1967

Periodic and Almost Periodic Vectors.

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PIERCE, Alan Carl, 1941-
PERIODIC AND ALMOST PERIODIC VECTORS.

Louisiana State University and Agricultural and Mechanical College, Ph.D., 1967
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
PERIODIC AND ALMOST PERIODIC VECTORS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University
and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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M.S., Louisiana State University, 1965
August, 1967
ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor James R. Dorroh for his advice and encouragement.
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ABSTRACT

In this paper the classical concept of periodic and almost periodic functions is generalized by considering, rather than translation, an arbitrary strongly continuous semi-group of operators acting on a complex Banach space.

Let $X$ be a complex Banach space and $[T(t); t \geq 0]$ a strongly continuous semi-group of bounded linear operators on $X$ with infinitesimal generator $A$. A vector $x \in X$ is said to be periodic of period $\lambda > 0$ (with respect to $[T(t); t \geq 0]$) if $T(\lambda)x = x$. Let $P(\lambda)$ be the set of vectors periodic of period $\lambda$ and let $P$ be the linear span of $\bigcup_{\lambda > 0} P(\lambda)$.

The first chapter contains basic results about the space $P(\lambda)$. In particular, each element $x \in P(\lambda)$ has a generalized Fourier series $\sum_{n=-\infty}^{\infty} x_n$ where $x_n = (1/\lambda) \int_{0}^{\lambda} T(t)x e^{-(2\pi i n/\lambda)t} dt$. The Cesaro means of the series $\sum e^{(2\pi i n/\lambda)t} x_n$ converge uniformly on $[0, \lambda]$ to $T(t)x$. For each $n$, $x_n$ is in the domain $\mathcal{D}(A)$ of $A$ and $Ax_n = (2\pi i n/\lambda)x_n$. In fact, $P(\lambda)$ is the closed linear span of $\{x \in \mathcal{D}(A) | Ax = (2\pi i n/\lambda)x\}$ and $P$ is the linear span of eigenvectors of $A$.

Several additional convergence theorems are proved. For instance, if $x \in \mathcal{D}(A) \cap P(\lambda)$ then $\sum x_n$ converges (in norm) to $x$.

A generalization of Parseval's Theorem is also given. Chapter I concludes with some applications of the theory.
The main results of Chapter II are partial generalizations of the classical theorems of Bohr and Bochner. In this chapter AP is the set of almost periodic vectors, i.e. the set of all \( x \in X \) such that for each \( \varepsilon > 0 \) there is a number \( \lambda > 0 \) such that each interval \([a, a+\lambda]\subset[0, \infty)\) contains a number \( \tau \) such that 
\[ \| T(t+\tau)x - T(t)x \| < \varepsilon \]
for all \( t \geq 0 \). In addition, \( K \) is the set of all vectors \( x \in X \) such that \( \{ T(t)x \mid t \geq 0 \} \) has compact closure.

The spaces \( P \) and \( AP \) have the same closures; in fact, if \( x \in AP \) and \( \varepsilon > 0 \) there is a polynomial \( \sum_{k=1}^{n} y_{k} e^{i\lambda_{k} t} \) with each \( y_{k} \in P \) such that \( \| T(t)x - \sum_{k=1}^{n} y_{k} e^{i\lambda_{k} t} \| < \varepsilon \) for all \( t \geq 0 \).

Also \( AP \subset K \).

An example is given at the end of Chapter II to show that neither \( AP \) nor \( K \) is, in general, closed. If \( \{ \| T(t) \| \mid t \geq 0 \} \) is a bounded set then both \( AP \) and \( K \) are closed. However, another example shows that, even if \( \{ \| T(t) \| \mid t \geq 0 \} \) is bounded, \( AP \) may be a proper subset of \( K \).

If \( \{ \| T(t) \| \mid t \geq 0 \} \) is bounded then the semi-group \( [T(t); t \geq 0] \) can be extended to a strongly continuous group of bounded operators on \( AP \).
Suppose \( C_u(-\infty, \infty) \) is the Banach space (with supremum norm) of bounded uniformly continuous complex-valued functions defined on \(( -\infty, \infty )\) and \([T(t); -\infty < t < \infty] \) is the translation group on 
\( C_u(-\infty, \infty) \), i.e. \([T(t)f](s) = f(t+s)\) for \( f \in C_u(-\infty, \infty) \) and \( t, s \in (-\infty, \infty) \). A function \( f \in C_u(-\infty, \infty) \) is then periodic (of period \( \lambda > 0 \)) if \( T(\lambda)f = f \) and almost periodic if for each \( \epsilon > 0 \) there is a number \( \delta > 0 \) such that each interval \([a, a + \delta] \subset (-\infty, \infty)\) contains a number \( \tau \) such that \( \|T(t + \tau)f - T(t)f\|_{\infty} < \epsilon \) for all \( t \). The function \( f \) is said to have compact orbit if \([T(t)f | -\infty < t < \infty]\) has compact closure in \( C_u(-\infty, \infty) \).

Some generalizations of these ideas are considered in \([2], [4], [7], \) and \([10]\). Here, by way of generalization, it is supposed only that \([T(t); t \geq 0]\) is a strongly continuous semi-group of bounded operators on a complex Banach space \( X \).

A vector \( x \in X \) is said to be periodic (with respect to the semi-group \([T(t); t \geq 0]\)) of period \( \lambda > 0 \) if \( T(\lambda)x = x \). Each such vector is shown, in Chapter I, to have a generalized Fourier series with properties similar to those of the classical Fourier series.

In Chapter II a vector \( x \in X \) is defined to be almost periodic with respect to \([T(t); t \geq 0]\) if for each \( \epsilon > 0 \) there
is a number $\varepsilon > 0$ such that each interval $[a, a + \varepsilon] \subset [0, \infty)$ contains a number $\tau$ such that $\|T(t+\tau)x - T(t)x\| < \varepsilon$
for all $t \geq 0$. A vector $x$ is said to have compact orbit with
respect to $[T(t); t \geq 0]$ if $\{T(t)x | t \geq 0\}$ has compact
closure. Each almost periodic vector has compact orbit and
is, at the same time, the limit of a sequence of periodic
vectors. If, in addition, $\{\|T(t)\| | t \geq 0\}$ is a bounded
set then the almost periodic vectors form a Banach space and
$[T(t); t \geq 0]$ has an extension to a strongly continuous group
of bounded operators on this space. Examples show, however,
that complete analogues to the classical theorems of Bohr
and Bochner are not possible in this setting.
CHAPTER 1
PERIODIC VECTORS

Throughout this chapter $X$ is a complex Banach space and $[T(t); t \geq 0]$ or $[T(t)]$ is a strongly continuous semi-group of bounded linear operators on $X$ with infinitesimal generator $A$ defined on a dense domain $\mathcal{D}(A)$ ([5], Chapter VIII, 1).

Definition. A vector $x \in X$ is periodic (with respect to $[T(t)]$) of period $\lambda > 0$ if $T(\lambda)x = x$.

If $\lambda > 0$ let $P(\lambda)$ be the set of all vectors periodic of period $\lambda$. For the sake of simplicity, most of the results of this chapter are stated and proved for $\lambda = 2\pi$.

1.1. Theorem. $P(2\pi)$ is a closed subspace of $X$ and is invariant under each $T(t)$.

Proof. Suppose $x^{(k)}$ is a sequence in $P(2\pi)$, $x \in X$, and $x^{(k)} \to x$ as $k \to \infty$. Then $T(2\pi)x = T(2\pi)\lim_{k \to \infty}x^{(k)} = \lim_{k \to \infty}T(2\pi)x^{(k)} = \lim_{k \to \infty}x^{(k)} = x$ and so $x \in P(2\pi)$.

Therefore $P(2\pi)$ is closed.

If $x \in P(2\pi)$ and $t \geq 0$ then $T(2\pi)T(t)x = T(t)T(2\pi)x = T(t)x$ so $T(t)x \in P(2\pi)$.

1.2. Corollary. $\mathcal{D}(A) \cap P(2\pi)$ is dense in $P(2\pi)$, $A(\mathcal{D}(A) \cap P(2\pi)) \subseteq P(2\pi)$, and $A|_{P(2\pi)}$ is a closed operator.
Proof. By Theorem 1.1 \( P(2\pi) \) is a Banach space and 
\[ [T(t)]_{P(2\pi)} \] is a strongly continuous semi-group on \( P(2\pi) \).
The infinitesimal generator of \([T(t)]_{P(2\pi)}\) is given by
\[ Bx = \lim_{h \to 0^+} \frac{1}{h} (T(h)x - x) \] when this limit exists. Thus
\[ B = A \big|_{P(2\pi)} \] and \( A \lfloor (A) \cap P(2\pi)) \subset P(2\pi) \). Since \( A \big|_{P(2\pi)} \) is
the infinitesimal generator of \([T(t)]_{P(2\pi)}\) it is a closed operator with dense domain ([5], page 620).

Definition. If \( x \in P(2\pi) \) and \( n \) is an integer then
\[ \frac{1}{2\pi} \int_0^{2\pi} T(t)x e^{-int} dt \] is called the \( n \)th Fourier coefficient of \( x \) and is denoted by \( x_n \). The corresponding series is called
the (generalized) Fourier series of \( x \) and this correspondence is denoted by \( x \sim \sum_{n=-\infty}^{\infty} x_n \).

By Theorem 1.1 if \( x \in P(2\pi) \) then \( x_n \in P(2\pi) \) for each \( n \).
The \( n \)th Fourier coefficient of \( T(t)x_n \) is \( \frac{1}{2\pi} \int_0^{2\pi} T(s)T(t)x e^{int} ds = e^{int} x_n \). Moreover, if \( x \in \mathcal{D}(A) \), the \( n \)th Fourier coefficient of \( Ax \) is \( \frac{1}{2\pi} \int_0^{2\pi} T(t)Ax e^{-int} dt = (in/2\pi) \int_0^{2\pi} T(t)x e^{-int} dt =inx_n \).

With only minor modification, the classical proof may be given for the following extension of Fejér's Theorem ([6],
pages 16-18).

