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Excluding a Weakly 4-connected Minor

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EXCLUDING A WEAKLY 4-CONNECTED MINOR

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I split dedication of this dissertation four ways. First, I dedicate this to my nieces Bethany Sierra and Melody Quinn. I hope you both dream big and always find encouragement on the path to achieving your dreams. Next, I dedicate this to the memory of my grandmother Agnes Perez (1915-2004). You always pushed me to learn and achieve more and I am grateful every day for your love and support. You are my inspiration in everything that I do. Finally, I dedicate this to my husband and best friend, Cassius D'souza. Thank you for being by my side through the many ups and downs of graduate school. I know I couldn't have done any of this without your never-ending support. Thank you for always believing in me.

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Abstract

A 3-connected graph G is called weakly 4-connected if min $(|E(G_1)|, |E(G_2)|) \leq 4$ holds for all 3-separations (G_1, G_2) of G. A 3-connected graph G is called quasi 4-connected if min $(|V(G_1)|, |V(G_2)|) \leq 4$. We first discuss how to decompose a 3-connected graph into quasi 4-connected components. We will establish a chain theorem which will allow us to easily generate the set of all quasi 4-connected graphs. Finally, we will apply these results to characterizing all graphs which do not contain the Pyramid as a minor, where the Pyramid is the weakly 4-connected graph obtained by performing a ΔY transformation to the octahedron. This result can be used to show an interesting characterization of quasi 4-connected, outer-projective graphs.

Chapter 1
Introduction

1.1 Overview

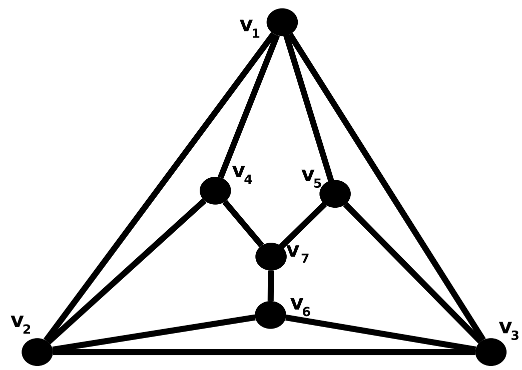
In graph theory, determining H minor-free graphs is an important problem. In Section 1.5, we will outline some of the known results in this area. The problem that we will address is how to characterize H minor-free graphs where H is a weakly 4-connected graph. A common approach to solve such problems is to reduce the problem to graphs of a comparable connectivity. In our case, we will decompose the graphs into quasi 4-connected components. In Section 1.3, we outline two decompositions that we will perform on the graph as well as results about minor relationships between the quasi 4-connected components and the original graph. Finally, we look at two applications of these results. First, we solve the problem of characterizing Pyramid minor-free graphs. In Section 1.5, we explain why this graph is interesting to study and overview the characterization results. Finally, in Section 1.6, we explain how the characterization of Pyramid minor-free graphs leads to a characterization of outer-projective graphs. We begin with Section 1.2 where we introduce some of the main terminology that will be used throughout the dissertation.

1.2 Preliminaries

We will begin by introducing some general graph terminology that will be utilized throughout the dissertation. A more thorough explanation of terminology used can be found in [6], [10], or [25].

A **graph** $G = (V, E)$ is an ordered pair consisting of a finite set V of **vertices** of G along with a finite multiset E of **edges** of G. We utilize the notation $V(G)$ and $E(G)$ to represent the vertices and edges respectively of graph G. The number of vertices in the graph is referred to as the **order** of the graph, while the number of edges is the **size** of the graph. Edges in $E(G)$ are usually denoted by $e = uv$, where $u, v \in V(G)$. If $e = uv$, then vertices u and v are said to be **incident** with edge e. Since the two vertices are joined by an edge, we also say that vertices u and v are **adjacent**. For convenience, we will usually refer to an edge by its endpoints, so in this instance we can refer to edge e by uv. It is possible to have multiple edges between the same pair of vertices. These edges are called **parallel edges**. We can also have an edge that starts and ends at the same vertex. These edges are called **loops**. If a graph contains neither parallel edges nor loops, then the graph is called **simple**. For distinction, graphs that do allow parallel edges and loops are sometimes called **multigraphs**. There are several ways that we can talk about the vertices that are adjacent to a given vertex v. First, we can discuss the number of edges incident to v. This is called the **degree** of vertex v, denoted either $deg_G(v)$ or simply $deg(v)$. Note that in a simple graph, the degree of a vertex is the same as the number of vertices adjacent to that vertex. In multigraphs, each parallel edge is counted once in computing the degree of its incident vertices; loops are counted twice. We can also talk about the vertices that are adjacent to v. If u is adjacent to v, we say that u and v are **neighbors**, or that u is in the **neighborhood** of vertex v. The neighborhood of a vertex v in a graph G, denoted $N_G(v)$ (or simply $N(v)$) is the set of vertices in G that are adjacent to v.

Graphs are very nicely represented pictorially. We use dots to represent each vertex. If two vertices are adjacent, we connect them with a line (an edge). We will use the following pictorial representations of graph P_1 and graph P_2 as shown in Figure 1.1 to illustrate some of the above concepts.



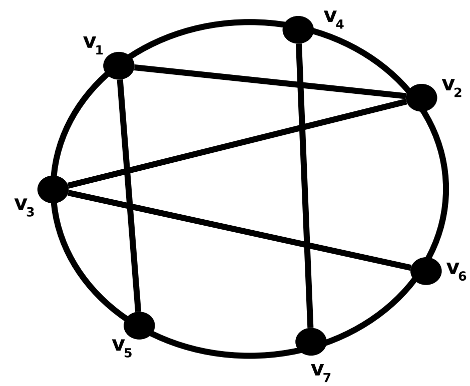


Figure 1.1: Graphs P_1 and P_2, respectively

We begin by noting that these are actually two drawings of the same graph. This graph is called the Pyramid and is a graph that we will see again later. To see that these two graphs are actually the same, we can check some of the properties that we discussed above. We can easily check that both graphs are of order seven and size twelve. We can also see that both graphs are simple as neither one contains loops or parallel edges. Now, let us consider the vertex labeled v_1. We can first check the degree of this vertex in both graphs. We see that $deg_{P_1}(v_1) = deg_{P_2}(v_1) = 4$. Further, we can see that $N_{P_1}(v_1) = N_{P_2}(v_1) = \{v_2, v_3, v_4, v_5\}$. We can perform similar checks for the other six vertices in the two graphs to see that they have matching degrees and neighborhoods in both representations. Therefore, our two drawings really are the same graph. In general, the drawing P_1 is preferable to the drawing P_2. First, P_1 is drawn in a symmetric manner. We can see that vertices v_1, v_2, and v_3 are all similar. We will discuss this similarity in more detail later. Additionally, there are no crossing edges in P_1, that is, the edges of P_1 only intersect at vertices. This means that P_1 actually shows a **plane embedding** of the Pyramid. A graph which can be drawn in the plane without crossing edges is called a **planar** graph. We note that although P_2 does not show a plane embedding of the Pyramid, it is still a planar graph since we can draw it in the plane without crossing edges.

There are several families of graphs which have a special structure. Here, we define some of the common ones that we will see later on. First, we consider **the path on** n **vertices,** P_n for $n \geq 1$. This graph has vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$. We can extend this graph to **the cycle on** n **vertices,** C_n by adding the single edge $v_1 v_n$ to P_n. We will also consider **the complete graph on** n **vertices,** K_n for $n \geq 1$. This graph has vertex set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ with edge $v_i v_j$ present in the graph for all $i \neq j$. Finally, we will consider the family of wheels. The **wheel graph** W_n, for $n \geq 3$ is the graph on $n+1$ vertices consisting of a cycle of order n for which every vertex in the cycle is adjacent to the single remaining vertex called the **hub**.

Next, we discuss some of the ways that two graphs can be related to each other. First, we discuss further the concept of two graphs being the same. Two graphs G and H are called **isomorphic** if there is a bijection $f : V(G) \to V(H)$ such that vertices u and v are adjacent in G if and only if the corresponding vertices $f(u)$ and $f(v)$ are adjacent in H. We note that if we consider the graph P_1 from Figure 1.1 and the same graph with labels v_1 and v_2 swapped, these two graphs are isomorphic. For simplicity, we will say that these two graphs are the same. Sometimes, we are only interested in considering a piece of graph. We call a graph H a **subgraph** of graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, but $H \neq G$, then H is called a **proper subgraph** of G. If we have a subgraph H of G such that $V(H) = V(G)$, then H is called **spanning subgraph** of G. A subgraph H of G is called an **induced subgraph** if for each pair of vertices u and v in H, $uv \in E(H)$ if and only if $uv \in E(G)$.

Again, we refer to graph P_1 from Figure 1.1 for illustration of some of these concepts. If we relabel the graph as shown in Figure 1.2 below, we note that there is an isomorphism between the graph P_1' as shown in Figure 1.2 and P_1 as shown in 1.1. We can simply map v_1 in P_1' to v_2 in P_1, map v_2 in P_1' to v_1 in P_1, and map all other v_i in P_1' to the corresponding v_i in P_1. Again, we note that for all practical purposes, both of the graphs are in fact the Pyramid graph so we make no distinction between the different labelings.

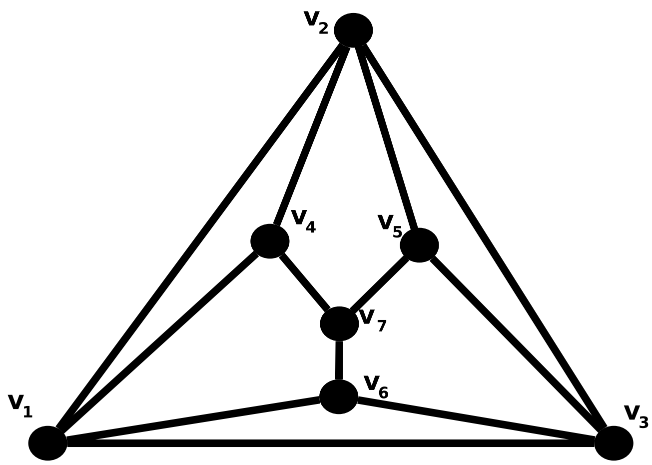


Figure 1.2: Graph P_1'

We can also consider subgraphs of P_1'. The graph shown in Figure 1.3 is a subgraph of P_1'. We can further note that this graph is also an induced subgraph of P_1'. To check this, we simply need to ensure that all edges that existed between vertices v_1, v_2, and v_3 in P_1' are also present in the subgraph.

In our discussion of results on graph decompositions, we will focus quite a bit on the connectivity of the graph to be decomposed. Here, we explain some of the terminology related to the connectivity of graphs. Let $k \geq 0$ be an integer. A **k-separation** of a graph G is a pair (G_1, G_2) of induced subgraphs of G such that $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) - V(G_2) \neq \emptyset$,

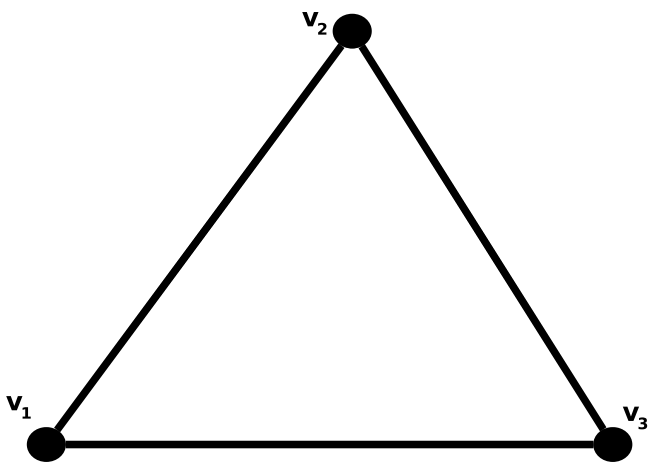


Figure 1.3: Subgraph of P_1'

$V(G_2) - V(G_1) \neq \emptyset$, and $|V(G_1) \cap V(G_2)| = k$. Let $|G|$ denote the order of graph G. Let $k \geq 0$ be an integer. A graph G is **k-connected** if $|G| > k$ and $G - X$ is connected for every $X \subset V$ with $|X| < k$. Equivalently, G is k-connected if $|G| > k$ and G has no k'-separation for all $k' < k$. For $k = 0, 1, 2, 3$, we define **k-sum** as follows. Let G_1, G_2 be disjoint graphs with more than k vertices. The **0-sum** of G_1 and G_2 is their disjoint union; a **1-sum** of G_1, G_2 is obtained by identifying a vertex of G_1 with a vertex of G_2; a **2-sum** of G_1, G_2 is obtained by identifying an edge of G_1 with an edge of G_2, where the common edge may or may not be deleted after the identification; a **3-sum** of G_1, G_2 is obtained by identifying a triangle of G_1 with a triangle of G_2, where some of the three identified edges may be deleted and some may be retained after the identification. It is clear that if G is a k-sum of G_1 and G_2, then G has a k-separation. The converse is also true. Let (G_1, G_2) be a k-separation ($k \leq 3$) of a graph G. For $i = 1, 2$, let G_i^+ be obtained from G_i by adding all edges between any two non-adjacent vertices in $V(G_1) \cap V(G_2)$. Let G_1' and G_2' be disjoint graphs which are isomorphic to G_1^+ and G_2^+ respectively. Then, G is isomorphic to a k-sum of G_1' and G_2'.

One important result that we can use to determine whether a graph is k-connected is Menger's Theorem. There are many versions of this theorem, so we state the version that will be utilized in later results. We must introduce the concept of internally disjoint $u - v$ paths for this result. A $u - v$ path in graph G is a sequence of vertices of G starting at u and ending at v such that each vertex is included at most once and consecutive vertices in the sequence are adjacent. Two $u - v$ paths are called **internally disjoint** if they have no vertices in common aside from u and v.

Theorem 1.2.1 (Menger's Theorem)**.** *Let G be a k-connected graph. Then, for any pair of vertices $u, v \in V(G)$, there are at least k pairwise-internally-disjoint $u - v$-paths in G.*

This theorem tells us that if G is a k-connected graph, then we should be able to find k paths from any vertex $u \in V(G)$ to any other vertex $v \in V(G)$ such that none of the paths have any vertices or edges in common (other than u and v of course).

Let us refer once again to the graph P_1 in Figure 1.1 for illustration of these concepts. We can find a 3-separation of this graph. Let (G_1, G_2) be the induced subgraphs of P_1 defined as follows. Let $V(G_1) = \{v_4, v_5, v_6, v_7\}$ and $E(G_1) = \{v_4v_7, v_5v_7, v_6v_7\}$. Let $V(G_2) = \{v_1.v_2, v_3, v_4, v_5, v_6\}$ and $E(G_2) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4, v_2v_6, v_3v_6\}$. We can easily check that both of these subgraphs are in fact induced. Further, we have included all vertices and edges from P_1. $V(G_1) \cap V(G_2) = \{v_4, v_5, v_6\}$ which is a set of cardinality three. Therefore, we have a 3-separation of P_1. It is impossible to find any smaller separations in the graph P_1. Therefore, we can also say that P_1 is 3-connected.

One final relationship that we want to explore among graphs is the minor relation. To talk about this relationship, we need to specify two graph operations. First, let G be a graph with edge $e = uv$. We can consider the graph G' formed from the **deletion** of e from G, denoted $G \backslash e$. The graph $G' = G \backslash e$ has the same vertex set as graph G, that is, $V(G') = V(G)$. The edge set $E(G') = \{e \in E(G) | e \neq uv\}$. For the second operation, we again consider a graph G with an edge $e = uv$. Now, we wish to **contract** the edge e, an operation denoted by G/e. To obtain G/e, we can delete the edge e and identify its two endpoints u and v. It is important to note that even if G is a simple graph, we are not guaranteed that the graph G/e will be simple. If we can obtain graph H through some series of edge deletions and edge contractions on graph G, then we say that H is a **minor** of G, denoted $H \leq_m G$. Sometimes, vertex deletion, removing a vertex and all of its incident edges from the graph, is included as a third allowable operation for forming graph minors. We note that this operation is not required. If we wish to delete vertex v from graph G, we can do so in two steps using edge deletion and edge contraction. First, we delete all of the edges incident to v except for one. Then, we contract the remaining edge that is incident to v. We would also like to note that if obtaining minor H from G requires multiple edge deletions and edge contractions, then the order in which we perform these operations is not important.

We can once again use graph P_1 from Figure 1.1 to illustrate some of these relationships. For example, if we consider $P_1 \backslash v_1 v_2$, we get the first graph shown in Figure 1.4. Note that all we have done here is remove the edge between vertices v_1 and v_2. The second graph shown in Figure 1.4 is $P_1/v_1 v_2$. The vertex labeled v_* is the vertex that we get from identifying v_1 with v_2. Note that there is an edge from v_* to every vertex that was adjacent to either v_1 or v_2. In the case of v_3 and v_4, there are two edges between each of these vertices and v_*. This is because each of these vertices was adjacent to both v_1 and v_2 in P_1. We can also say that each of the graphs pictured in Figure 1.4 is a minor of P_1.

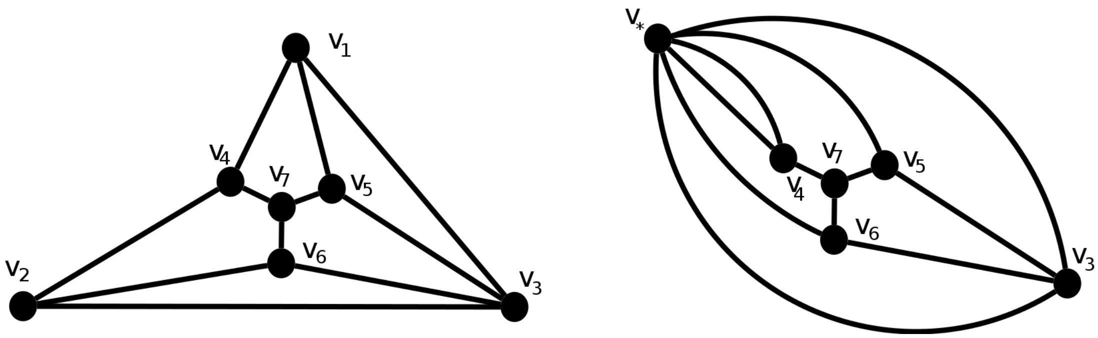


Figure 1.4: Graphs $P_1 \backslash v_1v_2$ and P_1/v_1v_2

One practical way to consider a graph $H \leq_m G$ is to model the minor H in G.

