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Connectivity of submodular functions

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Abstract

The notion of connectivity for submodular functions was introduced by Cunningham. This paper relates the connectivity of such a function f to that of certain submodular functions which are derived from f. In particular, we prove a generalisation of the well-known matroid result that, for every element x of a connected matroid M, either the deletion or contraction of x from M is connected.

1. Introduction

Submodular functions arise in a variety of combinatorial contexts, both explicitly as in [3, 6] and implicitly as in [4, 5, 8]. Moreover, as Lovász [6, p. 241] has noted, submodular functions ‘have sufficient structure so that a mathematically beautiful and practically useful theory can be developed’ for them. The purpose of this paper is to continue the development of this theory.

We shall focus here on the connectivity function of a submodular function as defined by Cunningham [3]. In particular, we shall investigate the behaviour of connectivity under the operations of deletion, contraction, and the formation of duals. These operations generalise the corresponding operations for matroids. However, the operation of contraction for submodular functions does not correspond to contraction in graphs. In this paper, we shall introduce a new operation on submodular functions, one which does correspond to contraction in
graphs. The properties of this operation and its dual operation will be investigated and we shall determine the effect of applying these operations to the rank function of a matroid. In each case, the matroid produced coincides with that obtained from a known matroid operation. The effect of these new operations on the connectivity of a submodular function will also be discussed.

The main results of this paper, Theorems 3.1 and 3.8, are stated and proved in Section 3. In Section 2, we present various preliminaries for submodular functions that will be used in proving these results. Most of our matroid terminology will follow Welsh [10] or White [11]. If $M$ is a matroid on the set $S$, we shall denote its rank function by $r$ or $r_M$, and its closure operator by $\text{cl}_M$. If $A \subseteq S$, then $M \setminus A$ and $M/A$ will denote the deletion and contraction of $A$ from $M$. Now suppose that $r_M(A) \geq 1$. Then the principal truncation, $T_A(M)$, of $M$ by $A$ is the matroid on $S$ whose rank function $r'$ is defined, for all subsets $X$ of $S$, by

$$r'(X) = \begin{cases} r_M(X) - 1 & \text{if } r_M(X \cup A) = r_M(X); \\ r_M(X) & \text{otherwise.} \end{cases}$$

Geometrically, $T_A(M)$ can be formed by freely placing a point $p$ on $\text{cl}_M(A)$ and then contracting $p$ from the extended matroid. It follows from this that $T_A(M) = T_{\text{cl}_M(A)}(M)$. We have, in fact, extended the usual definition [1] of principal truncation here by not requiring $A$ to be closed. Further properties of this operation and of the following modification of it may be found in [1, Section 4]. The complete principal truncation, $\bar{T}_A(M)$, of $M$ by $A$ is formed by freely placing $r_M(A) - 1$ independent points on $\text{cl}_M(A)$ and then contracting these points from the extended matroid. Both the principal truncation and the complete principal truncation will arise in Section 3 in connection with the new operations we shall introduce for submodular functions.

2. Submodular functions

An integer-valued set function is a function from the power set of a set into the integers. If $f : 2^S \to \mathbb{Z}$ is such a function, then the ground set of $f$ is $S$, and we say that $f$ is a function on $S$. All ground sets are assumed to be finite.

If $f$ is a function on $S$, then the dual of $f$ is the function $f^*$ on $S$ defined, for all subsets $A$ of $S$, by

$$f^*(A) = |A| + f(S - A) - f(S) + f(\emptyset).$$

It is straightforward to check that $f^{**} = f$ and that $f^*(\emptyset) = f(\emptyset)$.

If $A$ is a subset of $S$, then the deletion of $A$ from $f$, denoted $f \setminus A$, is the restriction of $f$ to the power set of $S - A$; that is, $f \setminus A(B) = f(B)$ for all subsets $B$ of $S - A$. The contraction of $A$ from $f$, denoted $f/A$, is the function on $S - A$ defined by $f/A = (f\setminus A)^*$.  

