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Symmetric spaces

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SYMMETRIC SPACES

A Dissertation
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Louisiana State University and
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in partial fulfillment of the
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Doctor of Philosophy
in
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by
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Abstract

We first review the basic theory of a general class of symmetric spaces with canonical reflections, midpoints, and displacement groups. We introduce a notion of gyrogroups established by A. A. Ungar and define gyrovector spaces slightly different from Ungar’s setting. We see the categorical equivalence of symmetric spaces and gyrovector spaces with respect to their corresponding operations.

In a smooth manifold with spray we define weighted means using the exponential map and develop the Lie-Trotter formula with respect to midpoint operation. Via the idea that we associate a spray with a Loos symmetric space, we construct an analytic scalar multiplication on a smooth gyrocommutative gyrogroup with unique square roots. We furthermore develop the concepts of parallel transport and parallelogram. Later we see that the exponential map associated with spray in the Finsler gyrovector space with seminegative curvature gives us a length minimizing geodesic.

Analogous to define a partial order on a vector space, we construct the partial order on the gyrovector space and investigate its properties related with what we call the gyrolines and the cogyrolines. Finally we apply the concept of gyrogroup structure to the setting of density matrices, especially qubits generated by Bloch vectors, and show the equivalence between the set of Bloch vectors and the set of Lorentz boosts.
Introduction

This story actually began with our interest in matrix means on the open convex cone $\Omega$ of positive definite matrices. We equip a partial order $\leq$ on $\Omega$ such as

$$A \leq B \text{ if and only if } B - A \in \overline{\Omega},$$

where $\overline{\Omega}$ is the set of positive semidefinite matrices. Over several decades there are many different approaches of matrix means, but we mainly consider the definition appeared in [6, Chapter 4].

Definition 0.1. A matrix mean is a binary operation $M : \Omega \times \Omega \to \Omega$ that satisfies

(i) $A \leq M(A, B) \leq B$ whenever $A \leq B$,

(ii) $M(A, B) = M(B, A)$,

(iii) $M(A, B) \leq M(A', B')$ whenever $A \leq A'$ and $B \leq B'$,

(iv) $M(X^*AX, X^*BX) = X^*M(A, B)X$ for any invertible $X$,

(v) $M(A, B)$ is continuous with respect to variables $A$ and $B$.

There are good examples of matrix means satisfying Definition 0.1; for instance, the arithmetic and harmonic mean

$$\frac{A + B}{2} \text{ and } \left(\frac{A^{-1} + B^{-1}}{2}\right)^{-1},$$

defined as same as those for positive numbers. It is not obvious to define the geometric mean of positive definite matrices $A$ and $B$ since the product $A^{1/2}B^{1/2}$ is not Hermitian. Using the property of the geometric mean $\sqrt{ab}$ for positive real numbers $a$ and $b$, the unique positive solution of the equation $xa^{-1}x = b$, we can define the geometric mean for positive definite matrices $A$ and $B$ as the unique
positive definite solution of the Riccati equation $XA^{-1}X = B$ and obtain its explicit form

$$A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2},$$

usually denoted by $A\# B$; see [9, Lemma 2.4]. Furthermore, R. Bhatia and J. Holbrook have identified the geometric mean of $A$ and $B$ as the midpoint of the geodesic joining $A$ and $B$ with respect to a natural Riemannian metric. They have also found the form of geodesic joining $A$ and $B$

$$A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for $t \in [0, 1]$, usually called the weighted geometric mean and denoted by $A\#_t B$, and have established its basic properties with respect to the Riemannian metric; see [5] and [6, Chapter 6].

We here focus on the operation $X \bullet A = XA^{-1}X$ in the Riccati equation, viewed as a symmetry or point reflection through $X$. This concept of symmetry gives us the algebraic theory of symmetric set, see [11]. In Chapter 1 we give a quick review of background work on symmetric means derived from the combined notions of symmetry and mean, the quadratic representation into the group of displacements, and lineated symmetric spaces modified symmetric means topologically.

We introduce in Chapter 3 a notion of spray developed by S. Lang and the exponential map associated with spray, see [20, Chapter IV]. Moreover, we define weighted means using the fundamental result of S. Lang that any two points in a normal neighborhood are joined by a unique geodesic connected with the exponential map. We also give a definition of locally midpoint-preserving map and show that the continuous and locally midpoint-preserving map is smooth on manifolds with spray using the Lie-Trotter formula in Theorem 3.77.
We consider in Chapter 4 symmetric spaces on Banach manifold satisfying the axioms of Loos that are additionally lineated symmetric spaces. We refer an important result of K.-H. Neeb that we associate a spray to a symmetric space with the same symmetries and it is uniquely determined by this property. The concept of a spray is central to our discussion below because it encodes the exponential function of the underlying manifold.

Attempts to find appropriate algebraic coordinatizations of more general geometric settings have led to the study of more general algebraic structures. A number of these have nonassociative binary operations. In recent years A. A. Ungar has introduced gyrocommutative gyrogroups as an appropriate structure for the development of an analytic hyperbolic geometry [1]. It has been shown that gyrocommutative gyrogroups are equivalent to Bruck loops [17] with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to $B$-loops. We show in Chapter 2 furthermore that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to pointed symmetric means, and investigate the homomorphisms in their setting. Later on, we see that gyrovectors are equivalent to pointed lineated symmetric spaces in the topological setting.

We show in Chapter 4 that a smooth gyrocommutative gyrogroup with unique square roots is a Loos symmetric space, and moreover, becomes a gyrovector space with a scalar multiplication derived from the exponential map associated with the spray. A. A. Ungar has defined the concept of parallel transport of rooted gyrovectors via certain condition of their values, but we show the condition using the well-known properties obtained by K.-H. Neeb, J. Lawson and Y. Lim. Then we study the equivalent conditions of vertices parallelogram, analogous to the Eu-
clidean parallelogram in vector spaces, and moreover, investigate several properties related with the gyroautomorphisms known as Thomas precession.

Finding the shortest path joining two points in a manifold is one of the interesting research subjects. In a differentiable manifold, for instance, we can find it by writing the equation for the length of a curve, and then minimizing this length using the calculus of variations. We consider in Chapter 5 a Finsler gyrovector space defined in Definition 5.115. Applying the result that a smooth gyrocommutative gyrogroup with unique square roots is a gyrovector space, we see that the exponential map associated with spray in the Finsler gyrovector space with semi-negative curvature gives us a length minimizing geodesic. Furthermore, we obtain the length minimizing geodesic is a gyrolines named by A. A. Ungar [1, Chapter 6]. On the open convex cone $\Omega$, for instance, the gyroline passing through $A$ and $B$ is the weighted geometric mean given by the equation (1).

On the other hand, Ungar has introduced in [1, Chapter 2] the gyrogroup cooperation that captures useful analogies between gyrogroups and groups, and uncovers dual symmetries with the gyrogroup operation. Under this cooperation he has defined a cogyroline and found many properties similar to the gyrolines. On the open convex cone $\Omega$, for instance, the cogyroline passing through $A$ and $B$ is given by

$$\left( A^{-1} \# B \right)^t A \left( A^{-1} \# B \right)^t. \quad (2)$$

At $t = 1/2$ this is known as the spectral geometric mean, and we may see its properties in [8]. In the first part of Chapter 6 we investigate more the gyroline and the cogyroline, and corresponding midpoints in the general setting of gyrogroups. We define a partial order on the gyrovector space based on the idea to define a partial order on the vector space via a proper convex cone. Moreover, we develop its
properties related with gyrolines and cogyrolines in the rest of Chapter 6 analogous to those for positive definite matrices.

In [1, Chapter 9] A. A. Ungar have talked Bloch vectors, mixed state qubits, and the Bures fidelity in the sense of gyro structure that are important concepts in quantum computation and quantum information. See his bibliography in [1] for more references. We first give a gyrogroup structure to the setting of density matrices that can provide useful algebraic tools in their study. Via the diagonalization of qubits, we see that the open unit ball of Bloch vectors with the Einstein vector addition is isomorphic to the set of invertible qubits with certain operation. Later on, we show an isomorphism between the set of Bloch vectors with the Einstein vector addition and the set of Lorentz boosts with certain operation through a polar decomposition. We finally see the several equivalent formulas of the Bures fidelity generated by two qubits.
1. Symmetric Spaces

The structure and theory of the set of positive operators we call the symmetric geometry of positive operators. The machinery we investigate has wider application than the study of positive operators. Hence we develop the various components of this theory in a broader generality.

The first basic notion that we introduce and study is a symmetry, sometimes called a canonical reflection. This is one of the novel aspects of our approach. This symmetry that we have in mind arises from a canonical reflection $S_x$ through the point $x$.

The second basic notion that we are interested is a midpoint or mean. We develop this notion from our symmetric structure: a point $m$ is the mean of $x$ and $y$, written $m = x\#y$, if the symmetry $S_m$ satisfies $S_m(x) = y$, and vice-versa.

In Section 1 we give a quick review of background work on the algebraic theory of symmetric means derived from the combined notions of symmetry and mean.

In Section 2 we introduce the quadratic representation into the group of displacements, which is a useful tool in the study of symmetric sets and spaces.

Repeated application of taking means and canonical reflections leads to means weighted by dyadic rational numbers. It is natural to extend these weighted means continuously to all real numbers by a topological setting. In Section 3 we do this in the structure of lineated symmetric spaces.

1.1 Symmetric Means

The basic notion of a symmetric space is that of a space equipped with a canonical reflection $S_x$ through each point $x$, called a symmetry or a point reflection. In the
presence of the existence of midpoints, the symmetry that carries \( x \) to \( y \) should be the symmetry \( S_m \) through the midpoint \( m \) of \( x \) and \( y \). This provides us the general framework for the notion of a midpoint of points \( x \) and \( y \) in the context of symmetry.

We consider a binary system \((X, \bullet)\) for which \( S_x : X \to X \) defined by \( S_x(y) = x \bullet y \) may be viewed as a symmetry or a point reflection through \( x \). We usually write \( S_x y \) for \( S_x(y) \). Note that \( S_x \) is a left translation by \( x \) with respect to the binary operation.

**Definition 1.2.** A symmetric mean consists of a binary system \((X, \bullet)\) satisfying for all \( a, b, c \in X \) the following list of axioms:

1. \( a \bullet a = a \) \((S_aa = a)\);
2. \( a \bullet (a \bullet b) = b \) \((S_aS_a = id_X)\);
3. \( a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \) \((S_aS_b = S_{S_aba};\)
4. the equation \( x \bullet a = b \) \((S_xa = b)\) has a unique solution \( x \in X \), called a midpoint or a mean of \( a \) and \( b \), and denoted \( a\#b \).

A binary system \((X, \bullet)\) satisfying Axioms (1)-(3) is called a symmetric set, and hence we occasionally refer to a symmetric mean as a symmetric set with midpoints. A pointed symmetric mean is a triple \((X, \bullet, \varepsilon)\), where \((X, \bullet)\) is a symmetric mean and \( \varepsilon \in X \) is some distinguished point, called the base point.

In verifying that specific examples satisfy the symmetric mean axioms, it is often helpful to have the axioms given in a weakened form.

**Lemma 1.3.** Let a symmetric set \((X, \bullet)\) have a distinguished point \( \varepsilon \) satisfying for all \( b \in X \)
(4') the equation $x \varepsilon = b$ has a unique solution $x \in X$.

Then $(X, \bullet)$ is a symmetric mean.

**Proof.** Let $a, b \in X$. By (4’) pick $u \in X$ such that $S_u \varepsilon = a \bullet \varepsilon = a$. By Axiom (2) $S_u$ is involutive, and hence $\varepsilon = S_u a$. Pick $x \in X$ such that $x \bullet \varepsilon = S_u b$. Applying $S_u$ to both sides, we have $S_u x \bullet S_u \varepsilon = S_u x \bullet a = b$. Thus $S_u x$ is a solution of the equation $(\bullet \bullet a = b$.