1.3. Theorem. If \( x \in P(2\pi) \) then the Cesaro means of the
Fourier series \( \sum x_n e^{int} \) converge to \( T(t)x \) uniformly on \([0, 2\pi]\).

1.4. Corollary. If \( x, y \in P(2\pi) \) have the same Fourier
series then \( x = y \).
1.5. **Theorem.** If \( x \in P(2\pi) \) then \( x_n \in \mathcal{L}(A) \) for each \( n \) and \( Ax_n = \text{inx} \).

**Proof.** If \( h > 0 \),
\[
(1/2\pi h) \int_0^{2\pi} (T(t+h)x - T(t)x)e^{-ient} dt = (1/h)(e^{inh} - 1)x_n. \]
Thus
\[
\lim_{h \to 0^+} \| (1/h)(T(h)x_n - x_n) - \text{inx} \|_n = \lim_{h \to 0^+} \| (1/h)(e^{inh} - 1) - \text{inx} \|_n = 0.
\]
Hence \( x_n \in \mathcal{L}(A) \) and \( Ax_n = \text{inx} \).

1.6. **Theorem.** If \( x \in \mathcal{L}(A) \) and \( Ax = \text{inx} \) for some integer \( n \) then \( T(t)x = e^{int}x \), \( x \in P(2\pi) \), and \( x = x_n \). Conversely, if \( x \in \mathcal{L}(A) \cap P(2\pi), x \neq 0 \), and \( Ax = \lambda x \) for some complex number \( \lambda \), then there is an integer \( n \) such that \( \lambda = in \).

**Proof.** Suppose \( x \in \mathcal{L}(A) \), \( Ax = \text{inx} \). Let \( h(t) = e^{-int}T(t)x \) for \( t \geq 0 \). Then \( h'(t) = -ine^{-int}T(t)x + e^{-int}T(t)Ax = -e^{-int}T(t)inx + e^{-int}T(t)inx = 0 \). Thus \( h \) is constant. Since \( h(0) = x \), \( T(t)x = e^{int}x \). Clearly then \( x \in P(2\pi) \) and \( x_n = (1/2\pi) \int_0^{2\pi} T(t)xe^{-int} dt = (1/2\pi) \int_0^{2\pi} xdt = x \).

Suppose now that \( x \in \mathcal{L}(A) \cap P(2\pi), x \neq 0 \), and \( Ax = \lambda x \). As above, \( T(t)x = e^{\lambda t}x \) so, if \( k \) is an integer,
\[
x_k = (1/2\pi) \int_0^{2\pi} T(t)xe^{-ikt} dt = \begin{cases} x & \text{if } \lambda = ik \\ (1-e^{2\pi(\lambda-ik)})x/2\pi(\lambda-ik) & \text{if } \lambda \neq ik. \end{cases}
\]

If \( \lambda = ik \) the proof is complete. If not, \( Ax_k = ikx_k = ik(1-e^{2\pi(\lambda-ik)})x/2\pi(\lambda-ik) \). But also
\[ A - \lambda \neq 0 \text{ and } \lambda \neq ik, \quad 1 - e^{2\pi(i/(\lambda + ik))} = 0 \text{ so } \lambda = im \text{ for some integer } m. \]

1.7. Theorem. For each \( \lambda > 0 \), \( P(\lambda) \) is the closed linear span of \{ \( x \in \mathcal{D}(A) \mid Ax = (2\pi \text{in/\lambda})x \text{ for some } n \} \).

Proof. If \( \lambda > 0 \) and \( T_\lambda(t) = T(\lambda t/2\pi) \) for \( t \geq 0 \) then \( [T_\lambda(t)] \) is a strongly continuous semi-group on \( X \) with infinitesimal generator \((\lambda/2\pi)A\). Moreover, \( T_\lambda(2\pi)x = x \) if and only if \( T(\lambda)x = x \). Thus it is sufficient to give the proof for \( \lambda = 2\pi \). But this follows immediately from Theorems 1.3, 1.5, and 1.6.

1.8. Theorem. The semi-group \( [T(t)] \) is uniformly continuous on \( P(2\pi) \) if and only if there is a positive integer \( N \) such that \( x = \sum_{n=-N}^{N} x_n \) for each \( x \in P(2\pi) \).

Proof. If \( [T(t)] \) is uniformly continuous on \( P(2\pi) \) then \( A \) is a bounded operator on \( P(2\pi) \) ([5], page 621). In particular, the spectrum of \( A|_{P(2\pi)} \) is bounded by some positive integer \( N \). Then, for \( n > N \) and \( x \in P(2\pi) \), \( Ax_n = inx_n \) so \( x_n = 0 \). Thus \( x = \sum_{n=-N}^{N} x_n \).

Conversely, suppose \( x = \sum_{n=-N}^{N} x_n \) for each \( x \in P(2\pi) \). Then \( P(2\pi) \subset \mathcal{D}(A) \). Since \( A \) is a closed operator, \( A|_{P(2\pi)} \) is bounded. Hence \( [T(t)] \) is uniformly continuous on \( P(2\pi) \).
1.9. **Theorem.** If \( x \in P(2\pi) \) the non-zero Fourier coefficients of \( x \) are linearly independent.

**Proof.** A single non-zero Fourier coefficient of \( x \) forms a linearly independent set. Suppose each set of \( k \) non-zero Fourier coefficients of \( x \) is linearly independent. Let \( x_{n_1}, x_{n_2}, \ldots, x_{n_{k+1}} \) be \( k+1 \) distinct non-zero Fourier coefficients of \( x \) and suppose that \( \sum_{j=1}^{k+1} a_j x_{n_j} = 0 \). Then

\[
\sum_{j=1}^{k+1} n_j a_j x_{n_j} = A(\sum_{j=1}^{k+1} a_j x_{n_j}) = 0.
\]

But \( \sum_{j=1}^{k+1} n_1 a_j x_{n_j} = 0 \) so\( (n_j - n_1) a_j = 0 \) and hence \( a_j = 0 \) for \( 2 \leq j \leq k+1 \). Then \( a_1 = 0 \) and the proof is complete.

1.10. **Theorem.** If \( x \in P(2\pi) \) then \( x_n \to 0 \) as \( |n| \to \infty \).

**Proof.** Suppose first that \( x \in \mathcal{A}(A) \cap P(2\pi) \). Then

\[
x_n = (1/2\pi) \int_0^{2\pi} T(t)x e^{-int} dt = (i/2\pi \text{Im}) \int_0^{2\pi} AT(t)x e^{-int} dt.
\]

So \( \| x_n \| \leq (1/|n|) \sup_{0 \leq t \leq 2\pi} \| T(t)Ax \| \). Hence \( x_n \to 0 \) as \( |n| \to \infty \).

Now suppose only that \( x \in P(2\pi) \). Let \( \epsilon > 0 \). By Corollary 1.2, there is a \( y \in \mathcal{A}(A) \cap P(2\pi) \) such that

\[
\| x - y \| < \epsilon/2 \sup_{0 \leq t \leq 2\pi} \| T(t) \| .
\]

Then there is an integer \( N \) such that \( \| y_n \| < \epsilon/2 \) if \( |n| > N \). Thus, if \( |n| > N \),

\[
\| x_n \| \leq \| x_n - y_n \| + \| y_n \| \leq (1/2\pi) \int_0^{2\pi} \| T(t) \| \| x - T(t)y \| dt + \| y_n \| \leq \| x - y \| \sup_{0 \leq t \leq 2\pi} \| T(t) \| + \| y_n \| < \epsilon/2 + \epsilon/2 = \epsilon.
\]
It follows from the proof of Theorem 1.10 that $\sum \|x_n\|^2 < \infty$ if $x \in \mathcal{S}(A) \cap P(2\pi)$. This need not be true if $x \in \mathcal{S}(A)$. For the translation semi-group is continuous on $L_1(0, 2\pi)$ and if $f(t) = \sum a_n e^{int}$ then $f_n(t) = a_n e^{int}$. But, by the Riesz-Fischer Theorem, if $\sum \|f_n\|^2 < \infty$ then $f \in L_2(0, 2\pi)$.

Let $F$ be the Banach space of bounded uniformly continuous functions from $[0, \infty)$ into $X$ with $\|f\|_\infty = \sup_{t \geq 0} \|f(t)\|$ for $f \in F$. If $f \in F$ let $[S(t)f](u) = f(t+u)$ for $t, u \geq 0$. Clearly $[S(t)]$ is a semi-group of bounded operators on $F$.

Suppose that $f \in F$ and that $\epsilon > 0$. Then there is a $\delta > 0$ such that $\|f(t) - f(u)\| < \epsilon$ if $t, u \geq 0$ and $|t-u| < \delta$. If $0 \leq t < \delta$ then $\|S(t)f - f\|_\infty = \sup_{u \geq 0} \|f(t+u) - f(u)\| \leq \epsilon$ so that $[S(t)]$ is strongly continuous. Let $B$ be its infinitesimal generator.

Lemma. If $f \in F$ then $\int_0^{2\pi} f(s) \sin(n+1/2) s \, ds \to 0$ as $|n| \to \infty$.

Proof. Use of an elementary trigonometric identity yields

$$\int_0^{2\pi} f(s) \sin(n+1/2) s \, ds = \int_0^{2\pi} f(s) \sin(ns) \cos(s/2) \, ds + \int_0^{2\pi} f(s) \cos(ns) \sin(s/2) \, ds.$$ 

Suppose first that $f \in \mathcal{D}(B)$.