Proposition 2.1. If $B$ is a subset of $S - A$, then $f/A(B) = f(A \cup B) - f(A) + f(\emptyset)$.

The following proposition summarises certain basic properties of deletion and contraction. The routine proof is omitted.

Proposition 2.2. If $X$ and $Y$ are disjoint subsets of $S$, then:

(i) $f/X(\emptyset) = f(\emptyset) = f\setminus X(\emptyset)$,
(ii) $(f\setminus X)\setminus Y = (f\setminus Y)\setminus X$,
(iii) $(f/X)/Y = (f/Y)/X$,
(iv) $(f\setminus X)/Y = (f/Y)\setminus X$.

A function $g$ on a subset of $S$ is a minor of the function $f$ on $S$ if there are disjoint subsets $X$ and $Y$ of $S$ such that $g = (f\setminus X)/Y$. A class of functions is minor-closed if every minor of every member of the class also belongs to the class; it is closed under duality if the dual of every member of the class also belongs to the class.

A function $f$ on $S$ is submodular if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all subsets $A$ and $B$ of $S$. The class of submodular functions is minor-closed and closed under duality. The function $f$ is normalised if $f(\emptyset) = 0$, and is increasing if $f(A) \geq f(B)$ whenever $A$ and $B$ are subsets of $S$ with $A \supseteq B$. The class of normalised functions is both minor-closed and closed under duality.

For the remainder of this paper we shall require functions to be submodular, but we will not expect them to be either increasing or normalised.

The connectivity function $c_f$ of the submodular function $f$ on $S$ is defined by

$$c_f(A) = f(A) + f(S - A) - f(S) - f(\emptyset)$$

for all subsets $A$ of $S$. If $f$ is normalised, then this definition agrees with that of Cunningham [3]. The following result is a routine generalisation of [3, Proposition 4].

Proposition 2.3. Suppose that $f$ is a submodular function. Then $c_f$ is submodular and nonnegative. Moreover, for all $A \subseteq S$, $c_f(S - A) = c_f(A)$.

Again following Cunningham [3], we define $c_f$ to be the minimum value that $c_f$ takes on non-empty proper subsets of $S$ unless $|S| \leq 1$. In the exceptional case, we take $c_f$ to be $\infty$. By Proposition 2.3, $c_f \geq 0$. We say that $f$ is disconnected if $c_f = 0$. A separator of $f$ is a subset $A$ of $S$ for which $c_f(A) = 0$.

We now consider the behaviour of the connectivity function in somewhat more detail. The height $h_f$ of a submodular function $f$ on $S$ is defined by $h_f = f(S) - f(\emptyset)$. This generalises the definition of the rank of a matroid, but note that $h_f$ may well be negative for a given submodular function. Now if $A$ is a subset of
\[ c_f(A) = f(A) + f(S - A) - f(S) - f(\emptyset) \]
\[ = f \backslash A(S - A) - f / A(S - A) \]
\[ = h_f \backslash A - h_f / A. \]

Therefore \( c_f(A) \) records the difference in height between the minor of \( f \) obtained by deleting \( A \) and that obtained by contracting \( A \). It is not difficult to check that \( A \) is a separator of \( f \) if and only if \( f \backslash A = f / A \).

Despite the fact that we do not require functions to be normalised, it will usually suffice, in proofs, to consider only normalised functions. We now justify this claim.

