Low Energy Excitations in Liquid Helium.

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ABSTRACT

A variational approximation method described by Valatin and Butler¹ is used as a basis for describing the low temperature behavior of liquid Helium. It is assumed that the Fourier transform of the interaction potential of the Helium atoms has a Gaussian form, specified by two parameters. For a particular choice of these two parameters, for which the corresponding Gaussian transform is in good qualitative agreement with a transform obtained from gas data, the ground state expectation value of the number of Helium atoms with zero momentum is calculated to be 8%, in agreement with the experimental value deduced by Penrose and Onsager.² For the same choice of parameters, a curve (called the dispersion curve) is calculated for the energy versus momentum relation for the elementary excitations in the liquid. The resulting dispersion curve correctly passes through the origin linearly with a slope equal to the experimental value of the speed of sound in the liquid. However, this dispersion curve diverges increasingly from the experimental results of Henshaw and Woods³ as the momentum (and energy) of the excitations is

increased from zero; and, in particular no minimum occurs. This result is interpreted as probably due to an inadequacy of the weak interaction assumption implied by the method of Valatin and Butler.
1.1 A Brief Historical Survey

About 1927 Keesom and Wolfke at Leiden discovered a discontinuity in the curve of the dielectric constant of liquid He\textsuperscript{4} as a function of temperature. This suggested that some sort of thermodynamic transition may occur at the temperature (about 2.2\textdegree K) at which the discontinuity was observed. Keesom and his co-workers were thus led to measure the specific heat vs temperature curve for the liquid to determine if there was a jump in the specific heat associated with the transition. The result demonstrated a discontinuity in the specific heat at about the expected temperature (called the "lambda point"). The designation "Helium II" was given to the fluid in the temperature region below this point. Subsequently further peculiar properties of liquid Helium II were discovered, including extremely low viscosity.\textsuperscript{1}

Landau,\textsuperscript{2} in 1941, proposed a two-fluid model of liquid Helium II in which two components of the fluid coexist at temperatures between absolute zero and the lambda point. He proposed that at 0\textdegree K the Helium is

\textsuperscript{1}Andronikashivili, E., J. Phys. (U.S.S.R.) \textbf{10}, 201 (1946).

that any number of them may be present with the same momentum. The ground state of the fluid is characterized by the absence of excitations, the low-lying energy states by the presence of few excitations.

Unfortunately, the Bogoliubov approximation cannot both yield the correct dispersion curve and be internally consistent. It follows therefore that the interaction among the Helium atoms must be stronger than required by the Bogoliubov theory. The present paper will deal with a particular modification of the Bogoliubov method.

1.2 The Experimental Dispersion Curve

For liquid Helium II, Henshaw and Woods\(^4\) (extending the experiments of Yarnell, et al.\(^5\) and acting on a suggestion of Feynman and Cohen\(^6\)) experimentally measured the dispersion curve. In this experiment neutrons were scattered from liquid Helium held at 1.1°K and the change in momentum of the neutrons at different scattering angles was determined. Feynman and Cohen had shown that in such an experiment there is an extremely low probability of the incident neutron's causing multiple excitations. Therefore the neutron normally produces a single

excitation of energy $E$ given by

$$E = \frac{p_i^2 - p_f^2}{2m},$$

(1.2.1)

where $p_i$ is the momentum of the incident neutron, $p_f$ is the momentum of the scattered neutron, and $m$ is the neutron mass. The momentum $p$ of the excitation is given by

$$p^2 = p_i^2 - p_f^2 - 2 p_i p_f \cos \phi,$$

(1.2.2)

where $\phi$ is the angle through which the neutron is scattered. Thus, from a measurement of $p_i$, $p_f$, and (and from the known neutron mass), the energy and momentum of the excitation are experimentally determined. The results of this experiment appear in Fig. 1. This dispersion curve has the behavior predicted by Landau.
FIGURE 1

The experimental dispersion curve for the energy $E_k$ vs wave vector $k$ of elementary excitations in liquid Helium II at 1.1°K from the experiment by Henshaw and Woods.
1.3 The Hamiltonian

The Hamiltonian for a collection of Helium atoms interacting via a two body interaction potential $V$ may be written as

$$H = \sum_k \mathcal{E}_k a^+_k a_k + \frac{1}{2\Omega} \sum_{k,k',q} \mathcal{V}_{k-k'} a^+_k a^+_q a_q a_{k'} , \tag{1.3.1}$$

wherein the symbols are defined as follows: $\mathcal{E}_k$ represents the energy of a free Helium atom of momentum $\hbar k$, that is, (for a particle of mass $m$)

$$\mathcal{E}_k = \frac{\hbar^2 k^2}{2m} , \tag{1.3.2}$$

$a^+_k$ and $a_k$ are the creation and annihilation operators for Helium atoms of momentum $\hbar k$ and satisfy the commutation relations for bosons:

$$a_y^+ a^+_{\bar{m}} - a^+_{\bar{m}} a_y = \delta_y^{\bar{m}} \quad , \tag{1.3.3a}$$

$$a_k^+ a_{\bar{m}} - a_{\bar{m}} a_k = 0 \quad , \tag{1.3.3b}$$

$$a_y^+ a^+_{\bar{m}} - a^+_{\bar{m}} a_y = 0 \quad . \tag{1.3.3c}$$

---

A development of the transformation of the Hamiltonian from configuration space to occupation number space representation is presented in Quantum Mechanics by Landau and Lifschitz, Addison Wesley, Reading, Mass. (1958).
\( \Omega \) represents the volume to which the atoms are constrained (This volume will be taken to approach infinity to describe the bulk properties of the fluid, neglecting surface effects.). \( \widetilde{V}_k \) represents the Fourier transform of the interparticle potential, that is,

\[
\widetilde{V}_k = \int V(r) e^{i k \cdot r} \, d^3r , \tag{1.3.4}
\]

where \( V(r) \) is the two body interaction potential for a pair of Helium atoms. The range of the summation indices in (1.3.1) is given by the set of all vectors obtained by substituting for \( n_x, n_y, \) and \( n_z \) independently all integers into the expression

\[
\bar{y} = \frac{2 \pi}{\Omega^{1/3}} (n_x \bar{i} + n_y \bar{j} + n_z \bar{k}) . \tag{1.3.5}
\]

It should be noted that the operator for the number of atoms with momentum \( \hbar k \) is expressed by

\[
N_k = \hat{a}_k^\dagger \hat{a}_k , \tag{1.3.6}
\]

The operator for the total number of atoms, \( N \), given by

\[
\hat{N} = \sum_k N_k , \tag{1.3.7}
\]

commutes with the Hamiltonian. Therefore the total number of atoms is a constant of the motion and may be chosen at liberty. We are considering a system with an experimentally specified particle density \( \varphi \); hence, as
the volume $\Omega$ is taken to approach infinity, the total number of atoms, $N$, must also be taken to approach infinity such that the ratio

$$\frac{N}{\Omega} = \varphi \quad (1.3.8)$$

To obtain a low temperature description of liquid Helium it is desired to find a description of the low lying energy eigenstates of the Hamiltonian that are simultaneously eigenstates of the total number operator with corresponding eigenvalue $N$, and to examine this description as $N$ and $\Omega$ are taken to approach infinity such that the ratio $\varphi = N/\Omega$ is constant. In this limit, sums of the form

$$\lim_{\Omega \to \infty} \sum_{\vec{q} \neq 0} f(\vec{q}) = \frac{\Omega}{(2\pi)^N} \int f(\vec{q}) \, d^3\vec{q} \quad (1.3.9)$$

where the range of the summation index $y$ is given by (1.3.5).
CHAPTER 2
THE BOGOLIUBOV APPROACH

2.1 Selection of the neglected terms

The Hamiltonian (1.3.1) may be expressed in the following form by rearranging its terms and by replacing the operator for the number of atoms,

\[ \sum_k a_k^{\dagger} a_k \]

by its constant value, \( N \):

\[ H = \frac{\hbar \omega}{2} N(N-1) + \sum_k \epsilon_k a_k^{\dagger} a_k + R + S, \] (2.1.1)

where

\[ R = \frac{\hat{N}_o}{2 \Omega} \sum_k \left[ a_k^{\dagger} a_k + a_k^{\dagger} a_k + a_k a_k^{\dagger} + a_k^{\dagger} a_k^{\dagger} \right], \] (2.1.2)

and

\[ S = \frac{1}{2 \Omega} \sum_{k,k',q} \left[ a_k^{\dagger} a_{q,k}^{\dagger} a_{q,k'} a_{k'} \right]. \] (2.1.3)

In this result,

\[ \hat{N}_o = a_0^{\dagger} a_0 \]

is the operator for the number of atoms with zero momentum and the prime on the summation in (2.1.3) indicates the exclusion of those terms for which any two subscripts of the creation and annihilation operators occurring therein
are zero. Hence, the terms of (2.1.3) contain at most one zero subscript.

In the Bogoliubov approximation, the interaction among the Helium atoms is assumed to be weak. Thus, in a low temperature description of the liquid, the number, \( N_0 \), of atoms with zero momentum is approximately the total number of atoms, \( N \), a constant of the motion. We therefore may replace \( \hat{N}_0 \equiv \hat{a}_o^\dagger a_o \) by the c-number \( N \). If, moreover, unity is neglected in comparison with \( N \), then the commutation relation

\[
1 = a_o a_o^\dagger - a_o^\dagger a_o \equiv a_o a_o^\dagger - \hat{N}_0 \quad (2.1.4)
\]

may be replaced by the approximation

\[
a_o^\dagger a_o \approx a_o a_o^\dagger \approx N_0 \approx N . \quad (2.1.5)
\]

By employing this result and perturbation theory, it then follows that if \( V \) is sufficiently small (that is, the interaction among the Helium atoms is sufficiently weak), then, in comparison with \( R \), \( S \) will contribute negligibly to the low lying energy eigenvalues of \( H \). Therefore, for a low temperature description of the fluid, if the interactions among the Helium atoms is sufficiently weak,

\[
H \approx H_b \equiv \frac{\tilde{V}_o}{2\Omega} N(N-1) + \sum_{\kappa} \xi_{\kappa} a_{\kappa}^\dagger a_{\kappa} + \sum_{\kappa} \tilde{V}_{\kappa} \left[ a_{\kappa} a_{\kappa}^\dagger + a_{\kappa}^\dagger a_{\kappa} + a_{\kappa}^\dagger a_{\kappa}^\dagger + a_{\kappa} a_{\kappa} \right] \quad (2.1.6)
\]
2.2 Diagonalization of the Bogoliubov Hamiltonian

To find the eigenfunctions and eigenvalues of an operator, $A$, one procedure is to seek a unitary operator, $U$, (that is, one for which $U^* = U^{-1}$) such that the transformed operator

$$A' = U^* A U$$

(2.2.1)

has a sufficiently simple form that it can be diagonalized easily. This approach works because if $\Psi$ and $\alpha$ are an eigenfunction and corresponding eigenvalue of $A'$, then

$$\Psi = U^{-1} \Psi$$

(2.2.2)

and $\alpha$ are an eigenfunction and corresponding eigenvalue of $A$.