As in the proof of Theorem 1.10, $\|\int_0^{2\pi} f(s) \sin(ns) \cos(s/2) \, ds\| \leq (1/n) (\|f\|_{\infty} (2\pi) + 2\pi \|B\|_{\infty})$ and $\|\int_0^{2\pi} f(s) \cos(ns) \sin(s/2) \, ds\| \leq (1/n) (\pi \|f\|_{\infty} + 2\pi \|B\|_{\infty})$. Thus $\int_0^{2\pi} f(s) \sin(n+1/2) s \, ds \to 0$ as $|n| \to \infty$. 
Suppose now only that $f \in \mathcal{F}(B)$ such that $\|f - g\|_\infty < \epsilon/4\pi$. By the first part of the proof there is an integer $N$ such that $\|\int_0^{2\pi} g(s) \sin(n+1/2)s \, ds\| < \epsilon/2$ if $|n| > N$. Then $\|\int_0^{2\pi} f(s) \sin(n+1/2)s \, ds\| \leq \|\int_0^{2\pi} (f(s) - g(s)) \sin(n+1/2)s \, ds\| + \|\int_0^{2\pi} g(s) \sin(n+1/2)s \, ds\| \leq \int_0^{2\pi} \|f(s) - g(s)\| \, ds + \|\int_0^{2\pi} g(s) \sin(n+1/2)s \, ds\| \leq 2\pi \|f - g\|_\infty + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon.$

1.12. Theorem. If $x \in \mathcal{F}(A) \cap P(2\pi)$ then $\Sigma x_n$ converges.

Proof. For each positive integer $n$, let $s_n(x) = \Sigma_{k=-n}^{n} x_k$. Then $s_n(x) - x = (1/2\pi) \int_0^{2\pi} D_n(t) [T(t)x-x] \, dt$ where $D_n(t) = [\sin(n+1/2)t]/(\sin t/2)$ is Dirichlet's kernel. Let $g(s) = [T(s)x-x]/(\sin(s/2)$ for $0 < s < 2\pi$, $g(0) = g(2\pi) = 2Ax$. Then $g$ is continuous on $[0, 2\pi]$ and hence has a continuous periodic extension (again denoted by $g$) to $[0, \infty)$. Thus $g \in \mathcal{F}$ and so, by Lemma 1.11, $s_n(x) - x = (1/2\pi) \int_0^{2\pi} D_n(t) [T(t)x-x] \, dt = (1/2\pi) \int_0^{2\pi} g(t) \sin(n+1/2)t \, dt \to 0$ as $|n| \to 0$.

Of course $x = \Sigma x_n$ by Theorem 1.3.

1.13. Theorem. If $x \in \mathcal{F}(A^2) \cap P(2\pi)$ then $\Sigma x_n$ converges absolutely.

Proof. For each integer $n$, $x_n = (1/2\pi) \int_0^{2\pi} T(t)x e^{-int} \, dt = (1/2\pi n) \int_0^{2\pi} A T(t)x e^{-int} \, dt = - (1/2\pi n^2) \int_0^{2\pi} A^2 T(t)x e^{-int} \, dt.$

Thus $\|x_n\| \leq (1/2\pi n^2) \int_0^{2\pi} \|T(t)A^2x\| \, dt \leq (1/n^2) \sup_{0 \leq \omega \leq 2\pi} \|T(t)A^2x\|$.
and $\sum\|x_n\|$ converges by comparison.

1.14. Theorem. If $x \in P(2\pi)$ and $x^* \in X^*$ (the adjoint of $X$) then $\sum_{n=-\infty}^{\infty} |x^*(x_n)|^2 = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} |x^* T(t)x|^2 \, dt$.

Proof. Let $g(t) = x^* T(t)x$. Then $g$ is continuous on $[0, 2\pi]$ so $g \in L^2(0, 2\pi)$. By Parseval's Theorem, if $a_n = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} |g(t)|^2 \, dt$, Then

$$a_n = \left(\frac{1}{2\pi}\right) \int_0^{2\pi} x^* T(t)x e^{-int} \, dt = x^* T(t)x e^{-int} \, dt = x^* (x_n)$$

the proof is complete.

1.15. Lemma. If $\{nt_n\} \in \ell_2$ then $\{t_n\} \in \ell_1$.

Proof. Suppose $\sum |nt_n|^2 < \infty$. If $|t_n| \neq |nt_n|^2$ then $|t_n| < (1/n^2)$. Thus $\sum |t_n|$ is dominated by the convergent series $\sum [a(n)|nt_n|^2 + (1-a(n))(1/n^2)]$ where $a(n) = 1$ if $|t_n| \leq |nt_n|^2$ and $a(n) = 0$ if $|t_n| \neq |nt_n|^2$. Consequently $\{t_n\} \in \ell_1$.

1.16. Theorem. If $x \in \mathcal{A}(A) \cap P(2\pi)$ and $x^* \in X^*$ then $\sum x^*(x_n)$ converges absolutely.

Proof. If $x \in \mathcal{A}(A) \cap P(2\pi)$ then $Ax \in P(2\pi)$ and $Ax \times \sum \in x_n$. By Theorem 1.14, $\sum |x^*(nx_n)|^2 < \infty$, i.e. $\sum |nx^*(x_n)|^2 < \infty$.

Thus by Lemma 1.15, $\sum |x^*(x_n)| < \infty$.

In the following three examples some interesting results are given as easy consequences of the preceding theory.

(a). Let $C_u [0, \infty)$ be the Banach space of bounded uniformly continuous complex-valued functions on $[0, \infty)$ with
\[ \|f\| = \sup_{t \geq 0} |f(t)| \] for \( f \in C_u [0, \infty) \). Let \([T(t); t \geq 0]\) be the translation semi-group on \( C_u [0, \infty) \), that is, \([T(t)f](u) = f(t + u)\) for \( f \in C_u [0, \infty) \), \( t, u \geq 0 \). Then \( P(2\pi) \) is the set of those functions in \( C_u [0, \infty) \) which are periodic (of period \( 2\pi \)) in the usual sense. If \( f \in P(2\pi) \), \( f(t) \sim \sum a_n e^{int} \), then the \( n \)th generalized Fourier coefficient of \( f \) is \( f_n(s) = \frac{1}{2\pi} \int_0^{2\pi} [T(t)f](s) e^{-ins} \, ds = e^{ins} \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt = a_n e^{ins} \).

1.17 Theorem. If \( f \in P(2\pi) \) and \( \phi \) is a function of bounded variation on \([a, b] \subset [0, \infty)\) then
\[ \sum_{n=-\infty}^{\infty} \left| \int_a^b f(t) e^{inx} \, dt \right|^2 = 2\pi \int_0^{2\pi} \left| \sum_{n=-\infty}^{\infty} a_n e^{int} \right|^2 \, dt. \]

Proof. If \( \phi \) is of bounded variation on \([a, b]\) then \( x^* (g) = \int_a^b g(x) \, d\phi(x) \) defines a continuous linear functional \( x^* \) on \( C_u [0, \infty) \). By Theorem 1.14, \[ \sum_{n=-\infty}^{\infty} \left| x^*(f_n) \right|^2 = (1/2\pi) \int_0^{2\pi} \left| x^* T(t)f \right|^2 \, dt. \] Since \( x^* (f_n) = (1/2\pi) \int_a^b \left[ \sum_{n=-\infty}^{\infty} a_n e^{int} \right] \, d\phi(x) \) and \( x^* T(t)f = \int_a^b [T(t)f](x) \, d\phi(x) \) the proof is complete.

1.18 Theorem. Suppose \( f \in P(2\pi) \) has a uniformly continuous derivative and that \( \phi \) is a function of bounded variation on \([a, b] \subset [0, \infty)\). Then, if \( f(t) \sim \sum a_n e^{int} \), \( \sum a_n \int_a^b e^{inx} \, d\phi(x) \) converges absolutely to \( \int_a^b f(x) \, d\phi(x) \). In particular, \( \sum |a_n| < \infty \).
Proof. The hypothesis insures that \( f \) is in the domain of the infinitesimal generator of \( [T(t)] \) so, by Theorem 1.16, \( \sum x(\phi_n) \) converges absolutely to \( x(\phi) \) for any \( x(\phi) \) in the adjoint of \( C_u[0, \infty) \). Finally, if \( \phi \) is a step function with a single step at \( c \in (a, b) \), \( \int_a^b e^{i \phi(x)} = e^{ic} \) so \( \sum |a_n| < \infty \).

(b). Let \([T(t); t \geq 0]\) be the translation (modulo \( 2\pi \)) semi-group on \( L_p(0, 2\pi) \). Then \([T(t)]\) is strongly continuous on \( L_p(0, 2\pi) \) for \( 1 \leq p < \infty \).

1.19. Theorem. Suppose \( 1 \leq p < \infty \), \( (1/p) + (1/q) = 1 \), \( f(t) \sim \sum a_n e^{int} \) is in \( L_p(0, 2\pi) \), \( g(t) \sim \sum c_n e^{int} \) is in \( L_q(0, 2\pi) \), and \((f \ast g)(\tau) = \int_0^{2\pi} f(\tau-t)g(t) \ dt \). Then \( f \ast g \in L_2(0, 2\pi) \) and \( \sum |a_n c_n|^2 = (1/2\pi) \int_0^{2\pi} |(f \ast g)(\tau)|^2 \ dt \).

Proof. Let \( F(t) = f(-t) \) and \( G(t) = \overline{g(t)} \) for \( t \in (0, 2\pi) \).

Then \( x(\phi) (h) = \int_0^{2\pi} h(t) \overline{G(t)} \ dt \) defines a continuous linear functional \( x(\phi) \) on \( L_p(0, 2\pi) \). By Theorem 1.14, \( \sum |x(\phi)(F_n)|^2 = (1/2\pi) \int_0^{2\pi} |x(\phi)(T(t)F_n)|^2 \ dt \) and the proof is easily completed.

(c). Let \( H \) be the Banach space (with supremum norm) of holomorphic functions on the open unit disc which have continuous extensions to the closed disc. Let \([T(t)f](z) = f(e^{it}z) \) for \( f \in H, |z| \leq 1 \), and \( t \geq 0 \). Then \([T(t)]\) is a strongly continuous semi-group on \( H \) with infinitesimal generator given by \( [\Lambda f](z) = izf'(z) \). In this case \( P(2\pi) = H \) and, if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) (for \( |z| < 1 \)) is in \( H \), \( f_n(z) = a_n z^n \)
for \( n \geq 0 \) and \( f_n(z) = 0 \) for \( n < 0 \). In particular, by Theorem 1.10, \( a_n \to 0 \) as \( n \to \infty \).

1.20. Theorem. If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) \((|z| < 1)\) is in \( H \) then the Cesaro means of \( \sum a_n z^n \) converge uniformly on the closed disc. If, in addition, \( zf'(z) \) is in \( H \) then \( \sum a_n z^n \) converges uniformly on the closed disc.

