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RIGID CONNECTIONS ON \mathbb{P}^1 VIA THE BRUHAT-TITS BUILDING

MASOUD KAMGARPOUR AND DANIEL S. SAGE

ABSTRACT. We apply the theory of fundamental strata of Bremer and Sage to find cohomologically rigid G -connections on the projective line, generalising the work of Frenkel and Gross. In this theory, one studies the leading term of a formal connection with respect to the Moy-Prasad filtration associated to a point in the Bruhat-Tits building. If the leading term is regular semisimple with centraliser a (not necessarily split) maximal torus S , then we have an S -toral connection. In this language, the irregular singularity of the Frenkel-Gross connection gives rise to the homogeneous toral connection of minimal slope associated to the Coxeter torus \mathcal{C} . In the present paper, we consider connections on \mathbb{G}_m which have an irregular homogeneous \mathcal{C} -toral singularity at zero of slope i/h , where h is the Coxeter number and i is a positive integer coprime to h , and a regular singularity at infinity with unipotent monodromy. Our main result is the characterisation of all such connections which are rigid.

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1. INTRODUCTION

1.1. Rigid connections. The study of rigid connections originates in the seminal work of Riemann on Gauss's hypergeometric function [Rie57]. His central insight was that one should study Gauss's function by the corresponding *local system* on $\mathbb{P}_{\mathbb{C}}^1 - \{0, 1, \infty\}$, defined as the system of holomorphic local solutions of the hypergeometric differential equation. This approach greatly clarified and expanded the invariant properties of the hypergeometric functions established by Euler, Gauss, and Kummer. Riemann's investigation was a stunning success largely because the hypergeometric local system is *physically rigid*, i.e., determined up to isomorphism as soon as one knows the local monodromies at the three singular points.

In modern times, the subject of rigid local systems was rejuvenated by the influential book of Katz [Kat96]. Katz took the point of view that a rigid rank n local system on $\mathbb{P}^1 - \{m \text{ points}\}$ corresponds to an $n \times n$ first order system of differential equations with singularities at the m missing points. He then proceeded to systematically study and classify these local systems using the notion of middle convolution. To facilitate the understanding of rigidity, he defined the notion of *cohomological rigidity* which, roughly speaking, means that the local system has no infinite dimensional deformations. Using deep results of Laumon on the ℓ -adic Fourier transform, he proved

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that physical and cohomological rigidity for ℓ -adic local systems are equivalent. Subsequently, Bloch and Esnault [BE04] proved an analogous result in the de Rham setting. Cohomological rigidity gives rise to a practical numerical criterion for rigidity, see §4.2.

1.2. Rigid G -connections. One can view an $n \times n$ first order system of differential equation as a connection on a rank n vector bundle which, in our case, lives on $\mathbb{P}^1 - \{m \text{ points}\}$. Using the equivalence between rank n vector bundles and principal GL_n -bundles, one obtains a principal GL_n -bundle equipped with a connection, or, for brevity, a GL_n -connection. This construction can be generalised to any algebraic group G , thereby obtaining G -connections.

Foundational results on G -connections were established by Babbitt and Varadarajan [BV83]. More recently, this subject has received renewed attention due to its application to the geometric Langlands program. Indeed, if G is reductive, G -connections on complex algebraic curves are precisely the *geometric Langlands parameters* [Fre07].¹ In particular, the geometric Langlands community has been interested in *rigid G -connections* because it is expected that in this case the Langlands correspondence can be described explicitly. Another source of interest in rigid G -connections is the expectation that they should be, in some sense, motivic. For instance, in the GL_n case, it is known that an irreducible connection is rigid if and only if it can be reduced to the trivial connection using Fourier transforms and convolutions [Kat96, Ari10].

Remark 1. Physical and cohomological rigidity make sense for G -connections, and it is known that physical rigidity implies cohomological rigidity; however, it is not known if the converse implication holds [Yun14b, §3.2]. Thus, while we have many examples of cohomologically rigid G -connections, many of which are conjectured to be physically rigid, we are unaware of any (proven) example of a physically rigid G -connection beyond type A .

Convention 2. Henceforth, we will simply call cohomologically rigid connections rigid.

1.3. Constructions of rigid G -connections. We now review some of the methods for constructing and analysing rigid G -connections.

1.3.1. *Opers.* Roughly speaking, a G -oper is a G -connection whose underlying bundle admits a reduction to a Borel subgroup satisfying a certain transversality condition with respect to the connection [Fre07, §4]. Using the notion of oper and building on the earlier work of Deligne [Del77] and Katz [Kat88, Kat96], Frenkel and Gross [FG09] constructed a remarkable G -connection ∇_{FG} on the trivial G -bundle on $\mathbb{P}^1 - \{0, \infty\}$ satisfying the following properties:

- (1) it has a regular singularity at 0 with principal unipotent monodromy;
- (2) it has an irregular singularity at ∞ with slope $\frac{1}{h}$, where h is the Coxeter number;
- (3) it is cohomologically rigid;
- (4) it has a large differential Galois group (equal to G in many cases).

1.3.2. *Rigid automorphic data.* The geometric Langlands program predicts a correspondence between G -local systems and certain automorphic data for the Langlands dual group \check{G} . One expects therefore that there is a notion of rigidity for automorphic data which corresponds to rigidity for local systems. The first example of rigid automorphic data was found by Heinloth, Ngô, and Yun [HNY13], who used this to define the analogue of Kloosterman sheaves for general reductive groups. They conjectured that the de Rham version of their local system is isomorphic to the Frenkel–Gross connection ∇_{FG} . This conjecture was subsequently proved by [Zhu17]. Lam and Templier [LT17] then used results of [HNY13, Zhu17] to settle mirror symmetry for minuscule flag varieties.

Yun subsequently systematised the study of rigid automorphic data [Yun14a, Yun16, Yun14b]. In particular, in [Yun14a], he settled an old question of Serre on inverse Galois theory by constructing

¹Another reason for interest in G -connections is that they arise naturally in G -analogues of non-abelian Hodge theory and the Riemann–Hilbert correspondence, cf. [Boa11] where Bruhat–Tits theory also makes an appearance.

sought-after Galois representations via rigid automorphic data, again highlighting the role of rigidity in the Langlands program and related areas.

1.3.3. *θ -connections.* A torsion automorphism $\theta \in \text{Aut}(\mathfrak{g})$ gives rise to a grading

$$(1) \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{m-1}.$$

The action of G_0 on \mathfrak{g}_1 and the resulting GIT quotient was studied extensively by Vinberg and his school under the name of *θ -groups*. Yun and Chen [Yun16, Che17] associate to a stable element $X \in \mathfrak{g}_1$ a twisted connection called a *θ -connection*.² In general, ∇_X is a connection on \mathbb{G}_m which is regular singular with unipotent monodromy at 0 and irregular at ∞ . If the grading is inner, so that the connection is untwisted, then the slope at the irregular point equals $1/m$. In [Che17], it is shown that these connections are usually rigid. For an appropriate choice of θ (specifically the one defined by the torsion element $\exp(\check{\rho}/h)$ in a fixed maximal torus T), the corresponding θ -connection ∇_X equals the Frenkel-Gross connection ∇_{FG} .

1.4. **Our goal.** In this paper, we construct a large class of connections satisfying properties analogous to the Frenkel-Gross connection. We found these connections by using the theory of fundamental strata for formal connections introduced by Bremer and Sage [BS12, BS13, BS14, BS18, Sag17]. A major theme in these works is that Moy-Prasad filtrations – certain filtrations on the loop algebra associated to points in the Bruhat-Tits building – can be used to define a new notion of a leading term for a formal connection. Moreover, if this leading term is regular semisimple in a suitable sense, then the connection enjoys favourable properties. If the centraliser of the leading term is the (not necessarily split) maximal torus S , we say that the connection is *S -toral*.

