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Existence and uniqueness of solutions to the backward 2D stochastic Navier–Stokes equations

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Abstract

The backward two-dimensional stochastic Navier–Stokes equations (BSNSEs, for short) with suitable perturbations are studied in this paper, over bounded domains for incompressible fluid flow. A priori estimates for adapted solutions of the BSNSEs are obtained which reveal a pathwise $L^\infty(H)$ bound on the solutions. The existence and uniqueness of solutions are proved by using a monotonicity argument for bounded terminal data. The continuity of the adapted solutions with respect to the terminal data is also established.

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1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a stochastic basis satisfying the usual conditions over which a Hilbert-space-valued Wiener process $\{W(t)\}$ with a nuclear covariance operator Q is defined. The two-dimensional stochastic Navier–Stokes system in a bounded domain $G \subset \mathbb{R}^2$ with no-slip condition is given by

$$\partial \mathbf{u} + \{(\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u}\}\mathrm{d}t = \{-\nabla p + \mathbf{f}(t)\}\mathrm{d}t + \sigma(t, \mathbf{u})\mathrm{d}W(t)$$
$$\nabla \cdot \mathbf{u} = 0$$

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with $\mathbf{u}(t,x) = 0,\ x \in \partial G$, and $\mathbf{u}(0,x) = \mathbf{u}_0(x),\ x \in G$. Here p denotes pressure, a real-valued random field, and $\mathbf{u}_0$ is the initial condition. The solution consists of $(\mathbf{u}, p)$, where $\mathbf{u}$ is a two-dimensional random field. It is well known (as explained in the next section) that the above system can be written in the abstract evolution equation setup as

$$\mathrm{d}\mathbf{u}(t) + \{\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))\}\mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \sigma(t,\mathbf{u}(t))\,\mathrm{d}W(t) \tag{1.1}$$

with $\mathbf{u}(0) = \mathbf{u}_0$. In this equation the pressure p doesn't appear. However, it can be determined from the solution $\mathbf{u}$ of (1.1) and the no-slip boundary condition. The backward stochastic Navier–Stokes equation corresponding to Eq. (1.1) is given by the following terminal value problem: for $0 \le t \le T$,

$$\begin{cases}\mathrm{d}\mathbf{u}(t) = -\nu\mathbf{A}\mathbf{u}(t)\mathrm{d}t - \mathbf{B}(\mathbf{u}(t))\mathrm{d}t + \mathbf{f}(t)\mathrm{d}t + Z(t)\mathrm{d}W(t)\\ \mathbf{u}(T) = \xi.\end{cases}$$

The above stochastic equation is understood in the integral form:

$$\mathbf{u}(t) = \xi + \int_t^T \{\nu\mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\}\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s).$$

The problem consists in finding an adapted solution $\{\mathbf{u}(t), Z(t)\}$ for $0 \le t \le T$.

A backward stochastic Navier–Stokes system can be viewed as an inverse problem wherein the velocity profile at a time T is observed and given, and the noise coefficient has to be ascertained from the given terminal data. Such a motivation arises naturally when one understands the importance of inverse problems in partial differential equations (see Lions [5, 6]). A systematic study of backward stochastic differential equations was initiated by Pardoux and Peng, Ma, Protter, Yong, Zhou, and several other authors. Ma and Yong [8] have studied linear degenerate backward stochastic differential equations, motivated by stochastic control theory. Later, Hu, Ma and Yong [3] considered the semi-linear equations as well. Backward stochastic partial differential equations were shown to arise naturally in stochastic versions of the Black–Scholes formula by Ma and Yong [7]. A nice introduction to backward stochastic differential equations is presented in the book by Yong and Zhou [13], with various applications.

In the present work, it is worthwhile to note that the drift coefficient in the backward stochastic Navier–Stokes equation (BSNSE) is nonlinear and unbounded, so the usual methods of proving existence and uniqueness of solutions do not apply. Therefore, we assume that the function $\mathbf{f}$ depends on $\mathbf{u}$ and satisfies certain conditions to ensure that useful estimates on $\mathbf{u}$ can be obtained. The fact that the operator in the Navier–Stokes equation is monotone on bounded $L^4(G)$ balls in V was first observed by Menaldi and Sritharan [9]. The method of monotonicity is used in this paper to prove the existence and uniqueness of solutions to BSNSEs. Much of the paper rests on a surprising a priori estimate on $\sup_{[0,T]} \|\mathbf{u}(t)\|_H^2$ that holds almost surely. Such an estimate seldom holds for stochastic Navier–Stokes equations that move forward in time. Continuity of the solution with respect to the data at the terminal time T is also established in this article.

The organization of the paper is as follows. In Section 2, the setup of the problem, the function spaces, and the background results are presented. The a priori estimates for the solutions are proved in Section 3. The existence and uniqueness of solutions to the projected finite dimensional systems with an external forcing term are established. The existence of solutions to the backward stochastic Navier–Stokes equations is shown by the Minty–Browder monotonicity argument in Section 4 for bounded terminal data when the forcing term depends on the solution. In Section 5, uniqueness of solutions is proved. In Section 6, the continuity property of the solution is studied.

2. Preliminaries

Let G be a bounded domain in $\mathbb{R}^2$ with a smooth boundary, and for $t \in [0, T]$, let $\mathbf{u}(t) = \binom{u_1(t)}{u_2(t)} \in H$ where

$$H = \{\mathbf{u} \in (L^2(G))^2 : \mathrm{div}(\mathbf{u}) = \nabla \cdot \mathbf{u} = 0 \text{ and } \gamma(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}_G = 0\},$$

where $\mathbf{n}_G$ stands for the outer normal to ∂G.

Let $V = \{\mathbf{u} \in (H^1(G))^2 : \nabla \cdot \mathbf{u} = 0 \text{ and } \gamma_0(\mathbf{u}) = \mathbf{u}|_{\partial G} = 0\}$ and V' be the dual of V. From the definition of V and H, we see that they are both separable Hilbert spaces, V is a dense subset of H, and the embedding $V \hookrightarrow H$ is dense, continuous and compact.

We identify H' with H. For any $\mathbf{h} \in H$, there exists an $\mathbf{h}' \in V'$ such that $\langle \mathbf{h}', \mathbf{v} \rangle_{V',V} = \langle \mathbf{h}, \mathbf{v} \rangle_H$. Then the mapping $\mathbf{h} \mapsto \mathbf{h}'$ is linear, injective, compact and continuous. We may identify $\mathbf{h}'$ with $\mathbf{h}$. In this sense, H is a dense subset of V'. Thus we have evolution triple

$$V \subset H \subset V'.$$

The following definitions are basic and standard [1].

Definition 2.1. Let A be a linear operator from a separable Hilbert space K to a separable Hilbert space H. Let $\{e_j\}_{j=1}^{\infty}$ denote a complete orthonormal system (CONS for short) in K.

(a) We denote by $L(K, H)$ the class of all bounded linear operators with the uniform operator norm $\|\cdot\|_L$.

(b) If $\|A\|_{L_1} = \sum_{k=1}^{\infty} \langle (A^*A)^{\frac{1}{2}} e_k, e_k \rangle_K < \infty$, then A is called a *trace class (nuclear) operator*. We denote by $L_1(K, H)$ the class of all trace class operators equipped with norm $\|\cdot\|_{L_1}$.

(c) Let $L_2(K, H)$ denote the class of *Hilbert–Schmidt operators* with norm $\|\cdot\|_{L_2}$ given by $\|A\|_{L_2} = (\sum_{k=1}^{\infty} \langle Ae_k, Ae_k \rangle_H)^{\frac{1}{2}}$. Sometimes $\|\cdot\|_{L_2}$ is also denoted by $\|\cdot\|_{H.S.}$.

(d) Let $Q \in L_1(K, K)$ be self-adjoint and positive definite. Let K_0 be the Hilbert subspace of K with inner product

$$\langle f, g \rangle_{K_0} = \langle Q^{-\frac{1}{2}} f, Q^{-\frac{1}{2}} g \rangle_K,$$

and we define $L_Q = L_2(K_0, H)$ with the inner product

$$\langle F, G \rangle_{L_Q} = \mathrm{tr}(FQG^*) = \mathrm{tr}(GQF^*), \quad F, G \in L_Q.$$

Definition 2.2. An adapted process $W(t)$ defined on a complete stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is an H-valued Q-Wiener process, where Q is a trace class operator on H, if $W(t)$ satisfies the following:

(a) $W(t)$ has continuous sample paths in H-norm with $W(0) = 0$.

(b) $(W(t), h)$ has stationary independent increments for all $h \in H$.

(c) $W(t)$ is a Gaussian process with mean zero and covariance operator Q, i.e. $E(W(t), g)(W(s), h) = (t \wedge s)(Qg, h)$ for all $g, h \in H$.

From now on, the filtration $\{\mathcal{F}_t\}$ will be taken to be the natural filtration of $\{W_t\}$, augmented by all the P-null sets in $\mathcal{F}$.