If $y$ is another solution of the equation $y \bullet a = b$, then applying $S_u$ to both sides yields $S_u y \bullet \varepsilon = S_u b$. So $S_u y = x$ since $(\bullet \bullet \varepsilon = S_u b$ has a unique solution $x \in X$. Therefore, $y = S_u x$ which establishes the uniqueness in general. □

It is easily to check the following lemma.

**Lemma 1.4.** For any $a, b$ in a symmetric mean $(X, \bullet)$,

(1) $a \# a = a$,

(2) $a \# b = b \# a$.

We recall from [10] that in this setting we define

$$x^{-1} := S_{\varepsilon} x, \quad x^2 := S_{\varepsilon} \varepsilon, \quad x^\frac{1}{2} := \varepsilon \# x,$$

and that inductively from these definitions all dyadic powers may be defined since the operators commute and expand a bit. In this case all basic laws of exponents also hold.

There is a category of symmetric sets with objects and morphism functions preserving the operation $\bullet$; we call such functions homomorphisms of symmetry or simply homomorphisms. The symmetric means with the same morphisms form a full subcategory.
Remark 1.5. In a symmetric set \((X, \bullet)\), the point \(m\) is called a midpoint of symmetry for \(x\) and \(y\) if \(m \bullet x = y\) (and hence \(m \bullet y = x\) by Axiom (2)). Symmetric means are then symmetric sets in which every two points have a unique midpoint of symmetry. It follows immediately that homomorphisms carry a midpoint of symmetry to a midpoint of symmetry. In particular homomorphisms of symmetric means are also homomorphisms with respect to the operation \(\#\). Hence the dyadic powers must be preserved by the base point preserving homomorphisms since these powers are defined from the base point and the two operations \(\bullet\) and \(\#\).

Conversely a function \(\beta : X \rightarrow Y\) between symmetric means that carries midpoints of symmetry to midpoints of symmetry must be a homomorphism since for \(m \bullet x = y\) implies \(\beta(m)\) is a midpoint of symmetry for \(\beta(x)\) and \(\beta(y)\), which means \(\beta(m) \bullet \beta(x) = \beta(y) = \beta(m \bullet x)\).

Remark 1.6. A twisted subgroup \(P\) of a group \(G\) is a subset that contains the identity and is closed under the core operation \(x \bullet y := xy^{-1}x\). Every twisted subgroup is a pointed symmetric set with respect to the operation \((x, y) \mapsto x \bullet y\) and the base point \(\varepsilon\), the identity. It is uniquely 2-divisible, which means every member of \(P\) has a unique square root in \(P\), if and only if it is a symmetric mean. In this case all dyadic powers in \(P\) computed from the group viewpoint or from the pointed symmetric mean viewpoint agree.

Proof. First let us prove that a twisted subgroup \(P\) with the identity \(\varepsilon\) is a pointed symmetric set. For any \(x, y, z \in P\)

(1) \(x \bullet x = xx^{-1}x = x\),

(2) \(x \bullet (x \bullet y) = x(xy^{-1}x)^{-1}x = x(x^{-1}yx)x^{-1} = y\),
(3) \( x \cdot (y \cdot z) = x(y^{-1}zy^{-1})x \)
\[ = (xy^{-1}x)(xz^{-1}x)^{-1}(xy^{-1}x) \]
\[ = (x \cdot y)(x \cdot z)^{-1}(x \cdot y) \]
\[ = (x \cdot y) \cdot (x \cdot z). \]

Next we show that \( P \) is uniquely 2-divisible if and only if it is a symmetric mean.

Assume that \( P \) is uniquely 2-divisible. By Lemma 1.3, it is enough to show that the equation \( x \cdot \varepsilon = b \) has a unique solution in \( P \) for all \( b \in P \). Since \( P \) is uniquely 2-divisible, \( x^2 = x \cdot \varepsilon = b \) has a unique solution in \( P \). So \( P \) is a symmetric mean.

The converse is obvious because we can consider \( a = \varepsilon \) in Axiom (4).

**Remark 1.7.** The unique solution of \( x \cdot a = b \) translates in the context of twisted subgroups to the unique solution of the basic Riccati equation \( xa^{-1}x = b \).

Every pointed symmetric mean is isomorphic to one arising from a uniquely 2-divisible twisted subgroup \( P \) of some group, as we shall see shortly.

A special case of Remark 1.6 is the additive group \((\mathbb{D}, +)\) of dyadic rational numbers, which may be viewed as a uniquely 2-divisible twisted subgroup of itself.

The operations on a symmetric mean are given by

\[ r \cdot s = 2r - s, \quad r \# s = \frac{r + s}{2}. \]

**Definition 1.8.** A homomorphism from \( \mathbb{D} \) into the symmetric set \( X \) is called a dyadic geodesic.

If the homomorphism is injective, then the image is called a dyadic line. The following result is proved in [11].

**Lemma 1.9.** Let \((X, \bullet)\) be a symmetric set. Then \( X \) is a symmetric mean if and only if given distinct \( x, y \in X \), there exists a unique dyadic geodesic \( \alpha : \mathbb{D} \to X \) such that \( \alpha(0) = x \) and \( \alpha(1) = y \).
For the special case that \( x = \varepsilon \), the unique homomorphism \( \alpha \) is given by \( \alpha(t) = y^t \). We thus have for all dyadic rational numbers \( r, s \in \mathbb{D} \)

\[
(y^r)^s = y^{rs}, \quad S_y^r y^s = y^{2r-s}, \quad y^r \# y^s = y^{\frac{r+s}{2}}.
\]

The last two equalities follow from the fact that the mapping is a homomorphism for the operations \( \bullet \) and \( \# \), and the first equality follows from the uniqueness of the homomorphism, since \( s \mapsto (y^r)^s \) and \( s \mapsto y^{rs} \) are both homomorphism carrying 0 to \( \varepsilon \) and 1 to \( y^r \).

### 1.2 The Displacement Group and Quadratic Representation

An important role in the theory of symmetric means is played by the displacement group \( G(X) \).

**Definition 1.10.** Let \( X \) be a pointed symmetric set. Each composition \( S_x S_y \) for any \( x, y \in X \) is called a displacement and the group \( G(X) \) generated by all \( S_x S_y \) under composition is called the displacement group. The quadratic representation of \( X \) is the map \( Q : X \to G(X) \) defined by \( Q(x) = S_x S_\varepsilon \).

From Axioms (2) and (3) of Definition 1.2, we know that each \( S_x \) is an involutive automorphism. Thus each member of \( G(X) \) is also an automorphism.

**Remark 1.11.** Note that \( Q(x)Q(y)^{-1} = (S_x S_\varepsilon)(S_\varepsilon S_y) = S_x S_y \) since \( (S_y)^{-1} = S_y \) from Axiom (2) of Definition 1.2. So \( G(X) \) is alternatively the group generated by \( Q(X) \).

**Remark 1.12.** Let \( P \) be a twisted subgroup with the operation \( x \bullet y = xy^{-1}x \). Then its quadratic representation is given by \( Q(x)y = S_x S_\varepsilon y = S_x y^{-1} = xyx \).
The next lemma gives the basic properties of the quadratic representation (see [18] and [16]).

**Lemma 1.13.** For any $a, b$ in a pointed symmetric set $(X, \cdot, \varepsilon)$, we have

1. $Q(Q(a)b) = Q(a)Q(b)Q(a)$;

2. $Q(a)^{-1} = Q(a^{-1})$, $(Q(a)b)^{-1} = Q(a^{-1})b^{-1}$, and $Q(\varepsilon) = id_X$;

3. $Q(a \cdot b) = Q(a)Q(b)^{-1}Q(a)$;

4. $Q(a)\varepsilon = a^2$, $Q(a)^2 = Q(a^2)$.

If $X$ is a symmetric mean, then $Q(x)\varepsilon = Q(y)\varepsilon$ implies that $x = y$, which means that the quadratic representation is injective.

**Theorem 1.14.** Let $(X, \cdot, \varepsilon)$ be a pointed symmetric set. Then $Q(X) \subseteq G(X)$ is a twisted subgroup with respect to the core operation, and the map $x \mapsto Q(x) : (X, \cdot) \to (Q(X), \cdot)$ is a homomorphism of pointed symmetric sets. If $X$ is a pointed symmetric mean, then $Q(X)$ is uniquely 2-divisible, and the map $x \mapsto Q(x) : (X, \cdot) \to (Q(X), \cdot)$ is an isomorphism.

*Proof.* For $x, y \in X$, $Q(x)Q(y)^{-1}Q(x) = Q(Q(x)y^{-1}) \in Q(X)$ by Lemma 1.13(1).

Thus $Q(X)$ is a twisted subgroup of $G(X)$. If the twisted subgroup is given the structure of a symmetric set, then Lemma 1.13(3) establishes that the map $x \mapsto Q(x)$ is a homomorphism of symmetric sets from $X$ onto $Q(X)$.

Assume that $X$ is a pointed symmetric mean. If $a = Q(x)\varepsilon = Q(y)\varepsilon$, then $x = \varepsilon \# a = y$ by Axiom (4). So the quadratic representation $Q : X \to Q(X)$ is an isomorphism of symmetric sets, and hence $Q(X)$ is a symmetric mean. By Remark 1.6 $Q(X)$ is uniquely 2-divisible. \hfill \square

We obtain immediately the following corollary.
**Corollary 1.15.** Every uniquely 2-divisible twisted subgroup of a group becomes a pointed symmetric mean when equipped with the core operation \( x \circ y = xy^{-1}x \) and the distinguished point \( \varepsilon \). Conversely, every pointed symmetric mean is isomorphic to one constructed in this manner.

In Remark 1.7 we have discussed the Riccati equation in a twisted subgroup; \( x \circ a = xa^{-1}x = b \). We can write in a general pointed symmetric mean

\[
b = x \circ a = S_xS_\varepsilon a^{-1} = Q(x)a^{-1}.
\]

Thus we have a quadratic version of the Riccati equation: \( Q(x)a^{-1} = b \). The following lemma is then immediate.

**Lemma 1.16.** (The Riccati Lemma) For each \( a, b \in X \), a pointed symmetric mean, the Riccati equation \( Q(x)a^{-1} = b \) has a unique solution in \( X \), namely \( x = a \# b \).

In a uniquely 2-divisible twisted subgroup, the unique solution of the Riccati equation, \( xa^{-1}x = b \), is given by

\[
a \# b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}.
\]

It is sometimes called the geometric mean of \( a \) and \( b \). The next theorem gives us an explicit formula for the mean in a symmetric mean, which is a generalization of the one derived from the twisted subgroup.

**Theorem 1.17.** In a pointed symmetric mean \( a \# b = Q(a^{1/2})(Q(a^{-1/2})b)^{1/2} \) is the unique point \( m \in X \) such that \( S_ma = b \).
Proof. We show that \( Q(a^{\frac{1}{2}})(Q(a^{-\frac{1}{2}})b)^{\frac{1}{2}} \) satisfies the quadratic version of the Riccati equation \( Q(x)a^{-1} = b \). Indeed

\[
Q(Q(a^{\frac{1}{2}})(Q(a^{-\frac{1}{2}})b)^{\frac{1}{2}})a^{-1} = Q(a^{\frac{1}{2}})Q(Q(a^{-\frac{1}{2}})b)^{\frac{1}{2}}Q(a^{\frac{1}{2}})a^{-1} \\
= Q(a^{\frac{1}{2}})Q(Q(a^{-\frac{1}{2}})b)^{\frac{1}{2}}\varepsilon \\
= Q(a^{\frac{1}{2}})Q(a^{-\frac{1}{2}})b \\
= b,
\]

where the first equality follows from (1) of Lemma 1.13, the second follows from the fact

\( Q(a^{\frac{1}{2}})a^{-1} = S_{\frac{3}{2}} S_{\varepsilon} a^{-1} = S_{\frac{3}{2}} a = \varepsilon \)

since \( x^{-1} = S_{\varepsilon} x \) and \( x^{\frac{1}{2}} = \varepsilon \# x \), the third follows from (4) of Lemma 1.13, and the last follows from (2) of Lemma 1.13. \( \square \)

1.3 Lineated Symmetric Spaces

In this section we consider a topological version of a symmetric mean. In Definition 1.8, we replace the set \( \mathbb{D} \) of dyadic rational numbers by the set \( \mathbb{R} \) of real numbers with its usual topology equipped with the operation \( s \bullet t = s + ((-t) + s) = 2s - t \), which makes the uniquely 2-divisible group \( \mathbb{R} \) a symmetric mean (or a pointed symmetric mean with the base point 0).