Given a point x in the Bruhat-Tits building \mathcal{B} , there is an associated Moy-Prasad filtration [MP94] on the loop algebra $\mathfrak{g}(\mathbb{C}((t)))$, and one can speak of the leading term of ∇ with respect to this filtration. If we restrict to the standard apartment (with respect to a fixed maximal torus $T \subset G$), then this filtration comes from a grading on the polynomial loop algebra $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$. We say that $\nabla = d + A dt$ is *homogeneous* if A is homogeneous with respect to the grading. In this case, $A \in \mathfrak{g}(\mathbb{C}[t, t^{-1}])$, and we can think of ∇ as a connection on the trivial bundle on $\mathbb{P}^1 - \{0, \infty\}$. We can then use properties of the leading term to investigate whether ∇ is rigid.

If x is chosen to be the barycentre of the fundamental alcove, then the resulting filtration comes from the principal grading.³ Our main result is the classification of all rigid connections that are homogeneous with respect to the principal grading and have a regular semisimple leading term. The corresponding formal connections at the irregular singularity are toral with respect to a certain elliptic maximal torus called the Coxeter torus. As we shall see, there are two different flavours of such rigid connections depending on whether the slope at the irregular point is less than or bigger than one.

If the slope is bigger than one, then we can determine the rigid connections in a uniform manner for all simple groups G . These connections are regular everywhere except at 0, where they have an irregular singularity of slope $(h+1)/h$. Explicitly, they are defined as follows. For each negative simple root α , let x_α be a generator of the corresponding root space. Let $N := \sum x_\alpha$ and let E be a generator of the highest root space. The connection in question is then given by⁴

$$(2) \quad \nabla := d + (N + t^{-1}E) \frac{dt}{t^2}.$$

Note that replacing t^2 by t gives the Frenkel–Gross connection.

²In fact, the constructions given by Yun and Chen are different. Conjecturally, they result in equivalent connections.

³This grading already appeared in the works of Kac on infinite dimensional Lie algebras [Kac90].

⁴In Type A, the ℓ -adic analogue of ∇ has been studied intensively by Katz, cf. [Kat18]. Finding the ℓ -adic analogue of ∇ for general G is an interesting open problem.

On the other hand, if the slope is less than one, we determine rigidity by a case-by-case analysis. This is necessary because, in contrast to the Frenkel-Gross situation, the monodromy at the regular singular point is no longer the principal unipotent class.⁵ Therefore, specific information about nilpotent conjugacy classes in each type is required. It turns out that if G is classical, such rigid connections are plentiful. In contrast, if G is exceptional, there is only one example aside from the Frenkel-Gross connection, and it appears for $G = E_7$. This connection is irregular at 0 of slope $7/18$ and is regular singular at ∞ with unipotent monodromy of type $A_2 + 3A_1$.

1.4.1. *Relationship to θ -connections.* As already noted, θ -connections have slope $1/m$ at the irregular point, whereas the connections considered here have slopes r/h , with r a positive integer relatively prime to h . However, there is a strong relationship between our construction and the approach of Yun and Chen. On the one hand, our methods can be used to construct connections equivalent to those of Yun and Chen. Indeed, in our language, (formal) θ -connections are toral connections of minimal depth associated to “elliptic regular” maximal tori in the loop group.⁶ In the Frenkel-Gross case (the simplest example of a θ -connection), one obtains a toral connection of minimal depth with respect to the Coxeter torus. On the other hand, the connections studied in this paper can also be obtained by considering the principal torsion automorphism. In future work, we will examine a common generalisation of the work of Yun and Chen and of this paper – the classification of homogeneous rigid connections whose irregular singularity is toral for other classes of elliptic regular maximal tori. These would correspond to variants of θ -connections where one starts with appropriate homogeneous elements of the grading (1) of degree $r > 1$ relatively prime to m .

1.5. **Organization of the paper.** In §2, we give a quick review of Moy-Prasad filtrations and their role in studying formal connections, following earlier works of Bremer and Sage. In §3, we discuss formal Coxeter connections. In particular, we write convenient expressions for these connections and determine such basic properties as slope, adjoint irregularity, and the local differential Galois group. In §4, we state and prove our main result giving the necessary and sufficient conditions for (global) Coxeter connections to be rigid. We conclude the paper by discussing the global differential Galois group of these connections.

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2. MOY-PRASAD FILTRATIONS AND FORMAL CONNECTIONS

Let G be a simple complex algebraic group with Lie algebra \mathfrak{g} . Let B be a Borel subgroup, and $T \subset B$ a maximal torus with Lie algebras \mathfrak{t} and \mathfrak{b} . We write \mathfrak{g}_α for the root space corresponding to a root α . Let $\mathcal{K} = \mathbb{C}((t))$ be the field of Laurent series with ring of integers $\mathcal{O} = \mathbb{C}[[t]]$. Let $G(\mathcal{K})$ and $\mathfrak{g}(\mathcal{K})$ denote the loop group and algebra, respectively.

⁵In particular, the connections studied here are not necessarily in oper form.

⁶One needs to extend the theory of toral connections to twisted loop groups to get those θ -connections which are twisted connections.

2.1. Fundamental strata. Let \mathcal{B} be the Bruhat-Tits building of G ; it is a simplicial complex whose facets are in bijective correspondence with the parahoric subgroups of the loop group $G(\mathcal{K})$. The standard apartment \mathcal{A} associated to the split maximal torus $T(\mathcal{K})$ is an affine space isomorphic to $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Given $x \in \mathcal{B}$, we denote by $G(\mathcal{K})_x$ (resp. $\mathfrak{g}(\mathcal{K})_x$) the parahoric subgroup (resp. subalgebra) corresponding to the facet containing x .

For any $x \in \mathcal{B}$, the Moy-Prasad filtration associated to x is a decreasing \mathbb{R} -filtration

$$\{\mathfrak{g}(\mathcal{K})_{x,r} \mid r \in \mathbb{R}\}$$

of $\mathfrak{g}(\mathcal{K})$ by \mathcal{O} -lattices. The filtration satisfies $\mathfrak{g}(\mathcal{K})_{x,0} = \mathfrak{g}(\mathcal{K})_x$ and is periodic in the sense that $\mathfrak{g}(\mathcal{K})_{x,r+1} = t\mathfrak{g}(\mathcal{K})_{x,r}$. Moreover, if we set $\mathfrak{g}(\mathcal{K})_{x,r+} = \bigcup_{s>r} \mathfrak{g}(\mathcal{K})_{x,s}$, then the set of r for which $\mathfrak{g}(\mathcal{K})_{x,r} \neq \mathfrak{g}(\mathcal{K})_{x,r+}$ is a discrete subset of \mathbb{R} .

For our purposes, it will suffice to give the explicit definition of the filtration for $x \in \mathcal{A}$. In this case, the filtration is determined by a grading on $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$, with the graded subspaces given by

$$\mathfrak{g}(\mathcal{K})_x(r) = \begin{cases} \mathfrak{t}t^r \oplus \bigoplus_{\alpha(x)+m=r} \mathfrak{g}_{\alpha}t^m, & \text{if } r \in \mathbb{Z} \\ \bigoplus_{\alpha(x)+m=r} \mathfrak{g}_{\alpha}t^m, & \text{otherwise.} \end{cases}$$

Remark 3. The gradings associated to $x \in \mathcal{A}_{\mathbb{Q}}$ appeared in early works of Kac on infinite dimensional algebras. Thus, these are sometimes called Kac-Moy-Prasad gradings, cf. [Che17].