Lemma 1.2.2. *Let H be a minor of G. Since vertices of H are obtained by contracting connected subgraphs of G, there must exist a set $\{W_v : v \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ and a set $\{f_e : e \in E(H)\}$ of edges of G such that the following two properties hold:*

1) $G[W_v]$, the subgraph of G induced by W_v, is connected for every $v \in V(H)$, and

2) If $e = uv \in E(H)$, then f_e is an edge between W_u and W_v.

We say that the minor H is modeled in G by $\{W_v\}$ and $\{f_e\}$. Often, when we are trying to show that G has an H minor it is convenient to find a model of H in G.

1.3 Graph Decompositions

Graph decompositions are a powerful tool commonly used in the study of graph theory. Decomposition results are very useful because they allow us to study the structure of the graph. The general purpose of performing a decomposition is to decompose the graph into smaller pieces that are generally better connected than the original graph and allow for more effective analysis of the original graph. We also hope to be able to have a unique decomposition. Sometimes, we can prove results on the components of the decomposition which in turn allows us to prove results about the original graph. Decomposition results have a long history in the study of graphs.

In 1932, Hassler Whitney proved the uniqueness of a decomposition of 1-connected graphs [26]. In this decomposition, the graph is uniquely decomposed into 2-connected pieces. Whitney called graphs **separable** if they had such a decomposition and **non-separable** otherwise. In his paper, he was able to prove many results about both separable and non-separable graphs. Each component of Whitney's decomposition was in fact a subgraph of the original graph.

Naturally, decomposition results have been explored for graphs of higher connectivity. In 1966, Tutte decomposed 2-connected graphs into **cleavage graphs** [24]. These cleavage graphs were either polygons, bonds (graph duals of polygons), or 3-connected graphs. In this decomposition, components were not necessarily subgraphs of the original graph. Each component was however a minor of the original graph. In 1980, Cunningham and Edmonds showed that a 2-connected graph G has a unique, minimal decomposition [9]. They further showed that each component of the decomposition was prime (a graph which cannot be further decomposed in a non-trivial manner), a polygon, or a bond. Their decomposition produced the same canonical decomposition of the original graph as Tutte's decomposition using a different set of theorems. While Tutte defined the decomposition and established some properties of the decomposition, Cunningham and Edmonds further showed that the decomposition is characterized by some of the stated properties.

In 1993, Coullard, Gardner, and Wagner proved several results about decomposition of 3-connected graphs [8]. Their decomposition was based on the 3-separations of the graph. Their decomposition result satisfied many properties, including uniqueness. Their first result showed that every minimally 3-connected graph has a unique minimal decomposition such that every component of the decomposition is either cyclically 4-connected, a twirl, or a wheel. They further showed that every 3-connected graph has a unique minimal decomposition with the property that no member has a good split. Finally, they showed that a minimally 3-connected graph does not have a good split if and only if it is either cyclically 4-connected, a twirl, or a wheel.

The decomposition that we will show will decompose a 3-connected graph into quasi 4-connected components. This decomposition is similarly based on the 3-separations in the starting graph. However, the components of the decomposition that we will be considering are different than those considered in the work of Coullard, Gardner, and Wagner. We will be decomposing the starting graph into components that are all quasi 4-connected. Additionally, we will see that it will be possible for two different graphs to decompose into the same set of components.

In addition to these decomposition results, there have been many others which decompose a graph into paths and cycles, decompose complete graphs into small graphs, and focus on decomposition of hypergraphs. These are just a few examples as graph decompositions is a widely studied area.

Aside from being interesting to mathematicians, these decomposition results have many practical purposes. There are some algorithmic benefits to graph decomposition. Decomposition of a graph allows us to "divide and conquer" to make certain algorithms run more efficiently. For our decomposition, we will see a practical application to the study of graph minors. We will be able to completely classify

H-minor-free graphs for a weakly 4-connected graph H in the event that we are able to find the complete set of quasi 4-connected H-minor-free graphs.

Here, we preview some of the main decomposition results that we will cover in detail in the next chapter. We will look at two operations that we can use to decompose a graph into its quasi 4-connected components. The first operation will be a fan reduction. This operation will reduce a large fan in our graph to a smaller one. We will be able to prove the following theorem in relation to fan reductions:

Theorem 1.3.1. *Let H be a weakly 4-connected graph such that $H \neq Prism$. Let G be a 3-connected graph and let G' be a fan-reduction of G. Then G is H minor-free if and only if G' is H minor-free.*

This result gives us that if H is a minor of a graph G with a 3-separation (G_1, G_2) where G_2 is an arbitrarily large fan of size $k \geq 4$, then it is also a minor of a graph G' with a 3-separation (G'_1, G'_2) where G'_1 is almost exactly G_1 (we may add at most two new edges) and G'_2 is a fan of size exactly three. The converse result also holds. This result will be very useful to our application to graph minors.

The second operation in our decomposition will be the K_4-split. There is a similar theorem for K_4-splits as relates to graph minors.

Theorem 1.3.2. *Let H be a weakly 4-connected graph. Let (G_1, G_2) be a 3-separation of a 3-connected graph G such that neither G_1 nor G_2 is a fan. For $i = 1, 2$, let G_i^+ be the graph formed from a K_4-split of G over $\{v_1, v_2, v_3\}$. Then, H is a minor of G if and only if H is a minor of G_1^+ or G_2^+.*

We note that performing a K_4-split actually decomposes the graph into two components unlike the fan reduction.

Since both of our decomposition operation have results related to graph minors, it is natural that we explore applications of these results to graph minors. We note that the Pyramid, which we saw earlier in Figure 1.1, is a weakly 4-connected graph. Therefore, the results of the previous two theorems apply to this graph.

If we can characterize quasi 4-connected H minor-free graphs for a weakly 4-connected graph H, then we will be able to characterize the 3-connected H minor-free graphs as well. The following theorem gives the construction.

Theorem 1.3.3. *Let H be a weakly 4-connected graph such that $H \neq$ Prism. Then, 3-connected H minor-free graphs are precisely those graphs that are constructed from the quasi 4-connected H minor-free graphs by fan-extensions and K_4-sums.*

1.4 A Chain Theorem

In 1996, Politof and Satyanarayana published many results on the structure of quasi 4-connected graphs [22]. Since we are decomposing our graphs into quasi 4-connected components, we wish to study the structure of quasi 4-connected graphs in more detail.

Suppose H is a quasi 4-connected minor of a quasi 4-connected graph G. Then, an (H, G) **-chain** is a sequence $G_1, G_2, \ldots G_k$ of quasi 4-connected graphs such that $G_1 \cong H$, $G_k \cong G$, and for every $i = 1, 2, \ldots, k-1$, G_{i+1} is a G_i-add, a G_i-split, or a G_i-straddle as defined below.

A G_i-add is the addition of a single edge to G_i. We add this edge in such a way that the resultant graph is still simple, that is, we do not add any loops or parallel edges. A G_i-split replaces a vertex v of degree at least four in G_i by two adjacent vertices v' and v'' and joins all neighbors of v to exactly one of v' or v'' such that both v' and v'' have degree at least three. The requirement that both vertices have degree at least three ensures that our graph remains 3-connected. A G_i-straddle replaces an edge uv where uv is contained in a triangle uvw of G_i in which u, v, and w all have degree at least four by a new vertex x which is joined to u, v, and w. Here again, we have specific degree requirements to ensure that we will produce a 3-connected graph after performing this operation.

In their paper Politof and Stayanarayana provided a recursive theorem for generating the set of all quasi 4-connected graphs from a set of base graphs using the three operations we described. We will look at their theorem later as well as prove the following refinement of their theorem:

Theorem 1.4.1. *For every quasi 4-connected graph* $G \notin \{W_3, W_4, W_5\} \cup \{L_n, M_n : n \geq 8\}$, *there exists a* (W_4, G)*-chain.*

The families of graphs L_n and M_n will be defined in Section 3.1.

1.5 Pyramid Minor-Free Graphs

The problem of characterizing H minor-free graphs is another long studied problem in graph theory. This problem has been solved for all 3-connected graphs with at most eleven edges [14]. If we consider graphs on twelve edges, this problem has been solved for three of them. First, it has been solved for the octahedron which is a 4-connected graph [11]. It has also been solved for the two internally 4-connected graphs on twelve edges, namely the Cube and the Wagner graph M_8. There are exactly two other graphs on twelve edges that are weakly 4-connected. One graph, which is a minor of the Petersen graph, was characterized by Adam Ferguson in [15]. The other is the Pyramid graph. This graph is pictured in Figure 1.1. One of the main goals of this work is to present a characterization of graphs which are Pyramid minor-free. Not only is the Pyramid one of the smallest 3-connected graphs H for which H minor-free graphs have not yet been characterized, it is a graph which we encounter in other areas as well. First, Geelen and Zhou [17] showed that we can construct all weakly 4-connected matroids from ladders and tridents. The Pyramid is the only trident which we can represent graphically meaning it is one of the most basic weakly 4-connected graphs. Further, the Pyramid represents an excluded minor for a special class of graphs called Feynman 5-splitting graphs [5]. Further, this graph also appears in [12] as a forbidden minor for the class of 3-connected graphs of pathwidth at most three. We will also see in the next section that the Pyramid is an excluded minor for outer-projectivity. We will use our decomposition results combined with the chain theorem for generating quasi 4-connected graphs to completely characterize the set of Pyramid minor-free graphs.

We will use the results to classify completely the set of graphs which are Pyramid-minor-free. We will classify these graph using two theorems.

Theorem 1.5.1. *Quasi 4-connected, Pyramid-minor-free graphs are graphs in $\mathcal{M}$ along with 31 isolated graphs.*

For now, we simply say that $\mathcal{M}$ is an infinite family of graphs. We will explore this family in great detail later on. We will also explicitly show all of the isolated graphs mentioned. This theorem is the first step to a complete classification.

Thus far, we have said nothing about Pyramid-minor-free graphs of lower connectivity. We address this issue in another theorem.

Theorem 1.5.2. *Pyramid-minor-free graphs are precisely those graphs formed from a series of $0, 1, 2$-sums, K_4-sums, and fan extensions performed on graphs in $\mathcal{M}$, the 31 isolated graphs from the previous theorem, K_1, K_2, C_2, and C_3.*

Now, we have a means of constructing all of the Pyramid-minor-free graphs if we can simply find all of the quasi 4-connected, Pyramid-minor-free graphs. We will do precisely this is a later chapter.

1.6 Outer-projective Graphs

A graph G is called **outer-projective** if it can be drawn in the projective plane so that there is a face which meets all vertices of G. Let $G + v$ denote the graph obtained from G by adding a single vertex v to the graph which is adjacent to all of the original vertices of G. It is easy to check that G is an outer-projective graph if and only if $G + v$ is a projective graph. Forbidden minors have been characterized for projective graphs. Therefore, it is interesting to study the forbidden minors for outer-projective graphs as well. We will be able to use our characterization of Pyramid minor-free graphs in order to characterize outer-projective graphs.

Theorem 1.6.1. *Let G be a quasi 4-connected graph. Then, the following are equivalent:*

a) G is outer-projective.

b) G is Pyramid, $\mathcal{G}_{12,4}$, $\mathcal{G}_{12,5}$, $\mathcal{G}_{12,7}$, and $\mathcal{G}_{14,2}$ minor-free.

c) G is Pyramid minor-free and G is not one of the thirty-one graphs listed in Theorem 4.3.8

The graphs $\mathcal{G}_{12,4}$, $\mathcal{G}_{12,5}$, $\mathcal{G}_{12,7}$, and $\mathcal{G}_{14,2}$ will be shown in Chapter 5 and are given by their edge listings in the Appendix.

Chapter 2
Quasi 4-connected Graphs

2.1 Preliminaries

In this chapter, we will discuss how to decompose a 3-connected graph into its quasi 4-connected components. We will also discuss how to reconstruct the 3-connected graph G from its quasi 4-connected components.

Before we start discussing quasi 4-connected graphs in great detail, we first define what it means for a graph to be quasi 4-connected.

Let G be a 3-connected graph. We say that G is **quasi 4-connected** if for every 3-separation (G_1, G_2) of G, either G_1 or G_2 has exactly four vertices.

Suppose G is a quasi 4-connected graph with a 3-separation (G_1, G_2). Without loss of generality, we may assume that G_2 has exactly four vertices. Since (G_1, G_2) is a 3-separation, we know that $|V(G_1) \cap V(G_2)| = 3$ Therefore, there is exactly one vertex that is contained in G_2 that is not also contained in G_1. Alternatively, there is no limit on the number of vertices in $G_1 \backslash G_2$. The graph shown in Figure 2.1 illustrates what a typical 3-separation of a quasi 4-connected graph looks like.

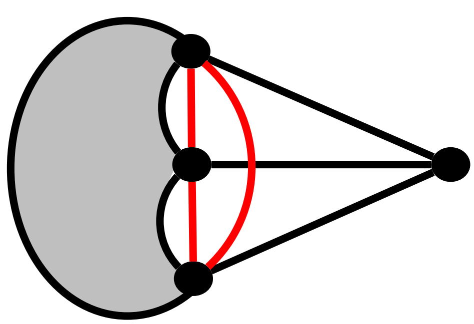


Figure 2.1: A typical 3-separation of a quasi 4-connected graph

We note a few additional properties of the separation. First, there are three edges incident to the vertex in $V(G_2)\backslash V(G_1)$. This vertex is adjacent to each of the vertices in $V(G_1) \cap V(G_2)$. All three of these edges must be present, otherwise our graph cannot possibly be quasi 4-connected as it is not even 3-connected. The red edges indicate edges that may be present in the graph. We can have any subset of these edges in the graph and our separation still satisfies the condition for quasi 4-connectivity.

Now that we know what a quasi 4-connected graph looks like, we will consider two operations that decompose a 3-connected graph into quasi 4-connected components.

2.2 Decomposition Operations

First, we will consider graphs that have a special 3-separation. Let G be a 3-connected graph with 3-separation (G_1, G_2). For $k \geq 3$, we call G_2 a **fan of size** k if $V(G_1) \cap V(G_2) = \{v_0, v_1, v_k\}$, $V(G_2)\backslash V(G_1) = \{v_2, v_3, \dots v_{k-1}\}$, and $E(G_2) = \{v_0 v_i | 2 \leq i \leq k-1\} \cup \{v_j v_{j+1} | 1 \leq j \leq k-1\} \cup F$, where F is a subset of $\{v_0 v_1, v_0 v_k, v_1 v_k\}$. We will call this special kind of 3-separation a **fan separation** of G. We can see in Figure 2.2 what a typical fan separation looks like in a graph. We note that any subset of the red edges in the graph may be present and the given separation is still a fan separation.

If a 3-connected graph G has a fan separation where the fan is of size $k \geq 4$, one decomposition that we can perform is actually reducing the size of the fan. Let x, y be cubic vertices of G such that the only five edges incident with them are xy, ux, vy, wx, wy. A **fan reduction** gives us a new graph which is obtained from $G - \{x, y\}$ by adding one new vertex z, three new edges uz, vz, wz and also

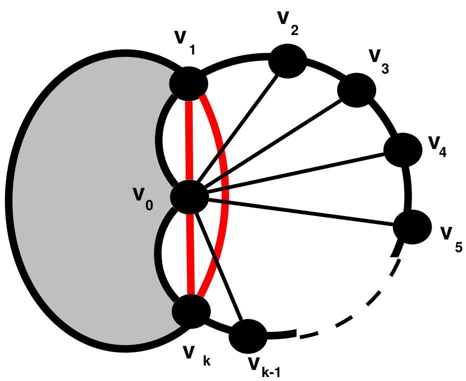


Figure 2.2: A fan-separation

two other edges wu, wv if they were not already edges of G. Figure 2.3 shows an application of fan reduction.

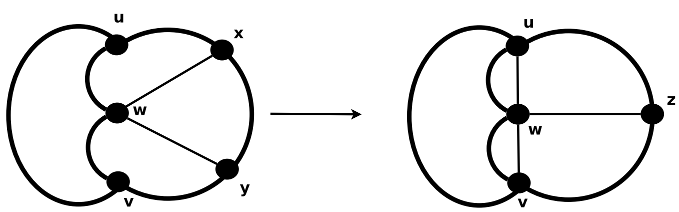


Figure 2.3: Application of Fan Reduction

This operation is actually related to fan separation. We call this operation a fan reduction since it can be used to reduce any fan of size $k \geq 4$ to a fan of size exactly three. Consider the graph G shown in Figure 2.4.