Definition 1.18. A lineated symmetric space consists of a symmetric mean \((X, \bullet)\) equipped with a Hausdorff topology such that

1. the map \((x, y) \mapsto x \bullet y : X \times X \to X\) is continuous;
2. for \(x, y \in X\), there exists a unique continuous homomorphism \( \alpha_{x, y} : (\mathbb{R}, \bullet) \to (X, \bullet) \) such that \( \alpha_{x, y}(0) = x \) and \( \alpha_{x, y}(1) = y \), where \((\mathbb{R}, \bullet)\) is given the core
operation $s \bullet t = 2s - t$. The image $\alpha_{x,y}(t)$ is also denoted $x\#_t y$, and is called the $t$-weighted mean of $x$ and $y$;

(3) the map $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times X \times X \to X$ is continuous.

**Remark 1.19.** Note that $\alpha_{x,y}$ must also be a homomorphism under the operation $\#$ from Remark 1.5, and thus

$$x\#_{\frac{1}{2}} y = \alpha_{x,y}\left(\frac{1}{2}\right) = \alpha_{x,y}(0\#1) = \alpha_{x,y}(0)\#\alpha_{x,y}(1) = x\# y.$$ 

In particular the map $(x, y) \mapsto x\# y$ is continuous in a lineated symmetric space by property (3) of Definition 1.18.

We call the map $t \mapsto x\#_t y : \mathbb{R} \to X$ in Definition 1.18 a geodesic of symmetry, or geodesic for short.

**Lemma 1.20.** Let $X$ and $Y$ be lineated symmetric spaces, and let $\beta : X \to Y$ be a continuous homomorphism under the operation $\bullet$. Then

$$\beta \circ \alpha_{x,y} = \alpha_{\beta(x),\beta(y)}$$

for any $x, y \in X$.

**Proof.** Note that $\alpha_{x,y}$ is the unique continuous homomorphism such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$ for any $x, y \in X$. Then $\beta \circ \alpha_{x,y} : \mathbb{R} \to Y$ is also a continuous homomorphism such that $\beta \circ \alpha_{x,y}(0) = \beta(\alpha_{x,y}(0)) = \beta(x)$ and $\beta \circ \alpha_{x,y}(1) = \beta(\alpha_{x,y}(1)) = \beta(y)$. By the uniqueness of $\alpha_{\beta(x),\beta(y)}$ in $Y$, we obtain $\beta \circ \alpha_{x,y} = \alpha_{\beta(x),\beta(y)}$. $\square$

**Remark 1.21.** Applying Lemma 1.20 to a map

$$\alpha_{x,y} \circ \alpha_{s,t} : (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet) \to (X, \bullet)$$
for fixed $s, t \in \mathbb{R}$, we obtain

\[ \alpha_{x,y} \circ \alpha_{s,t} = \alpha_{\alpha_{x,y}(s),\alpha_{x,y}(t)}. \]

In other words, for any $u \in \mathbb{R}$

\[ \alpha_{x,y}((1 - u)s + ut) = \alpha_{x,y}(s\#u t) = \alpha_{x,y}(s)\#\alpha_{x,y}(t). \]

This means that a geodesic $\alpha_{x,y}$ is a $\#_t$ homomorphism.

Note that a symmetry $S_x$ is a continuous homomorphism under the operation $\bullet$ from Axiom (3) of Definition 1.2 and Axiom (1) of Definition 1.18. Therefore, we have the following lemma.

**Lemma 1.22.** For any $x \in X$, a lineated symmetric space, $S_x$ distributes over $\#_t$ for all $t \in \mathbb{R}$. Hence if $X$ is pointed, then any member $Q(x)$ of the displacement group does.

The next proposition is already proved in [11].

**Proposition 1.23.** The following hold in a lineated symmetric space $X$:

1. each nontrivial continuous homomorphism $\alpha : \mathbb{R} \rightarrow X$ is injective;

2. if distinct points $x$ and $y$ belong to the images of two continuous homomorphisms $\alpha, \beta : \mathbb{R} \rightarrow X$, then $\alpha$ and $\beta$ are linear reparametrization of each other, and hence, have the same image.

**Proof.** (1) If $\alpha(s) = \alpha(t)$ for $t < s$, then

\[ \alpha(0) = \alpha \left( \frac{s}{2} \bullet s \right) = \alpha \left( \frac{s}{2} \right) \bullet \alpha(s) = \alpha \left( \frac{s}{2} \right) \bullet \alpha(t) = \alpha(s - t). \]

Let $u = s - t$ and choose the base point $\varepsilon := \alpha(0)$. Since $\alpha$ preserves dyadic powers from Remark 1.5, we have $\varepsilon = \varepsilon^q = (\alpha(u))^q = \alpha(qu)$ for each $q \in \mathbb{Z}$. 


Since \( \{qu : q \in \mathbb{D}\} \) is dense in \( \mathbb{R} \), we conclude that \( \alpha \) is a constant map, a contradiction.

(2) Suppose that \( x = \alpha(0) = \beta(b) \) and \( y = \alpha(1) = \beta(a) \). Then

\[
\gamma(t) := \beta((a - b)t + b) : (\mathbb{R}, \bullet) \to (X, \bullet)
\]

is a continuous homomorphism since it is the composition of two such homomorphisms. Since \( \gamma(0) = \beta(b) = x \) and \( \gamma(1) = \beta(a) = y \), we have \( \alpha = \gamma \) by (2) of Definition 1.18. Thus \( \alpha \) and \( \beta \) differ by a linear reparameterization of \( \mathbb{R} \). Now if \( \phi : \mathbb{R} \to X \) is another continuous homomorphism, then \( \phi \) and \( \alpha \) are linear reparametrizations of each other by the same argument, and so are \( \beta \) and \( \phi \) since the composition of linear maps is linear.

\[\square\]

**Remark 1.24.** For a pointed lineated symmetric space \((X, \bullet, \varepsilon)\), the geodesic

\[ t \mapsto x^t := \varepsilon \#_t x : \mathbb{R} \to X \]

restricted to the set \( \mathbb{D} \) of dyadic rational numbers is a dyadic geodesic carrying 0 to \( \varepsilon \) and 1 to \( x \) as appeared in Definition 1.8. By the uniqueness of dyadic geodesics from Lemma 1.9, the restriction must be the map \( t \mapsto x^t : \mathbb{D} \to X \). This observation establishes that the definition of all real powers is the unique continuous extension of the dyadic powers. By continuity we have that

\[
(x^r)^s = x^{rs}, \quad S_x x^s = x^{2r-s}, \quad x^r \# x^s = x^{r+s}
\]

for all \( r, s \in \mathbb{R} \).

Lawson and Lim have shown in [11] that an apparently weakened axiom system in the pointed context yields a pointed lineated symmetric space.
Theorem 1.25. Let $(X, \bullet, \varepsilon)$ be a pointed symmetric mean equipped with a Hausdorff topology such that

(1) the map $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous;

(2) the function $(q, x) \mapsto x^q : \mathbb{D} \times X \to X$, where $\mathbb{D}$ is the set of all dyadic rational numbers, extends to a continuous map $(t, x) \mapsto x^t : \mathbb{R} \times X \to X$.

Then $X$ is a pointed lineated symmetric space.
2. Gyrovector Spaces

In [1] A. Ungar has introduced a notion of a gyrogroup and has developed the study of analytic hyperbolic geometric with the study of gyrogroups. The gyrogroup is a most natural extension of a group into the nonassociative algebra. The associativity of group operations is salvaged in a suitably modified form, called a gyroassociativity.

The reason for starting this chapter with gyrogroups is that some gyrocommutative gyrogroups give rise to gyrovector spaces just as some commutative groups give rise to vector spaces. We will see it in later chapter. To elaborate a precise language, we prefix a gyro to terms that describe concepts in Euclidean geometry to mean the analogous concepts in hyperbolic geometry. The prefix gyro stems from Thomas gyration, which is the mathematical abstraction of a special relativistic effect known as Thomas precession.

In Section 1 we start with the weaken axioms of gyrogroups, and then derive the extensive set of axioms using certain properties of gyrations. Moreover, we give a definition of gyrocommutativity and its equivalence conditions.

In Section 2 we show the equivalence between uniquely 2-divisible (gyrocommutative) gyrogroups and pointed symmetric means, and investigate the homomorphisms in their setting.

In Section 3 we give the definition of a gyrovector space. Furthermore we show the one-to-one correspondence between topological gyrovector spaces and pointed lineated symmetric spaces.
2.1 Gyrogroups

The notion of a gyrogroup has been introduced by A. Ungar, primarily as an appropriate structure for the study of Einstein velocity addition in special relativity and related topics (see [1]). The prefix gyro that we use stems from analogies shared with the Thomas gyration of special theory of relativity.

Definition 2.26. A gyrogroup consists of a set $G$ equipped with a binary operation $\oplus$ satisfying the following axioms:

(G1) There exists some element $0$ in $G$, called a left identity, satisfying for all $a \in G$

$$0 \oplus a = a,$$

(G2) For each $a \in G$ there exists an element $x$ in $G$, called a left inverse of $a$, satisfying

$$x \oplus a = 0,$$

(G3) For any $a, b, c \in G$ there exists a unique element $\text{gyr}[a, b]c$ in $G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c,$$

(G4) The map $\text{gyr}[a, b] : G \to G$ given by $z \mapsto \text{gyr}[a, b]z$ is an automorphism of $G$ under the operation $\oplus$, called the Thomas gyration or the gyroautomorphism of $G$ generated by $a, b \in G$,

(G5) The gyroautomorphism $\text{gyr}[a, b]$ generated by $a, b \in G$ satisfies the left loop property:

$$\text{gyr}[a, b] = \text{gyr}[a \oplus b, b].$$

We usually denote the automorphism group of $(G, \oplus)$ by $\text{Aut}(G, \oplus)$. We recall some elementary facts about gyrogroups (see [1]).
Lemma 2.27. In a gyrogroup $(G, \oplus)$, the following properties hold.

(1) $a \oplus b = a \oplus c$ implies $b = c$.

(2) $\text{gyr}[0, a] = I$, the identity map on $G$.

(3) $\text{gyr}[x, a] = I$, where $x$ is the left inverse of $a$.

(4) $\text{gyr}[a, a] = I$.

(5) A left identity is a right identity and is unique,

(6) Every left inverse is a right inverse and is unique, and so we denote $\ominus a$ for the inverse of $a$.

(7) $\text{gyr}[a, 0] = I$.

(8) $\ominus a \oplus (a \oplus b) = b$.

(9) $\text{gyr}[a, b]x = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x))$.

(10) $\text{gyr}[a, b](\ominus x) = \ominus \text{gyr}[a, b]x$.

The first statement (1) is called the left cancellation law.

Proof. For any elements $a, b, x$ in a gyrogroup $(G, \oplus)$, the proofs follow.

(1) Assume that $a \oplus b = a \oplus c$. Adding the left inverse $x$ of $a$ on the left hand side, we have

$$x \oplus (a \oplus b) = (x \oplus a) \oplus \text{gyr}[x, a]b = 0 \oplus \text{gyr}[x, a]b = \text{gyr}[x, a]b.$$ 

Similarly we have $x \oplus (a \oplus c) = \text{gyr}[x, a]c$, and so $\text{gyr}[x, a]b = \text{gyr}[x, a]c$. Since the gyration is an automorphism, we obtain $b = c$. 

