_\nu f(z), \quad X \in \mathfrak{p}^+.
\]

Let \( A = D_{v(a,z)} \mathcal{L}_\nu f(z) \). We will calculate \( A \):

\[
A = D_{v(a,z)} \mathcal{L}_\nu f(z) = D_{-az-a-az-za} \int_{\Omega} e^{-z|x|} f(x) \det(x)^m dx
\]

\[
= \int_{\Omega} D_{-az-a-az-za} e^{-z|x|} f(x) \det(x)^m dx
\]

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\[
K = -D_{az+za} \mathcal{L}_v f(z) = - \int_{\Omega} D_{az+za} e^{-|z|^2} f(x) \det(x)^m dx
\]

\[
= - \int_{\Omega} e^{-|z|^2} (-az - a - az - za|x|) f(x) \det(x)^m dx
\]

\[
= \int_{\Omega} e^{-|z|^2} (az + za|x|) f(x) \det(x)^m dx
\]

\[
+ \int_{\Omega} e^{-|z|^2} (a|x|) f(x) \det(x)^m dx
\]

\[
+ \int_{\Omega} e^{-|z|^2} (za|x|) f(x) \det(x)^m dx
\]

Denote the last three integrals by \(K, L,\) and \(M.\) We will calculate now each one of them:

\[
K = -D_{az+za} \mathcal{L}_v f(z)
\]

\[
= - \int_{\Omega} D_{az+za} e^{-|z|^2} f(x) \det(x)^m dx
\]

\[
= \int_{\Omega} e^{-|z|^2} (az + za|x|) f(x) \det(x)^m dx
\]

\[
= \int_{\Omega} e^{-|z|^2} \text{tr}((az + za)x)f(x) \det(x)^m dx
\]

\[
= \sum_{i,j,k} \int_{\Omega} e^{-|z|^2} (a_{ik}z_{kj} + z_{ik}a_{kj})x_{ji} f(x) \det(x)^m dx
\]

\[
= \sum_{i,j,k} \int_{\Omega} e^{-|z|^2} (a_{ik}D_{kj} + a_{kj}D_{ik})(x_{ji}f(x) \det(x)^m) dx,
\]

by Prop. 2.4.4(2) \((v = \tilde{E}_{kj})\)

\[
= \sum_{i,j,k} \int_{\Omega} e^{-|z|^2} (a_{ik}\frac{1}{2}(\delta_{kj}\delta_{ji} + \delta_{ki}\delta_{jj}) + a_{kj}\frac{1}{2}(\delta_{kj}\delta_{ii} + \delta_{ij}\delta_{ki}))f(x) \det(x)^m dx
\]

\[
+ \sum_{i,j,k} \int_{\Omega} e^{-|z|^2} (a_{ik}x_{ji}D_{kj}f(x) + a_{kj}x_{ji}D_{ik}f(x)) \det(x)^m dx
\]

\[
+ \sum_{i,j,k} \int_{\Omega} e^{-|z|^2} f(x)m \det(x)^m(a_{ik}x_{ji}\text{tr}(x^{-1}\tilde{E}_{kj}) + a_{kj}x_{ji}\text{tr}(x^{-1}\tilde{E}_{ik}))dx,
\]

by \((\ast)\) of p. 56

\[
= (n + 1)\text{tr}(a) \int_{\Omega} e^{-|z|^2} f(x) \det(x)^m dx
\]

\[
+ \int_{\Omega} e^{-|z|^2} ((ax + xa)_{ik}D_{ik}f(x)) \det(x)^m dx
\]

\[
+ \int_{\Omega} e^{-|z|^2} f(x)m \det(x)^m\text{tr}(x^{-1}(xa + ax))dx
\]

\[
= (2m + n + 1)\text{tr}(a) \int_{\Omega} e^{-|z|^2} f(x) \det(x)^m dx
\]
\[
\begin{align*}
&+ \int_{\Omega} e^{-(z|x)}(\text{tr}((ax + xa)\nabla f)(x)) \det(x)^{m} \, dx \\
&= 2\nu\text{tr}(a) \int_{\Omega} e^{-(z|x)} f(x) \det(x)^{m} \, dx + \int_{\Omega} e^{-(z|x)}(\text{tr}((ax + xa)\nabla f)(x)) \det(x)^{m} \, dx \\
&= 2\nu\text{tr}(a) \mathcal{L}_{\nu}(f)(x) + \mathcal{L}_{\nu}((\text{tr}(ax + xa)\nabla) f)(x)
\end{align*}
\]

For \( L \) we have:

\[
L = \int_{\Omega} e^{-(z|x)}(a|x) f(x) \det(x)^{m} \, dx = \int_{\Omega} e^{-(z|x)}\text{tr}(ax) f(x) \det(x)^{m} \, dx = \mathcal{L}_{\nu}(\text{tr}(ax)f)(z).
\]

Finally, using the fact that \( \det(x) = \sum_{ij} x_{ij} \text{cof}_{ij}(x) \), we have:

\[
\begin{align*}
M &= \int_{\Omega} e^{-(z|x)}(za|x) f(x) \det(x)^{m} \, dx \\
&= \int_{\Omega} e^{-(z|x)}\text{tr}((za|x)f(x) \det(x)^{m} \, dx \\
&= \int_{\Omega} e^{-(z|x)} \sum_{i,j,k,l} z_{ik} a_{kl} z_{lj} x_{ji} f(x) \det(x)^{m} \, dx \\
&= \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} z_{lj} x_{ji} f(x) \det(x)^{m} \, dx \\
&= \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} D_{lj}(x_{ji} f(x) \det(x)^{m}) \, dx \\
&= \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} \frac{1}{2}(\delta_{ij}\delta_{ji} + \delta_{ij}\delta_{jj}) f(x) \det(x)^{m} \, dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} x_{ji} (D_{lj} f)(x) \det(x)^{m} \, dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} x_{ji} f(x) m \det(x)^{m-1} \text{cof}_{ij}(x) \, dx \\
&= \sum_{i,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} \left(\frac{n}{2} + \delta_{li}\right) f(x) \det(x)^{m} \, dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} D_{ik}(x_{ji} D_{lj} f)(x) \det(x)^{m} \, dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-(z|x)} a_{kl} z_{ik} \delta_{li} x_{ji} f(x) m \det(x)^{m-1} \text{cof}_{ij}(x) \, dx
\end{align*}
\]
\[
\begin{align*}
&= \frac{n + 1}{2} \sum_{i,k} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{ki} z_{ik} f(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} \frac{1}{2} (\delta_{ij} \delta_{ki} + \delta_{ii} \delta_{kj}) (D_{ij} f)(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} x_{ji} (D_{ik} D_{lj} f)(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} x_{ji} D_{ij} f(x) m \det(x)^{m-1} \text{cof}_{ik}(x) dx \\
&\quad + m \sum_{j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{lk} z_{kl} x_{lj} f(x) \det(x)^{m-1} \text{cof}_{lj}(x) dx \\
&\quad = \frac{n + 1}{2} \int_{\Omega} e^{-\langle z \mid x \rangle} (\sum_{i,k} a_{ki} z_{ik}) f(x) \det(x)^m dx \\
&\quad + \sum_{j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} (\frac{n + 1}{2} \delta_{kj}) (D_{ij} f)(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} x_{ji} (D_{ik} a_{kl} D_{lj} f)(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} \delta_{kj} x_{ik} D_{lj} f(x) m \det(x)^{m-1} \text{cof}_{ik}(x) dx \\
&\quad + m \sum_{j,l} \int_{\Omega} e^{-\langle z \mid x \rangle} (a \nabla)_{ll} x_{lj} f(x) \det(x)^{m-1} \text{cof}_{lj}(x) dx \\
&= \frac{n + 1}{2} \int_{\Omega} e^{-\langle z \mid x \rangle} \text{tr}(a z) f(x) \det(x)^m dx \\
&\quad + \frac{n + 1}{2} \sum_{j,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{jl} (D_{ij} f)(x) \det(x)^m dx \\
&\quad + \sum_{i,j,k} \int_{\Omega} e^{-\langle z \mid x \rangle} x_{ji} (D_{ik} (a D)_{kj} f)(x) \det(x)^m dx \\
&\quad + m \sum_{i,k,l} \int_{\Omega} e^{-\langle z \mid x \rangle} a_{kl} x_{ik} D_{ik} f(x) \det(x)^{m-1} \text{cof}_{ik}(x) dx \\
&\quad + m \int_{\Omega} e^{-\langle z \mid x \rangle} \text{tr}(a z) f(x) \det(x)^m dx \\
&= \left[ \frac{n + 1}{2} + m \right] \int_{\Omega} e^{-\langle z \mid x \rangle} \text{tr}(a z) f(x) \det(x)^m dx \\
&\quad + \frac{n + 1}{2} \sum_{j} \int_{\Omega} e^{-\langle z \mid x \rangle} (a \nabla f)_{jj}(x) \det(x)^m dx
\end{align*}
\]
+ \sum_{i,j} \int_{\Omega} e^{-|x|} x_{ji} (D(aD)_{ij} f)(x) \det(x)^m dx \\
+ m \sum_{i,k} \int_{\Omega} e^{-|x|} (a \nabla f)_{kk}(x) x_{ik} \det(x)^{m-1} \text{cof}_{ik}(x) dx \\
= \nu \text{tr}(az) \mathcal{L}_\nu(f)(z) + \frac{n + 1}{2} \int_{\Omega} e^{-|x|} (\text{tr}(a \nabla f))(x) \det(x)^m dx \\
+ \sum_{j} \int_{\Omega} e^{-|x|} (x(D(aD))_{jj} f)(x) \det(x)^m dx \\
+ m \int_{\Omega} e^{-|x|} (\text{tr}(a \nabla f))(x) \det(x)^m dx \\
= \nu \text{tr}(az) \mathcal{L}_\nu(f)(z) + \frac{n + 1}{2} \int_{\Omega} e^{-|x|} (\text{tr}(a \nabla f))(x) \det(x)^m dx \\
+ \int_{\Omega} e^{-|x|} (\text{tr}(x \nabla a \nabla f))(x) \det(x)^m dx \\
= \nu \text{tr}(az) \mathcal{L}_\nu(f)(z) + \nu \mathcal{L}_\nu(\text{tr}(a \nabla f))(z) + \mathcal{L}_\nu(\text{tr}(a \nabla x \nabla f))(z).

Collecting the results, we have:

\begin{align*}
K &= 2 \nu \text{tr}(a) \mathcal{L}_\nu(f)(z) + \mathcal{L}_\nu(\text{tr}((ax + xa) \nabla f))(z) \\
L &= \mathcal{L}_\nu(\text{tr}(ax) f)(z) \\
M &= \nu \text{tr}(az) \mathcal{L}_\nu(f)(z) + \nu \mathcal{L}_\nu(\text{tr}(a \nabla f))(z) + \mathcal{L}_\nu(\text{tr}(a \nabla x \nabla f))(z).
\end{align*}

Finally, recall that:

\begin{align*}
A &= K + L + M \\
&= 2 \nu \text{tr}(a) \mathcal{L}_\nu(f)(z) + \mathcal{L}_\nu(\text{tr}((ax + xa) \nabla f))(z) + \mathcal{L}_\nu(\text{tr}(ax) f)(z) + \nu \text{tr}(az) \mathcal{L}_\nu(f)(z) \\
&\quad + \nu \mathcal{L}_\nu(\text{tr}(a \nabla f))(z) + \mathcal{L}_\nu(\text{tr}(a \nabla x \nabla f))(z).
\end{align*}

Since,

\[ \pi_\nu(X) \mathcal{L}_\nu(f)(z) = -\nu \text{tr}(az + a) \mathcal{L}_\nu(f)(z) + A, \quad X \in p^+ \]
we get that:

$$\pi_\nu(X)\mathcal{L}_\nu(f)(z) = \nu\text{tr}(a)\mathcal{L}_\nu(f)(z) + \mathcal{L}_\nu(\text{tr}(af))(z) + \nu\mathcal{L}_\nu(\text{tr}(a\nabla f))(z) + \mathcal{L}_\nu(\text{tr}(a\nabla \nabla f))(z).$$

Taking now $\mathcal{L}_\nu^*$ in both sides, and considering (2.4), we get:

$$\lambda_\nu(X)f(x) = \text{tr}(\nu a + ax + (ax + xa + \nu a)\nabla + a\nabla \nabla) f(x), \; X \in \mathfrak{p}^+.$$

Case (2): Let $X \in \mathfrak{p}^-$. Just like case (1), we know how the Lie algebra acts on functions of $\mathcal{H}_\nu(T(\Omega))$, by Proposition 2.3.3. We first calculate the part $\pi_\nu(X)\mathcal{L}_\nu(f)(z), \; X \in \mathfrak{p}^-$, of equation (2.4) since $\mathcal{L}_\nu f(z) \in \mathcal{H}_\nu(T(\Omega))$. We have,

$$\pi_\nu(X)\mathcal{L}_\nu(f)(z) = -\nu\text{tr}(-az + a)\mathcal{L}_\nu f(z) + D_{\nu(-a,-z)}\mathcal{L}_\nu f(z), \; X \in \mathfrak{p}^-.$$