We note that this graph has a fan separation where the fan is of size k. We first consider the 3-separation of G over $\{v_0, v_1, v_4\}$. We note that this too is a fan separation where the fan is of size exactly four. Further, this 3-separation has precisely the structure described in the definition of fan reduction. Therefore, we may apply the fan reduction operation. In doing so, we delete vertices v_2 and v_3 from G. We add a new vertex v_* to G and add edges v_0v_*, v_1v_*, and v_4v_* to G. We note that the original 3-separation of G over the vertices $\{v_0, v_1, v_4\}$ is still a fan separation, but now the fan has size exactly $k-1$. If $k-1$ is still at least four,

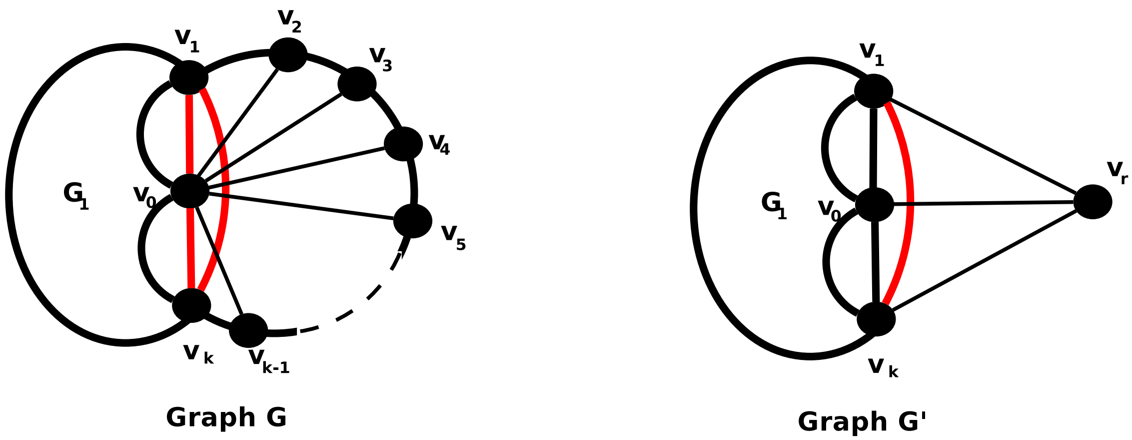


Figure 2.4: Reduction of a Large Fan

we may continue to perform fan reductions in exactly the same manner. When we can no longer apply fan reductions, our graph will be exactly the graph G' shown in Figure 2.4. In both G and G', the red edges denote edges that may or may not be present in the graph. The edge v_1v_k will be an edge in G' if and only if it is an edge in G.

The second decomposition operation that we will use will be applied to any 3-separation which is not a fan separation. Let G be a graph with 3-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{v_1, v_2, v_3\}$. For $i = 1, 2$, let G_i^+ be the graph obtained from G_i by adding a new vertex v_4, adding three new edges v_1v_4, v_2v_4, v_3v_4, and adding edges v_1v_2, v_1v_3, v_2v_3 if they were not already present in G_i. We call this operation a K_4**-split** of G over $\{v_1, v_2, v_3\}$. Note that unlike fan reduction where we started with a single graph and still had a single graph after the reduction, K_4-split takes a single graph and decomposes it into two graphs. It is worth noting that G_i^+ is not necessarily a minor of the original graph G.

We illustrate the process for performing a K_4-split on a graph in a series of figures. First note the graph in Figure 2.5.

Let us consider the 3-separation shown in Figure 2.6.

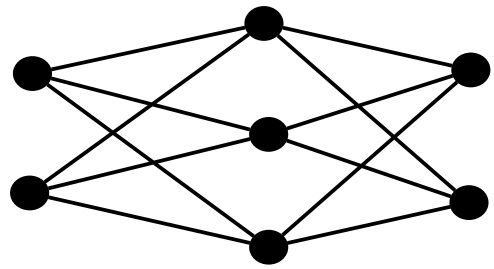


Figure 2.5: Performing a K_4-split

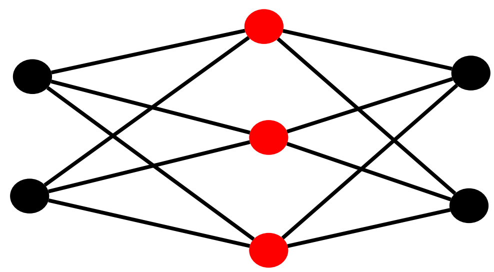


Figure 2.6: A 3-separation

Both sides of this 3-separation have exactly five vertices. Therefore, this graph is not quasi 4-connected. We also note that neither side of the 3-separation is a fan. If we did have a fan on one side of the separation, we would simply perform a fan reduction first. We will perform a K_4 split of this graph over the three red vertices.

First, we separate the graph into two components. A copy of each of the red vertices is present in both components as shown in Figure 2.7.

Figure 2.7: Splitting the Graph into Two Pieces

Finally, we add the required new vertex to each component as well as the required new edges. The two components generated from this K_4-split can be seen in Figure 2.8. It can be easily verified that each of these components is quasi 4-connected. If we had one or more components that were not quasi 4-connected, we would simply perform another K_4-split to the necessary component(s).

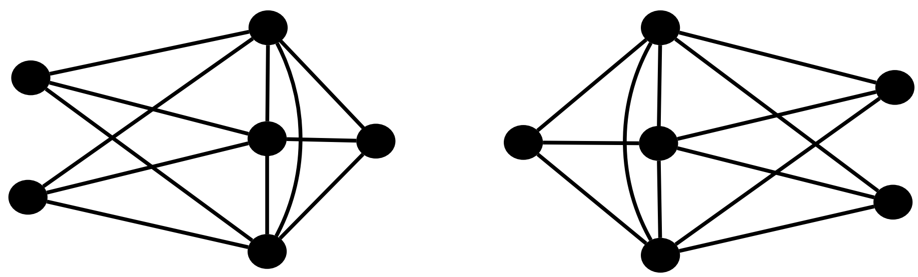


Figure 2.8: Two Components generated from the K_4-split

Now that we have these two operations, we are ready to decompose a 3-connected graph into its quasi 4-connected components. We follow the steps as listed to achieve the decomposition:

1) Identify a fan separation in graph G.

2) If the size of the fan is greater than three, perform fan reductions until it is reduced to a fan of size exactly three.

3) Repeat steps 1 and 2 until the only fan separations left are those with fans of size exactly three.

4) Identify a 3-separation (G_1, G_2) in the graph where both sides of the separation have more than four vertices.

5) Perform a K_4-split over the vertices in $V(G_1) \cap V(G_2)$.

6) Repeat steps 4 and 5 until all components of the decomposition are quasi 4-connected.

The components that are generated from Steps 1-6 are the quasi 4-connected components of graph G. For any 3-connected graph G, we achieve a unique decomposition from 1-6.

We would like to note that it is possible to reconstruct graph G from its quasi 4-connected components. This also requires two operations. First, let w be a cubic vertex of G such that $N_G(w) = \{x, y, z\}$ and both xy and yz are both edges of G. A **fan extension** of G is obtained by splitting the vertex w into two adjacent vertices u and v such that u is adjacent to both x and y and v is adjacent to both y and z. Edges xy and yz may be preserved in the extended graph or we may choose to delete them, provided none of x, y, or z has degree less than three after the edge deletions. We note that when performing fan extensions, we may produce more than one graph as the result. We refer back to Figure 2.3 for illustration. If we start from the graph on the right and apply a fan extension, the graph on the left is one possible graph which can be generated by this operation. We note that there are three other graphs that could possibly be generated by performing this fan extension.

We also have an operation that can reverse a K_4-split. Let G be a graph with 3-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{v_1, v_2, v_3\}$, $V(G_2) - V(G_1) = \{v_4\}$, and all possible edges between v_1, v_2, v_3, v_4 are present. Let H be a graph with 3-separation (H_1, H_2) such that $V(H_1) \cap V(H_2) = \{u_1, u_2, u_3\}$, $V(H_2) - V(H_1) = \{u_4\}$, and all possible edges between u_1, u_2, u_3, u_4 are present. The K_4**-sum** of G and H, denoted $G \oplus_{K_4} H$, is obtained from $G\backslash\{v_4\}$ and $H\backslash\{u_4\}$ by identifying u_i with v_i for $i = 1, 2, 3$ and possibly deleting some of the identified edges. Again, we may be required to keep some of the edges to ensure the identified vertices all have degree at least three. This operation almost reverses a K_4-split. Note that a K_4-sum could actually produce more than one graph. In fact, there are six possibilities for the resultant graph depending on which of the identified edges we choose to delete. Therefore, if we start from a graph G, perform a K_4-split, and then perform a K_4-sum on the two components, we have six possible graphs that our result could be. G is one of those six graphs.

We can see the process of performing a K_4-sum by reversing Figures 2.8 through 2.5. First, we start with two graphs that each contain a 3-separation where one side of the separation is K_4 as seen in Figure 2.8. Then, we delete vertex from each component that is contained solely on the K_4 side of the separation. Note, we actually get the graphs shown in Figure 2.9.

Figure 2.9: Reconstructing a Graph via K_4-sum

We do not remove any of the edges between the three remaining vertices of the K_4 yet. Then, we identify the remaining K_4 vertices of one component with the remaining K_4 vertices from the second component. Then, we may remove any of the identified edges. One possible graph that we could generate in this way is the graph pictured in Figure 2.5.

There are many ways we could choose to decompose a 3-connected graph. Why choose this way? It turns out that the graphs generated from either a fan reduction or a K_4-split have some interesting properties in relation to the original starting graph. We will explore some of these properties now.

The property that we will be exploring relates to finding a weakly 4-connected graph H as a minor in both the original graph G and the components of the decomposition. We call a 3-connected graph G **weakly 4-connected** if for every 3-separation (G_1, G_2) of G, either G_1 or G_2 has at most four edges.

Theorem 2.2.1. *Let H be a weakly 4-connected graph such that $H \neq$ Prism (shown in Figure 2.10).*

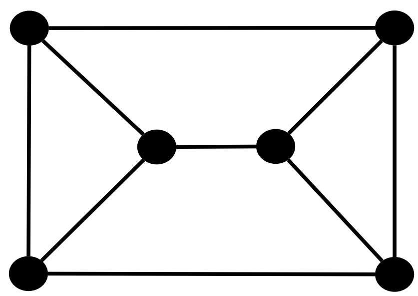


Figure 2.10: The Prism

Let G be a 3-connected graph and let G' be a fan reduction of G. Then, G is H minor-free if and only if G' is H minor-free.

We will require several lemmas to prove this result.

First, since we will be concerned with 3-connected graphs for these results, the following theorems of Seymour and Tutte will be useful. The first result of Seymour will help us ensure that our graphs remain 3-connected. The second result of Tutte gives an easy way to identify whether certain graphs are 3-connected.

Lemma 2.2.2. *Let e be an edge of a 3-connected graph G with $|G| \geq 5$. Then, either G/e is obtained from a 3-connected graph by adding parallel edges or $G \backslash e$ is obtained from a 3-connected graph by subdividing edges. [23]*

Lemma 2.2.3. *A graph is 3-connected if and only if it is obtained from a wheel by repeatedly adding edges and splitting vertices. [14]*

The next two lemmas can help us find a specific minor in a given graph provided that the graph has a special structure.

Lemma 2.2.4. *Let H be a graph with all vertices of degree at least three. If H is a minor of a graph G with a vertex v of degree two whose adjacent edges are e_1 and e_2, then H is a minor of G/e_i for $i = 1$ or 2.*

Proof. Suppose neither e_1 nor e_2 can be contracted to obtain an H-minor. Then we must delete either e_1 or e_2 to form the minor since H has no vertices of degree two. Suppose, without loss of generality, that we delete edge e_1. Now v has degree one. Therefore, we must delete or contract e_2 to form the minor since H has no vertices of degree one. However, deletion or contraction of e_2 will produce the same graph. Therefore, we can form the H-minor by contracting e_2. □

Lemma 2.2.5. *Let H be a simple graph. If H is a minor of a graph G with parallel edges e_1, e_2, then H is a minor of $G \backslash e_i$ for $i = 1$ or $i = 2$.*

Proof. Suppose neither e_1 nor e_2 can be deleted to obtain an H-minor. Then, we must contract either e_1 or e_2 to form the minor since H has no parallel edges or loops. Suppose, without loss of generality, that we contract edge e_1. Now e_2 is a loop. Therefore, we must delete or contract e_2 to form the minor since H has no loops. However, deletion or contraction of e_2 will produce the same graph. Therefore, we can form the H-minor by deleting e_2. □

Before proving the theorem, we consider finding a weakly 4-connected minor in two graphs both of which have fan separations with a specific structure.

Lemma 2.2.6. *Let $H \neq W_3$ be a weakly 4-connected graph. Let w be a cubic vertex of a graph G such that $N_G(w) = \{x, y, z\}$ and $xy, yz \in E(G)$. If H is a minor of*

G, then at least one of wx, wy, wz, xy, xz must be deleted or contracted to form the minor H.

Proof. Suppose none of wx, wy, wz, xy, xz is deleted or contracted to form the minor H. Then, we can consider w, x, y, z as vertices of H. The vertices x, y, z are all still distinct vertices since H is a simple graph. Let $H_1 = H \backslash w$. We consider two cases.

Case 1: $|H_1| = 3$. If the only vertices in H_1 are x, y, z, then the only edges which can be in $E(H_1)$ are $\{xy, xz, yz\}$ which means H would have to be W_3. All of these edges must be contained in H_1, otherwise H would not be 3-connected, so $H = W_3$ is the only possibility.

Case 2: $|H_1| \geq 4$. Let H_2 be the subgraph of H induced by $\{w, x, y, z\}$. Then, (H_1, H_2) is a 3-separation of H. Since H is a weakly 4-connected graph, side H_1 must have four or fewer edges since H_2 has at least five edges. If a single vertex u is in $V(H_1)$ but not in $V(H_2)$, then u must have degree at least three since H is 3-connected. But this would mean that $|E(H_1)| \geq 5$ which contradicts H being weakly 4-connected. Similarly, it is not possible for H_1 to contain more than four vertices.

Therefore, we must delete or contract at least one of wx, wy, wz, xy, xz in order to obtain the H-minor. □

Lemma 2.2.7. *Let $H \notin \{W_3,$ Prism$\}$ be a weakly 4-connected graph. Let u, v be cubic vertices of a graph G such that $F = \{uv, ux, uy, vy, vz\}$ is the set of edges in G that are incident with u or v. If H is a minor of G, then at least one of the edges from F must be deleted or contracted to form the minor H.*

Proof. Suppose no edge from F is deleted or contracted to form the minor H. Then, we can consider u, v, x, y, z as vertices of H. We first note that x, y, z are distinct vertices in H. The pairs x, y and y, z are distinct since H is simple. The only other possibility is if there is a xz-path which is contracted to form new vertex w. In this case, $\{w, y\}$ forms a 2-separation of H, which contradicts H being 3-connected, unless $H = W_3$ which is disallowed by assumption.

Now, let $H_1 = H \backslash \{u, v\}$. We consider two cases.

Case 1: $|H_1| = 3$. If the only vertices in H_1 are x, y, z, then the only edges which may be contained in $E(H_1)$ are xy, xz, yz. In fact, all of these edges must contained in $E(H_1)$, or else H would not be 3-connected. However, in this case $H = W_4$ which is not a weakly 4-connected graph.

Case 2: $|H_1| \geq 4$. Let H_2 be the subgraph of H induced by $\{u, v, x, y, z\}$. Then, (H_1, H_2) is a 3-separation of H. Since H is a weakly 4-connected graph, side H_1 must have four or fewer edges since H_2 has at least five edges. If a single vertex q is in $V(H_1)$ but not it $V(G_2)$, then q must have degree at least 3 since H is 3-connected. This means that $|E(H_1)| = 3$ or 4. If $|E(H_1)| = 3$, x has degree 2, which would mean that H is not 3-connected. Therefore, it must be that $|E(H_1)| = 4$. The only graph of this type which is 3-connected (and also weakly 4-connected) is the Prism. It is not possible for H_1 to contain more than four vertices, as the resulting graph H would not be weakly 4-connected as both H_1 and H_2 would need to have at least five edges to maintain 3-connectedness.

Therefore, at least one edge in F must be deleted or contracted to form the minor H.

□

Proof of Theorem 2.2.1. Case 1: $H = W_3$: Since G is 3-connected, by Theorem 2.2.3, G must contain a W_3-minor.

Let G' be the graph generated from a fan reduction of G. We will see later that this graph is also guaranteed to be 3-connected. Therefore G' must also always contain a W_3 minor.

Therefore, we can conclude that the theorem must hold for $H = W_3$.

Case 2: Now, we may assume that $H \neq W_3$. Suppose H is a minor of G. Let x and y be cubic vertices of G such that the only five edges incident with either of them are xy, ux, vy, wx, wy. According to Lemma 2.2.7, at least one edge of $\{wx, wy, ux, xy, vy\}$ must be deleted or contracted to form the H-minor. We consider the possibilities. Contraction of ux or vy yields a graph which is a minor of G'. Contraction of wx, wy, or xy yields parallel edges. By Lemma 2.2.5, we may delete one of the parallel edges and still contain the H-minor. The resulting graph is a minor of G'. Therefore, we assume that we cannot contract any edges in $\{wx, wy, ux, xy, vy\}$ to form the H-minor. However, regardless of which edge we delete, we get at least one vertex of degree two. Then, by Lemma 2.2.4, we can contract one of the edges adjacent to the degree two vertex to form the minor, a contradiction. Therefore, H must be a minor of G'.

Let G' be the graph generated from a fan reduction of G. Now assume that H is a minor of G'. According to Lemma 2.2.6, at least one edge of $\{uw, vw, uz, vz, wz\}$ must be deleted or contracted to form the H-minor. We consider the possibilities. Deletion of uw or vw yields a graph which is a minor of G. Deletion of uz, vz, or wz will result in z being a degree two vertex. By Lemma 2.2.4, we may contract one of the adjacent edges and still contain the H-minor. By Lemma 2.2.5, if we form any parallel edges, we may delete one and still contain the H-minor. In any case, the resulting graph will be a minor of G. Therefore, we assume that we cannot delete any edges to form the H minor. However, regardless of which edge we contract, we get a set of parallel edges. Then, by Lemma 2.2.5, we can delete one of the edges to form the H-minor, a contradiction. Therefore H must be a minor of G.

$\square$

We note that H = Prism is actually an exception to Theorem 2.2.1, with the counterexample shown in Figure 2.11 where the first graph is G and the second is G'.