It is straightforward to demonstrate that the operator $U$ defined by

$$U^* a_k U = \frac{a_k + g_k a_k^*}{\sqrt{1 - g_k^2}}$$

(2.2.3)

where $g_k$ is an arbitrary real function of $k$, is unitary provided $g_k^2 < 1$, and to demonstrate that

$$H'_b \equiv U^* H_b U = W_0 + \sum_{k \neq 0} E_k a_k^* a_k$$

(2.2.4)

where $W_0$ is a constant and

$$E_k = \sqrt{E_k^2 + \rho \bar{\nu}_k E_k}$$

(2.2.5)
provided \( g_k \) is taken as

\[
g_k = \begin{cases} 
0 & \text{if } k = 0 \\
\frac{\sqrt{\varepsilon_k^2 + \frac{1}{2} \rho \tilde{V}_k} - \left( \varepsilon_k + \frac{1}{2} \rho \tilde{V}_k \right)}{\frac{1}{2} \rho \tilde{V}_k} & \text{if } k \neq 0
\end{cases}
\]  \hspace{1cm} (2.2.6)

By inspection of (2.2.4), the eigenvalues of \( H_b \) are seen to be

\[
\mathcal{W}_0 + \sum_{k \neq 0} \mathcal{E}_k n_k
\]  \hspace{1cm} (2.2.7)

where

\[n_k = 0, 1, 2, 3, \ldots \]

It is also straightforward to demonstrate similarly that the eigenfunctions of \( H_b \) are also eigenfunctions of the total momentum of the system and that the corresponding eigenvalues are

\[
\sum_{k} \hbar \mathcal{K}_k n_k
\]  \hspace{1cm} (2.2.8)

2.3 Results of the Bogoliubov Method

In view of (2.2.7) and (2.2.8), we may say that in the Bogoliubov approximation, the (internal) motion of the fluid is described as a system of non-interacting collective modes of excitation, each excitation possessing a definite momentum, \( \hbar \mathcal{K} \), and a definite energy \( \mathcal{E}_k \).
determined by its momentum in accordance with the dispersion curve (2.2.5). An examination of this curve reveals that it may, on suitable choice of interatomic potential transform, $\tilde{V}$, have the required shape of the Landau curve (Fig. 1), that is, the curve may have a "roton minimum" and a linear slope at the origin ($h_k = 0$). However, Jackson\(^8\) has shown that, in a calculation fitting the experimental results of Henshaw and Woods to the Bogoliubov dispersion curve, the quantity

$$\frac{N - N_0}{N} \approx 2.6$$ \hspace{1cm} (2.3.1)

In light of the approximation assumption (2.1.5), this result shows that the Bogoliubov approach fails to both give the required dispersion curve and, at the same time, maintain internal consistency. In fact, the Bogoliubov assumption that most of the atoms have zero momentum is now known to be false. From experimental data Onsager and Penrose\(^9\) calculate that (on the average) only 8% of the atoms have zero momentum.


CHAPTER 3
THE VALATIN APPROACH

3.1 The Approximation Method

We have seen that, on the basis of an assumption of weak interactions among the Helium atoms, the Bogoliubov approximation consisted of diagonalizing an approximate Hamiltonian for the fluid. In principle, a more accurate procedure would involve transforming the exact Hamiltonian by a transformation of the Bogoliubov form and employing the variational principle to find the best such transformation for describing the ground state of the fluid. Such a procedure has the advantages that it must give an improved evaluation of the ground state energy and that it avoids the false assumption that most of the atoms have zero momentum. This procedure was the original motivation for the present paper.

According to the Ritz variational principle and to the method of Lagrange multipliers, for the system of \( N \) Helium atoms, the exact value of the ground state energy, \( E_0 \), and the exact value (namely zero) of the ground state r.m.s. deviation, \( \Delta N \), of the number of atoms from its mean value are given by

\[
E_0 = \min_{\psi} \left\{ \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \right\}
\]  

(3.1.1a)
and

$$\langle \Delta N \rangle^2 = \min_{\lambda (\Psi \text{ fixed})} \left\{ \frac{\langle \Psi | (\hat{N} - N)^2 | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right\} , \quad (3.1.1b)$$

where $\Psi$ represents an arbitrary state function of a quantum mechanical system of Helium atoms (not necessarily a system of specifically $N$ Helium atoms),

$$\tilde{H}(\lambda) = H - \lambda (\hat{N} - N) , \quad (3.1.2)$$

$\lambda$ is a Lagrange multiplier, and $\hat{N}$ is the operator for the total number of atoms, $N$. The exact ground state eigenfunction, $\Psi_e$, and the exact Lagrange multiplier, $\lambda_e$, are those for which

$$E_o = \frac{\langle \Psi_e | \tilde{H} | \Psi_e \rangle}{\langle \Psi_e | \Psi_e \rangle} \quad (3.1.3a)$$

and

$$\langle \Delta N \rangle^2 = \frac{\langle \Psi_e | (\hat{N} - N)^2 | \Psi_e \rangle}{\langle \Psi_e | \Psi_e \rangle} , \quad (3.1.3b)$$

where $E_o$ and $\Delta N$ are determined from (3.1.1). The approximation procedure described in the previous paragraph can be implemented by carrying out the minimization process just described except that $\Psi$ is to be varied
over state functions only of the Bogoliubov form.\textsuperscript{10}

Since $\Psi_e$ is not exactly of the Bogoliubov form, the resulting value of $E_0$ will be only approximate. Moreover, since state functions of the Bogoliubov form are not eigenfunctions of $\hat{N}$, the resulting value of $\Delta N$ will not be exactly zero. However, a result for $\Delta N$ such that

\[
\frac{\Delta N}{N} \ll 1 \tag{3.1.4}
\]

is still a satisfactory one.

Valatin and Butler\textsuperscript{11} introduced a modification of the above procedure whereby the minimization (3.1.1b) of $\Delta N$ with respect to $\lambda$ is replaced by the requirement of the known experimental fact that the dispersion curve, $E_k$, approaches zero as $k$ approaches zero. Failure to minimize $\Delta N$ with respect to $\lambda$ requires that the condition (3.1.4) be demonstrated \textit{a posteriori}.

To accomplish the minimization (3.1.1a) with respect to state functions of only the Bogoliubov form,

\[
\Psi_b = \bigcup_b |0\rangle \tag{3.1.5}
\]

\textsuperscript{10}A discussion of this procedure is found, for example, in "Applications of Field Theoretical Techniques to Many-Boson Systems", by N. Hugenholtz, an article in the collection entitled "1962 Cargese Lectures in Theoretical Physics", M. Levy (ed.), W. Benjamin, N.Y. (1963).

we substitute \( \psi_b \) into (3.1.1a) and obtain (by using
\( u^+ = u^{-1} \) and (3.1.2))

\[
E_\circ = \min_{\lambda \text{ fixed}} \left\{ \frac{\langle \psi_b | H | \psi_b \rangle}{\langle \psi_b | \psi_b \rangle} \right\}
\] (3.1.6)

or
\[
E_\circ = \min_{\lambda \text{ fixed}} \left\{ \langle 0 | u^+ [H - \lambda(\hat{N} - N)] u \rangle \right\}.
\] (3.1.7)

For convenience, we define the so-called statistical
operator, \( H' \), as
\[
H' = H - \lambda \hat{N}
\] (3.1.8)

so that, from (3.1.2),
\[
H' = \tilde{H} - \lambda N
\] (3.1.9)

Combining (3.1.7) and (3.1.8) gives
\[
E_\circ - \lambda N = \min_{\lambda \text{ fixed}} \left\{ \langle 0 | u^+_b H' u_b | 0 \rangle \right\}.
\] (3.1.10)

We note from (2.2.3) that \( u_b \) is determined by a single
function, \( g_k \). Therefore, (3.1.10) may be written as
\[
E_\circ - \lambda N = \min_{\lambda \text{ fixed}} \left\{ W_\circ (g_k) \right\},
\] (3.1.11)

where
\[
W_\circ (g_k) = \langle 0 | u^+_b (g_k) H' u_b (g_k) | 0 \rangle
\] (3.1.12)
Hence, to obtain the best approximation for $E_o$ possible within the Valatin scheme, it is required that

$$E_o = W_o(q_k) + \lambda N,$$  \hspace{1cm} (3.1.13)

where $q_k$ must satisfy

$$\frac{\partial W_o(q_k)}{\partial q_k} = 0 \quad \text{for all } k.$$  \hspace{1cm} (3.1.14)

In Appendix I, the operator $U_b^{-1} H' U_b$ is expressed explicitly; and, in Appendix II, the result is simplified to the form

$$U^{-1} H' U = W'_o + \sum_k E_k a_k^{+} a_k + C + D$$  \hspace{1cm} (3.1.15)

where $D$ is a sum of terms of higher order than quadratic in creation and annihilation operators, $C$ is a sum of quadratic terms of form different from $a_k^{+} a_k$, and

$$W'_o = \frac{1}{2} \sum_k \{ (E_k - \lambda) h_k + \nu_k h_k - \mu_k \chi_k \}.$$  \hspace{1cm} (3.1.16)

and

$$E_k = \nu_k (1 + 2 h_k) - 2 \nu_k \chi_k$$  \hspace{1cm} (3.1.17)

where we have defined

$$h_k = \frac{g_k^2}{(1 - g_k^2)}$$  \hspace{1cm} (3.1.18)

and

$$\chi_k = \frac{g_k}{(1 - g_k^2)}$$  \hspace{1cm} (3.1.19)
\[ \mathcal{V}_k = (\epsilon_k - \lambda) + \frac{1}{2\Omega} \sum_{k'} \left( \sqrt{V}_{k-k'} + V_0 \right) \hat{h}_{k'} , \]  

(3.1.20)

and

\[ \mathcal{U}_k = -\frac{1}{\Omega} \sum_{k'} \sqrt{V}_{k-k'} \chi_{k'} . \]  

(3.1.21)

Also, by inspection of (3.1.12) and (3.1.15),

\[ W_0 = W'_0 \]  

(3.1.22)

Minimization of \( W_0 \) with respect to \( g_k \) may be accomplished by employing (3.1.14) and (3.1.16) through (3.1.21). The result is shown in Appendix II to be

\[ \mathcal{U}_k g_k^2 - 2 \nu_k g_k + \mathcal{U}_k = 0 , \]  

(3.1.23)

which in principle determines \( g_k \), which, in turn determines an approximate ground state function via equations (3.1.5) and (2.2.3).