Let κ be the Killing form for \mathfrak{g} . Any element $X \in \mathfrak{g}(\mathcal{K})$ gives rise to a continuous \mathbb{C} -linear functional on $\mathfrak{g}(\mathcal{K})$ via $Y \mapsto \text{Res} \kappa(Y, X) \frac{dt}{t}$. This identification induces an isomorphism

$$(\mathfrak{g}(\mathcal{K})_{x,r}/\mathfrak{g}(\mathcal{K})_{x,r+})^{\vee} \cong \mathfrak{g}(\mathcal{K})_{x,-r}/\mathfrak{g}(\mathcal{K})_{x,-r+}.$$

Definition 4. A G -stratum of depth r is a triple (x, r, β) with $x \in \mathcal{B}$, $r \geq 0$, and $\beta \in (\mathfrak{g}(\mathcal{K})_{x,r}/\mathfrak{g}(\mathcal{K})_{x,r+})^{\vee}$. We say that this strata is based at x .

Any element of the corresponding $\mathfrak{g}(\mathcal{K})_{x,-r+}$ -coset is called a representative of β . If $\hat{\beta}$ is a representative, we will abuse notation to refer to the stratum as $(x, r, \hat{\beta})$. If $x \in \mathcal{A}$, there is a unique homogeneous representative $\beta^b \in \mathfrak{g}(\mathcal{K})_x(-r)$.

Definition 5. The stratum is called fundamental if every representative is non-nilpotent.

When $x \in \mathcal{A}$, it suffices to check that β^b is non-nilpotent.

2.2. Leading term of formal connections. A formal flat G -bundle (\mathcal{E}, ∇) (or simply a formal G -connection) is a principal G -bundle \mathcal{E} on the formal punctured disk $\mathcal{D}^{\times} := \text{Spec}(\mathcal{K})$ endowed with a connection ∇ (which is automatically flat). Upon choosing a trivialisation, the connection may be written in terms of its matrix

$$[\nabla]_{\phi} \in \Omega_F^1(\mathfrak{g}(\mathcal{K}))$$

via $\nabla = d + [\nabla]_{\phi}$. If one changes the trivialisation by an element $g \in G(\mathcal{K})$, the matrix changes by the gauge action:

$$(3) \quad [\nabla]_{g\phi} = \text{Gauge}_g([\nabla]_{\phi}) = \text{Ad}_g([\nabla]_{\phi}) - (dg)g^{-1}.$$

Accordingly, the set of isomorphism classes of flat G -bundles on \mathcal{D}^{\times} is isomorphic to the quotient $\mathfrak{g}(\mathcal{K}) \frac{dt}{t} / G(\mathcal{K})$, where the loop group $G(\mathcal{K})$ acts by the gauge action.

Definition 6. If $x \in \mathcal{A} \cong \mathfrak{t}_{\mathbb{R}}$, we say that (\mathcal{E}, ∇) contains the stratum (x, r, β) with respect to the trivialisation ϕ if $[\nabla]_{\phi}/(dt/t) - x \in \mathfrak{g}(\mathcal{K})_{x,-r}$ and is a representative for β .

We refer the reader to [BS18] for the definition for an arbitrary point $x \in \mathcal{B}$. If ∇ contains the stratum (x, r, β) , then we can think of β as the leading term of ∇ with respect to x . The following theorem shows that a non-nilpotent leading term contains meaningful information about ∇ :⁷

Theorem 7 ([BS18, Theorem 2.14]). *Every flat G -bundle (\mathcal{E}, ∇) contains a fundamental stratum (x, r, β) , where x is in the closure of the fundamental alcove $C \subset \mathcal{A}$ and $r \in \mathbb{Q}$; the depth r is positive if and only if (\mathcal{E}, ∇) is irregular singular. Moreover, the following statements hold.*

- (1) *If (\mathcal{E}, ∇) contains the stratum (y, r', β') , then $r' \geq r$.*
- (2) *If (\mathcal{E}, ∇) is irregular singular, a stratum (y, r', β') contained in (\mathcal{E}, ∇) is fundamental if and only if $r' = r$.*

As a consequence, one can define the slope of ∇ as the depth of any fundamental stratum it contains.

2.3. Regular strata and toral flat G -bundles. We will also need some results on flat G -bundles which contain a *regular stratum*, a kind of stratum that satisfies a graded version of regular semisimplicity. For convenience, we will only describe the theory for strata based at points in \mathcal{A} .

Let $S \subset G(\mathcal{K})$ be a (in general, non-split) maximal torus, and let $\mathfrak{s} \subset \mathfrak{g}(\mathcal{K})$ be the associated Cartan subalgebra. We denote the unique Moy-Prasad filtration on \mathfrak{s} by $\{\mathfrak{s}_r\}$. More explicitly, we first observe that if S is split, then this is just the usual degree filtration. In the general case, if \mathcal{K}_b is a splitting field for S , then \mathfrak{s}_r consists of the Galois fixed points of $\mathfrak{s}(\mathcal{K}_b)_r$. Note that $\mathfrak{s}_r \neq \mathfrak{s}_{r+}$ implies that $r \in \mathbb{Z}_{\frac{1}{b}}$. We remark that the filtration on \mathfrak{s} can be defined in terms of a grading, whose graded pieces we denote by $\mathfrak{s}(r)$.

Definition 8. (i) *A point $x \in \mathcal{A}$ is called compatible (resp. graded compatible) with \mathfrak{s} if $\mathfrak{s}_r = \mathfrak{g}(\mathcal{K})_{x,r} \cap \mathfrak{s}$ (resp. $\mathfrak{s}(r) = \mathfrak{g}(\mathcal{K})_{x,r} \cap \mathfrak{s}$) for all $r \in \mathbb{R}$.*
(ii) *A fundamental stratum (x, r, β) with $x \in \mathcal{A}$ and $r > 0$ is an S -regular stratum if x is compatible with S and S equals the connected centralizer of β^b .*

In fact, every representative of β will be regular semisimple with connected centralizer a conjugate of S .

Definition 9. *If (\mathcal{E}, ∇) contains the S -regular stratum (x, r, β) , we say that (\mathcal{E}, ∇) is S -toral.*

Recall that the conjugacy classes of maximal tori in $G(\mathcal{K})$ are in one-to-one correspondence with conjugacy classes in the Weyl group W [KL88]. It turns out that there exists an S -toral flat G -bundle of slope r if and only if S corresponds to a regular conjugacy class of W and $e^{2\pi ir}$ is a regular eigenvalue for this class [BS14, Corollary 4.10]. Equivalently, $\mathfrak{s}(-r)$ contains a regular semisimple element.

An important feature of S -toral flat G -bundles is that they can be “diagonalised” into \mathfrak{s} . To be more precise, suppose that there exists an S -regular stratum of depth r . Let $\mathcal{A}(S, r)$ be the open subset of $\bigoplus_{j \in [-r, 0]} \mathfrak{s}(j)$ whose leading component (i.e., the component in $\mathfrak{s}(-r)$) is regular semisimple. This is called the set of *S -formal types* of depth r . Let $W_S^{\text{aff}} = N(S)/S_0$ be the relative affine Weyl group of S ; it is the semidirect product of the relative Weyl group W_S and the free abelian group S/S_0 . The group W_S^{aff} acts on $\mathcal{A}(S, r)$. The action of W_S is the restriction of the obvious linear action while S/S_0 acts by translations on $\mathfrak{s}(0)$.

