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On Circuit Exchange Properties for Matroids

JAMES G. OXLEY*

Fournier has characterized binary materials in terms of a certain circuit exchange property. This paper examines several related circuit exchange properties and gives excluded-minor characterizations of the matroids having these properties.

1. INTRODUCTION

Fournier [3] has shown that a matroid M is binary if and only if, whenever C_1 and C_2 are distinct circuits of M and x and y are elements of $C_1 \cap C_2$, the set $(C_1 \cup C_2) - \{x, y\}$ contains a circuit of M. In this paper, we extend this result in several directions. The following general property is useful in relating these extensions.

Let n and k be positive integers with $n \geqslant 2$. A matroid M has the *(n, k)-exchange property for circuits*, if, whenever $C_1, C_2, \ldots, C_n$ are distinct circuits of M and $x_1, x_2, \ldots, x_k$ are distinct elements of $C_1 \cap C_2 \cap \ldots \cap C_n$, there is a circuit contained in $(C_1 \cup C_2 \cup \ldots \cup C_n) - \{x_1, x_2, \ldots, x_k\}$. It is straightforward to check that if M has the (n, k)-exchange property, then:

(1.1) *Every minor of M has the (n, k)-exchange property*;
and
(1.2) *M has the (n', k)-exchange property for all $n' \geqslant n$.*

In this new terminology, the usual circuit exchange axiom is just the (2, 1)-exchange property, so by (1.2), all matroids satisfy the $(n, 1)$-exchange property for all n. For all but the last section, this paper will be exclusively concerned with when one of n and k is 2. Fournier's result characterizes binary matroids as those matroids with the (2, 2)-exchange property. In Section 2 of this paper, we give excluded minor characterizations of the matroids having the $(2, k)$-exchange property for $k = 3, 4, 5$. The motivation here is to determine how far one can get from the class of binary matroids by using successively weaker exchange properties. In Section 3 we discuss the conjecture that, if $k \geqslant 2$, a matroid with the $(2, k)$-exchange property has the $(2, k + t)$-exchange property for all $t \geqslant 0$. In Section 4 we turn attention to when k rather than n is 2 and characterize those matroids with the $(n, 2)$-exchange property. The last section uses a weakening of the (3, 3)-exchange property to give a new characterization of when a 3-connected matroid is binary.

The matroid terminology used here will, in general, follow Welsh [5]. The ground set and rank of a matroid M will be denoted by $E(M)$ and $\operatorname{rk} M$ respectively. If $T \subseteq E(M)$, then $\operatorname{rk} T$ will denote the rank of T. The deletion and contraction of T from M will be denoted by $M \backslash T$ and M/T respectively. Flats of M of ranks one and two will be called *points* and *lines*. A *coline* of M is a flat whose rank equals $\operatorname{rk} M - 2$. The matroid N is a *single-element extension* of M if N has an element e such that $N \backslash e = M$. If, instead, $N/e = M$, then N is a *lift* of M. The matroid M_1 is an *upper minor* of M if, for some subsets U and T of $E(M)$ such that $\operatorname{rk}(M/U) = \operatorname{rk} M_1$, we have $M_1 \simeq M/U \backslash T$. The following fundamental result will be used frequently throughout this paper.

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THEOREM 1.3 (THE SCUM THEOREM) [2, Theorem 9.4]. *If a matroid M has a minor isomorphic to a matroid M_1, then M has an upper minor isomorphic to M_1.*

2. THE MATROIDS WITH THE (2, k)-EXCHANGE PROPERTY

In this section we prove characterizations of those matroids with the (2, k)-exchange property when $k = 3$ or 4 and state the corresponding result for $k = 5$. In addition, we note that a pattern established by these three results does not continue for all k. Since a matroid has the (n, k)-exchange property if and only if each of its connected components has this property, it suffices to characterize those matroids with the (n, k)-exchange property which are connected. We now distinguish a class of lines which will occur frequently throughout this section. For $m \geqslant 4$, a *special m-element line* is an m-element line having no loops and at least four points, at least two of which contain just one element. There are precisely two special five-element lines, $L_{5,1}$ and $L_{5,2}$, the first of which has five points.

