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INFINITE GRAPHS AND BICIRCULAR MATROIDS

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B-matroids are a class of pre-independence spaces which retain many important properties of independence spaces. Higgs has shown that an infinite generalization of the cycle matroid of a finite graph which admits two-way infinite paths as circuits need not be a B-matroid. In this note it is shown that a similar generalization of the finite bicircular matroid is a stays a B-matroid.

1. Infinite matroids and graphs

The operator approach to infinite matroids has been examined by Klee [5] and Higgs [1–3] and the link between their investigations is discussed in [7]. As far as possible Klee's terminology will be followed here and, where necessary, results of Higgs will be restated.

Let $S$ be a non-empty set and $f$ be a function from $2^S$, the power set of $S$, into $2^S$. Then $f$ is an operator on $S$ if and only if the following conditions are satisfied for all $X \subseteq Y \subseteq S$.

(i) $X \subseteq f(X)$.

(ii) $f(X) \subseteq f(Y)$.

If $f$ is an operator on $S$, define a function $f^*$ by $f^*(X) = X \cup \{x : x \notin f(S\setminus(X \cup x))\}$ for all $X \subseteq S$, where a one-element subset $\{x\}$ of $S$ is denoted $x$. Then $f^*$ is an operator on $S$ called the dual of $f$, and $(f^*)^* = f$. If $T \subseteq S$ and $f_T$ is defined by $f_T(X) = f(X) \cap T$ for all $X \subseteq T$, then $f_T$ is an operator on $T$ called the restriction of $f$ to $T$.

The closure operator of a finite matroid is an operator in the above sense. Such operators satisfy certain additional conditions including the following, which hold for all elements $p, x$ of $S$ and all subsets $X, Y$ of $S$:

(w1) If $x \in f(Y)$, then $f(x \cup Y) = f(Y)$.

(l) If $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$.

(E) If $p \in f(Y)$ and $p \notin f(Y \setminus X)$, then $x \in f((Y \setminus x) \cup p)$ for some $x$ in $X$.

Any operator satisfying (w1) and (E) will be called simply a wIE-operator. Similar abbreviations will be used for other types of operators.

Note that (l) implies (w1) and that (l) is equivalent to the condition that
\( f(f(Y)) = f(Y) \). Moreover, (I) and (E,) form a dual pair in the sense that \( f \) satisfies (I) if and only if \( f^* \) satisfies (E).

Again, for \( f \) an operator on \( S \) and \( X \) a subset of \( S \), \( X \) is \( f \)-independent provided that, for all \( x \in X \), \( x \notin f(X \setminus x) \); \( X \) is \( f \)-spanning if \( f(X) = S \) and \( X \) is an \( f \)-base if \( X \) is both \( f \)-independent and \( f \)-spanning. An \( f \)-circuit is a minimal \( f \)-dependent subset of \( S \). The operator prefix in the above terms will be dropped where ambiguity will not arise.

A \( B \)-matroid, \((S, f)\), is an IE-operator \( f \) on a non-empty set \( S \) such that, for all \( T \subseteq S \), every \( f \)-independent subset of \( T \) is contained in an \( f_T \)-base. If \((S, f)\) is a \( B \)-matroid, then so is \((S, f^*)\), the dual of \((S, f)\) \([1]\). Although \( B \)-matroids need not have finite circuits, they retain many familiar properties of independence spaces (see \([1]\) and \([2]\)).

This note will be concerned with \( wIE \)-operators satisfying the further condition:

\( (C) \) If \( p \in f(Y) \) and \( p \notin Y \), then there is a circuit \( X \) contained in \( Y \cup p \) such that \( p \in X \).

From \([1]\), if \((S, f)\) is a \( B \)-matroid, then \( f \) is a \( wIE \)-operator on \( S \). The next result \([5]\) characterizes \( wIE \)-operators in terms of their collections of circuits. This should be compared with a similar, more familiar result for independence spaces.

**Theorem 1.1.** The collection \( \Gamma \) of circuits of a \( wIE \)-operator satisfies the following conditions.

1. No element of \( \Gamma \) properly contains another.
2. If \( C_1 \) and \( C_2 \) are in \( \Gamma \), \( p \in C_1 \setminus C_2 \) and \( q \in C_1 \cap C_2 \), then there is an element \( C_3 \) of \( \Gamma \) such that \( p \in C_3 \subseteq (C_1 \cup C_2) \setminus q \).