For an integer \( k \), let \( \mathcal{F}_k \) denote the class of integer-valued set functions which take the value \( k \) on the empty set, and let \( \alpha_k : \mathcal{F}_0 \rightarrow \mathcal{F}_k \) be defined as follows: If the function \( f \) on \( S \) is a member of \( \mathcal{F}_0 \), and if \( A \) is a subset of \( S \), then \( \alpha_k(f)(A) = f(A) + k \). Now it is easily seen that \( c_f = c_{\alpha_k(f)} \) and that, if \( X \) and \( Y \) are disjoint subsets of \( S \), then \( \alpha_k(f \backslash X / Y) = \alpha_k(f) \backslash X / Y \). Thus a theorem on connectivity and minors of normalised functions—that is, of members of \( \mathcal{F}_0 \)—will usually give, straightforwardly via \( \alpha_k \), a theorem which holds for members of \( \mathcal{F}_n \). The simplicity achieved by dealing with normalised functions is only that one does not have to carry the term \( f(\emptyset) \) through the proof.

The examples of submodular functions used in this paper to illustrate the theory arise from matroids and graphs. We conclude this section with a brief discussion of these.

Much of our motivation here derives from the fact that matroids have submodular rank functions. Moreover, one easily checks that a matroid \( M \) is connected if and only if its rank function \( r \) is a connected submodular function. Indeed, the separators of \( M \) are precisely the separators of \( r \). If \( e \) is an element of the ground set of \( M \), then \( c_r(\{e\}) = 1 \) unless \( e \) is either a loop or a coloop, in which case, \( c_r(\{e\}) = 0 \). If \( M \) is connected having at least two elements, then clearly \( c_r = 1 \).

Given an increasing integer-valued submodular function on a set \( S \), it is well known (see, for example, [10, Section 8.1]) that one can construct a matroid \( M_f \) on \( S \) by taking the independent sets to be those subsets \( X \) of \( S \) for which \( f(Y) \geq |Y| \) for all non-empty subsets \( Y \) of \( X \). In particular, if \( f \) is the rank function of a matroid \( M \), then \( M_f = M \). In that case, as noted above, \( M_f \) is connected if and only if \( f \) is connected. In general, however, \( M_f \) can be disconnected when \( f \) is connected, and \( f \) can be disconnected when \( M_f \) is connected. To see the first of these claims, let \( S = \{1, 2\} \) and \( f(X) = 2 \) if \( X \) is a non-empty subset of \( S \), and 0 otherwise. Then \( M_f \cong U_{2,2} \), so \( M_f \) is disconnected, yet \( f \) is clearly connected. On the other hand, if \( S = \{1, 2\} \) and \( f(X) = 1 \) for all subsets \( X \) of \( S \), then \( M_f \cong U_{1,2} \), so \( M_f \) is connected. This time, however, \( f \) is disconnected.

Let \( G \) be a graph with edge set \( E \). Define the function \( f_G \) on \( E \) by setting \( f_G(E') \)
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Then it is well known and, indeed, easily checked that $f_G$ is submodular. We shall call a function graphic if it is equal to $f_G$ for some graph $G$. (Of course, a number of other submodular functions can be derived from graphs; see for example [3].) If $f = f_G$ and $e$ is an edge of $G$, then $f \setminus e = f_{G \setminus e}$. Typically, however, contraction in $f_G$ does not correspond to contraction in $G$.

If $G$ is a graph without isolated vertices, then $G$ is connected if and only if $f_G$ is connected. Moreover, $\xi_G = 2$ if and only if $G$ is 2-connected in the sense of Tutte [9]. An edge $e$ of a connected graph $G$ has $c_G(e) = 1$ if $e$ is a loop or a pendant edge; otherwise $c_G(e) = 2$. Evidently, adding an isolated vertex to a graph $G$ or deleting such a vertex from $G$ does not alter $f_G$ and therefore does not affect whether or not $f_G$ is connected. For the remainder of the paper, by a connected graph, we shall mean one that is connected except for the possible presence of isolated vertices; that is, we shall call $G$ connected exactly when $f_G$ is connected.

3. Main results

It was proved by Tutte [7] that if $x$ is an element of a connected matroid $M$, then at least one of $M \setminus x$ and $M / x$ is connected. This result is a very powerful tool in inductive arguments for connected matroids. The following theorem, one of the two main results of this paper, generalises Tutte’s result to submodular functions. The proof is a straightforward extension of Tutte’s proof of his result.