Proof. The Cesaro means of \( \sum a_n z^n \) converge in the norm of \( H \) by Theorem 1.3. If \( zf'(z) \) is in \( H \) then \( f \) is in the domain of the infinitesimal generator of \([T(t)]\) and so \( \sum a_n z^n \) converges in norm by Theorem 1.12.

In particular, it follows from Theorem 1.20 that \( H \) is the set of holomorphic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) defined in the open disc for which the Cesaro means of \( \sum_{n=0}^{\infty} a_n z^n \) converge uniformly on the closed disc.

1.21. Theorem. Suppose \( M \) is a closed subspace of \( H \) which is closed under rotation and such that \( zf'(z) \) is in \( M \) if \( f \in M \). Then there is an integer \( N \) such that each element of \( M \) is a polynomial of degree less than or equal to \( N \).

Proof. Since \( M \) is closed under rotation, \( T(t)M \subseteq M \). Since \( f \in M \) implies that \( zf'(z) \) is in \( M \), the domain of the infinitesimal generator contains \( M \). Consequently, \([T(t)]_M\) has a bounded infinitesimal generator and is therefore uniformly continuous ([5], page 621). By Theorem 1.8 there is an integer \( N \) such that \( f = \sum_{n=-N}^{N} f_n \) if \( f \in M \). Thus, if \( f \in M \), \( f(z) = \sum_{n=0}^{N} a_n z^n \).
Again let $X$ be a complex Banach space and $[T(t); t \geq 0]$ a strongly continuous semi-group of bounded operators on $X$ with infinitesimal generator $A$.

Definition. A set $D$ is said to be relatively dense in an interval $I$ if there is a number $\delta > 0$ such that $D$ intersects each subinterval of $I$ of length $\delta$.

Definition. Let $Y$ be a Banach space and $I$ either $(-\infty, \infty)$ or $[0, \infty)$. Then $f: I \rightarrow Y$ is almost periodic if $f$ is continuous and for each $\varepsilon > 0$ there is a set $D(\varepsilon)$ relatively dense in $I$ such that $\|f(t + \tau) - f(t)\| < \varepsilon$ for $t \in I$ and $\tau \in D(\varepsilon)$.

Definition. A vector $x \in X$ is said to be almost periodic with respect to $[T(t)]$ if the function $f(t) = T(t)x$ is almost periodic on $[0, \infty)$. Let $AP$ denote the set of almost periodic vectors in $X$.

The elements of the set $D(\varepsilon)$ referred to above are called $\varepsilon$-translation numbers for (the function) $f$ or for (the vector) $x$ as the case may be.

Definition. Let $P$ be the linear span of $\cup_{\lambda > 0} P(\lambda)$.

2.1. Lemma. If $x \in AP$ then $T(\cdot)x$ is uniformly continuous on $[0, \infty)$, i.e. if $\varepsilon > 0$ there is a $\delta > 0$ such that $\|T(t)x - T(s)x\| < \varepsilon$ if $|t - s| < \delta$.

Proof. Let $\varepsilon > 0$ and choose $\delta > 0$ such that in each interval $[a, a + \delta] \subseteq [0, \infty)$ there is an $\varepsilon/3$-translation number for $x$. 

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Then there is a $\delta > 0$, $\delta < \ell$, such that if $t_1$, $t_2 \in [0, 2\ell]$ and 
$|t_1 - t_2| < \delta$ then $\|T(t_1)x - T(t_2)x\| < \epsilon/3$.

Suppose that $0 \leq s_1 \leq s_2$, $|s_1 - s_2| < \delta$. If $s_1 \leq \ell$ then 
$s_1, s_2 \in [0, 2\ell]$. If $s_1 > \ell$ there is an $\epsilon/3$-translate 
$\tau \in [s_1 - \ell, s_1]$, $s_1 - \tau, s_2 - \tau \in [0, 2\ell]$ and 
$|(s_1 - \tau) - (s_2 - \tau)| < \delta$. Hence 
$\|T(s_1)x - T(s_2)x\| \leq \|T(s_1)x - T(s_1 - \tau)x\| + 
\|T(s_1 - \tau)x - T(s_2 - \tau)x\| + \|T(s_2 - \tau)x - T(s_2)x\| < \epsilon$.

As above in Lemma 2.1, the following two results may be established by making minor modifications in the proofs given 
by Bohr ([3]) for complex-valued functions on $(-\infty, \infty)$.

2.2. Lemma. If $x \in AP$ then for each $\epsilon > 0$ there exists 
a $\delta > 0$ such that each number in $[0, \delta]$ is an $\epsilon$-translation 
number for $x$.

2.3. Lemma. If $x \in AP$ then for each $\epsilon > 0$ there exist 
an $\lambda > 0$ and a $\delta > 0$ such that each interval $[a, a + \lambda] \subset [0, \infty)$ 
contains an interval of length $\delta$ composed entirely of 
$\epsilon$-translation numbers for $x$.

2.4. AP is a linear subspace of $X$.

Proof. Clearly AP is closed under scalar multiplication.

Suppose $x, y \in AP$. Let $\epsilon > 0$. Following [3] again, choose 
$\lambda_x, \delta_x > 0$ such that if $0 \leq \eta \leq \delta_x$ each interval $[a, a + \lambda_x]$ 
contains an integral multiple of $\eta$ which is an $\epsilon/2$-translation 
number for $x$. Choose $\lambda_y, \delta_y$ in a similar fashion. Let
$L = \max \{ l_x, l_y \}$ and $\eta = \min \{ \delta_x, \delta_y \}$. Then each interval $[a, a+L]$ contains $\epsilon/2$-translation numbers $\tau_x, \tau_y$ which are integral multiples of $\eta$.

Consider the set of all pairs $(\tau_x, \tau_y)$ of such numbers, where there are integers $n', n''$ such that $\tau_x = n'\eta, \tau_y = n''\eta$ and $|\tau_x - \tau_y| < L$. Thus $|(n' - n'')\eta| < L$ so there can be only finitely many values of $n' - n''$, say $n_1, n_2, \ldots, n_q$. Corresponding to $n_1, n_2, \ldots, n_q$ choose pairs $(\tau_x', \tau_y'), (\tau_x^2, \tau_y^2), \ldots, (\tau_x^q, \tau_y^q)$. Let $\ell = \max_{1 \leq k \leq q} \frac{\epsilon}{\tau_x^k}$.

Suppose now that $[a, a+L+2\ell]$ is any interval of length $L+2\ell$ contained in $[0, \infty)$. Choose $\tau_x = n'\eta$ and $\tau_y = n''\eta$ in $[a+\ell, a+L+\ell]$. Then, for some $k$ ($1 \leq k \leq q$), $\tau_x - \tau_y = n_k\eta$ so that $\tau_x - \tau_y = n_k\eta = \tau_x^k - \tau_y^k$. Then $\tau_x - \tau_y = n_k\eta = \tau_x^k - \tau_y^k$ is a common $\epsilon/2$-translation number of $x$ and $y$. Thus each interval of length $L+2\ell$ in $[0,\infty)$ contains an $\epsilon$-translation number for $x+y$ and $x+y \in AP$.

It is convenient here to make a lengthy digression (which will not be given) and follow Bohr ([3]) to prove the following lemma. Where the proofs of the necessary results deviate from those given for functions defined on $(-\infty, \infty)$, nothing more difficult is involved than the modifications required for Lemma 2.1.

2.5. Lemma. Suppose $f$ is a complex-valued function defined and almost periodic on $[0, \infty)$. Then for each $\epsilon > 0$ there is a trigonometric polynomial $\sum_{n=1}^{m} a_n e^{i\lambda_n t}$ such that $|f(t) - \sum_{n=1}^{m} a_n e^{i\lambda_n t}| < \epsilon$.
for all \( t \geq 0 \). Indeed, the numbers \( \lambda_n \) may be chosen to be Fourier exponents of \( f \), i.e., from those numbers \( \lambda \) for which
\[
\lim_{K \to \infty} (1/K) \int_0^K f(t) e^{-i\lambda t} dt
\]
is non-zero.

### 2.6. Lemma
If \( x \in \mathcal{A} \) then the mean value
\[
M_{\lambda} x = \lim_{K \to \infty} \frac{1}{K} \int_0^K T(t) x e^{-i\lambda t} dt
\]
exists and \( (1/K) \int_0^a T(t) x e^{-i\lambda t} dt \)
converges uniformly in \( a \geq 0 \) to \( M_{\lambda} x \).

Again, a proof of Bohr ([3]) may be used, virtually unchanged, to prove Lemma 2.6.

### Definition
The semi-group \([T(t)]\) is said to be a bounded semi-group if \( \sup_{t \geq 0} \|T(t)\| < \infty \).

### 2.7. Theorem
For each real \( \lambda \) let \( \mathcal{L}_\lambda = \{ x \in \mathcal{A} | Ax = i\lambda x \} \).

Then

1) \( \mathcal{L}_\lambda \) is a closed subspace of \( X \),

2) if \( \lambda \neq 0 \) then \( A \) is a topological isomorphism of \( \mathcal{L}_\lambda \) onto \( \mathcal{L}_\lambda \),

3) \( M_{\lambda}(\mathcal{A}) = \mathcal{L}_\lambda \),

4) if \([T(t)]\) is a bounded semi-group then \( M_{\lambda} \) is a projection of \( \mathcal{A} \) onto \( \mathcal{L}_\lambda \) and \( \|M_{\lambda}\| \leq \sup_{t \geq 0} \|T(t)\| \).

**Proof.**
1) Clearly \( \mathcal{L}_\lambda \) is a linear space. Suppose \( \{ x_n \} \) is a sequence in \( \mathcal{L}_\lambda \), \( x \in X \), and \( \|x - x_n\| \to 0 \) as \( n \to \infty \). Then \( Ax_n = i\lambda x_n \) converges to \( i\lambda x \). Since \( A \) is a closed operator, \( x \in \mathcal{D}(A) \) and \( Ax = i\lambda x \). Thus \( x \in \mathcal{L}_\lambda \) so \( \mathcal{L}_\lambda \) is closed.