As for determining the low lying excited states of the fluid, we have to revert to an assumption of weak interactions among the Helium atoms, as was also done in the Bogoliubov approximation. What we need to assume is that the interaction is sufficiently weak that (from perturbation theory) the terms C and D of equation (3.1.15) may be neglected. Thence, (by comparison of (3.1.15) with (2.2.4)) \( E_k \) as given by (3.1.17) is, in the present approximation, the dispersion curve for the elementary excitations in the fluid at low temperature. Two
**a priori** indications that the weak interaction assumption is more appropriate in the present context than it was in the Bogoliubov method are:

1. We do not have to assume that the interaction is so weak that most of the atoms have (on the average) zero momentum.

2. Providing the condition (3.1.4) is well met (as will be demonstrated in the present calculations), then according to the variational principle, the ground state energy, \( E_0 \), will be more accurately obtained in the present method. We can also say that if the system is in a state of lowest energy, \( W(k) \), possible for it under the circumstance that its momentum is specified as some value \( h_k \), then as \( k \to 0 \), \( W(k) \to W(0) = E_0 \), the ground state energy of the system. In other words, if there is only one phonon present and its momentum is \( h_k \) and its energy is \( E_k \), then as \( k \to 0 \), \( W(k) = (E_0 + E_k) \to E_0 \). It then also similarly follows from the variational principle that for sufficiently small \( k \), \( E_0 + E_k \) must be more accurately determined by the present method. But one cannot exclude the seemingly remote possibility that nevertheless the difference \( E_k = (E_k + E_0) - E_0 \) may be less accurately determined even if both \( (E_k + E_0) \) and \( E_0 \) are more accurately determined. Ignoring this possibility, we expect to obtain a superior dispersion curve for excitations of sufficiently low energy.

The results (3.1.16) through (3.1.23) need to be organized into a form convenient for calculation. This organization can be accomplished by algebraic manipulations yielding the following intermediate but useful results:

\[
1 + 2h_k = \frac{\nu_k}{E_k} \tag{3.1.24}
\]

and

\[
2 \chi_k = \frac{\mu_k}{E_k} \tag{3.1.25}
\]

and then the following required results:

\[
(1 + 2h_k)^2 - (2\chi_k)^2 = 1 \tag{3.1.26}
\]
Equations (3.1.26) through (3.1.28) form a set of 3 coupled equations (one algebraic and two integral) to be solved for the unknown desired dispersion curve $E_k$ by eliminating the two unknown functions $h_k$ and $\chi_k$.

Before attempting such a solution, we may simplify (3.1.27) somewhat by explicating the physical interpretation of the quantity

\[ \sum_k h_k \]

It is the trial ground state expectation value of the number of Helium atoms, $N$, as may be seen by using (3.1.5), (2.2.3), and (3.1.8) as follows

\[
\frac{\langle \psi_b | \hat{N} | \psi_b \rangle}{\langle \psi_b | \psi_b \rangle} = \frac{\langle 0 | \hat{U} \hat{N} \hat{U}^\dagger | 0 \rangle}{\langle 0 | 0 \rangle} = \sum_k \langle \psi_b | \psi_b \rangle = \sum_k \langle 0 | \hat{U}^\dagger \hat{a}_k \hat{a}_k^\dagger \hat{U} | 0 \rangle = \sum_k h_k ,
\]

which, in view of the consistency condition (4.1.4), may be replaced by $N$ to give

\[ N = \sum_k h_k \]
4.1 The interparticle potential

In order to solve equations (3.2.14), (3.2.15) and (3.2.16) it is necessary to assume an interatomic potential whose mathematical form possesses a Fourier transform. This necessity immediately excludes from consideration potentials of the Lennard-Jones six-n type, and other similar forms. A reasonable choice for a potential is one which has the same form as one which describes Helium gas. One such choice is the Frost-Musulin form of the potential,\(^{12}\)

\[ V = e^{-a r} \left[ \frac{c}{r^4} - b \right] \quad (4.1.1) \]

where \( r \) is the interparticle separation, and \( a, b, \) and \( c \) are parameters to be chosen to fit the experimental situation. This expression can be shown to fit closely other appropriate forms of the Helium gas potential, such as the Yntema-Schneider,\(^{13}\) the Slater-Kirkwood,\(^{14}\) and the DeBoer-Michels\(^{15}\) forms, for the following choice of parameters:


\(^{13}\)Yntema and Schneider, J. Chem. Phys. 18, 641 (1950).

\(^{14}\)Slater and Kirkwood, Phys. Rev. 37, 682 (1931).

\(^{15}\)DeBoer and Michels, Physica 6, 12 (1939).
parameters: \( a = 2.97 \, \text{Å}^{-1} \), \( b = 6.87 \times 10^{-11} \, \text{erg} \), \( c = 1.79 \times 10^{-10} \, \text{erg} \cdot \text{Å} \). These values were found by fitting the Frost-Musulin form to a curve representing the average of the three curves just mentioned. These values of the parameters are not likely to be the exact parameters appropriate for a description of liquid Helium. The Fourier transform of the Frost-Musulin potential is, according to (1.3.4),

\[
\tilde{\nabla}_k = \frac{2}{\alpha^2 + k^2} \left[ c - \frac{2ab}{\alpha^2 + k^2} \right]. \tag{4.1.2}
\]

A plot of this transform for the parameters mentioned is given in Fig. 2. This Frost-Musulin potential will not be convenient for the calculations of this paper; so for convenience, we shall choose a potential transform of the form of a Gaussian,

\[
\tilde{\nabla}_k = A e^{-\alpha k^2} \tag{4.1.3}
\]

where \( A \) and \( \alpha \) are parameters to be chosen in order to best fit experimental evidence for the liquid state. A Gaussian transform may be made to fit the Frost-Musulin transform fairly well, as illustrated in Fig. 2. For the sake of comparison, Fig. 2 also shows the Gaussian with parameters chosen to fit the liquid data on the basis of
results of the present paper. The choice of the Gaussian transform has the advantage of reducing the number of arbitrary parameters from three to two.
Fourier transforms for the interatomic potential of Helium.
4.2 Approximate Solution of the Integral Equations

We wish to solve (3.1.26) through (3.1.28) with (3.1.29) for $E_k$, eliminating the two functions, $h_k$ and $\chi_k$.

Taking the sums occurring in (3.1.26) and (3.1.27) to integrals as in (1.3.9) gives, for these three equations

\[ E - \lambda + \mathcal{V}_0 \Phi + \frac{i}{2(2\pi)^3} \int \mathcal{V}_{1k'k} (2h_{k'}) d^3k' = E_k (1 + 2h_k) \]  \hspace{1cm} (4.2.1)

\[ -\frac{i}{2} \mathcal{V}_k \chi_k - \frac{i}{2(2\pi)^3} \int \mathcal{V}_{1k'k} (2\chi_{k'}) d^3k' = E_k (2\chi_k) \] \hspace{1cm} (4.2.2)

and, (simply rewriting (3.1.2) for reference),

\[ (1 + 2h_k)^2 - (2\chi_k)^2 = 1 \] \hspace{1cm} (4.2.3)

According to (4.1.3), the two integrals occurring here have the form

\[ I = \int \mathcal{V}_{1k'k} f(k') d^3k' = 4\pi A \mathcal{E} \int_0^\infty k'^2 f(k') F(\alpha k k') dk' \] \hspace{1cm} (4.2.4)

where

\[ F(\alpha k k') = \frac{\sinh(2\alpha k k')}{(2\alpha k k')} = 1 + O(\alpha k k') \] \hspace{1cm} (4.2.5)

and where

\[ f(k') = \begin{cases} 
2h_{k'} & \text{if } \alpha k k' \\
2\chi_{k'} & \text{if } 2\chi_k 
\end{cases} \] \hspace{1cm} (4.2.6)
To reduce the integral equations (4.2.1) and (4.2.2) to the form of algebraic equations, an approximation Ansatz is made that $f(k')$ is negligible except for values of $k'$ such that $\alpha k k' \ll 1$. For such values of $k'$,

$$F(\alpha k k') \approx 1.$$  \hspace{1cm} (4.2.7)

Then, from (4.2.4),

$$I \approx 4\pi A e^{-\alpha k^2} \int_{k'}^{\infty} k' f(k') \, dk'.$$  \hspace{1cm} (4.2.8)

This approximation Ansatz is not generally valid \textit{a priori}. It will be necessary, therefore, to verify it \textit{a posteriori}.