Remark 2.3. Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal system in H consisting of the eigenfunctions of the Stokes operator $\mathbf{A}$ that is defined below. The corresponding eigenvalues of the operator are an increasing sequence of positive numbers $\{\lambda_j\}_{j=1}^{\infty}$. In this paper, we take the covariance operator Q of the H-valued Wiener process to satisfy $Qe_k = q_k e_k$. If $\{b^k(t)\}$ is a sequence of iid Brownian motions in $\mathbb{R}$, the Wiener process $\{W(t)\}$ can be written as $W(t) = \sum_{k=1}^{\infty} \sqrt{q_k} b^k(t) e_k$ with $\sum_{k=1}^{\infty} q_k < \infty$.

Consider the stochastic Navier–Stokes equation for a viscous incompressible flow with no-slip condition at the boundary. Displaying the external forces on the right side of the equation, we have, for $\nu > 0$,

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) + \sigma(t) \dfrac{\mathrm{d}W(t)}{\mathrm{d}t} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{2.1}$$

with $\mathbf{u}(t, x) = 0$ for $x \in \partial G$, and $\mathbf{u}(0) = \mathbf{u}_0 \in H$. In the above, p denotes pressure and is a scalar-valued function. The external body force $\mathbf{f}(t)$ is assumed to be V'-valued for all t. The process $\{W_t\}$ is an H-valued Q-Wiener process, and ν is the coefficient of viscosity. The solution of the above system is $(\mathbf{u}, p)$ where $\mathbf{u}$ is a two-dimensional vector.

The stochastic Navier–Stokes equation can be written in the abstract evolution equation setup (see Temam [12]) for bounded domains. Let P be the orthogonal Leray projector $\mathsf{P} : (L^2(G))^2 \to H$. By applying the Leray projection to each term of the Navier–Stokes system, and invoking the result of Helmholtz that $(L^2(G))^2$ admits an orthogonal decomposition into divergence free and irrotational components, namely $(L^2(G))^2 = H + H^{\perp}$ where $H^{\perp}$ can be characterized by

$$H^{\perp} = \{\mathbf{g} \in (L^2(G))^2 : \mathbf{g} = \nabla h \text{ where } h \in W^{1,2}(G)\}, \tag{2.2}$$

we can write the system (2.1) as

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)\mathrm{d}t - \mathbf{B}(\mathbf{u}(t))\mathrm{d}t + \mathbf{f}(t)\mathrm{d}t + \mathsf{P}\sigma(t)\mathrm{d}W(t) \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\mathbf{B}(\mathbf{u}, \mathbf{v}) \doteq \mathsf{P}((\mathbf{u} \cdot \nabla)\mathbf{v})$ with the notation $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \mathbf{u})$, and $\mathbf{A}\mathbf{u} \doteq -\mathsf{P}(\Delta \mathbf{u})$. The operator $\mathbf{A}$ is known as the Stokes operator.

Since $V = \mathbf{D}(\mathbf{A}^{1/2})$, we can endow V with the norm $\|\mathbf{u}\|_V = \|\mathbf{A}^{1/2}\mathbf{u}\|_H$. The V-norm is equivalent to the $W^{1,2}$-norm by the Poincaré inequality. Then we get $\|\mathbf{v}\|_H \le C\|\mathbf{v}\|_V$, $\|\mathbf{h}\|_{V'} \le C\|\mathbf{h}\|_H$, and $\langle \mathbf{h}, \mathbf{v}\rangle_{V',V} = \langle \mathbf{h}, \mathbf{v}\rangle_H$ for all $\mathbf{v} \in V$, $\mathbf{h} \in H$ and some constant C. From now on, we may consider $\mathbf{A}$ and $\mathbf{B}$ as mappings that map V into V'.

Suppose the terminal value is specified as $\mathbf{u}(T) = \xi \in H$. Then the backward stochastic Navier–Stokes equation is given by

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)\mathrm{d}t - \mathbf{B}(\mathbf{u}(t))\mathrm{d}t + \mathbf{f}(t)\mathrm{d}t + Z(t)\mathrm{d}W(t) \\ \mathbf{u}(T) = \xi. \end{cases} \tag{2.3}$$

Notation: For any Banach space $\mathbb{K}$, let $L^p_{\mathcal{F}}(\Omega; L^r(0, T; \mathbb{K}))$ be the set of all $\mathcal{F}_t$-adapted $\mathbb{K}$-valued processes $X(\cdot)$ such that $E\{(\int_0^T \|X(t)\|^r_{\mathbb{K}} \mathrm{d}t)^p\} < \infty$.

Definition 2.4. A pair of $\mathcal{F}_t$-adapted processes $(\mathbf{u}(t), Z(t))$ is called a solution of backward stochastic differential equation (2.3) if the following hold:

(1) $\mathbf{u}(t) = \xi + \int_t^T \{\nu \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s)) - \mathbf{f}(s)\}\mathrm{d}s - \int_t^T Z(t)\mathrm{d}W(t)$ holds P-a.s. in V',
(2) $\mathbf{u}(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; V)) \cap L^2_{\mathcal{F}}(\Omega; L^{\infty}(0, T; H))$,
(3) $Z(\cdot) \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_Q))$.

The results stated below are quite standard, and the proofs can be found in Temam [12], and Constantin and Foias [2].

Proposition 2.5 (*[12]*). *For* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$*, we have*

(1) $\langle \mathbf{Au}, \mathbf{w} \rangle_{V',V} = \langle \mathbf{Aw}, \mathbf{u} \rangle_{V',V} = \langle \mathbf{u}, \mathbf{w} \rangle_V$.
(2) $\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{V',V} = -\langle \mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle_{V',V}$.

The next corollary follows readily:

Corollary 2.6. *For any* $\mathbf{u}, \mathbf{v} \in V$, $\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle_{V',V} = 0$.

Proposition 2.7 (*[2]*). *Let* $G \subset \mathbb{R}^n$ *be bounded, open and of class* C^l *where* $l \geq 1$*. Then there exists a constant* $C_G > 0$*, a scale invariant constant, such that*

$$|\langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle_{V',V}| \leq C_G \|\mathbf{u}\|_H^{1/2} \|\mathbf{u}\|_V^{1/2} \|\mathbf{v}\|_V \|\mathbf{w}\|_H^{1/2} \|\mathbf{w}\|_V^{1/2}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

Lemma 2.8 (*[12]*). *Assume that* $G \subset \mathbb{R}^2$ *is bounded and of class* $\mathcal{C}^2$*. If a function* $\mathbf{u} \in V \cap H^2(G)$*, then* $\mathbf{B}(\mathbf{u}) \in H \subset L^2(G)$ *and*

$$\|\mathbf{B}(\mathbf{u})\|_H \leq C_1 \|\mathbf{u}\|_H^{\frac{1}{2}} \|\mathbf{u}\|_V \|\mathbf{Au}\|_H^{\frac{1}{2}}.$$

3. A priori estimates

Let $P_N : H \to H_N$ be the projection where $H_N = \text{span}\{e_1, e_2, \ldots, e_N\}$. Notice the fact that $V_N = H_N = V'_N$ for all N. We introduce the following projections:

$$\mathbf{A}^N = P_N \mathbf{A}, \qquad \mathbf{B}^N = P_N \mathbf{B}, \quad \text{and} \quad \mathbf{f}^N = P_N \mathbf{f}.$$

Then the projected backward Navier–Stokes equation is defined as

$$\begin{cases} \mathrm{d}\mathbf{u}^N(t) = -\nu \mathbf{A}^N \mathbf{u}^N(t)\mathrm{d}t - \mathbf{B}^N(\mathbf{u}^N(t))\mathrm{d}t + \mathbf{f}^N(t)\mathrm{d}t + Z^N(t)\mathrm{d}W^N(t) \\ \mathbf{u}^N(T) = \xi^N \end{cases} \tag{3.1}$$

for $0 \leq t \leq T$, where $W^N(t) = \sum_{k=1}^N \sqrt{q_k} b^k(t) e_k$, $\xi^N = E(P_N \xi | \mathcal{F}_T^N)$ where the filtration $\{\mathcal{F}_t^N\}$ pertains to the natural filtration of the process $\{W^N(t)\}$, and $Z^N(t) : [0, T] \times \Omega \to L(H_N, H_N)$. For simplicity of notation, we write $\mathcal{F}_t^N$ as simply $\mathcal{F}_t$ in this paper wherever Galerkin approximations are used.

Below is the backward version of the Gronwall inequality, and the proof is straightforward.