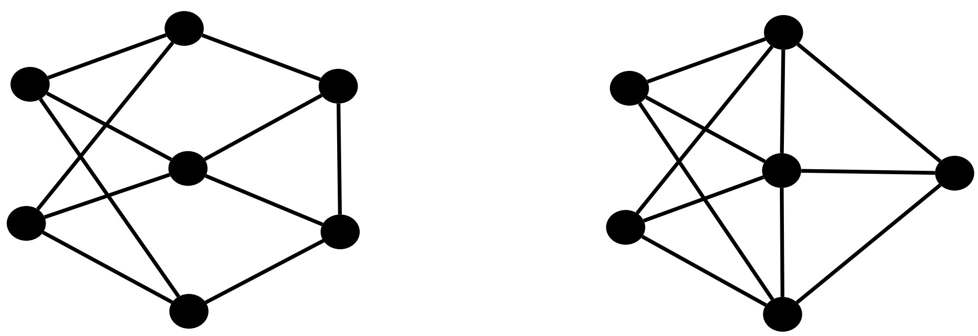


Figure 2.11: Prism as a Counter-example to Theorem 2.2.1

It is easy to check that the Prism is a minor of G but is not a minor of G'.

There is a similar minor result for our second decomposition operation.

Theorem 2.2.8. *Let H be a weakly 4-connected graph. Let G be a 3-connected graph with a 3-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{v_1, v_2, v_3\}$ and neither G_1 nor G_2 is a fan. For $i = 1, 2$, let G_i^+ be the graphs formed from a K_4-split of G over $\{v_1, v_2, v_3\}$. Then, H is a minor of G if and only if H is a minor of G_1^+ or G_2^+. (Graphs G_1^+ and G_2^+ are illustrated in Figure 2.12.)*

We will need the following lemma to prove this result.

Lemma 2.2.9. *Let G be a 3-connected graph with 3-separation (G_1, G_2) such that $G_1 \cap G_2 = \{x, y, z\}$. If G_2 does not contain the graph F (as shown in Figure 2.13 as a minor, with vertices x, y, and z preserved, then $G_2 \backslash z$ is an xy-path. [12]*

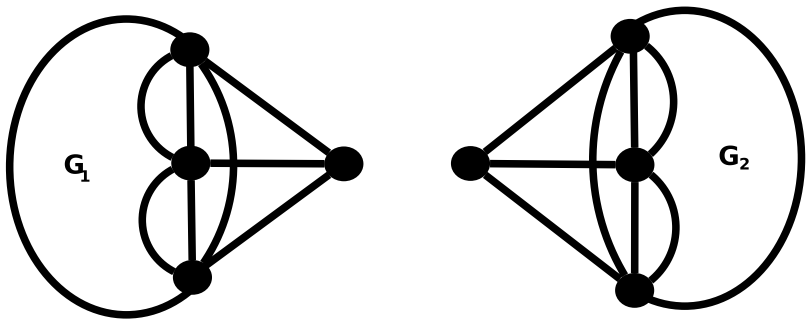


Figure 2.12: Graphs G_1^+ and G_2^+

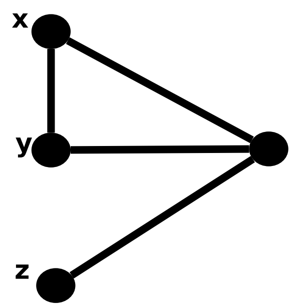


Figure 2.13: Graph F

Proof of Theorem 2.2.8. Case 1: $H = W_3$ Since G is 3-connected, it contains $H = W_3$ by Theorem 2.2.3. In both G_1^+ and G_2^+, the subgraph induced by v_1, v_2, v_3, and v_4 is exactly W_3, so both of these graphs contain W_3 as a minor.

Case 2: $H \neq W_3$ Suppose H is a minor of either G_1^+ or G_2^+. Without loss of generality, we may assume H is a minor of G_1^+. Since H is weakly 4-connected, we know that one side of any 3-separation of H may contain at most four edges. Since H is not W_3, at least one vertex from G_1 must be used to form the minor. To avoid having both sides of the corresponding 3-separation of H from having too many edges, at least one of $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$ must be either deleted or contracted. Suppose, first, that we delete either v_1v_4, v_2v_4, or v_3v_4 to form the H-minor. Now v_4 is a degree two vertex and by Lemma 2.2.4, we may contract one of the adjacent edges and still have a graph which contains that H-minor. Contracting either of the adjacent edges will result in a set of parallel edges. By

Lemma 2.2.5, we may delete one of the parallel edges and still contain the H-minor. The resulting graph, $G_1^+\backslash v_4$, is always a minor of G unless G_2 is exactly the fan of size three which it cannot be by assumption. Therefore, we may assume that we cannot delete any of v_1v_4, v_2v_4, or v_3v_4 to form the H-minor. Contraction of any of the six edges will always result in a set of parallel edges. By Lemma 2.2.5, we can delete one of the parallel edges and still contain the H-minor. We could have found the H-minor by deleting the edge first. One edge in the parallel edge pair will be either v_1v_4, v_2v_4, or v_3v_4 regardless of which edge in the graph was contracted. Therefore, we assume that we do not contract any edges to form the H-minor. If we delete either v_1v_2, v_2v_3, or v_1v_3, we must delete or contract something else from $\{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$ since otherwise H is not weakly 4-connected. This means we can only delete a second edges of v_1v_2, v_2v_3, or v_1v_3. Suppose the resulting graph is also not a minor of G. Suppose without loss of generality that the edge of v_1v_2, v_2v_3, and v_1v_3 which remains is v_1v_2. Then, by Lemma 2.2.9, since G is 3-connected $G_2\backslash v_3$ is a path from v_1 to v_2. If $G_2\backslash v_3$ consists on only a single edge from v_1 to v_2, then G_1^+ is always a minor of G unless G_2 is exactly the fan of size 3 which it cannot be by assumption. Suppose then that there are $n \geq 1$ vertices other than v_1 and v_2 on this path. Then, since G is 3-connected, G_2 would have to be the fan of size $n + 2$ which it cannot be by assumption. Therefore, H must be a minor of G.

Now, suppose H is a minor of G. By Lemma 1.2.2, we can find a model $\{\{W_v\}, \{f_e\}\}$ of H in G.

Let $V_0 = V(G_1) \cap V(G_2)$ and let Z be the set of vertices v of H with $W_v \cap V_0 \neq \emptyset$. Then $|Z| \leq |V_0| = 3$. We first observe that there do not exist vertices $w, x, y, z \in V(H)$ with both W_w and W_x contained in $V(G_1)\backslash V_0$ and both W_y and W_z contained in $V(G_2)\backslash V_0$. Suppose otherwise. Since V_0 separates W_w and W_x from W_y and W_z

in G, Z separates w and x from y and z in H which means that H has an l-separation with $l = |Z| \leq 3$ which separates w and x from y and z. If $l < 3$, then this contradicts H being 3-connected. If $l = 3$, then H has a 3-separation which has at least two vertices on each side of the separation and thus at least five edges on each side of the separation (otherwise H is not 3-connected). This contradicts H being weakly 4-connected. Therefore, we may assume, without loss of generality, that there is at most one vertex w such that W_w is contained in $V(G_1)\backslash V_0$.

Now, we consider two possibilities for $|Z|$.

$|Z| < 3$: If $|Z| < 3$, then we note that there can be no vertex w in H such that W_w is contained in $V(G_1)\backslash V_0$. Otherwise, H would have a k-separation with $k = |Z| < 3$ which separates w from the rest of H, contradicting H being 3-connected. Therefore, no vertices of the H-minor are contained entirely in $G_1\backslash G_2$. The only thing which may be contributed to the minor from $G_1\backslash G_2$ are edges between vertices in Z. Since all edges between vertices in Z are accounted for in G_2^+,the H-minor is contained in G_2^+.

$|Z| = 3$: We will now show that we can find a model of H in G_2^+. Any W_v that are contained in G_2 will also be contained in G_2^+. Any edges f_e that are contained in G_2 are again also contained in G_2^+. The vertices v_1, v_2, v_3 in G can be modeled by the vertices v_1, v_2, v_3 in G_2^+. Finally, if we do have a W_w contained in $V(G_1)\backslash V_0$, then it is modeled by the single vertex w in $G_2^+\backslash V(G_2)\backslash\{v_1, v_2, v_3\}$. All possible edges between w, v_1, v_2 and v_3 are present in G_2^+. Therefore, H is a minor of G_2^+. (Note: For this direction we did not need that G is a 3-connected graph). $\square$

This theorem tells us that if we perform any K_4-split on a 3-connected graph G, then any weakly 4-connected minors H of G are contained in a component of the

K_4-split. Likewise, if we can find an H minor in one of the components generated from a K_4-split, this means there was an H minor in the original graph G.

Finally, we wish to show that by performing fan extensions and K_4-sums on the quasi 4-connected H minor-free graphs, we can in fact generate all of the 3-connected H minor-free graphs. We do so with the following theorem.

Theorem 2.2.10. *Let H be a weakly 4-connected graph such that $H \neq Prism$. Then, 3-connected, H minor-free graphs are precisely those graphs that are constructed from the quasi 4-connected, H minor-free graphs by fan-extensions and K_4-sums.*

Proof. Let G be any 3-connected, H minor-free graph. Then, we can easily decompose G into its quasi 4-connected components as previously described. We are guaranteed by Theorems 2.2.1 and 2.2.8 that each of the quasi 4-connected components that we generate will also be H minor-free. We know that we can reform G through some series of the operations fan extension and K_4-sum.

Now, we wish to show that performing some series of fan extension and K_4-sum on quasi 4-connected, H minor-free graphs will always yield a 3-connected, H minor-free graph. Again, we are guaranteed by Theorems 2.2.1 and 2.2.8 that the resultant graph will be H minor-free. We now show that the resultant graph will also be 3-connected.

First, we consider performing a fan extension on a quasi 4-connected graph as shown in Figure 2.14.

We will show that there are still three internally disjoint paths between every pair of vertices. First, we consider w_2 to w_3. There is an edge between those two vertices, so that constitutes one path between them. There is also the path $w_2w_0w_3$. Finally, there is a path starting at w_2 going to w_1 which passes through G_1 to w_4

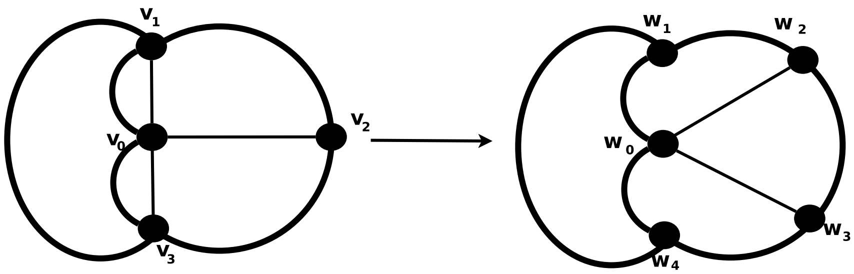


Figure 2.14: Fan Extension

and ending at w_3. It is easy to check that these paths are in fact internally disjoint. Now, we check w_0 to w_2. There is an edge between these two vertices, so that is one path between them. There is also the path $w_0w_3w_2$. Finally, there is a path starting at w_0, going through vertices in G_1 to w_1 which is adjacent to w_2. Again, we may easily verify that these paths are disjoint. A similar argument shows that there are three internally disjoint paths between w_0 and w_3. Next, we check w_2 to any vertex contained in G_1. We note that in the original graph, there were three internally disjoint paths from v_2 to any vertex in G_1. We can easily replicate those paths in the extended graph. A similar argument holds to show three internally disjoint paths from w_3 to a vertex in G_1. Now, we consider two vertices both contained in G_1. There were three internally disjoint paths between any such pair in the original graph. If all three of these paths were contained entirely in G_1, then those same paths exist in the extended graph. If any of the paths leave G_1, they must travel through one of the center vertices out of G_1 and back through a second of the center vertices. Therefore, there can be only one such path of this type. Whichever path it took out of G_1 and back can be easily replicated in the extended graph. Now, we consider w_0 to any vertex in G_1. We note that there were three internally disjoint paths from v_0 to any vertex in G_1 in the original graph. If all of the edges on the path were contained exclusively in G_1, then we can easily replicate those paths in the extended graph. There can be at most two such paths that were had edges not exclusively contained in G_1, one which passed through v_1 and one which passed through v_3. Since in the extended graph, we have similar paths from w_0 to w_1 and w_0 to w_4, we could replicate either of these paths. Therefore, there are still three internally disjoint paths in the extended graph. Now, we consider w_1 to w_4. If both edges w_0w_1 and w_0w_4 exist, then finding three internally disjoint paths is easy. Therefore, we may assume that one of these edges does not exist, say w_0w_4. Then, w_4 must have two neighbors u_1 and u_2 in G_1. We note that there must be two internally disjoint paths P_1 and P_2 from w_1 to u_1 and two internally disjoint paths Q_1 and Q_2 from w_1 to u_2. Therefore, starting at u_1, we may travel along P_1 until the first time we intersect one of the Q_i paths. Without loss of generality, suppose we intersect Q_1. Then, we travel on Q_1 the rest of the way until we reach w_1. The second path is Q_2 which is completely disjoint from the first path. Finally, we may take the path through w_2 and w_3. This type of argument will be the basis for the remainder of our arguments. Now we consider w_1 to any vertex in G_1. At most two of the paths between these vertices can pass through edges not exclusively contained in G_1. If, the edge w_0w_1 is present, we can replicate both of those paths in the extended graph. Otherwise, w_1 must have two neighbors in G_1 and we can form two internally disjoint paths as in the previous case. This argument also works to show paths from w_4 to any vertex in G_1. Again for w_1 to w_0, we may assume the edge between them is not present, otherwise finding the three paths is easy. Therefore, w_1 must have two neighbors in G_1 and we can therefore find two internally disjoint paths which are both disjoint from the $w_1w_2w_0$ path. This argument also works for w_0 and w_4. Finally, we consider w_1 to w_2. One path between them is the edge between them. If w_0w_1 is an edge, then finding the paths is easy. Otherwise w_1 must have two neighbors in G_1. Each of these neighbors has two internally disjoint paths from it to w_0 and two internally disjoint paths from it to w_4. Using these paths we can find a path from w_1 to w_0 and a path from w_1 to w_4 such that the pair of paths is internally disjoint. This method also can be used to show w_2 to w_4, w_1 to w_3, and w_3 to w_4.

Now, we show that after performing a K_4-sum, on two quasi 4-connected graphs that the resulting graph is 3-connected. Consider $K = G \oplus_{K_4} H$. We show that there are three internally disjoint paths between every pair of vertices in K. Consider two vertices from G that were not a part of the K_4 that we summed over. There were three internally disjoint paths between these two vertices in G. At most one such path could have used edges from the K_4. This path can be replicated using edges from H. Therefore, three paths still exist. A similar argument holds for two vertices in H not contained in the K_4. Now consider a vertex u from G and v from H, neither of which was contained in the K_4. We can find three disjoint paths in G from u to each of the three remaining K_4 vertices, and can find similar paths in H from v to each of the K_4 vertices. If we link these paths at the three K_4 vertices, then we have three internally disjoint paths from u to v. Now, we consider paths that begin, end, or both at the K_4 vertices. Consider one vertex u that is one of the K_4 vertices and a vertex v that is not. We may assume that none of the edges between the K_4 vertices is present since otherwise finding the paths is easy. Therefore, u must have degree at least three. It may have either two neighbors in G and one in H or vice versa. Either way, we can find internally disjoint paths through these three vertices as in the fan extension case. Finally we consider that u and v are both K_4 vertices. First, we show that if none of the three edges is deleted during the K_4-sum, then we can find the three internally disjoint paths. By preserving all three edges, we automatically have two internally disjoint paths between u and v. Also, we are guaranteed a path from u to v contained exclusively in G which is disjoint from the other two, giving us the required three internally disjoint paths. Now, if we assume that one or more of the edges was deleted when performing the K_4-sum, we must make use of the result of Lemma 2.2.2 to show 3-connectivity. Let K be the graph $G \oplus_{K_4} H$ without any of the edges deleted. We know K is 3-connected. Lemma 2.2.2 says that for any edge e in K, either the simplification of K/e is 3-connected or $K \backslash e$ is the subdivision of a 3-connected graph. If we let e be one of the center edges, consider K/e. This graph has a clear 2-separation and therefore we know $K \backslash e$ must be the subdivision of a 3-connected graph. Since the operation of K_4-sum ensures all vertices have degree at least three before an edge may be deleted, we know there can be no subdivided edges and $K \backslash e$ must still be 3-connected. The same analysis holds for any of the other center edges we wish to delete.

$\square$

Since our decomposition results yield quasi 4-connected graphs as components, we want to explore the structure of quasi 4-connected graphs further. In the next section, we will see how we can algorithmically generate quasi 4-connected graphs.

Chapter 3

A Chain Theorem for Generating Quasi 4-connected Graphs

3.1 The Chain Theorem

Chain theorems enable us to construct "all" of the graphs of a given connectivity using a set of base graphs and one or more operations. To generate the entire set of graphs of the given connectivity, we perform the given operations on the known graphs in the set which generates more graphs in the set. We can then apply the given operations to those graphs to generate more graphs which are in the set. This process could go on infinitely if we keep generating new graphs by performing the given operations. One well-known example of a chain theorem is for 3-connected graphs.

Lemma 3.1.1. *A graph G is 3-connected if and only if G can be constructed from a wheel by repeatedly performing the two operations of adding a non-parallel edge and splitting a vertex.*

To generate 3-connected graphs, our base graphs are wheels. We have two operations that we can use to generate more 3-connected graphs. We can either add an edge or split a vertex.