Employing this approximation, we make (4.2.1) and (4.2.2) become the following algebraic equations:

$$\varepsilon - \lambda + \frac{\hbar}{2 \omega} A e^{-\alpha k^2} + \frac{i}{2 \pi^2} A e^{-\alpha k^2} Z(\omega) = E_k (1 + 2 \hbar \omega),$$  \hspace{1cm} (4.2.9)

and

$$-\frac{\hbar}{2} A_k e^{-\alpha k^2} - \frac{i}{2 \pi^2} A e^{-\alpha k^2} W(\omega) = E_{-k} (2 \chi_k),$$  \hspace{1cm} (4.2.10)

where

$$Z(\omega) = \int_{0}^{\infty} e^{-\alpha k'^2} (2 \hbar \omega) k'^2 \, dk'$$  \hspace{1cm} (4.2.11)

and

$$W(\omega) = \int_{0}^{\infty} e^{-\alpha k'^2} (2 \chi_k) \, dk'.$$  \hspace{1cm} (4.2.12)
Equations (4.2.9), (4.2.10), and (4.2.3) now may be solved easily for $E_k$. The result is

$$
E_k = \left\{ \left[ (E_{k0} - \lambda \cdot \nabla \phi) + \left( \frac{h_0}{\alpha \tau} + \frac{1}{4\pi^2} Z(\pi) \right) \nabla e^{-\alpha k^2} \right] \right\}^{\frac{1}{2}} \, (4.2.13)
$$

where $Z(\pi)$ and $W(\pi)$ are given by (4.2.11) and (4.2.12), with

$$2 \chi_k = \frac{1}{2} \frac{1}{\lambda_k \sqrt{X^2 - 1}} \, (4.2.14)
$$

and

$$1 + 2 \lambda_k = \frac{X}{\lambda \sqrt{X^2 - 1}} \, (4.2.15)
$$

$$X = \frac{1 + 2 \lambda_k}{2 \chi_k} = \left\{ \left[ (E_{k0} - \lambda \cdot \nabla \phi) \nabla e^{-\alpha k^2} + \left( \frac{h_0}{\alpha \tau} + \frac{1}{4\pi^2} Z(\pi) \right) \nabla A \right] \right\} \, (4.2.16)
$$

and

$$\lambda_k = \pm 1 \, (4.2.17)
$$

As discussed in section 3.1, we shall now choose the Lagrange multiplier, $\lambda$, so that $E_k = 0$ at $k = 0$. Setting $k = 0$ in (4.2.13) gives

$$\left[ (E_{0k} - \lambda \cdot \nabla \phi) + \left( \frac{h_0}{\alpha \tau} + \frac{1}{4\pi^2} Z(\pi) \right) \nabla A \right] = \left[ \left( \frac{h_0}{\alpha \tau} + \frac{1}{4\pi^2} W(\pi) \right) \nabla A \right] \, (4.2.18)
$$

Solving this for $\lambda$ gives
\[
\lambda = \frac{\hbar}{\sqrt{2}} \left( \frac{\hbar^2}{2m} + \frac{1}{2}\mathcal{Z}(\chi) \right) \Delta - j \left( \frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi) \right) \quad (4.2.19)
\]

where \( j = \pm 1 \). Using this value of \( \lambda \) in (4.2.16) and (4.2.13) yields

\[
X = \left\{ \frac{\hbar e^{-\chi^2}}{\sqrt{2}} \frac{(\frac{\hbar^2}{2m} + \frac{1}{2}\mathcal{Z}(\chi)) \Delta (e^{-\chi^2} - 1) - \left( \frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi) \right) \Delta}{(\frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi)) \Delta} \right\} \quad (4.2.20)
\]

and

\[
E_k^2 = \left\{ \left[ \frac{\hbar e^{-\chi^2}}{\sqrt{2}} \frac{(\frac{\hbar^2}{2m} + \frac{1}{2}\mathcal{Z}(\chi)) \Delta (e^{-\chi^2} - 1) - \left( \frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi) \right) \Delta}{(\frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi)) \Delta} \right]^2 - \left[ - \left( \frac{\chi^2}{\Delta^2} + \frac{1}{4\pi^2} \mathcal{W}(\chi) \right) \Delta e^{-\chi^2} \right]^2 \right\} \quad (4.2.21)
\]

Equations (4.2.20) and (4.2.21) contain \( \hbar_0 \) and \( \chi_0 \), which may be related by the following arguments:

From the derivation of (3.1.29), we note that

\[
\hbar_k = N_k \quad (4.2.22)
\]

where \( N_k \) is the ground state expectation value of the number of Helium atoms of momentum \( \hbar k \). Thus, from (4.2.3),

\[
\chi_0^2 = \hbar_0 (\hbar_0 + 1) = N_0 (N_0 + 1)
\]
But it is certainly true that

\[ \lim_{\nu \to \infty} \langle N_{\nu}, 1 \rangle = N_0, \]

so that, since we are concerned only with results calculated in this limit,

\[ \chi_o^2 = \hbar_o^2. \]

Thus

\[ \chi_o = \pm j \hbar_o \quad (4.2.23) \]

since \( j = \pm 1 \). The lower sign occurring here is spurious as may be seen as follows by reducing equation (4.2.21) with (4.2.23) under the further Bogoliubov assumption that \( N_o \gg N \): From (3.1.29), (4.2.23), and (4.2.3) we have

\[ N = \sum \sum N_{\nu \kappa} \approx N_o \approx \sum N_{\kappa} \approx 0 \Rightarrow \hbar_k = \chi_k \approx 0. \]

Then this result, together with (4.2.4) and (4.2.12) yields

\[ Z(\alpha) \approx W(\alpha) \approx 0, \quad (4.2.24) \]

which, with the aid of (4.2.11), (4.2.12), (4.2.21), (4.2.23), and (1.3.8), yields the Bogoliubov expression (2.2.6) for the dispersion curve, provided the upper sign is chosen in (4.2.23). The choice of the lower sign does not yield the correct Bogoliubov result in
the degenerate case that the interaction is so weak that \( N_0 \approx N \). Therefore we take

\[
\chi_0 = j h_0 .
\] (4.2.25)

It is not necessary in the present approximation to assume (falsely) that \( N_0 \approx N \). Instead, we shall employ the Bogoliubov result (4.2.24) only as a first step in an iterative solution of (4.2.20), (4.2.11), (4.2.12), (4.2.14), and (4.2.15) for \( Z(\alpha) \) and \( W(\alpha) \). In so doing, we shall, in the following treatment, proceed to obtain only a first order iterative approximate solution for the dispersion curve (4.2.21). Once a suitable choice is made for the parameters, \( \alpha \) and \( \Lambda \), which specify the interaction potential, then a second iteration will be made to validate the first order iteration approximation.

Upon substituting the Bogoliubov result (4.2.24) into (4.2.20), we obtain (in light of (4.2.25))

\[
\chi = - j \left\{ 1 + \frac{\Omega}{h_0 \Lambda} \sum_k E_k e^{-\omega k^2} \right\} .
\] (4.2.26)

To determine \( h_0 \), we employ (4.2.23) and (1.3.8) to obtain

\[
\rho \equiv \frac{N}{\Omega} = \frac{1}{\Omega} \sum_k N_k = \frac{h_0}{\Omega} + \frac{1}{\Omega} \sum_{k \neq 0} h_k ,
\]
or

\[
h_0 = \Omega \rho - \sum_{k \neq 0} h_k .
\] (4.2.27)

where, from (4.2.22), since
we have

\[ N_k \geq 0 \quad k \neq 0 \]

Finally, by combining (4.2.21), (1.3.2), (4.2.23), (4.2.28), (4.2.27), (4.2.15), (4.2.26), (4.2.11), (4.2.12), (4.2.14) and (4.2.15), we obtain the following summary for calculating the dispersion curve \( E_k \) for liquid Helium for any choice of the parameters \( A \) and \( \sigma \) (which characterize the assumed Gaussian two-body interaction potential for the Helium atoms):
\[ (4.2.29a) \]
\[ 2 - \left( \frac{1}{2} \left( \chi_{10} a^2 \right) - \frac{1}{2} \right) \Delta \left( e^{-\xi_k^2} + \left( \rho_0 - \frac{1}{\frac{\kappa}{\xi_k}} \right) \Delta \right)^2 \]
\[ - \left[ \left( \rho_0 - \frac{1}{\frac{\kappa}{\xi_k}} \right) \Delta e^{-\xi_k^2} \right]^2 \] 

with

\[ \xi_k = \frac{1}{\epsilon m} (\hbar k)^2 , \]  \hspace{1cm} (4.2.29b)

\[ \rho_0 \equiv \frac{\rho}{\rho_0} = - \frac{1}{\epsilon m^2 \xi_k} \int_0^\infty \left\{ \frac{e^{\frac{\epsilon}{\xi_k}}}{\sqrt{\left( \epsilon \xi_k \right)^2 + \gamma}} - \frac{1}{\gamma} \sqrt{\gamma} \right\} \epsilon d \xi , \]  \hspace{1cm} (4.2.29c)

\[ \Omega' \equiv \frac{\gamma}{\epsilon m} \int_0^\infty \frac{e^{-\frac{\epsilon}{\xi_k}}}{\sqrt{\left( \epsilon \xi_k \right)^2 + \gamma}} \frac{d \xi}{\xi} + \frac{1}{\epsilon m \xi} \int_0^\infty \frac{\epsilon \xi_k}{\sqrt{(\epsilon \xi_k)^2 + \gamma}} - \frac{1}{\epsilon m \xi} \sqrt{\gamma} \]  \hspace{1cm} (4.2.29d)

\[ W' \equiv \frac{\gamma}{\epsilon m} \int_0^\infty \frac{e^{-\frac{\epsilon}{\xi_k}}}{\sqrt{\left( \epsilon \xi_k \right)^2 + \gamma}} \frac{d \xi}{\xi} , \]  \hspace{1cm} (4.2.29e)

\[ \gamma = \frac{4 \omega n \Delta \rho_0}{\epsilon m} , \]  \hspace{1cm} (4.2.29f)

where \( m \) is the mass of a Helium atom and \( \rho \) is the density of the liquid.
5.1 Reduction to Dimensionless Form

It will be convenient for calculational purposes to express the functions to be calculated in terms of dimensionless quantities. We, therefore, make the following definitions:

\[ W'' = \frac{1}{4\pi^2\rho} W' \]  \hspace{1cm} (5.1.1)

\[ Z'' = \frac{1}{4\pi^2\rho} Z' \]  \hspace{1cm} (5.1.2)

\[ \rho_0'' = \frac{1}{\rho} \rho_0 \]  \hspace{1cm} (5.1.3)

\[ t = \sqrt{\alpha} k \]  \hspace{1cm} (5.1.4)

\[ E_k'' = \left( \frac{2m\alpha}{\hbar^2} \right) E_k \]  \hspace{1cm} (5.1.5)

\[ \Lambda'' = \left( \frac{\rho_0^2m\alpha'}{\hbar^2} \right) \Lambda \]  \hspace{1cm} (5.1.6)

With these equations (4.2.29a), (4.2.29c), (4.2.29d), and (4.2.29e) can be written in the following summary form
As it is of interest to examine the slope of the dispersion curve at the origin in order to compute the theoretical sound velocity, we expand (5.1.10) for small t, retaining only terms up to order \( t^2 \). The result of this expansion is that, for small \( t \),

\[
E_t \approx \left\{ 2A'' \left( \rho_0''' - W'' \right) \left[ 1 - \left( \rho_0''' + Z'' \right) A''' - \left( \rho_0''' - W'' \right) A'' \right] \right\} t^{1/2} \tag{5.1.11}
\]