THEOREM 2.1. *The following statements are equivalent for a connected matroid M:*
(*i*) *M has the* (2, 3)*-exchange property*;
(*ii*) *M^* has no minor isomorphic to $L_{5,1}$ or $L_{5,2}$*;
(*iii*) *M is binary or is a line with at least four points.*

The proof of this theorem uses the following result, the routine proof of which is omitted.

LEMMA 2.2. *If M^* has a special* ($k + 2$)*-element line as a minor, then M does not have the* (2, k)*-exchange property.* □

PROOF OF THEOREM 2.1. The fact that (i) implies (ii) is an immediate consequence of the preceding lemma. To prove that (ii) implies (iii), assume that no minor of M^* is isomorphic to $L_{5,1}$ or $L_{5,2}$. Then either (iii) holds or M^* is non-binary. In the latter case, by (1.3) and the excluded-minor characterization of binary matroids, $U_{2,4} \simeq M\backslash X/Y$ for some subsets X and Y of $E(M)$, where Y is independent, having $\operatorname{rk} M - 2$ elements. If Y is empty, then (iii) is immediate, so assume that Y contains an element y. Then $M\backslash X/(Y - y)$ has rank three and is a lift of $U_{2,4}$. Thus $M\backslash X/(Y - y)$ is isomorphic to $L^*_{5,1}$, $L^*_{5,2}$ or $U_{2,4} \oplus U_{1,1}$. But, in the first two cases, we obtain the contradiction that M^* has either $L_{5,1}$ or $L_{5,2}$ as a minor. Therefore y is a coloop of $M\backslash X/(Y - y)$ and hence of $M\backslash X$. Thus $M\backslash X \simeq U_{2,4} \oplus U_{|Y|,|Y|}$. As M is connected having the same rank as $M\backslash X$, there are an element x of X and a circuit C of $M\backslash(X - x)$ so that $x \in C$ and C meets both Y and $E(M) - (X \cup Y)$. Now delete $Y - C$ from $M\backslash(X - x)$. Then choose an element z from $C \cap Y$ and consider $M\backslash[(X - x) \cup (Y - C)/[(C \cap Y) - z]$. This rank-three matroid has $\{x, z\}$ as a cocircuit and $U_{2,4}$ as the corresponding hyperplane. Hence it has $L^*_{5,2}$ as a restriction. We conclude that M has $L^*_{5,2}$ as a minor, and this contradiction completes the proof that (ii) implies (iii). Finally, it is straightforward to check that (iii) implies (i). □

The next two theorems characterize those connected matroids having the (2, 4)-exchange property. It is not difficult to check that there are exactly four special six-element lines. The following lemma will be used in the proof of the next theorem. We omit its straightforward proof.

LEMMA 2.3. *If M is a matroid which does not satisfy the* (2, k)*-exchange property for circuits, then M has a minor N and a set $\{x_1, x_2, \ldots, x_k\}$ of k distinct elements such that N has two distinct spanning circuits C_1 and C_2 for which $C_1 \cup C_2 = E(N)$, $C_1 \cap C_2 = \{x_1, x_2, \ldots, x_k\}$, and $C_1 \triangle C_2$ is independent.* □

THEOREM 2.4. *Let M be a connected matroid. Then M has the* (2,4)*-exchange property for circuits if and only if M^* has no minor isomorphic to a special six-element line.*

PROOF. By Lemma 2.2, if M^* has a minor isomorphic to a special six-element line, then M does not satisfy the (2, 4)-exchange property.