Lemma 3.1. *Suppose that* $g(t)$*,* $\alpha(t)$*,* $\beta(t)$ *and* $\gamma(t)$ *are integrable functions, and* $\beta(t), \gamma(t) \geq 0$*. For* $0 \leq t \leq T$*, if*

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \gamma(\rho) g(\rho) \mathrm{d}\rho,$$

then

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \alpha(\eta)\gamma(\eta) \mathrm{e}^{\int_t^\eta \beta(\rho)\gamma(\rho)\mathrm{d}\rho} \mathrm{d}\eta. \tag{3.2}$$

In particular, if $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ *and* $\gamma(t) \equiv 1$, *then*

$$g(t) \le \alpha \mathrm{e}^{\beta(T-t)}.$$

Proposition 3.2. *Assume that* $\mathbf{f} \in L^2(0, T; V')$, *and* $\|\xi\|_H^2 \le K$ *for almost all* $\omega \in \Omega$ *and some constant* K. *If* $(\mathbf{u}^N(t), Z^N(t))$ *is an adapted solution for the projected system* (3.1), *then there exists a constant* $K_0(N)$ *such that*

$$\sup_{0\le t\le T} \|\mathbf{u}^N(t)\|_H^2 + E\int_0^T \|\mathbf{u}^N(t)\|_V^2 \mathrm{d}t + E\int_0^T \|Z^N(t)\|_{L_Q}^2 \mathrm{d}t \le K_0(N), \quad P\text{-a.s.} \tag{3.3}$$

Proof. An application of the multidimensional Itô formula to $\|\mathbf{u}^N(t)\|_H^2$ yields

$$\|\mathbf{u}^N(t)\|_H^2 + \int_t^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s = \|\xi^N\|_H^2 - 2\int_t^T \langle -\nu \mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} \mathrm{d}s$$
$$- 2\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} \mathrm{d}s - 2\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), \mathrm{d}W^N(s)\rangle_H,$$

where $(Z^N)^*$ is the adjoint of Z^N, and the duality pairing $\langle\cdot,\cdot\rangle_{V',V}$ is simply the H-inner product. For $0 < r \le t \le T$, we take the conditional expectation

$$E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s = E^{\mathcal{F}_r}\|\xi^N\|_H^2$$
$$+ 2E^{\mathcal{F}_r}\int_t^T \langle \nu \mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} \mathrm{d}s - 2E^{\mathcal{F}_r}\int_t^T \langle \mathbf{f}^N(s), \mathbf{u}^N(s)\rangle_{V',V} \mathrm{d}s.$$

Thus

$$E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s + E^{\mathcal{F}_r}\int_t^T \langle \mathbf{A}^N \mathbf{u}^N(s), \mathbf{u}^N(s)\rangle_{V',V} \mathrm{d}s$$
$$\le E^{\mathcal{F}_r}\|\xi^N\|_H^2 + (2\nu+1)\lambda_N \int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|_H^2 \mathrm{d}s + 2\int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 \mathrm{d}s$$
$$+ \frac{1}{2}\int_t^T E^{\mathcal{F}_r}\|\mathbf{u}^N(s)\|_V^2 \mathrm{d}s,$$

where λ_N is the Nth eigenvalue of $\mathbf{A}$. By Gronwall's inequality (3.2) and Proposition 2.5, we have

$$E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s + \frac{1}{2}E^{\mathcal{F}_r}\int_t^T \|\mathbf{u}^N(s)\|_V^2 \mathrm{d}s$$
$$\le \left\{ E^{\mathcal{F}_r}\|\xi^N\|_H^2 + 2\int_0^T \|\mathbf{f}^N(s)\|_{V'}^2 \mathrm{d}s \right\} \mathrm{e}^{(2\nu+1)\lambda_N T} \le K_N, \tag{3.4}$$

for some constant K_N depending on N. Omitting the first term on the left hand side of the above inequality and taking the expectation on both sides, one gets

$$E\int_0^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s + \frac{1}{2}E\int_0^T \|\mathbf{u}^N(s)\|_V^2 \mathrm{d}s \le K_N.$$

Taking r to be t and omitting the last two terms on the left hand side of inequality (3.4), we know that $\|\mathbf{u}^N(t)\|_H^2 \le K_N$, P-a.s. Thus we have shown

$$\sup_{0\le t\le T}\|\mathbf{u}^N(t)\|_H^2 + E\int_0^T \|\mathbf{u}^N(t)\|_V^2 \mathrm{d}t + E\int_0^T \|Z^N(t)\|_{L_Q}^2 \mathrm{d}t \le K_0(N), \quad P\text{-a.s.}$$

where $K_0(N) = 3K_N$. □

The following corollary can be shown in a similar manner upon observing that

$$\begin{aligned} &E^{\mathcal{F}_r}\left(\int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), \mathrm{d}W^N(s)\rangle_H\right)^2 \\ &= E^{\mathcal{F}_r}\left(\sum_{i=1}^N \int_t^T \langle (Z^N(s))^*(\mathbf{u}^N(s)), \sqrt{q_i}e_i\rangle_H \mathrm{d}b^i(s)\right)^2 \\ &\le K_0(N)E^{\mathcal{F}_r}\int_t^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s. \end{aligned}$$

Corollary 3.3. *Let the conditions in Proposition 3.2 hold. Additionally, we assume that* $\mathbf{f} \in L^4(0, T; V')$. *Then there exists a constant* $K_1(N)$ *such that*

$$\sup_{0\le t\le T}\|\mathbf{u}^N(t)\|_H^4 + E\left\{\int_0^T \|\mathbf{u}^N(t)\|_V^2 \mathrm{d}t\right\}^2 \le K_1(N), \quad P\text{-a.s,}$$

i.e. $\{\mathbf{u}^N(t)\}$ *is bounded in* $L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$.

With slight variations in the above proof, one can easily obtain

Corollary 3.4. *Let the conditions in Proposition 3.2 hold. Also let* $\|\xi\|_V^2 \le K$ *for almost all* $\omega\in\Omega$ *and some constant* K. *Then there exists a constant* $K_2(N)$ *such that*

$$\sup_{0\le t\le T}\|\mathbf{u}^N(t)\|_V^2 \le K_2(N), \quad P\text{-a.s.}$$

A proposition similar to Corollary 3.4 is used in Section 5 and a detailed proof is given there.

Remark 3.5. It is quite simple to show that under the conditions of Corollary 3.4,

$$E\int_0^T \mathrm{tr}[A^N Z^N(s)Q(Z^N(s))^*]\mathrm{d}s \le K_2(N).$$

Now for every $M\in\mathbb{N}$, we define L_M to be a Lipschitz C^∞ function as follows:

$$L_M(\|\mathbf{u}\|_V) = \begin{cases} 1 & \text{if } \|\mathbf{u}\|_V < M, \\ 0 & \text{if } \|\mathbf{u}\|_V > M+1, \\ 0\le L_M(\|\mathbf{u}\|_V)\le 1 & \text{otherwise.}\end{cases}$$

It is easy to show the Lipschitz property of L_M:

$$\|L_M(\|x\|_V)\mathbf{B}^N(x) - L_M(\|y\|_V)\mathbf{B}^N(y)\|_H \le C_{N,M}\|x-y\|_V$$

for any $x, y\in H_N$ and $M\in\mathbb{N}$.

Theorem 3.6. *Assume that* $\mathbf{f}\in L^2(0,T;V')$, *and* $\|\xi\|_H^2 \le K$ *for almost all* $\omega\in\Omega$ *and some constant* K. *Then the projected system (3.1) admits a unique adapted solution* $(\mathbf{u}^N(t), Z^N(t))$

for each N and

$$E(\sup_{0\le t\le T} \|\mathbf{u}^N(t)\|_H^2) + E\int_0^T \|Z^N(s)\|_{L_Q}^2 \mathrm{d}s < \infty. \tag{3.5}$$

Proof. First, we introduce some notation. For $1 \le i \le N$, let $\langle \mathbf{u}^N(t), e_i\rangle_H = \hat{u}_i^N(t)$ and we define $\hat{\mathbf{u}}^N(t) = \begin{pmatrix} \hat{u}_1^N(t) \\ \vdots \\ \hat{u}_N^N(t)\end{pmatrix}$.

For $\mathbf{A}^N$, we define $\langle \mathbf{A}^N\mathbf{u}^N(t), e_i\rangle_{V',V} = \langle \sum_{i=1}^N \lambda_i \hat{u}_i^N(t)e_i, e_i\rangle_{V',V} = \lambda_i \hat{u}_i^N(t)$.