It is evident from this expression that the dispersion curve passes through the origin with a constant slope, a distinct advantage of this theoretical approach.
For reference we list below the designation of the functions (5.1.7) through (5.1.11) that occur in the computer program of Appendix 3:

\[ W' = \text{FINRO} \]
\[ Z' = \text{FINZ} \]
\[ \varphi'_{o} = \text{FINRO} \]
\[ A' = \text{FINAMP} \]
\[ E_t' = \text{ETPLS} \]
\[ \lim_{t \to 0} \left( \frac{\partial E_t}{\partial t} \right) = \text{EPLSL} \]

and the following combinations of these functions:

\[ (\varphi'_{o} - W')A' = V1 \]
\[ (\varphi'_{o} + Z')A' = V2 \]

5.2 The Two Parameter Search

By substituting into (5.1.5) the mass of the Helium atom, \( m = 6.65 \times 10^{-24} \) gm, and substituting into (5.1.7) through (5.1.11) the particle density of Helium II at 1.1°K, \( \varphi = 2.2 \times 10^{-22} \) cm\(^{-3}\), values of the dispersion curve, \( E_k \), were computed for various choices of parameters \( \alpha \) and \( \gamma \). To insure a clear picture of the behavior of \( E_k \) as these parameters change, approximately 10,000 pairs of parameters were employed, with \( \alpha \) ranging from \( 1 \times 10^{-18} \) cm\(^2\) to \( 1 \times 10^{-13} \) cm\(^2\) and \( \gamma \) ranging from \( 1 \times 10^{-4} \) to \( 1 \times 10^6 \). (These computations were performed with the aid of an IBM 7040 computer. A description of
the computer programs employed is found in Appendix III.)

The results of these computations are presented in the following paragraphs. The theoretical dispersion curves were compared with the experimental result of Henshaw and Woods (Fig. 1). Only those theoretical results which exhibited a dispersion curve with a slope at the origin (zero momentum) within 10% of the experimental value are presented here. For each value of the parameter \( \gamma \) chosen, there will be only a single value of \( \alpha \) for which \( E_k \) has the proper slope at zero momentum. Hence, restricting the discussion to those \( E_k \) for which the slope is fixed, effectively reduces the two parameter (\( \gamma \) and \( \alpha \)) search to a single parameter (\( \gamma \)) search.

For reference, the definitions of some of the terms used in the following presentation of the results are reviewed here:

1) From (5.1.13) and (4.2.27), we find that

\[
\rho_0'' = \frac{1}{\mathcal{Q}} \rho_0 = \frac{N_0}{N}
\]  

(5.2.1)

is the fractional occupation of the zero momentum state. Any choice of \( \gamma \) that yields a resulting \( \rho_0'' \) less than zero will be excluded as leading to a non-physical situation.

2) From (4.2.29e),

\[
\gamma = \frac{4\alpha \rho_0 h A}{\hbar^2}
\]  

(5.2.2)
a dimensionless parameter. For any trial value of \( \chi \), once \( \varrho_0 \) is calculated, this equation (5.2.2) allows us to calculate the corresponding value of the parameter \( A \) which was originally introduced as one of the two parameters (\( A \) and \( \alpha \)) which specify our choice of an interaction potential transform.

In order that the theoretical and experimental slopes of the dispersion curve at the origin be equal (to within 10\%), the parameter \( \varrho_0'' \) increases (monotonically) as the parameter \( \chi \) increases. This result appears in Fig. 3. The value for \( \alpha \) increases with increasing \( \chi \) as well. For \( \chi \) less than 2 \( \times \) 10\(^{-2} \) (\( \alpha \) less than 1.2 \( \times \) 10\(^{-17} \) cm\(^2\)), \( \varrho_0'' \) is negative. These results appear in Fig. 4. As will be discussed in the next chapter, the results of greatest interest are those for small \( \varrho_0'' \) (below 0.1). This region corresponds to a choice of \( \varrho_0'' \) below 6 \( \times \) 10\(^{-2} \) (\( \alpha \) below 2 \( \times \) 10\(^{-17} \) cm\(^2\)).

For various choices of \( \chi \) the resulting dispersion curves, \( E_k \) vs \( k \) (with approximately the correct experimental slope) have the following features:

1) For \( \chi \) less than 1.5 (\( \varrho_0'' \) less than 0.880) the curve has a positive curvature near the origin, as shown by the typical curves in Fig. 5.

2) For \( \chi \) greater than 1.5 the curve has a negative curvature near the origin.
3) For $\xi$ Greater than 4 ( $\xi_0$ greater than 0.965) a minimum in the curve occurs. As $\xi$ is subsequently increased the minimum moves toward the origin and the height of the minimum decreases. This result is also indicated by the typical curves in Fig. 5.

For $\psi = 0.040$ and $\kappa = 1.58 \times 10^{-17}$ cm$^2$, $\xi_0$ has the value 0.08, corresponding to the experimentally determined value reported by Penrose and Onsager. The trial potential transform for these values of $\psi$ and $\kappa$ is indicated in Fig. 2.
FIGURE 3

The parameter $\gamma$ vs the fractional occupation number of the zero momentum state, $\rho_0''$, (for the slopes of the theoretical and experimental dispersion curves at the origin approximately equal).
The parameter $\gamma$ vs the parameter $\alpha$, (for the slopes of the theoretical and experimental dispersion curves at the origin approximately equal).
FIGURE 5

Theoretical dispersion curves for various choices of parameters $\alpha$ and $\gamma$. 
6.1 The Consistency Checks

In the foregoing presentation several approximation assumptions were made which were to be investigated a posteriori. One of these is (4.2.8). The approximation (4.2.8) was employed in obtaining the integrands (FZ and FW) of the integrals Z" (5.1.7) and W" (5.1.8). To determine the extent of the resulting error in $E_k$ (5.1.10) for a specific value of $k$, the integral Z" from (5.1.8) was compared to a corrected integral (one whose integrand was multiplied by

$$\frac{\sinh(2\alpha'k')}{(2\alpha'k')^{1/2}} = \frac{\sinh(2\sqrt{\alpha}k'z')}{(2\sqrt{\alpha}k'z')}$$

(6.1.1)

as indicated by (4.2.5) and by the change in variables of integration, $Z' = \sqrt{\alpha'} k)$. This calculation was repeated for several choices of $\chi$ and $\alpha$ (such that $E_k$ had approximately the correct experimental slope at the origin). The results of these calculations of errors introduced from error in FZ are summarized in the following table, where "%ERR" represents the percentage error in the dispersion curve at the value, $k_p$, of $k$ for which the experimental curve goes through a maximum, and where 10% CUT represents the value of the fraction

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for which the error in $E_k$ is 10%.

\[
\frac{k}{k_p} \tag{6.1.2}
\]

The foregoing argument applies only to errors introduced by the approximation in FZ. However, further calculations have shown that errors introduced by the approximation in FW are much smaller.

The second approximation assumption to be investigated is the validity of (3.1.3), \( \frac{\Delta N}{N} \ll 1 \). We shall determine whether the approximation was justified by computing directly the root mean square deviation \( \Delta N \) of the total number of atoms from the average number, \( N \).

\[
\Delta N = \sqrt{\frac{\langle N^2 \rangle - N}{N}}, \tag{6.1.3}
\]

where, according to (1.3.7),

\[
N = \langle 0 | U^{-1} \sum_k c_k^\dagger c_k U | 0 \rangle, \tag{6.1.4}
\]
and, similarly

$$\langle N^2 \rangle = \langle 0 | U^{-1} \sum_{\mathbf{k}, \mathbf{k}'} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} a_{\mathbf{k}'}^\dagger a_{\mathbf{k}} U | 0 \rangle \quad (6.1.5)$$

From the Bogoliubov transformation and from (3.3.17), equations (6.1.4) and (6.1.5) become

$$N = \sum_{\mathbf{k}, \mathbf{k}'} h_{\mathbf{k}} h_{\mathbf{k}'} \quad (6.1.6)$$

and

$$\langle N^2 \rangle = \sum_{\mathbf{k}, \mathbf{k}'} h_{\mathbf{k}} h_{\mathbf{k}'} + \sum_{\mathbf{k}} \chi_k^2 \quad (6.1.7)$$

where $h_{\mathbf{k}}$ and $\chi_k$ are defined by (3.2.6) and (3.2.7), respectively. From (6.1.6), (6.1.7), and (6.1.3), we find

$$\left( \Delta N \right)^2 = \sum_k \chi_k^2 \quad (6.1.8)$$

which with the aid of

$$\frac{\chi_0}{\mathcal{N}} = \varrho_0 = \frac{N_0}{\Omega} \quad (6.1.9)$$

implies

$$\left( \Delta N \right)^2 = \sum_k \chi_k^2 = N_0^2 + \sum_{k \neq 0} \chi_k^2 \quad (6.1.10)$$

In the limit as $\Omega$ approaches infinity,

$$\sum_{k \neq 0} \chi_k^2 = \frac{\Omega}{8 \pi^2} \int \chi_k^2 \, d^3 k = \frac{\Omega}{4 \pi^2} \int_0^{\infty} k^2 \chi_k^2 \, d k \quad (6.1.11)$$

Thus, from (4.2.14)
\[ \sum_{k} x_k^2 = \frac{\omega}{4\pi^2} \int_{0}^{\infty} \frac{k^2}{4(x^2-1)} \, dk \quad (6.1.12) \]

where \( X \) is given by (4.2.20) in the Valatin scheme independently of any further approximations. We note that the integral

\[ B \equiv \int_{0}^{\infty} \frac{k^2}{4(x^2-1)} \, dk \quad (6.1.13) \]

is independent of \( \omega \) and \( N \). Thus, from (5.2.2), (6.1.8), (6.1.12), and (6.1.13),

\[ \left( \frac{\Delta N}{N} \right)^2 = \left( \frac{N_0}{N} \right)^2 + \frac{\omega B}{4\pi^2 N^2} = \phi_0'' + \frac{B}{4\pi^2 \phi^2} \cdot \frac{1}{\omega} \]

so that, as \( \omega \) approaches \( \infty \),

\[ \frac{\Delta N}{N} = \phi_0'' \quad (6.1.14) \]

We shall now examine the effect of employing, rather than a first iteration approximation (as discussed in Section 4.2), a second iteration to determine the slope of the dispersion curve at the origin. For the particular choice of parameters \( \gamma = 0.040, \, \alpha = 1.58 \times 10^{-17} \text{erg-cm}^2 \), direct computation of the slope of the dispersion curve at the origin for the second iteration differed from the slope calculated from the first iteration by approximately 4%. 