Let $A = D_{\nu(-a,-z)}\mathcal{L}_\nu f(z)$. We will calculate $A$:

$$A = D_{\nu(-a,-z)}\mathcal{L}_\nu f(z) = D_{az + az - za} \int e^{-|z|x} f(x) \det(x)^m \, dx$$

$$= \int e^{-|z|x} D_{az + az - za} f(x) \det(x)^m \, dx$$

$$= - \int e^{-|z|x} D_{az + az - za} (z|x)f(x) \det(x)^m \, dx$$

$$= - \int e^{-|z|x} (-az + a + az - za |x)f(x) \det(x)^m \, dx$$

$$= \int e^{-|z|x} (az + za \, |x) f(x) \det(x)^m \, dx$$

$$- \int e^{-|z|x} (a \, |x) f(x) \det(x)^m \, dx$$

$$- \int e^{-|z|x} (za \, |x) f(x) \det(x)^m \, dx$$

Denote the last three integrals by $K, L,$ and $M$. We calculate again each integral:
\[ K = -D_{az + za} L_{\nu} f(z) \]
\[ = -\int_{\Omega} D_{az + za} e^{-\langle z|x \rangle} f(x) \det(x)^m \, dx \]
\[ = \int_{\Omega} e^{-\langle z|x \rangle} (az + za) f(x) \det(x)^m \, dx \]
\[ = \int_{\Omega} e^{-\langle z|x \rangle} \text{tr}((az + za) f(x) \det(x)^m \, dx \]
\[ = \sum_{i,j,k} \int_{\Omega} e^{-\langle z|x \rangle} (a_{ik} z_{kj} + z_{ik} a_{kj}) x_{ji} f(x) \det(x)^m \, dx \]
\[ = \sum_{i,j,k} \int_{\Omega} e^{-\langle z|x \rangle} (a_{ik} D_{kj} + a_{kj} D_{ik}) (x_{ji} f(x) \det(x)^m \, dx \]
\[ = \sum_{i,j,k} \int_{\Omega} e^{-\langle z|x \rangle} \left( a_{ik} \frac{1}{2} (\delta_{kj} \delta_{ji} + \delta_{ki} \delta_{jj}) + a_{kj} \frac{1}{2} (\delta_{kj} \delta_{ii} + \delta_{ji} \delta_{ki}) \right) f(x) \det(x)^m \, dx \]
\[ + \sum_{i,j,k} \int_{\Omega} e^{-\langle z|x \rangle} (a_{ik} x_{ji} D_{kj} f(x) + a_{kj} x_{ji} D_{ik} f(x)) \det(x)^m \, dx \]
\[ + \sum_{i,j,k} \int_{\Omega} e^{-\langle z|x \rangle} f(x) m \det(x)^m (a_{ik} x_{ji} \text{tr}(x^{-1} \tilde{E}_{kj}) + a_{kj} x_{ji} \text{tr}(x^{-1} \tilde{E}_{ik})) \, dx \]
\[ = (n + 1) \text{tr}(a) \int_{\Omega} e^{-\langle z|x \rangle} f(x) \det(x)^m \, dx \]
\[ + \int_{\Omega} e^{-\langle z|x \rangle} (ax + xa) a_{ik} D_{ik} f(x)) \det(x)^m \, dx \]
\[ + \int_{\Omega} e^{-\langle z|x \rangle} f(x) m \det(x)^m \text{tr}(x^{-1} (xa + ax)) dx \]
\[ = (2m + n + 1) \text{tr}(a) \int_{\Omega} e^{-\langle z|x \rangle} f(x) \det(x)^m \, dx \]
\[ + \int_{\Omega} e^{-\langle z|x \rangle} (\text{tr}((ax + xa) \nabla f)) f(x)) \det(x)^m \, dx \]
\[ = 2\nu \text{tr}(a) \int_{\Omega} e^{-\langle z|x \rangle} f(x) \det(x)^m \, dx + \int_{\Omega} e^{-\langle z|x \rangle} (\text{tr}((ax + xa) \nabla f)) f(x)) \det(x)^m \, dx \]
\[ = 2\nu \text{tr}(a) L_{\nu} (f(x)) + L_{\nu} (\text{tr}(ax + xa) \nabla f)(x) \]

With similar calculations, we find that:

\[ L = L_{\nu} (\text{tr}(ax) f)(z) \]
and

\[ M = \nu \text{tr}(az)\mathcal{L}_\nu(f)(z) + \nu \mathcal{L}_\nu(\text{tr}(a\nabla)f)(z) + \mathcal{L}_\nu(\text{tr}(a\nabla x \nabla)f)(z) \]

Hence,

\[ A = K - L - M \]

\[ = 2\nu \text{tr}(a)\mathcal{L}_\nu(f)(z) + \mathcal{L}_\nu(\text{tr}(ax + xa)\nabla)f)(z) - \mathcal{L}_\nu(\text{tr}(ax)f)(z) - \nu \text{tr}(az)\mathcal{L}_\nu(f)(z) \]

\[ - \nu \mathcal{L}_\nu(\text{tr}(a\nabla)f)(z) - \mathcal{L}_\nu(\text{tr}(a\nabla x \nabla)f)(z). \]

Since,

\[ \pi_\nu(X)\mathcal{L}_\nu(f)(z) = -\nu \text{tr}(-az + a)\mathcal{L}_\nu(f)(z) + A, \; X \in \mathfrak{p}^+ \]

we get that:

\[ \pi_\nu(X)\mathcal{L}_\nu(f)(z) = \nu \text{tr}(a)\mathcal{L}_\nu(f)(z) - \mathcal{L}_\nu(\text{tr}(ax)f)(z) + \mathcal{L}_\nu(\text{tr}(ax + xa)\nabla)f)(z) \]

\[ - \nu \mathcal{L}_\nu(\text{tr}(a\nabla)f)(z) - \mathcal{L}_\nu(\text{tr}(a\nabla x \nabla)f)(z). \]

Taking now \( \mathcal{L}_\nu^* \) in both sides, and considering (2.4), we get:

\[ \lambda_\nu(X)f(x) = \text{tr} \left( \nu a - ax + (ax + xa - \nu a)\nabla - a\nabla x \nabla \right) f(x), \; X \in \mathfrak{p}^- \]

Case (3): Let \( X \in \mathfrak{k}_C \). Just like cases (1) and (2) we know how the Lie algebra acts on functions of \( \mathcal{H}_\nu(T(\Omega)) \), by Proposition 2.3.3 As before, we first calculate the part \( \pi_\nu(X)\mathcal{L}_\nu f(z), \; X \in \mathfrak{p}^- \), of equation (2.4) since \( \mathcal{L}_\nu f(z) \in \mathcal{H}_\nu(T(\Omega)) \). We have,

\[ \pi_\nu(X)\mathcal{L}_\nu f(z) = \nu \text{tr}(bz)\mathcal{L}_\nu f(z) + D_{w(a,b,z)}\mathcal{L}_\nu f(z), \; X \in \mathfrak{k}_C. \]
Let $A = D_{w(a,b,z)}L_vf(z)$. We will calculate $A$:

\[
A = D_{w(a,b,z)}L_vf(z) = D_{-az-b+zb+z|z} \int e^{-(z|x)} f(x) \det(x)^m dx
\]

\[
= \int_\Omega D_{-az-b+zb+z|z} e^{-(z|x)} f(x) \det(x)^m dx
\]

\[
= - \int_\Omega e^{-(z|x)} D_{-az-b+zb+z|z} (z|x) f(x) \det(x)^m dx
\]

\[
= - \int_\Omega e^{-(z|x)} (-az - b + zb + za|x) f(x) \det(x)^m dx
\]

\[
= \int_\Omega e^{-(z|x)} (az - za|x) f(x) \det(x)^m dx
\]

\[+ \int_\Omega e^{-(z|x)} (b|x) f(x) \det(x)^m dx
\]

\[\]

Denote the last three integrals by $K$, $L$, and $M$. Let us calculate again each integral:

\[
K = D_{az-za}L_vf(z)
\]

\[
= \int_\Omega D_{az-za} e^{-(z|x)} f(x) \det(x)^m dx
\]

\[
= \int_\Omega e^{-(z|x)} (az - za|x) f(x) \det(x)^m dx
\]

\[
= \int_\Omega e^{-(z|x)} \text{tr}((az - za)x)f(x) \det(x)^m dx
\]

\[
= \sum_{i,j,k} \int_{\Omega} e^{-(z|x)} (a_{ik}z_{kj} - z_{ik}a_{kj})x_{ji}f(x) \det(x)^m dx
\]

\[
= \sum_{i,j,k} \int_{\Omega} e^{-(z|x)} (a_{ik}D_{kj} - a_{kj}D_{ik})(x_{ji}f(x) \det(x)^m) dx
\]

\[
= \sum_{i,j,k} \int_{\Omega} e^{-(z|x)} (a_{ik} \frac{1}{2}(\delta_{kj}\delta_{ji} + \delta_{ki}\delta_{jj}) - a_{kj} \frac{1}{2}(\delta_{kj}\delta_{ii} + \delta_{ji}\delta_{ki})) f(x) \det(x)^m dx
\]

\[+ \sum_{i,j,k} \int_{\Omega} e^{-(z|x)} (a_{ik}x_{ji}D_{kj}f(x) - a_{kj}x_{ji}D_{ik}f(x)) \det(x)^m dx
\]

\[+ \sum_{i,j,k} \int_{\Omega} e^{-(z|x)} f(x) \det(x)^m (a_{ik}x_{ji}\text{tr}(x^{-1}\tilde{E}_{kj}) - a_{kj}x_{ji}\text{tr}(x^{-1}\tilde{E}_{ik})) dx
\]

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\[(n + 1)\text{tr}(a) \int_{\Omega} e^{-|z|} f(x) \det(x)^m \, dx \]
\[+ \int_{\Omega} e^{-|z|} ((ax - xa)_{ik} D_{ik} f(x)) \det(x)^m \, dx \]
\[+ \int_{\Omega} e^{-|z|} f(x)m \det(x)^m \text{tr}(x^{-1}(ax - xa)) \, dx \]
\[= (2m + n + 1)\text{tr}(a) \int_{\Omega} e^{-|z|} f(x) \det(x)^m \, dx \]
\[+ \int_{\Omega} e^{-|z|} (\text{tr}(ax - xa) \nabla f(x)) \det(x)^m \, dx \]
\[= 2\nu \text{tr}(a) \int_{\Omega} e^{-|z|} f(x) \det(x)^m \, dx \]
\[+ \int_{\Omega} e^{-|z|} (\text{tr}(ax - xa) \nabla f(x)) \det(x)^m \, dx \]
\[= 2\nu \text{tr}(a) L_\nu(f)(x) + L_\nu((\text{tr}(ax - xa) \nabla) f)(x) \]

Furthermore, we have that:

\[L = L_\nu(\text{tr}(bx) f(z))\]

and

\[M = \nu \text{tr}(bz) L_\nu(f)(z) + \nu L_\nu(\text{tr}(b \nabla)f)(z) + L_\nu(\text{tr}(b \nabla x \nabla) f)(z) \]

Hence,

\[A = K + L - M \]
\[= L_\nu(\text{tr}(ax - xa) \nabla) f)(z) + L_\nu(\text{tr}(bx) f)(z) - \nu \text{tr}(bz) L_\nu(f)(z) \]
\[-\nu L_\nu(\text{tr}(b \nabla)f)(z) - L_\nu(\text{tr}(b \nabla x \nabla) f)(z). \]

Since,

\[\pi_\nu(X) L_\nu(f)(z) = \nu \text{tr}(bz) L_\nu(f)(z) + A, \quad X \in \mathfrak{g} \]

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we have that:

\[
\pi_{\nu}(X)\mathcal{L}_{\nu}(f)(z) = \mathcal{L}_{\nu}(\text{tr}(bx)f)(z) + \mathcal{L}_{\nu}(\text{tr}((ax - xa)\nabla)f)(z) - \nu \mathcal{L}_{\nu}(\text{tr}(b\nabla)f)(z) - \mathcal{L}_{\nu}(\text{tr}(b\nabla x)f)(z).
\]

Finally, taking \(\mathcal{L}^*_\nu\) in both sides, and considering (2.4), we get:

\[
\lambda_{\nu}(X)f(x) = \text{tr}(\nu bx + (ax - xa - \nu b)\nabla - b\nabla x\nabla)f(x), \quad X \in \mathfrak{k}_C.
\]

\[\square\]

### 2.5 Recursion Relations for \(\ell_{\nu}^m\)

Recall the functions defined in Theorem 2.3.1 given by \(q_{m}^{\nu}(z) = \Delta(z + e)^{-\nu}\psi_{m}^{\nu}\left(\frac{z - \nu}{\nu}\right)\). As shown, these functions form an orthogonal basis of \(\mathcal{H}_{\nu}(T(\Omega))^L\). Recall also, that \(\mathfrak{k}_C\) and \(\mathfrak{h}_C\) denote the Lie algebras of \(K_C\) and \(H_C\) respectively.

**Proposition 2.5.1.** The Laguerre functions \(\ell_{\nu}^m\) relate with \(q_{m}^{\nu}(z)\) as follows:

\[
\mathcal{L}(\ell_{m}^{\nu})(z) = \Gamma_{\Omega}(m + n)q_{m}^{\nu}(z)
\]


From Proposition 2.3.3 and Theorem 2.4.5, we can actually obtain the corresponding relations for \(q_{m}^{\nu}\) and \(\ell_{m}^{\nu}\) respectively. Concerning \(q_{m}^{\nu}\) we know (by Lemma 5.5 in [4], p. 182) that, for \(\xi \in Z(\mathfrak{k}_C)\) and \(Z_0 \in Z(\mathfrak{h}_C)\) (can take \(\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\)), where \(Z(\mathfrak{k}_C)\) and \(Z(\mathfrak{h}_C)\) denote the centers of \(\mathfrak{k}_C\) and \(\mathfrak{h}_C\) respectively, we have:

\[
\pi_{\nu}(\xi)q_{m}^{\nu}(z) = (r\nu + 2|m|)q_{m}^{\nu}(z)
\] (2.5)
and

\[ \pi_\nu(-2z_0)q^\nu_m(z) = \sum_{j=1}^r \left( \frac{m}{m-e_j} \right) q^\nu_{m-e_j} - \sum_{j=1}^r \left( \nu + n_j - \frac{s}{2}(j-1) \right) c_m(j)q^\nu_{m+e_j}(z) \quad (2.6) \]

where:

\[ e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^t, \text{ with 1 in the } j^{th} \text{ position} \]

and

\[ c_m(j) = \prod_{j \neq k} \frac{n_j - n_k - \frac{s}{2}(j-1)}{n_j - n_k - \frac{s}{2}(j-k)}. \]

**Theorem 2.5.2.** The Laguerre functions satisfy the following differential recursion relations:

1. \( (r \nu + 2|m|) \ell^\nu_m = \text{tr}(x - \nu \nabla - \nabla x \nabla) \ell^\nu_m \)
2. \[ \sum_{j=1}^r c_m(j)\ell^\nu_{m+e_j} = \text{tr}(\nu e - x + (2x - \nu)\nabla - \nabla x \nabla) \ell^\nu_m \]
3. \[ -\sum_{j=1}^r \left( \frac{m}{m-e_j} \right) (m_j - 1 + \nu - (j-1)\frac{s}{2}) \ell^\nu_{m-e_j} = \text{tr}(\nu e + x + (2x + \nu)\nabla + \nabla x \nabla) \ell^\nu_m \]

**Proof.** Transferring representation (2.5) above onto \( L^2(\Omega, d\mu_\nu)^L \), we get:

\[ \lambda_\nu(\xi) \ell^\nu_m(x) = (r \nu + 2|m|) \ell^\nu_m(x). \quad (2.7) \]

Now, combining Proposition 2.4.5(3) (for \( f = \ell^\nu_m, X = \xi, \text{ i.e. } a = 0, b = 1 \)) and (2.7), we get the first recursion relation for \( \ell^\nu_m \):

\[ (r \nu + 2|m|) \ell^\nu_m = \text{tr}(x - \nu \nabla - \nabla x \nabla) \ell^\nu_m \quad (2.8) \]

Let \( L^2_{k}(\Omega, d\mu_\nu) = \{ f \in L^2(\Omega, d\mu_\nu) \mid \lambda_\nu f = (r \nu + 2k)f \} \). Then, as \( \lambda_\nu \) are highest weight representations, we have:
\( L(\Omega, d\mu_\nu) = \bigoplus L^2_k(\Omega, d\mu_\nu), \)

where \( L^2_k(\Omega, d\mu_\nu) \neq \{0\} \) if \( k \geq 0 \). Observe also that \( \ell_m^\nu \in L^2_{|m|}(\Omega, d\mu_\nu) \), by (2.7).