Theorem 10. [BS14, Corollary 5.14] *If (\mathcal{E}, ∇) is S -toral of slope r , then ∇ is gauge-equivalent to a connection with matrix in $\mathcal{A}(S, r) \frac{dt}{t}$. Moreover, the moduli space of S -toral flat G -bundles of slope r is given by $\mathcal{A}(S, r)/W_S^{\text{aff}}$.*

⁷This theorem and Theorem 10 remain true for connected reductive G .

3. FORMAL COXETER CONNECTIONS

Recall that G is a simple group over \mathbb{C} , B a Borel subgroup, and T a maximal torus. Let Φ denote the set of roots of G and $\Delta \subset \Phi$ the subset of simple roots. Let Φ^+ and Φ^- denote the set of positive and negative roots, respectively. For each root α , let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote the corresponding root subspace. Following [FG09, §5], we choose, once and for all, an “affine pinning” for G , i.e.,

- (1) a generator $X_\alpha \in \mathfrak{g}_\alpha$ for each negative simple root $\alpha \in -\Delta$;
- (2) a generator E for the root space \mathfrak{g}_θ associated to the highest root θ .

Using this data, we construct formal connections that turn out to be homogeneous toral connections (Definition 9) associated to a certain torus, called the Coxeter torus. For brevity, we refer to these connections as formal Coxeter connections. As we shall see, the gauge equivalence class of these connections depends only mildly on the above choices.

3.1. Recollections on some results of Kostant. The aim of this subsection is to recall some results of [Kos59], in a format convenient for our purposes. Let $\check{\rho} \in \mathfrak{t}$ denote the sum of fundamental coweights and $x_0 := \check{\rho}/h \in \mathfrak{t}$. Consider the adjoint action of the torsion element $\exp(x_0) = e^{2\pi i x_0} \in T$ on \mathfrak{g} . The resulting eigenspaces define a periodic grading

$$(4) \quad \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/h\mathbb{Z}} \mathfrak{g}_i,$$

where $\mathfrak{g}_0 = \mathfrak{t}$, and for $i \in \{1, 2, \dots, h-1\}$, \mathfrak{g}_i is direct sum of root spaces

$$(5) \quad \mathfrak{g}_i = \left(\bigoplus_{\text{ht}(\alpha)=i-h} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus_{\text{ht}(\alpha)=i} \mathfrak{g}_\alpha \right).$$

Next, let

$$(6) \quad N_1 := \sum_{\alpha \in -\Delta} X_\alpha \in \mathfrak{g}_{-1}$$

and $E_1 = E$, where we recall that E is a generator of the root space for the highest root θ . Then N_1 is a principal (=regular) nilpotent element in \mathfrak{g} . Kostant proved that the element $N_1 + E_1 \in \mathfrak{g}$, is regular semisimple, so its centraliser is a maximal torus, denoted by S . Let $\mathfrak{s} := \text{Lie}(S)$ denote the corresponding Cartan subalgebra of \mathfrak{g} . The grading (4) induces a grading

$$(7) \quad \mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_{h-1}.$$

Kostant proved that $\mathfrak{s}_0 = 0$ and \mathfrak{s}_i contains a regular semisimple element if and only if $\gcd(i, h) = 1$, in which case, $\dim(\mathfrak{s}_r) = 1$.

In particular, $N_1 + E_1$ is the unique, up to scalar, nonzero element in \mathfrak{s}_{-1} . Moreover, if A_r is a (regular semisimple) generator of \mathfrak{s}_{-r} , where $\gcd(r, h) = 1$, then there exists generators X_α of root spaces \mathfrak{g}_α comprising \mathfrak{g}_{-r} (5), and a decomposition $A_r = N_r + E_r$ where

$$(8) \quad N_r = \sum_{\text{ht}(\alpha)=-r} X_\alpha, \quad E_r = \sum_{\text{ht}(\alpha)=h-r} X_\alpha.$$

Example 11. For $\mathfrak{g} = \mathfrak{sl}_n$, let N_1 denote the matrix with 1’s on the subdiagonal and zeros everywhere else, and let E_1 be the matrix with a 1 on the top right hand corner and 0’s everywhere else. We can take A_r to be A_1^r . Then N_r is the matrix with 1’s on the r^{th} subdiagonal and 0’s everywhere else, and E_r is the matrix with 1’s on the $(n-r)^{\text{th}}$ superdiagonal and 0’s everywhere else. For

instance, for $n = 5$ we have

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also remark that given any generators X_α for the root spaces of heights $-r$ and $h - r$, the resulting $N_r + E_r$ can be obtained from an appropriate affine pinning. Indeed, Kostant shows that a choice of such X_α 's is just an affine pinning with respect to a different choice of positive roots.

3.2. The Coxeter Cartan subalgebra and its graded pieces. We now transport the above results to the setting of loop algebras. Let x_I denote the barycentre of the fundamental alcove of the standard apartment \mathcal{A} . The associated Kac-Moy-Prasad grading is given by

$$(9) \quad \mathfrak{g}(\mathbb{C}[t, t^{-1}]) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\mathcal{K})_{x_I}(i/h).$$

In fact, this is a renormalisation of the *principal grading* of the polynomial loop algebra [Kac90, §14], [Che17, Example 3.1], where the degrees have been divided by h .

Convention 12. By Kostant's theorem, the element $N_1 + t^{-1}E_1 \in \mathfrak{g}(\mathcal{K})_{x_I}(-1/h)$ is regular semisimple. Following [GKM06], we call its centraliser the *Coxeter torus* and denote it by \mathcal{C} . We call $\mathfrak{c} := \text{Lie}(\mathcal{C})$ the *Coxeter Cartan* subalgebra. The reason for this terminology is that under the bijection between conjugacy classes of maximal tori in the loop group to conjugacy classes in the Weyl group (cf. [KL88]), the class of \mathcal{C} maps to the Coxeter class.

Next, the grading 9 induces a grading

$$\mathfrak{c} \cap \mathfrak{g}(\mathbb{C}[t, t^{-1}]) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{c}(i/h).$$

This is in fact the canonical Kac-Moy-Prasad grading of \mathfrak{c} . In particular, we see that x_I is graded compatible with \mathfrak{c} in the sense of Definition 8.

3.2.1. Relationship to the grading (4). As noted in [RY14] and [Che17], there is a dictionary between Moy-Prasad filtrations and periodic gradings of \mathfrak{g} . In the present setting, this means that we have a canonical isomorphism between the graded spaces of the principal grading and the grading discussed in the previous subsection, i.e.,

$$\mathfrak{g}_i \simeq \mathfrak{g}(\mathcal{K})_{x_I}(i/h) \quad i \in \mathbb{Z}.$$

This observation allows us to transport the results of the previous subsection to the current setting. Thus, we find that $\mathfrak{c}(i/h)$ contains a regular semisimple element if and only if $\gcd(i, h) = 1$, in which case $\dim(\mathfrak{c}(i/h)) = 1$. Moreover, $N_r + t^{-1}E_r$ is a generator for $\mathfrak{g}(\mathcal{K})_{x_I}(-r/h)$ for all r satisfying $1 \leq r \leq h$ and $\gcd(r, h) = 1$.