Now that we know how to decompose a graph into its quasi 4-connected components, we will take a closer look at the structure of a quasi 4-connected graph. In particular, we will look at an algorithm that allows us to recursively generate the set of quasi 4-connected graphs from a set of base graphs using three operations.

Suppose H is a quasi 4-connected minor of a quasi 4-connected graph G. Then, an (H, G) **-chain** is a sequence $G_1, G_2, \ldots G_k$ of quasi 4-connected graphs such that $G_1 \cong H$, $G_k \cong G$, and for every $i = 1, 2, \ldots, k-1$, $G_i + 1$ is a G_i-add, a G_i-split, or a G_i-straddle as defined below.

A G_i-add is the addition of a single edge to G_i. We add this edge in such a way that the resultant graph is still simple, that is, we do not add any loops or parallel edges. A G_i-split replaces a vertex v of degree at least four in G_i by two adjacent vertices v' and v'' and joins all neighbors of v to exactly one of v' or v'' such that both v' and v'' have degree at least three. The requirement that both vertices have degree at least three ensures that our graph remains 3-connected. A G_i-straddle replaces an edge uv where uv is contained in a triangle uvw of G_i in which u, v, and w all have degree at least four by a new vertex x which is joined to u, v, and w. Here again, we have specific degree requirements to ensure that we will produce a 3-connected graph after performing this operation.

In generating quasi 4-connected graphs, we will encounter two infinite families of graphs, all of whose members are quasi 4-connected. First, we have $\{M_n : n \geq 8\}$, where M_n is the **Möbius ladder** on n vertices. This graph is the graph formed from an even cycle of length n by adding edges $\{ij | 1 \leq i \leq \frac{n}{2}, j = i + \frac{n}{2}\}$. We will also have $\{L_n : n \geq 8\}$ or the **circular ladder** on n vertices. This graph is the graph formed by taking two cycles of length $\frac{n}{2}$ and adding an edge between corresponding vertices of the two cycles. These two graphs can be seen in more detail in Figure 3.1 where we provide drawings of the circular ladder L_8 and the Möbius ladder M_8 respectively.

Now we are able to look at the main structure theorem for quasi 4-connected graphs. In 1996. Politof and Satyanarayana proved the following theorem:

Theorem 3.1.2. *Suppose G is a quasi 4-connected graph. Then exactly one of the following holds:*

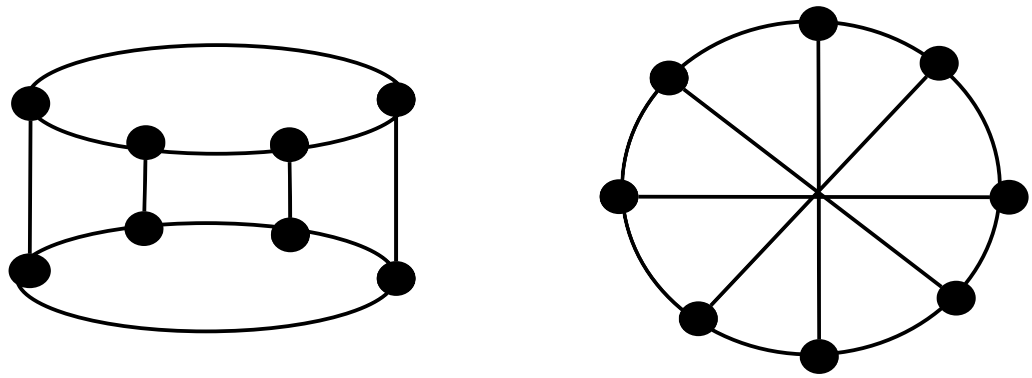


Figure 3.1: A circular ladder and a Möbius ladder

(a) G is either W_3, W_4, or W_5, or a circular ladder or Möbius ladder on n vertices for some even $n \geq 8$.

(b) G is obtained from a quasi 4-connected graph H by an application of one of the following three operations.

1. *The addition of an edge to H.*
2. *The replacement of a vertex u of degree ≥ 4 in H by two adjacent vertices u' and u'' and joining all vertices in the neighborhood of u to exactly one of u', u'', such that both u' and u'' will have degree ≥ 3.*
3. *The replacement of an edge uv, where uv is contained in a triangle uvw of H with $deg_H(z) > 3$ for all $z \in \{u, v, w\}$, by a new vertex x and joining it to u, v, and w. [22]*

Basically, theorem 3.1.2 tells us that for every quasi 4-connected graph G, there exists an (H, G)-chain with $H \in \{W_3, W_4, W_5\} \cup \{L_n, M_n : n \geq 8\}$. The three graphs W_3, W_4, and W_5 indicated in the theorem are the three smallest wheels.

Using this result, we can establish the following:

Theorem 3.1.3. *For every quasi 4-connected graph $G \notin \{W_3, W_5\} \cup \{L_n, M_n : n \geq 8\}$, there exists a (W_4, G)-chain.*

Proof. Suppose there is a counterexample. Let G be a counterexample with the least number of edges. By Theorem 3.1.2, there exists an (H, G)-chain with $H \in \{W_3, W_5\} \cup \{L_n, M_n : n \geq 8\}$. It follows from the minimality of G that G must be an H-add, an H-split, or an H-straddle. We consider the possibilities.

If $H = W_3$, there are no edges that can be added. There are also no vertices of degree four, so we may not perform a split or straddle. Thus, G does not exist.

If $H = W_5$, G could be an H-add or an H-split. Neither of the two H-splits are quasi 4-connected. Therefore, G must be an H-add. Let v_0 be the hub vertex of H and let v_1, v_2, v_3, v_4, v_5 by the vertices on the cycle of H. We cannot add any edges with v_0 as an endpoint. Therefore, we may assume that the edge added is v_1v_3. Then, $G' = G \backslash v_0v_2$ is weakly 4-connected and is not a graph in $\{W_3, W_5\} \cup \{L_n, M_n : n \geq 8\}$. By minimality of G, there must be a (W_4, G)-chain, a contradiction.

If $H = L_n$ or M_n, then G must be an H-add since H is cubic. In this case, we make the following observation. Let $e = xy$ be a rim edge of H and let x', x'', y, y'' be as shown in Figure 3.2.

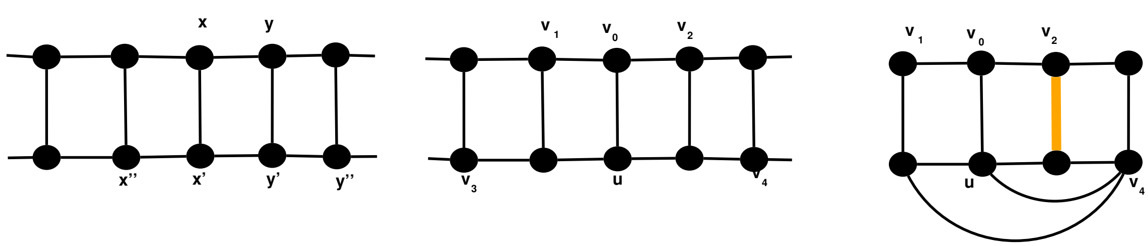


Figure 3.2: Structure of Graph H

Let z denote the new vertex of H/e. Then, $\{z, x'', y''\}$ is a 3-cut of H/e which separates x' and y' from the rest of the graph. This means that H/e cannot be quasi 4-connected. However, it is straightforward to show that $\{z, x'', y''\}$ is the only 3-cut of H/e that separates H/e into two pieces which each have at least two vertices not contained in the other part.

Now suppose $G = H + uv$. We prove that H has a rim edge $e = xy$ such that, under the above labeling, uv is between $x'y'$ and $H - \{x, y, x', y', x'', y''\}$. Then, the uniqueness of the 3-cut $\{z, x'', y''\}$ implies that G/e is quasi 4-connected.

If $v \notin \{v_1, v_3\}$, then $xy = v_0v_1$ satisfies our requirement. If $v = v_1$ or $v = v_3 \neq v_4$, then $xy = v_0v_2$ satisfies our requirement. Finally, if $v = v_3 = v_4$, then H is the cube and $e = v_2v_5$ is a rim edges which satisfies our requirement.

Since G/e has a degree four vertex, G/e cannot belong to $\{W_3, W_5\} \cup \{L_n, M_n : n \geq 8\}$. By minimality of G, there must be a (W_4, G)-chain, a contradiction.

□

3.2 Small Quasi 4-connected Graphs

We can apply the chain theorem to begin generating quasi 4-connected graphs. Here, we show a few lemmas which find the sets of quasi 4-connected graphs on a small number of edges. We show quasi 4-connected graphs on up to eleven edges and explain the process for finding the larger quasi 4-connected graphs.

Lemma 3.2.1. *The quasi 4-connected graphs containing nine or fewer edges are W_3, W_4, $K_5\backslash e$, Prism, and $K_{3,3}$ as shown in Figure 3.3.*

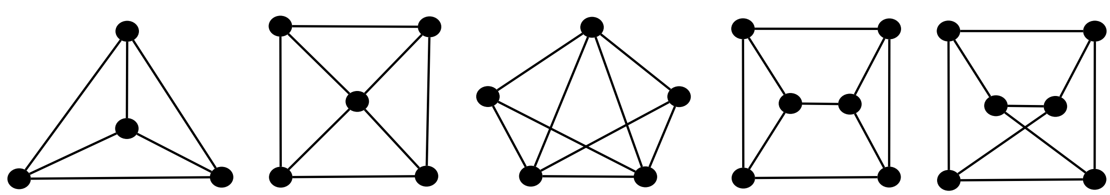


Figure 3.3: Quasi 4-connected Graphs on Nine or fewer edges

Proof. From Theorem 3.1.2, both W_3 and W_4 must be included on this list. We may generate further graphs on this list by performing adds, splits, or straddles to W_4. There is only one (non-isomorphic) way to add an edge to W_4. It produces the graph above labeled $K_5 \backslash e$. There is only one vertex of degree at least four in W_4. There are two (non-isomorphic) ways in which we can split that vertex. This yields the two graphs above labeled Prism and $K_{3,3}$. $\square$

We will continue in the same way to determine the quasi 4-connected graphs with a higher number of edges. As a general first step when we are trying to find the list of quasi 4-connected graphs on n edges, we will apply adds and splits to the list of quasi 4-connected graphs on $n-1$ edges and apply straddles to the list of quasi 4-connected graphs on $n-2$ edges. It is possible that doing so may generate isomorphic graphs, in which case we only keep one copy. It is also possible that we may produce graphs which are not quasi 4-connected in which case we remove them from the list.

Lemma 3.2.2. *The quasi 4-connected graphs containing ten edges are K_5, $Prism+e$, $K_{3,3}+e$, and W_5 as shown in Figure 3.4.*

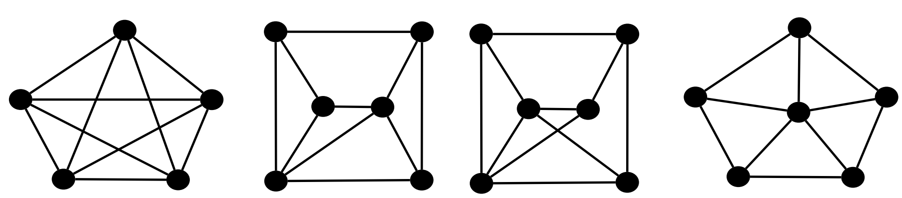


Figure 3.4: Quasi 4-connected Graphs on Ten edges

Proof. To generate the quasi 4-connected graphs on ten edges, we will need to perform adds and splits to the three nine edge graphs from the previous lemma. We will also need to perform straddles on W_4. Finally, we add W_5 to our list. There is only one edge which can be added to $K_5 \backslash e$. This yields the graph labeled

K_5 above. There is only one (non-isomorphic) way to add an edge to the Prism. It is given by the graph above labeled Prism $+e$. There is also only one (non-isomorphic) way to add an edge to $K_{3,3}$. It is given by the graph above labeled $K_{3,3}+e$. All vertices in both the Prism $K_{3,3}$ have degree three, so we cannot split vertices in either of these graphs. We may, however, split a vertex in $K_5\backslash e$. There are two (non-isomorphic) ways to do so. One yields Prism $+e$ and the other yields $K_{3,3}+e$. Since both of these graphs are already in the list, we do not need to add them again. We note that since W_4 only has one vertex of degree at least four, so we cannot perform any straddles on this graph. Finally, we add W_5 to complete our list. $\square$

Lemma 3.2.3. *The quasi 4-connected graphs containing eleven edges are W_5+e, $K_{3,3}^{\perp}$, $Oct\backslash e$, and $K_{3,3}^{\ddagger}$ as shown in Figure 3.5.*

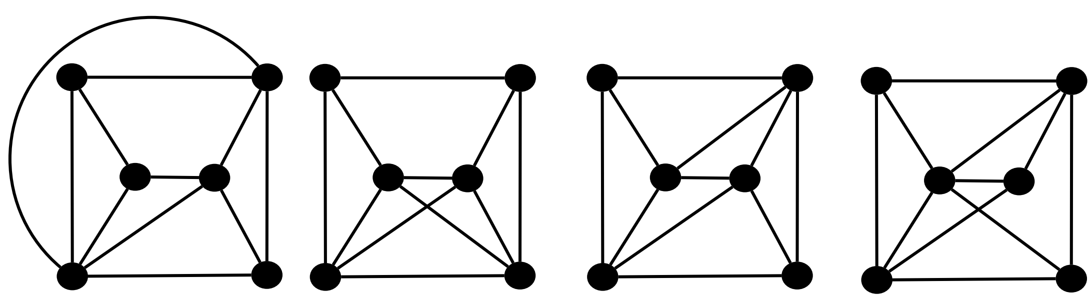


Figure 3.5: Quasi 4-connected Graphs on Eleven edges

Proof. To generate the quasi 4-connected graphs on eleven edges, we will need to perform adds and splits to the ten edge graphs from the previous lemma. Since K_5 is a complete graph, we cannot add any edges to it. There are three (non-isomorphic) ways to add an edge to Prism $+e$. They are W_5+e, $K_{3,3}^{\perp}$, and $Oct\backslash e$. There are two (non-isomorphic) ways to add an edge to $K_{3,3}+e$. One is a graph which we have already generated, namely $K_{3,3}^{\perp}$. The other is $K_{3,3}^{\ddagger}$. There is one (non-isomorphic) way to add an edge to W_5. It produces $W_5 + e$. We now move on to splitting vertices in the ten edge graphs. There is only one (non-isomorphic) way to split a vertex in K_5. It yields $K_{3,3}^{\perp}$. There are three (non-isomorphic) ways to split a vertex in Prism $+e$. However, none of these three graphs is quasi 4-connected. Therefore, we do not include them in our list. There is one (non-isomorphic) way to split a vertex in $K_{3,3} + e$. However, it also produces a graph which is not quasi 4-connected and is also left off of our list. Finally, there are two (non-isomorphic) ways to split a vertex in W_5. Again, neither of these graphs is quasi 4-connected, so they are also left off of our list. There is one way to perform a straddle in $K_5 \backslash e$. It produces $K_{3,3}^{\ddagger}$. The straddle operation cannot be applied to either Prism or $K_{3,3}$, so our list is complete. $\square$

Now that we have generated a few sets of small quasi 4-connected graphs, we can easily extend the same process to finding larger quasi 4-connected graphs. We note that we can write computer programs implementing the recursive algorithm to find these sets of graphs for us. In some sense this is more efficient as it keeps track of the graphs for us, can be made to automatically remove isomorphic graph copies, can be made to remove the graphs that are not quasi 4-connected from the list, and ensures that we do not omit any graphs from our list.

Chapter 4
Pyramid Minor-Free Graphs

4.1 Introduction

Decomposition results are generally very useful when it comes to trying to characterize H-free graphs for a graph H of a given connectivity. For example, we can consider the following result related to connected graphs.

Lemma 4.1.1. *If H is connected, then H minor-free graphs are precisely 0-sums of connected H minor-free graphs.*

We can consider each connected component of a disconnected graph as a component of the graph decomposition. If we consider this decomposition of a disconnected graph G, then G contains an H minor, where H is a connected graph, precisely when H is contained in at least one of the connected components of G. This simplifies the problem of finding the H minor to finding it in a single connected component.

There are similar results for graphs of higher connectivity.

Lemma 4.1.2. *(i) If H is 2-connected, then H minor-free graphs are precisely* 0-, 1- *sums of loops, K_1, K_2, and 2-connected H minor-free graphs.*

(ii) If H is 3-connected, then H minor-free graphs are precisely 0-,1-, 2-*sums of loops, K_1, K_2, C_2, C_3, and 3-connected H minor-free graphs.*

These results are very useful to us. If we want to be able to characterize Pyramid minor-free graphs, these lemmas tell us that it is sufficient to characterize all of the 3-connected, Pyramid minor-free graphs. Using the results of the previous two chapters, it is actually sufficient for us to be able to characterize all of the quasi 4-connected, Pyramid minor-free graphs. We can use the operations of K_4-sum and fan extension to generate from those all of the 3-connected, Pyramid minor-free graphs.

In the previous section, we generated all quasi 4-connected graphs on at most eleven edges. We note that since the Pyramid is a twelve edge graph, all of these quasi 4-connected graphs are necessarily Pyramid minor-free. In generating quasi 4-connected graphs, we noted two infinite families, namely L_n and M_n. One of these infinite families, M_n, gives rise to an infinite family of quasi 4-connected, Pyramid minor-free graphs which we will explore in detail.

4.2 An Infinite Family of Pyramid Minor-Free Graphs

When describing quasi 4-connected graphs with no Pyramid minor, we have one infinite family of graphs, namely the graphs in the $\mathcal{M}$ class of graphs. We say that a graph G belongs to $\mathcal{M}$ if it is a Möbius ladder or a quasi 4-connected minor of any Möbius ladder. Recall that we defined a Möbius ladder as an even cycle $1, 2, \ldots, n$ of **rim edges** plus **chord edges** ij where $1 \leq i \leq \frac{n}{2}$ and $j = i + \frac{n}{2}$.