Conversely, if \( \Gamma \) is a collection of subsets of \( S \) satisfying (1.1) and (1.2), then there is a unique \( wIE \)-operator on \( S \) having \( \Gamma \) as its collection of circuits. This operator is given by

\[ \gamma_\Gamma(X) = X \cup \{x : x \in C \subseteq X \cup x \text{ for some } C \in \Gamma\} \text{ for all } X \subseteq S. \]

If, in the above, every element of \( \Gamma \) is finite, then the collection \( \mathscr{J} \) of \( \gamma_\Gamma \)-independent subsets of \( S \) is an independence structure on \( S \). Moreover, for all subsets \( X \) of \( S \), \( \gamma_\Gamma(X) \) is the closure or span (see, for example, \([4]\)) of \( X \) in \((S, \mathscr{J})\).

Let \( G \) be an infinite undirected graph which we allow to have loops and parallel edges. We may use Theorem 1.1 to define four \( wIE \)-operators on the edge set \( E(G) \) of \( G \).

**Example 1.2.** From the set \( \mathcal{M}(G) \) of finite cycles of \( G \), we obtain a \( wIE \)-operator \( \gamma_{\mathcal{M}(G)} \) on \( E(G) \), and, as above, an independence structure — the familiar cycle matroid of an infinite graph.

**Example 1.3.** From the set of subgraphs of \( G \) homeomorphic to one of the graphs (a), (b) or (c) in Fig. 1, we obtain an independence structure on \( E(G) \). This is an obvious extension of the bicircular matroid of a finite graph (see \([8]\) or \([6]\)).

The next two examples generalize the first two by admitting infinite circuits.
Example 1.4. (see Klee [5] or Higgs [3]). If $C(G)$ consists of all finite cycles of $G$ together with all two-way infinite paths, then $\gamma_{C(G)}$ is a wIEC-operator on $E(G)$.

Example 1.5. (see Klee [5]). Let $B(G)$ be the collection of subgraphs of $G$ homeomorphic to one of the five graphs shown in Fig. 1 (where an arrow denotes a one-way infinite path). Then $\gamma_{B(G)}$ is a wIEC-operator on $E(G)$ and $(E(G), \gamma_{B(G)})$ will be called the infinite-bicircular matroid of $G$.

![Fig. 1.](image_url)

Higgs [3] has given an excluded subgraph characterization of graphs $G$ for which $(E(G), \gamma_{C(G)})$ is a $B$-matroid, showing further that $(E(G), \gamma_{C(G)})$ is a $B$-matroid if and only if $\gamma_{C(G)}$ is an IE-operator. These results prompt the questions as to when $\gamma_{B(G)}$ is an IE-operator and when $(E(G), \gamma_{B(G)})$ is a $B$-matroid. The main result of this note answers these questions. (A similar problem was posed by Simões-Pereira [9].)

**Theorem 1.6.** Let $G$ be a graph. Then the infinite-bicircular matroid $(E(G), \gamma_{B(G)})$ is a $B$-matroid.

### 2. Infinite-bicircular matroids and $B$-matroids

The proof of Theorem 1.6 will use two lemmas.

**Lemma 2.1.** Let $G$ be a graph. Then $\gamma_{B(G)}$ is an IE-operator on $E(G)$.

**Proof.** We need only show that $\gamma_{B(G)}$ satisfies (I). In this proof we shall write $\gamma$ for $\gamma_{B(G)}$ and $B$ for $B(G)$.

Suppose that $X \subseteq E(G)$ and that $x \in \gamma(\gamma(X)) \setminus \gamma(X)$. Then $x \in C \subseteq \gamma(X) \cup x$ for some $C$ in $B$. We distinguish two cases: when $C \cap (\gamma(X) \setminus X)$ is finite, and when $C \cap (\gamma(X) \setminus X)$ is infinite. In each case we shall get a contradiction, thereby showing that $\gamma(\gamma(X)) \setminus \gamma(X)$ is empty and hence that $\gamma(\gamma(X)) = \gamma(X)$.

If $C \cap (\gamma(X) \setminus X)$ is finite and $y \in C \cap (\gamma(X) \setminus X)$, then, since $y \in \gamma(X) \setminus X$, there is an element $C_y$ of $B$ such that $y \in C_y \subseteq X \cup y$. Now $y \in C \cap C_y$ and $x \in C \setminus C_y$. 


hence, by (1.2), there is an element $C'$ of $B$ such that $x \in C' \subseteq (C \cup C') \setminus y$. Since $|C \cap (\gamma(X) \setminus X)| < |C \cap (\gamma(X) \setminus X)|$, after finitely many steps we get an element $C''$ of $B$ such that $x \in C'' \subseteq X \cup x$. Therefore, $x \in \gamma(X)$; a contradiction.