3.3. Homogeneous \mathcal{C} -toral connections. In the previous subsection, we explained that the canonical Kac-Moy-Prasad grading of the Coxeter Cartan subalgebra \mathfrak{c} is compatible with the point $x_I \in \mathcal{A}_{\mathbb{Q}}$ and that the regular semisimple elements in the graded pieces are exactly of the form $t^{-m}(N_r + t^{-1}E_r)$ with $m, r \in \mathbb{Z}$, $1 \leq r \leq h$ and $\gcd(r, h) = 1$. Thus, if we further assume $m \geq 0$, then the strata

$$\left(x_I, m + r/h, t^{-m}(N_r + t^{-1}E_r) \right)$$

are, up to rescaling the third entry by an element of \mathbb{C}^\times , the \mathcal{C} -regular strata based at x_I (Definition 8). In view of Definition 9, we find:

Lemma 13. *The homogeneous \mathcal{C} -toral connections based at x_I are the formal connections defined by*

$$(10) \quad \boxed{\nabla_{r,m}^\lambda := d + t^{-m} \lambda (N_r + t^{-1} E_r) \frac{dt}{t}}$$

where

$$\lambda \in \mathbb{C}^\times, m \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z} \cap [1, h), \gcd(r, h) = 1.$$

Here, the term homogeneous \mathcal{C} -toral connection means that the \mathcal{C} -formal type (as defined in §2.3) is homogeneous. We remark that N_r and E_r are only determined by the affine pinning up to a common nonzero scalar. Thus, in our notation, we have fixed one such choice. (As explained in §3.1, an appropriate choice of affine pinning allows one to take N_r and E_r to be any elements of heights $-r$ and $h - r$ with nonzero component in each root space of these heights.)

Convention 14. For brevity, we refer to the $\nabla_{r,m}^\lambda$'s as *formal Coxeter connections*, and write $\nabla_{r,m}$ for $\nabla_{r,m}^1$.

We note that the restriction of the Frenkel-Gross connection ∇_{FG} to the formal neighbourhood of the irregular singular point is $\nabla_{1,0}^{-1}$.

Proposition 15. *The connection $\nabla_{r,m}^\lambda$ is irregular of slope $r/h + m$ with adjoint irregularity $(r + mh)\ell$, where ℓ is the rank of G .*

Proof. Since the connection contains a fundamental stratum of depth $m + r/h$, Theorem 7 implies that the slope is $m + r/h$. The statement about adjoint irregularity follows from [KS19, Lemma 19].⁸ □

3.4. Uniqueness. Recall that to define the formal Coxeter connections $\nabla_{r,m}^\lambda$, we fixed an affine pinning of G . We will now discuss to what extent the connections depend on this choice. Let us choose another affine pinning T' , B' , and X'_α , $\alpha \in -\Delta \cup \{\theta\}$. Let $N'_1 + E'_1$ denote the resulting regular semisimple element. By [Kos59, Theorem 6.2], there exists $g \in G$ and $\lambda \in \mathbb{C}^\times$ such that the adjoint action by g takes T' to T , B' to B , the root spaces with respect to T' to those with respect to T , and X'_α to λX_α . This implies that

$$(11) \quad \text{Ad}_g(N'_1 + t^{-1} E'_1) = \lambda(N_1 + t^{-1} E_1).$$

Moreover, for each r relatively prime to h , we can choose N'_r and E'_r such that the equation

$$\text{Ad}_g(N'_r + t^{-1} E'_r) = \lambda(N_r + t^{-1} E_r)$$

holds as well.

Now, let $\nabla'_{r,m} := d + t^{-m} (N'_r + t^{-1} E'_r) \frac{dt}{t}$. The above discussion implies:

Lemma 16. *There exists $g \in G$ and $\lambda \in \mathbb{C}^\times$ such that*

$$\text{Gauge}_g(\nabla'_{r,m}) = \nabla_{r,m}^\lambda$$

It remains to find when two connections $\nabla_{r,m}^\lambda$ and $\nabla_{r,m}^{\lambda'}$ are gauge equivalent. To this end, we determine the relevant part of the moduli space of \mathcal{C} -toral connections of slope $i/h = r/h + m$ (cf. [KS19, Example 18], where the case $i = 1$ is discussed).

⁸One can also prove these facts using similar considerations to [FG09, §5].

3.4.1. *Determination of the moduli space.* By Theorem 10, the moduli space of \mathcal{C} -toral connections of slope i/h is $\mathcal{A}(\mathcal{C}, i/h)/W_{\mathcal{C}}^{\text{aff}}$. Here $\mathcal{A}(\mathcal{C}, i/h)$ is the open subset of $\mathfrak{c}(-i/h) \oplus \cdots \oplus \mathfrak{c}(0)$ for which the component in degree $-i/h$ is regular semisimple. This subset is nonempty if and only if i is coprime to h , in which case its homogeneous degree $-i/h$ part is isomorphic to \mathbb{C}^\times . Recall that $W_{\mathcal{C}}^{\text{aff}}$ is the semidirect product of $W_{\mathcal{C}} = N_{G(\mathcal{K})}(\mathcal{C})/\mathcal{C}$ by $\mathcal{C}/\mathcal{C}_0$. Moreover, $\mathcal{C}/\mathcal{C}_0$ acts by translation on $\mathfrak{c}(0)$. But the latter space is zero, which implies the action is trivial. Thus, the homogeneous part of the moduli space is well-defined and isomorphic to $\mathbb{C}^\times/W_{\mathcal{C}}$.

Finally, $W_{\mathcal{C}}$ is isomorphic to a subgroup of the centralizer of a Coxeter element in W [BS14, Proposition 5.9], so $W_{\mathcal{C}}$ is a cyclic group of order h' dividing h . In fact, in the present case, we have $h' = h$. To see this, let ζ be a primitive h^{th} root of unity. Kostant [Kos59] proved that there is an element $s \in T$ for which

$$\text{Ad}(s)x = \zeta^k x, \quad \forall x \in \mathfrak{g}_\alpha, \quad |\alpha| = k.$$

In particular, the standard cyclic element $N_1 + E_1$ is an eigenvector of $\text{Ad}(s)$ with eigenvalue ζ^{-1} . It follows that $s \in N(\mathcal{C})$ and $\text{Ad}(s)(N_1 + t^{-1}E_1) = \zeta^{-1}(N_1 + t^{-1}E_1)$, so the image of s in $W_{\mathcal{C}}$ has order h . We summarise the above discussion in the following:

Proposition 17. *If r is a positive integer coprime to h , and m is a nonnegative integer, then the moduli space of formal Coxeter connections of slope $i/h = m + r/h$ is isomorphic to \mathbb{C}^\times/μ_h .*

Corollary 18. *The formal connections $\nabla_{r,m}^\lambda$ and $\nabla_{r,m}^\mu$ are gauge equivalent if and only if λ/μ is an h^{th} -root of unity.*

3.5. **Local differential Galois group.** Let \mathcal{J}_0 denote the differential Galois group of $\text{Spec}(\mathcal{K})$. Let \mathcal{P}_0 be the subgroup of \mathcal{J}_0 classifying irregular connections. In some references, \mathcal{J}_0 is referred to as the inertia group and \mathcal{P}_0 the wild inertia subgroup. In view of the Tannakian definition of \mathcal{J}_0 , every formal connection ∇ determines a homomorphism $\phi : \mathcal{J}_0 \rightarrow G$. The group $I_0 = I_0^\nabla := \phi(\mathcal{J}_0)$ is called the differential Galois group of ∇ .

In this subsection, we follow [FG09, §13] to determine I_0 for the formal Coxeter connection $\nabla_{r,m}^\lambda$. Let $\phi : \mathcal{J}_0 \rightarrow G$ be the homomorphism associated to $\nabla_{r,m}^\lambda$. Then the image $P_0 := \phi(\mathcal{P}_0)$ is the smallest torus S in G whose Lie algebra contains the regular semisimple element $N_r + E_r$. The full image $I_0 = \phi(\mathcal{J}_0)$ normalises S , and the quotient is generated by the element $n = (2\rho)(e^{\pi i/h})$. As n normalises S , it also normalises $T' = C_G(S)$ which is a maximal torus in G . The image of n in $N(T')/T'$ is a Coxeter element.