We note that all graphs in $\mathcal{M}$ can be formed from Möbius ladders by contracting rim edges. Therefore, we may still talk about rim and chord edges for any graph M' in $\mathcal{M}$ by simply considering the smallest Möbius ladder M which contains it as a minor. We simply identify the rim and chord edges in M and perform the necessary contractions to yield M'. Any rim edges from M that remain are rim edges in M' and likewise any chord edges from M that remain are chord edges in M'.

We will first show that the Pyramid is not contained in this class of graphs. Then, we will show that performing any of the three operations listed in Theorem 3.1.2 will either yield a larger graph in the $\mathcal{M}$ class of graphs or will yield a graph that is outside of the class because it is no longer quasi 4-connected or contains a Pyramid minor.

Lemma 4.2.1. *Let M be a graph in the class $\mathcal{M}$. Let xx' and yy' be two distinct chords in M. Then either xx' and yy' cross or they are incident.*

Proof. Let us call any two distinct chords **parallel** if they do not cross and are not incident. Suppose xx' and yy' are parallel in M. Since M was formed by contracting rim edges of some M_n, it follows that if we reverse the contraction operation on M, we should get M_n. However, xx' and yy' will remain parallel regardless of how many rim edges we uncontract. We know that M_n has no parallel chords, thus a contradiction. Chords xx' and yy' must either cross or intersect. □

Using this result, we can very easily see that the Pyramid is not a graph in $\mathcal{M}$. The Pyramid has exactly one Hamiltonian cycle up to symmetry. With respect to this cycle, the Pyramid has two pairs of parallel chords as seen in Figure 4.1. Therefore, it cannot be in $\mathcal{M}$.

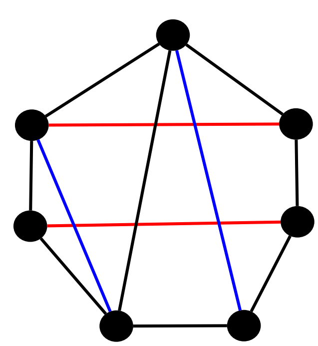


Figure 4.1: Hamiltonian Cycle of Pyramid showing Parallel Chords

Now, we will explore a few properties of graphs in $\mathcal{M}$ which will be useful in later proofs.

Consider any vertex u in a graph M in $\mathcal{M}$. We will denote the degree of vertex u by d_u. Then, we can let uu_i for $i = 1, 2, \ldots k$ denote all of the chords incident with vertex u, where $u_1, u_2, \ldots u_k$ are enumerated in the order they appear clockwise on the rim. There are a few facts we can easily discuss about these vertices.

Lemma 4.2.2. *Let u be a vertex of a graph M in $\mathcal{M}$ as described above. The following hold for M:*

a) $\{u_1, u_2, \ldots, u_k\}$ is a consecutive set and $d_{u_i} = 3$ for all $1 < i < k$.

b) $d_u \leq 5$ for every u.

c) If $d_u = 5$, then $d_{u_1} \geq 4$ and $d_{u_3} \geq 4$.

d) If $d_u = 4$, then $d_{u_1} \geq 4$ or $d_{u_2} \geq 4$.

Proof. a) First suppose $\{u_1, u_2, \ldots, u_k\}$ is not a consecutive set. Then, we may assume without loss of generality that there is a vertex v between u_1 and u_2 on the rim on M that is not adjacent to u. Then, vv_i would be parallel to either uu_1 or uu_2 for any chord vv_i which is impossible by Lemma 4.2.1. Now suppose that there exists a u_i for $1 < i < k$ such that $d_{u_i} > 3$. We may assume that $d_{u_2} > 3$. We know u_2 is adjacent to u, u_1, and u_3. Let w be another neighbor of u_2. Then, wu_2 is parallel to either uu_1 or uu_3 which again contradicts Lemma 4.2.1.

b) Otherwise, $\{u, u_1, u_k\}$ is a 3-cut which has at least two vertices on each side of the separation violating M being quasi 4-connected.

c) Otherwise either $\{u, u_0, u_3\}$ or $\{u, u_1, u_4\}$ is a 3-cut which again violates M being quasi 4-connected.

d) Otherwise $\{u, u_0, u_3\}$ is a 3-cut which again violates M being quasi 4-connected.

□

In describing graphs in $\mathcal{M}$ we will read vertices clockwise around the rim. We will use $u \to v \to w$ to indicate that after we read u from the rim we will read v before reading w. We will use (u, w) to denote the set of all vertices v such that $u \to v \to w$.

Lemma 4.2.3. *Let $x \to u \to y' \to x' \to v \to y \to x$ for a graph M in $\mathcal{M}$. Let xx' and yy' also be chords in M. If there are no chords between vertices in (x, y') and vertices in (x', y), then $(x, y') = \{u\}$ and $(x', y) = \{v\}$.*

Proof. Suppose there is a vertex v' such that $x' \to v \to v' \to y$. Suppose there are no chords between vertices in (x, y') and vertices in (x', y). Then u must be adjacent to either x' or y. By symmetry, we assume ux' is a chord in M. Then both v and v' must both be adjacent to y' since the only other option is that they are adjacent to x and thus parallel to ux' which we know is impossible. This means y' has degree five by Lemma 4.2.2 part b. Therefore, by Lemma 4.2.2 part c, v must have degree at least four. However, to avoid parallel chords v must either be adjacent to u or a vertex in (u, y') and would thus be adjacent to a vertex in (x, y'), a contradiction. □

Now that we have established some basic properties of graphs in $\mathcal{M}$, we want to establish what happens when we apply any of the operations from part b of Theorem 2.1 to a graph in $\mathcal{M}$.

Theorem 4.2.4. *Let M be a graph in the class $\mathcal{M}$ with at least 8 edges such that M is not the graph N shown in Figure 4.2. Let M' be a quasi 4-connected graph generated by adding an edge to M. Then, either M' is also a graph in $\mathcal{M}$ or M' contains a Pyramid minor.*

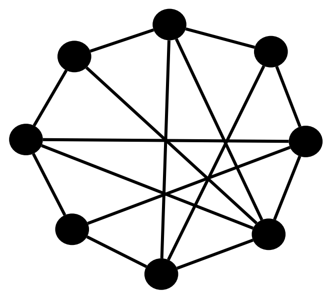


Figure 4.2: Graph N

Proof. Let M be a graph in $\mathcal{M}$. Let $M' = M + xy$. First, there must be chords xx' and yy' such that $x \to y' \to x' \to y$. Choose chord xx' such that (x', y) is as short as possible. If no such pair of chords exists, then by Lemma 4.2.1, $x'y$ must be a chord. Since xy is not an edge of M, there must be a vertex w such that $y \to w \to x$ which means that zx' is an edge. By Lemma 4.2.2 parts b and c, there exists another chord xx'' which will either contradict Lemma 4.2.1 or minimality of (x', y).

Now, we will choose x' and y' such that xx' and yy' are chords and (x, y') and (x', y) are minimized.

We can first establish that there must be a chord zz' such that $x \to y' \to z' \to x' \to y \to z \to x$. Suppose there is no such chord. Since x and y are non-adjacent, let z be an vertex in (y, x). By Lemma 4.2.1, z must be adjacent to either x' or y', say x'. Now, any chord incident with x must be incident with either y or a vertex in (x', y) by Lemma 4.2.1. We already know xy is not an edge and having x adjacent to a vertex in (x', y) would contradict the minimality of (x', y). Therefore,

x must have degree exactly three. Therefore, by Lemma 4.2.2, part d, z must have degree at least four. This means that zy' must also be an edge and y similarly has degree exactly three. Since z was an arbitrary vertex in (y, x), it follows that z is the only vertex in (y, x). It also follows that $x'y'$ is an edge of G. Since G has at least eight vertices, we may assume that there exist vertices u and v such that $x' \to u \to v \to y$. Edges in (x, y') can be contracted so that ux and vy' are edges. Then, M' has a Pyramid minor.

Next we show that either (x, y') or (x', y) must be empty. Suppose neither is empty. There can be no edge from a vertex in (x, y') to a vertex in (x', y), otherwise M' contain a Pyramid minor. By Lemma 4.2.3, $(x, y') = \{u\}$ and $(x', y) = \{v\}$. Note that u is adjacent to either x' or y. By minimality of (x, y'), we know that ux' is a chord. Then vy' must also be a chord. Further, both u and v must have degree exactly three which means both x and y must have degree at least four. If (y', z') is non-empty, then it contains a vertex w which must be adjacent to y. This graph has a Pyramid minor. Thus, both $y'z'$ and $x'z'$ are edges. Therefore, xz' and yz' must also be edges. By Lemma 4.2.2 part a, zx and zy are also edges. By assumption, M is not this graph, a contradiction.

Finally, we will show that both (x', y) and (x, y') are empty and therefore M' is a graph in $\mathcal{M}$. We will begin by assuming that xy' is an edge and (x', y) contains a vertex v. Then, v must be adjacent to either x or y'. By minimality of (x', y), vy' must be a chord and v has degree exactly three. If (y', z') contains a vertex u, then u must be adjacent to a vertex u' in (y, z). Note that u' can be contracted to either y or z. In either case, M' contains a Pyramid minor. Therefore, $y'z'$ must be an edge. By a similar argument, $z'x'$ is also an edge. If we choose v with (x', v) minimal, then $x'v$ must be an edge by minimality of (x', y). Furthermore, vy is also an edge because otherwise y' would have degree five which would mean v must have degree at least four which it does not. Therefore, y has degree at least four and $z'y$ is a chord. If (y, z) is non-empty, it contains a vertex w which must be adjacent to z' which implies z' has degree five and zx' must be a chord. In this case, we again have a Pyramid minor in M'. Thus yz must be an edge. Since our graph has at least eight vertices, (z, x) must contain a vertex u which must be adjacent to either z' or x'. In either case, we again get a Pyramid minor in M'. □

Now, we will consider a similar result for the operation of splitting a vertex.

Theorem 4.2.5. *Let M be a graph in $\mathcal{M}$ with at least nine vertices. Let M' be a quasi 4-connected graph generated from M by splitting a vertex x. Then, either M' is also a graph in $\mathcal{M}$ or M' contains a Pyramid minor.*

Proof. We will first consider splitting a vertex x of degree four in M. Suppose the neighbors of x are x_0, x_1, x_2, and x_3 arranged counterclockwise around the rim. We can split vertex x into x and x^* in three ways.

If x is adjacent to x_0 and x_2 and x^* is adjacent to x_1 and x_3 in M', then M' is in $\mathcal{M}$.

Now suppose x is adjacent to x_0 and x_1 and x^* is adjacent to x_2 and x_3 in M'. Since x has degree four in M, then at least one of x_1 and x_2 also has degree four. We let x_1 have degree four. Therefore, x_1x_3 must be an edge. There must be a chord yy' in M that crosses both xx_1 and xx_2. Otherwise, since x_1 could have at most one other neighbor and x_2 could also have at most one other neighbor, M could have at most seven vertices, a contradiction. We will choose the chord yy' such that neither y nor y' is a neighbor of x. Such a chord exists, otherwise M could not possibly have nine vertices. We can assume that y' is in (x_0, x_1) and y is in (x_2, x_3). First, we assume x_3 has degree at least four. Any chords from x_3 must be adjacent to a vertex in (y', x_1) or to y' itself. If x_3 is adjacent to a vertex in

(y', x_1), then M' contains a Pyramid minor. Therefore, x_3y' is a chord and there can be no vertices in (y', x_1). Any vertex in (y, x_3) must be adjacent to y'. If such a vertex exists, then y' has degree five, so y must have degree at least four. Then, M' will contain a Pyramid minor. Therefore, there can be no vertices in (y, x_3). There must be at least one vertex u in $x_0, y')$ and at least one vertex v in (x_2, y), otherwise M cannot have nine vertices. If there exists a pair u, v such that u and v are adjacent, then M' contains a Pyramid minor. If there is no such pair, then u could be adjacent to y making v adjacent to x_0 or u could be adjacent to x_2 making v adjacent to y'. In either case, M' contains a Pyramid minor. Therefore, x_3 must have degree exactly three. Any vertex z in (y', x_1) must be adjacent to either y or a vertex in (y, x_3). If z is adjacent to y, then y has degree four in M. Therefore, either y' or z must have degree at least four. Any valid edge from y' or z results in M' having a Pyramid minor. If z is adjacent to a vertex z' contained in (y, x_3), then M' again contains a Pyramid minor. Therefore, there can be no vertices in (y', x_1). Any vertex w in (y, x_3) must be adjacent to either y' or x_1. If w is adjacent to x_1, then x_1 has degree five in M requiring w to have degree at least four. So w must also be adjacent to y'. Then, M' again contains a Pyramid minor. So, w must be adjacent to y' and w has degree exactly three. This means y' has degree four in M and so y must also have degree at least four. Whichever vertex y is adjacent to results in M' having a Pyramid minor. Therefore, there can be no vertices in (y, x_3). Now, we can again choose a vertex u and v as before and will be able to similarly find a Pyramid minor.

Finally, we assume that x is adjacent to x_0 and x_3 and x^* is adjacent to x_1 and x_2. Similar to the previous case, we may assume x_1x_3 is an edge. By the same argument, we again have a chord yy' which cross both xx_1 and xx_2 such that neither y nor y' is a neighbor of x and y' is in (x_0, x_1) and y is in (x_2, x_3). There must be at least one chord edge leaving x_0. It can either go to x_2 or to a vertex in (y, x_2). In either case, we have a Pyramid minor.

Now, we consider splitting a vertex of degree five. Suppose the neighbors of x are x_0, x_1, x_2, x_3, and x_4 arranged counterclockwise around the rim. We can split x into x and x^* six non-isomorphic ways. If x is adjacent to x_0 and x_3 and x^* is adjacent to x_1, x_2, and x_4, then M' is still in $\mathcal{M}$.

For the other five cases, we note that since x has degree five in M, both x_1 and x_3 must have degree at least four. Therefore, both x_0x_3 and x_1x_4 must be chords in M.

We also note that M must have a chord yy' which crosses all of xx_1, xx_2, and xx_3, otherwise M could not have nine vertices. We also note that we can find yy' such that neither y nor y' is a neighbor of x for the same reason. We let y' be in (x_0, x_1) and let y be in (x_3, x_4). Now, we consider M' as the following splits:

(1) x is adjacent to x_0 and x_1; x^* is adjacent to x_2, x_3 and x_4

(2) x is adjacent to x_0 and x_2; x^* is adjacent to x_1, x_3 and x_4

(3) x is adjacent to x_0 and x_4; x^* is adjacent to x_1, x_2 and x_3

(4) x is adjacent to x_1 and x_2; x^* is adjacent to x_0, x_3 and x_4

(5) x is adjacent to x_1 and x_3; x^* is adjacent to x_0, x_2 and x_4

In all of these cases, M' contains a Pyramid minor. □

Theorem 4.2.6. *Let M be a graph in $\mathcal{M}$ with at least eight vertices. Let M' be a quasi 4-connected graph generated from M by straddling a triangle in M. Then, either M' is also a graph in $\mathcal{M}$ or M' contains a Pyramid minor.*

Proof. It is impossible that we could have a triangle formed by three rim edges, two rim edges and a rung edge, or three rung edges. Therefore, our triangle must have two rung edges x_1x_2 and x_1x_3 and one rim edge x_2x_3. Since x_2 and x_3 have degree at least four, there must be chords x_3x_4 and x_2x_5 such that xx_4 and xx_5 are also edges. There must be a chord yy' such that y is in (x_4, x_2) and y' is in (x_3, x_5). Otherwise, M can have at most seven vertices. We can split the triangle in three ways. If we subdivide x_2x_3 with vertex v with x_1 adjacent to v, then M' is still in $\mathcal{M}$. If we subdivide x_1x_2 with v where x_3 adjacent to v or if we subdivide x_1x_3 with v where v is adjacent to x_2, then M' contains a Pyramid minor found by contraction of yy'. □

4.3 Pyramid-free Graphs

We can continue generating quasi 4-connected graphs using adds, splits, and straddles from the graphs in Lemmas 3.2.2 and 3.2.3. Now, in addition to generating isomorphic graphs and graphs which are not quasi 4-connected, we may also generate graphs that contain a Pyramid-minor. We may also delete those, since any larger graph generated from a graph containing the Pyramid will itself contain the Pyramid. The remainder of the results in this paper were verified using two independently written Mathematica programs. The results of this programming are outlined in a few lemmas.