Now suppose that M fails to satisfy the (2, 4)-exchange property. Then, by Lemma 2.3, M has a minor N and a set $\{x_1, x_2, x_3, x_4\}$ of four distinct elements such that N has two distinct spanning circuits C_1 and C_2 for which $C_1 \triangle C_2$ is independent, $C_1 \cup C_2 = E(N)$ and $C_1 \cap C_2 = \{x_1, x_2, x_3, x_4\}$. Let $\operatorname{rk} N^* = r$, $H_1 = E(N) - C_1$ and $H_2 = E(N) - C_2$. Then H_1 and H_2 are disjoint independent hyperplanes of N^*. Moreover, as $C_1 \triangle C_2$ is independent in N, $\{x_1, x_2, x_3, x_4\}$ spans N^* and so $2 \leqslant r \leqslant 4$. If $r = 2$, it is easy to check that N^* is isomorphic to a special six-element line.

Now let $H_1 = \{a_1, a_2, \ldots, a_{r-1}\}$ and $H_2 = \{b_1, b_2, \ldots, b_{r-1}\}$. Then, if $r = 3$, $E(N^*) = \{a_1, a_2, b_1, b_2, x_1, x_2, x_3, x_4\}$ and $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are disjoint lines of N^*. As $\{x_1, x_2, x_3, x_4\}$ spans N^*, we may assume that $\{x_1, x_2, x_3\}$ is a basis of N^*. Consider N^*/a_1. This has $\{a_2\}$ as a rank-one flat. If it has at least three other rank-one flats, then as $N^*/a_1\backslash a_2$ has six elements, it follows easily that N^*/a_1 has a special six-element line as a minor. Thus we may assume that each of N^*/a_1, N^*/a_2, N^*/b_1 and N^*/b_2 has at most three rank-one flats. Then, from considering N^*/a_1, we can assume, without loss of generality, that $N^* | \{a_1, x_1, x_2, x_3, b_1, b_2\}$ is as shown in Figure 1. But now N^*/b_2 has at least four rank-one flats; a contradiction. We conclude that $r \neq 3$.

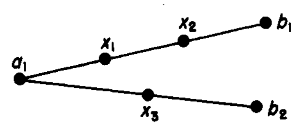

FIGURE 1

If $r = 4$, then $E(N^*) = \{a_1, a_2, a_3, b_1, b_2, b_3, x_1, x_2, x_3, x_4\}$. As N^* is not binary, it has a coline X such that N^*/X is a line with at least four points. If $|E(N^*/X)| \geqslant 6$, then N^*/X has a special six-element line as a restriction. Hence we may assume that $|E(N^*/X)| \leqslant 5$ and so $|X| \geqslant 5$. As N^* has rank four, X is a line of N^*. Because $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are independent hyperplanes of N^*, X contains at most one element of each of these hyperplanes. Moreover, since $\{x_1, x_2, x_3, x_4\}$ is a basis of N^*, X contains at most two elements of $\{x_1, x_2, x_3, x_4\}$. Thus, $|X| \leqslant 4$, a contradiction to the fact that $|X| \geqslant 5$. □

The next theorem uses the operation of parallel connection [1] to give another characterization of the matroids with the (2, 4)-exchange property. We have omitted a number of details of the proof.

THEOREM 2.5. *A connected matroid M has the* (2, 4)*-exchange property if and only if either*:

(*i*) *M is binary; or*

(*ii*) $\operatorname{rk} M \leqslant 3$; *or*

(*iii*) *M is isomorphic to P_n or $P_n\backslash p$, where P_n is the parallel connection of n lines each having at least three points, and p is the basepoint of the parallel connection.*

PROOF. It is routine to check that a connected matroid satisfying (i), (ii) or (iii) has the (2, 4)-exchange property. Now, suppose that M has the (2, 4)-exchange property but that none of (i), (ii) or (iii) occurs. Using Theorems 2.1 and 2.4 and the Scum Theorem, it can be shown that M has a restriction N that is isomorphic to $L^*_{5,1}$ or $L^*_{5,2}$. Let $E(N) = \{1, 2, 3, 4, 5\}$, and $\{1, 2, 3\}$ be a basis for N. In addition, let F be the flat of M spanned by