Next, note that the composition $I_0 \rightarrow G \rightarrow \text{Aut}(\mathfrak{g})$ gives rise to an action of I_0 on \mathfrak{g} . For future reference, we record the following lemma:

Lemma 19. *For formal Coxeter connections $\nabla_{r,m}^\lambda$, we have $\mathfrak{g}^{I_0} = 0$.*

Proof. Indeed, $\mathfrak{g}^{P_0} \subset \mathfrak{g}^{N_r + E_r} = \mathfrak{t}' := \text{Lie}(T')$ implying that $\mathfrak{g}^{I_0} \subset (\mathfrak{t}')^n = 0$. \square

4. RIGIDITY

Let X be a smooth projective curve over \mathbb{C} . Let $j : U \rightarrow X$ denote the inclusion of a nonempty open subset.

Definition 20. *A connection ∇ on U is said to be (cohomologically) rigid if*

$$H^1(\mathbb{P}^1, j_{!*}\text{ad}_\nabla) = 0.$$

As explained in [Yun14b, §3.2], this notion of rigidity is closely related to having no infinitesimal deformations. In more detail, if ∇ satisfies

$$H^i(\mathbb{P}^1, j_{!*}\text{ad}_\nabla) = 0, \quad i \in \{0, 1, 2\}$$

then ∇ has no infinitesimal deformations. In practice, H^0 and H^2 vanish for many connections of interest, e.g., for irreducible ones.

4.1. Main theorem. In the previous section, we defined the formal Coxeter connections $\nabla_{r,m}^\lambda$ as connections on the punctured disk. Now observe that the element of the loop algebra representing this formal connection, i.e. $t^{-m}(N_r + t^{-1}E_r)$, lies in $\mathfrak{g}(\mathbb{C}[t, t^{-1}])$. Thus, we can think of $\nabla_{r,m}^\lambda$ as a connection on the trivial bundle on \mathbb{G}_m . We refer to these global connections simply as *Coxeter connections* and, by an abuse of notation, also denote them by $\nabla_{r,m}^\lambda$. The aim of this section is to determine when $\nabla_{r,m}^\lambda$ is rigid.

Theorem 21. *Let $\lambda \in \mathbb{C}^\times$. A Coxeter connection $\nabla = \nabla_{r,m}^\lambda$ is rigid if and only if we are in one of the following cases:*

- (i) $m = 1$ and $r = 1$ in which case ∇ is regular at infinity;
- (ii) $m = 0$ and $r = 1$ in which case ∇ is regular singular at infinity with principal unipotent monodromy⁹;
- (iii) $m = 0$ and r satisfies the following conditions

Root system	Conditions on r
A_{n-1}	$r n \pm 1$
B_n	$r n + 1, r 2n + 1$
C_n	$r 2n \pm 1$
D_n	$r 2n, r 2n - 1$
E_7	$r = 7$

in which case ∇ is regular singular at infinity with unipotent monodromy $\exp(N_r)$.

For Coxeter connections, it is easy to see that

$$H^0(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla) = H^2(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla) = 0.$$

Thus, being rigid in this case is equivalent to having no infinitesimal deformations.

4.2. A numerical criterion for rigidity. Let $\mathcal{J} = \pi_1^{\mathrm{diff}}(\mathbb{G}_m)$ be the differential Galois group of \mathbb{G}_m . Then a connection ∇ on \mathbb{G}_m defines a homomorphism $\mathcal{J} \rightarrow G$ whose image is called the (global) differential Galois group of ∇ and denoted by I . It contains the local differential Galois groups I_0 and I_∞ .

Now suppose ∇ is irregular at 0 and regular singular at ∞ . Define

$$(12) \quad n(\nabla) := \mathrm{Irr}_0(\mathrm{ad}_\nabla) - \dim(\mathfrak{g}^{I_0}) - \dim(\mathfrak{g}^{I_\infty}) + 2 \dim(\mathfrak{g}^I);$$

here, Irr_0 denotes the irregularity at 0. According to [FG09, prop. 11], $\dim(H^1(\mathbb{P}^1, j_{!*}\mathrm{ad}_\nabla)) = n(\nabla)$. Thus, we find that ∇ is rigid if and only if $n(\nabla) = 0$. We call this the *numerical criterion for rigidity*.

We now use this criterion to establish when the Coxeter connections $\nabla_{r,m}^\lambda$ are rigid. Since λ plays no role in the analysis, we set $\lambda = 1$ and omit it from the notation.

Proposition 22. *If $\nabla_{r,m}$ is rigid, then $m = 0$ or $m = 1$.*

Proof. By Lemma 19, we have $\mathfrak{g}^{I_0} = 0$. As I contains I_0 , this implies that \mathfrak{g}^I also vanishes. Next, by Proposition 15, $\mathrm{Irr}(\mathrm{ad}_{\nabla_r}) = (r + mh)\ell$. Thus, we have

$$n(\nabla) = (r + mh)\ell - \dim(\mathfrak{g}^{I_\infty}) \geq (r + mh)\ell - (h + 1)\ell = (r + mh - h - 1)\ell.$$

Note that the first inequality is using the fact that $\dim(\mathfrak{g}^{I_\infty}) \leq \dim(\mathfrak{g}) = (h + 1)\ell$. As $m, r \in \mathbb{Z}_{\geq 0}$ and $h \geq 2$, it is clear that if $n(\nabla) = 0$ then either $m = 0$ or $m = 1$. \square

⁹This is essentially the Frenkel-Gross connection.

Proposition 23. *The connection $\nabla_{r,1}$ is rigid if and only if $r = 1$.*

Proof. It is easy to see that the connection is regular at ∞ , thus, $I_\infty = 0$ and $\mathfrak{g}^{I_\infty} = \mathfrak{g}$ and therefore $n(\nabla) = 0$. Thus, by the numerical criterion for rigidity, ∇ is rigid. \square

Next we treat the case $m = 0$ of the theorem and write ∇_r for $\nabla_{r,0}$. It is easy to see that the connection ∇_r is regular singular at infinity with monodromy $\exp(N_r)$. Let \mathcal{O}_r denote the nilpotent orbit containing N_r and $C_{\mathfrak{g}}(N_r)$ the centraliser of N_r in \mathfrak{g} .

Proposition 24. *The connection ∇_r is rigid if and only if*

$$(13) \quad \dim(\mathcal{O}_{N_r}) = (h + 1 - r)\ell;$$

or equivalently,

$$(14) \quad \dim C_{\mathfrak{g}}(N_r) = r\ell.$$

Proof. We have

$$\dim(\mathfrak{g}^{I_\infty}) = \dim(C_G(\text{Exp}(N_r))) = \dim(C_{\mathfrak{g}}(N_r)) = \dim(\mathfrak{g}) - \dim(\mathcal{O}_{N_r}).$$

Thus,

$$n(\nabla) = r\ell - \dim(\mathfrak{g}^{I_\infty}) = (\dim(\mathfrak{g}) - (h + 1 - r)\ell) - (\dim(\mathfrak{g}) - \dim(\mathcal{O}_{N_r})) = \dim(\mathcal{O}_{N_r}) - (h + 1 - r)\ell.$$

By the numerical criterion, ∇ is rigid if and only if $\dim(\mathcal{O}_{N_r}) = (h + 1 - r)\ell$. The second equality follows from this and the fact that $\dim(\mathfrak{g}) = (h + 1)\ell$. \square

Note that if $r = 1$, then N_r is the principal nilpotent element. Therefore, $\dim(\mathcal{O}_{N_r}) = h\ell$ and (13) holds. Thus, as proved in [FG09], the connections ∇_1 are always rigid. On the other hand, when $r > 1$, $\dim(\mathcal{O}_{N_r})$ depends on the type of the Lie algebra \mathfrak{g} ; thus, a type by type analysis is required to determine the rigidity of ∇_r . As we have already discussed the case $r = 1$, we will not consider it as one of the possibilities.