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(2) Using that 0 is the left identity, we have

\[ a \oplus b = 0 \oplus (a \oplus b) = (0 \oplus a) \oplus \text{gyr}[0, a]b = a \oplus \text{gyr}[0, a]b. \]

By (1), we obtain \( b = \text{gyr}[0, a]b \), and hence \( \text{gyr}[0, a] = I \).

(3) The left loop property gives us the following:

\[ \text{gyr}[x, a] = \text{gyr}[x \oplus a, a] = \text{gyr}[0, a] = I. \]

(4) The left loop property gives us the following:

\[ I = \text{gyr}[0, a] = \text{gyr}[0 \oplus a, a] = \text{gyr}[a, a]. \]

(5) Let \( x \) be the left inverse of \( a \) for the left identity 0. Then

\[ x \oplus (a \oplus 0) = (x \oplus a) \oplus \text{gyr}[x, a]0 = 0 \oplus 0 = 0 = x \oplus a. \]

By (1), we obtain \( a \oplus 0 = a \). Suppose that 0 and \( 0^* \) are left identities, one of which is also a right identity. Then \( 0 = 0 \oplus 0^* = 0^* \).

(6) Let \( x \) be a left inverse of \( a \). Then

\[ x \oplus (a \oplus x) = (x \oplus a) \oplus \text{gyr}[x, a]x = 0 \oplus x = x = x \oplus 0. \]

By (1), we have \( a \oplus x = 0 \). Suppose \( x \) and \( y \) are left inverses of \( a \). From the above, we have \( a \oplus x = 0 = a \oplus y \). By (1), we obtain \( x = y \).

(7) Using that 0 is a right identity as well as a left identity, we have

\[ a \oplus b = a \oplus (0 \oplus b) = (a \oplus 0) \oplus \text{gyr}[a, 0]b = a \oplus \text{gyr}[a, 0]b. \]

By (1), we have \( b = \text{gyr}[a, 0]b \), and hence \( \text{gyr}[a, 0] = I \).
(8) By (G3) of Definition 2.26 and (3),

\[ \ominus a \oplus (a \oplus b) = (\ominus a \oplus a) \oplus \text{gyr}[\ominus a, a]b = 0 \oplus b = b. \]

(9) From (G3) of Definition 2.26 and (8), we obtain (9) easily.

(10) Since \( \text{gyr}[a, b] \) is an automorphism of \((G, \oplus)\), we have

\[ \text{gyr}[a, b](\ominus x) \oplus \text{gyr}[a, b]x = \text{gyr}[a, b](\ominus x \oplus x) = \text{gyr}[a, b]0 = 0. \]

So \( \text{gyr}[a, b](\ominus x) \) is the left inverse of \( \text{gyr}[a, b]x \), denoted \( \ominus \text{gyr}[a, b]x \).

\[ \square \]

**Remark 2.28.** We define a left translation \( L_a \) by \( L_a(x) = a \oplus x \). From (8) we have \( L_{\ominus a} = L_a^{-1} \). In particular, the left translation is bijective.

The preceding list of axioms is minimal in nature. We typically work with the more extensive, but equivalent, set of axioms.

**Definition 2.29.** A gyrogroup consists of a set \( G \) equipped with a binary operation \( \oplus \) satisfying the following axioms:

1. There exists some element 0 in \( G \) satisfying for all \( a \in G \)

\[ 0 \oplus a = a \oplus 0 = a, \]

2. For each \( a \in G \) there exists a unique inverse \( \ominus a \) in \( G \) satisfying

\[ \ominus a \oplus a = a \oplus (\ominus a) = 0, \]

The map \( \text{gyr}[a, b] : G \to G \) given by \( z \mapsto \text{gyr}[a, b]z \) for all \( z \in G \) is an automorphism of \( G \) under the operation \( \oplus \) satisfying the following ;
(3) The left and right gyroassociative laws; For any \(a, b, c \in G\),
\[
a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c
\]
\[
(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b, a]c),
\]

(4) The left and right loop properties;
\[
\text{gyr}[a, b] = \text{gyr}[a \oplus b, b]
\]
\[
\text{gyr}[a, b] = \text{gyr}[a, b \oplus a].
\]

From Lemma 2.27 we have seen that the left identity is the right identity, and every left inverse is also the right inverse. Thus it is enough to derive the right gyroassociative law and the right loop property. We will prove them later.

We usually write \(-a\) instead of \(\ominus a\), and moreover, write \(a \ominus b\) instead of \(a \oplus (\ominus b)\).

Let us give an interesting result between the gyroautomorphisms and the automorphisms.

**Lemma 2.30.** Let \(a\) and \(b\) be any elements in a gyrogroup \((G, \oplus)\). For any \(A \in \text{Aut}(G, \oplus)\),
\[
A \text{gyr}[a, b] = \text{gyr}[A(a), A(b)]A.
\]

**Proof.** For any \(x \in G\) we have by the left gyroassociativity
\[
(A(a) \oplus A(b)) \oplus A \text{gyr}[a, b]x = A((a \oplus b) \oplus \text{gyr}[a, b]x)
\]
\[
= A(a \oplus (b \oplus x))
\]
\[
= A(a) \oplus (A(b) \oplus A(x))
\]
\[
= (A(a) \oplus A(b)) \oplus \text{gyr}[A(a), A(b)]A(x).
\]

By the left cancellation we proved. \(\square\)

**Remark 2.31.** In particular, if \(A = \text{gyr}[a, b]\) then we have
\[
\text{gyr}[a, b] = \text{gyr}[\text{gyr}[a, b]a, \text{gyr}[a, b]b].
\]

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From a gyrogroup, we can obtain a group structure which is very useful and helpful to prove many properties in the gyrogroup.

**Proposition 2.32.** Let \((G, \oplus)\) be a gyrogroup and let \(\text{Aut}_0(G)\) be a subgroup of its automorphism group containing all gyroautomorphisms \(\text{gyr}[a, b]\) for any \(a, b \in G\). Then \(G \times \text{Aut}_0(G)\) is a group with respect to the operation

\[
(a, A)(b, B) = (a \oplus A(b), \text{gyr}[a, A(b)]AB). \tag{2.3}
\]

**Proof.** Let us define a set \(S = \{L_aA : a \in G, A \in \text{Aut}_0(G)\}\), where \(L_a\) is a left translation by \(a\). Then for any \(a, b, c \in G\) and \(A, B \in \text{Aut}_0(G)\),

\[
L_aA(L_bB(c)) = L_aA(b \oplus B(c))
= a \oplus A(b \oplus B(c))
= a \oplus (A(b) \oplus AB(c))
= (a \oplus A(b)) \oplus \text{gyr}[a, A(b)]AB(c)
= L_{a \oplus A(b)}\text{gyr}[a, A(b)]AB(c).
\]

This means that \(S\) is closed under composition. Moreover, we obtain two binary systems, \(G \times \text{Aut}_0(G)\) under the operation (2.3) and \(S\) under composition, which are isomorphic because the map

\[G \times \text{Aut}_0(G) \to S: (a, A) \mapsto L_aA\]

is an isomorphism. Since the associative law is obviously satisfied in \(S\) under composition, \(G \times \text{Aut}_0(G)\) satisfies the associative law under the operation (2.3).

Next we show that \((0, I)\) is a left identity in \(G \times \text{Aut}_0(G)\). For any \((a, A) \in G \times \text{Aut}_0(G)\),

\[
(0, I)(a, A) = (0 \oplus I(a), \text{gyr}[0, I(a)]IA) = (0 \oplus a, \text{gyr}[0, a]A) = (a, A)
\]

since \(\text{gyr}[0, a] = I\) from (2) of Lemma 2.27.
Finally we show that \((\ominus A^{-1}(a), A^{-1})\) is the left inverse of \((a, A) \in G \times \text{Aut}_0(G)\), where \(A^{-1}\) is the inverse automorphism of \(A\).

\[
(\ominus A^{-1}(a), A^{-1})(a, A) = (\ominus A^{-1}(a) \oplus A^{-1}(a), \text{gyr}[\ominus A^{-1}(a), A^{-1}(a)]A^{-1}A) = (0, I)
\]

since \(\text{gyr}[x, a] = I\) for the left inverse \(x\) of \(a\) from (3) of Lemma 2.27.

We list some basic properties of the Thomas gyrations (see [1]).

**Lemma 2.33.** For any elements \(a, b\) in a gyrogroup \((G, \oplus)\), the gyrations satisfy the following properties;

1. \(\text{gyr}[b, a] = \text{gyr}^{-1}[a, b]\), the inverse of \(\text{gyr}[a, b]\),

2. \(\text{gyr}[\ominus a, \ominus b] = \text{gyr}[a, b]\).

**Proof.** In order to prove (1), consider the group \(G \times \text{Aut}_0(G)\) discussed in the previous proposition. For any two elements \((a, I), (b, I) \in G \times \text{Aut}_0(G)\), the associative law gives us

\[
(\ominus a, I)((a, I)(b, I)) = ((\ominus a, I)(a, I))(b, I) = (0, I)(b, I) = (b, I).
\]

However,

\[
(\ominus a, I)((a, I)(b, I)) = (\ominus a, I)(a \oplus b, \text{gyr}[a, b]) = (b, \text{gyr}[\ominus a, a \oplus b]\text{gyr}[a, b]).
\]

So we obtain

\[
\text{gyr}[\ominus a, a \oplus b]\text{gyr}[a, b] = I,
\]

or equivalently

\[
\text{gyr}^{-1}[a, b] = \text{gyr}[\ominus a, a \oplus b] = \text{gyr}[b, a \oplus b]
\]

by the left loop property and (8) of Lemma 2.27. Using the left loop property on the left hand side, we have

\[
\text{gyr}[b, a \oplus b] = \text{gyr}^{-1}[a, b] = \text{gyr}^{-1}[a \oplus b, b].
\]
By change of variable we obtain
\[ \text{gyr}^{-1}[c \oplus b, b] = \text{gyr}[b, c \oplus b]. \]

By Proposition 2.41, there exist a unique element \( c \) in \( G \) such that \( c \oplus b = a \). Thus, \( \text{gyr}^{-1}[a, b] = \text{gyr}[b, a] \).

Moreover, we have
\[ ((a, I)(b, I))^{-1} = (b, I)^{-1}(a, I)^{-1}. \]

However,
\[ ((a, I)(b, I))^{-1} = (a \oplus b, \text{gyr}[a, b])^{-1} = (\ominus \text{gyr}^{-1}[a, b](a \oplus b), \text{gyr}^{-1}[a, b]), \]
and
\[ (b, I)^{-1}(a, I)^{-1} = (\ominus b, I)(\ominus a, I) = (\ominus b \ominus a, \text{gyr}[\ominus b, \ominus a]). \]

By comparing two extreme right hand sides we have
\[ \text{gyr}^{-1}[a, b] = \text{gyr}[\ominus b, \ominus a]. \]

By (1), we get \( \text{gyr}[\ominus b, \ominus a] = \text{gyr}^{-1}[\ominus a, \ominus b] \), and so \( \text{gyr}^{-1}[a, b] = \text{gyr}^{-1}[\ominus a, \ominus b] \).

This implies that \( \text{gyr}[a, b] = \text{gyr}[\ominus a, \ominus b] \). \( \square \)

Now let us verify the right gyroassociative law and the right loop property.

**Remark 2.34.** Let \((G, \oplus)\) be a gyrogroup with the weakened axioms. We have seen that \( G \times \text{Aut}_0(G) \) is a group. So for \((a, I), (b, I), (\text{gyr}[b, a]c, I) \in G \times \text{Aut}_0(G)\), we have
\[ ((a, I)(b, I))(\text{gyr}[b, a]c, I) = (a, I)((b, I)(\text{gyr}[b, a]c, I)). \]

However,
\[ ((a, I)(b, I))(\text{gyr}[b, a]c, I) = (a \oplus b, \text{gyr}[a, b])(\text{gyr}[b, a]c, I) \]
\[ = ((a \oplus b) \oplus c, \text{gyr}[a \oplus b, c]\text{gyr}[a, b]), \]
and

\[(a, I)((b, I)(\text{gyr}[b, a]c, I)) = (a, I)(b \oplus \text{gyr}[b, a]c, \text{gyr}[b, \text{gyr}[b, a]c]) \]

\[= (a \oplus (b \oplus \text{gyr}[b, a]c), \text{gyr}[a, b \oplus \text{gyr}[b, a]c]\text{gyr}[b, \text{gyr}[b, a]c]).\]

Comparing the first component we have \((a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b, a]c), the right gyroassociative law.