$E(N)$ and 6 be an arbitrary element of $E(M) - F$. M has a circuit C containing 6 and meeting F. Choose C so that $|C \cap (F - E(N))|$ is minimal. Then it can be shown that $|C \cap (F - E(N))| = 0$. We now suppose that, subject to the other restrictions, $|C - F|$ is minimal. As F is a flat of M, $|C - F| \geq 2$. Choose an element 7 from $C - F - 6$ and let $M' = M/(C - F - \{6, 7\})$. Consider $M' | \{1, 2, \ldots, 7\}$. As $N \simeq L^*_{5,1}$ or $L^*_{5,2}$ but M^* does not have a special six-element line as a minor, it is routine to check that $M' | \{1, 2, \ldots, 7\} \simeq P(L_1, L_2, L_3)\backslash p$, where L_2 and L_3 are simple three-point lines and L_1 is a simple four-point line (see Figure 2, noting that the basepoint p of the parallel connection has also been marked).

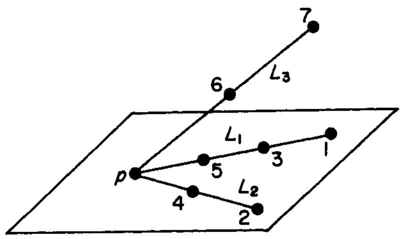

FIGURE 2

Now let F_1 be the flat of M' spanned by $\{1, 2, \ldots, 7\}$. Then it can be shown that $F_1 \subseteq L_1 \cup L_2 \cup L_3$ and $C - F = \{6, 7\}$. It follows from the latter that $M' = M$. Since the element 6 was arbitrarily chosen in $E(M) - F$, for every element e of $E(M) - F$, M has an element f such that $M | \{1, 2, \ldots, 5, e, f\} \simeq M | \{1, 2, \ldots, 7\}$. The final step of the argument is to establish the contradiction that M is isomorphic to P_n or $P_n\backslash p$, where P_n is as defined in (iii), and this is not difficult. □

The proof of the next result is similar to the proof of Theorem 2.4 although the case-by-case analysis is considerably longer. We omit the tedious details.

THEOREM 2.6. *A matroid M has the* (2, 5)*-exchange property for circuits if and only if no minor of M^* is isomorphic to a special seven-element line.* □

Theorems 2.1, 2.4 and 2.6 suggest the result that for all $k \geq 3$, a matroid M has the $(2, k)$-exchange property if and only if no minor of M^* is isomorphic to a special $(k + 2)$-element line. To see that this is not true in general, let M_q be the matroid that is obtained as follows. Take two distinct lines L_1 and L_2 of $PG(2, q)$ and let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be subsets of $L_1 - L_2$ and $L_2 - L_1$ respectively. We obtain M_q from $PG(2, q)$ by deleting the elements of $(L_1 \cup L_2) - \{a_1, a_2, b_1, b_2\}$. In M_q^*, the sets $C_1 = E(M_q) - \{a_1, a_2\}$ and $C_2 = E(M_q) - \{b_1, b_2\}$ are circuits having exactly $q^2 - q$ common elements. But, provided $q \geq 3$, $C_1 \triangle C_2$ is independent in M_q^*. Thus M_q^* does not have the $(2, q^2 - q)$-exchange property. It is routine to check that for $q \geq 4$, M_q has no minor isomorphic to a special $(q^2 - q + 2)$-element line.

3. A CONJECTURE

If a matroid M has the (2, 2)-exchange property for circuits, then M is binary and so M has the $(2, k)$-exchange property for all k. Next, suppose that M has the (2, 3)-exchange property and is connected and non-binary. Then, by Theorem 2.1, M has rank at most two, so no two distinct circuits of M meet in more than two elements. It follows that a matroid having the (2, 3)-exchange property has the $(2, k)$-exchange property for all $k \geq 3$.