2) If \( x \in \mathcal{L}_\lambda \) then \( A(Ax) = A(i\lambda x) = i\lambda(Ax) \) so \( A(L_{\lambda}) \subset \mathcal{L}_\lambda \).

Moreover, \( \|Ax\| = |\lambda| \|x\| \) so \( A \) is bounded. Suppose \( \lambda \neq 0 \).
If \( x \in L_\lambda \) then \((1/i\lambda)x \in L_\lambda\) and \(A(1/i\lambda)x = x\) so \(A\) is onto. Finally, if \(Ax = 0\) then \(i\lambda x = 0\) and hence \(x = 0\). Thus \(A\) is one-to-one and, clearly, \(A^{-1}x = (1/i\lambda)x\) is a bounded operator on \(L_\lambda\).

iii) Suppose that \(x \in AP\). Then, if \(x^{(K)} = (1/K)\int_0^K T(t)xe^{-i\lambda t}dt\), \(x^{(K)} \in \mathcal{S}(A)\) and \(Ax^{(K)} = i\lambda x^{(K)}\). Since \(A\) is a closed operator, \(M_\lambda x \in \mathcal{S}(A)\) and \(AM_\lambda x = i\lambda M_\lambda x\). Hence \(M_\lambda x \in L_\lambda\). Conversely, if \(x \in L_\lambda\) then \(x \in AP\) and \(M_\lambda x = \lim_{K \to \infty} (1/K)\int_0^K T(t)xe^{-i\lambda t}dt = \lim_{K \to \infty} (1/K)\int_0^K xze^{-i\lambda t}dt = \lim_{K \to \infty} (1/K)\int_0^K xdt = x\). Thus \(M_\lambda(L_\lambda) = L_\lambda\).

iv) It follows at once from the proof of part iii) that \(M_\lambda^2 x = M_\lambda x\) for \(x \in AP\). Suppose then that \([T(t)]\) is bounded. Then
\[
\|M_\lambda x\| \leq \lim_{K \to \infty} (1/K)\int_0^K \|T(t)x\|dt \leq \sup_{t \geq 0} \|T(t)\| \|x\|\text{ so that }
\|M_\lambda\| \leq \sup_{t \geq 0} \|T(t)\|.
\]

2.8. Theorem. If \(x \in AP\) then for each \(\varepsilon > 0\) there is a trigonometric polynomial \(\sum_{k=1}^n \gamma_k e^{ik\lambda t}\) with \(\gamma_k \in P\) such that
\[
\|T(t)x - \sum_{k=1}^n \gamma_k e^{ik\lambda t}\| < \varepsilon \text{ for all } t \geq 0.
\]
Again, the numbers \(\lambda_k\) may be chosen from the countable set of real numbers \(\lambda\) such that \(M_\lambda x \neq 0\).

Proof. The following is an outline of the proof given by Kopec ([7]). Because of Lemma 2.5 no substantial changes are necessary in dealing with functions defined on \([0, \infty)\).

Since \([T(t)x| t \geq 0]\) is separable it may be assumed, for the purpose of this proof, that \(X\) is separable. By a theorem of Banach ([1], page 124) there is a sequence \(\{x_n^*\}\) in \(X^*\) such
that if \( x^* \in X^* \) then there is a subsequence \( \{x^*_n\} \) of \( \{x^*_m\} \) such that \( \lim_{m \to \infty} x^*_n(x) = x^*(x) \) for each \( x \in X \).

It follows, with this assumption, that there are at most countably many real numbers \( \lambda \) for which \( M_\lambda x \neq 0 \).

Let \( \{\beta_j\} \) be a basis ([3]) of the Fourier exponents of \( x \) (i.e. those real numbers \( \lambda \) such that \( M_\lambda x \neq 0 \)). For each positive integer \( q \) let \( q! = Q \) and \( R = qQ \). Then let \( K^{(q)}(u) = \Sigma_{v_1}^{R} \Sigma_{v_2}^{R} \cdots \Sigma_{v_q}^{R} (1 - |v_1|/R)(1 - |v_2|/R) \cdots (1 - |v_q|/R) \exp(-iuv_k/Q) \beta_k \) and \( S_q(t) = \Sigma_{k=1}^{N} v_k e^{i\lambda_k t} = \lim_{W \to \infty} (1/W) \int_{0}^{W} K^{(q)}(u) T(t+u)x \, du \).

For each positive integer \( n \) there are non-negative numbers \( a_{1n}, a_{2n}, \ldots, a_{mn} \) ([8], page 81) such that \( V_n(t) = \Sigma_{j=1}^{m} a_j S_j(t) \) converges to \( T(t)x \) uniformly on each finite subinterval of \([0, \infty)\). Since \( T(t)x \) is almost periodic on \([0, \infty)\) the sequence \( \{V_n\} \) converges uniformly on \([0, \infty)\). In fact it then follows that \( \{S_n\} \) converges uniformly on \([0, \infty)\) to \( T(t)x \).

2.9. Corollary. The spaces \( P \) and \( AP \) have the same closure.

In fact, by Corollary 2.9, Theorem 1.7, and Theorem 2.7 the closure of \( AP \) is the closed linear span of \( \bigcup_\lambda M_\lambda(AP) \).

Definition. A vector \( x \in X \) is said to have compact orbit (with respect to \([T(t)]\)) if \( \{T(t)x|t \geq 0\} \) has compact closure.

Let \( K \) be the set of elements of \( X \) which have compact orbit.

2.10. Theorem. \( \mathcal{O}(A) \) is dense in \( AP \) and in \( K \).
Proof. For \( x \in \mathcal{A} P \) and \( a > 0 \) let \( x_a = (1/a) \int_0^a T(t)x \, dt \). Then \( x_a \in \mathcal{A}(A) \) and \( \lim_{a \to 0^+} x_a = x \) ([5], page 620). Moreover, if \( \varepsilon > 0 \) and \( \tau \) is an \( \varepsilon \)-translation number for \( x \), then 

\[
\|T(t+\tau)x_a - T(t)x_a\| \leq (1/a) \int_0^a \|T(t+\tau+s)x - T(t+s)x\| \, ds \leq \sup_{0 \leq s \leq a} \|T(t+\tau+s)x - T(t+s)x\| \leq \varepsilon.
\]

Thus \( x_a \in \mathcal{A}P \) so \( \mathcal{A}(A) \) is dense in \( \mathcal{A}P \).

Again, if \( x \in K \) and \( a > 0 \), let \( x_a = (1/a) \int_0^a T(t)x \, dt \). As before, \( x_a \in \mathcal{A}(A) \) and \( \lim_{a \to 0^+} x_a = x \). Suppose that \( \{t_n\} \) is a sequence in \([0, \infty)\). Then 

\[
\|T(t_n)x - T(t_m)x\| = (1/a) \int_0^a \|T(t+t_n)x - T(t+t_m)x\| \, dt \leq (1/a) \int_0^a \sup_{0 \leq s \leq a} \|T(t+t+s)x - T(t+s)x\| \, dt \leq \sup_{t \geq 0} \|T(t)\| \sup_{t \geq 0} \|T(t)x - T(t_m)x\|.
\]

Since \( x \in K \) some subsequence of \( \{T(t_n)x\} \) converges and hence is Cauchy. Thus \( \{T(t_n)x \}_a \) has a Cauchy subsequence. Therefore \( x_a \in K \) and \( \mathcal{A}(A) \) is dense in \( K \).

2.11 Lemma. Suppose \( x_n, x \in X \) and \( \sup_{t \geq 0} \|T(t)x - T(t)x_n\| \to 0 \) as \( n \to \infty \). Then 

i) \( x \in \mathcal{A}P \) if \( x_n \in \mathcal{A}P \) for each \( n \),

ii) \( x \in K \) if \( x_n \in K \) for each \( n \).

Proof. i) Let \( \varepsilon > 0 \). Then there is a positive integer \( N \) such that, if \( n \geq N \), \( \|T(t)x - T(t)x_n\| < \varepsilon/3 \) for all \( t \geq 0 \). Then, if \( \tau \) is an \( \varepsilon/3 \)-translation number for \( x_N \), 

\[
\|T(t+\tau)x - T(t)x\| \leq \|T(t+\tau)x - T(t+\tau)x_N\| + \|T(t+\tau)x_N - T(t)x_N\| + \|T(t)x_N - T(t)x\| < \varepsilon.
\]

Thus \( x \in \mathcal{A}P \).
ii) Let \( \{t_n\} \) be a sequence in \([0, \infty)\). Since \( x \in K \) there is a subsequence \( \{t_{n(1)}^j\} \) of \( \{t_n\} \) such that \( \{T(t_{n(1)}^j)x\} \) converges. In general there is a subsequence \( \{t_{n(k)}^j\} \) of \( \{t_{n(k-1)}^j\} \) such that \( \{T(t_{n(k)}^j)x_k\} \) converges. Finally, \( \{t_{n(n)}^j\} \) is a subsequence of \( \{t_n\} \).

If \( \varepsilon > 0 \), choose \( n \) such that \( \sup_{t \geq 0} ||T(t)x - T(t)x_n|| < \varepsilon/3 \). Then there is an integer \( N \) such that \( ||T(t_{j+1}^j)x_n - T(t_{j+1}^j)x^j|| < \varepsilon/3 \) if \( j, k > N \). Hence, \( ||T(t_{j+1}^j)x - T(t_{j+1}^j)x^j|| \leq ||T(t_{j+1}^j)x_n - T(t_{j+1}^j)x^j|| + \varepsilon/3 \). Thus \( \{T(t_{n(n)}^j)x\} \) converges and \( x \in K \).

2.12. Theorem. \( K \) is a linear subspace of \( X \) invariant under each \( T(t) \).

Proof. If \( x \in K \) and \( \lambda \) is a scalar then \( \{T(t)\lambda x | t \geq 0\} = \lambda \{T(t)x | t \geq 0\} \) so \( \{T(t)\lambda x | t \geq 0\} \) has compact closure and hence \( \lambda x \in K \). If \( y \in K \) also, then \( \{T(t)(x+y) | t \geq 0\} \subseteq \{T(t)x | t \geq 0\} + \{T(t)y | t \geq 0\} \) and so \( \{T(t)(x+y) | t \geq 0\} \) has compact closure. Thus \( x+y \in K \) and so \( K \) is a linear space.