6.2 Conclusions

The results of the consistency checks indicate that the approximation methods of this paper are internally consistent only for the range of the parameters \( \alpha \leq 1.7 \times 10^{-17}\text{cm}^2 \) and \( \gamma \leq 0.5 \). With the choice of parameters \( \alpha = 1.58 \times 10^{-17}\text{cm}^2 \) and \( \gamma = 0.040 \), a dispersion curve was obtained which passed through the origin linearly with a slope which fit experimental data. In Section 6.1 it was demonstrated that the first iteration approximation employed to obtain this result for the slope does not introduce significant error. Further, for the same choice of parameters, a value of \( \rho_0 \), the fractional occupation number of the zero momentum state, was obtained which was in agreement with that value (8%) calculated by Penrose and Onsager from experimental data. For the same choice of parameters, the form of the interaction potential transform is similar to the form of the potential transform obtained from gas data (see Fig. 2). However, for values of momentum increasing from zero, the dispersion curve obtained diverged increasingly from experimental results. In particular, no roton minimum was achieved.

The failure of this theory to yield a good dispersion curve (except for excitation with energy less than about \( 10^{-16} \text{ ergs} \)) in the region in which internal consistency is maintained could be taken to indicate one (or both) of two conclusions: Firstly, the form of the trial potential may
be inappropriate. The second, more significant conclusion is that the interaction among atoms may be too strong to allow the weak coupling approximation implied in neglecting the higher order perturbation terms in the statistical operator. Such a conclusion would indicate that the present improvement on the Bogoliubov approximation (in particular the avoidance of the assumption that the number of atoms in the zero momentum state approximately equals the total number of atoms is insufficient to correct the inability of the theory to yield both a dispersion curve with a satisfactory roton dip and a correct fractional occupation number of the zero momentum state. The foregoing remarks on the inadequacy of this approximation method apply only to the attempt to describe the excited states of the liquid with energy higher than the ground state energy by more than about $10^{-16}$ ergs.

The basic method may still be adequate for a reasonable description of the ground state wave function. As the justifying remarks of the first four chapters were directed primarily toward obtaining a ground state function, further investigations of this approach directed toward the ground state function may be warranted. A specifically suggested approach involves the following modifications:

1) Effort should be directed toward the determination of the liquid structure factor. (The liquid structure factor is the Fourier transform of the two-particle
distribution function.\textsuperscript{16} It is property of only the ground state of the liquid).

2) The Lagrange multiplier, $\lambda$, should then be determined such that the liquid structure factor, rather than the dispersion curve, passes through the origin.

3) The approximation by which the integral equations resulting from minimization of the ground state energy were converted to algebraic equations may be circumvented by an iteration process beginning with the approximation employed in this paper. If such an iterative process is used, a wider class of trial potential functions could as well be chosen.

BIBLIOGRAPHY

APPENDIX I
THE BOGOLIUBOV TRANSFORMATION

We shall here develop an explicit expression for the operator

\[ U_b^{-1} H' U_b \]  \hspace{1cm} (Al.0)

occurring in Chapter 3. In this expression, \( H' \) represents the statistical operator of Section 3.1, which may be written, upon combining (3.1.9), (3.1.2), and (1.3.1), as

\[ \sum_k (\varepsilon_k - \lambda) a_k^+ a_k + \frac{1}{2\Omega} \sum_{\Omega, \Omega', \Omega''} \tilde{Q}_{\Omega, \Omega', \Omega''} a_k^+ a_{\Omega', \Omega''} a_{\Omega, \Omega'} a_\Omega. \]  \hspace{1cm} (Al.1)

In (Al.0), \( U_b \) represents a unitary operator (transformation) defined (as in (2.2.3)) by

\[ U_b^{-1} a_k U_b = \frac{a_k + g_k a_{-k}^+}{\sqrt{1 - g_k^2}} \]  \hspace{1cm} (Al.2)

From (Al.1) and (Al.2), we find that (Al.0) may be written as

\[ U_b^{-1} H' U_b = \sum_k (\varepsilon_k - \lambda)(1 - g_k^2) (a_k^+ + g_k a_{-k})(a_{-k}^+ + g_k a_k) \]  \hspace{1cm} (Al.3)

\[ + \frac{1}{2\Omega} \sum_{\Omega, \Omega', \Omega''} \tilde{Q}_{\Omega, \Omega', \Omega''} \frac{(a_k^+ + g_k a_{-k})(a_{\Omega, \Omega', \Omega''}^+ + g_{\Omega', \Omega''} a_{\Omega, \Omega'})(a_{\Omega', \Omega''} + g_{\Omega', \Omega''} a_{\Omega, \Omega'}^+)}{(1 - g_k^2)(1 - g_{\Omega', \Omega''})(1 - g_{\Omega, \Omega'})(1 - g_{\Omega', \Omega''})} \]
We may express the right member of equation (A1.3) in the
following useful form by applying the commutation relations
(1.3.3) and by writing each term so that no creation
operator stand to the right of any annihilation operator:

\frac{1}{2!} \sum_{\mu, \nu} \left\{ (1 - q_{\mu}^2)^{1/2} (1 - q_{\nu}^2)^{1/2} (1 - q_{\nu}^2)^{1/2} (1 - q_{\mu}^2)^{1/2} \right\} \cdot (A1.4.0)

\cdot \left\{ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}) + (A1.4.1)

a_{-q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}) + (A1.4.2)

a_{q-k}^+ a_{q-k}^+ a_{-q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}) + (A1.4.3)

a_{q-k}^+ a_{q-k}^+ a_{-q-k}^+ (q_{q-k}, q_{q-k}) + (A1.4.4)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}, q_{q-k}) + (A1.4.5)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}, q_{q-k}) + (A1.4.6)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}, q_{q-k}) + (A1.4.7)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}, q_{q-k}) + (A1.4.8)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}, q_{q-k}) + (A1.4.9)

a_{q-k}^+ a_{q-k}^+ a_{q-k}^+ (q_{q-k}) + (A1.4.10)
\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.11) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.12) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.13) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.14) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.15) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.16) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.17) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.18) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.19) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.20) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.21) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.22) \]

\[ a_{\bar{\nu}}^+ a_{\bar{\nu}}^+ a_{\bar{\nu}} a_{\bar{\nu}} (g_{\bar{\nu} \bar{\nu}} g_{\bar{\nu} \bar{\nu}}) + \]  
\[ (Al.4.23) \]
\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 \right) \delta_{q_{\kappa},q_{\kappa'}} + \]  
(A1.4.24)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa} q_{\kappa'} \right) \delta_{\bar{q}_i,0} + \]  
(A1.4.25)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa} q_{\kappa'} \right) \delta_{q_{\kappa},0} + \]  
(A1.4.26)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa} q_{\kappa'} \right) \delta_{\bar{q}_i,0} + \]  
(A1.4.27)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa} q_{\kappa'} \right) \delta_{q_{\kappa},0} + \]  
(A1.4.28)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.29)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.30)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.31)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.32)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.33)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.34)

\[ a_{\kappa}^t a_{\kappa'} \left( q_{\kappa}^2 q_{\kappa} \right) \delta_{\bar{q}_{j},k_{i}} + \]  
(A1.4.35)
\begin{align*}
\alpha_{k'q} \alpha_{q'k} (q_{k+}^2 q_{q+}^2) \delta_{k,k'}^+ \\
\alpha_{k-} \alpha_{k^-} (q_{k-}^2 q_{k-}^2) \delta_{k,k}^+ \\
\alpha_{k'q} \alpha_{k'^{-}} (q_{k+}^2 q_{k'^{-}}^2) \delta_{k',k'^{-}}^+ \\
\alpha_{k-} \alpha_{k'^{-}} (q_{k-}^2) \delta_{k',0}^+ \\
\alpha_{k-} \alpha_{k'^{-}} (q_{k-}^2) \delta_{q',0}^+ \\
(q_{k+}^2 q_{k'^{-}}^2) \delta_{k',k'^{-}}^+ \\
(q_{k+}^2 q_{k'^{-}}^2) \delta_{k'-k',k'^{-}}^+ \\
(q_{k+} q_{k'^{-}}) \delta_{q',0}^+ \\
(q_{k+} q_{k'^{-}}) \delta_{q',0} \\
\end{align*}
In this section, we shall develop from expression (3.1.15) the explicit expressions for $W_0$ and $E_k$, equations (3.1.16) and (3.1.17). Further, we shall perform the minimization of $W_0$, as discussed in Section 3.1 and obtain the resulting equation (3.1.23).

In Section 3.1, it was stated that the operator $U^{-1}H'U$ (which is written explicitly in Appendix I) may be written (in light of (3.1.22)) as

$$U^{-1}H'U = W_0 + \sum_{k} E_k a_k^+ a_k + C + D,$$  \hspace{1cm} (A2.1)

where

$$W_0 = \langle 0 | U_b^{-1} H U_b | 0 \rangle,$$  \hspace{1cm} (A2.2)

and $E_k$ represents the coefficient of all terms in (A2.1) of the form $a_k^+ a_k$. Comparing (A2.1) and (A1.4) reveals that

$$W_0 = \sum_{k} (E_k - \lambda) \frac{g_k^2}{1 - g_k^2} + \frac{\nabla_0}{2 \Omega} \sum \frac{g_k^2 g_{k'}^2}{(1 - g_k^2)(1 - g_{k'}^2)}$$  \hspace{1cm} (A2.3)

$$+ \frac{1}{2 \Omega} \sum_{k,k'} \frac{g_k^2 g_{k'}^2}{(1 - g_k^2)(1 - g_{k'}^2)} + \frac{1}{2 \Omega} \sum_{k,k'} \frac{g_k g_k'}{(1 - g_k^2)(1 - g_{k'}^2)} (A2.4)$$

Into this expression, substitute the definitions (as in (3.1.18) and (3.1.19))
\[ h_k = \frac{g_k^2}{1 - g_k^2} \]  \tag{A2.4}

and

\[ \chi_k = \frac{g_k^2}{1 - g_k^2} \]  \tag{A2.5}

and (as in (3.1.20) and (3.1.21)),

\[ \psi_k = (\varepsilon_k - \gamma) + \frac{1}{\hbar} \sum_{k'} \left( \bar{\psi}_{k-k'} \bar{\psi}_o \right) h_k' \]  \tag{A2.6}

and

\[ \psi_k = -\frac{1}{\hbar} \sum_{k'} \bar{\psi}_{k-k'} \chi_{k'} \]  \tag{A2.7}

so that (A2.3) becomes

\[ W_0 = \frac{1}{2} \sum_k \left( \varepsilon_k - \gamma \right) h_k \psi_k \chi_k \]  \tag{A2.8}

which is in the form stated in (3.1.16).