Now, for \( X \in p^+ \), we have:

\[
\lambda_\nu(\xi) \lambda_\nu(X)f = \lambda_\nu(X)\lambda_\nu(\xi)f + \lambda_\nu([X, \xi])f = \lambda_\nu(X)(r\nu + 2k)f + \lambda_\nu(\text{ad}(\xi)X)f = (r\nu + 2k)\lambda_\nu(X)f + \lambda_\nu(2X)f.
\]

That is, \( \lambda_\nu(X)f \in L^2_{k+1}(\Omega, d\mu_\nu), X \in p^+ \). Similarly, \( \lambda_\nu(X)f \in L^2_{k-1}(\Omega, d\mu_\nu), X \in p^- \).

This says, that \( \lambda_\nu(X) \), for \( X \) in \( p^+ \) and \( p^- \) respectively act as \( k+1 \) and \( k-1 \) projections.

Let \( X^+ = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in p^+ \), \( X^- = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in p^- \) (take \( a = 1 \)). Then,

\[
Z_0 = \frac{1}{2}(X^+ + X^-) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and \( Z_0 \in Z(\mathfrak{h}_C) \), where \( Z(\mathfrak{h}_C) \) denotes the center of \( \mathfrak{h}_C \). Transferring representation (2.6) onto \( L^2(\Omega, d\mu_\nu)^L \) we get:

\[
-\lambda_\nu(-2Z_0)\ell_m^\nu(x) = \sum_{j=1}^r \binom{m}{m-e_j} \left(m_j - 1 + \nu - (j-1)\frac{s}{2}\right) \ell_{m-e_j}^\nu(x) - \sum_{j=1}^r c_m(j)\ell_{m+e_j}^\nu(x).
\]

(2.9)

So, for each one of the projections, (2.9) gives:

\[
\lambda_\nu(X^-)\ell_m^\nu(x) = -\sum_{j=1}^r \binom{m}{m-e_j} \left(m_j - 1 + \nu - (j-1)\frac{s}{2}\right) \ell_{m-e_j}^\nu(x) \tag{2.10}
\]
\[ \lambda_{\nu}(X^+)\ell_{m}^\nu(x) = \sum_{j=1}^{r} c_{m}(j)\ell_{m+e_j}^\nu(x). \]  

(2.11)

Finally, combining Proposition 2.4.5 (2,1) (for \( f = \ell_{m}^\nu, X = X^- \) and \( X = X^+, a = 1 \)) and (2.10),(2.11) respectively, one obtains the remaining recursion relations for \( \ell_{m}^\nu \):

\[ -\sum_{j=1}^{r} \binom{m}{m-e_j} (m_j - 1 + \nu - (j - 1)\frac{s}{2}) \ell_{m-e_j}^\nu = \text{tr} (\nu - x + (2x - \nu)\nabla - \nabla x \nabla) \ell_{m}^\nu \]  

(2.12)

and

\[ \sum_{j=1}^{r} c_{m}(j)\ell_{m+e_j}^\nu = \text{tr} (\nu + x + (2x + \nu)\nabla + \nabla x \nabla) \ell_{m}^\nu. \]  

(2.13)

Notice that when one restricts down to \( \mathbb{R} \), equations (2.8), (2.12) and (2.13) correspond to the classical relations (2.1), (2.2) and (2.3) respectively.

Remark 2.5.3. We conclude with the following remarks:

(a) Note that \( X_0, X^+, X^- \in \mathfrak{g}_C^L \), where \( \mathfrak{g}_C^L = \{ X \in \mathfrak{g}_C \mid \text{Ad}(l)X = X, \forall l \in L \} \).

(b) Starting from the unit disc \( D \), one could obtain the polynomials \( q_{m}^\nu(z) \) as follows:

Consider the functions \( \psi_{m}(z) = \int_{L} \Delta_{m}(lz) \, dl \). Then, \( \psi_{m}(z) \in \mathcal{H}_{\nu}(D) \), where \( \mathcal{H}_{\nu}(D) = \{ F \in \mathcal{O}(D) \mid \|F\|^2 < \infty \} \), and \( \|F\|^2 = \beta_{\nu} \int_{D} |F(z)|^2 dm(z) \). \( \mathcal{O}(D) \) denotes the space of holomorphic functions on the disc \( D \), and \( \beta_{\nu} = \frac{\Gamma_{\nu}(\nu)}{\pi^{\nu} \Gamma_{\nu}(\nu - \frac{s}{2})} \). Note that \( \psi_{m} \) are \( L \)-invariant, and furthermore that \( \mathcal{H}_{\nu}(D)^L \cong \bigoplus_{m \in \Lambda} \mathbb{C} \psi_{m} \). That is, the \( \psi_{m} \)'s span the 1-dimensional eigenspaces of the highest weight representation space. It is known that \( D \cong T(\Omega) \) through the Cayley transform \( c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \), so we have \( \pi_{\nu}(c)\mathcal{H}_{\nu}(D) = \mathcal{H}_{\nu}(T(\Omega)) \).

Hence, \( \pi_{\nu}(c)\psi_{m}(z) = \Delta(z + c)^{-\nu}\psi_{m\frac{z+c}{z+c}} \in \mathcal{H}_{\nu}(T(\Omega)) \). Denote the right-hand side of the above equation by \( q_{m}^\nu(z) \). Then, \( \{q_{m}^\nu(z)\} \) form an orthogonal basis for \( \mathcal{H}_{\nu}(T(\Omega))^{L} \).
Chapter 3

General Recursion Relations for Laguerre Functions

In this Chapter we describe the action of $\mathfrak{g}$, the Lie algebra of $G(T(\Omega))$, on $H_\nu(T(\Omega))$ and $L^2(\Omega, d\mu_\nu)$ respectively. The group $G(T(\Omega))$ acts on $T(\Omega) = \Omega + iV$, where $\Omega$ is any symmetric cone and $V$ any simple Euclidean Jordan algebra. Finally, we give the general recursion relations for $\ell_\nu^m$.

3.1 $G(T(\Omega))$ and Its Lie Algebra $\mathfrak{g}$

In this section we describe some important subgroups of $G(T(\Omega))$. Then, we introduce some subalgebras of $\mathfrak{g}_\mathbb{C}$, the complexification of the Lie algebra of $G$.

The first subgroup of $G(T(\Omega))$ is the familiar group $H = G(\Omega)^0$, the identity component of the group that fixes the cone. Another is the Abelian subgroup $N^+$ generated by the translation $\tau_u(z) = z + iu$, $u \in V$. Notice that $N^+ \cong V$. The group $K = G(T(\Omega))_e$, the stabilizer of the identity, is also a subgroup of $G(T(\Omega))$. Finally, from $N^+$ and the inversion map $j(z) = z^{-1}$ of $G(T(\Omega))$, we have the subgroup $N^- = j \circ N^+ \circ j$ that is generated by the element $j_v(z) = (z^{-1} + iv)^{-1}$, $v \in V$.

Proposition 3.1.1. The group $G(T(\Omega))$ decomposes as follows: $G(T(\Omega)) = N^+HK$.

Proof. We follow the same idea as in the proof in [8], p.207, with a few modifications.
Let \( g \in G(T(\Omega)) \) and set \( z = ge = x + iy, \ x \in \Omega, \ y \in V \). Since the action of \( H \) on \( \Omega \) is transitive, there is an \( h \in H \) such that \( x = he \). Hence, \( z = \tau_y \circ he \). Let \( g_0 = h^{-1} \circ \tau^{-1} \circ g \).

Then, \( g_0e = e \). Indeed, \( g_0e = h^{-1} \circ \tau^{-1} \circ ge = h^{-1} \circ \tau^{-1}(x + iy) = h^{-1} \circ x = e \). Since \( g_0e = e \), we have that \( g_0 \in K \).

**Theorem 3.1.2.** The group \( G(T(\Omega)) \) is generated by \( N^+, H \) and the element \( j \).

**Proof.** See [8], p.207

**Theorem 3.1.3.** The group \( G(T(\Omega)) \) is generated by \( N^+, H \) and the element \( j \_v \). Furthermore, the set \( N^+HN^- \) is open and dense in \( G(T(\Omega)) \).

**Proof.** Since, by Theorem 3.1.2, the group \( G(T(\Omega)) \) contains all products from \( N^+, H \) and \( j \), it will also contain all products from \( N^+, G(\Omega) \) and \( jN^+j \), i.e. \( N^+HN^- \). The fact that \( N^+HN^- \) is open and dense in \( G(T(\Omega)) \) is proved in [13], p.141.

Let \( g_t \) be a one-parameter subgroup of \( G \). Then, the map \( \tilde{X} : C^1(T(\Omega)) \rightarrow C^0(T(\Omega)) \)
defined by:

\[
\tilde{X}f(z) = \left. \frac{d}{dt}f(g_t(z)) \right|_{t=0}
\]
is vector field on \( T(\Omega) \). The set of all such vector fields, namely the set:

\[
\mathcal{V} = \left\{ \tilde{X} \mid \tilde{X}f(z) = \left. \frac{d}{dt}f(g_t(z)) \right|_{t=0}, \ f \in C^\infty(T(\Omega)) \right\}
\]
forms a real Lie algebra with the usual bracket. Since we can view a vector field \( \tilde{X} \), also, as a vector of a Lie algebra, we can always write:

\[
\tilde{X}f(z) = D_{X(z)}f(z),
\]
where \( X(z) \in V_C \). Hence, we can always identify the vector field \( \tilde{X} \) with the vector \( X(z) \).
Lemma 3.1.4. The vector fields associated to one-parameter subgroups of $N^+, H$ and $N^-$ are given, respectively, as follows:

1. $\tilde{X}f(z) = D_{iu}f(z), \ u \in V$
2. $\tilde{X}f(z) = D_{Tz}f(z), \ T \in \mathfrak{h}$
3. $\tilde{X}f(z) = D_{P(z)v}f(z), \ v \in V$

where $\mathfrak{h}$ is the Lie algebra of $H$ and $P$ is the quadratic representation of $V$.

Proof. See [14], p.76.

Let $g$ be the set of vectors $X(z)$ of the form $iu + Tz + iP(z)v$, where $u, v \in V$ and $T \in \mathfrak{h}$.

Theorem 3.1.5. The set $g$ is the Lie algebra of the group $G(T(\Omega))$.

Proof. See [8], p.211.

Let $\theta$ be a Cartan involution on $g$ defined by:

$\theta : g \rightarrow g$

$\theta(X) = -X^*$.

Let $\mathfrak{k} = \{X \in g \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in g \mid \theta(X) = -X\}$. Then, $g = \mathfrak{k} \oplus \mathfrak{p}$ and, consequently, $g_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$. Now, taking $X_0 \in Z(t_{\mathbb{C}})$, $\text{ad}(X_0) : g_{\mathbb{C}} \rightarrow g_{\mathbb{C}}$ has eigenvalues 0, 2, $-2$, where 0 corresponds to the subalgebra $\mathfrak{k}_{\mathbb{C}}$, and 2 and $-2$ correspond to the subalgebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. In other words, $g_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$

There is an involution $\tau$, that commutes with $\theta$, with which one can produce a similar decomposition of $g_{\mathbb{C}}$, taking the central element this time from $\mathfrak{h}_{\mathbb{C}}$ instead of $\mathfrak{k}_{\mathbb{C}}$. So, taking $Z_0 \in Z(h_{\mathbb{C}})$, $\text{ad}(Z_0) : g_{\mathbb{C}} \rightarrow g_{\mathbb{C}}$ has eigenvalues 0, 2, $-2$, where 0 corresponds
to the subalgebra $h_\mathbb{C}$, and 2 and $-2$ correspond to the subalgebras $q^+$ and $q^-$. Hence, $g_\mathbb{C} = h_\mathbb{C} \oplus q^+ \oplus q^-$. One can also show, that the subalgebras $q^+$ and $q^-$ are both isomorphic to $V$.

**Example 3.1.6.** Let $G \cong P^*\text{Sp}(2n, \mathbb{R})P$, where $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Then for the Lie algebra we have $g = P^*\text{sp}(2n, \mathbb{R})P$, and hence $g$ is given by:

$$g = \left\{ \begin{pmatrix} a & ib \\ -ic & -a^t \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b, c \text{ real}, b = b^t, c = c^t \right\}.$$ 

The involution $\theta$ is given by:

$$\theta : g \rightarrow g$$

$$\theta(X) = -X^*.$$ 

Let $\mathfrak{k} = \{ X \in g \mid \theta(X) = X \}$ and $\mathfrak{p} = \{ X \in g \mid \theta(X) = -X \}$. Then,

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = -a^t, b = b^t, \text{tr}(a) = 0 \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & ib \\ -ib & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ real}, a = a^t, b = b^t \right\}.$$ 

Their complexification gives:

$$\mathfrak{k}_\mathbb{C} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = -a^t, b = b^t, \text{tr}(a) = 0 \right\}$$

and
\[
p_C = \left\{ \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a, b \text{ complex}, a = a^t, b = b^t \right\}.
\]

Hence, \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) and, consequently, \( \mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_C \). Taking \( X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in Z(\mathfrak{k}_C) \), \( \text{ad}(X_0) : \mathfrak{g}_C \longrightarrow \mathfrak{g}_C \) has eigenvalues 0, 2, −2. The 0 eigenvalue corresponds to the subalgebra \( \mathfrak{k}_C \), and the eigenvalues 2 and −2 correspond to the subalgebras \( \mathfrak{p}^+ \) and \( \mathfrak{p}^- \) that are are given by:

\[
p^+ = \left\{ \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \in \mathfrak{g}_C \mid a = a^t \right\} \quad \text{and} \quad p^- = \left\{ \begin{pmatrix} a & -a \\ a & -a \end{pmatrix} \in \mathfrak{g}_C \mid a = a^t \right\}.
\]

Observe also that \( \mathfrak{p}^+ \cong V_C \), by the map \( \begin{pmatrix} a & a \\ -a & -a \end{pmatrix} \longmapsto a \). Let \( \tau \) be an involution given by \( \tau(X) = c^2 X c^2 \), where \( c = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) is the Caley transform. That is \( \tau(X) = c^2 X c^2 = \begin{pmatrix} a & -ib \\ ic \ & -a^t \end{pmatrix} \). Let \( \mathfrak{h} = \{ X \in \mathfrak{g} \mid \tau(X) = X \} \) and \( \mathfrak{q} = \{ X \in \mathfrak{g} \mid \tau(X) = -X \} \).