If \( x \in K \) and \( s \geq 0 \) then \( \{T(t)T(s)x | t \geq 0\} = \{T(t+s)x | t \geq 0\} \subset \{T(u) | u \geq 0\} \) so \( \{T(t)T(s)x | t \geq 0\} \) has compact closure. Therefore \( T(s)x \in K \).

2.13. Theorem. \( AP \subseteq K \).

Proof. If \( x \in P(\lambda) \) for some \( \lambda > 0 \) then \( \{T(t)x | t \geq 0\} \) is the continuous image of \( [0, \lambda] \) and so is compact. Thus \( x \in K \). Therefore \( P(\lambda) \subseteq K \) for each \( \lambda > 0 \) and \( K \) is a linear space \( P \subseteq K \).
By Theorem 2.8, if $x \in \text{AP}$ there is a sequence $\{p_n\}$ in $P$ such that $\sup_{t \geq 0} \|T(t)x - T(t)p_n\| \to 0$ as $n \to \infty$. By Lemma 2.11, $x \in K$. Therefore $\text{AP} \subseteq K$.

2.14. Theorem. If $[T(t)]$ is a bounded semi-group then $P = \text{AP} = \overline{\text{AP}} \subseteq K = \overline{K}$.

Proof. If $x_n, x \in X$ then $\|T(t)x - T(t)x_n\| \leq \|T(t)\| \|x - x_n\| \leq \|x - x_n\| \sup_{t \geq 0} \|T(t)\|$. Thus $\|x - x_n\| \to 0$ as $n \to \infty$ if and only if $\sup_{t \geq 0} \|T(t)x - T(t)x_n\| \to 0$ as $n \to \infty$. By Lemma 2.11, then, $\text{AP}$ and $K$ are both closed. The other relations have already been proved.

As an example will be given later to show that Theorem 2.14 cannot, in general, be strengthened to a complete generalization of Bochner's Theorem ([5], page 283).

However, the restriction of the domain of the almost periodic functions involved here to $[0, \infty)$, may, in certain cases, be removed. It is convenient to establish two lemmas before proceeding to this extension.

2.15. Lemma. Suppose $f, g: (-\infty, \infty) \to X$ are almost periodic and $f(t) = g(t)$ for $t \geq 0$. Then $f = g$.

Proof. For $x^* \in X^*$, $x^*f$ and $x^*g$ are complex-valued functions equal on $[0, \infty]$ and almost periodic on $(-\infty, \infty)$. Since then

$|\mathbb{M}[|x^*f - x^*g|^2]| = \lim_{K \to \infty} \int_0^K \int_0^K |x^*f(t) - x^*g(t)|^2 \, dt = 0$ it follows that $x^*f = x^*g$. Thus $f = g$. 

2.16. Lemma. Suppose that, for each positive integer $n$, $f_n: (-\infty, \infty) \to X$ is almost periodic on $(-\infty, \infty)$, $f:[0, \infty) \to X$, and $\sup_{t \geq 0} ||f(t)-f_n(t)|| \to 0$ as $n \to \infty$. Then $f$ is almost periodic on $[0, \infty)$ and has a unique extension to an almost periodic vector-valued function on $(-\infty, \infty)$.

Proof. In view of Lemma 2.15 and the easily established fact that the uniform limit of almost periodic functions is almost periodic, it is sufficient to show that $\sup_t ||f_n(t)-f_m(t)|| \to 0$ as $m, n \to \infty$. To that end let $\epsilon > 0$ and choose $N$ such that $\sup_{t \geq 0} ||f_n(t)-f_m(t)|| \leq \epsilon/3$ for $m, n > N$. Fix $m, n > N$. Suppose $s < 0$. Then there is a number $r > -s$ which is a common $\epsilon/3$-translation number for $f_m$ and $f_n$. Then $||f_n(s)-f_m(s)|| \leq ||f_n(s)-f_n(\tau+s)|| + ||f_n(\tau+s)-f_m(\tau+s)|| + ||f_m(\tau+s)-f_m(s)|| \leq \epsilon$. Hence $\sup_t ||f_n(t)-f_m(t)|| \leq \epsilon$ if $m, n > N$.

2.17. Theorem. If $[T(t); t \geq 0]$ is a bounded semi-group then it has an extension to a strongly continuous group of bounded operators on AP.

Proof. Suppose $x \in AP$. Then there is a sequence $\{p_n\}$ of trigonometric polynomials (with coefficients in $X$) such that $\sup_{t \geq 0} ||T(t)x-p_n(t)|| \to 0$ as $n \to \infty$. Moreover, $T(s)p_n(t) = p_n(t+s)$ for $t, s \geq 0$. Clearly each $p_n$ has an extension to an almost periodic vector-valued function (again denoted by $p_n$) on $(-\infty, \infty)$. Let $T(t)x = \lim_{n \to \infty} p_n(t)$ for all $t$. By Lemma 2.16 this is an almost periodic extension of the function $T(\cdot)x$ to $(-\infty, \infty)$. In particular, $T(t)x \in AP$. 
Now if $x \in \mathcal{A}P$ and $\{q_n\}$ is a sequence of trigonometric polynomials converging uniformly on $[0, \infty)$ to $T(t)y$ then $\{p_n + q_n\}$ is a similar sequence converging uniformly on $[0, \infty)$ to $T(t)(x + y)$. Hence, for any $t$, $T(t)(x + y) = \lim_{n \to \infty} [p_n(t) + q_n(t)] = \lim_{n \to \infty} p_n(t) + \lim_{n \to \infty} q_n(t) = T(t)x + T(t)y$. It is equally easy to see that $T(t)\lambda x = \lambda T(t)x$ for any scalar $\lambda$. Thus each $T(t)$ is a linear operator.

Now, if $s > 0$, $p_n(t + s) \to T(t)T(s)x$ for $t \geq 0$. Hence, by definition, $T(-s)T(s)x = \lim_{n \to \infty} p_n(-s + s) = \lim_{n \to \infty} p_n(0) = T(0)x = x$. Also by definition, $T(-s)x = \lim_{n \to \infty} q_n(-s)$. Let $q_n(t) = p_n(t-s)$. If $t \geq 0$ then, since $[T(t); t \geq 0]$ is a bounded semi-group, $\sup_{t \geq 0} \|T(t)T(-s)x - q_n(t)\| = \sup_{t \geq 0} \|T(t)T(-s)x - p_n(t-s)\| = \sup_{t \geq 0} \|T(t)T(-s)x - T(t)p_n(-s)\| \leq \sup_{t \geq 0} \|T(t)| |T(-s)x - p_n(-s)| |$ and so $\{q_n\}$ converges uniformly on $[0, \infty)$ to $T(t)T(-s)x$. Hence $T(s)T(-s)x = \lim_{n \to \infty} q_n(s) = \lim_{n \to \infty} p_n(0) = T(0)x = x$.

Since $[T(t); t \geq 0]$ is bounded, by Theorem 2.14 $\mathcal{A}P$ is a Banach space. Moreover, for $t \geq 0$, $T(t)$ is a bounded one-to-one linear operator on $\mathcal{A}P$. Finally $T(t)$ is onto $\mathcal{A}P$, for if $x \in \mathcal{A}P$, $T(-t)x \in \mathcal{A}P$ and $T(t)T(-t)x = x$. By the Banach Open Mapping Theorem ([5], page 57) $T(-t) = T(t)^{-1}$ is a bounded operator so that $[T(t); -\infty < t < \infty]$ is a group of bounded operators on $\mathcal{A}P$.

This group is strongly continuous: for suppose $x \in \mathcal{A}P$ and let $\varepsilon > 0$. There is a trigonometric polynomial $p$ such that $\|p(t) - T(t)x\| < \varepsilon/3$ for all $t$. There is then a $\delta > 0$ such that $\|p(t) - p(s)\| < \varepsilon/3$ if $|s - t| < \delta$. Then $\|T(t)x - T(s)x\| \leq \|T(t)x - p(t)\| + \|p(t) - p(s)\| + \|p(s) - T(s)x\| < \varepsilon$. Thus $[T(t)]$ has an
extension to a strongly continuous group on \( AP \).

If \([T(t)]\) has an extension to a bounded \((\sup_t \|T(t)\| < \infty)\) group on \( X \) then the argument from \([5]\), pages 283-84, may be adapted as follows to generalize the Theorem of Bochner.

2.18. Theorem. If \([T(t); -\infty < t < \infty]\) is a bounded strongly continuous group on \( X \), \( x \in X \), and \([T(t)x; -\infty < t < \infty]\) has compact closure then \( T(\cdot)x \) is almost periodic on \((-\infty, \infty)\).

Proof. If \([T(t)x; -\infty < t < \infty]\) has compact closure then it is totally bounded (pre-compact). Let \( \varepsilon > 0 \). Then there are numbers \( \lambda_1, \lambda_2, \ldots, \lambda_m \) such that, if \( t \in (-\infty, \infty) \) there is an integer \( k \) \((1 \leq k \leq m)\) such that \( \|T(t)x - T(\lambda_k)x\| < \varepsilon/M \), where \( M = \sup_t \|T(t)\| \). Consequently, if \( t \) and \( s \) are real,

\[
\|T(t+s)x - T(t+\lambda_k)x\| < \|T(t)\| \|T(s)x - T(\lambda_k)x\| < \varepsilon/M \text{ for some } k.
\]

Hence \( \|T(t)x - T(t-s+\lambda_k)x\| = \|T(-s)T(t+s)x - T(-s)T(t+\lambda_k)x\| \leq \|T(-s)\| \|T(t+s)x - T(t+\lambda_k)x\| < \varepsilon. \)

Let \( \lambda = 2 \max_{1 \leq i \leq m} |\lambda_i| \) and suppose \([a, a+\lambda] \subset (-\infty, \infty)\). For any real number \( s \), one of the numbers \( \lambda_1-s, \lambda_2-s, \ldots, \lambda_m-s \) is an \( \varepsilon \)-translation number for \( T(\cdot)x \). But if \( s = a+\lambda/2 \) each of these numbers is in \([a, a+\lambda]\). Therefore \( T(\cdot)x \) is almost periodic on \((-\infty, \infty)\).