Similarly, from Theorem 2.5, we have the result that a matroid having the (2, 4)-exchange property has the (2, k)-exchange property for all $k \geqslant 4$. The above observations verify the cases $k = 2$, 3 and 4 in the following:

CONJECTURE 3.1. Suppose that $k \geqslant 2$. If a matroid M has the (2, k)-exchange property for circuits, then M has the (2, $k + t$)-exchange property for all $t \geqslant 1$.

The next theorem establishes this conjecture for $k = 5$. We omit the relatively long proof since it does not appear to shed any light on the conjecture for larger values of k.

THEOREM 3.2. *If a matroid M has the* (2, 5)*-exchange property for circuits, then M has the* (2, h)*-exchange property for all $h \geqslant 5$.* □

4. Uniform Matroids of Corank Two

Fournier's proof that a matroid M is binary if and only if it satisfies the (2, 2)-exchange property showed that the latter is true if and only if M does not have $U_{2,4}$ as an upper minor. It is not difficult to generalize this proof to give the following circuit-exchange characterization of the matroids with no $U_{n,n+2}$-minor. We omit the details.

THEOREM 4.1. *A matroid has the* (n, 2)*-exchange property for circuits if and only if it has no minor isomorphic to $U_{n,n+2}$.* □

5. Another Exchange Property

The class of 3-connected matroids has been the focus of much recent work. In this section we use a weakened form of the (3, 3)-exchange property to give a new characterization of the binary members of this class.

The matroid M has the *weak* (3, 3)*-exchange property for circuits* if, whenever C_1, C_2, C_3 are circuits of M, none of which is contained in the union of the others, and x_1, x_2, x_3 are distinct elements of $C_1 \cap C_2 \cap C_3$, there is a circuit contained in $(C_1 \cup C_2 \cup C_3) - \{x_1, x_2, x_3\}$. The following result characterizes the matroids with this property. We omit the relatively straightforward proof. The matroids $\mathcal{W}^3$, P_6 and Q_6 are shown in Figure 3.

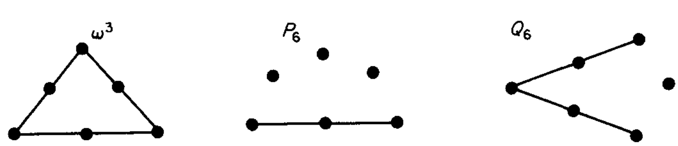

FIGURE 3

THEOREM 5.1. *A matroid has the weak* (3, 3)*-exchange property for circuits if and only if it has no minor isomorphic to any of $U_{3,6}$, $\mathcal{W}^3$, P_6 or Q_6.* □

The same four matroids that are excluded here arise in the following result.

THEOREM 5.2 [4, Theorem 3.1]. *A* 3*-connected matroid having rank and corank at least* 3 *is non-binary if and only if it has a minor isomorphic to one of $U_{3,6}$, $\mathcal{W}^3$, P_6 or Q_6.* □

Evidently, a 3-connected matroid of rank or corank two is binary if and only if it has three elements. On combining the last two theorems we get the following circuit-exchange characterization of the binary 3-connected matroids whose rank and corank both exceed two.

COROLLARY 5.3. *A 3-connected matroid having rank and corank at least* 3 *is binary if and only if it has the weak* (3, 3)*-exchange property for circuits.* □

REFERENCES

1. T. Brylawski, A combinatorial model for series-parallel networks, *Trans. Amer. Math. Soc.* **154** (1971), 1–22.
2. H. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries* (Preliminary edn), M.I.T. Press, Cambridge, Mass., 1970.
3. J.-C. Fournier, A characterization of binary geometries by a double elimination axiom, *J. Comb. Theory, Ser. B* **31** (1981), 249–250.
4. J. G. Oxley, On nonbinary 3-connected matroids, *Trans. Amer. Math. Soc.* **300** (1987), 663–679.
5. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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