For $\mathbf{B}^N$, we have

$$\begin{aligned}
\langle \mathbf{B}^N(\mathbf{u}^N(t)), e_i\rangle_{V',V} &= \left\langle P_N \mathbf{B}(\mathbf{u}^N(t)), e_i\right\rangle_{V',V} \\
&= \left\langle \sum_{j=1}^N \langle \mathbf{B}(\mathbf{u}^N(t)), e_j\rangle_{V',V} e_j, e_i \right\rangle_{V',V} \\
&= \left\langle \sum_{j=1}^N b(\mathbf{u}^N(t), \mathbf{u}^N(t), e_j)e_j, e_i \right\rangle_{V',V} \\
&= b(\mathbf{u}^N(t), \mathbf{u}^N(t), e_i) \\
&= b\left(\sum_{k=1}^N \hat{u}_k^N(t)e_k, \sum_{l=1}^N \hat{u}_l^N(t)e_l, e_i\right) \\
&= \sum_{k,l=1}^N b(e_k, e_l, e_i)\hat{u}_k^N(t)\hat{u}_l^N(t).
\end{aligned}$$

Since we have

$$\begin{aligned}
\|\mathbf{u}^N(t)\|_V &= \langle \mathbf{A}\mathbf{u}^N(t), \mathbf{u}^N(t)\rangle_{V',V}^{\frac{1}{2}} \\
&= \left\langle \sum_{j=1}^N \lambda_j \hat{u}_j^N(t)e_j, \sum_{i=1}^N \hat{u}_i^N(t)e_i \right\rangle_{V',V}^{\frac{1}{2}} = \sqrt{\sum_{j=1}^N \lambda_j(\hat{u}_j^N(t))^2},
\end{aligned}$$

we define $\hat{\mathbf{A}}^N(\hat{\mathbf{u}}^N(t)) = \begin{pmatrix} \lambda_1\hat{u}_1^N(t) \\ \lambda_2\hat{u}_2^N(t) \\ \vdots \\ \lambda_N\hat{u}_N^N(t)\end{pmatrix}$. For $\mathbf{B}^N$, we define

$$\hat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t)) = \begin{pmatrix} \sum_{k,l=1}^N b(e_k, e_l, e_1)\hat{u}_k^N(t)\hat{u}_l^N(t) \\ \sum_{k,l=1}^N b(e_k, e_l, e_2)\hat{u}_k^N(t)\hat{u}_l^N(t) \\ \vdots \\ \sum_{k,l=1}^N b(e_k, e_l, e_N)\hat{u}_k^N(t)\hat{u}_l^N(t) \end{pmatrix}.$$

We also define $\langle \mathbf{f}^N(t), e_i\rangle_{V',V} = \hat{f}_i^N(t)$, $\hat{\mathbf{f}}^N(t) = \begin{pmatrix} \hat{f}_1^N(t) \\ \vdots \\ \hat{f}_N^N(t) \end{pmatrix}$, and $\hat{\xi}^N = \begin{pmatrix} \langle \xi^N, e_1\rangle_H \\ \vdots \\ \langle \xi^N, e_N\rangle_H \end{pmatrix}$.

Since

$$\begin{aligned} \left\langle \int_t^T Z^N(s)\mathrm{d}W^N(s), e_i \right\rangle_H &= \left\langle \sum_{k=1}^N \sqrt{q_k} \int_t^T Z^N(s)(e_k)\mathrm{d}b^k(s), e_i \right\rangle_H \\ &= \left\langle \sum_{k=1}^N \sqrt{q_k} \sum_{l=1}^N \int_t^T \langle Z^N(s)(e_k), e_l\rangle_H e_l \mathrm{d}b^k(s), e_i \right\rangle_H \\ &= \sum_{k=1}^N \sqrt{q_k} \int_t^T \langle Z^N(s)(e_k), e_i\rangle_H \mathrm{d}b^k(s) \\ &= \sum_{k=1}^N \int_t^T \langle \sqrt{q_k} e_k, (Z^N(s))^*(e_i)\rangle_H \mathrm{d}b^k(s) \\ &= \sum_{k=1}^N \int_t^T \langle Q^{\frac{1}{2}}(e_k), (Z^N(s))^*(e_i)\rangle_H \mathrm{d}b^k(s), \end{aligned} \tag{3.6}$$

we define $\hat{Z}^N(t)$ as

$$\begin{pmatrix} \langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_1)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_1)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_1)\rangle_H \\ \langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_2)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_2)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_2)\rangle_H \\ \vdots & \vdots & \vdots & \vdots \\ \langle Q^{\frac{1}{2}}(e_1), (Z^N(s))^*(e_N)\rangle_H, & \langle Q^{\frac{1}{2}}(e_2), (Z^N(s))^*(e_N)\rangle_H, & \cdots, & \langle Q^{\frac{1}{2}}(e_N), (Z^N(s))^*(e_N)\rangle_H \end{pmatrix}$$

and $\hat{\mathcal{W}}^N(t) = \begin{pmatrix} b^1(t) \\ \vdots \\ b^N(t) \end{pmatrix}$ where $\{b^j(t) : 1 \le j \le N\}$ are N independent standard one-dimensional Brownian motions.

Thus for any $N \in \mathbb{N}$, the projected system (3.1) is equivalent to

$$\begin{cases} \mathrm{d}\hat{\mathbf{u}}^N(t) = -\nu \hat{\mathbf{A}}^N \hat{\mathbf{u}}^N(t)\mathrm{d}t - \hat{\mathbf{B}}^N(\hat{\mathbf{u}}^N(t))\mathrm{d}t + \hat{\mathbf{f}}^N(t)\mathrm{d}t + \hat{Z}^N(t)\mathrm{d}\hat{\mathcal{W}}^N(t) \\ \hat{\mathbf{u}}^N(T) = \hat{\xi}^N. \end{cases} \tag{3.7}$$

Define the associated truncated system as follows:

$$\begin{cases} \mathrm{d}\hat{\mathbf{u}}^{N,M}(t) = -\nu \hat{\mathbf{A}}^N \hat{\mathbf{u}}^{N,M}(t)\mathrm{d}t - L_M(\|\mathbf{u}^{N,M}(t)\|_V)\hat{\mathbf{B}}^N(\hat{\mathbf{u}}^{N,M}(t))\mathrm{d}t \\ \quad +\hat{\mathbf{f}}^N(t)\mathrm{d}t + \hat{Z}^{N,M}(t)\mathrm{d}\hat{\mathcal{W}}^N(t) \\ \hat{\mathbf{u}}^{N,M}(T) = \hat{\xi}^N. \end{cases} \tag{3.8}$$

Let $h^{N,M}(t,x) = -\nu \hat{\mathbf{A}}^N x - L_M(\|x\|_V)\hat{\mathbf{B}}^N(x) + \hat{\mathbf{f}}^N(t)$. Then $h^{N,M}(t,x)$ is Lipschitz with respect to x on $\mathbb{R}^N$. Thus it is clear that (3.8) admits a unique adapted solution $(\hat{\mathbf{u}}^{N,M}, \hat{Z}^{N,M}) \in \mathcal{M}[0,T]$ (see Yong and Zhou, p. 355 [13]), where $\mathcal{M}[0,T]$ is equipped with the norm $\|Y(\cdot), Z(\cdot)\|_{\mathcal{M}} = \{E(\sup_{0\le t\le T}|Y(t)|^2) + E\int_0^T |Z(t)|^2\mathrm{d}t\}^{\frac{1}{2}}$ and here $|Z|^2 = \mathrm{tr}(ZZ^T)$.

As in the proof of Proposition 3.2, it can be shown that $\sup_{0\le t\le T}\|\mathbf{u}^{N,M}(t)\|_H^2 \le K(N)$ for a constant $K(N)$ independent of M. For a fixed N, we make use of the fact that $V_N = H_N$ and $\|\cdot\|_V$ and $\|\cdot\|_H$ are equivalent to each other for the finite dimensional case. So $\sup_{0\le t\le T}\|\mathbf{u}^{N,M}(t)\|_V^2 \le K^*(N)$ for a constant $K^*(N)$ independent of M. Thus for $M > K^*(N)$, $L_M(\|\mathbf{u}^{N,M}(t)\|_V) = 1$, and the solution of (3.8) is also the solution of (3.7). The existence of a solution of (3.7) has thus been shown. Let $(\hat{\mathbf{u}}^N(t), \hat{Z}^N(t))$ and $(\hat{\mathbf{v}}^N(t), \hat{Y}^N(t))$ be two pairs of solutions of (3.7). We know that there exists an M_0 such that $\sup_{0\le t\le T}|\hat{\mathbf{u}}^N(t)|^2 \le K(N)$ and $\sup_{0\le t\le T}|\hat{\mathbf{v}}^N(t)|^2 \le K(N)$. Since (3.7) and (3.8) are the same for $M > K(N)$, we know that $(\hat{\mathbf{u}}^N(t), \hat{Z}^N(t))$ and $(\hat{\mathbf{v}}^N(t), \hat{Y}^N(t))$ are also solutions of (3.8). The uniqueness of the solution of (3.8) implies the uniqueness of the solution of (3.7).