**Theorem 3.1.** Let $A$ be a subset of the ground set $S$ of a connected submodular function $f$. If $c_f(A) < 2\xi_f$, then at least one of $f \setminus A$ and $f / A$ is connected.

**Proof.** As noted in the last section, we lose no generality in assuming that $f$ is normalised. Suppose that $f \setminus A$ is disconnected. Then there is a partition of $S - A$ into non-empty subsets $X_1$ and $X_2$ such that

$$(f \setminus A)(X_1) + (f \setminus A)(X_2) - (f \setminus A)(S - A) = 0,$$

that is

$$f(X_1) + f(X_2) - f(S - A) = 0. \quad (1)$$

Assume also that $f / A$ is disconnected. Then there is a partition of $S - A$ into non-empty subsets $Y_1$ and $Y_2$ such that

$$(f / A)(Y_1) + (f / A)(Y_2) - (f / A)(S - A) = 0,$$

that is,

$$f(Y_1 \cup A) + f(Y_2 \cup A) - f(S) - f(A) = 0. \quad (2)$$

Adding (1) and (2), we get

$$f(X_1) + f(X_2) + f(Y_1 \cup A) + f(Y_2 \cup A) - f(S - A) - f(S) - f(A) = 0.$$
Now $c_f(A) < 2\epsilon_f$, so that $f(S - A) + f(A) - f(S) < 2\epsilon_f$. Therefore
\[ f(X_1) + f(X_2) + f(Y_1 \cup A) + f(Y_2 \cup A) - 2f(S) < 2\epsilon_f. \] (3)

By the submodularity of $f$, if $i$ is in $\{1, 2\}$, then
\[ f(X_i) + f(Y_i \cup A) \geq f(X_i \cup Y_i \cup A) + f(X_i \cap Y_i). \]

Substituting the last inequality into (3), first using $i = 1$ and then using $i = 2$, we get:
\[ f(X_1 \cup Y_1 \cup A) + f(X_1 \cap Y_1) + f(X_2 \cup Y_2 \cup A) + f(X_2 \cap Y_2) - 2f(S) < 2\epsilon_f. \] (4)

But both $\{X_1 \cup Y_1 \cup A, X_1 \cap Y_1\}$ and $\{X_2 \cup Y_2 \cup A, X_1 \cap Y_1\}$ are partitions of $S$. It therefore follows from (4) that
\[ c_f(X_1 \cap Y_1) + c_f(X_2 \cap Y_2) > 2\epsilon_f. \]

Clearly this is possible only if either $X_1 \cap Y_1$ or $X_2 \cap Y_2$ is empty.

Interchanging $Y_1$ and $Y_2$ in (4), we also deduce that either $X_1 \cap Y_2$ or $X_2 \cap Y_1$ is empty. Therefore, one of $X_1, X_2, Y_1$ or $Y_2$ is empty, contradicting the assumption that all of these sets are non-empty. \(\square\)

If $M$ is a connected matroid on $S$ and $|S| \geq 2$, then $c_r(\{x\}) = 1$ for every element $x$ of $S$ and $\bar{c}_r = 1$. Hence Theorem 3.1 implies that at least one of $M\backslash x$ and $M/x$ is connected. In fact, for matroids, Theorem 3.1 says more. Suppose that $\{A, S - A\}$ is a $2$-separation of $M$, that is, $|S - A|, |A| \geq 2$ and $c_r(A) = 1$. Then at least one of $M/A$ and $M\backslash A$ is connected.

For graphs, the application of Theorem 3.1 is somewhat limited since, as noted earlier, contraction in a graphic submodular function does not correspond to contraction in the graph. Nevertheless, the following proposition does give us some leverage. For a subset $X$ of the edges of a graph, let $V(X)$ denote the set of vertices of the graph incident with at least one edge in $X$.