Then,

\[
\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid a \text{ real} \right\}.
\]

and

\[
\mathfrak{q} = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \mathfrak{sl}(2n, \mathbb{C}) \mid b, c \text{ imaginary} \right\}.
\]
Taking \( Z_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) \( \in \mathcal{Z}(\mathfrak{h}_\mathbb{C}) \), \( \text{ad}(Z_0) : \mathfrak{g}_\mathbb{C} \rightarrow \mathfrak{g}_\mathbb{C} \) has eigenvalues 0, 2, -2. The 0 eigenvalue corresponds to the subalgebra \( \mathfrak{h}_\mathbb{C} \), and the eigenvalues 2 and -2 correspond to the subalgebras \( \mathfrak{q}^+ \) and \( \mathfrak{q}^- \), which are given by:

\[
\mathfrak{q}^+ = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_\mathbb{C} \mid a = a' \right\} \quad \text{and} \quad \mathfrak{q}^- = \left\{ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathfrak{g}_\mathbb{C} \mid a = a' \right\}.
\]

Finally, observe that \( \mathfrak{q}^+ \cong V_\mathbb{C} \), by the map \( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mapsto a \).

### 3.2 Representations of \( \mathfrak{g}_\mathbb{C} \) on \( H_\nu(T(\Omega)) \)

In this section we want to build representations of \( G(T(\Omega)) \), and its Lie Algebra \( g \), on a Hilbert space of holomorphic functions on \( T(\Omega) \).

Consider the following Hilbert space of holomorphic functions:

\[
\mathcal{H}_\nu(T(\Omega)) = \left\{ F : T(\Omega) \rightarrow \mathbb{C}, \|F\|^2 < \infty, \nu \in \mathbb{R} \right\},
\]

where

\[
\|F\|^2 = \alpha_\nu \int_{T(\Omega)} |F(x + iy)|^2 \Delta(x)^{\nu-p} \, dx dy,
\]

with \( \Delta(x) = \det(x) \) and \( \alpha_\nu = \frac{2^{\nu}}{(4\pi)^{p/2} \Gamma(\nu - \frac{p}{2})} \). Recall that in general:

\[
\Gamma_\Omega(m) = \int_{\Omega} e^{-t(x)} \Delta_m(x) \Delta(x)^{-\frac{d}{2}} \, dx
\]

and

\[
\Delta_m(x) = \Delta_1^{m_1-m_2}(x) \Delta_2^{m_2-m_3}(x) \cdots \Delta_n^{m_n}(x).
\]
In addition, the space $H_\nu(T(\Omega)) \neq \{0\}$ if and only if $\nu \geq p$. For $\nu = p$, we get the Bergman space.

Now, the representation of $G(T(\Omega))$ on $H_\nu(T(\Omega))$ is given by a multiplier representation as follows:

$$\pi_\nu(g)F(z) = J(g^{-1}, z) F(g^{-1} \cdot z),$$

where $J(g, z)$ is complex Jacobian of the action of $G(T(\Omega))$ on $T(\Omega)$, i.e. $J(g, z) = \det D(g \cdot z)$, and $p = \frac{2d}{r}$. The Lie algebra representation is given as follows:

$$\pi_\nu(X)F(z) = \frac{d}{dt}\pi_\nu(\exp(tX))F(z)|_{t=0} = \frac{d}{dt}J(\exp(-tX), z) F(\exp(-tX) \cdot z)|_{t=0}$$

for all $z \in T(\Omega)$. One can extend this action by complex linearity to $g_C$.

Define $D_w$ by:

$$D_wF(z) = \frac{d}{dt}F(z + tw)|_{t=0} = F'(z)w,$$

where $F'$ denotes the derivative of $F$.

**Lemma 3.2.1.** On the generators $\tau_u$, $h$ and $j_v$ of $G(T(\Omega))$, the complex Jacobian $J(g, z)$ is given by:

1. $J(\tau_u, z) = 1$
2. $J(h, z) = \det h$
3. $J(j_v, z) = \det(e + izv)^{-p}$

**Proof.** The generators $\tau_u$, $h$ and $j_v$ of the group $G(T(\Omega))$, act on $T(\Omega)$ as follows:

$$\tau_u \cdot z = \tau_u(z) = z + iu$$
\[ h \cdot z = h(z) = hz \]
\[ j_v \cdot z = j_v(z) = (z^{-1} + iv)^{-1}. \]

(1) For the first generator, we have:

\[
D(\tau_u \cdot z)(w) = \frac{d}{dt}(\tau_u \cdot (z + tw))|_{t=0} = \frac{d}{dt}(z + tw + iu)|_{t=0} = w,
\]
which says that \( D(\tau_u \cdot z) = Id \). Hence, \( J(\tau_u, z) = \det D(\tau_u \cdot z) = \det(Id) = 1 \).

(2) Recall first that, by the definition of \( H \), any \( h \in H \) is an element of \( GL(T(\Omega)) \) because \( h(x + iy) = h(x) + ih(y) \) and \( h(x) \in \Omega \). In other words, \( h \) is linear and acts on \( T(\Omega) \). Now,

\[
D(h \cdot z)(w) = \frac{d}{dt}(h \cdot (z + tw))|_{t=0} = \frac{d}{dt}(h(z + tw))|_{t=0} = \frac{d}{dt}(hz + thw)|_{t=0} = hw,
\]
which implies that \( D(h \cdot z) = h \). Hence, \( J(h, z) = \det D(h \cdot z) = \det h \).

(3) For the third generator, we have:

\[
D(j_v \cdot z)(w) = \frac{d}{dt}(j_v \cdot (z + tw))|_{t=0} = \frac{d}{dt}((z + tw)^{-1} + iv)^{-1}|_{t=0} = \frac{d}{dt}[-P((z + tw)^{-1} + iv)|_{t=0} -1([-P(z + tw)|_{t=0}]^{-1}w - P(z^{-1} + iv)^{-1}P(z)^{-1}w = P(e + izv)^{-1}w.
\]

Hence, \( D(j_v \cdot z) = P(e + izv)^{-1} \). Therefore, \( J(j_v, z) = \det D(j_v \cdot z) = \det P(e + izv)^{-1} = \det(e + izv)^{-p} \), by Prop. III.4.2, p.52 in [8].
The following proposition expresses the relevant formulas on \( \mathfrak{h}, \mathfrak{q}^+ \) and \( \mathfrak{q}^- \). Its proof is a straightforward calculation using Lemma 3.2.1.

**Proposition 3.2.2.** For each piece of the Lie algebra of \( \mathfrak{g}_C \) we have:

1. \( \pi_{\nu}(T)F(z) = -\frac{\nu}{p} \text{tr}(T)F(z) - D_{Tz}F(z), \quad T \in \mathfrak{h}_C \)
2. \( \pi_{\nu}(u)F(z) = -D_{iu}F(z), \quad u \in \mathfrak{q}^+ \)
3. \( \pi_{\nu}(v)F(z) = i\nu \text{tr}(zv)F(z) + D_{P(z)i}F(z), \quad v \in \mathfrak{q}^- \)

**Proof.** Recall that the representation is given by:

\[
\pi_{\nu}(g)F(z) = J(g^{-1},z)\frac{\nu}{p}F(g^{-1} \cdot z), \quad \text{where } J(g^{-1},z) = \det D(g^{-1} \cdot z).
\]

Notice that \( \tau_{u}^{-1} \cdot z = z - iu, \quad h^{-1} \cdot z = h^{-1}z \) and \( j_{v}^{-1} \cdot z = (z^{-1} - iv)^{-1} \) respectively.

Case (1): Let \( \gamma_t = \exp(tT) \) be a one-parameter subgroup of \( H \), where \( T \in \mathfrak{h} \), that acts on \( T(\Omega) \) by \( \gamma_t(z) = \exp(tT)z \). Then, the Lie algebra action is given by:

\[
\pi_{\nu}(T)F(z) = \left. \frac{d}{dt} \pi_{\nu}(\gamma_t)F(z) \right|_{t=0} = \left. \frac{d}{dt} \left[ J(\gamma_t^{-1},z)\frac{\nu}{p}F(\gamma_t^{-1} \cdot z) \right] \right|_{t=0}
\]

\[
= \left. \frac{d}{dt} \left[ \det(\exp(-tT))\frac{\nu}{p}F(\exp(-tT)z) \right] \right|_{t=0}, \quad \text{by Lemma 3.2.1(2)}
\]

\[
= \frac{\nu}{p} \det(e)^{\frac{\nu}{p}-1} \text{det}'(\exp(-tT))\left|_{t=0}F(z) + \det(e)^{\frac{\nu}{p}}F'(z)[-Tz]
\]

\[
= \frac{\nu}{p} \text{tr}(-T)F(z) + D_{-Tz}F(z)
\]

\[
= -\frac{\nu}{p} \text{tr}(T)F(z) - D_{Tz}F(z).
\]

Case (2): Similarly, let \( \gamma_t \) be a one-parameter subgroup of \( \mathfrak{q}^+ \), with \( u \in V \), described by the action \( \gamma_t(z) = z + tu \) on \( T(\Omega) \). Then, the Lie algebra action is given by:
\[ \pi_\nu(u)F(z) = \frac{d}{dt} [\pi_\nu(\gamma_t)F(z)]|_{t=0} \]
\[ = \frac{d}{dt} [J(\gamma_t^{-1}, z)^{\nu/p}F(\gamma_t^{-1}(z) \cdot z)]|_{t=0} \]
\[ = \frac{d}{dt} [(1)^{\nu/p}F(z - t i v)]|_{t=0}, \text{ by Lemma 3.2.1(1)} \]
\[ = \frac{d}{dt} F(z - t i v)|_{t=0} \]
\[ = F'(z)[-i v] \]
\[ = D_{-i v}F(z). \]

Case (3): Finally, let \( \gamma_t \) be a one-parameter subgroup of \( Q^- \), where \( v \in V \), that acts on \( T(\Omega) \) by \( \gamma_t(z) = (z^{-1} + t i v)^{-1} \). Then, the Lie algebra action is given by:

\[ \pi_\nu(v)F(z) = \frac{d}{dt} [\pi_\nu(\gamma_t)F(z)]|_{t=0} \]
\[ = \frac{d}{dt} [J(\gamma_t^{-1} \cdot z)^{\nu/p}F(\gamma_t^{-1}(z) \cdot z)]|_{t=0} \]
\[ = \frac{d}{dt} [\det(e - t i z v)^{-\nu/p}F((z^{-1} - t i v)^{-1})]|_{t=0}, \text{ by Lemma 3.2.1(3)} \]
\[ = -\nu \det(e - t i z v)^{-\nu-1}|_{t=0} \det(e - t i z v)|_{t=0} F((z^{-1} - t i v)^{-1})|_{t=0} \]
\[ + \det(e - t i z v)^{-\nu}|_{t=0} F((z^{-1} - t i v)^{-1})'|_{t=0} \]
\[ = \nu \det(e)^{-\nu-1}|_{t=0} \text{tr}(i z v)F(z) + \det(e)^{-\nu}F'(z)((z^{-1} - t i v)^{-1})'|_{t=0} \]
\[ = i \nu \text{tr}(z v)F(z) + F'(z)[P(z^{-1} - t i v)]^{-1}i v|_{t=0} \]
\[ = i \nu \text{tr}(z v)F(z) + F'(z)[P(z) i v] \]
\[ = i \nu \text{tr}(z v)F(z) + D_{P(z)i v}F(z). \]

\[ \square \]

**Example 3.2.3.** In [1] we calculated the action of each piece, with respect to \( \theta \), of the Lie algebra \( \mathfrak{sp}(2n, \mathbb{R})_C \) over \( T(\Omega) = Sym^+(n, \mathbb{R}) + i Sym(n, \mathbb{R}) \). Now, we calculate the
action of each piece of \( \mathfrak{sp}(2n, \mathbb{R})_C \) with respect to \( \tau \). We claim that the action is given as follows:

(1) \( \pi_\nu(X)F(z) = D_{-a}F(z), \quad X = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{q}^+ \)

(2) \( \pi_\nu(X)F(z) = \nu \text{tr}(az)F(z) + D_{P(z)a}F(z), \quad X = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \in \mathfrak{q}^- \)

(3) \( \pi_\nu(X)F(z) = -\nu \text{tr}(a)F(z) - D_{az+za^t}F(z), \quad X = \begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix} \in \mathfrak{h}_C \)

**Proof.** Consider each case:

Case (1): Let \( X \in \mathfrak{q}^+ \). Then, \( X = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \). Now, as \( X^n = 0 \), for \( n \geq 2 \), we have:

\[
\exp(-tX) = \begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix}.
\]

As \( J(g, z) = \det(Cz + D)^{-(n+1)} \), whenever \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), we also have:

\[
J(\exp(-tX), z) = \det(0z + 1)^{-(n+1)} = 1.
\]

Hence,

\[
\pi_\nu(X)F(z) = \frac{d}{dt} J(\exp(-tX), z)^{\frac{x}{\nu+1}} F(\exp(-tX) \cdot z)|_{t=0} = \frac{d}{dt} F(z - ta)|_{t=0}
\]

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\[ F'(z)[-a] = D_{-a}F(z). \]

Case (2): Let \( X \in \mathfrak{q}^- \). Then, \( X = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \). Again, since \( X^n = 0 \), for \( n \geq 2 \), we have:

\[
\exp(-tX) = \begin{pmatrix} 1 & 0 \\ -ta & 1 \end{pmatrix}.
\]

Furthermore,

\[
J(\exp(-tX), z) = \det(-taz + 1)^{-(n+1)},
\]

so we have:

\[
\pi_\nu(X)F(z) = \frac{d}{dt}J(\exp(-tX), z)\pi_\nu F(\exp(-tX) \cdot z)|_{t=0}
\]

\[
= \frac{d}{dt} \left[ \det(-taz + 1)^{-\nu}F \left( \frac{z}{-taz + 1} \right) \right]_{t=0}
\]

\[
= -\nu \det(1)^{-\nu-1} \det'(1-taz)F \left( \frac{z}{-taz + 1} \right) |_{t=0}
\]

\[
+ \det(1)^{-\nu}F'(z) \left[ \frac{z}{-taz + 1} \right]' |_{t=0}
\]

\[
= -\nu \text{tr}(-az)F(z) + F'(z)[zaz]
\]

\[
= \nu \text{tr}(az)F(z) + D_{P(z)a}F(z), \quad \text{by Prop.1.2.18(a)}.
\]