4.3. Proof of Theorem 21 for classical groups. A convenient reference for nilpotent orbits in semisimple Lie algebras is [CM93]. In particular, in §5 of *op. cit.*, it is shown that Jordan canonical form leads to a bijection between nilpotent orbits in classical groups and certain partitions. This in turn leads to formulae for the dimension of nilpotent orbits in terms of the dual partition (cf. appendix of [MY16]). Below we will use these formulae to determine rigidity of ∇_r for classical groups. .

4.3.1. Type A_{n-1} . Let us write $n = kr + n'$ where $0 \leq n' < r$. Since N_1 is principal nilpotent and one can take $N_r = N_1^r$, it is easy to see that the Jordan form of N_r has n' parts of size $k + 1$ and $r - n'$ parts of size k . Thus, the dual partition has k parts of size r and 1 part of size n' and

$$\dim(\mathcal{O}_{N_r}) = n^2 - kr^2 - (n')^2.$$

The equality (14) therefore takes the form

$$kr^2 + (n')^2 - 1 = r(n - 1).$$

Using the equality $n = kr + n'$, we find that the above equality holds if and only if

$$(n' - 1)(r - (n' + 1)) = 0.$$

Thus, either $n' = 1$ or $n' = r - 1$. This is in turn equivalent to $r|(n - 1)$ or $r|(n + 1)$.

4.3.2. *Type B_n .* Nilpotent orbits in \mathfrak{so}_{2n+1} are in bijection with those partitions of $2n+1$ where even parts appear with even multiplicity. Let us write $2n+1 = kr + n'$ where $0 \leq n' < r$. In this case, N_1 is also principal nilpotent viewed as an element of \mathfrak{sl}_{2n+1} , and N_r can again be taken to be N_1^r . Accordingly, the partition associated to the nilpotent element N_r has n' parts of size $k+1$ and $r-n'$ parts of size k . To see that this partition satisfies the required constraint, note that r is odd, since it is coprime to the Coxeter number $2n$. Now if n' is even, then we must have that kr is odd, thus k is odd, which means that $k+1$ is even. On the other hand, if n' is odd, then $r-n'$ and k are even.

The dual partition has k parts of size r and 1 part of size n' . Therefore, we have

$$\dim C_{\mathfrak{g}}(N_r) = \frac{1}{2} \left(kr^2 + (n')^2 - \begin{cases} n' & \text{if } k \text{ is even} \\ r - n' & \text{if } k \text{ is odd} \end{cases} \right)$$

Equating this with $r(2n) = \frac{1}{2}r(kr + n' - 1)$, equality (14) takes the form

$$(n')^2 - \begin{cases} n' & \text{if } k \text{ is even} \\ r - n' & \text{if } k \text{ is odd} \end{cases} = r(n' - 1).$$

If k is even, then we obtain $n' = 1$ or $r = n'$. Both cases are impossible. Indeed, $n' = 1$ implies that $r|2n$ and that contradicts $\gcd(r, 2n) = 1$. On the other hand, $r = n'$ is not allowed because r is assumed to be less than n' . On the other hand, if k is odd, we obtain that either $n' = 0$ or $r = n' + 1$. In the first case, we obtain $r|(2n+1)$ and in the second case $r|(n+1)$.

Finally we check that if $r|(2n+1)$ or $r|(n+1)$ then k cannot be even. If $r|2n+1$, then $n' = 0$, so $2n+1 = kr$, thus k is odd. If $rs = n+1$, and $k = 2t$, then $2n+1 = 2rs - 1 = 2tr + n'$ implying that $n' = 2r(s-t) - 1$. But now $n' \geq 0$ implies $s > t$. Then $n' \geq 2r - 1$ which is impossible since $n' < r$.

4.3.3. *Type C_n .* Nilpotent orbits in \mathfrak{sp}_{2n} are in bijection with partitions of $2n$ in which odd parts appear with even multiplicity. Let us write $2n = kr + n'$ with $0 < n' < r$. Recall that $\gcd(r, 2n) = 1$; in particular r is odd. As in types A and B , the N_r 's are powers of a single Jordan block. Thus, the nilpotent element N_r corresponds to the partition of $2n$ consisting of n' parts of size $k+1$ together with $r-n'$ parts of size k . The fact that this partition satisfies the desired constraint follows by similar considerations to those in type B_n .

The dual partition has k parts of size r and 1 part of size n' . Now, we have

$$\dim C_{\mathfrak{g}}(N_r) = \frac{1}{2} \left(kr^2 + n'^2 + \begin{cases} n' & \text{if } k \text{ is even} \\ r - n' & \text{if } k \text{ is odd} \end{cases} \right).$$

Equating this with $rn = r(kr + n')/2$, we find that the equality (14) takes the form

$$(n')^2 + \begin{cases} n' & \text{if } k \text{ is even} \\ r - n' & \text{if } k \text{ is odd} \end{cases} = rn'.$$

If k is even, we have

$$(n')^2 + n' = rn' \implies r = n' + 1 \implies 2n = kr + n' = kr + r - 1 \implies 2n + 1 = r(k + 1) \implies r|(2n + 1).$$

This is indeed consistent with our assumption because $2n+1 = r(k+1)$ implies k is even.

On the other hand, if k is odd, we obtain

$$(n')^2 + r - n' = rn' \implies (n')^2 - n' = (n' - 1)r \implies r = n' \text{ or } n' = 1.$$

Note, however, that the case $r = n'$ is impossible. Thus, we obtain,

$$2n = kr + n' = kr + 1 \implies 2n - 1 = kr \implies r|(2n - 1).$$

This is also consistent with our assumption because $2n - 1 = kr$ implies that k has to be odd.

Finally, note that if $r|2n - 1$, then $2n = rs + 1$, so $s = k$. Now r is odd and therefore so is k . On the other hand, if $r|2n + 1$, then $2n = rs - 1$, so $k = s - 1$, i.e. $2n = r(s - 1) + (r - 1)$. Since $r - 1$ is even and r is odd, need $s - 1$ even, i.e. k is even.

4.3.4. *Type D_n .* Nilpotent orbits in \mathfrak{so}_{2n} are in bijection with partitions of $2n$ in which even parts appear with even multiplicity, except that very even partitions, i.e., those with only even parts each having even multiplicity, correspond to two nilpotent orbits.¹⁰ The Coxeter number is $2n - 2$ and by assumption $\gcd(r, 2n - 2) = 1$. Let us write $2n = kr + n' + 1$ with $0 \leq n' < r$. Here, N_1 is no longer a single Jordan block. Instead, it is subregular as an element of \mathfrak{sl}_{2n} , i.e., it corresponds to the partition $(2n - 1, 1)$. It is easy to check that one can take $N_r = N_1^r$. It follows that the partition associated to N_r has n' parts of size $k + 1$, $r - n'$ parts of size k , and 1 part of size 1. The dual partition has 1 part of size $r + 1$, 1 part of size n' , and $k - 1$ parts of size r . Therefore, we have

$$\dim C_{\mathfrak{g}}(N_r) = \frac{1}{2} \left((r + 1)^2 + (k - 1)r^2 + (n')^2 - \begin{cases} n' + 1 & \text{if } k \text{ is even} \\ r - n' + 1 & \text{if } k \text{ is odd} \end{cases} \right).$$

Equating this with $rn = \frac{1}{2}r(n' + kr + 1)$, we find that the equation (14) takes the form

$$(n')^2 - \begin{cases} n' & \text{if } k \text{ is even} \\ r - n' & \text{if } k \text{ is odd} \end{cases} = r(n' - 1).$$

If k is even, we obtain that either $n' = 1$ or $n' = r$. Neither of this is a possibility because the first case contradicts $\gcd(2n - 2, r) = 1$ and the second contradicts $r < n'$. On other hand, if k is odd, we obtain that $n' = 0$ or $r = n' + 1$, which implies $r|(2n - 1)$ or $r|(2n)$.