Two of the following three examples forestall strengthening certain points in the preceding theory. The third deals with a special case of this theory.

(a) The spaces \( AP \) and \( K \) need not be closed. If \( f \in C_u[0, \infty) \)
let \([T(t)f](s) = e^{tf(t+s)}\) (see example (a), Chapter I). Then \([T(t); t \geq 0]\) is a strongly continuous semi-group on \(C_u[0, \infty)\).

2.19. Lemma. If \([T(t)]\) is defined as above, then \(x \in \mathcal{A}P\) if and only if \(y(s) = e^s x(s)\) is almost periodic on \([0, \infty)\).

Proof. Suppose \(x \in \mathcal{A}P\), \(y(s) = e^s x(s)\). Then \(y\) is continuous on \([0, \infty)\) and 

\[|y(t+\tau) - y(t)| = |e^{t+\tau}x(t+\tau) - e^tx(t)| = |[T(t+\tau)x](0) - [T(t)x](0)| \leq \|T(t+\tau)x - T(t)x\|_\infty.\]

Thus each \(\varepsilon\)-translation number for the vector \(x\) is an \(\varepsilon\)-translation number for the function \(y\).

Suppose that \(y\) is a complex-valued almost periodic function on \([0, \infty)\) and that \(x(s) = e^{-s}y(s)\). Then \(\|T(t+\tau)x - T(t)x\|_\infty = \sup_{s \geq 0} |T(t+\tau)x(s) - [T(t)x](s)| = \sup_{s \geq 0} |e^{t+\tau}x(t+\tau) - e^tx(t)| = \sup_{s \geq 0} |e^{t+\tau}e^{-\tau}-s y(t+s) - e^t e^{-s} y(t+s)| = \sup_{s \geq 0} e^{-s}|y(t+\tau+s) - y(t+s)| \leq \sup_{u \geq 0} |y(u+\tau) - y(u)|.\)

Consequently, every \(\varepsilon\)-translation number for the function \(y\) is an \(\varepsilon\)-translation number for the vector \(x\). Moreover, \(y\) is bounded since it is almost periodic so \(x\) is uniformly continuous and bounded on \([0, \infty)\). Thus \(x \in \mathcal{A}P\).

Now let \(f_m(t) = \sum_{n=1}^{m} (1/n) |\sin(t/n)|\) for each positive integer \(m\) and real number \(t\). If \(t \in [0,K]\) then \((1/n)|\sin(t/n)| \leq (1/n)(t/n) \leq (1/n)(K/n) = K/n^2\). Thus, by the Weierstrass M-test ([9], page 119), \(f_m\) converges uniformly to \(f(t) = \sum_{n=1}^{\infty} (1/n) |\sin(t/n)|\) on each bounded subinterval of \([0, \infty)\). In particular, then, \(f\) is continuous on \([0, \infty)\).
Let \( g_m(s) = e^{-s}f_m(s) \), \( g(s) = e^{-s}f(s) \). Then \( g \) is bounded, for \( g(s) = e^{-s}f(s) = e^{-\sum_{n=1}^{\infty} (1/n^2) |\sin(s/n)|} \leq e^{-\sum_{n=1}^{\infty} (1/n^2)} = e^{-s} \sum_{n=1}^{\infty} (1/n^2) \) and \( \lim_{s \to \pm \infty} e^{-s} = 0 \). Each \( g_m \) is also bounded since \( g_m(s) \leq g(s) \). Clearly \( g \) and each \( g_m \) is continuous. Since \( f_m \) is almost periodic, \( g_m \in \text{AP} \) by Lemma 2.19.

Now let \( \epsilon > 0 \) and choose \( K > 0 \) such that \( 2e^{-s} \sum_{n=1}^{\infty} (1/n^2) < \epsilon/2 \) for \( s \geq K \). Then there is an integer \( M \) such that \( |f_m(s) - f(s)| < \epsilon/2 \) if \( m > M \) and \( s \in [0, K] \). Now \( \|g_m - g\|_{\infty} = \sup_{s \geq 0} |g_m(s) - g(s)| = \sup_{s \geq 0} e^{-s} |f_m(s) - f(s)| \). If \( s \in [0, K] \) then \( e^{-s} |f_m(s) - f(s)| < \epsilon/2 \) for \( m > M \) and, if \( s > K \), \( e^{-s} |f_m(s) - f(s)| \leq e^{-s}(|f_m(s)| + |f(s)|) \leq 2e^{-s} \sum_{n=1}^{\infty} (1/n^2) < \epsilon/2 \). Thus \( \|g_m - g\|_{\infty} \to 0 \) as \( m \to \infty \) and so \( g \in \text{AP} \).

But \( g \notin \text{AP} \), for if \( g \in \text{AP} \) then \( f(s) = e^s g(s) \) is almost periodic (Lemma 2.19) and must be bounded. As will now be shown, \( f \) is not bounded. Suppose that \( M > 0 \). Since the harmonic series is divergent there is an integer \( m \) such that \( \sum_{1 \leq n \leq m} \frac{1}{n} > M \). Let \( \bar{m} \) be the product of all odd positive integers less than or equal to \( m \) and let \( t = \bar{m} \pi/2 \). Then \( f(t) = \sum_{m=1}^{\infty} (1/n) |\sin(t/n)| \geq \sum_{1 \leq n \leq m} (1/n) |\sin(t/n)| = \sum_{1 \leq n \leq m} (1/n) |\sin(\bar{m} \pi/2)| = \sum_{1 \leq n \leq m} (1/n) > M \). Thus \( f \) is not bounded and \( g \notin \text{AP} \) although \( g \in \text{AP} \).

Let \( [T(t)] \), \( f, f_m, g, g_m \) be as before. Since \( g_m \in \text{AP} \) for each \( m \), \( g_m \in K \) by Theorem 2.13. As above, \( \|g_m - g\|_{\infty} \to 0 \) as \( m \to \infty \) so \( g \in K \). If \( \{t_n\} \) is a sequence in \([0, \infty)\) then \( \|T(t_n)g - T(t_m)g\|_{\infty} = \sup_{s \geq 0} |[T(t_n)g](s) - [T(t_m)g](s)| = \sup_{s \geq 0} |t_n g(t_n + s) - t_m g(t_m + s)| = \sup_{s \geq 0} |t_{n} e^{-t_{n}} f(t_n + s) - t_{m} e^{-t_{m}} f(t_m + s)| = \).
\[ \sup_{s \geq 0} e^{-s} |f(t_n + s) - f(t_m + s)| \geq |f(t_n) - f(t_m)|. \]

Since \( f \) is not bounded it is possible to choose \( \{t_n\} \) such that \( f(t_n) > n \) for each \( n \). In this case \( \{f(t_n)\} \) has no convergent subsequence and, a fortiori, \( \{T(t_n)g\} \) has no convergent subsequence. Thus \( \{T(t)g|t \geq 0\} \) does not have compact closure, i.e., \( g \notin K \).

In this example, not only is \( AP \neq \overline{AP} \) and \( K \neq \overline{K} \) but \( AP \) is a proper subset of \( K \). Let \( h(s) = \tan^{-1}s = \int_0^s (1 + \lambda^2)^{-1} \, d\lambda \) for \( s \geq 0 \). Then \( h \in C_u [0, \infty) \).

Moreover, \( h \) has compact orbit on \([0, \infty)\). Suppose \( \{t_n\} \) is a sequence in \([0, \infty)\). If there is a subsequence \( \{t_{n_j}\} \) of \( \{t_n\} \) such that \( t_{n_j} \to t_0 \in [0, \infty) \) then \( h(t_{n_j} + s) \to h(t_0 + s) \) uniformly in \( s \) by virtue of the uniform continuity of \( h \). On the other hand if \( \{t_{n_j}\} \) is a subsequence of \( \{t_n\} \) such that \( t_{n_j} \to \infty \), choose \( M > 0 \) such that \( \pi/2 - h(s) < \varepsilon \) if \( s > M \). Then choose \( N \) such that \( t_{n_j} > M \) if \( j > N \). Then, if \( j > N \), \( \pi/2 - h(t_{n_j} + s) < \varepsilon \) and hence \( \pi/2 - h(t_{n_j} + s) < \varepsilon \) for all \( s \geq 0 \). Thus \( h \) has compact orbit on \([0, \infty)\).

Now let \( \ell(s) = e^{-s} h(s) \). Then \( \ell \in C_u [0, \infty) \) and

\[
\|T(t_n)\ell - T(t_m)\ell\|_\infty = \sup_{s \geq 0} |[T(t_n)\ell](s) - [T(t_m)\ell](s)| = \sup_{s \geq 0} |e^{t_n} \ell(s + t_n) - e^{t_m} \ell(s + t_m)| = \sup_{s \geq 0} e^{-s} |h(s + t_n) - h(s + t_m)| \leq \sup_{s \geq 0} |h(s + t_n) - h(s + t_m)| \text{ for any sequence } \{t_n\} \text{ in } [0, \infty).
\]

Thus \( \ell \in K \).

But \( h \) is not almost periodic for if it were there would be a sequence \( \{\tau_n\} \) such that \( \tau_n \to \infty \) and \( \sup_{s \geq 0} |h(s + \tau_n) - h(s)| < \pi/4 \).
In particular, then, it would follow that \(|h(T_n)| = |h(\tau_n) - h(o)| < \pi/4\) for all \(n\). This is impossible since \(\lim_{s \to \infty} h(s) = \pi/2\).

Thus \(\not\in\text{AP}\) since if \(\not\in\) were in \(\text{AP}\) then \(h(s) = e^{s/2}\) would be almost periodic on \([0, \infty)\).

(b). Theorem 2.14 may not, in general, be strengthened. That is, even if \([T(t); t \geq 0]\) is a bounded semi-group it may happen, as the following example shows, that \(\text{AP}\) is a proper subset of \(K\).