**Proposition 3.2.** If $f$ is a graphic submodular function and $f/A$ is connected, then $f \backslash A$ is connected.

**Proof.** Since $f$ is graphic, $f = f_G$ for some graph $G$. Let $\{X_1, X_2, A\}$ be a partition of the edges of $G$. A routine computation shows that
\[ |V(X_1)| + |V(X_2)| + |V(A)| \]
\[ \geq |V(X_1 \cup X_2)| + |V(X_1 \cup A)| - |V(X_2 \cup A)| - |V(X_1 \cup X_2 \cup A)|, \]

that is,
\[ |V(X_1)| + |V(X_2)| - |V(X_1 \cup X_2)| \]
\[ \geq |V(X_1 \cup A)| + |V(X_2 \cup A)| - |V(X_1 \cup X_2 \cup A)| - |V(A)|. \]
But the left and right sides of this inequality are easily seen to be equal to $c_{f \setminus A}(X_i)$ and $c_{f/A}(X_i)$ respectively. Therefore $c_{f \setminus A}(X_i) \geq c_{f/A}(X_i)$. The result follows readily from this observation. \qed

The next result is obtained by combining Theorem 3.1 and Proposition 3.2. Recall that, by a connected graph, we mean one which is connected up to isolated vertices.

**Corollary 3.3.** Let $G$ be a 2-connected graph with edge set $E$, and $A$ be a subset of $E$ for which $|V(A) \cap V(E - A)| \leq 3$. Then $G \setminus A$ is connected.

**Proof.** Since $G$ is 2-connected, $\delta_{f_G} = 2$. Therefore, by Theorem 3.1, if $A$ is a subset of $E$ and $c_{f_G}(A) \leq 3$, then either $f_G/A$ or $f_G \setminus A$ is connected. But it is easily seen that $c_f(A) = |V(A) \cap V(E - A)|$ and, by Proposition 3.2, if $f_G/A$ is connected, then $f_G \setminus A$ is connected. Therefore if $|V(A) \cap V(E - A)| \leq 3$, then $G \setminus A$ is connected. \qed

Since contraction in graphic submodular functions does not correspond to contraction in graphs, one is led to seek an operation on submodular functions which does correspond to contraction in graphs. If $f$ is a submodular function on $S$ and $A$ is a subset of $S$, let $f \square A$ be the function defined, for all subsets $X$ of $S - A$, by

$$f \square A(X) = \begin{cases} f(X) & \text{if } f/A(X) = f(X); \\ f/A(X) + 1 & \text{otherwise.} \end{cases}$$

Equivalently,

$$f \square A(X) = \begin{cases} f(X) & \text{if } f(A \cup X) - f(A) + f(\emptyset) = f(X); \\ f(A \cup X) - f(A) + f(\emptyset) + 1 & \text{otherwise.} \end{cases}$$

It is easily seen that if $e$ is an edge of the graph $G$, then $f_{G/e} = f_G \square \{e\}$. Note, however, that if $\{e_1, \ldots, e_n\}$ is a set of edges of $G$, then $f_{G/(e_1, \ldots, e_n)}$ typically differs from $f_G \square \{e_1, \ldots, e_n\}$. Rather,

$$f_{G/(e_1, \ldots, e_n)} = (\cdots (f_G \cup e_i) \cup \cdots) \cup e_n.$$

It is not difficult to prove that this new operation has the following natural interpretation for matroids.

**Proposition 3.4.** If $M$ is a matroid with rank function $r$, and $A$ is a subset of the ground set of $M$, then $r \square A$ is the rank function of $T_A(M) \setminus A$. \qed

In order to show that $f \square A$ is submodular whenever $f$ is, we shall use the following lemma. For the remainder of this section, we lose no generality by assuming that $f$ is normalised.
Lemma 3.5. Let $f$ be a submodular function on $S$, and let $A$ and $X$ be disjoint subsets of $S$.