Finally note that in the first case, $n' = 0$ and $k|(2n - 1)$, so k is odd. In the second case, $n' = r - 1$, thus $2n = r(k + 1)$, implying that k is odd.

4.4. **Proof of Theorem 21 for exceptional groups.** We now investigate rigidity of ∇_r for exceptional groups.

4.4.1. *Type G_2 .* We have $h = 6$; thus, the conditions $\gcd(r, h) = 1$ and $1 < r < 6$ implies that $r = 5$ is the only case to consider. By Proposition 24, for ∇_r to be rigid, we need $\dim(\mathcal{O}_{N_r}) = 4$. However, by inspecting the table for nilpotent orbits in type G_2 [CM93, §8], one finds that \mathfrak{g} has no nilpotent orbit of dimension 4. Thus, ∇_5 is not rigid.

4.4.2. *Type F_4 .* In this case, $h = 12$. As $\gcd(r, 12) = 1$, we see that possibilities for r are 5, 7, and 11. For ∇_r to be rigid, the dimension of the orbit containing N_r must be 32, 24, and 8, respectively. But \mathfrak{g} has no nilpotent orbits of these sizes [CM93], so the ∇_r 's are not rigid.

4.4.3. *Type E_6 .* In this case, $h = 12$; thus $r \in \{5, 7, 11\}$. For ∇_r to be rigid, the dimension of the orbit containing N_r must be 48, 36, and 12 respectively. There are no nilpotent orbits of size 12 or 36. There is, however, a nilpotent orbit of size 48. Table 4 in [dG10] (last line of page 8), however, shows that the nilpotent orbit containing N_5 has label $2A_1 + A_2$. According to 8.4 of [CM93], this nilpotent orbit has size 50. Thus, ∇_5 is not rigid.

¹⁰These two orbits are identified if one uses O_{2n} instead of SO_{2n} .

4.4.4. *Type E_7 .* In this case $h = 18$; thus, $r \in \{5, 7, 11, 13, 17\}$. For ∇_r to be rigid, the dimension of the orbit containing N_r must be 98, 84, 56, 42, and 14 respectively. Note that in this case, we do not have a nilpotent orbit of size 14, 42, or 56. Moreover, as W. de Graaf explained to us, using a GAP computation, one can show that $\dim(\mathcal{O}_{N_5}) = 100$. In fact, this nilpotent orbit has Dynkin representative

$$[0, 0, 0, 0, 2, 0, 0]$$

and is denoted by $A_3 + A_2 + A_1$ (see page 10 of [dG10]). Thus ∇_5 is not rigid. On the other hand, the table in [dG10] shows that in fact $\dim(\mathcal{O}_{N_7}) = 84$, as required. This is the orbit $A_2 + 3A_1$ with Dynkin representative

$$[2, 0, 0, 0, 0, 0, 0].$$

Thus, ∇_7 is rigid.

4.4.5. *Type E_8 .* In this case, $h = 30$; thus, $r \in \{7, 11, 13, 17, 19, 23, 29\}$. For ∇_r to be rigid, the dimension of the orbit containing N_r must be 192, 160, 144, 112, 96, 64, 16, respectively. Amongst these, only 192 and 112 are dimensions of some nilpotent orbits. However, one can check [dG10] that the dimension of $\dim(\mathcal{O}_{N_7}) = 196$ and $\dim(\mathcal{O}_{N_{17}}) = 128$. Thus, the ∇_r 's are not rigid.

4.5. Global differential Galois group. In §3.5, we gave a description of the local differential Galois group I_0 of $\nabla_{r,m}$. In this subsection, we discuss the global differential Galois group I . It is easy to see that the proof of [FG09, Proposition 8] generalises to our setting to show that I is reductive. Moreover, I contains I_0 and I_∞ as subgroups. Recall that I_∞ is generated by the monodromy $\exp(N_r)$ if $m = 0$ and is trivial otherwise. In the case of ∇_1 , the monodromy is the principal unipotent class. This puts a severe restriction on the reductive group I and leads to the classification of I in each type [FG09, §13]. However, when $r > 1$, the unipotent monodromy $\exp(N_r)$ of ∇_r is not the principal class; thus, we do not have such a strong restriction on the type of the reductive group I . For instance, as W. de Graaf explained to us, there are at least 46 semisimple subalgebras of E_7 of different types containing the nilpotent element N_7 .

Nevertheless, we expect that the global differential Galois groups of the $\nabla_{r,m}$'s are “large”, i.e., almost equal to G . Our expectation is motivated by the following computation of the differential Galois groups of the GL_n -analogue of these connections. In type A_{n-1} , consider the connection $\text{std}_{\nabla_{r,m}}$ associated to the standard representation. Note that this is just the trivial vector bundle of rank n on \mathbb{G}_m equipped with the connection $\nabla_{r,m}$.

Proposition 25. *The differential Galois group of $\text{std}_{\nabla_{r,m}}$ is equal to $SL_n(\mathbb{C})$ if n is odd and to $Sp_n(\mathbb{C})$ if n is even.*

Proof. Let I denote the differential Galois group of $\text{std}_{\nabla_{r,m}}$. Let I° denote the identity component of I and $I^{\circ'}$ the derived subgroup of I° . The connection $\nabla_{r,m}$ is irreducible as its restriction to 0 is irreducible. Moreover, as $\gcd(r, n) = 1$ and the irregularity is an integer, the slope $m + \frac{r}{n}$ in the slope decomposition at the irregular point must appear with multiplicity n . Therefore, we are in a position to apply Theorem 2.8.1 of [Kat90]¹¹ to conclude that $I^{\circ'}$ equals SL_n if n is odd, and it equals one of SL_n , Sp_n or SO_n if n is even.

Next, observe that the defining matrix $t^{-m}(N_r + t^{-1}E_r)$ is in $\mathfrak{sl}_n(\mathbb{C}[t, t^{-1}])$. Thus, by [vdPS03, Proposition 1.31], we have $I \subseteq SL_n$. Now for odd n , the previous paragraph implies that $I \supseteq SL_n$ and so $I = SL_n$. For even n , the nontrivial pinned automorphism σ of \mathfrak{sl}_n fixes N_r and E_r . Thus, I is a subgroup of $SL_n^\sigma = Sp_n$. In view of the previous paragraph, we have $I = Sp_n$. \square

¹¹This result depends on Gabber’s “torus trick” [Kat90, §1], which appears to be a type A phenomenon.

Remark 26. The Coxeter connection $\nabla_{1,1}$ associated to $G = G_2$ appears in [Kat90, §2.10.6].¹² Indeed, a computation shows that the realisation of $\nabla_{1,1}$ in the seven-dimensional representation of G_2 is given by the differential equation

$$(15) \quad \partial^7 - 2t\partial - t,$$

where $\partial = t \frac{d}{dt}$. Moreover, Katz proved that the global differential Galois of the above differential equation is G_2 .

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