Case (3): Let \( X \in \mathfrak{h}_C \). Then, \( X = \begin{pmatrix} a & 0 \\ 0 & -a' \end{pmatrix} \), and for \( \exp(-tX) \) we have:

\[
\exp(-sX) = \begin{pmatrix} 1 - sa + \frac{s^2}{2!}a^2 - ... & 0 \\ 0 & 1 + sa' + \frac{s^2}{2!}(a')^2 + ... \end{pmatrix}.
\]
Finally,

\[ J(\exp(-sX), z) = \det(1 + sa + \frac{s^2}{2!}(a')^2 + ...)^{-(n+1)}, \]

so:

\[
\pi_\nu(X)F(z) = \frac{d}{ds} J(\exp(-sX), z) \pi_\nu^\nu F(\exp(-sX) \cdot z)|_{s=0} \\
= \frac{d}{ds} \left[ \det(1 + sa + \frac{s^2}{2!}(a')^2 + ...)^{-\nu} F\left( \frac{(1 - sa + \frac{s^2}{2}a^2 - ...)}{1 + sa' + \frac{s^2}{2}(a')^2 + ...} \right) \right]_{s=0} \\
= -\nu \det(1)^{-\nu-1} \det'(1 + sa')|_{s=0} F(z) \\
+ \det(1)^{-\nu} F'(z) \left[ \frac{(1 - sa + \frac{s^2}{2}a^2 - ...)}{1 + sa' + \frac{s^2}{2}(a')^2 + ...} \right]'_{s=0} \\
= -\nu \text{tr}(a') F(z) + F'(z)[-az - za'] \\
= -\nu \text{tr}(a) F(z) + D_{-(az + za')} F(z), \quad \text{as } \text{tr}(a') = \text{tr}(a). 
\]

\[
\square
\]

### 3.3 Representations of \( \mathfrak{g}_C \) on \( L^2(\Omega, d\mu_\nu) \)

In the previous section we have seen representations of \( \mathfrak{g} \) on \( \mathcal{H}_\nu(T(\Omega)) \). In this section we want to build representations of \( \mathfrak{g} \) on \( L^2(\Omega, d\mu_\nu) \). We will actually transfer the previous representations that are on \( \mathcal{H}_\nu(T(\Omega)) \) to \( L^2(\Omega, d\mu_\nu) \) with the help of the Laplace transform.

#### 3.3.1 Preliminaries

Let \( \{c_1, c_2, ..., c_r\} \) be a Jordan frame and \( V = \bigoplus_{i \leq j} V_{ij}, \ 1 \leq i \leq j \leq d \) the Pierce decomposition (see [8], p.68). Let \( \dim V_{ii} = 1 \) and \( \dim V_{ij} = s, \ i < j \). For \( i < j \), choose an orthonormal basis \( \{E_{ij}^1, E_{ij}^2, ..., E_{ij}^a\} \) for \( V_{ij} \), and since \( V_{ii} = \mathbb{R}c_i \) let \( \{c_1, c_2, ..., c_r\} \) be an orthonormal basis of \( V_{ii} \). Then, \( \{c_1, c_2, ..., c_r\} \cup \bigcup_{i < j} \{E_{ij}^1, E_{ij}^2, ..., E_{ij}^a\} \), denote it by

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\{e_1, e_2, ..., e_d\}, is an orthonormal basis of \(V\).

**Lemma 3.3.1.** Let \(\{e_i\}\) be an orthonormal basis of a Jordan algebra \(V\), with identity element \(e\). Then, the following are true:

1. \(\sum_i e_i^2 = \frac{d}{r} e\)
2. \(\sum_i (P(x, e_i) e_i | y) = \frac{d}{r} (x | y), \ \forall x, y \in V\)
3. If \(x \in V\) is invertible, then \(\sum_{i,j} (z | e_i) (P(e_i, e_j) y | x^{-1} e_j) = (z | y), \ \forall z \in V_C\)
4. \(P(z) = \sum_i (z | e_i) (z | e_j) P(e_i, e_j), \ \forall z \in V_C\).

**Proof.** (1) Let \(k \in K \subset O(V)\). Then,

\[
k(\sum_i e_i^2) = \sum_i k(e_i^2) = \sum_i ke_i ke_i
\]

\[
= \sum_{i,j,l} (ke_i | e_j)(ke_i | e_l)e_l
\]

\[
= \sum_{i,j,l} (ke_i | e_j)(ke_i | e_l)e_j e_l
\]

\[
= \sum_{j,l} (ke_j)(ke_l)e_j e_l, \ \text{by Parseval identity}
\]

\[
= \sum_{j,l} (e_j)(e_l)e_j e_l, \ \text{because } k \text{ is unitary}
\]

\[
= \sum_{j,l} \delta_{jl} e_j e_l
\]

\[
= \sum_j e_j^2,
\]

which implies \(\sum_i e_i^2 = \lambda e\), by Prop.III(4.1) in [8], p.51. We now calculate \(\lambda\). Taking the inner product with \(e\) in both sides, we have:

\[
(\sum_i e_i^2|e) = (\lambda e|e).
\]

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Since \((\sum_i e_i^2 | e) = d\) and \((e | e) = \text{tr}(e) = r\), the equation above gives:

\[
\lambda = \frac{d}{r}.
\]

Substituting \(\lambda\) in \(\sum_i e_i^2 = \lambda e\), we get \(\sum_i e_i^2 = \frac{d}{r} e\).

(2) With similar calculations we get,

\[
\begin{align*}
\sum_i (P(x, e_i) e_i | y) &= \sum_i (P(x, e_i) e_i | y) \\
&= \sum_i (x e_i^2 | y) \\
&= (x \sum_i e_i^2 | y) \\
&= (x - e | y) \\
&= \frac{d}{r} (x | y).
\end{align*}
\]

(3) Similarly we have,

\[
\begin{align*}
\sum_{i, j} (z | e_i) (P(e_i, e_j) y|x)(x^{-1} | e_j) &= \sum_i (z | e_i) (P(e_i, \sum_j e_j (x^{-1} | e_j)) y|x) \\
&= \sum_i (z | e_i) (P(e_i, x^{-1}) y|x) \\
&= (P(\sum_i (z | e_i) e_i, x^{-1}) y|x) \\
&= (P(z, x^{-1}) y|x) \\
&= (y | P(z, x^{-1}) x) \\
&= (y | P(z, e) \\
&= (y | z).
\end{align*}
\]

(4) Finally,
\[ P(z) = P(z, z) = P\left(\sum_i (z|e_i)e_i, \sum_j (z|e_j)e_j\right) = \sum_{i,j} (z|e_i)(z|e_j)P(e_i, e_j). \]

Recall Prop.2.4.4 which stated that for \( f, g \in L^2(\Omega, d\mu_\nu) \) smooth, with \( f \) vanishing on \( \Omega \), the following relations hold:

\[
\int_{\Omega} D_\nu f(s)g(s)ds = -\int_{\Omega} f(s)D_\nu g(s)ds \\
\int_{\Omega} e^{-\langle z|s \rangle} (z|v)f(s)ds = \int_{\Omega} e^{-\langle z|s \rangle} D_\nu f(s)ds
\]

Finally, consider the space \( L^2(\Omega, d\mu_\nu) \), where \( d\mu_\nu(x) = \Delta(x)^{\nu-\frac{d}{2}} \). Notice that \( d\mu_\nu(x) \) generalizes the measure \( x^{\nu-1}dx \) on \( \mathbb{R}^+ \). The Laplace transform is defined by:

\[
\mathcal{L}_\nu(f)(z) = \int_{\Omega} e^{-\langle z|x \rangle} f(x) d\mu_\nu(x) \\
= \int_{\Omega} e^{-\langle z|x \rangle} f(x)\Delta(x)^m dx,
\]

where \( m = \nu - \frac{d}{r} \). Recall that \( \mathcal{L}_\nu \) is a unitary isomorphism between \( L^2(\Omega, d\mu_\nu) \) and \( \mathcal{H}_\nu(T(\Omega)) \).

### 3.3.2 Transferred Representations

We transfer now the representations of \( g \) on \( \mathcal{H}_\nu(T(\Omega)) \) to \( L^2(\Omega, d\mu_\nu) \). This can be done via:

\[
\lambda_\nu(X)f(x) = (\mathcal{L}_\nu^* \pi_\nu(X)\mathcal{L}_\nu)f(x). \tag{3.1}
\]
Recall that we determined the action of $\mathfrak{h}_C$, $\mathfrak{q}^+$ and $\mathfrak{q}^-$ on $\mathcal{H}_\nu(T(\Omega))^\infty$ in Proposition 3.2.2. We denote the subspace of smooth vectors in $L^2_\nu(\Omega, d\mu_\nu)$ by $L^2_\nu(\Omega)^\infty$. Thus $f \in L^2_\nu(\Omega)^\infty$ if and only if the map

$$\mathbb{R} \ni t \mapsto \lambda_\nu(\exp tX)f \in L^2_\nu(\Omega, d\mu_\nu)$$

is smooth for all $X \in \mathfrak{g}$. Thus,

$$L^2_\nu(\Omega)^\infty = \mathcal{L}^{-1}_\nu(\mathcal{H}_\nu(T(\Omega))^\infty).$$

The action of $\mathfrak{g}$ on $L^2_\nu(\Omega)^\infty$ is, as usual, defined by:

$$\lambda_\nu(X)f(x) = \frac{d}{dt}\lambda_\nu(\exp(tX))f(x)|_{t=0}, \forall X \in \mathfrak{g},$$

where by complex linearity the action extends to $\mathfrak{g}_C$. The following theorem expresses the corresponding action of $\mathfrak{g}_C$ on $L^2_\nu(\Omega)^\infty$.

**Theorem 3.3.2.** For $f \in L^2_\nu(\Omega)^\infty$, we have:

1. $\lambda_\nu(T)f(x) = \frac{\nu}{p}\text{tr}(T)f(x) + D_{Tx}f(x), \quad T \in \mathfrak{h}_C$
2. $\lambda_\nu(u)f(x) = \text{tr}(iu x)f(x), \quad u \in \mathfrak{q}^+$
3. $\lambda_\nu(v)f(x) = -i\nu D_v f(x) - i \sum_{i,j} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(x), \quad v \in \mathfrak{q}^-$

**Proof.** We prove the Theorem case by case:

Case (1): Let $T \in \mathfrak{h}_C$. By Proposition 3.2.2 we know how the Lie algebra acts on functions of $\mathcal{H}_\nu(T(\Omega))$. So, at first, it is better if we calculate the part $\pi_\nu(T)\mathcal{L}_\nu f(z)$, $T \in \mathfrak{h}$, of equation (3.1) as $\mathcal{L}_\nu f(z) \in \mathcal{H}_\nu(T(\Omega))$. Hence,
\[ \pi_{\nu}(T)\mathcal{L}_{\nu}f(z) = -\frac{\nu}{p} \text{tr}(T)\mathcal{L}_{\nu}f(z) - D_{Tz}\mathcal{L}_{\nu}f(z), \quad T \in \mathfrak{h}_{\mathbb{C}}. \]

We calculate \( D_{Tz}\mathcal{L}_{\nu}f(z) \) first:

\[
D_{Tz}\mathcal{L}_{\nu}f(z) = D_{Tz} \int_{\Omega} e^{-(z|x)}f(x)\Delta(x)^m dx \\
= \frac{d}{ds} \int_{\Omega} e^{-(z+Ts|x)}f(x)\Delta(x)^m dx |_{s=0} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z+Ts|x)}f(x)\Delta(x)^m dx |_{s=0} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z+sT^2+\frac{s^2}{2}T^2+\ldots|x)}f(x)\Delta(x)^m dx |_{s=0} \\
= \frac{d}{ds} \int_{\Omega} e^{-(e^{sT^2}x)|x}f(x)\Delta(x)^m dx |_{s=0} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z|e^{sT^2}x)}f(x)\Delta(x)^m dx |_{s=0} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z|e^{sT^2}x)}f(x)\Delta(x)^m dx |_{s=0}, \quad \text{because } m = \nu - \frac{d}{r} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z|e^{sT^2}x)}f(e^{-sT^2}x)\Delta(e^{-sT^2}x)^\nu\Delta(e^{-sT^2}x)^{-\frac{d}{r}}d(e^{-sT^2}x) |_{s=0}, \quad \text{by letting } x \to e^{-sT^2}x \\
= \frac{d}{ds} \int_{\Omega} e^{-(z|x)}f(e^{-sT^2}x)\Delta(e^{-sT^2}x)^\nu\Delta(x)^{-\frac{d}{r}}dx |_{s=0}, \quad \text{because } \Delta(x)^{-\frac{d}{r}}dx \\
is \ H^- \text{-invariant} \\
= \frac{d}{ds} \int_{\Omega} e^{-(z|x)}f(e^{-sT^2}x)\text{Det}(e^{-sT^2})^{\frac{\nu}{2}}\Delta(x)^\nu\Delta(x)^{-\frac{d}{r}}dx |_{s=0}, \quad \text{by Prop. III(4.3) in [8], p.53} \\
= \int_{\Omega} e^{-(z|x)}f(x)[-T^t x]\text{Det}(I)^{\frac{\nu}{2}}\Delta(x)^m dx \\
+ \int_{\Omega} e^{-(z|x)}f(x)\frac{d}{\nu}\text{Det}(I)^{\frac{\nu}{2}+1}\text{Det'}(e^{-sT^2})\Delta(x)^m dx \\
= \int_{\Omega} e^{-(z|x)}D_{-T^tx}f(x)\Delta(x)^m dx + \int_{\Omega} e^{-(z|x)}f(x)\frac{2\nu}{p}\text{Tr}(-T^t)\Delta(x)^m dx, \\
as \ p = \frac{2d}{r} \\
= - \int_{\Omega} e^{-(z|x)}D_{T^tx}f(x)\Delta(x)^m dx - \frac{2\nu}{p} \int_{\Omega} e^{-(z|x)}f(x)\text{Tr}(T^t)\Delta(x)^m dx
\[ - \mathcal{L}(D_{T_t}x f(z)) - \frac{2\nu}{p} \text{Tr}(T) \mathcal{L} f(z), \quad \text{as } \text{Tr}(T^t) = \text{Tr}(T). \]