(i) $f(A(X)) \leq f(X)$.
(ii) $f(A(X)) \leq f(X)$.
(iii) If $f(A(X)) = f(X)$, then $f(A(X')) = f(X')$ for all subsets $X'$ of $X$.

Proof. Part (i) is immediate from the submodularity of $f$, and (ii) follows from (i). Consider (iii). Say $X'$ is a subset of $X$. Then, by the submodularity of $f$,

$$f(X' \cup A) + f(X) \geq f(X \cup A) + f(X').$$

But $f(A(X)) = f(X)$, so $f(X \cup A) = f(X) + f(A)$. Therefore,

$$f(X' \cup A) = f(X') + f(A),$$

that is, $f(A(X')) \geq f(X')$. But, by (i), $f(A(X')) \leq f(X')$, so $f(A(X')) = f(X')$. □

Proposition 3.6. If $f$ is a submodular function on $S$ and $A$ is a subset of $S$, then $f \Box A$ is submodular.

Proof. Let $X$ and $Y$ be subsets of $S - A$, and consider $f \Box A(X) + f \Box A(Y)$. It suffices to consider the following three cases:

(i) $f(A(X)) = f(X)$ and $f(A(Y)) = f(Y)$;
(ii) $f(A(X)) = f(A(X)) + 1$ and $f(A(Y)) = f(Y)$; and
(iii) $f(A(X)) = f(A(X)) + 1$ and $f(A(Y)) = f(A(Y)) + 1$.

Assume that (i) holds. Then, by Lemma 3.5(ii), $f(X \cup Y) \geq f(A(X \cup Y))$ and $f(X \cap Y) \geq f \Box A(X \cap Y)$. Therefore,

$$f \Box A(X) + f \Box A(Y)
= f(X) + f(Y)
\geq f(X \cap Y) + f(X \cup Y)
\geq f \Box A(X \cap Y) + f \Box A(X \cup Y).$$

In case (ii), by Lemma 3.5(iii), $f \Box A(X \cap Y) = f(X \cap Y)$. Therefore,

$$f \Box A(X) + f \Box A(Y)
= f(A \cup X) - f(A) + 1 + f(Y)
\geq f(A \cup X \cup Y) - f(A) + 1 + f(X \cap Y)
\geq f \Box A(X \cup Y) + f \Box A(X \cap Y).$$

Finally, in case (iii),

$$f \Box A(X) + f \Box A(Y)
= f(A \cup X) + f(A \cup Y) - 2f(A) + 2
\geq f(A \cup X \cup Y) + f(A \cup (X \cap Y)) - 2f(A) + 2
\geq f \Box A(X \cup Y) + f \Box A(X \cap Y).$$

In some cases $f \Box A$ reduces to a familiar operation.
**Proposition 3.7.** Let \( A \) be a subset of the ground set \( S \) of a submodular function \( f \).

(i) If \( A \) is a separator of \( f \), then \( f \triangle A = f \setminus A = f \cap A \).

(ii) If \( c_f(A) = 1 \), then \( f \triangle A = f \setminus A \).

**Proof.** Part (i) is immediate. Consider (ii). Let \( X \) be a subset of \( S - A \). Noting that \( c_f(A) = 1 \) implies that \( f(A) = 1 - f(S - A) + f(S) \), and using the submodularity of \( f \), we see that

\[
f(A) = f(A \cup X) + f(S - A) - f(S) - 1
\]

By Lemma 3.5(i), \( f(A) \leq f(X) \). Therefore, either \( f(A) = f(X) \), or \( f(A) = f(X) - 1 \). Either case implies that \( f \triangle A(X) = f(X) \) and the proposition is proved. \( \square \)

Theorem 3.1 is one of the two main results of this paper. The second such result is the following.

**Theorem 3.8.** Let \( f \) be a connected submodular function on \( S \), and let \( A \) be a subset of \( S \).

(i) If \( c_f(A) = 1 \), then either \( f \cap A \) or \( f \triangle A \) is connected.