Since (3.1) and (3.7) are equivalent, we have shown that there is a unique adapted solution $(\mathbf{u}^N(t), Z^N(t))$ to the projected system (3.1). Eq. (3.5) is proved by the following:

$$\begin{aligned}
E\left(\int_0^T \hat{Z}^N(s)\mathrm{d}\hat{\mathcal{W}}^N(s)\right)^2 &= E\int_0^T |\hat{Z}^N(s)|^2\mathrm{d}s = E\int_0^T \mathrm{tr}(\hat{Z}^N(s)\hat{Z}^{N^T}(s))\mathrm{d}s\\
&= E\sum_{k,l=1}^N \int_0^T \langle Q^{\frac12}(e_k), (Z^N(s))^*(e_l)\rangle_H^2\mathrm{d}s = E\left(\int_0^T Z^N(s)\mathrm{d}W^N(s)\right)^2 \quad \text{by (3.6)}\\
&= E\sum_{k,l=1}^N \int_0^T \langle e_k, Q^{\frac12}(Z^N(s))^*(e_l)\rangle_H^2\mathrm{d}s = E\sum_{k,l=1}^\infty \int_0^T \langle e_k, Q^{\frac12}(Z^N(s))^*(e_l)\rangle_H^2\mathrm{d}s\\
&= E\sum_{l=1}^\infty \int_0^T \|Q^{\frac12}(Z^N(s))^*(e_l)\|_H^2\mathrm{d}s = E\int_0^T \|Q^{\frac12}(Z^N(s))^*\|_{H.S.}^2\mathrm{d}s\\
&= E\int_0^T \mathrm{tr}(Z^N(s)Q(Z^N(s))^*)\mathrm{d}s = E\int_0^T \|Z^N(s)\|_{L_Q}^2\mathrm{d}s. \quad \square
\end{aligned}$$

4. Existence of solutions

In order to establish the existence of a solution to the backward stochastic Navier–Stokes equations, we need a priori estimates that are uniform for the sequence of Galerkin approximations. Therefore, we assume that **f** depends on the solution **u** and satisfies the following hypotheses. Such an approach is commonly taken in the study of stochastic Euler equations by several authors so that a dissipative effect arises.

Hypotheses H. H.1 (Continuity): $\mathbf{f} : V \to V'$ is a continuous operator.
H.2 (Coercivity): There exist positive constants α and β such that

$$\langle \nu\mathbf{A}\mathbf{u} - \mathbf{f}(\mathbf{u}), \mathbf{u}\rangle_{V',V} \le \alpha\|\mathbf{u}\|_H^2 - \beta\|\mathbf{u}\|_V^2;$$
$$\langle \nu\mathbf{A}\mathbf{u} - \mathbf{f}(\mathbf{u}), \mathbf{A}\mathbf{u}\rangle_{V',V} \le \alpha\|\mathbf{u}\|_V^2 - \beta\|\mathbf{A}\mathbf{u}\|_V^2.$$

H.3 (Monotonicity): For any **u** and **v** in V, a constant $\kappa > \nu$ and a positive constant α,

$$\langle \kappa\mathbf{A}(\mathbf{u}-\mathbf{v}) - (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})), \mathbf{u}-\mathbf{v}\rangle_{V',V} \le \alpha\|\mathbf{u}-\mathbf{v}\|_H^2.$$

H.4 (Linear growth): For any $\mathbf{u}\in V$ and some positive constant α,

$$|\langle \mathbf{f}(\mathbf{u}), \mathbf{u}\rangle_{V',V}| \le \alpha\|\mathbf{u}\|_V^2.$$

In the rest of the paper, $\mathbf{f}$ is assumed to satisfy Hypotheses H, which is quite standard (see, for example, Chow (p. 177 [1]), Kallianpur and Xiong [4]). The BSNSE and the projected BSNSE become

$$\begin{cases} \mathrm{d}\mathbf{u}(t) = -\nu \mathbf{A}\mathbf{u}(t)\mathrm{d}t - \mathbf{B}(\mathbf{u}(t))\mathrm{d}t + \mathbf{f}(\mathbf{u}(t))\mathrm{d}t + Z(t)\mathrm{d}W(t) \\ \mathbf{u}(T) = \xi, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \mathrm{d}\mathbf{u}^N(t) = -\nu \mathbf{A}^N\mathbf{u}^N(t)\mathrm{d}t - \mathbf{B}^N(\mathbf{u}^N(t))\mathrm{d}t + \mathbf{f}^N(\mathbf{u}^N(t))\mathrm{d}t + Z^N(t)\mathrm{d}W^N(t) \\ \mathbf{u}^N(T) = \xi^N. \end{cases} \tag{4.2}$$

The proof of existence and uniqueness of an adapted solution of (4.2) follows along similar lines to that of Theorem 3.6. One can also refer to the paper by Rong [10]. Throughout the proof, we shall use the following proposition.

Proposition 4.1. *Assume that $\|\xi\|_H^2 \leq K$ for almost all $\omega \in \Omega$ and some constant K. If $(\mathbf{u}^N(t), Z^N(t))$ is an adapted solution for the projected system (4.2), then there exists a constant K_0, independent of N, such that*

$$\sup_{0\le t\le T} \|\mathbf{u}^N(t)\|_H^2 + E\int_0^T \|\mathbf{u}^N(t)\|_V^2\mathrm{d}t + E\int_0^T \|Z^N(t)\|_{L_Q}^2\mathrm{d}t \le K_0, \quad P\text{-a.s.} \tag{4.3}$$

Proof. We make use of the coercivity assumption on $\mathbf{f}$ and apply the Gronwall inequality to obtain the result. The proof is similar to the proof of Proposition 3.2. □

The proof of the following proposition uses techniques similar to those for the a priori estimates established earlier. However, it is given in full since some variations are needed.

Proposition 4.2. *Let the conditions in Proposition 4.1 hold. Also let $\|\xi\|_V^2 \leq K$ for almost all $\omega \in \Omega$ and some constant K. Then there exists a constant K_1 such that*

$$\sup_{0\le t\le T} \|\mathbf{u}^N(t)\|_V^2 \le K_1, \quad P\text{-a.s.}$$

Proof. First, an application of the Itô formula to $\|u^N(t)\|_V^2$ yields

$$\begin{aligned} \mathrm{d}\|\mathbf{u}^N(t)\|_V^2 &= -2\langle \nu\mathbf{A}^N\mathbf{u}^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H\mathrm{d}t \\ &\quad - 2\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H\mathrm{d}t + 2\langle \mathbf{f}^N(\mathbf{u}^N(t)), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H\mathrm{d}t \\ &\quad + 2\langle Z^N(t)\mathrm{d}W^N(t), \mathbf{A}^N\mathbf{u}^N(t)\rangle_H + \sum_{i=1}^N \langle Z^N(t)Q(Z^N(t))^*e_i, e_i\rangle_V\mathrm{d}t. \end{aligned}$$

Note that

$$\sum_{i=1}^N \langle Z^N(t)Q(Z^N(t))^*e_i, e_i\rangle_V = \sum_{i=1}^N \langle Z^N(t)Q(Z^N(t))^*e_i, \mathbf{A}^N e_i\rangle_H \ge \|Z^N(t)\|_{L_Q}^2. \tag{4.4}$$

Integrating from t to T, and taking the conditional expectation for $0 \le r \le t \le T$, we get

$$\begin{aligned} E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_V^2 &\le E^{\mathcal{F}_r}\|\xi^N\|_V^2 + 2E^{\mathcal{F}_r}\int_t^T \langle \nu \mathbf{A}^N\mathbf{u}^N(s) - \mathbf{f}^N(\mathbf{u}^N(s)), \mathbf{A}^N\mathbf{u}^N(s)\rangle_H \mathrm{d}s \\ &\quad + 2E^{\mathcal{F}_r}\int_t^T \langle \mathbf{B}^N(\mathbf{u}^N(s)), \mathbf{A}^N\mathbf{u}^N(s)\rangle_H \mathrm{d}s \\ &\le E^{\mathcal{F}_r}\|\xi^N\|_V^2 + 2E^{\mathcal{F}_r}\int_t^T (\alpha\|\mathbf{u}^N(s)\|_V^2 - \beta\|\mathbf{A}^N\mathbf{u}^N(s)\|_V^2)\mathrm{d}s \\ &\quad + 2E^{\mathcal{F}_r}\int_t^T \|\mathbf{u}^N(s)\|_H\|\mathbf{u}^N(s)\|_V\|\mathbf{A}^N\mathbf{u}^N(s)\|_V \mathrm{d}s \\ &\le E^{\mathcal{F}_r}\|\xi^N\|_V^2 + 2E^{\mathcal{F}_r}\int_t^T (\alpha\|\mathbf{u}^N(s)\|_V^2 - \beta\|\mathbf{A}^N\mathbf{u}^N(s)\|_V^2)\mathrm{d}s \\ &\quad + E^{\mathcal{F}_r}\int_t^T \left(\frac{1}{2\beta}\|\mathbf{u}^N(s)\|_H^2\|\mathbf{u}^N(s)\|_V^2 + 2\beta\|\mathbf{A}^N\mathbf{u}^N(s)\|_V^2\right)\mathrm{d}s \\ &\le E^{\mathcal{F}_r}\|\xi^N\|_V^2 + \int_t^T E^{\mathcal{F}_r}\left(2\alpha + \frac{1}{2\beta}\sup_{0\le t\le T}\|\mathbf{u}^N(t)\|_H^2\right)\|\mathbf{u}^N(s)\|_V^2 \mathrm{d}s. \end{aligned}$$

An application of Proposition 4.1 and the Gronwall inequality shows that

$$E^{\mathcal{F}_r}\|\mathbf{u}^N(t)\|_V^2 \le K_1,$$

Setting $r = t$, and then taking the supremum,

$$\sup_{0\le t\le T}\|\mathbf{u}^N(t)\|_V^2 \le K_1, \quad P\text{-a.s.} \quad \square$$

Now let us list some simple results. The proofs can be found in the paper by Menaldi and Sritharan [9].