For \(f \in C_u[0, \infty)\) and \(t, s \geq 0\) let \([T(t)f](s) = e^{-t}f(t+s)\). Then \([T(t)]\) is a bounded strongly continuous semi-group on \(C_u[0, \infty)\) with infinitesimal generator \(Af = f' - f\).

Suppose that \(f \in C_u[0, \infty)\) and \(Af = i\lambda f\) for some real number \(\lambda\). Then \(f' - f = i\lambda f\) so \(f(s) = ae^{(1+i\lambda)s}\) for some complex number \(a\). But if \(a \neq 0\) then \(f\) is not bounded. Consequently there are no non-zero periodic vectors (Theorem 1.7) and, by Theorem 2.8, \(\text{AP} = \{0\}\).

However \(K = C_u[0, \infty)\). Suppose that \(f \in C_u[0, \infty)\) and that \(\{t_n\}\) is a sequence in \([0, \infty)\). Then a subsequence of \(\{t_n\}\) converges to a real number or to \(\infty\). There is no loss of generality in assuming that this subsequence is \(\{t_n\}\) itself.

Suppose first of all that \(\lim_{n \to \infty} t_n = t_o \in [0, \infty)\). Let \(\varepsilon > 0\). Then there is a \(\delta > 0\) such that \(|e^{-t} - e^{-s}| < \varepsilon/(2\|f\|_{\infty} + 1)\) and \(|f(t) - f(s)| < \varepsilon/2\) if \(|t-s| < \delta\). Choose \(N\) such that \(|t_n - t_o| < \delta\) if \(n > N\). Then, if \(n > N\), \(\|T(t_0) f - T(t_n) f\|_{\infty} = \sup_{s \geq 0} |e^{-t} f(t+s) - e^{-t} f(t+s)| = \sup_{s \geq 0} e^{-t} |f(t_0+s) - f(t_0+s)| + \sup_{s \geq 0} \|f\|_{\infty} e^{-t_o} e^{-t_n} < \varepsilon\). Thus \(\lim_{n \to \infty} T(t_n) f = T(t_0) f\).
On the other hand suppose that \( \lim_{n \to \infty} t_n = \infty \). Let \( \varepsilon > 0 \) and choose \( N \) such that \( e^{-t_n} < \varepsilon/(\|f\|_\infty+1) \) if \( n > N \). Then, if \( n > N \),
\[
\|T(t_n)f\|_\infty = \sup_{s \geq 0} |[T(t_n)f](s)| = \sup_{s \geq 0} e^{-t_n}f(t_n+s) 
\leq e^{-t_n} \sup_{u \geq 0} |f(u)| = e^{-t_n}\|f\|_\infty < \varepsilon.
\]
Thus \( \lim_{n \to \infty} T(t_n)f = 0 \).

In either case \( \{T(t_n)f\} \) converges so \( f \in K \). Therefore \( K = C_u[0, \infty) \) and \( \text{AP} \neq K \).

(c). The Stepanoff theory of almost periodic functions ([2], page 77) is a special case of the preceding theory.

For \( \lambda > 0 \) and \( p \geq 1 \) let \( S^p \) be the set of all measurable complex-valued functions \( f \) defined on \([0, \infty)\) for which
\[
\sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t)|^p \, dt \right]^{1/p} < \infty.
\]
Let \( \|f\| = \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t)|^p \, dt \right]^{1/p} \). Then this defines a norm on \( S^p \) which is a linear space with the usual identification of functions \( f \) and \( g \) for which \( \|f-g\| = 0 \).

Suppose that \( \{f_n\} \) is a sequence in \( S^p \) such that
\[
\|f_n - f_m\| \to 0 \quad \text{as} \quad m, n \to \infty.
\]
Then \( \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_n(t)-f_m(t)|^p \, dt \right]^{1/p} \to 0 \)
uniformly in \( x \) as \( m, n \to \infty \). Thus there exists a function \( f \) defined on \([0, \infty)\) such that \( f \in L_p(x, x+\lambda) \) for each \( x \geq 0 \) and such that, if \( \varepsilon > 0 \), there is an integer \( N \) such that \( \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_n(t)-f(t)|^p \, dt \right]^{1/p} < \varepsilon \) for all \( x \geq 0 \) if \( n > N \). Indeed, suppose \( \varepsilon > 0 \). Then there is an integer \( N \) such that \( \|f_n - f_m\| < \varepsilon/2 \) if \( m, n > N \). Fix \( n > N \).

If \( x \geq 0 \) then there is an integer \( m > N \) such that
\[
\left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_m(t)-f(t)|^p \, dt \right]^{1/p} < \varepsilon/2.
\]
Then \( \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_n(t)-f(t)|^p \, dt \right]^{1/p} \leq \|f_n - f_m\| + \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_m(t)-f(t)|^p \, dt \right]^{1/p} < \varepsilon \). Thus
\[
\sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_n(t) - f(t)|^p dt \right]^{1/p} < \epsilon. \quad \text{In particular } f \in S^P_{\lambda} \text{ for, if } n > N, \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t)|^p dt \right]^{1/p} \leq \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t) - f_n(t)|^p dt \right]^{1/p} + \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f_n(t)|^p dt \right]^{1/p} < \infty. \quad \text{Consequently, } \|f_n - f\| = \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t) - f_n(t)|^p dt \right]^{1/p} \to 0 \text{ as } n \to \infty. \quad \text{Thus } S^P_{\lambda} \text{ is a Banach space.}
\]

Let \([T(t); t \geq 0]\) be the translation semi-group on \(S^P_{\lambda}\). This semi-group is not strongly continuous on \(S^P_{\lambda}\). For each positive integer \(n\) let \(f(x) = \sin(2\pi nx/\lambda)\) if \(2\pi(n-1)x < 2\pi n x \leq 2\pi n x\). Clearly \(f \in S^P_{\lambda}\) but \(\|T(t)f - f\| = \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |T(t)f(s) - f(s)|^p ds \right]^{1/p} = \sup_{x \geq 0} \left[ \frac{1}{\lambda} \int_x^{x+\lambda} |f(t+s) - f(s)|^p ds \right]^{1/p} = 1\) if \(t\) is any member of the sequence \(\{\lambda/n\}\) which converges to 0. Thus \(\|T(t)f - f\|\) does not converge to 0 as \(t\) approaches 0 from the right.

However \(L = \{f \in S^P_{\lambda} : \|T(t)f - f\| \to 0 \text{ as } t \to 0^+\}\) is a Banach space. Clearly it is a linear space. If \(\{f_n\}\) is a sequence in \(L\) such that \(\|f_n - f_m\| \to 0\) as \(m, n \to 0\) then there is a function \(f \in S^P_{\lambda}\) such that \(\|f_n - f\| \to 0\) as \(n \to \infty\). Suppose \(\epsilon > 0\). Then there is an integer \(N\) such that \(\|f_n - f\| < \epsilon/3\) if \(n > N\) and there is a \(\delta > 0\) such that \(\|T(t)f_N - f_N\| < \epsilon/3\) if \(0 \leq t < \delta\). Thus, if \(0 \leq t < \delta\),

\[
\|T(t)f - f\| \leq \|T(t)f - T(t)f_N\| + \|T(t)f_N - f_N\| + \|f_N - f\|
\leq \|T(t)f - T(t)f_N\| + \|T(t)f_N - f_n\| + \|f_N - f\|
= 2\|f_N - f_n\| + \|T(t)f_N - f_n\| < \epsilon.
\]

It follows then that \(f \in L\). The space \(L\) is thus shown to be closed and hence is a Banach space.

Obviously \([T(t)]\) is strongly continuous on \(L\). Suppose \(\lambda\) is a real number and that \(g(t) = e^{i\lambda t}\) for \(t \geq 0\). Then

\[
\left[ \frac{1}{\lambda} \int_x^{x+\lambda} |g(t)|^p dt \right]^{1/p} = \left[ \frac{1}{\lambda} \int_x^{x+\lambda} dt \right]^{1/p} = 1 \quad \text{so } g \in S^P_{\lambda}.
\]
If $\varepsilon > 0$ let $\delta > 0$ such that $|e^{i\lambda t} - e^{i\lambda s}| < \varepsilon$ if $|t-s| < \delta$.

Then, if $0 \leq t < \delta$, $\|T(t)g - g\| = \sup_{X \geq 0} \left[ (1/\varepsilon) \int_X^{X+\varepsilon} |g(t+s) - g(s)|^p ds \right]^{1/p} = \sup_{X \geq 0} \left[ (1/\varepsilon) \int_X^{X+\varepsilon} |e^{i\lambda(t+s)} - e^{i\lambda s}|^p ds \right]^{1/p} \leq \sup_{X \geq 0} (1/\varepsilon) \int_X^{X+\varepsilon} \varepsilon^{1/p} ds \right]^{1/p} = \varepsilon$.

Thus $g \in L$.

Now $L$ contains all functions of the form $g(t) = e^{i\lambda t}$ with $\lambda$ real, i.e., all the eigenvectors of the infinitesimal generator of $[T(t)]$ corresponding to pure imaginary eigenvalues. Thus, by Theorem 1.7, Corollary 2.9, and Theorem 2.14, the closure in $L$ (and hence in $S_P^\lambda$) of the trigonometric polynomials is the set of all almost periodic (with respect to the norm on $S_P^\lambda$) functions in $L$. 
BIBLIOGRAPHY


BIOGRAPHY

Alan Carl Pierce was born on June 28, 1941 in McComb, Mississippi. He attended elementary school in Kentwood, Louisiana and graduated from Kentwood High School in 1959. In the fall of 1959 he entered Louisiana State University and received a Bachelor of Science degree from that school in 1963. In the fall of that year he entered graduate school at Louisiana State University where he held a National Defense Education Act fellowship. He received a Master of Science degree in 1965. Mr. Pierce currently holds a teaching assistantship at Louisiana State University where he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics.
EXAMINATION AND THESIS REPORT

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Major Field: Mathematics

Title of Thesis: Periodic and Almost Periodic Vectors

Approved:

[Signatures of Major Professor and Chairman and Dean of the Graduate School]

EXAMINING COMMITTEE:

[Signatures of Committee Members]

Date of Examination:

July 20, 1967