We have shown in the proof of Lemma 2.33 that

\[\text{gyr}^{-1}[a, b] = \text{gyr}[b, a \oplus b].\]

Since \(\text{gyr}^{-1}[a, b] = \text{gyr}[b, a]\), we have

\[\text{gyr}[b, a] = \text{gyr}[b, a \oplus b].\]

By change of variable, we obtain the right loop property.

**Remark 2.35.** By comparing the second component in the previous remark, we obtain

\[\text{gyr}[a \oplus b, c]\text{gyr}[a, b] = \text{gyr}[a, b \oplus \text{gyr}[b, a]c]\text{gyr}[b, \text{gyr}[b, a]c].\]

Replacing \(c\) by \(\text{gyr}[a, b]c\) and (1) of Lemma 2.33 give us

\[\text{gyr}[a \oplus b, \text{gyr}[a, b]c]\text{gyr}[a, b] = \text{gyr}[a, b \oplus c]\text{gyr}[b, c]. \hspace{1cm} (2.4)\]

In the special case when \(c = \ominus b\), the left loop property and the left gyroassociativity imply

\[I = \text{gyr}[a \oplus b, \ominus \text{gyr}[a, b]b]\text{gyr}[a, b] \]

\[= \text{gyr}[(a \oplus b) \ominus \text{gyr}[a, b]b, \ominus \text{gyr}[a, b]b]\text{gyr}[a, b] \]

\[= \text{gyr}[a \oplus (b \ominus b), \ominus \text{gyr}[a, b]b]\text{gyr}[a, b] \]

\[= \text{gyr}[a, \ominus \text{gyr}[a, b]b]\text{gyr}[a, b].\]

Equivalently,

\[\text{gyr}[a, b] = \text{gyr}^{-1}[a, \ominus \text{gyr}[a, b]b] = \text{gyr}[\ominus \text{gyr}[a, b]b, a]. \hspace{1cm} (2.5)\]
The next proposition is very useful to prove the equivalence between a gyrogroup and a symmetric mean, as we shall see later.

**Proposition 2.36.** Let \( (G, \oplus) \) be a gyrogroup. Then \( P := \{(a, I) : a \in G\} \) is a twisted subgroup of a group \( G \times \text{Aut}_0(G) \) with respect to the core operation \( \bullet \). In particular, \( (P, \bullet) \) is isomorphic to \( (G, \bullet) \), where \( a \bullet b = a \oplus (\ominus b \oplus a) \) for any \( a, b \in G \).

**Proof.** We have seen that \( G \times \text{Aut}_0(G) \) is a group in Proposition 2.32. Obviously the identity \((0, I)\) belongs to a subset \( P \). It is enough to show that \( P \) is closed under the core operation.

\[
(a, I) \bullet (b, I) = (a, I)(b, I)^{-1}(a, I)
= (a, I)((\ominus b, I)(a, I))
= (a, I)(\ominus b \oplus a, \text{gyr}[\ominus b, a])
= (a \oplus (\ominus b \oplus a), \text{gyr}[a, \ominus b \oplus a]\text{gyr}[\ominus b, a]).
\]

Since \( \text{gyr}[a, \ominus b \oplus a] = \text{gyr}[a, \ominus b] = \text{gyr}^{-1}[\ominus b, a] \) from the right loop property and (1) of Lemma 2.33, we have \( \text{gyr}[a, \ominus b \oplus a]\text{gyr}[\ominus b, a] = I \). Thus, \( P \) is closed under the core operation, and so is a twisted subgroup.

Moreover, define a map

\[
(P, \bullet) \rightarrow (G, \bullet) : (a, I) \mapsto a,
\]

where \( a \bullet b = a \oplus (\ominus b \oplus a) \) for any \( a, b \in G \). Then it is clearly a bijection and a homomorphism from the above equations. Thus this map is an isomorphism, which means that \( (P, \bullet) \) is isomorphic to \( (G, \bullet) \). \( \square \)

**Lemma 2.37.** For all \( a, b, c \) in a gyrogroup \( (G, \oplus) \),

\[
\ominus(a \oplus b) \oplus (a \oplus c) = \text{gyr}[a, b](\ominus b \oplus c).
\]
Proof. Note that gyr\([a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)). Replacing\( c \) by \( \ominus b \oplus c \) and using (8) of Lemma 2.27, we obtain this lemma.

**Proposition 2.38.** A gyrogroup satisfies the left Bol identity

\[
a \oplus (b \oplus (a \oplus c)) = (a \oplus (b \oplus a)) \oplus c.
\]

Proof. We have

\[
a \oplus (b \oplus (a \oplus c)) = a \oplus ((b \oplus a) \oplus \text{gyr}[b, a]c)
\]

\[
= (a \oplus (b \oplus a)) \oplus \text{gyr}[a, b \oplus a] \text{gyr}[b, a]c.
\]

Noting from Lemma 2.33 that \( \text{gyr}[a, b \oplus a] \text{gyr}[b, a] = \text{gyr}[a, b] \text{gyr}[b, a] = I \), we obtain the result.

In the next two propositions we show that a gyrogroup is a loop.

**Proposition 2.39.** In a gyrogroup \((G, \oplus)\) the equation \(a \oplus x = b\) in unknown \(x\) has the unique solution \(x = \ominus a \oplus b\).

Proof. From (8) of Lemma 2.27, we have that \(x = \ominus a \oplus b\) is a solution of the equation \(a \oplus x = b\).

We check the uniqueness. Suppose that \(y\) is another solution of the equation \(a \oplus y = b\). Then we have \(a \oplus x = b = a \oplus y\). By the left cancellation law, we obtain \(x = y\).

**Definition 2.40.** In a gyrogroup \((G, \oplus)\) we define the coaddition \(\boxplus\) by

\[
a \boxplus b = a \oplus \text{gyr}[a, \ominus b]b.
\]

We set \(a \boxminus b := a \boxplus (\ominus b) = a \oplus \text{gyr}[a, b](\ominus b) = a \oplus \text{gyr}[a, b]b\).

**Proposition 2.41.** For any elements \(a, b, c\) in a gyrogroup \((G, \oplus)\), \(a \oplus b = c\) if and only if \(a = c \boxminus b\).
Proof. If \( a \oplus b = c \), then

\[
\begin{align*}
  a &= a \oplus (b \ominus b) \\
  &= (a \oplus b) \ominus \text{gyr}[a, b](\ominus b) \\
  &= (a \oplus b) \ominus \text{gyr}[a, b]b \\
  &= (a \oplus b) \ominus \text{gyr}[a \oplus b, b]b \\
  &= c \ominus \text{gyr}[c, b]b \\
  &= c \ominus b.
\end{align*}
\]

Conversely we first note that

\[
\begin{align*}
  c \ominus b &= c \ominus (\ominus b) \\
  &= c \ominus \text{gyr}[c, b](\ominus b) \\
  &= c \ominus [\ominus(c \ominus b) \ominus (c \ominus (b \ominus (\ominus b))))] \\
  &= c \ominus [\ominus(c \ominus b) \ominus (c \ominus 0)] \\
  &= c \ominus [\ominus(c \ominus b) \ominus c].
\end{align*}
\]

Therefore,

\[
(c \ominus b) \ominus b = (c \ominus [\ominus(c \ominus b) \ominus c]) \ominus b = c \ominus [\ominus(c \ominus b) \ominus (c \ominus b)] = c,
\]

where the second equality follows from the left Bol identity.

Corollary 2.42. In a gyrogroup \((G, \oplus)\) the equation \( x \oplus a = b \) has the unique solution \( x = b \ominus a \).

Proof. We obtain this corollary easily by changing the variables in Proposition 2.41.

Lemma 2.43. For any elements \( a, b, \) and \( c \) in a gyrogroup \((G, \oplus)\),

\[
(a \ominus b) \ominus c = a \ominus \text{gyr}[a, \ominus b](b \ominus c).
\]
This property is called a mixed gyroassociative law.

Proof. By the left gyroassociativity and the Equation (2.5) we have

\[(a \triangledown b) \oplus c = (a \ominus \text{gyr}[a, b]b) \oplus c = a \ominus (\ominus \text{gyr}[a, b]b \ominus \text{gyr}[a, b]c) = a \ominus \text{gyr}[a, b](\ominus b \ominus c).\]

Replacing \(b\) by \(\ominus b\) gives us the desired identity. \(\square\)

Lemma 2.44. For any elements \(a, b,\) and \(x\) in a gyrogroup \((G, \oplus),\)

\[(a \oplus \text{gyr}[a, b]x) \triangledown (b \oplus x) = a \triangledown b.\]

Proof. By the left gyroassociativity, Corollary 2.42, and the left loop property, we have

\[(a \triangledown b) \ominus (b \ominus x) = \{(a \triangledown b) \ominus b\} \ominus \text{gyr}[a \triangledown b, b]x = a \ominus \text{gyr}[a \triangledown b, b]x = a \ominus \text{gyr}[a, b]x.\]

By Proposition 2.41, we obtain

\[a \triangledown b = (a \oplus \text{gyr}[a, b]x) \triangledown (b \ominus x).\]

Remark 2.45. In particular, if \(\text{gyr}[a, b] = I\) holds, then

\[(a \ominus x) \triangledown (b \ominus x) = a \triangledown b = a \ominus b.\]

Definition 2.46. A gyrogroup is called gyrocommutative if it satisfies

\[a \ominus b = \text{gyr}[a, b](b \ominus a).\]
We consider the basic alternative characterizations of gyrocommutative gyrogroups.

Lemma 2.47. Let \((G, \oplus)\) be a gyrogroup. Then the following are equivalent;

(1) \(G\) is gyrocommutative,

(2) \(G\) satisfies the automorphic inverse property ; \(\ominus(a \oplus b) = \ominus a \ominus b\),

(3) \(G\) satisfies the Bruck identity ; \((a \oplus b) \oplus (a \ominus b) = a \oplus (b \ominus (b \ominus a))\),

(4) The coaddition \(\boxplus\) is commutative ; \(a \boxplus b = b \boxplus a\).

Proof. Our plan of proof is to show that (1) is equivalent to (2) and is equivalent to (3).

(1) \(\Rightarrow\) (2) Assume that a gyrogroup \((G, \oplus)\) is gyrocommutative. From the left gyroassociative law and (8) of Lemma 2.27, we have

\[
(a \oplus b) \oplus \text{gyr}[a, b](\ominus b \ominus a) = a \oplus (b \ominus (\ominus b \ominus a)) = a \oplus (\ominus a) = 0.
\]

Thus,

\[
\ominus(a \oplus b) = \text{gyr}[a, b](\ominus b \ominus a) = \text{gyr}[a, b]\text{gyr}[\ominus b, \ominus a](\ominus a \ominus b) = \ominus a \ominus b.
\]

The last equality follows from Lemma 2.33.