Lemma 4.3. *For all* $\mathbf{u}, \mathbf{v} \in V$ *and a constant* C_G *depending on the domain* G*,*

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle_{V',V}| \le (\kappa - \nu)\|\mathbf{u} - \mathbf{v}\|_V^2 + \frac{C_G^2}{4(\kappa - \nu)}\|\mathbf{u} - \mathbf{v}\|_H^2\|\mathbf{v}\|_V^2.$$

Corollary 4.4. *For all* $\mathbf{u}, \mathbf{v} \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; V))$*, let*

$$r_1(t) = \int_t^T \left\{2\alpha + \frac{C_G^2}{2(\kappa - \nu)}\|\mathbf{u}(s)\|_V^2\right\}\mathrm{d}s$$

and

$$r_2(t) = \int_t^T \left\{2\alpha + \frac{C_G^2}{2(\kappa - \nu)}\|\mathbf{v}(s)\|_V^2\right\}\mathrm{d}s.$$

Then

$$\langle \nu\mathbf{A}\mathbf{w} + \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}) - (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})) + \frac{1}{2}\dot{r}_i(t)\mathbf{w}, \mathbf{w}\rangle_{V',V} \le 0, \quad i = 1, 2,$$

where $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Proof. It is a direct application of the monotonicity assumption of $\mathbf{f}$ and Lemma 4.3. □

Lemma 4.5. *For any* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w} \in V$, *we have*

$$|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle_{V',V}| \le C(\|\mathbf{u}\|_H^{\frac{1}{2}}\|\mathbf{u}\|_V^{\frac{1}{2}} + \|\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{v}\|_V^{\frac{1}{2}})\|\mathbf{u}-\mathbf{v}\|_H^{\frac{1}{2}}\|\mathbf{u}-\mathbf{v}\|_V^{\frac{1}{2}}\|\mathbf{w}\|_V.$$

Lemma 4.6 (*[14]*). *Let* $A : K \to K'$ *be linear and monotone, where* K *is a Banach space and* K' *is the dual. Then* A *is continuous.*

Theorem 4.7. *Assume that* $\|\xi\|_V^2 < K$ *for some constant* K, P*-a.s. Then the backward stochastic Navier–Stokes equation* (4.1) *admits an adapted solution* $(\mathbf{u}(t), Z(t)) \in L^\infty(\Omega \times [0,T]; H) \cap L^4_{\mathcal{F}}(\Omega; L^2(0,T;V)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Proof. We will prove the theorem in the following steps.

Step 1: First, let us find some bounds for the projected system. By Proposition 4.1, we know that $\{\mathbf{u}^N\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V))$. Hence $\{\mathbf{u}^N\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$.

Since $\mathbf{A}$ is linear and monotone, i.e.,

$$\langle \mathbf{A}(\mathbf{u}-\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle_{V',V} = \|\mathbf{u}-\mathbf{v}\|_V \ge 0,$$

by Lemma 4.6, we know that $\mathbf{A}$ is continuous. So there exists a constant C such that $\|\mathbf{A}\mathbf{u}\|_{V'} \le C\|\mathbf{u}\|_V$ for all $\mathbf{u} \in V$. Thus from (4.3), we know that $\{\mathbf{A}^N\mathbf{u}^N\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$.

By Proposition 2.7, for any $\mathbf{v} \in V$,

$$|\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{v}\rangle_{V',V}| \le C_G\|\mathbf{u}^N(t)\|_V\|\mathbf{u}^N(t)\|_H\|\mathbf{v}\|_V.$$

By Proposition 4.1,

$$\|\mathbf{B}^N(\mathbf{u}^N(t))\|_{V'} = \sup_{\|\mathbf{v}\|_V=1}|\langle \mathbf{B}^N(\mathbf{u}^N(t)), \mathbf{v}\rangle_{V',V}| \le C_G\sqrt{K_0}\|\mathbf{u}^N(t)\|_V.$$

Since $\{\mathbf{u}^N(t)\}_{N=1}^\infty$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$, so is $\{\mathbf{B}^N(\mathbf{u}^N(t))\}$.

It readily follows from Proposition 4.1 that $\{Z^N\}$ is bounded in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$.

Step 2: Clearly we have the following strong convergence:

$$\xi^N \to \xi \text{ in } L^2_{\mathcal{F}_T}(\Omega; V').$$

Indeed, for any $A \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_T^n$, there is an m such that $A \in \mathcal{F}_T^m$. For all $N \ge m$, we have

$$\int_A E(P_N\xi|\mathcal{F}_T^N)\mathrm{d}P = \int_A P_N\xi\mathrm{d}P.$$

By the Lebesgue dominated convergence theorem, one gets

$$\begin{aligned}\int_A \lim_{N\to\infty} E(P_N\xi|\mathcal{F}_T^N)\mathrm{d}P &= \lim_{N\to\infty}\int_A E(P_N\xi|\mathcal{F}_T^N)\mathrm{d}P \\ &= \lim_{N\to\infty}\int_A P_N\xi\mathrm{d}P = \int_A \lim_{N\to\infty} P_N\xi\mathrm{d}P = \int_A \xi\mathrm{d}P.\end{aligned}$$

Thus

$$\int_A \lim_{N\to\infty} E(P_N\xi|\mathcal{F}_T^N)\mathrm{d}P = \int_A \xi\mathrm{d}P, \quad \forall A \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_T^n.$$

Hence the above equality holds on $\sigma(\bigcup_{n\in\mathbb{N}} \mathcal{F}_T^n) = \mathcal{F}_T$. Thus

$$\lim_{N\to\infty} E(P_N\xi|\mathcal{F}_T^N) = \xi \quad P-\text{a.s.}$$

Since $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$ and $L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$ are Hilbert spaces, we infer from Step 1 and the coercivity assumption on $\mathbf{f}$ that along a subsequence $\{N_k\}_{k=1}^{\infty}$,

$$\mathbf{u}^{N_k} \xrightarrow{w} \mathbf{u},\ \nu\mathbf{A}^{N_k}\mathbf{u}^{N_k} - \mathbf{f}^{N_k}(\mathbf{u}^{N_k}) \xrightarrow{w} Y, \quad \text{and} \quad \mathbf{B}^{N_k}(\mathbf{u}^{N_k}) \xrightarrow{w} G$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$, and

$$Z^{N_k} \xrightarrow{w} Z \quad \text{in } L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)).$$

For every t, we define $\mathcal{L}_t : L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q)) \to L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$ as follows: $\mathcal{L}_t(M(\cdot)) = \int_t^T M(s)\mathrm{d}W(s)$, $t\in[0,T]$. Then by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}
E\int_0^T \|\mathcal{L}_t(M(\cdot))\|_{V'}^2\mathrm{d}t &\le TE\sup_{0\le t\le T}\|\mathcal{L}_t(M(\cdot))\|_H^2 \\
&\le 2TE\|\int_0^T M(s)\mathrm{d}W(s)\|_H^2 + 2TE\sup_{0\le t\le T}\|\int_0^t M(s)\mathrm{d}W(s)\|_H^2 \\
&\le 4TE\sup_{0\le t\le T}\|\int_0^t M(s)\mathrm{d}W(s)\|_H^2 \le 4TCE\int_0^T \|M(s)\|_{L_Q}^2\mathrm{d}s
\end{aligned}$$

for some constant C. This shows that $\mathcal{L}_t$ is a bounded linear operator. Hence $\mathcal{L}_t$ maps weakly convergent sequence $\{Z^{N_k}\}_{k=1}^{\infty}$ to a weakly convergent sequence $\{\int_t^T Z^{N_k}(s)\mathrm{d}W^{N_k}(s)\}_{k=1}^{\infty}$ with limit $\int_t^T Z(s)\mathrm{d}W(s)$. Here we have used the fact that $\int_t^T Z^N(s)\mathrm{d}W(s) = \int_t^T Z^N(s)\mathrm{d}W^N(s)$ by letting $Z^N(t)(e_i) = 0$ for $i > N$. Similarly, it can be shown that

$$\int_t^T \{\nu\mathbf{A}^{N_k}\mathbf{u}^{N_k}(s) - \mathbf{f}^{N_k}(\mathbf{u}^{N_k}(s)) + \mathbf{B}^{N_k}(\mathbf{u}^{N_k}(s))\}\mathrm{d}s \xrightarrow{w} \int_t^T (Y(s)+G(s))\mathrm{d}s$$

in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$. Let $F^{N_k}(t)$ denote

$$\xi^{N_k} + \int_t^T \{\nu\mathbf{A}^{N_k}\mathbf{u}^{N_k}(s) + \mathbf{B}^{N_k}(\mathbf{u}^{N_k}(s)) - \mathbf{f}^{N_k}(\mathbf{u}^{N_k}(s))\}\mathrm{d}s - \int_t^T Z^{N_k}(s)\mathrm{d}W^{N_k}(s).$$

Then $\mathbf{u}^{N_k}(t) = F^{N_k}(t)P$-a.s. for every k, and they both converge weakly in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;V'))$. Hence the weak limits agree P-a.s, i.e.,

$$\mathbf{u}(t) = \xi + \int_t^T (Y(s)+G(s))\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s), \quad P\text{-a.s.} \tag{4.5}$$

Step 3: Now let us prove the existence. From now on, we will denote the index of those convergent subsequences by N again, instead of N_k.