Next suppose that $C \cap (\gamma(X) \setminus X)$ is infinite. Then $C$ is homeomorphic to one of the graphs (d) or (e) in Fig. 1. Let the edge $x$ have endpoints $u$ and $v$. Then either there is an edge $y$ in $C \cap (\gamma(X) \setminus X)$ and a finite path $P_y$ in $X \cap C$ joining $u$ to an endpoint of $y$, or there is no such path in $X \cap C$ from $u$ to an edge of $C \cap (\gamma(X) \setminus X)$. In the latter case, $X \cap C$ contains either a one-way infinite path from $u$, or a cycle joined to $u$ by a path of finite (possibly zero) length. In the former case, since $y \in \gamma(X) \setminus X$, there is an element $C' \subseteq B$ such that $y \in C' \subseteq X \cup y$. But both $P_y$ and $C' \setminus y$ are contained in $X$. A routine check of the five possibilities for $C'$ shows that this time $X$ contains either a one-way infinite path from $u$, or a cycle joined to $u$ by a finite path in $X$.

The argument for $u$ may be repeated for $v$ and then a check of the various possibilities (including the case $u = v$) gives that $B$ contains an element $C$ containing $x$ such that $C \subseteq X \cup x$. Thus, when $C \cap (\gamma(X) \setminus X)$ is infinite, $x \in \gamma(X)$; the required contradiction.

The following result is proved by Higgs [3.(5)(i)].

**Lemma 2.2.** Let $G$ be a graph. Then every $\gamma_{\alpha(G)}$-independent set is contained in a $\gamma_{\alpha(G)}$-base.

We now complete the proof of the main theorem.

**Proof of Theorem 1.6.** By Lemma 2.1, $\gamma_{\alpha(G)}$ is an $iE$-operator. Since every restriction of $(E(G), \gamma_{\alpha(G)})$ is isomorphic to $(E(F), \gamma_{\alpha(F)})$ for some subgraph $F$ of $G$, it suffices to show that every independent set of $(E(G), \gamma_{\alpha(G)})$ is contained in a base.

Suppose that $A$ is a $\gamma_{\alpha(G)}$-independent subset of $E(G)$ and let $\{A_k : k \in K\}$ be the set of connected components of the subgraph of $G$ induced by $A$. Let

$$H = \{k \in K : A_k \text{ contains a cycle of } G\}.$$

Modify $G$ to obtain a graph $G'$ as follows. For each $h$ in $H$, using the Axiom of Choice, select an edge $e_h$ of the cycle contained in $A_k$; delete the edge $e_h$ and insert a one-way infinite path $P_h$ (of new vertices and edges) from an endpoint of $e_h$. Let

$$A' = \left( \bigcup_{h \in H} A'_k \right) \cup \left( \bigcup_{k \in K \setminus H} A_k \right).$$

Now apply Lemma 2.2 in $G'$. As $A'$ is a $\gamma_{\alpha(G')}$-independent set, there is a $\gamma_{\alpha(G')}$-base $T'$ containing $A'$. Let $T$ be the subgraph of $G$ corresponding to $T'$, that is, $T$ is
obtained from $T'$ by deleting the path $P_h$ and inserting the edge $e_h$ for each $h$ in $H$.

By construction, $T$ is $\gamma_{\mathcal{G}}$-independent, and moreover, if $e$ is an edge of $G$ joining vertices in distinct components of the subgraph $T$, then $T \cup e$ is $\gamma_{\mathcal{G}}$-dependent. Let $\{T_i : i \in I\}$ be the set of components of $T$ and $J = \{i \in I : T_i$ contains no finite cycle or one-way infinite path$\}$. Now recall that $\gamma_{\mathcal{G}}(T_i)$ is the closure of $T_i$ in the cycle matroid of $G$ (see Example 1.2). If $j \in J$ and $\gamma_{\mathcal{G}}(T_j) \setminus T_j$ is non-empty, then a single element from $\gamma_{\mathcal{G}}(T_j) \setminus T_j$ may be added to $T_j$ without forming a $\gamma_{\mathcal{G}}$-circuit. For each such $j$, using the Axiom of Choice, select such an element $f_j$. Finally let

$$B = T \cup \{f_j : j \in J \text{ and } \gamma_{\mathcal{G}}(T_j) \setminus T_j \text{ is non-empty}\}.$$ 

It is easy to check that $B$ is $\gamma_{\mathcal{G}}$-independent and $\gamma_{\mathcal{G}}$-spanning; hence, as required, $B$ is a $\gamma_{\mathcal{G}}$-base containing $A$.

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