We shall now consider all terms, (A1.4.17) through (A1.4.28), in \( \langle o | U'H'U | o \rangle \) of the form \( a_k^+ a_k \). This collection of terms is denoted \( E_k \) in (3.2.3). As before, substitute into this collection

\[ h_k = \frac{g_k^2}{1 - g_k^2} \]  \text{,} \quad \chi_k = \frac{g_k^2}{1 - g_k^2}  \tag{A2.9}

whereupon, obtain that
\[
\sum_{k} E_k a_{ik}^t c_{ik} = \sum_{k} \frac{1 + g_{k}^{2}}{1 - g_{k}^{2}} (E_k - \lambda) a_{ik}^t c_{ik}
\]

\[
+ \frac{1}{2\Omega} \sum_{k,k'} \tilde{V}_{o, k} h_{k} c_{ik} a_{ik}^t + \frac{1}{2\Omega} \sum_{k'} \tilde{V}_{1k, k'} h_{k'} c_{ik} a_{ik}^t
\]

\[
+ \frac{1}{2\Omega} \sum_{k,k'} \tilde{V}_{1k, k'} a_{ik} a_{ik}^t + \frac{1}{2\Omega} \sum_{k} \tilde{V}_{o, k} h_{k} a_{ik}^t c_{ik}
\]

\[
+ \frac{1}{2\Omega} \sum_{k,k'} \tilde{V}_{o, k} h_{k'} a_{ik}^t a_{ik} + \frac{1}{2\Omega} \sum_{k} \tilde{V}_{1k, k'} h_{k'} a_{ik}^t a_{ik}
\]

\[
+ \frac{1}{\Omega} \sum_{k,k'} \tilde{V}_{1k, k'} a_{ik} a_{ik}^t + \frac{1}{\Omega} \sum_{k} \tilde{V}_{1k, k'} a_{ik} a_{ik}^t
\]

The first term of this expression takes a convenient form if we add and subtract \(g_{k}^{2}\) in the numerator; whereupon, by employing (A2.9), we obtain

\[
\sum_{k} \frac{1 + g_{k}^{2}}{1 - g_{k}^{2}} (E_k - \lambda) c_{ik}^t c_{ik} = \sum_{k} \frac{(1 - g_{k}^{2}) + 2 g_{k}^{2}}{(1 - g_{k}^{2})} (E_k - \lambda) c_{ik}^t c_{ik} (A2.10)
\]

\[
= \sum_{k} (1 + 2 h_{k})(E_k - \lambda) c_{ik}^t c_{ik}
\]

By combining this result with (A2.10) and substituting from (3.1.20) and (3.1.21), we find

\[
\nu_k = (E_k - \lambda) + \frac{1}{\Omega} \sum_{k'} (\tilde{V}_{1k, k'} + \tilde{V}_{o}) h_{k'} (A2.11)
\]
and

\[ \mathcal{W}_k = -\frac{1}{\sigma^2} \sum_{k'} \tilde{\mathcal{V}}_{kk'} \chi_{kk'} \]

the form

\[ E_k = \mathcal{W}_k (1 + 2 h_k) - 2 \mu_k \chi_k \]

which is in the form of (3.1.17).

We shall now minimize \( W_o \), equation (A2.8), with respect to \( g_k \). To accomplish this, we require (using A2.8) that

\[ 0 = \frac{\partial \mathcal{W}_o}{\partial g_{k'}} = \frac{1}{2} \sum_k \left\{ (\varepsilon_{kk'} - \lambda) \frac{\partial h_k}{\partial g_{k'}} + \gamma_k \frac{\partial h_k}{\partial g_{k'}} + \mu_k \frac{\partial \chi_k}{\partial g_{k'}} - \chi_k \frac{\partial h_k}{\partial g_{k'}} - \mathcal{W}_k \frac{\partial \mu_k}{\partial g_{k'}} \right\} \]

which can be rearranged as follows:

\[ 0 = (\varepsilon_{kk'} - \lambda) \frac{\partial h_k}{\partial g_{k'}} + \gamma_k \frac{\partial h_k}{\partial g_{k'}} - \mu_k \frac{\partial h_k}{\partial g_{k'}} + \sum_k \left( h_k \frac{\partial \chi_k}{\partial g_{k'}} - \mathcal{W}_k \frac{\partial \mu_k}{\partial g_{k'}} \right) \]

It will be necessary to find explicit expressions for the four derivatives occurring in this expression. These derivatives may be written from their definitions as follows:

1) From the definition of \( h_k \) in (A2.1), we find that

\[ \frac{\partial h_k}{\partial g_{k'}} = \frac{2 g_k}{(1 - g_k)} \]

2) From the definition of \( \chi_k \) in (A2.5), we find that
\[ \frac{\partial Y_k}{\partial g_k} = \frac{1+g_k^2}{(1-g_k^2)^2} \]  

(A2.17)

3) From the definition of \( Y_k \) in (A2.6), we find that

\[ \frac{\partial Y_k}{\partial g_k'} = \frac{1}{\Omega} (\nabla_{(k-k')1}^2 \nabla_o) \frac{2g_k'}{(1-g_k^2)^2} \]  

(A2.18)

4) From the definition of \( Y_k \) in (A2.7), we find that

\[ \frac{\partial Y_k}{\partial g_k'} = -\frac{1}{\Omega} \nabla_{(k-k')1} \left( \frac{1+g_k^2}{1-g_k^2} \right) \]  

(A2.19)

Now substituting (A2.16) through (A2.19) into (A2.15) yields

\[ O = (\epsilon_k - \lambda) \frac{2g_k'}{(1-g_k^2)} + \frac{2g_k'}{(1-g_k^2)} \frac{1+g_k^2}{(1-g_k^2)^2} \]  

(A2.20)

By multiplying this expression by \((1-g_k)^2\), rearranging, and employing definitions (A2.4) through (A2.7), we obtain

\[ Y_k g_k^2 - 2 Y_k g_k + Y_k = O \]  

(A2.21)

which represents the minimizing condition on \( g_k \), as in (3.1.23).
Several of the computations of this paper were performed with the aid of an IBM 7040 computer. In this appendix we shall briefly describe the program employed.

The calculations included computing, from equations (5.1.7) through (5.1.10), a dispersion curve, (5.1.10), for various choices of parameters $\alpha$ and $\gamma$, with $1 \times 10^{-18}$ cm$^2 < \alpha < 1 \times 10^{-13}$ cm$^2$ and $1 \times 10^{-4} < \gamma < 1 \times 10^{6}$. (The integrations indicated in (5.1.7), (5.1.8), and (5.1.9) do not involve the parameter $\alpha$.) In the program, expressions for the integrands in (5.1.7), (5.1.8), and (5.1.9) are denoted $F_z$, $F_w$, and $F_{RO}$. The variable of integration, $z$, is denoted by $X$ in the program. The integration was accomplished by Simpson's rule, integrating from $z$ equal zero to a value $z'$ of $z$ for which the integral from $z'$ to infinity (overestimated by an approximate asymptotic form) was less than a fraction, (EFR), of the value of integral integrated up to $z'$. The fraction EFR was specified by an input card. For $\gamma$ very small (less than 0.01), a power series expansion was employed for calculating $FW$. Results for several choices of the fraction (EFR) and several choices for the increment, $DX$, of $X$ (involved in Simpson's rule) were compared to insure

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accurate results. This procedure resulted in a final choice of $10^{-4}$ for EFR and 0.01 for DX.

Values of $\gamma$ and $\omega$ were then used in the computation of (5.1.1), (5.1.2), (5.1.3), (5.1.6), and (5.1.5) (designated FINW, FINZ, FINRO, FINAMP, ETPLS, respectively). An option was provided to have printed only those dispersion curves which have slopes within 10% of the experimental value. Other options allowed the operator to avoid having printed the integrands or the dispersion curves. This allowed a quick printout of the test parameters described below.