Let $r(t)=\int_t^T\{2\alpha+\frac{C_G^2}{2(\kappa-\nu)}K_1\}\mathrm{d}s$. Applying Itô's formula to $\mathrm{e}^{-r(t)}\|\mathbf{u}^N(t)\|_H^2$ and taking the expectation, we get

$$
\begin{aligned}
&E\|\xi^N\|_H^2-E\mathrm{e}^{-r(0)}\|\mathbf{u}^N(0)\|_H^2-E\int_0^T\mathrm{e}^{-r(t)}\|Z^N(t)\|_{L_Q}^2\mathrm{d}t\\
&=-2E\int_0^T\mathrm{e}^{-r(t)}\langle\nu\mathbf{A}^N\mathbf{u}^N(t)+\mathbf{B}^N(\mathbf{u}^N(t))-\mathbf{f}^N(\mathbf{u}^N(t))\\
&\quad+\frac{1}{2}\dot{r}(t)\mathbf{u}^N(t),\mathbf{u}^N(t)\rangle_{V',V}\mathrm{d}t.
\end{aligned}\tag{4.6}
$$

Taking the limit, (4.6) becomes

$$
\begin{aligned}
&\varliminf_{N\to\infty}\Big\{2E\int_0^T\mathrm{e}^{-r(t)}\langle\nu\mathbf{A}^N\mathbf{u}^N(t)+\mathbf{B}^N(\mathbf{u}^N(t))-\mathbf{f}^N(\mathbf{u}^N(t))\\
&\quad+\frac{1}{2}\dot{r}(t)\mathbf{u}^N(t),\mathbf{u}^N(t)\rangle_{V',V}\mathrm{d}t\Big\}\\
&=\varliminf_{N\to\infty}\Big\{E\mathrm{e}^{-r(0)}\|\mathbf{u}^N(0)\|_H^2-E\|\xi^N\|_H^2+E\int_0^T\mathrm{e}^{-r(t)}\|Z^N(t)\|_{L_Q}^2\mathrm{d}t\Big\}\\
&\geq E\mathrm{e}^{-r(0)}\|\mathbf{u}(0)\|_H^2-E\|\xi\|_H^2+E\int_0^T\mathrm{e}^{-r(t)}\|Z(t)\|_{L_Q}^2\mathrm{d}t.
\end{aligned}\tag{4.7}
$$

Likewise, Eq. (4.5) and the Itô formula applied to $\mathrm{e}^{-r(t)}\|\mathbf{u}(t)\|_H^2$ yield

$$
\begin{aligned}
&E\mathrm{e}^{-r(0)}\|\mathbf{u}(0)\|_H^2+E\int_0^T\mathrm{e}^{-r(t)}\|Z(t)\|_{L_Q}^2\mathrm{d}t\\
&=E\|\xi\|_H^2+2E\int_0^T\mathrm{e}^{-r(t)}\langle Y(t)+G(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{u}(t)\rangle_{V',V}\mathrm{d}t.
\end{aligned}\tag{4.8}
$$

Combining (4.7) and (4.8), we get

$$
\begin{aligned}
&\varliminf_{N\to\infty}\Big\{E\int_0^T\mathrm{e}^{-r(t)}\langle\nu\mathbf{A}^N\mathbf{u}^N(t)+\mathbf{B}^N(\mathbf{u}^N(t))-\mathbf{f}^N(\mathbf{u}^N(t))\\
&\quad+\frac{1}{2}\dot{r}(t)\mathbf{u}^N(t),\mathbf{u}^N(t)\rangle_{V',V}\mathrm{d}t\Big\}\\
&\geq E\int_0^T\mathrm{e}^{-r(t)}\langle Y(t)+G(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{u}(t)\rangle_{V',V}\mathrm{d}t.
\end{aligned}\tag{4.9}
$$

Now by Corollary 4.4, we have

$$
\begin{aligned}
&E\int_0^T\mathrm{e}^{-r(t)}\langle\nu\mathbf{A}(\mathbf{v}(t)-\mathbf{u}^N(t))+\mathbf{B}(\mathbf{v}(t))-\mathbf{B}(\mathbf{u}^N(t))-(\mathbf{f}(\mathbf{v}(t))-\mathbf{f}(\mathbf{u}^N(t)))\\
&\quad+\frac{1}{2}\dot{r}(t)(\mathbf{v}(t)-\mathbf{u}^N(t)),\mathbf{v}(t)-\mathbf{u}^N(t)\rangle_{V',V}\mathrm{d}t\leq 0
\end{aligned}
$$

for any adapted $\mathbf{v}$ with values in V_N and $\|\mathbf{v}(t)\|_V^2 < K_1$ for all t. Hence

$$
\begin{aligned}
&E\int_0^T \mathrm{e}^{-r(t)}\langle \nu\mathbf{A}\mathbf{u}^N(t)+\mathbf{B}(\mathbf{u}^N(t))-\mathbf{f}(\mathbf{u}^N(t))+\frac{1}{2}\dot{r}(t)\mathbf{u}^N(t),\mathbf{u}^N(t)-\mathbf{v}(t)\rangle_{V',V}\,\mathrm{d}t\\
&\quad\le E\int_0^T \mathrm{e}^{-r(t)}\langle \nu\mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))-\mathbf{f}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{u}^N(t)-\mathbf{v}(t)\rangle_{V',V}\,\mathrm{d}t.
\end{aligned}
$$

Taking the limit, and by (4.9), we get

$$
\begin{aligned}
&E\int_0^T \mathrm{e}^{-r(t)}\langle Y(t)+G(t)+\frac{1}{2}\dot{r}(t)\mathbf{u}(t),\mathbf{u}(t)-\mathbf{v}(t)\rangle_{V',V}\,\mathrm{d}t\\
&\quad\le E\int_0^T \mathrm{e}^{-r(t)}\langle \nu\mathbf{A}\mathbf{v}(t)+\mathbf{B}(\mathbf{v}(t))-\mathbf{f}(\mathbf{v}(t))+\frac{1}{2}\dot{r}(t)\mathbf{v}(t),\mathbf{u}(t)-\mathbf{v}(t)\rangle_{V',V}\,\mathrm{d}t. \qquad (4.10)
\end{aligned}
$$

By a density argument as in [9,11], (4.10) is true for all functions $\mathbf{v}\in L^\infty(\Omega\times[0,T];V)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$. Now we take $\mathbf{v}(t)=\mathbf{u}(t)+\lambda\mathbf{w}(t)$ for any $\mathbf{w}\in L^\infty(\Omega\times[0,T];V)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$ and sufficiently small $\lambda>0$. Like for Corollary 3.3, one can show that $\mathbf{u}$ is also in $L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$. Eq. (4.10) becomes

$$
\begin{aligned}
&E\int_0^T \mathrm{e}^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t))+\mathbf{f}(\mathbf{u}(t)+\lambda\mathbf{w}(t)),\lambda\mathbf{w}(t)\rangle_{V',V}\,\mathrm{d}t\\
&\quad\ge E\int_0^T \mathrm{e}^{-r(t)}\langle \lambda\nu\mathbf{A}\mathbf{w}(t)+\frac{\lambda}{2}\dot{r}(t)\mathbf{w}(t),\lambda\mathbf{w}(t)\rangle_{V',V}\,\mathrm{d}t.
\end{aligned}
$$

Canceling λ, and using the fact that

$$
\begin{aligned}
\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t)),\mathbf{w}(t)\rangle_{V',V} &= -\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)+\lambda\mathbf{w}(t)\rangle_{V',V}\\
&= -\langle\mathbf{B}(\mathbf{u}(t)+\lambda\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}\\
&= -\langle\mathbf{B}(\mathbf{u}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}-\lambda\langle\mathbf{B}(\mathbf{w}(t),\mathbf{w}(t)),\mathbf{u}(t)\rangle_{V',V}\\
&= \langle\mathbf{B}(\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V}+\lambda\langle\mathbf{B}(\mathbf{w}(t),\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V},
\end{aligned}
$$

we get

$$
\begin{aligned}
&E\int_0^T \mathrm{e}^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t))+\mathbf{f}(\mathbf{u}(t)+\lambda\mathbf{w}(t)),\mathbf{w}(t)\rangle_{V',V}\,\mathrm{d}t\\
&\quad\ge \lambda E\int_0^T \mathrm{e}^{-r(t)}\langle \nu\mathbf{A}\mathbf{w}(t)+\mathbf{B}(\mathbf{w}(t),\mathbf{u}(t))+\frac{1}{2}\dot{r}(t)\mathbf{w}(t),\mathbf{w}(t)\rangle_{V',V}\,\mathrm{d}t.
\end{aligned}
$$

Recall the continuity assumption on $\mathbf{f}$. Letting $\lambda\to 0$, since the right hand side of the last inequality is finite, we get

$$
E\int_0^T \mathrm{e}^{-r(t)}\langle Y(t)+G(t)-\nu\mathbf{A}\mathbf{u}(t)-\mathbf{B}(\mathbf{u}(t))+\mathbf{f}(\mathbf{u}(t)),\mathbf{w}(t)\rangle_{V',V}\,\mathrm{d}t\le 0
$$

for all $\mathbf{w}\in L^\infty(\Omega\times[0,T];H)\cap L^4_{\mathcal{F}}(\Omega;L^2(0,T;V))$.