(2) \(\Rightarrow\) (1) Assume that a gyrogroup \((G, \oplus)\) satisfies the automorphic inverse property. Then using (2) of Lemma 2.33 and the left gyroassociative law, we obtain

\[
\ominus(a \oplus b) \oplus \text{gyr}[a, b](b \ominus a) = (\ominus a \ominus b) \oplus \text{gyr}[\ominus a, \ominus b](b \ominus a)
\]

\[
= \ominus a \ominus (b \ominus (b \ominus a))
\]

\[
= \ominus a \ominus a
\]

\[
= 0.
\]
It follows that $\text{gyr}[a, b](b \oplus a)$ is the inverse of $\ominus(a \oplus b)$, and hence must equal to $a \oplus b$.

\[(1) \Rightarrow (3)\] Assume that a gyrogroup $(G, \oplus)$ is gyrocommutative. Then $a \oplus b = \text{gyr}[a, b](b \oplus a)$ for any $a, b \in G$. From (9) of Lemma 2.27 we have $\text{gyr}[a, b] = L_{a \oplus b}^{-1}L_aL_b$, where $L_x$ is a left translation by $x$. Thus,

\[
(a \oplus b) \oplus (a \oplus b) = L_{a \oplus b}(a \oplus b)
\]
\[
= L_{a \oplus b}\text{gyr}[a, b](b \oplus a)
\]
\[
= L_{a \oplus b}L_{a \oplus b}^{-1}L_aL_b(b \oplus a)
\]
\[
= L_aL_b(b \oplus a)
\]
\[
= a \oplus (b \oplus (b \oplus a)).
\]

\[(3) \Rightarrow (1)\] Assume that a gyrogroup $(G, \oplus)$ satisfies the Bruck identity ;

\[
(a \oplus b) \oplus (a \oplus b) = a \oplus (b \oplus (b \oplus a)).
\]

It can be replaced by

\[
L_{a \oplus b}(a \oplus b) = L_aL_b(b \oplus a).
\]

From Remark 2.37 we know that a left translation is bijective. So we have

\[
a \oplus b = L_{a \oplus b}^{-1}L_aL_b(b \oplus a).
\]

By (9) of Lemma 2.27 we obtain $a \oplus b = \text{gyr}[a, b](b \oplus a)$.

A. Ungar has proved the equivalence between (2) and (4) in Theorem 3.4, [1].

**Example 2.48.** Let $(V, +, <, >)$ be a real (or complex) inner product space, and let

\[
V_s = \{v \in V : \|v\| < s\}
\]
be its open $s$-ball, where $s$ is an arbitrary fixed positive constant.

Define the binary operation $\oplus_E$ in $V_s$ by

$$u \oplus_E v = \frac{1}{1 + \frac{<u,v>}{s^2}} \left\{ u + \frac{1}{\gamma_u} v + \frac{1}{s^2} \gamma_u \frac{<u,v>}{1 + \gamma_u} <u,v> u \right\}$$

for any $u, v \in V_s$, where $\gamma_u$ is the well-known Lorentz factor such that

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{<u,u>}{s^2}}}.$$

The binary system $(V_s, \oplus_E)$ forms a gyrocommutative gyrogroup called the standard real (or complex) relativistic gyrogroup or an Einstein gyrogroup.

**Example 2.49.** In the open $s$-ball $V_s$ of a real (or complex) inner product space $(V, +, <, >)$, define the binary operation $\oplus_M$ by

$$u \oplus_M v = \frac{1}{1 + \frac{2}{s^2} <u,v> + \frac{1}{s^2} \|u\|^2 \|v\|^2}$$

$$\times \left\{ \left( \frac{1}{s^2} <u,v> + \frac{1}{s^2} \|v\|^2 \right) u + \left( 1 - \frac{1}{s^2} \|u\|^2 \right) v \right\},$$

where $\|u\| = <u,u>^{\frac{1}{2}}$. The binary system $(V_s, \oplus_M)$ forms another gyrocommutative gyrogroup called the nonstandard real (or complex) relativistic gyrogroup or Möbius gyrogroup.

### 2.2 The Correspondence Structures

In this section we show the relation between a gyrogroup and a symmetric set.

**Theorem 2.50.** A gyrogroup $(G, \oplus)$ is a pointed symmetric set with respect to the operation

$$a \bullet b = a \oplus (\ominus b \oplus a).$$

It is a symmetric mean if and only if the gyrogroup is uniquely 2-divisible.
Proof. Let \((G, \oplus)\) be a gyrogroup. From Proposition 2.36 we know that \(P := \{(a, I) : a \in G\}\) is a twisted subgroup of a group \(G \times \text{Aut}_0(G)\). From Remark 1.6 we have that \((P, \bullet)\) is a pointed symmetric set with respect to the operation \(x \bullet y = xy^{-1}x\) and the base point \((0, I)\). Since \((P, \bullet)\) is isomorphic to \((G, \bullet)\), where

\[
a \bullet b = a \oplus (\ominus b \oplus a),
\]

\((G, \bullet)\) is also a pointed symmetric set with respect to the operation \(\bullet\) and the base point 0.

To check for the existence of the mean, we apply Lemma 1.3. Thus we need that the equation \(b = x \bullet 0 = x \oplus (\ominus 0 \oplus x) = x \ominus x\) has a unique solution. This is equivalent to the existence of unique square roots in the gyrogroup. \(\square\)

In a gyrogroup \((G, \oplus)\), we define \(n \otimes a\) for any \(a \in G\) and any positive integer \(n\) inductively as

\[
(n + 1) \otimes a = a \ominus (n \otimes a), \quad 1 \otimes a = a.
\]

The following theorem is an alternative version for a gyrocommutative gyrogroup to be a pointed symmetric set.

**Theorem 2.51.** A gyrocommutative gyrogroup \((G, \oplus)\) is a pointed symmetric set with respect to the operation

\[
a \bullet b = a \ominus (\ominus a \oplus b) = a \ominus (a \ominus b) = (2 \otimes a) \ominus b. \tag{2.6}
\]

It is a symmetric mean if and only if the gyrogroup is uniquely 2-divisible.

**Proof.** Let \((G, \oplus)\) be a gyrocommutative gyrogroup. Define a binary operation \(\bullet : G \times G \to G\) by

\[
a \bullet b = a \ominus (\ominus a \oplus b)
\]
for any $a, b \in G$. Note that

$$a \bullet b = a \ominus (\ominus a \oplus b) = a \ominus (a \ominus b) = 2 \otimes a \ominus b.$$  

We may use these equalities in the following proof. The second equality follows from the automorphic inverse property, and the third equality follows from the left gyroassociative law and (4) of Lemma 2.27.

We show that $(G, \bullet, 0)$ is a pointed symmetric set. For any $a, b, c \in G$,

1. $a \bullet a = a \oplus (a \ominus a) = a \ominus 0 = a$.

   The second equality follows from the fact that $\ominus a$ is the inverse of $a$, and the third equality follows from the fact that 0 is the identity.

2. $a \bullet (a \bullet b) = 2 \otimes a \ominus (2 \otimes a \ominus b) = 2 \otimes a \oplus (\ominus 2 \otimes a \oplus b) = b$.

   The second equality follows from the automorphic inverse property, and the third equality follows from (8) of Lemma 2.27.

3. $(a \bullet b) \bullet (a \bullet c) = (2 \otimes a \ominus b) \oplus ((2 \otimes a \ominus b) \ominus (2 \otimes a \ominus c))$

   $= (2 \otimes a \ominus b) \oplus ((2 \otimes a \ominus b) \ominus (\ominus 2 \otimes a \oplus c))$

   $= (2 \otimes a \ominus b) \oplus \text{gyr}[\ominus 2 \otimes a, b](\ominus b \ominus c)$

   $= (2 \otimes a \ominus b) \oplus \text{gyr}[2 \otimes a, \ominus b](\ominus b \ominus c)$

   $= 2 \otimes a \ominus (\ominus b \ominus (\ominus b \ominus c))$

   $= 2 \otimes a \ominus (\ominus 2 \otimes b \ominus c)$

   $= 2 \otimes a \ominus (2 \otimes b \ominus c)$

   $= a \bullet (b \ominus c)$.

The second equality follows from the automorphic inverse property, the third follows from Lemma 2.37, the fourth follows from (2) of Lemma 2.33, the fifth and the sixth follow from the left gyroassociative law, and the seventh follows from the automorphic inverse property.
Thus, \((G, \bullet, 0)\) is a pointed symmetric set. The proof of the last assertion is similar to that of Theorem 2.50. □

The next theorem gives us the inverse construction from a symmetric mean to a gyrogroup.

**Theorem 2.52.** A pointed symmetric mean \((X, \bullet, \varepsilon)\) is a uniquely 2-divisible gyrocommutative gyrogroup with respect to the operation

\[
x \oplus y = Q(x^{\frac{1}{2}})y = x^{\frac{1}{2}} \bullet (\varepsilon \bullet y).
\]

(2.7)

**Proof.** First, we show that \((X, \oplus)\) is a gyrogroup.

1. \(\varepsilon \oplus x = \varepsilon^{\frac{1}{2}} \bullet (\varepsilon \bullet x) = \varepsilon \bullet (\varepsilon \bullet x) = x\). The last equality follows from (2) of Definition 1.2. So \(\varepsilon\) is a left identity.

2. Consider \(x \oplus a = \varepsilon\) for any element \(a\) in \(X\). That is,

\[
x^{\frac{1}{2}} \bullet (\varepsilon \bullet a) = \varepsilon.
\]

This implies that \(\varepsilon \bullet a = x^{\frac{1}{2}} \bullet \varepsilon = x\), which means \(\varepsilon \bullet a\) is a left inverse of \(a\).

Let us define a gyration generated by \(a\) and \(b\) as

\[
\text{gyr}[a, b] = Q(Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}}Q(a^{\frac{1}{2}})Q(b^{\frac{1}{2}}),
\]

where \(Q(Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}} = Q((Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}})\).

In Section 1.2, we have seen that each member of \(G(X)\) is an automorphism under the operation \(\bullet\). So each \(Q(a)\) is an automorphism under the operation \(\bullet\), and moreover, under the operation \# from Remark 1.5. Thus the gyration \(\text{gyr}[a, b]\), the composition of several members of \(Q(X)\), is also an automorphism under the operations \(\bullet\) and \#. However, we must show that the gyration is a homomorphism under the gyroaddition \(\oplus\).
Note that
\[ \text{gyr}[a, b] \varepsilon = Q(Q(a^\frac{1}{2})b)^{-\frac{1}{2}}Q(a^\frac{1}{2})Q(b^\frac{1}{2}) \varepsilon \]
\[ = Q(Q(a^\frac{1}{2})b)^{-\frac{1}{2}}Q(a^\frac{1}{2})b \]
\[ = \varepsilon \]

since \( Q(a^{-\frac{1}{2}})a = \varepsilon \) for any \( a \in X \). Moreover,
\[ \text{gyr}[a, b] x^\frac{1}{2} = \text{gyr}[a, b](\varepsilon \# x) \]
\[ = \text{gyr}[a, b] \varepsilon \# \text{gyr}[a, b] x \]
\[ = \varepsilon \# \text{gyr}[a, b] x \]
\[ = (\text{gyr}[a, b] x)^{\frac{1}{2}}. \]

Thus, for any \( x, y \in X \)
\[ \text{gyr}[a, b](x \oplus y) = \text{gyr}[a, b](x^{\frac{1}{2}} \bullet (\varepsilon \bullet y)) \]
\[ = \text{gyr}[a, b] x^{\frac{1}{2}} \bullet (\text{gyr}[a, b] \varepsilon \bullet \text{gyr}[a, b] y) \]
\[ = (\text{gyr}[a, b] x)^{\frac{1}{2}} \bullet (\varepsilon \bullet \text{gyr}[a, b] y) \]
\[ = \text{gyr}[a, b] x \oplus \text{gyr}[a, b] y. \]

This gives us that the gyration is an automorphism under the operation \( \oplus \).

(3) From the definition of \( \text{gyr}[a, b] \), we have for any \( c \in X \)
\[ \text{gyr}[a, b] c = Q(Q(a^\frac{1}{2})b)^{-\frac{1}{2}}Q(a^\frac{1}{2})Q(b^\frac{1}{2})c. \]

This is equivalent that \( Q(Q(a^\frac{1}{2})b)^{\frac{1}{2}} \text{gyr}[a, b] c = Q(a^\frac{1}{2})Q(b^\frac{1}{2})c \), which means
\( (a \oplus b) \oplus \text{gyr}[a, b] c = a \oplus (b \oplus c) \).