To compare the theoretical results with the experimental data, several test parameters were computed and printed. The value of $t$, occurring in (5.1.10), at which the experimental maximum of $E_t$ occurred was compared with the computed value and their ratio, TPTST, was printed. The values of the computed and experimental dispersion curve at the maximum and at the minimum of $E_t$ (if any) were compared and their resulting ratios, PKTST and DPTST, were printed. The theoretical and experimental slope of the dispersion curve at zero momentum were compared and their ratio, ESLTST, was printed. If no theoretical maximum in the dispersion curve occurred (up to a value of $k$ or 1.4 times the value of $k$ for which the experimental dispersion curve has a maximum) "NO PEAK" was printed, and if no minimum occurred, "NO DIP" was printed.
Further details of the computations are illustrated in the flow chart of the computer program, Fig. 6. The complete FORTRAN IV Program follows the flow chart.
FORTRAN IV program flow chart (continued on next page)
FORTRAN IV program flow chart (continued on next page)
FÖRTRAN IV program flow chart
The following is the computer program employed in the computations of the present paper:

```plaintext
C1=-1.0
C2=45./48.
C3=-84./32.
C4=105./128.
C5=-198./256
C6=3003./4096.
C7=-17160./24576.
Y=-1.

10 PAUSE
2 READ 1,INDX,VAR,DEL,NV
1 FORMAT (II,2E10.2,13)
INDX=INDX+1
GO TO (2,3,4,7,8,91),INDX
3 G=VAR
GI=G
DG=DEL
NG=NV
IG=0
GO TO 2
4 AL=VAR
ALI=VAR
DAL=DEL
NAL-NV
IAL=0
GO TO 2
7 EFR=VAR
DX=DEL
GO TO 2
8 RNGMT=VAR
NT=NV
GO TO 2
91 CALL SSWTCH (3,I)
IF (I-1) 74,92,74
92 PRINT 72,AL,G,EFR
72 FORMAT(1H1INTEGRANDS,5X,7H ALPHA=,E13.6,4X,7H $\Gamma$=,E13.6,2X,5H EFR=,E10.2/3X,1HX,10X,2HFZ,17X, $\text{FW}=,14X,3HFRO,12X,4H SRO,10X,1HY)
74 MTZ=1
MTW=1
MTRO=1
WT=1.
SZ=0.
SW=0.
SRO=0.
X=0.
```
11 Ul=X*EXP(X)+G
   Ul=SQRT(U1)
   U2=EXP(-X/2.)
   IF(MTZ) 17,17,12
12 FZ=X*U2/U1
   SZ=SZ+WT*FZ
17 IF(MTW) 23,23,18
18 FW=EXP(-3.*X/2.)/U1
   SW=SW+WT*FW
23 IF(MTRO) 29,29,24
24 IF(X) 76,75,76
75 FRO-U2/U1
   GO TO 88
76 Y=G*U2*U2/X
   IF(Y=0.1) 80,77,77
77 FRO=(U2+2.*X/(G*U2))/(U1)-2.*SQRT(X)/G
   GO TO 88
80 Q=1.
   Y2=Y*Y
   IF(Y=1.0E-08) 87,85,85
85 Y3=Y2*Y
   Q=Q+C1*Y+C2*Y2+C3*Y3
   IF(Y=0.01) 87,86,86
86 Y4=Y3*Y
   Y5=Y4*Y
   Y6=Y5*Y
   Y7=Y6*Y
   Q=Q+C4*Y4+C5*Y5+C6*Y6+C7*Y7
87 FRO=SQRT(X)*Y2*Q/(4.*G)
88 SRO=SRO+WT*FRO
29 IF(X) 31,30,31
30 WT=4.0
   GO TO 48
31 IF(X=1.) 45,45,32
32 IF(MTZ) 36,36,33
33 IF(WT=2.) 35,34,36
34 ERZ=3.*(1.+X)*EXP(-X)/DX-EFR*SZ
   IF(ERZ) 35,35,36
35 MTZ=0.
   SZ=SZ-FZ
   FZ=0.
36 IF(MTW) 40,40,37
37 IF(WT=2.) 40,38,40
38 ERW=3.*EXP(-2.*X)/(2.*SQRT(X)*DX)-EFR*SW
   IF(ERW) 39,39,40
39 MTW=0.
   SW=SW-FW
   FW=0.
40 IF(MTRO) 44,44,41
41 IF(WT=2.) 44,42,44
42 ERRG=3.*C*EXP(-2.*X)/(8.*X*SQRT(X)*DX)-EFR*SRO
   IF(ERRG) 43,43,44
ATRO=0.
  SRO=SRO-FRO
  FRO=0.
IF(MTZ+MTW+MTRO) 45,49,45
IF(WT-2.) 47,46,47
WT=4.
  GO TO 48
WT=2.
CALL SSWTCH (3,I)
  IF(I-1) 20,21,20
21 PRINT 25,X,FZ,FW,FRO
25 FORMAT (IX,F7.3,4(2X,E13.6),2X,E10.2)
  CALL SSWTCH (1,J)
  IF(J-1) 20,10,20
X=X+DX
  GO TO 11
ENZ=DX*SZ/3.
  ENW=DX*SW/3.
  ENRO=DX*SRO/3.
  PRINT 78
50 TMIN=SQRT(AL)*1.930E+8
  TMAX=SQRT(AL)*1.100E+8
  FINRO=1.-ENRO/(8.*9.86588*2.1832E+22*AL*SQRT(AL))
IF(FINRO) 53,53,61
53 PRINT 52,AL,G,FINRO
52 FORMAT(1& RH0 OUT OF RANGE ,6H ALPHA=,E13.6,2X,7H $  GAMMA=,E13.6,2X,4HRHO=, El3.6)
  CALL SSWTCH (1,J)
  IF(J-1) 99,10,99
99 CALL SSWTCH (6,12)
  IF(I2-1) 61,90,61
61 FINW=G*ENW/(16.*9.86588*2.1832E+22*AL*SQRT(AL))
  FINZ=FINW+ENZ/(8.*9.86588*2.1832E+22*AL*SQRT(AL))
  $-1.77245/(16.*9.86588*2.1832E+22*AL*SQRT(AL))
  IF(FINRO) 63,62,63
62 FINAMP=0.
  GO TO 64
63 FINAMP=2.*G/FINRO
64 AMP=1.11195E-16*FINAMP/(2.*6.6690*AL*2.1832E+26)
  V1=(FINRO+FINZ)*FINAMP
  V2=(FINRO-FINW)*FINAMP
  ETMAX=2.2180E+16*AL
  ETMIN=1.3730E+16*AL
  EPLDR=2.*(V2-V2*V1+V2*V2)
  IF(EPLDR) 131,130,131
130 EPLSL=SQRT(EPLDR)
  GO TO 135
131 EPLSL=0.0
135 ETHSL=3.700E+08*SQRT(AL)
  ESLTST=EPLSL/ETHSL
  IF(ESLTST-1.1) 201,201,202
201 IF (0.5-ESLTST) 203, 203, 202
202 CALL SWITCH (4, J2)
   IF (J2-1) 203, 90, 203
203 TSTOP=RNGMT*TMIN
   FNT=XT
   DT=TSTOP/FNT
   T=0.
56 FORMAT (26H DISPERSION CURVE ALPHA=, E13.6, 4X, 6H
$\alpha$=, E13.6/10H RHOPRIME=, E13.6, 4X, 5H AMP=, E13.6/
$\gamma$=, E13.6, 4X, 6H TMIN=, E13.6, 1X, 6H ETMIN=, E13.6,
$\delta$=, E13.6/7H EPLSL=, E13.6, 4X, 7H ETHSL=,
$\eta$=, E13.6, 4X, 7H ESLTST= , E13.6/7X, 1HT, 13X, 5HETPLS, 1X)
57 PRINT 56, AI, G, FINRO, AMP, TMIN, ETMIN, TMAX, ETMAX, EPLSL,
$\text{ETHSL, ESLTST}$
   PRINT 555, ENZ, ENW, ENRO, FINZ, FINW
555 FORMAT (2X, 6NH/4 NZ=, E13.6, 2X, 6HENW=, E13.6, 2X, 5HENRO=,
$\text{Finw}=, E13.6, 2X, 5FINW, E13.6)$
PSTP=0.0
   IDXP=0
   IDXD=0
120 IF (6.-T) 110,110,65
110 V3=T*T-V1
   V4=0.0
   GO TO 104
65 V3=(T*T)+V1*(EXP(-T*T)-1.)
   V4=V2*EXP(-T*T)
104 V5=V4*V4
   EPLSQ=(V3+V2) *(V3+V2) -V5
   IF (EPLSQ) 13, 13, 14
13 ETPLS=0.0
   GO TO 68
14 ETPLS=SQRTEP (EPLSQ)
   IF (T-TMAX) 332, 331, 331
331 TENDST=ETPLS/ETMAX
332 IF (IDXP) 153, 153, 161
151 IF (IDXD) 171, 171, 68
153 S=T+DT
   IF (S-TSTOP) 162, 162, 68
162 IF (ETFORM-ETPLS) 68, 68, 164
164 PEAK=ETFORM
   TPTP=T-DT
   TPTST=TP/TMAX
   IDXP=1
   GO TO 68
171 U=T+DT
   IF (U=TSTOP) 150, 150, 68
150 IF (ETFORM-ETPLS) 152, 68, 68
152 DIP=ETFORM
   TD=T-DT
   TDTST=TD/TMIN
IDXD=1
68 C.LL SSWITCH (2,13)
   IF (I3-1) 180, 69, 180
180 PRINT 67, T, ETPLS
67 FORMAT (1X, E13.6, 3X, E13.6)
69 T=T+DT
   ETFORM=ETPLS
   IF (T-TSTOP) 120, 120, 155
155 IF (IDX(P) ) 158, 158, 156
156 PKTST=PEAK/ETMAX
   PRINT 157, PLAK, TP, PKTST, TPTST
157 FORMAT (1X, 5HPEAK=, E13.6, 3X, 3HTP=, E13.6, 3X, 6HPKTST=,
      $E13.6, 3X, 6HTPTST=, E13.6)
   IF (IDXD) 165, 165, 166
166 DPTST=DIP/ETMIN
   PRINT 167, DIP, TD, DPTST, TDTST
167 FORMAT (2X, 4'.iDIP=, E13.6, 3X, 3HTD=, E13.6, 3X, 6HDPTST=,
      $E13.6, 3X, 6HTDTST=, E13.6)
   GO TO 79
165 PRINT 163, TENDST
168 FORMAT (1O, 6HNO DIP, 3X, 7HTENDST=, E13.6)
   GO TO 79
158 PRINT 159, TENDST
159 FORMAT (10X, 7HNO PEAK, 3X, 7HTENDST=, E13.6)
79 IF (I3-1) 181, 132, 181
182 PRINT 183
183 FORMAT (1X//)
   GO TO 90
181 PRINT 78
78 FORMAT (1H1)
90 IAL=IAL+1
   AL=AL+DAL
   IF (NAL-IAL) 6, 6, 50
6 AL=AL
   IAL=0
   IG=IG+1
   G=G+DG
   IF (NG-IG) 58, 58, 91
58 G=GI
   IG=0
   GO TO 2
LND
VITA

Peter Duane Skiff was born in Pittsburgh, Pennsylvania, on December 16, 1938. He attended elementary and secondary schools in Euclid, Ohio, and Houston, Texas, and was graduated from C. H. Milby High School, Houston, Texas, in 1955. He pursued undergraduate studies for two years at the University of Houston and for two years at the University of California, receiving the Bachelor of Arts degree in Physics from the University of California in June, 1959. He pursued graduate studies at the University of Houston, where he held teaching and research assistantships. He received the Master of Science degree in Physics from the University of Houston in August, 1961, and enrolled in Graduate School of Louisiana State University in September, 1961. At Louisiana State University he held a teaching assistantship from 1961 until 1963, and the position of Instructor in Physics from 1963 until 1966. In August, 1966, he was appointed to the position of Assistant Professor of Physics, Bard College. He is now a candidate for the Doctor of Philosophy degree in the Department of Physics and Astronomy, Louisiana State University.
EXAMINATION AND THESIS REPORT

Candidate: Peter Duane Skiff

Major Field: Physics

Title of Thesis: Low Energy Excitations in Liquid Helium

Approved:

[Signature]

Major Professor and Chairman

[Signature]

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signature]

[Signature]

[Signature]

[Signature]

Date of Examination: August 11, 1966