Hence,

\[
\pi_\nu(T) f(x) = -\frac{\nu}{p} \text{Tr}(T) \mathcal{L} f(z) + \frac{2\nu}{p} \text{Tr}(T) \mathcal{L} f(z) + \mathcal{L}(D_{T_t}x f(z)) \]

Taking now \( \mathcal{L}_\nu^* \) in both sides, and considering (3.1), we get:

\[
\lambda_\nu(T) f(x) = \frac{\nu}{p} \text{tr}(T) f(x) + D_{T_t}x f(x), \quad T \in \mathfrak{h}_\mathbb{C}
\]

Case (2): Let \( u \in \mathfrak{q}^+ \). Just like case (1), we know how the Lie algebra acts on functions of \( \mathcal{H}_\nu(T(\Omega)) \) by Proposition 3.2.2. We calculate the part \( \pi_\nu(X) \mathcal{L}_\nu f(z), u \in \mathfrak{q}^+ \), of equation (3.1) since \( \mathcal{L}_\nu f(z) \in \mathcal{H}_\nu(T(\Omega)) \). We have,

\[
\pi_\nu(u) \mathcal{L}_\nu f(z) = D_{-iu} \mathcal{L}_\nu f(z) = -\int_{\Omega} D_{iu} \left[ e^{-\langle z|x \rangle} f(x) \Delta^m(x) \right] dx
\]

Taking now \( \mathcal{L}_\nu^* \) in both sides, and considering (3.1), we get:

\[
\lambda_\nu(u) f(x) = \text{tr}(iu x) f(x), \quad u \in \mathfrak{q}^+.
\]

Case (3): Let \( v \in \mathfrak{q}^- \). Just like cases (1) and (2), we calculate the part \( \pi_\nu(X) \mathcal{L}_\nu f(z) \),
\[ v \in q^-, \text{ of equation (3.1) since } L_{\nu} f(z) \in H_{\nu}(T(\Omega)), \text{ and then take } L^\ast. \text{ We have,} \]

\[ \pi_{\nu}(v) L_{\nu} f(z) = i \nu \text{tr}(zv) L_{\nu} f(z) + D_{P(z)iv} L_{\nu} f(z), \quad v \in q^- . \]

We calculate \( D_{P(z)iv} L_{\nu} f(z), \quad v \in q^- . \text{ We have} \]

\[
D_{P(z)iv} L_{\nu} f(z) = D_{P(z)iv} \int_{\Omega} e^{-(z|x)} f(x) \Delta^m(x) dx \\
= \int_{\Omega} [D_{P(z)iv}] f(x) \Delta^m(x) dx \\
= - \int_{\Omega} e^{-(z|x)} (P(z) iv|x) f(x) \Delta^m(x) dx \\
= -i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(z|e_j)(P(e_i, e_j)v|x) f(x) \Delta^m(x) dx, \quad \text{by Lemma 3.3.1 (4)} \\
= -i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i) D_{e_j} [(P(e_i, e_j)v|x) f(x) \Delta^m(x)] dx, \quad \text{by Prop. 2.4.4 (2)} \\
= -i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(P(e_i, e_j)v|e_j) f(x) \Delta^m(x) dx \\
- i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)[(P(e_i, e_j)v|x)D_{e_j} f(x)] \Delta^m(x) dx \\
- i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(P(e_i, e_j)v|x) f(x) D_{e_j} \Delta^m(x) dx \\
= -i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(P(e_i, e_j)e_j|v) f(x) \Delta^m(x) dx \\
- i \int_{\Omega} \int e^{-(z|x)} (z|e_i)(P(e_i, e_j)v|x)D_{e_j} f(x) \Delta^m(x) dx \\
- i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(P(e_i, e_j)v|x)f(x)m(x^{-1}|e_i)\Delta^m(x) dx, \\
\text{by (\ast) in p.56} \\
= -i \frac{d}{r} (z|v) \int_{\Omega} e^{-(z|x)} f(x) \Delta^m(x) dx \\
- i \sum_{i,j} \int_{\Omega} e^{-(z|x)} (z|e_i)(P(e_i, e_j)v|x)[D_{e_j} f(x)] \Delta^m(x) dx \\
\]

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\[-im(z|v) \int_\Omega e^{-(z|x)} f(x) \Delta^m(x) dx, \text{ by Lemma 3.3.1 (3)}\]
\[= -i(\frac{d}{r} + m) \text{tr}(zv) \int_\Omega e^{-(z|x)} f(x) \Delta^m(x) dx \]
\[= -i \sum_{i,j} \int_\Omega e^{-(z|x)} D_{e_i}[(P(e_i, e_j)v|x)D_{e_j}f(x)\Delta^m(x)] dx\]
\[= \nu \text{tr}(zv) \int_\Omega e^{-(z|x)} f(x) \Delta^m(x) dx \]
\[-i \sum_{i,j} \int_\Omega e^{-(z|x)} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(x)\Delta^m(x) dx\]
\[= -i \nu \text{tr}(zv) \int_\Omega e^{-(z|x)} f(x) \Delta^m(x) dx - i \nu \int_\Omega e^{-(z|x)} D_vf(x)\Delta^m(x) dx \]
\[-i \sum_{i,j} \int_\Omega e^{-(z|x)} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(x)\Delta^m(x) dx\]

Hence,

\[\pi_\nu(v) f(x) = i \nu \text{tr}(zv) \mathcal{L} f(z) - i \nu \text{tr}(zv) \mathcal{L} f(z) - i \nu \mathcal{L}(D_vf(z)) \]
\[= -i \mathcal{L} \sum_{i,j} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(z) \]
\[= -i \nu \mathcal{L}(D_vf(z)) - i \mathcal{L} \sum_{i,j} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(z) \]

Taking now $\mathcal{L}_\nu^*$ in both sides, and considering (3.1), we get:

\[\lambda_\nu(v) f(x) = -i \nu D_vf(x) - i \sum_{i,j} (P(e_i, e_j)v|x)D_{e_i}D_{e_j}f(x), \quad v \in q^-\]

\[\square\]

**Remark 3.3.3.** The gradient of $f$ is defined as $(\nabla f(x)|u) = D_u f(x)$, for $u \in V$, which actually implies that $D_u f(x) = \text{tr}(u \nabla) f(x)$. Let also the sum in Theorem 3.3.2(3) be
denoted by \( \text{tr}(v \nabla x \nabla) \). Considering now these observations, we can write Theorem 3.3.2 in a more elegant form as the following Corollary suggests.

**Corollary 3.3.4.** For a smooth function \( f \in L^2_\nu(\Omega) \) we have:

1. \( \lambda_\nu(T) f(x) = \text{tr}(\nu T + T' x \nabla) f(x), \quad T \in \mathfrak{h}_C \)
2. \( \lambda_\nu(u) f(x) = i \text{tr}(u x) f(x), \quad u \in \mathfrak{q}^+ \)
3. \( \lambda_\nu(v) f(x) = -i \text{tr}(\nu v \nabla + v \nabla x \nabla) f(x), \quad v \in \mathfrak{q}^- \)

### 3.4 General Recursion Relations

In this section, we will decompose the Lie algebra \( \mathfrak{g} \) into the spaces \( \mathfrak{k}, \mathfrak{p}^+ \) and \( \mathfrak{p}^- \) and show how the action of the \( L \)-invariant basic elements of these spaces give rise to some general recursion relations for Laguerre functions.

#### 3.4.1 The Cartan Decomposition

Let \( D = \{ z \in V \mid \|z\|_s < 1 \} \subset V \) be an irreducible bounded symmetric domain, where \( \|\cdot\|_s \) denotes the spectral norm on \( V \). \( D \) is isomorphic to \( T(\Omega) \) via the Cayley transform:

\[
c(z) = (z + e)(-z + e)^{-1}.
\]

For the groups \( G(D) \) and \( G(T(\Omega)) \) that act on \( D \) and \( T(\Omega) \) we also have that \( G(D) = cG(T(\Omega))c^{-1} \). Let \( G(D)_0 \) denote the subgroup of \( G(D) \) that fixes 0, i.e. \( G(D)_0 = \{ g \in G(D) \mid g0 = 0 \} \). We have \( G(D)_0 = cKc^{-1} \), where \( K = G(T(\Omega))c \). Finally, from Prop.X.3.1 in [8] we know that \( G(\Sigma) = G(D)_0 \), with Lie algebra \( \mathfrak{g}(G(\Sigma)) = \mathfrak{g}(G(D)_0) = \mathfrak{k}_1 + i\mathfrak{p}_1 \), where \( \mathfrak{k}_1 \) is the Lie algebra of \( G(D)_0 \) and \( \Sigma \) is the Shilov boundary of \( D \).

In the next proposition, using the above information, we calculate \( \mathfrak{k} \), the Lie algebra of \( K \). The proof is step by step identical with the proof of Prop.X.5.9 in [8] but is
modified for the right-half plane.

**Proposition 3.4.1.** The associated vector fields that correspond to \( \mathfrak{k} \) have the form \( X(z) = u + Tz - P(z)u \), where \( T = -T^t \).

*Proof.* From Prop. X.3.1 in [8], we know that an \( A \in \mathfrak{g}(G(D)_0) \) is \( A = T + iL(u) \); \( T \in \mathfrak{k}_1 \), \( u \in V \). Let \( \gamma_t(z) = \exp(tA)w \) be a one-parameter subgroup in \( G(D)_0 \). Then, \( \gamma_t(z) = c\gamma_t^{-1}(z) \) is a one-parameter subgroup of \( K \).

Consider the first piece of \( A \), that is \( T \). Notice that, since \( g \in G(D)_0 \) is also in \( \text{Aut}(V_C) \), we have \( c(g^{-1}(z)) = c(c^{-1}(gz)) = gz \). Now, \( \gamma_t = \exp(tT) \in G(D)_0 \), for \( T \in \mathfrak{k}_1 \), hence \( \gamma_t(z) = c\gamma_t^{-1}(z) = \exp(tS)z \). The corresponding associated vector field in \( \mathfrak{k} \) can be found by:

\[
\tilde{X}_1 f(z) = \frac{d}{dt} f(\gamma_t(z))|_{t=0} = \frac{d}{dt} f(\exp(tT)z)|_{t=0} = f'(z)[Tz] = D_Tz f(z)
\]

Hence, the corresponding vector field is \( X_1(z) = Tz \).

For the second piece of \( A \), that is \( iL(u) \), consider the one-parameter group \( h_t(z) = c \exp(tL(u))c^{-1}(z) \). Notice that \( c(z) = (z + e)(e - z)^{-1} = (z - e + 2e)(e - z)^{-1} = (z - e)(e - z)^{-1} + 2e(z + e)^{-1} = -e + 2(e - z)^{-1} \) (**) Then, we can find the corresponding associated vector field in \( \mathfrak{k} \) as follows:

\[
\tilde{X}_2 f(z) = \frac{d}{dt} f(h_t(z))|_{t=0} = \frac{d}{dt} f(c \exp(tL(u))c^{-1}(z)|_{t=0} = f'(z)[c'(c^{-1}(z))iL(u)c^{-1}(z)]
\]

\[
= f'(z)[2P(e - c^{-1}(z))iL(u)c^{-1}(z)],
\]

since \( c^{-1}(z) = e - 2(z + e)^{-1} \) by (** above

\[
= f'(z)[2P(2(z + e)^{-1})^{-1}iL(u)c^{-1}(z)]
\]
\[ f'(z)[\frac{i}{2} P((z + e)) L(c^{-1}(z))(u)] \]
\[ = f'(z)[\frac{i}{2} P((z + e)c^{-1}(z), z + e)(u)], \]

since \( P(a)L(b) = P(ab,a) \) from the proof of Prop.X.5.9 in [8], p.211
\[ f'(z)[\frac{1}{2} \{ P(z - e + z + e) = P(z - e) - P(z + e) \}(u)], \]

by Prop.1.17(b)
\[ f'(z)[\frac{1}{2} \{ P(2z) - 2P(z) - 2I \}(u)] \]
\[ = f'(z)[\frac{1}{2} \{ 2P(z) - 2I \}(u)] \]
\[ = f'(z)[-\frac{i}{2} u + P(z) \frac{i}{2} u] \]
\[ = D_{-\frac{i}{2} u + P(z) \frac{i}{2} u} f(z) \]

Hence, the corresponding vector field is \( X_2(z) = -\frac{i}{2} u + P(z) \frac{i}{2} u \). Adding now the two pieces together we get that the associated vector fields of \( \mathfrak{k} \) are of the form \( X(z) = -\frac{i}{2} u + Tz + P(z) \frac{i}{2} u \). Using the transformation \( u \mapsto 2iu \), we can write \( X \) as \( X(z) = u + Tz - P(z)u \).

Since every vector \( X \) in \( \mathfrak{g}(G(T(\Omega))) \) can be identified with \((u, T, v)\) in \( V \times \mathfrak{g} \times V \), we can write \( X \) in \( \mathfrak{k} \) as \( X = (u, T, u) \). We know that there is an involution \( \theta \) on \( \mathfrak{g}(G(T(\Omega))) \) that decomposes \( \mathfrak{g}(G(T(\Omega))) \) into a direct sum of \( \mathfrak{k} \) and \( \mathfrak{p} \), where \( \mathfrak{p} = \{ X \in \mathfrak{g}(G(T(\Omega))) \mid \theta(X) = -X \} \). Notice that \( \theta(X) = X \), for \( X \in \mathfrak{k} \). Considering the relations:

\[ \mathfrak{k} = (1 + \theta)q^\perp + \mathfrak{h} \cap \mathfrak{k} \tag{3.2} \]
\[ \mathfrak{p} = (1 - \theta)q^\perp + \mathfrak{h} \cap \mathfrak{p} \tag{3.3} \]

we can actually identify \( \mathfrak{p} \) and calculate \( \theta \).
Proposition 3.4.2. The associated vector fields that correspond to \( p \) have the form 
\[ X = (u, T, -u), \]
where \( T = T^u \).