Hence $Y(t)+G(t)=\nu\mathbf{A}\mathbf{u}(t)+\mathbf{B}(\mathbf{u}(t))-\mathbf{f}(\mathbf{u}(t))\,P$-a.s. and this completes the proof of the existence of the solution. □

5. Uniqueness of solutions

In order to obtain uniqueness of solutions to the backward Navier–Stokes equations introduced in the previous section, we again use the uniform bound on the V-norm of the solution. Such a situation arises in certain other nonlinear stochastic partial differential equations (stochastic Euler equations, equations for compressible flow, etc.) as well.

Theorem 5.1. *Assume the conditions in Theorem 4.7 and assumptions on* **f**. *The adapted solution of (4.1) is unique in* $L^{\infty}(\Omega \times [0,T]; V) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_Q))$.

Proof. The existence of an adapted solution is guaranteed by Theorem 4.7. Proposition 4.2 shows that the solution lies in $L^{\infty}(\Omega \times [0,T]; V) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T; L_Q))$. Let $(\mathbf{u}, Z)$ and $(\mathbf{v}, \sigma)$ be two adapted solutions in the corresponding solution space.

Let $\mathbf{w}(t) = \mathbf{u}(t) - \mathbf{v}(t)$; then

$$\begin{cases} \mathrm{d}\mathbf{w}(t) = -\nu \mathbf{A}\mathbf{w}(t)\mathrm{d}t - (\mathbf{B}(\mathbf{u}(t)) - \mathbf{B}(\mathbf{v}(t)))\mathrm{d}t \\ \quad +(\mathbf{f}(\mathbf{u}(t)) - \mathbf{f}(\mathbf{v}(t)))\mathrm{d}t + (Z(t) - \sigma(t))\mathrm{d}W(t) \\ \mathbf{w}(T) = 0. \end{cases}$$

The Itô formula yields

$$\begin{aligned} &\|\mathbf{w}(t)\|_H^2 + \int_t^T \|Z(s) - \sigma(s)\|_{L_Q}^2 \mathrm{d}s \\ &= 2\int_t^T \langle \nu \mathbf{A}\mathbf{w}(s) + (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{v}(s))) - (\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))), \mathbf{w}(s)\rangle_{V',V} \mathrm{d}s \\ &\quad - 2\int_t^T \langle (Z(s) - \sigma(s))\mathrm{d}W(s), \mathbf{w}(s)\rangle_H \end{aligned} \tag{5.1}$$

First,

$$\begin{aligned} &|\langle \mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{v}(s)), \mathbf{w}(s)\rangle_{V',V}| \\ &= |\langle \mathbf{B}(\mathbf{u}(s), \mathbf{w}(s)), \mathbf{w}(s)\rangle_{V',V} + \langle \mathbf{B}(\mathbf{u}(s), \mathbf{v}(s)), \mathbf{w}(s)\rangle_{V',V} \\ &\quad - \langle \mathbf{B}(\mathbf{v}(s), \mathbf{v}(s)), \mathbf{w}(s)\rangle_{V',V}| \\ &= |\langle \mathbf{B}(\mathbf{w}(s), \mathbf{v}(s)), \mathbf{w}(s)\rangle_{V',V}| \\ &\leq \|\mathbf{w}(s)\|_H \|\mathbf{w}(s)\|_V \|\mathbf{v}(s)\|_V \\ &\leq \frac{1}{4\beta}\|\mathbf{v}(s)\|_V^2 \|\mathbf{w}(s)\|_H^2 + \beta\|\mathbf{w}(s)\|_V^2. \end{aligned}$$

By hypotheses H.3, the above equality and Eq. (5.1), one gets

$$\begin{aligned} &E^{\mathcal{F}_r}\|\mathbf{w}(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|Z(s) - \sigma(s)\|_{L_Q}^2 \mathrm{d}s \\ &\leq \int_t^T E^{\mathcal{F}_r}\left(2\alpha + \frac{1}{2\beta}\sup_{0\leq t\leq T}\|\mathbf{v}(s)\|_V^2\right)\|\mathbf{w}(s)\|_H^2 \mathrm{d}s \end{aligned}$$

for $0 \leq r \leq t \leq T$. Note that $\mathbf{v} \in L^{\infty}(\Omega \times [0,T]; V)$. An application of the Gronwall inequality yields that $(\mathbf{u}, Z) = (\mathbf{v}, \sigma) P$-a.s. □

6. Continuity of the solution

The continuity of the solution of stochastic Navier–Stokes equations with respect to initial data and the external body force is well known (see [9]). Likewise, we prove below a continuity result for BSNSEs with respect to *terminal* data.

Theorem 6.1. *Let the conditions in Theorem 5.1 hold. Then the solution* $(\mathbf{u}, Z)$ *is continuous in* $L^\infty(\Omega \times [0, T]; H) \cap L^2_{\mathcal{F}}(\Omega; L^2(0, T; V)) \times L^2_{\mathcal{F}}(\Omega; L^2(0, T; L_Q))$*, with respect to the terminal value.*

Proof. Let $\xi_1, \xi_2 \in L^\infty_{\mathcal{F}_T}(\Omega; V)$ be such that $\|\xi_1 - \xi_2\|_V^2 < \epsilon$. Let $(\mathbf{u}(t), Z(t))$ be the solution of (4.1) with respect to terminal value ξ_1, and let $(\mathbf{v}(t), Y(t))$ be the solution of (4.1) with respect to terminal value ξ_2. Define $\mathbf{w}(t) = \mathbf{u}(t) - \mathbf{v}(t)$ and $\sigma(t) = Z(t) - Y(t)$. Then

$$\begin{cases} \mathrm{d}\mathbf{w}(t) = -\nu\mathbf{A}\mathbf{w}(t)\mathrm{d}t - (\mathbf{B}(\mathbf{u}(t)) - \mathbf{B}(\mathbf{v}(t)))\mathrm{d}t + (\mathbf{f}(\mathbf{u}(t)) - \mathbf{f}(\mathbf{v}(t)))\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \\ \mathbf{w}(T) = \xi_1 - \xi_2. \end{cases}$$

An application of the Itô formula implies

$$\begin{aligned} \|\mathbf{w}(t)\|_H^2 + \int_t^T \|\sigma(s)\|_{L_Q}^2 \mathrm{d}s = {} & \|\xi_1 - \xi_2\|_H^2 + 2\int_t^T \langle \nu\mathbf{A}\mathbf{w}(s) + (\mathbf{B}(\mathbf{u}(s)) - \mathbf{B}(\mathbf{v}(s))) \\ & - (\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))), \mathbf{w}(s)\rangle_{V',V} \mathrm{d}s \\ & - 2\int_t^T \langle \sigma(s)\mathrm{d}W(s), \mathbf{w}(s)\rangle_H. \end{aligned} \tag{6.1}$$

Like for Theorem 5.1, one gets

$$\begin{aligned} & E^{\mathcal{F}_r}\|\mathbf{w}(t)\|_H^2 + E^{\mathcal{F}_r}\int_t^T \|\sigma(s)\|_{L_Q}^2 \mathrm{d}s + \beta E^{\mathcal{F}_r}\int_t^T \|\mathbf{w}(s)\|_V^2 \mathrm{d}s \\ & \le E^{\mathcal{F}_r}\|\xi_1 - \xi_2\|_H^2 + \left(2\alpha + \frac{C}{\beta}\right)\int_t^T E^{\mathcal{F}_r}\|\mathbf{w}(s)\|_H^2 \mathrm{d}s \end{aligned}$$

for $0 \le r \le t \le T$. Using the standard arguments as in Proposition 3.2, one obtains that

$$\sup_{0\le t\le T} \|\mathbf{w}(t)\|_H^2 + E\int_0^T \|\sigma(s)\|_{L_Q}^2 \mathrm{d}s + E\int_0^T \|\mathbf{w}(s)\|_V^2 \mathrm{d}s \le \epsilon\left(\frac{1}{\beta} + 1\right)\mathrm{e}^{(2\alpha + \frac{C}{\beta})T}$$

P-almost surely, and this completes the proof. □

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