(4) Let \( x = a \oplus b = Q(a^\frac{1}{2})b \). By Riccati Lemma, we have \( x^{-1} \# b = a^{-\frac{1}{2}} \). So
\( Q(x^{-1} \# b) = Q(a^{-\frac{1}{2}}) \), equivalently \( Q(x \# b^{-1}) = Q(a^\frac{1}{2}) \). Note that \( a \# b = Q(a^\frac{1}{2})(Q(a^{-\frac{1}{2}})b)^{\frac{1}{2}} \) in a pointed symmetric mean. So
\[ Q(x \# b^{-1}) = Q(Q(x^\frac{1}{2})(Q(x^{-\frac{1}{2}})b^{-1})^{\frac{1}{2}}) = Q(x^\frac{1}{2})Q(Q(x^{-\frac{1}{2}})b^{-1})^{\frac{1}{2}}Q(x^\frac{1}{2}) = Q(a^\frac{1}{2}). \]
The second equality follows from (1) of Lemma 1.13. The above equality implies that $Q(Q(x^{-\frac{1}{2}})b^{-\frac{1}{2}})Q(x^{\frac{1}{2}}) = Q(x^{-\frac{1}{2}})Q(a^{\frac{1}{2}})$. Therefore,

$$gyr[a \oplus b, b] = gyr[x, b]$$

$$= Q(Q(x^{-\frac{1}{2}})b^{-\frac{1}{2}})Q(x^{\frac{1}{2}})Q(b^{\frac{1}{2}})$$

$$= Q(x^{-\frac{1}{2}})Q(a^{\frac{1}{2}})Q(b^{\frac{1}{2}})$$

$$= Q(Q(a^{\frac{1}{2}})b)^{\frac{1}{2}}Q(a^{\frac{1}{2}})Q(b^{\frac{1}{2}})$$

$$= gyr[a, b]$$

This means that the left loop property holds.

Next we show that the gyrogroup $(X, \oplus)$ is gyrocommutative.

(5) $gyr[a, b](b \oplus a) = Q(Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}}Q(a^{\frac{1}{2}})Q(b^{\frac{1}{2}})Q(b^{\frac{1}{2}})a$

$$= Q(Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}}Q(a^{\frac{1}{2}})Q(b)Q(a^{\frac{1}{2}})\varepsilon$$

$$= Q(Q(a^{\frac{1}{2}})b)^{-\frac{1}{2}}Q(Q(a^{\frac{1}{2}})b)\varepsilon$$

$$= Q(Q(a^{\frac{1}{2}})b)^{\frac{1}{2}}\varepsilon$$

$$= Q(a^{\frac{1}{2}})b$$

$$= a \oplus b.$$

The second and fifth equalities follow from (4) of Lemma 1.13, and the third equality follows from (1) of Lemma 1.13.

Therefore, $(X, \oplus)$ is a gyrocommutative gyrogroup. By Theorem 2.51, $X$ is uniquely 2-divisible since $(X, \bullet, \varepsilon)$ is a symmetric mean. □

**Remark 2.53.** From Proposition 2.36, Theorem 2.51, and Theorem 2.52, we have shown the following equivalences;

(1) A uniquely 2-divisible (gyrocommutative) gyrogroup,

(2) a uniquely 2-divisible twisted subgroup,
We next show the correspondence among the homomorphisms under the binary operations $\oplus$ and $\bullet$. In the following we denote $(G, \bullet)$ and $(G, \oplus)$ symmetric mean and uniquely 2-divisible gyrocommutative gyrogroup, corresponding one to the other.

**Lemma 2.54.** Let $f : G_1 \to G_2$ be a map with $f(0_1) = 0_2$, where $0_1$ and $0_2$ are the distinguished points of $G_1$ and $G_2$, respectively. Then the following are equivalent.

1. $f : (G_1, \oplus_1) \to (G_2, \oplus_2)$ is an $\oplus$ homomorphism.
2. $f : (G_1, \bullet_1) \to (G_2, \bullet_2)$ is a $\bullet$ homomorphism.

**Proof.** We use the equations (2.6) and (2.7). Let $a, b \in G_1$.

(1) $\Rightarrow$ (2) Assume that the map $f$ is an $\oplus$ homomorphism. Then

$$0_2 = f(0_1) = f(a \oplus_1 (\ominus_1 a)) = f(a) \oplus_2 f(\ominus_1 a).$$

So $f(\ominus_1 a) = \ominus_2 f(a)$. This implies

$$f(a \bullet_1 b) = f(a \oplus_1 (a \ominus_1 b)) = f(a) \oplus_2 f(a \ominus_1 b) = f(a) \oplus_2 (f(a) \ominus_2 f(b)) = f(a) \bullet_2 f(b).$$

(2) $\Rightarrow$ (1) Assume that the map $f$ is an $\bullet$ homomorphism. By Remark 1.5 we know that $f$ is also a $\#$ homomorphism. Then

$$f(a^{\frac{1}{2}}) = f(0_1 \#_1 a) = 0_2 \#_2 f(a) = f(a)^{\frac{1}{2}}.$$
This implies

\[ f(a \oplus_1 b) = f(a^{\frac{1}{2}} \bullet_1 (0_1 \bullet_1 b)) \]
\[ = f(a^{\frac{1}{2}}) \bullet_2 f(0_1 \bullet_1 b) \]
\[ = f(a)^{\frac{1}{2}} \bullet_2 (0_2 \bullet_2 f(b)) \]
\[ = f(a) \oplus_2 f(b). \]

Remark 2.55. From Lemma 2.54 and Remark 1.5 we obtain the equivalence among the homomorphisms under the binary operations \( \oplus, \bullet, \) and \#.

2.3 Topological Gyrovector Spaces

In this section our goal is to equip a gyrocommutative gyrogroup with enough structure, called a gyrovector space, to carry out analytic hyperbolic geometry. Moreover, we show that a topological gyrovector space is equivalent to a pointed lineated symmetric space.

Definition 2.56. A gyrovector space consists of a gyrocommutative gyrogroup \((G, \oplus)\) equipped with a scalar multiplication \((t, x) \mapsto t \otimes x : \mathbb{R} \times G \to G\) satisfying the following: for any \(s, t \in \mathbb{R}\) and \(a, b, x \in G\)

1. \(1 \otimes x = x,\quad 0 \otimes x = t \otimes 0 = 0,\) and \((-1) \otimes x = \ominus x;\)

2. \((s + t) \otimes x = s \otimes x \oplus t \otimes x;\)

3. \(s \otimes (t \otimes x) = (st) \otimes x;\)

4. \(\text{gyr}[a, b](t \otimes x) = t \otimes \text{gyr}[a, b]x.\)
Example 2.57. In the open $s$-ball $V_s$ of a real (or complex) inner product space $(V, +, <, >)$, define a scalar multiplication by

$$t \otimes v = s \cdot \left(1 + \frac{\|v\|}{s}\right)^t - \left(1 - \frac{\|v\|}{s}\right)^t \frac{v}{\|v\|}$$

$$= s \tanh \left(t \tanh^{-1} \frac{\|v\|}{s}\right) \frac{v}{\|v\|},$$

where $t \in \mathbb{R}$, $v \in V_s$, $v \neq 0$, and define $t \otimes 0 := 0$.

We call $(V_s, \oplus, \otimes)$ the Einstein gyrovector space, where the binary operation $\oplus$ is defined in Example 2.48. On the other hand, we call $(V_s, \oplus_M, \otimes)$ the Möbius gyrovector space, where the binary operation $\oplus'$ is defined in Example 2.49.

Definition 2.58. A topological gyrovector space is a gyrovector space $G$ equipped with a Hausdorff topology such that both $\oplus : G \times G \to G$ and $\otimes : \mathbb{R} \times G \to G$ are continuous.

We have an interesting property of gyrations in the topological gyrovector space $G$.

Lemma 2.59. Let $a \in G$, and let $s,t \in \mathbb{R}$. Then

$$\text{gyr}[s \otimes a, t \otimes a] = I,$$

where $I$ is the identity map.

Proof. From Theorem 4.1 and Lemma 5.1 in [2] A. Ungar has proved for a uniquely 2-divisible gyrogroup $G$ that

$$\text{gyr}[p \otimes a, q \otimes a] = I$$

for any two dyadic rational numbers $p$ and $q$. This implies

$$p \otimes a \oplus (q \otimes a \oplus b) = (p + q) \otimes a \oplus b.$$
for any $b \in G$.

In the topological gyrovector space $G$, both the gyroaddition $\oplus : G \times G \to G$ and the scalar multiplication $\otimes : \mathbb{R} \times G \to G$ are continuous. So, by the density of dyadic rational numbers we have

$$s \otimes a \oplus (t \otimes a \oplus b) = (s + t) \otimes a \oplus b.$$ 

By applying the left gyroassociativity on the left hand side and using the left cancellation law, we obtain

$$\text{gyr}[s \otimes a, t \otimes a]b = b$$

for any $b \in G$. 

We derive the main theorem of this section, which is a one-to-one correspondence between a topological gyrovector space and a pointed lineated symmetric space.

**Theorem 2.60.** There is a one-to-one correspondence between a topological gyrovector space and a pointed lineated symmetric space.

**Proof.** Let $(X, \oplus, \otimes)$ be a topological gyrovector space. Define a binary operation $\bullet : X \times X \to X$ by

$$x \bullet y = x \oplus (\otimes x \oplus y)$$

for any $x, y \in X$. Then from Theorem 2.51, we have shown that $(X, \bullet, 0)$ is a pointed symmetric set. From the condition (3) of Definition 2.56 we know that a gyrovector space $X$ is uniquely 2-divisible. Thus $(X, \bullet, 0)$ is a pointed symmetric mean equipped with a Hausdorff topology since the topological gyrovector space $X$ already has the Hausdorff topology.

Let us define $x^q := q \otimes x$ for all dyadic rational numbers $q$. Then obviously the operation extends to a continuous map $x^t = t \otimes x : \mathbb{R} \times X \to X$ since scalar
multiplication is a continuous function. Moreover, the binary operation • is also continuous because both the gyroaddition ⊕ and the scalar multiplication ⊗ are continuous in a topological gyrovector space \( X \). Therefore, \( X \) is a pointed lineated symmetric space by Theorem 1.25.

Let \( (X, \cdot, \varepsilon) \) be a pointed lineated symmetric space. Define a binary operation \( \oplus : X \times X \to X \) by

\[
x \oplus y = Q(x^{\frac{1}{2}})y = x^{\frac{1}{2}} \cdot (\varepsilon \cdot y).
\]

From Theorem 2.52, we have shown that \( (X, \oplus) \) is a gyrocommutative gyrogroup.

Define a scalar multiplication \( \otimes : \mathbb{R} \times X \to X \) by

\[
t \otimes x = \alpha_{\varepsilon,x}(t) = \varepsilon \#_t x = x^t,
\]

where \( \alpha_{\varepsilon,x} \) is a unique continuous homomorphism such that \( \alpha_{\varepsilon,x}(0) = \varepsilon \) and \( \alpha_{\varepsilon,x}(1) = x \).

From Remark 1.24, the conditions (1), (2), and (3) of Definition 2.56 hold. Moreover, the condition (4) follows from Lemma 1.22. Thus, \( (X, \oplus, \otimes) \) is a gyrovector space.

From the condition (3) of Definition 1.18 we have that the map \( (t, \varepsilon, x) \mapsto \varepsilon \#_t x = x^t : \mathbb{R} \times X \times X \to X \) is continuous. So the scalar multiplication \( \otimes : \mathbb{R} \times X \to X \) is continuous. Since the binary operation \( \oplus : X \times X \to X \) is a composition of • operations, it is also continuous from the definition of the pointed lineated symmetric space \( X \). Therefore, \( (X, \oplus, \otimes) \) is a topological gyrovector space with a Hausdorff topology.

Note that the \( t \)-weighted mean of \( x \) and \( y \) in the gyrovector space \( X \) is given by

\[
x \#_t y = x \oplus t \otimes (\ominus x \oplus y),
\]

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where \( t \in [0, 1] \). In the special case when \( t = \frac{1}{2} \), we call it the midpoint or mean of \( x \) and \( y \), and denote \( x \# y \).