Proof. From (3.2), we have that
\[
(u, T, u) = (1 + \theta)(u, 0, 0) + (0, T, 0).
\]
This implies that
\[
(u, 0, 0) + (0, T, 0) + (0, 0, u) = (u, 0, 0) + \theta(u, 0, 0) + (0, T, 0),
\]
which gives
\[ \theta(u, 0, 0) = (0, 0, u). \]
In other words, \( \theta \) acts on \( q^+ \) by
\[
\theta(u, 0, 0) = (0, 0, u). \quad (3.4)
\]
From (3.3), we also get that
\[
(u, T, v) = (1 - \theta)(u, 0, 0) + (0, T, 0)
\]
This implies that
\[
(u, 0, 0) + (0, T, 0) + (0, 0, v) = (u, 0, 0) - \theta(u, 0, 0) + (0, T, 0),
\]
which gives
\[ \theta(u, 0, 0) = (0, 0, -v), \] on \( q^+ \). \quad (3.5)
From (3.4) and (3.5), we get \( v = -u \). Therefore, any element in \( p \) is of the form 
\[ X = (u, T, -u). \]

**Proposition 3.4.3.** The Cartan involution \( \theta \) on \( g \) is given by 
\[ \theta(u, T, v) = (v, -T^t, u). \]

**Proof.** Let \( X = (u, T, v) \in g \). Then, \( (u, T, v) \) decomposes as: 
\[ (u, T, v) = \left( \frac{u + v}{2}, \frac{T - T^t}{2}, \frac{u + v}{2} \right) + \left( \frac{u - v}{2}, \frac{T + T^t}{2}, -\frac{u - v}{2} \right). \]

Recall that \( \theta(X) = X \), for \( X \in \mathfrak{k}_C \) and \( \theta(X) = -X \), for \( X \in p \). Taking now \( \theta \) in both sides we have:
\[
\theta(u, T, v) = \theta \left( \left( \frac{u + v}{2}, \frac{T - T^t}{2}, \frac{u + v}{2} \right) + \left( \frac{u - v}{2}, \frac{T + T^t}{2}, -\frac{u - v}{2} \right) \right)
\]
\[
= \theta\left( \frac{u + v}{2}, \frac{T - T^t}{2}, \frac{u + v}{2} \right) + \theta\left( \frac{u - v}{2}, \frac{T + T^t}{2}, -\frac{u - v}{2} \right)
\]
\[
= \left( \frac{u + v}{2}, \frac{T - T^t}{2}, \frac{u + v}{2} \right) - \left( \frac{u - v}{2}, \frac{T + T^t}{2}, -\frac{u - v}{2} \right)
\]
\[
= (v, -T^t, u)
\]

**Lemma 3.4.4.** Let \( X_1 = (u_1, T_1, v_1) \), \( X_2 = (u_2, T_2, v_2) \in g \). Then, the Lie bracket of \( X_1, X_2 \) is given by
\[
[X_1, X_2] = (T_1 u_2 - T_2 u_1, [T_1, T_2] + 2(u_1 \Box v_2 + u_2 \Box v_1), -T_1^t v_2 + T_2^t v_1)
\]

**Proof.** We use the other notation for the vector field \( X \), i.e. \( X(z) = u + Tz - P(z)v \), as it is more convinient for the calculations. Recall, also, that for \( x, y \in V \) the \( \Box \) is defined by:
\[
x \Box y = L(xy) + [L(x), L(y)],
\]
and that for vector fields we have:

$$[X_1, X_2] = (DX_1)X_2 - (DX_2)X_1.$$ 

Thus,

$$[X_1, X_2](z) = [u_1 + T_1 z - P(z)v_1, u_2 + T_2 z - P(z)v_2]$$

$$= [u_1, u_2] + [u_1, T_2 z] + [u_1, -P(z)v_2] + [T_1 z, u_2] + [T_1 z, T_2 z]$$

$$+ [T_1 z, -P(z)v_2] + [-P(z)v_1, u_2] + [-P(z)v_1, T_2 z]$$

$$+ [-P(z)v_1, -P(z)v_2]$$

$$= 0 - T_2 u_1 + P(z, v_2)u_1 + T_1 u_2 + [T_1, T_2](z) - T_1 P(z) v_2 + P'(z)v_2 T_1 z$$

$$- P(z, v_1)u_2 - P'(z)v_1 T_2 z - T_2 P(z) v_1 + 0$$

$$= -T_2 u_1 + P(z, v_2)u_1 + T_1 u_2 + [T_1, T_2](z) - T_1 P(z) v_2$$

$$+ 2P(z, T_1 z)v_2 - P(z, v_1)u_2 - 2P(z, T_2 z)v_1 + T_2 P(z) v_1$$

$$= T_1 u_2 - T_2 u_1 + [T_1, T_2](z) + P(z, v_2)u_1 - P(z, v_1)u_2$$

$$- T_1 P(z) v_2 + 2P(z, T_1 z)v_2 - 2P(z, T_2 z)v_1 + T_2 P(z) v_1$$

$$= T_1 u_2 - T_2 u_1 + [T_1, T_2](z) + P(v_2, z)u_1 - P(v_1, z)u_2$$

$$- T_1 P(z) v_2 + 2P(z, T_1 z)v_2 - 2P(z, T_2 z)v_1 + T_2 P(z) v_1, \text{ by Prop.1.2.17(d)}$$

$$= T_1 u_2 - T_2 u_1 + [T_1, T_2](z) + P(v_2, z)u_1 - P(v_1, z)u_2$$

$$- P(z)T_1 v_2 + T_2 P(z) v_1, \text{ by Prop. III.5.3 in [8]}$$

$$= T_1 u_2 - T_2 u_1 + [T_1, T_2](z) + (v_2 □ u_1)z - (v_1 □ u_2)z - P(z)T_1 v_2 + T_2 P(z) v_1,$$

combining parts (a), (b) from Prop.1.2.17, and the definition of □.

Hence, $[X_1, X_2] = (T_1 u_2 - T_2 u_1, [T_1, T_2] + 2(u_1 □ v_2 + u_2 □ v_1), -T_1^4 v_2 + T_2^4 v_1)$.  

Proposition 3.4.5. Let $X_0 = (−iε, 0, −iε) ∈ g$. Then, $X_0 = (−iε, 0, −iε) ∈ Z(t_C)$ and
the map \( \text{ad}(X_0) : \mathfrak{g}_C \rightarrow \mathfrak{g}_C \) has eigenvalues 0, 2 and \(-2\). The eigenvalues 0, 2 and \(-2\) correspond to \( \mathfrak{k}_C, \mathfrak{p}^+ \) and \( \mathfrak{p}^- \) respectively, where \( \mathfrak{p}^+ \) and \( \mathfrak{p}^- \) are subalgebras of \( \mathfrak{p}_C \) given by

\[
\mathfrak{p}^+ = \{(u, L(-2iu), -u) | u \in V_C\}
\]

and

\[
\mathfrak{p}^- = \{(u, L(2iu), -u) | u \in V_C\}
\]

Proof. Let \( X_0 = (-ie, 0, -ie) \in \mathfrak{k}_C \). Then, \( X_0 = (-ie, 0, -ie) \in \mathcal{Z}(\mathfrak{k}_C) \) since for every \( X = (u, T, u) \in \mathfrak{k}_C \), we have:

\[
[X_0, X] = [(-ie, 0, -ie), (u, T, u)]
\]

\[
= (0u - T(-ie, [0, T] - 2((-ie)\Box u - u\Box(-ie)), -0u + T^i(-ie)),
\]

by Lemma 3.4.4

\[
= (iT e, -2i(e\Box u + e\Box u), iT e), \text{ because } e\Box u = u\Box e = L(u), \text{ in the special case that } v = e
\]

\[
= (0, 0, 0), \text{ because } ge = e \text{ for } g \in L, \text{ so } Te = 0 \text{ by differentiation.}
\]

Now, since \( \text{ad}(X_0)(X) = [X_0, X] = 0 = 0X \), this says that the 0 eigenvalue corresponds to \( \mathfrak{k}_C \).

Let \( X = (u, T, -u) \in \mathfrak{p}_C \). Recall, that every linear transformation in \( \mathfrak{h} \) is actually an \( L(a), a \in V \) (see [8], p.55 and Remark 1.3.18). So, \( T = L(w), w \in V_C \). Hence, every element in \( \mathfrak{p}_C \) is written as \( X = (u, L(w), -u) \). Therefore,

\[
[X_0, X] = [(-ie, 0, -ie), (u, L(w), -u)] = (iw, -4iL(u), -iw)
\]

(3.6)

We claim that the vectors \( X_1 = (u, L(-2iu), -u) \) and \( X_2 = (u, L(2iu), -u) \) are eigenvectors of \( \text{ad}(X_0) \).
Indeed,

\[
\text{ad}(X_0)(X_1) = [X_0, X_1] = [(-ie, 0, -ie), (u, L(-2iu), -u)]
\]
\[
= (i(-2iu), -4iL(u), -i(-2iu)), \text{ where } w = -2iu \text{ in (3.6)}
\]
\[
= (2u, 2L(-2iu), -2u)
\]
\[
= 2(u, L(-2iu), -u)
\]
\[
= 2X_1
\]

which says that the eigenvalue 2 corresponds to \( p^+ \).

Similarly,

\[
\text{ad}(X_0)(X_2) = [X_0, X_2] = [(-ie, 0, -ie), (u, L(2iu), -u)]
\]
\[
= (i(2iu), -4iL(u), -i(2iu)), \text{ where } w = 2iu \text{ in (3.6)}
\]
\[
= (-2u, -2L(2iu), 2u)
\]
\[
= -2(u, L(2iu), -u)
\]
\[
= -2X_2
\]

which means that the eigenvalue \(-2\) corresponds to \( p^- \). \(\square\)

### 3.4.2 The Action of \( g_L^C \) on \( \mathcal{H}_\nu(T(\Omega)) \) and \( L^2(\Omega, d\mu_\nu) \)

In this section, we calculate the action of \( g_L^C \) on \( \mathcal{H}_\nu(T(\Omega)) \) and \( L^2(\Omega, d\mu_\nu) \) respectively.

Consider the \( L \)-invariant elements \( X_0 = (-ie, 0, -ie) \in \mathfrak{g}_C^L, X^+ = (e, -2iI, -e) \in \mathfrak{p}^+ \) and \( X^- = (e, 2iI, -e) \in \mathfrak{p}^- \). From Prop.3.3.2, we know the action of the elements \( u, T \) and \( v \). These elements can also be written as \( u = (u, 0, 0), T = (0, T, 0) \) and \( v = (0, 0, v) \). Therefore, we know how the basic elements \( Z^+ = (e, 0, 0), Z_0 = (0, I, 0) \) and \( Z^- = (0, 0, e) \) act. Notice now that:
\[ X_0 = -(iZ^+ + iZ^-) \quad (3.7) \]
\[ X^+ = Z^+ - 2iZ_0 - Z^- \quad (3.8) \]
\[ X^- = Z^+ + 2iZ_0 - Z^- \quad (3.9) \]

**Proposition 3.4.6.** For each piece of the Lie algebra of \( g_C \) we have:

1. \( \pi_\nu(X_0)F(z) = -\nu \text{tr}(z)F(z) - D_{-e+z^2}F(z), \quad X_0 \in \mathfrak{k}_C \)
2. \( \pi_\nu(X^+)F(z) = i\nu \text{tr}(\frac{2}{p}I - z)F(z) + D_{-ie + 2iz - iz^2}F(z), \quad X^+ \in \mathfrak{p}^+ \)
3. \( \pi_\nu(X^-)F(z) = -i\nu \text{tr}(\frac{2}{p}I + z)F(z) - D_{ie + 2iz + iz^2}F(z), \quad X^- \in \mathfrak{p}^- \)

**Proof.** (1) Using (3.7) and Prop.3.2.2, we have

\[
\pi_\nu(X_0)F(z) = \pi_\nu(-iZ^+ + iZ^-)F(z) \\
= -i\pi_\nu(Z^+)F(z) - i\pi_\nu(Z^-)F(z) \\
= -i(-D_{ie}F(z)) - i(i\nu \text{tr}(z)F(z) + D_{P(z)ie}F(z)) \\
= \nu \text{tr}(z)F(z) + D_{-e}F(z) + D_{z^2}F(z) \\
= \nu \text{tr}(z)F(z) + D_{-e+z^2}F(z)
\]

(2) Similarly, using (3.8) and Prop.3.2.2, we have

\[
\pi_\nu(X^+)F(z) = \pi_\nu(Z^+ - 2iZ_0 - Z^-)F(z) \\
= \pi_\nu(Z^+)F(z) - 2i\pi_\nu(Z_0)F(z) - \pi_\nu(Z^-)F(z) \\
= -D_{ie}F(z) + 2i\nu \text{tr}I + 2iD_{Iz} - i\nu \text{tr}(z)F(z) - D_{P(z)ie}F(z) \\
= i\nu \text{tr}(\frac{2}{p}I - z)F(z) + D_{-ie + 2iz - iz^2}F(z)
\]
(3) Finally, using (3.9) and Prop.3.2.2, we have

\[\pi_{\nu}(X^-)F(z) = \pi_{\nu}(Z^+ + 2iZ_0 - Z^-)F(z)\]
\[= \pi_{\nu}(Z^+)F(z) + 2i\pi_{\nu}(Z_0)F(z) - \pi_{\nu}(Z^-)F(z)\]
\[= -D_{ie}F(z) - (2i\nu - \text{tr} - 2iD_{Iz} - i\nu \text{tr}(z)F(z) - D_{p(z)ie}F(z)\]
\[= -i\nu \text{tr}(\frac{2}{p}I + z)F(z) - D_{ie + 2iz + iz^2}F(z)\]

\[\square\]

Now, Corollary 3.3.4 enables us to calculate representation \(\lambda_{\nu}\) without having to transfer representation \(\pi_{\nu}\) of Prop.3.2.2 on \(L^2_\nu(\Omega)\) via the Laplace transform. That is captured in the following Theorem.