(ii) If \( c_f(A) > 1 \), then \( f \cap A \) is connected.

**Proof.** By Proposition 3.7, if \( c_f(A) = 1 \), then \( f \triangle A = f \setminus A \). Therefore (i) is a special case of Theorem 3.1. Assume that \( c_f(A) > 1 \). Let \( \{X_1, X_2\} \) be a partition of \( S - A \) into non-empty subsets. Now

\[
c_{\cap \cup A}(X_i) = f \cap \cup A(X_1) + f \cup \cap A(X_2) - f \cup \cap A(S - A).
\]

Since \( c_f(A) > 1 \), it follows that \( f \cap A(S - A) \neq f(S - A) \) and so \( f \cap A(S - A) = f(S) - f(A) + 1 \). Therefore,

\[
c_{\cap \cup A}(X_i) = f \cap \cup A(X_1) + f \cap \cup A(X_2) - f(S) + f(A) - 1.
\]

To prove that \( c_{\cap \cup A}(X_1) \) is positive and hence that \( f \cap A \) is connected, it suffices to consider the following three cases:

(i) \( f \cap \cup A(X_i) = f(X_i) \) for \( i = 1, 2 \);

(ii) \( f \cap \cup A(X_1) = f(X_1 \cup A) - f(A) + 1 \) and \( f \cap \cup A(X_2) = f(X_2) \); and

(iii) \( f \cap \cup A(X_i) = f(X_i \cup A) - f(A) + 1 \) for \( i = 1, 2 \).

In case (i),

\[
c_{\cap \cup A}(X_1) = f(X_1) + f(X_2) - f(S) + f(A) - 1
\]

\[
= c_f(A) - 1 > 0.
\]
In case (ii),
\[ c_{f \cap A}(X_1) = f(X_1 \cup A) - f(A) + 1 + f(X_2) - f(S) + f(A) - 1 \]
\[ = c_f(X_1 \cup A) > 0. \]

Finally, in case (iii),
\[ c_{f \cap A}(X_1) = f(X_1 \cup A) - f(A) + 1 + f(X_2 \cup A) - f(A) + 1 - f(S) + f(A) - 1 \]
\[ = f(X_1 \cup A) + f(X_2 \cup A) - f(A) - f(S) + 1 \]
\[ \geq f(S) + f(A) - f(S) + f(S) + 1 = 1. \]

On applying the last result to matroids, we obtain the following.

**Corollary 3.9.** Let \( A \) be a subset of the ground set of a connected matroid \( M \). If \( c_f(A) \geq 2 \), then \( T_A(M) \setminus A \) is connected.

In fact, it is also not hard to see that \( T_A(M) \) is connected.

A well-known theorem in graph theory (see, for example, [9, Theorem III.33]) states that if \( e \) is an edge of the 2-connected graph \( G \), then either \( G \setminus e \) or \( G/e \) is 2-connected. It is natural to ask if this theorem has a generalisation to submodular functions.

Consider an example. Let \( S = \{a, b, c\} \), and let \( f \) be defined as follows: If \( S' \) is a subset of \( S \), then \( f(S') \) is equal to 0, 2, 4 or 4 according to whether \( S' \) has cardinality 0, 1, 2 or 3 respectively. Clearly \( f \) is submodular. This function arises naturally where \( a, b \) and \( c \) are three mutually skew lines in rank-4 space; \( f(S') \) then measures the rank of the span of \( S' \). Now \( \tilde{c}_f = 2 \) and \( c_f(\{a\}) = c_f(\{b\}) = c_f(\{c\}) = 2 \), so \( f \) acts, in some ways, like the graphic submodular function of a 2-connected graph. But, if \( x \) is an element of \( S \), then \( f \setminus x \) is disconnected, while \( \tilde{c}_{f \cap x} = 1 \). It would appear that the abovementioned property of graphs does not have an analogue for submodular functions in general.