Lemma 4.3.1. *There are seven quasi 4-connected, Pyramid minor-free graphs on twelve edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the eleven edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the ten edge graphs. Between the two lists, we have eighteen total graphs. We must ensure that both the circular ladder and the Möbius ladder on twelve edges are present, so we add them to the list, giving us twenty graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to nineteen graphs. We delete any graphs which contain a Pyramid minor which leaves eighteen total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of seven unique, quasi 4-connected, Pyramid minor-free graphs on twelve edges. □

Lemma 4.3.2. *There are eight quasi 4-connected, Pyramid minor-free graphs on thirteen edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the twelve edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the eleven edge graphs. Between the two lists, we have thirty-two total graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to twenty-eight graphs. We delete any graphs which contain a Pyramid minor which leaves twenty-three total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of eight unique, quasi 4-connected, Pyramid minor-free graphs on thirteen edges. □

Lemma 4.3.3. *There are fourteen quasi 4-connected, Pyramid minor-free graphs on fourteen edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the thirteen edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the twelve edge graphs. Between the two lists, we have sixty-four total graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to fifty-six graphs. We delete any graphs which contain a Pyramid minor which leaves thirty-three total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of fourteen unique, quasi 4-connected, Pyramid minor-free graphs on fourteen edges. □

Lemma 4.3.4. *There are fifteen quasi 4-connected, Pyramid minor-free graphs on fifteen edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the fourteen edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the thirteen edge graphs. Between the two lists, we have one hundred and fifty total graphs. We must ensure that both the circular ladder and the Möbius ladder on fifteen edges are present, so we add them to the list, giving us one hundred and fifty-two graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to one hundred and thirty-one graphs. We delete any graphs which contain a Pyramid minor which leaves forty-six total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of fifteen unique, quasi 4-connected, Pyramid minor-free graphs on fifteen edges. □

Lemma 4.3.5. *There are thirteen quasi 4-connected, Pyramid minor-free graphs on sixteen edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the fifteen edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the fourteen edge graphs. Between the two lists, we have two hundred and forty-eight total graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to two hundred and twenty-one graphs. We delete any graphs which contain a Pyramid minor which leaves fifty-four total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of thirteen unique, quasi 4-connected, Pyramid minor-free graphs on sixteen edges. □

Lemma 4.3.6. *There are ten quasi 4-connected, Pyramid minor-free graphs on seventeen edges as listed in Appendix A.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the sixteen edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the fifteen edge graphs. Between the two lists, we have two hundred and ninety-two total graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to two hundred and sixty-six graphs. We delete any graphs which contain a Pyramid minor which leaves fifty total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of ten unique, quasi 4-connected, Pyramid minor-free graphs on seventeen edges. □

Lemma 4.3.7. *There are fifteen quasi 4-connected, Pyramid minor-free graphs on eighteen edges as listed in Appendix A. Further, all fifteen of these graphs are in the class $\mathcal{M}$.*

Proof. We begin by generating the list of unique graphs which can be formed from adding an edge or splitting a vertex in one of the seventeen edge graphs. We also generate the list of unique graphs which can be formed from straddling a triangle in one of the sixteen edge graphs. Between the two lists, we have two hundred and ninety-eight total graphs. We must ensure that both the circular ladder and the Möbius ladder on eighteen edges are present, so we add them to the list, giving us three hundred graphs. We delete duplicate copies of isomorphic graphs, bringing the total number to two hundred and seventy-five graphs. We delete any graphs which contain a Pyramid minor which leaves forty-five total graphs. Finally, we remove those graphs which are not quasi 4-connected giving a final list of fifteen unique, quasi 4-connected, Pyramid minor-free graphs on eighteen edges. Further, we can verify that each of these graphs is in fact a minor of some Möbius ladder and is therefore in $\mathcal{M}$. □

We can combine the results of the Lemmas 4.3.1 through 4.3.7 to state the following theorem:

Theorem 4.3.8. *Quasi 4-connected, Pyramid-free graphs are graphs in* $\mathcal{M}$ *along with 31 isolated graphs as shown in Figure 4.3.*

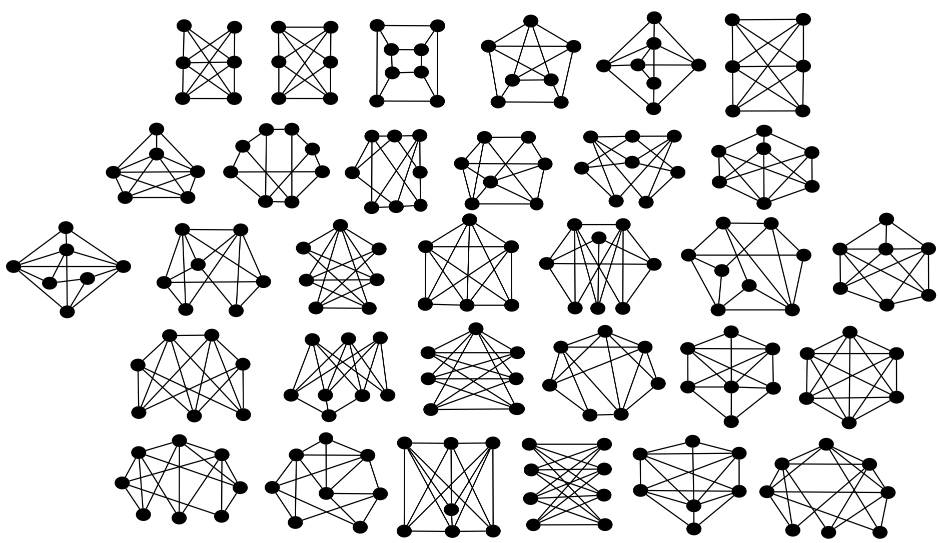


Figure 4.3: Thirty-one Pyramid minor-free graphs not in $\mathcal{M}$

All quasi 4-connected, Pyramid-free graphs on eighteen edges were determined to be graphs in $\mathcal{M}$. Additionally, each of these graphs has at least nine vertices.

Therefore, the previous section tells us that any larger graph generated from these graphs that are quasi 4-connected and Pyramid-free will also be in $\mathcal{M}$. Since there is one graph on seventeen edges on our list that is not in $\mathcal{M}$, we check any straddles which can be applied to this graph. Any quasi 4-connected, Pyramid minor-free graph resulting from a straddle of this graph is a graph on nineteen edges which belongs to the family $\mathcal{M}$ with at least nine vertices. Therefore, we do not need to generate larger graphs as all of the quasi 4-connected, Pyramid-free graphs with more than seventeen edges will always be in $\mathcal{M}$.

Theorem 4.3.9. *Pyramid-minor-free graphs are precisely those graphs formed from a series of* $0, 1, 2$*-sums,* K_4*-sums, and fan extensions performed on graphs in* $\mathcal{M}$*, the 31 isolated graphs from Theorem 4.3.8,* K_1*,* K_2*,* C_2*, and* C_3*.*

We note that all of the quasi 4-connected, Pyramid-free graphs up to eighteen edges generated by our Mathematica programs are given in Appendix A by their edge listings. In this appendix, graphs denoted with (*) represent the thirty-one isolated graphs not in the family $\mathcal{M}$ that are depicted in Figure 4.3.

Chapter 5
Outer-projective Graphs

5.1 Introduction

A graph G is called **outer-projective** if it admits a drawing on the projective plane so that there is a face meeting all vertices of G. It is easy to check that the class of outer-projective graphs is minor-closed. Therefore, we have interest in determining the set of forbidden minors for the class of outer-projective graphs.

A related problem is to determine the forbidden minors for outer-planar graphs. This problem can be easily solved by applying Kuratowski's Theorem. For any graph G, let $G + v$ denote the graph obtained from G by adding a single vertex v to the vertex set and adding an edge from each of the vertices of G to v. That is, $V(G + v) = V(G) \cup \{v\}$ and $E(G + v) = E(G) \cup \{vv_i | v_i \in G\}$. Then, G is outer-planar if and only if $G + v$ is planar. From Kuratowski's Theorem, we can therefore say that G is outer-planar if and only if $G + v$ is $\{K_5, K_{3,3}\}$ minor-free. This is equivalent to saying that G is $\{K_4, K_{2,3}\}$ minor-free. This gives us the set of forbidden minors for the class of outer-planar graphs.

We can use the same approach to characterize the forbidden minors for the class of outer-projective graphs. It is easy to check that G is an outer-projective graph if and only if $G + v$ is projective. Glover, Huneke, and Wang [18] found 103 graphs that were irreducible for the projective plane in 1979. Archdeacon [2] showed that this list was complete in 1980. Mahader showed in [20] that 35 of these graphs are minor-minimal, which was also implicitly stated in Archdeacon's work.

Therefore, it is straightforward to determine the forbidden minors for outer-projective graphs. This problem has been solved, in this way, by Archdeacon, Hartsfield, Little, and Mohar [3]. In this chapter, we will prove a stronger result. We will determine the forbidden minors for k-connected ($k = 1, 2, 3$) and quasi 4-connected outer-projective graphs.

5.2 3-connected, Outer-projective Graphs

Let $\mathcal{A}$ be the set of all minor-minimal, non-projective graphs. For $k = 1, 2, 3$, let $\mathcal{A}_k$ denote the set of k-connected members of $\mathcal{A}$. As noted in the previous section, $|\mathcal{A}| = 35$. We point out that $|\mathcal{A}_1| = 32$, $|\mathcal{A}_2| = 29$, and $|\mathcal{A}_3| = 23$. Robertson, Seymour, and Thomas proved the following result which is unpublished. A short proof can be found in [13].

Lemma 5.2.1. *For $k = 1, 2, 3$, a k-connected graph G is projective if and only if G is $\mathcal{A}_k$ minor-free.*

This result will allow us to easily determine forbidden minors for 1- and 2-connected outer-projective graphs. We will do so using two lemmas.

Lemma 5.2.2. *Suppose $|G| > k$. Then G is k-connected if and only if $G + v$ is $(k + 1)$-connected.*

Proof. Let G be a graph such that $|G| > k$. Suppose G is k-connected. This means that G has k pairwise internally disjoint paths between every pair of vertices x and y in $V(G)$. Now, consider $G + v$. This graph still has the same set of k pairwise internally disjoint paths between x and y. There is also a new xvy path which is clearly internally disjoint from each of the other k paths. Therefore, $G+v$ has $k+1$ pairwise internally disjoint paths between every pair of vertices x, y in $V(G)$. We must also determine $k+1$ internally disjoint paths from v to any vertex $u \in V(G)$. One simple path is simply the edge uv. We note that since G was k-connected, u must have at least k neighbors in G. Therefore, we have at least k paths from v to each neighbor of u to u. These are all clearly internally disjoint paths. Therefore, we may conclude that $G + v$ is $(k + 1)$-connected.

Now assume that $G+v$ is $(k+1)$-connected. This means $G+v$ has $k+1$ pairwise internally disjoint paths between each pair of vertices x and y. Now, we consider the graph G obtained by deleting v. Of the $k+1$ pairwise internally disjoint paths between x and y at most one could have passed through vertex v. Therefore, there are still at least k pairwise internally disjoint paths between each pair of vertices in G, and therefore G is k-connected. □

Lemma 5.2.3. *Suppose $G + v$ contains a minor H. Then, H has a vertex x such that G contains $H - x$ as a minor. Conversely, if G contains $H - x$ as a minor for some vertex x of H, then $G + v$ contains H as a minor.*

Proof. Suppose $G + v$ contains an H minor. If G also contains an H minor, it clearly contains an $H - x$ minor for any $x \in H$. Otherwise, we wish to consider a model $(\{G_u\}, \{f_e\})$ of H in $G+v$. Choose x such that v is a vertex in W_x. We can find W_u in G for all $u \in V(H) - \{x\}$. All necessary edges f_e between these W_u are also present. Therefore G must contain an $H - x$ minor.

Now suppose G contains an $H-x$ minor for some vertex $x \in H$. Now, we consider a model $(\{G_u\}, \{f_e\})$ of $H - x$ in G. We can easily replicate in $G + v$ all W_u such that $u \in V(H - x)$. All edges f_e between these W_u are also still present. Finally, we can let W_x be the single vertex v. Since this vertex is adjacent to everything, we can keep the edges necessary for forming the H minor and simply delete the rest giving us an H minor in $G + v$. □

For $k = 0, 1, 2$, let $\mathcal{B}_k$ denote the set of minor minimal graphs in $\{G - v : G \in \mathcal{A}_{k+1}$ and $v \in V(G)\}$. We can easily check that $|\mathcal{B}_0| = 32$, $|\mathcal{B}_1| = 29$, and $|\mathcal{B}_2| = 23$. Additionally, $\mathcal{B}_1$ and $\mathcal{B}_2$ are precisely the 1-connected and 2-connected members of $\mathcal{B}_0$ respectively.

Theorem 5.2.4. *A graph G is outer-projective if and only if G is $\mathcal{B}_0$ minor-free. [2]*

Proof. G is not outer-projective. $\Leftrightarrow$ $G + v$ is not projective. $\Leftrightarrow$ $G + v$ contains a graph $H \in \mathcal{A}_1$ as a minor by Lemma 5.2.1. $\Leftrightarrow$ G contains a graph $H - x$ as a minor where $H \in \mathcal{A}_1$ and $v \in V(H)$, by Lemma 5.2.3. $\Leftrightarrow$ G contains a graph in $\mathcal{B}_0$ as a minor. □

Theorem 5.2.5. *A connected graph G is outer-projective if and only if G is $\mathcal{B}_1$ minor-free.*

Proof. Let G be a connected graph. Then, $G + v$ is 2-connected by Lemma 5.2.2. G is not outer-projective. $\Leftrightarrow$ $G + v$ is not projective. $\Leftrightarrow$ $G + v$ contains a graph $H \in \mathcal{A}_2$ as a minor by Lemma 5.2.1. $\Leftrightarrow$ G contains a graph $H - x$ as a minor where $H \in \mathcal{A}_2$ and $v \in V(H)$, by Lemma 5.2.3. $\Leftrightarrow$ G contains a graph in $\mathcal{B}_1$ as a minor. □

Theorem 5.2.6. *A 2-connected graph G is outer-projective if and only if G is $\mathcal{B}_2$ minor-free.*

Proof. Let G be a 2-connected graph. Then, $G+v$ is 3-connected by Lemma 5.2.2. G is not outer-projective. $\Leftrightarrow$ $G + v$ is not projective. $\Leftrightarrow$ $G + v$ contains a graph $H \in \mathcal{A}_3$ as a minor by Lemma 5.2.1. $\Leftrightarrow$ G contains a graph $H - x$ as a minor where $H \in \mathcal{A}_3$ and $v \in V(H)$, by Lemma 5.2.3. $\Leftrightarrow$ G contains a graph in $\mathcal{B}_2$ as a minor. □

Ding and Iverson [13] identified a set $\mathcal{A}_4$ of 23 internally 4-connected graphs and proved the following result:

Lemma 5.2.7. *An internally 4-connected graph G is projective if and only if G is $\mathcal{A}_4$ minor-free.*

We let $\mathcal{B}_3'$ denote the set of minor-minimal graphs in the set $\{G - v | G \in \mathcal{A}_4$ and $v \in V(G)\}$. It is routine to determine all the members of $\mathcal{B}_3'$. We note that $\mathcal{B}_3'$ has 18 graphs. These graphs are shown in Figure 5.1.

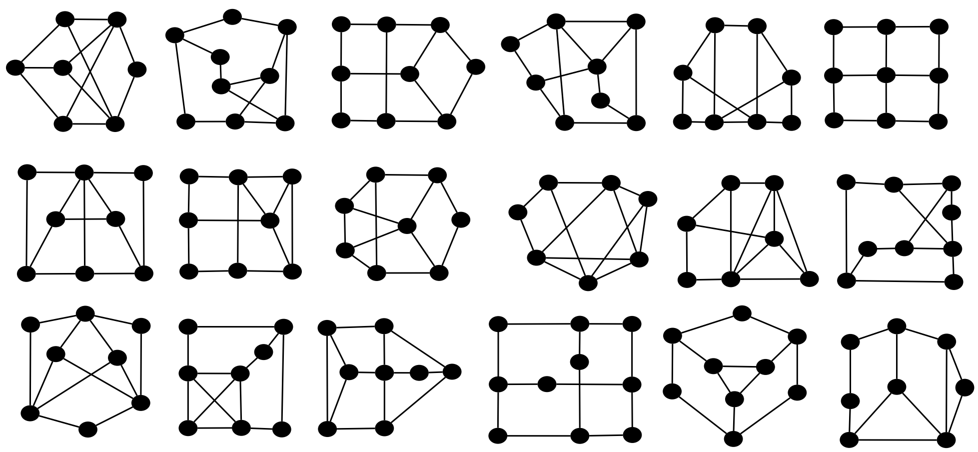


Figure 5.1: Graphs in $\mathcal{B}_3'$

Since graphs in $\mathcal{A}_4$ are internally 4-connected, graphs in $\mathcal{B}_3'$ are internally 3-connected, that is that such a graph G is 2-connected and for every 2-separation (G_1, G_2) of G, at least one of G_1, G_2 is $K_{1,2}$.

Theorem 5.2.8. *An internally 3-connected graph G is outer-projective if and only if G is $\mathcal{B}_3'$ minor-free.*

Proof. Let G be an internally 3-connected graph. Then, $G + v$ is internally 4-connected. G is not outer-projective. $\Leftrightarrow$ $G + v$ is not projective. $\Leftrightarrow$ $G + v$ contains a graph $H \in \mathcal{A}_4$ as a minor by Lemma 5.2.7. $\Leftrightarrow$ G contains a graph $H - x$ as a minor where $H \in \mathcal{A}_4$ and $v \in V(H)$, by Lemma 5.2.3. $\Leftrightarrow$ G contains a graph in $\mathcal{B}_3'$ as a minor. $\square$

In order to determine all of the 3-connected forbidden minors, we require the following lemma:

Lemma 5.2.9. *Let G be a 3-connected graph with a minor H such that $|H| \geq 4$. Suppose $x \in V(H)$ is incident with precisely two edges xx_1 and xx_2 where $x_1 \neq x_2$ and $x_1x_2 \notin E(H)$. Then, G has a minor isomorphic to one of the following two graphs H':*

(i) H' is obtained from H by adding an edge xy where $y \in V(H) - \{x, x_1, x_2\}$;

(ii) For some $i \in \{1, 2\}$, H' is obtained from $H + xx_i$ by splitting vertex x_i such that the two edges between x, x_i are no longer in parallel and both of the two new vertices have degree at least four.