**Theorem 3.4.7.** For each piece of the Lie algebra of \(g_C^L\) we have:

1. \(\lambda_{\nu}(X_0)f(x) = \text{tr}(x - \nu \nabla - \tilde{\nabla}x \tilde{\nabla})f(x), \quad X_0 \in k_C\)
2. \(\lambda_{\nu}(X^+)f(x) = i\text{tr}(\frac{-2\nu}{p}I + x + \nu \nabla - 2x \nabla + \tilde{\nabla}x \tilde{\nabla})f(x), \quad X^+ \in p^+\)
3. \(\lambda_{\nu}(X^-)f(x) = i\text{tr}(\frac{2\nu}{p}I + x + \nu \nabla + 2x \nabla + \tilde{\nabla}x \tilde{\nabla})f(x), \quad X^- \in p^-\)

**Proof.** (1) Using (3.7) and Corollary 3.3.4 we have:

\[\lambda_{\nu}(X_0)f(x) = \lambda_{\nu}(-(iZ^+ + iZ^-))f(x)\]
\[= -i\lambda_{\nu}(Z^+)f(x) - i\lambda_{\nu}(Z^-)f(x)\]
\[= -i(i\text{tr}(x)f(x)) - i \left(-i\text{tr}(\nu \nabla + \tilde{\nabla}x \tilde{\nabla})f(x)\right)\]
\[= \text{tr}(x)f(x) - \text{tr}(\nu \nabla + \tilde{\nabla}x \tilde{\nabla})f(x)\]
\[= \text{tr}(x - \nu \nabla - \tilde{\nabla}x \tilde{\nabla})f(x)\]
(2) Similarly, using (3.8) and Corollary 3.3.4 we have:

\[
\lambda_\nu(X^+)f(x) = \lambda_\nu(Z^+ - 2iZ_0 - Z^-)f(x) \\
= \lambda_\nu(Z^+)f(x) - 2i\lambda_\nu(Z_0)f(x) - \lambda_\nu(Z^-)f(x) \\
= \text{itr}(x)f(x) - 2\text{itr}(\nu I + I^\prime x\nabla)f(x) - (\nu \nabla + \nabla x \nabla f(x)) \\
= \text{itr}(x)f(x) - \text{itr}(2\nu I + x\nabla)f(x) + \nu \nabla + \nabla x \nabla f(x) \\
= \text{itr}(\frac{-2\nu}{p} I + x + \nu \nabla - 2x\nabla + \tilde{\nabla} x \tilde{\nabla} f(x)
\]

(3) Finally, using (3.9) and Corollary 3.3.4 we have:

\[
\lambda_\nu(X^-)f(x) = \lambda_\nu(Z^+ + 2iZ_0 - Z^-)f(x) \\
= \lambda_\nu(Z^+)f(x) + 2i\lambda_\nu(Z_0)f(x) - \lambda_\nu(Z^-)f(x) \\
= \text{itr}(x)f(x) + 2\text{itr}(\nu I + I^\prime x\nabla)f(x) - (\nu \nabla + \nabla x \nabla f(x)) \\
= \text{itr}(x)f(x) + \text{itr}(2\nu I + x\nabla)f(x) + \nu \nabla + \nabla x \nabla f(x) \\
= \text{itr}(\frac{2\nu}{p} I + x + \nu \nabla + 2x\nabla + \tilde{\nabla} x \tilde{\nabla} f(x)
\]

\[
\square
\]

3.4.3 General Recursion Relations for \( \ell^\nu_m \)

We are ready now to calculate the recursion relations for \( \ell^\nu_m(x) \) that are derived by the action of the \( L \)-invariant elements \( X_0 = (-ie, 0, -ie) \in \mathfrak{f}_C, X^+ = (e, -2iI, -e) \in \mathfrak{p}^+ \) and \( X^- = (e, 2iI, -e) \in \mathfrak{p}^- \).

Recall that the functions defined in Theorem 2.3.1, namely the ones given by:

\[
q^\nu_m(z) = \Delta(z + e)^{-\nu} \psi_m \left( \frac{z - e}{z + e} \right),
\]

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form an orthogonal basis for $\mathcal{H}_\nu(T(\Omega))^L$. The Laguerre functions relate with $q^\nu_m(z)$ via $L^\nu$, by:

$$L^\nu(\ell^\nu_m)(z) = \Gamma_\nu(m + n)q^\nu_m(z),$$

and the former form an orthogonal basis in $L^2(\Omega, d\mu^\nu_L)$.

Proposition 2.3.1 and Theorem 2.4.2, actually enable us to obtain the corresponding relations for $q^\nu_m$ and $\ell^\nu_m$ respectively. For $q^\nu_m$, we know (by Lemma 5.5 in [4], p. 182) that for $\xi \in \mathcal{Z}(t_C)$ and $Z_0 \in \mathcal{Z}(h_C)$ (can take $\xi = (-ie, 0, -ie)$ and $Z_0 = (0, I, 0)$), we have:

$$\pi^\nu(\xi)q^\nu_m(z) = (r\nu + 2|m|)q^\nu_m(z) \quad (3.10)$$

and

$$\pi^\nu(-2Z_0)q^\nu_m(z) = \sum_{j=1}^r \left(\frac{m}{m - \varepsilon_j}\right)q^\nu_{m-\varepsilon_j} - \sum_{j=1}^r \left(\nu + n_j - \frac{s}{2}(j - 1)\right) c_m(j)q^\nu_{m+\varepsilon_j}(z) \quad (3.11)$$

where: $\varepsilon_j = (0, ..., 0, 1, 0, ..., 0)^t$, with 1 in the $j^{th}$ position

and

$$c_m(j) = \prod_{j \neq k} \frac{n_j - n_k - \frac{s}{2}(j + 1 - k)}{n_j - n_k - \frac{s}{2}(j - k)}.$$

**Theorem 3.4.8.** The Laguerre functions satisfy the following differential recursion relations:

1. $$(r\nu + 2|m|)\ell^\nu_m = \text{tr}(x - \nu \nabla - \tilde{\nabla} x \tilde{\nabla})\ell^\nu_m$$
2. $$2 \sum_{j=1}^r c_m(j)\ell^\nu_{m+\varepsilon_j} = \text{tr}\left(\frac{2\nu}{p}I - x + (2x - \nu)\nabla - \tilde{\nabla} x \tilde{\nabla}\right)\ell^\nu_m$$
3. $$-2 \sum_{j=1}^r \left(\frac{m}{m - \varepsilon_j}\right) \left(m_j - 1 + \nu - (j - 1)\frac{s}{2}\right) \ell^\nu_{m-\varepsilon_j} =$$

$$\text{tr}\left(\frac{2\nu}{p}I + x + (2x + \nu)\nabla + \tilde{\nabla} x \tilde{\nabla}\right)\ell^\nu_m$$
Proof. Transferring representation (3.10) above onto \(L^2(\Omega, d\mu_\nu)\), we get:

\[
\lambda_\nu(X_0)\ell_m^\nu(x) = (r \nu + 2|m|)\ell_m^\nu(x).
\] (3.12)

Now, combining Theorem 3.3.7(1) and (3.12), we get the first recursion relation for \(\ell_m^\nu\):

\[
(r \nu + 2|m|)\ell_m^\nu = \text{tr}(x - \nu \nabla - \tilde{\nabla}x \tilde{\nabla}x)\ell_m^\nu.
\]

Let \(L^2_k(\Omega, d\mu_\nu) = \{ f \in L^2(\Omega, d\mu_\nu) \mid \lambda_\nu(X_0)f = (r \nu + 2k)f \}\), where \(\lambda_\nu\) is the representation of the compact group \(L\) on \(L^2(\Omega, d\mu_\nu)\). Then, as \(\lambda_\nu\) are highest weight representations, we have:

\[
L(\Omega, d\mu_\nu) = \bigoplus L^2_k(\Omega, d\mu_\nu),
\]

where \(L^2_k(\Omega, d\mu_\nu) \neq \{0\}\) if \(k \geq 0\). Observe also that \(\ell_m^\nu \in L^2_{|m|}(\Omega, d\mu_\nu)\), by (3.12). Now, for \(X \in \mathfrak{p}^+\), we have:

\[
\lambda_\nu(\xi)\lambda_\nu(X)f = \lambda_\nu(X)\lambda_\nu(X_0)f + \lambda_\nu([X, X_0])f = \lambda_\nu(X)(r \nu + 2k)f + \lambda_\nu(ad(X_0)X)f = (r \nu + 2k)\lambda_\nu(X)f + \lambda_\nu(2X)f = (r \nu + 2(k + 1))\lambda_\nu(X)f.
\]

That is, \(\lambda_\nu(X)f \in L^2_{k+1}(\Omega, d\mu_\nu)\), \(X \in \mathfrak{p}^+\). Similarly, \(\lambda_\nu(X)f \in L^2_{k-1}(\Omega, d\mu_\nu)\), \(X \in \mathfrak{p}^-\). This says that \(\lambda_\nu(X)\), for \(X\) in \(\mathfrak{p}^+\) and \(\mathfrak{p}^+\) respectively, act as \(k+1\) and \(k-1\) projections. Let \(X^+ = (e, -2iI, -e) \in \mathfrak{p}^+\) and \(X^- = (e, 2iI, -e) \in \mathfrak{p}^-\). Then,

\[
Z_0 = \frac{1}{4i}(X^- - X^+) = (0, I, 0),
\]

and \(Z_0 \in \mathcal{Z}(\mathfrak{h}_\mathbb{C})\), where \(\mathcal{Z}(\mathfrak{h}_\mathbb{C})\) denotes the center of \(\mathfrak{h}_\mathbb{C}\). Transferring representation
(3.11) onto $L^2(\Omega, d\mu, )$ we get:

$$-2i\lambda_\nu(-2Z_0)\ell^\nu_m = \sum_{j=1}^r \left( \frac{m}{m-e_j} \right) \left( m_j - 1 + \nu - (j-1)\frac{s}{2} \right) \ell^\nu_{m-e_j} - \sum_{j=1}^r c_m(j)\ell^\nu_{m+e_j},$$

(3.13)

So, for each one of the projections, we have:

$$\lambda_\nu(X^-)\ell^\nu_m = -2i \sum_{j=1}^r \left( \frac{m}{m-e_j} \right) \left( m_j - 1 + \nu - (j-1)\frac{s}{2} \right) \ell^\nu_{m-e_j}$$

(3.14)

and

$$\lambda_\nu(X^+)\ell^\nu_m = -2i \sum_{j=1}^r c_m(j)\ell^\nu_{m+e_j}.$$  

(3.15)

Finally, combining Theorem 3.3.7(2),(3) and (3.14),(3.15) respectively, one obtains the remaining recursion relations for $\ell^\nu_m$:

$$2 \sum_{j=1}^r c_m(j)\ell^\nu_{m+e_j} = \text{tr} \left( \frac{2\nu_p}{p} I - x + (2x - \nu)\nabla - \nabla \tilde{x} \hat{\nabla} \right) \ell^\nu_m$$

and

$$-2 \sum_{j=1}^r \left( \frac{m}{m-e_j} \right) \left( m_j - 1 + \nu - (j-1)\frac{s}{2} \right) \ell^\nu_{m-e_j} = \text{tr} \left( \frac{2\nu_p}{p} I + x + (2x + \nu)\nabla + \hat{\nabla} x \hat{\nabla} \right) \ell^\nu_m.$$  

Notice that the recursion relations in Theorem 3.4.8, generalize the recursion relations of Theorems 6.3, 3.4 and 6.1 in [1], [3] and [4] respectively.
3.5 Further Projects

In this section, we discuss some further projects related to Laguerre functions and the results obtained in the dissertation.

3.5.1 Relation to Classical Laguerre Functions

One project is to express the generalized Laguerre functions in terms of classical Laguerre functions in one variable. Every symmetric cone \( \Omega \) can be written as \( \Omega = L \cdot \Omega_1 \), where \( \Omega_1 \cong (\mathbb{R}^+)^r \). For example, every \( A \in \text{Sym}^+(n, \mathbb{R}) \) can be written as \( A = gDg^{-1} \), where \( g \in SO(n) \) and \( D = \text{diag}(t_1, t_2, \ldots, t_r) \) is a diagonal matrix with entries \( t_j > 0 \). Thus, if \( \Omega_1 = \{ \text{diag}(t) \mid t \in (\mathbb{R}^+)^r \} \cong (\mathbb{R}^+)^r \), then \( \Omega = L \cdot \Omega_1 \). In general, the isomorphism \( \Omega_1 \cong (\mathbb{R}^+)^r \) corresponds to a homomorphism of \( \mathfrak{sl}(n, \mathbb{R})^r \) into the Lie algebra \( \mathfrak{g} \) and to an embedding \( (\mathbb{C}^+)^r \hookrightarrow T(\Omega) \), where \( \mathbb{C}^+ = \mathbb{R}^+ + i\mathbb{R} \). Note that \( \mathfrak{sl}(n, \mathbb{R})^r \) acts transitively on \( (\mathbb{C}^+)^r \). Denote by \( S \) the subgroup of \( G \) corresponding to \( \mathfrak{sl}(n, \mathbb{R})^r \). The restriction \( \ell^\nu_m|_{\Omega_1} \) decomposes into a sum of highest weight representations of \( S \). It follows, that the restriction \( \ell^\nu_m|_{\Omega_1} \) is a sum of classical Laguerre functions. The goal is to determine this sum explicitly.

3.5.2 Relations in the \( \nu \)-parameter

It is well known, that the classical Laguerre polynomials satisfy the following recursion relations:

\[
\begin{align*}
x L^\nu_m & = (m + \nu + 1)L^\nu_{m-1} - (m + 1)L^\nu_{m+1} \\
x L^\nu_m & = (m + \nu)L^\nu_{m-1} - (m - x)L^\nu_m \\
x L^\nu_{m-1} & = L^\nu_m - L^\nu_{m-1}.
\end{align*}
\]
It was shown, in [3], that these relations follow directly from the representation theory of \( \mathfrak{sl}(2, \mathbb{R}) \). It is therefore natural to look for similar relations for the generalized Laguerre polynomials and functions.

### 3.5.3 Relations in the \( x, y \)-parameters

Several other classical relations should be extended to the general case. We mention here only the following:

\[
L_{m}^{\nu+\mu+1}(x + y) = \sum_{n=0}^{m} L_{n}^{\nu}(x)L_{m-n}^{\mu}(y).
\]

This relation is closely related to the decomposition of the tensor product of two highest weight representations. The expectation is that a similar relation can be derived also for the general case.

### 3.5.4 Physical Interpretation

Finally, it is well known that the classical Laguerre functions are solutions of the radial part of the Schrödinger equation for the hydrogen atom in three dimensions. Namely, the equation:

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2\mu}{\hbar} (Er^2 + ke^2r) - l(l+1) = 0,
\]

where \( \mu \) is the reduced mass, \( \hbar \) is the reduced Plank constant, \( E \) is the energy state and \( k, l \) are constants, has solutions:

\[
R^{l}_{n}(r) = r^{l}e^{-\frac{r}{a_{0}}} L^{l}_{n}(r),
\]

where \( n \) is the principal quantum number and \( a_{0} \) is a constant. Since the classical Laguerre functions have a physical interpretation as hydrogen radial wavefunctions, then
a natural question would be the following: what is the natural interpretation (if any) of the generalized Laguerre functions and what use could the generalized recursion relations have in this higher dimensional setting?
Bibliography


Vita

Michael Aristidou was born on June 7th 1975, in Athens, Greece. He finished his undergraduate studies in mathematics at Aristotle University of Thessaloniki, Greece, in July 1999. He earned a master of science degree in mathematics from Louisiana State University in May 2001. In August 2001 he started his doctorate in mathematics, under the direction of professor Gestur Ólafsson. He earned a master of arts degree in philosophy from Louisiana State University in May 2004, under the direction of professor Jon Cogburn. He was awarded the Focused Research Award in spring 2005 and is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005.