We now consider duality, beginning by noting that a submodular function is connected if and only if its dual is connected.

**Proposition 3.10.** If \( f \) is a submodular function on \( S \), then \( c_f = c_{f^*} \).

**Proof.** Let \( X \) be a subset of \( S \). Then
\[ c_{f^*}(X) = f^*(X) + f^*(S - X) + f^*(S) \]
\[ = |X| + f(S - X) - f(S) + |S - X| + f(X) - f(S) - |S| + f(S) \]
\[ - f(X) + f(S - X) - f(S) = c_f(X). \]
\[ \square \]

It follows from the last result that the separators of \( f^* \) are equal to those of \( f \), and that \( \tilde{c}_f = \tilde{c}_{f^*} \).
Next we examine the dual of $f \cap A$, defining $f^{\square} A$ by $f^{\square} A = (f^* \cap A)^*$. Since the class of submodular functions is closed under duality, it is evident that $f^{\square} A$ is submodular. Moreover, a routine computation from duality gives the following characterisation of $f^{\square} A$.

**Proposition 3.11.** Let $f$ be a submodular function on $S$ and let $A$ be a subset of $S$. If $A$ is a separator of $f$, then $f^{\square} A = f \cap A = f \setminus A = f / A$. If $A$ is not a separator of $f$, then $f^{\square} A$ is defined, for all subsets $X$ of $S - A$, by

$$ f^{\cap} A = \begin{cases} f(X) - 1 & \text{if } f(X \cup A) - f(X) = f(S) - f(S - A) \\ f(X) & \text{otherwise.} \end{cases} $$

Dualising Theorem 3.8(ii) we obtain the following.

**Corollary 3.12.** If $A$ is a subset of the ground set of a connected submodular function $f$ and $c_r(A) > 1$, then $f^{\square} A$ is connected.

We now consider the interpretation of Corollary 3.12 for matroids. Upon comparing Proposition 3.11 with the rank definition of the principal truncation $T_A(M)$, we see that if $M$ and $M \setminus A$ have the same rank, then $r^{\square} A$ is the rank function of $T_A(M) \setminus A$. The following result follows on combining this observation with Corollary 3.12.

**Corollary 3.13.** Let $A$ be a subset of the ground set of a connected matroid $M$. Assume that $M$ and $M \setminus A$ have the same rank and that $c_r(A) \geq 2$. Then $T_A(M) \setminus A$ is connected.

For graphs, Corollary 3.12 gives us no new information, since it is easily seen that if $e$ is an edge of a graph $G$ and $e$ is neither a loop nor a pendant edge, then $f_G \cap e = f_G e$. This is a special case of the following.

**Proposition 3.14.** If $f$ is a submodular function on $S$ and $A$ is a subset of $S$ with $c_r(A) = 2$, then $f \cap A = f^{\square} A$.

**Proof.** Let $X$ be a subset of $S - A$. Since $c_r(A) = 2$, it is easily seen that $f(X) - f / A(X)$ is an element of $\{0, 1, 2\}$. Therefore either (i) $f \cap A(X) = f(X)$, or (ii) $f \cap A(X) = f(X) - 1$. Assume that (i) holds. Then $f(X) - f / A(X) \leq 1$; that is, $f(X) - f(A \cup X) + f(A) \leq 1$. But $c_r(A) = 2$, so $f(A) + f(S - A) - f(S) = 2$. Therefore $f(X \cup A) - f(X) = f(S) - f(S - A)$ and it follows, in case (i), that $f^{\square} A(X) = f(X) = f \cap A(X)$. Now assume that (ii) holds. Then $f(X) - f(A \cup X) + f(A) = 2$ and, since $c_r(A) = 2$, it follows that $f(A \cup X) - f(X) = f(S) - f(S - A)$. Hence $f^{\square} A(X) = f(X) - 1$. Thus in case (ii), as in case (i), $f \cap A(X) = f^{\square} A(X)$. □
References