Proof. Let $(\{G_u\}, \{f_e\})$ be a model of H in G. Without loss of generality, we may assume that each G_u is a tree in which each leaf is incident with some f_e. For $i = 1, 2$, let the end of f_{xx_i} in G_x be t_i. Since G is a 3-connected graph and $|H| \geq 4$, we know that $G - \{t_1, t_2\}$ has a path P with one end in G_x and the other end p in the union of G_u over all $u \in V(H - x)$. If p belongs to some G_u with $u \in V(H) - \{x, x_1, x_2\}$, then (i) holds. Therefore, we may assume that G_{x_i} contains p for one of $i \in \{1, 2\}$. Let T be the component of $G_{x_i} - t_i$ that contains p. If T is incident with a single f_e, then (i) holds once again since $x_1x_2 \notin E(H)$. Therefore, T is incident with at least two edges f_e. In this case a minor satisfying (ii) can be obtained by contracting each G_u for $u \neq x_i$, and contracting the two components of $G_{x_i} \backslash t_it_i'$, where t_it_i' is the edge between t_i and T. The vertex corresponding to

T clearly has degree at least four. By symmetry, the other vertex must also have degree at least four. $\square$

This lemma allows us to determine all forbidden minors for 3-connected outer-projective graphs.

Theorem 5.2.10. *Let $\mathcal{B}_3$ consist of the nine graphs shown in Figure 5.2 . A 3-connected graph G is outer-projective if and only if G is $\mathcal{B}_3$ minor-free.*

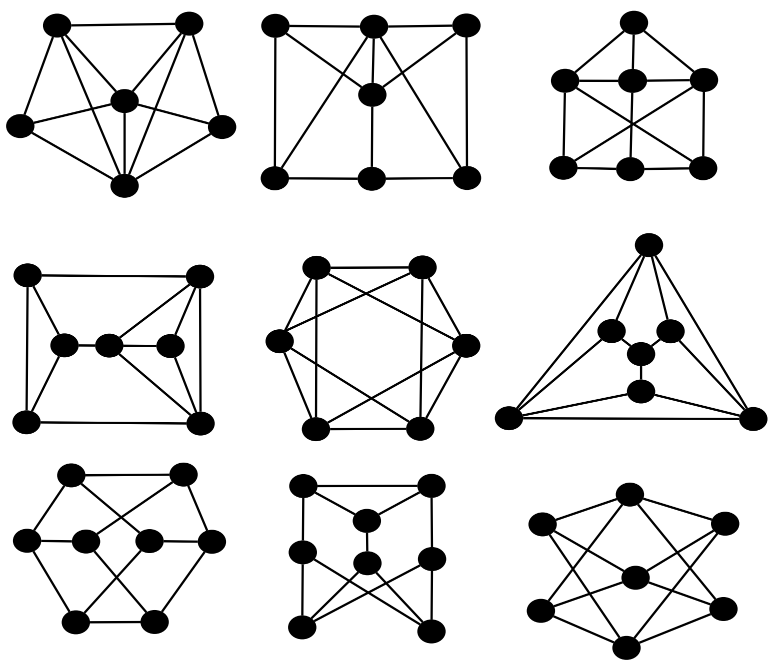


Figure 5.2: Graphs in $\mathcal{B}_3$

Proof. We can easily check that each graph G in Figure 5.2 is not outer-projective. For each of these graphs we consider $G + v$. In each of these graphs we can find one of Archdeacon's obstructions to projectivity as a minor. Since $G + v$ is not projective, G cannot be outer-projective. Therefore, every outer-projective graph is $\mathcal{B}_3$ minor-free.

Conversely, suppose G is $\mathcal{B}_3$ minor-free. If G is not outer-projective by Theorem 5.2.8, G must contain a graph $H \in \mathcal{B}'_3$ as a minor. For each vertex of degree two in H, we can apply Lemma 5.2.9. Examining all possible cases shows that G must contain a graph from $\mathcal{B}_3$ as a minor. □

5.3 Quasi 4-connected, Outer-projective Graphs

In the preceding section, we saw that increasing connectivity gave us a shorter list of excluded minors for outer-projectivity. In this section, we show that for quasi 4-connected, outer-projective graphs, there is essentially only one forbidden minor.

Theorem 5.3.1. *Let G be a quasi 4-connected graph. Then, the following are equivalent:*

a) G is outer-projective.

b) G is $\mathcal{B}_4$ minor-free (Graphs in $\mathcal{B}_4$ are shown in Figure 5.3).

c) G is Pyramid minor-free and G is not one of the thirty-one graphs listed in Theorem 4.3.8

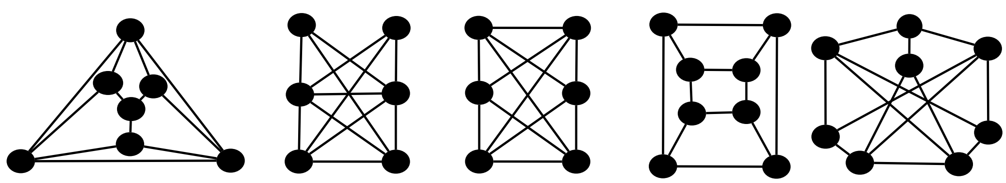


Figure 5.3: Graphs in $\mathcal{B}_4$: Pyramid, $\mathcal{G}_{12,4}$, $\mathcal{G}_{12,5}$, $\mathcal{G}_{12,7}$, and $\mathcal{G}_{14,2}$

Proof. First, we show that (a) implies (b). We show that all five graphs mentioned in part (b) are not outer-projective by showing that those graphs plus a vertex contain one of Archdeacon's obstructions to being projective. First, consider the Pyramid. If we consider Pyramid $+v$, this graph contains B_7 from Archdeacon's list. Since Pyramid $+v$ is not projective, Pyramid cannot be outer-projective. We consider the same for the other four graphs. $\mathcal{G}_{12,4}+v$ contains B_1, $\mathcal{G}_{12,5}+v$ contains

A_2, $\mathcal{G}_{12,7} + v$ contains C_4, and $\mathcal{G}_{14,2} + v$ contains D_{12}, where B_1, A_2, C_4, and D_{12} are all on Archdeacon's list of obstructions to being projective. Therefore, none of the graphs listed in part (b) is outer-projective. This means that since G is an outer-projective graphs it cannot contain any of those five graphs since the family of outer-projective graphs is minor closed.

Now, we show (b) implies (c). It is easy to check that every graph on our list of thirty-one graphs contains either $\mathcal{G}_{12,4}$, $\mathcal{G}_{12,5}$, $\mathcal{G}_{12,7}$, or $\mathcal{G}_{14,2}$ as a minor. Thus if our graph is free of those four graph plus the Pyramid, it must be Pyramid-free and cannot be on our list of thirty-one graphs.

Finally, we show (c) implies (a). If G is Pyramid-free and not one of the thirty-one isolated graphs, then G must be in $\mathcal{M}$. This implies that G is outer-projective. $\square$

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Appendix: Figure 4.3 Graphs by Edge Listing

In this appendix, we list all of the quasi 4-connected graphs generated by our Mathematica programs up to graphs on eighteen edges. We list each graph $\mathcal{G}_{i,j}$, the j^{th} graph of size i, by listing all of the edges that are present in that graph. The graphs denoted with (*) represent the thirty-one isolated graphs that are not in the family $\mathcal{M}$.

$\mathcal{G}_{6,1} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

$\mathcal{G}_{8,1} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}$

$\mathcal{G}_{9,1} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\}\}$

$\mathcal{G}_{9,2} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{2,5\},\{4,5\},\{1,6\},\{3,6\},\{5,6\}\}$

$\mathcal{G}_{9,3} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\},\{1,3\}\}$

$\mathcal{G}_{10,1} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\}\}$

$\mathcal{G}_{10,2} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{2,5\},\{4,5\},\{1,6\},\{3,6\},\{5,6\},\{1,3\}\}$

$\mathcal{G}_{10,3} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{1,5\},\{2,5\},\{3,5\},\{4,5\},\{1,3\},\{2,4\}\}$

$\mathcal{G}_{10,4} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\},\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\}\}$

$\mathcal{G}_{11,1} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{3,5\}, \{4,5\}, \{1,6\}, \{2,6\}, \{5,6\}, \{1,3\}, \{1,5\}\}$

$\mathcal{G}_{11,2} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{3,5\},\{4,5\}, \{1,6\}, \{2,6\}, \{5,6\}, \{1,3\}, \{2,4\}\}$

$\mathcal{G}_{11,3} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{3,5\}, \{4,5\}, \{1,6\}, \{2,6\}, \{5,6\}, \{1,3\}, \{2,5\}\}$

$\mathcal{G}_{11,4} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{2,5\}, \{4,5\}, \{1,6\}, \{3,6\}, \{5,6\}, \{1,3\}, \{1,5\}\}$

$\mathcal{G}_{12,1} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\}, \{4,7\},\{1,7\}\}$

$\mathcal{G}_{12,2} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,4\}\}$

$\mathcal{G}_{12,3} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,5\}\}$

$(*)\mathcal{G}_{12,4} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{2,5\},\{4,6\}\}$

$(*)\mathcal{G}_{12,5} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{2,5\},\{4,5\},\{1,6\},\{3,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{3,5\}\}$

$\mathcal{G}_{12,6} = \{\{1,2\},\{2,3\},\{3,4\},\{5,6\},\{6,7\},\{7,8\},\{1,5\},\{2,6\},\{3,7\},\{4,8\},$
$\{1,4\},\{5,8\}\}$

$(*)\mathcal{G}_{12,7} = \{\{1,2\},\{2,3\},\{3,4\},\{5,6\},\{6,7\},\{7,8\},\{1,5\},\{2,6\},\{3,7\},\{4,8\},$
$\{1,8\},\{4,5\}\}$

$\mathcal{G}_{13,1} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\}\}$

$(*)\mathcal{G}_{13,2} = \{\{2,3\},\{3,4\},\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{2,7\},\{4,7\},\{1,7\},$
$\{1,8\},\{4,8\},\{5,8\}\}$

$\mathcal{G}_{13,3} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\}\}$

$(*)\mathcal{G}_{13,4} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,4\}\}$

$(*)\mathcal{G}_{13,5} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,5\}\}$

$(*)\mathcal{G}_{13,6} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{4,6\}\}$

$(*)\mathcal{G}_{13,7} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,4\},\{2,5\}\}$

$\mathcal{G}_{13,8} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,4\},\{3,6\}\}$

$\mathcal{G}_{14,1} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\}\}$

$(*)\mathcal{G}_{14,2} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,5\}\}$

$\mathcal{G}_{14,3} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{2,4\}\}$

$\mathcal{G}_{14,4} = \{\{2,3\},\{3,4\},\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{2,7\},\{4,7\},\{1,7\},$
$\{1,8\},\{4,8\},\{5,8\},\{2,8\}\}$

$(*)\mathcal{G}_{14,5} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\}\}$

$\mathcal{G}_{14,6} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,4\}\}$

$(*)\mathcal{G}_{14,7} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,5\}\}$

$(*)\mathcal{G}_{14,8} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{3,7\}\}$

$\mathcal{G}_{14,9} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{4,6\}\}$

$(*)\mathcal{G}_{14,10} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{5,7\}\}$

$\mathcal{G}_{14,11} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,4\},\{6,7\}\}$

$(*)\mathcal{G}_{14,12} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,5\},\{4,6\}\}$

$(*)\mathcal{G}_{14,13} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,5\},\{5,7\}\}$

$(*)\mathcal{G}_{14,14} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,4\},\{2,5\},\{3,6\}\}$

$\mathcal{G}_{15,1} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\},\{1,5\}\}$

$\mathcal{G}_{15,2} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\},\{2,4\}\}$

$\mathcal{G}_{15,3} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\},\{6,8\}\}$

$(*)\mathcal{G}_{15,4} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{3,9\},\{4,9\},\{2,9\}\}$

$(*)\mathcal{G}_{15,5} = \{\{2,3\},\{3,4\},\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{2,7\},\{4,7\},\{1,7\},$
$\{1,8\},\{4,8\},\{5,8\},\{2,8\},\{4,6\}\}$

$\mathcal{G}_{15,6} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\},\{1,2\}\}$

$(*)\mathcal{G}_{15,7} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\},\{3,7\}\}$

$(*)\mathcal{G}_{15,8} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,4\},\{4,6\}\}$

$(*)\mathcal{G}_{15,9} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,5\},\{3,7\}\}$

$(*)\mathcal{G}_{15,10} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,5\},\{5,7\}\}$

$(*)\mathcal{G}_{15,11} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{3,7\},\{2,8\},\{7,8\},\{1,8\}\}$

$(*)\mathcal{G}_{15,12} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{4,6\},\{5,7\}\}$

$\mathcal{G}_{15,13} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{2,5\},\{4,6\},\{5,7\}\}$

$(*)\mathcal{G}_{15,14} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},$
$\{1,5\},\{2,4\},\{2,5\},\{3,6\},\{4,6\}\}$

$\mathcal{G}_{15,15} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{6,7\},\{7,8\},\{8,9\},\{9,10\},\{1,6\},\{2,7\},$
$\{3,8\},\{4,9\},\{5,10\},\{1,10\},\{5,6\}\}$

$\mathcal{G}_{16,1} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\},\{1,5\},\{2,4\}\}$

$\mathcal{G}_{16,2} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{3,9\},\{4,9\},\{2,9\}\}$

$\mathcal{G}_{16,3} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\}\}$

$\mathcal{G}_{16,4} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},$
$\{3,8\},\{5,8\},\{1,8\},\{1,2\},\{2,4\},\{6,8\}\}$

$\mathcal{G}_{16,5} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{6,8\},\{1,9\},\{6,9\},\{2,9\}\}$

$\mathcal{G}_{16,6} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{6,8\},\{1,9\},\{2,9\},\{6,9\}\}$

$\mathcal{G}_{16,7} = \{\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{5,10\},\{4,10\}\}$

$(*)\mathcal{G}_{16,8} = \{\{2,3\},\{3,4\},\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{2,7\},\{4,7\},\{1,7\},$
$\{1,8\},\{4,8\},\{5,8\},\{2,8\},\{4,6\},\{5,7\}\}$

$(*)\mathcal{G}_{16,9} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\},\{1,2\},\{3,7\}\}$

$(*)\mathcal{G}_{16,10} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\},\{3,7\},\{5,7\}\}$

$(*)\mathcal{G}_{16,11} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,4\},\{4,6\},\{5,7\}\}$

$(*)\mathcal{G}_{16,12} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{1,7\},\{1,2\},\{2,5\},\{3,7\},\{5,7\}\}$

$\mathcal{G}_{16,13} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,3\},\{1,5\},\{2,7\},$
$\{4,7\},\{3,7\},\{2,8\},\{7,8\},\{1,8\},\{1,2\}\}$

$\mathcal{G}_{17,1} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{1,5\},\{3,9\},\{4,9\},\{2,9\}\}$

$\mathcal{G}_{17,2} = \{\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{5,10\},\{4,10\}\}$

$\mathcal{G}_{17,3} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{6,8\}\}$

$\mathcal{G}_{17,4} = \{\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},\{1,2\},$
$\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{2,10\},\{4,10\},\{3,10\}\}$

$\mathcal{G}_{17,5} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{3,5\}\}$

$\mathcal{G}_{17,6} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{6,8\}\}$

$\mathcal{G}_{17,7} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{8,9\}\}$

$\mathcal{G}_{17,8} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{6,8\},\{1,9\},\{6,9\},\{2,9\},\{6,10\},\{8,10\},\{1,10\}\}$

$\mathcal{G}_{17,9} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,9\},\{6,9\},\{2,9\},\{1,10\},\{6,10\},\{8,10\}\}$

$(*)\mathcal{G}_{17,10} = \{\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{1,5\},\{2,7\},\{4,7\},$
$\{1,7\},\{2,8\},\{3,8\},\{1,8\},\{1,2\},\{3,7\},\{5,7\}\}$

$\mathcal{G}_{18,1} = \{\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{1,2\},\{1,5\},\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{5,10\},\{4,10\}\}$

$\mathcal{G}_{18,2} = \{\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{1,2\},\{1,5\},\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{4,10\},\{5,10\}\}$

$\mathcal{G}_{18,3} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{1,5\},\{3,9\},\{4,9\},\{2,9\},\{2,4\}\}$

$\mathcal{G}_{18,4} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{1,5\},\{3,9\},\{4,9\},\{2,9\},\{7,9\}\}$

$\mathcal{G}_{18,5} = \{\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{5,10\},\{4,10\},\{6,8\}\}$

$\mathcal{G}_{18,6} = \{\{3,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},$
$\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{3,10\},\{5,10\},\{4,10\},\{7,9\}\}$

$\mathcal{G}_{18,7} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{1,10\},\{6,10\},\{8,10\}\}$

$\mathcal{G}_{18,8} = \{\{3,4\},\{3,5\},\{4,5\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},$
$\{5,8\},\{1,8\},\{1,2\},\{3,9\},\{4,9\},\{2,9\},\{6,8\},\{7,9\}\}$

$\mathcal{G}_{18,9} = \{\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},\{1,2\},$
$\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{2,10\},\{4,10\},\{3,10\},\{6,8\}\}$

$\mathcal{G}_{18,10} = \{\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{1,8\},\{1,2\},$
$\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{2,10\},\{4,10\},\{3,10\},\{8,9\}\}$

$\mathcal{G}_{18,11} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{3,5\},\{6,8\}\}$

$\mathcal{G}_{18,12} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{1,10\},\{6,10\},\{8,10\}\}$

$\mathcal{G}_{18,13} = \{\{2,3\},\{3,4\},\{1,6\},\{2,6\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},$
$\{1,8\},\{1,2\},\{2,4\},\{3,9\},\{4,9\},\{5,9\},\{6,8\},\{8,9\}\}$

$\mathcal{G}_{18,14} = \{\{3,5\},\{4,5\},\{5,6\},\{2,7\},\{4,7\},\{1,7\},\{3,8\},\{5,8\},\{6,8\},\{1,9\},$
$\{6,9\},\{2,9\},\{6,10\},\{8,10\},\{1,10\},\{2,11\},\{4,11\},\{3,11\}\}$

$\mathcal{G}_{18,15} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{6,7\},\{7,8\},\{8,9\},\{9,10\},\{10,11\},$
$\{11,12\},\{1,12\},\{1,7\},\{2,8\},\{3,9\},\{4,10\},\{5,11\},\{6,12\}\}$

Vita

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