**Proposition 2.61.** The midpoint \( x \# y \) of \( x \) and \( y \) in the gyrovector space \( X \) holds

\[
x \# y = \frac{1}{2} \otimes (x \oplus y).
\]

*Proof.* By the Bruck identity of Lemma 2.47

\[
2 \otimes (a \oplus b) = a \oplus (b \oplus (b \oplus a)) = a \oplus (2 \otimes b \oplus a),
\]

the left gyroassociativity, the loop properties, and (4) of Lemma 2.47, we have

\[
2 \otimes (x \# y) = 2 \otimes \left( x \oplus \frac{1}{2} \otimes (\ominus x \oplus y) \right)
\]

\[
= x \oplus \left( 2 \otimes \frac{1}{2} \otimes (\ominus x \oplus y) \oplus x \right)
\]

\[
= x \oplus \{(\ominus x \oplus y) \oplus x\}
\]

\[
= \{x \oplus (\ominus x \oplus y)\} \oplus \text{gyr}[x, \ominus x \oplus y] x
\]

\[
= y \oplus \text{gyr}[y, \ominus x] x
\]

\[
= y \oplus x
\]

\[
= x \oplus y.
\]

Therefore, \( x \# y = \frac{1}{2} \otimes (x \oplus y) \). \( \square \)

In [1] A. Ungar calls \( x \oplus_t y \) for all \( t \in \mathbb{R} \) a gyroline passing through \( x \) at \( t = 0 \) and \( y \) at \( t = 1 \). We will see more detail about the gyrolines in Section 5.1.
3. The Local Flow of a Spray

In differential geometry, a spray is a vector field on the tangent bundle $T(X)$ that encodes a quasilinear second order system of ordinary differential equations on the base manifold $X$. Sprays arise naturally in Riemannian and Finsler geometry as the geodesic sprays, whose integral curves are precisely the tangent curves of locally length minimizing curves. We focus on a notion of spray and the exponential map associated with spray developed by S. Lang in Chapter IV, [20].

In Section 1 we introduce the concepts of a local flow and the connection with spray to the exponential map. Moreover, we refer S. Lang’s result that any two points in a normal neighborhood are joined by a unique geodesic.

In Section 2 we define weighted means using the exponential map and find their properties. We furthermore investigate the Lie-Trotter formula in a manifold with spray.

In Section 3 we give a definition of locally midpoint-preserving map and show that the continuous and locally midpoint-preserving map is smooth on manifolds with spray.

3.1 Sprays and the Exponential Map

Let $X$ be a smooth manifold modeled on a Banach space $E$. Let $\pi : T(X) \to X$ be its tangent bundle.

By a vector field on $X$ we mean a smooth map

$$\xi : X \to T(X)$$

such that $\xi(x)$ lies in the tangent space $T_x(X)$ for each $x \in X$. In other words, $\pi \circ \xi = id_X$. 

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An integral curve for the vector field $\xi$ with initial condition $x_0 \in X$ is a curve

$$\alpha : J \to X$$

mapping an open interval $J$ of $\mathbb{R}$ containing $0$ into $X$ such that for all $t \in J$

$$\alpha'(t) = \xi(\alpha(t)), \ \alpha(0) = x_0.$$

From Theorem 2.2 in Chapter IV. 2 of [20], we obtain the following result.

**Lemma 3.62.** Let $\xi : X \to T(X)$ be a smooth vector field. Then there exists a smooth map $\Phi : U \to X$, called a local flow, where $U$ is an open subset of $\mathbb{R} \times X$ such that $\{0\} \times X \subseteq U$, satisfying

1. $\Phi(0, x) = x$ (identity property),

2. $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$ provided the left-hand side is defined (semigroup property),

3. Let $J(x) = (t^-(x), t^+(x))$ be the time interval for $x$, that is, the largest interval containing $0$ such that $(t^-(x), t^+(x)) \times \{x\} \subseteq U = \text{dom}(\Phi)$. Then $\varphi_x(t)$ on $J(x)$ defined by $\varphi_x(t) = \Phi(t, x)$ is a maximal integral curve, in particular $\varphi'_x(t) = \xi(\varphi_x(t))$. Furthermore, the solution is unique on any interval on which it is defined with $\varphi_x(0) = x$.

**Remark 3.63.** In Lemma 1.15 of Chapter IV. 1 and Theorem 2.2 of Chapter IV. 2 in [20] S. Lang has shown

$$J(tx) = J(x) - t,$$

where $J(tx)$ is the time interval of the local flow $\Phi$ with initial condition $tx = \Phi(t, x)$. This identity provides the time constraints that we implicitly assume in our calculations with local flows.
**Definition 3.64.** Let $\pi: T(X) \to X$ be a tangent bundle. A second-order vector field is a vector field $F: T(X) \to T(T(X))$ satisfying

$$T(\pi) \circ F = id_{T(X)}.$$ 

Applying Lemma 3.62 to a second-order vector field $F$, we obtain a local flow $\Phi_F: U_F \to T(X)$, where $U_F$ is an open subset of $\mathbb{R} \times T(X)$ containing $\{0\} \times T(X)$, such that for a vector $v \in T(X)$

1. $\Phi_F(0, v) = v,$

2. $\Phi_F(s, \Phi_F(t, v)) = \Phi_F(s + t, v)$ provided the left-hand side is defined.

There is an equivalent condition to be a second-order vector field, which is related with the local flow. See Chapter IV. 3 of [20].

**Proposition 3.65.** Let $F: T(X) \to T(T(X))$ be a vector field. Then $F$ is a second-order vector field if and only if the local flow $\Phi_F$ on the tangent space $T(X)$ satisfies

$$\left(\pi \circ \Phi_F\right)' = \Phi_F,$$

more specifically,

$$\frac{\partial}{\partial t} (\pi \circ \Phi_F)(t, x) = \Phi_F(t, x).$$

**Definition 3.66.** Let $s \in \mathbb{R}$ and $s_*: T(X) \to T(X)$ denote the scalar multiplication by $s$ in each tangent space. A second-order vector field $F$ on $T(X)$ is called a spray if for all $s \in \mathbb{R}$ and $v \in T(X)$

$$F(sv) = T(s_*)(sF(v)).$$

(3.8)

In Chapter IV. 4 of [20] S. Lang presents equivalent conditions of a spray involving the integral curves of the second-order vector field $F$. We translate it to conditions on the local flow.
Proposition 3.67. For a vector \( v \) in \( T(X) \), let \( \Phi_F \) be the local flow of the second-order vector field \( F \) with initial condition \( v \). Then the following conditions are equivalent to the condition (3.8):

1. A pair \((t, sv)\) is in the domain of \( \Phi_F \) if and only if \((st, v)\) is in the domain of \( \Phi_F \), and then
   \[
   \Phi_F(t, sv) = s\Phi_F(st, v).
   \]

2. A pair \((st, v)\) is in the domain of \( \Phi_F \) if and only if \((s, tv)\) is in the domain of \( \Phi_F \), and then
   \[
   \pi\Phi_F(s, tv) = \pi\Phi_F(st, v).
   \]

3. A pair \((t, v)\) is in the domain of \( \Phi_F \) if and only if \((1, tv)\) is in the domain of \( \Phi_F \), and then
   \[
   \pi\Phi_F(t, v) = \pi\Phi_F(1, tv).
   \]

Next we consider connections of the local flow of a spray to the exponential map.

Let \( F \) be a spray on \( X \). As above, we let \( \Phi_F \) be the local flow of \( F \). Let \( D \) be the set of vectors \( v \) in \( T(X) \) such that \( \Phi_F \) is defined at least on the domain \([0, 1] \times \{v\}\).

We know from Theorem 2.6 of Chapter IV. 2 in [20] that \( D \) is an open set in \( T(X) \), and the map \( v \mapsto \Phi_F(1, v) \) is a morphism of \( D \) into \( T(X) \).

We define the exponential map
\[
\exp : D \to X, \quad \exp(v) = \pi\Phi_F(1, v).
\]

Then \( \exp \) is a morphism. We also call \( D \) the domain of the exponential map associated with \( F \).

If \( x \in X \) and \( 0_x \) denotes the zero vector in \( T_x(X) \), then we obtain \( \exp(0_x) = x \).

Indeed,
\[
\exp(0_x) = \pi\Phi_F(1, 0_x) = \pi\Phi_F(1, 0 \cdot 0_x) = \pi\Phi_F(0, 0_x) = \pi(0_x) = x.
\]
Thus our exponential map coincides with $\pi$ on the zero section, and so induces an isomorphism of the cross section onto $X$. We denote the restriction of $\exp$ to the tangent space $T_x(X)$ by $\exp_x$. Thus

$$\exp_x : T_x(X) \to X, \quad \exp_x(0_x) = x.$$  

**Remark 3.68.** From the above we can define $\exp_x(tv)$ for $v$ in the domain of $\exp_x$ and $t \in [0, 1]$ as

$$\exp_x(tv) = \pi\Phi_F(1, tv) = \pi\Phi_F(t, v).$$

Furthermore, we have

$$\exp_x(s(tv)) = \pi\Phi_F(s, tv) = \pi\Phi_F(st, v) = \exp_x((st)v)$$

by (2) of Proposition 3.67.

Moreover, there exists an open neighborhood $V$ of $0_x$ in $T(X)$ on which we can define a map $G : V \to X \times X$ such that

$$G(v) = (\pi v, \exp_{\pi v}(v)).$$

It is sometimes useful to express this map in different notation. Specifically, if we denote a point in the tangent bundle by a pair $(x, v)$ for $v \in T_x(X)$, then

$$G(x, v) = (x, \exp_x(v)).$$

For the next local result, it is convenient to express certain uniformities in a chart, where we can measure distances uniformly in the model Banach space $E$ with a given norm. It is irrelevant to know whether this norm has any smoothness properties or not. It will be used just to describe neighborhoods of a zero vector in the tangent bundle.
Let $x_0 \in X$. For $\epsilon > 0$ we let $B(\epsilon)$ denote the open ball of elements $v \in E$, the modeled Banach space, with $\|v\| < \epsilon$. Arbitrarily small open neighborhoods of $(x_0, 0_{x_0})$ in a chart for $T(X)$ is of the form

$$U_0 \times B(\epsilon),$$

where $U_0$ is an open neighborhood of $x_0$ in $X$ and $\epsilon$ is arbitrarily small.

From Corollary 5.2 in Chapter VIII. 5 of [20], we know that each point of $X$ has a small neighborhood such that any two points in it are joined by a unique geodesic.

**Theorem 3.69.** Given $x_0 \in X$, let $V$ be an open neighborhood of $(x_0, 0_{x_0})$ in $T(X)$ such that $G$ induces an isomorphism of $V$ with its image. We may pick $V$ such that for some $\epsilon > 0$

$$V = U_0 \times B(\epsilon) \subseteq X \times E.$$

Let $W$ be a neighborhood of $x_0$ in $X$ such that $G(V) \supset W \times W$. Then

1. Any two points $x, y \in W$ are joined by a unique geodesic in $X$ lying in $U_0$, and this geodesic depends $C^\infty$ on the pair $(x, y)$.

   In other words, if
   $$t \mapsto \exp_x(tv), \ 0 \leq t \leq 1$$
   is the geodesic joining $x$ and $y$ with $y = \exp_x(v)$, then the correspondence
   $$(x, v) \leftrightarrow (x, y)$$
   is a $C^\infty$-diffeomorphism.

2. For each $x \in W$ the exponential $\exp_x$ maps the open set in $T_x(X)$ represented by $(x, B(\epsilon))$ isomorphically onto an open set $U(x)$ containing $W$.  

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We call $W$ a normal neighborhood of $x_0$ in $X$ and $B(\epsilon)$ a normal ball around $0_{x_0}$ in $T(X)$.

### 3.2 The Lie-Trotter Formula for Sprays

In this section we consider a manifold $X$ with a spray $F$. We define the weighted mean of any pair $(x, y)$ and derive the Lie-Trotter formula with respect to the midpoint operation.

From Theorem 3.69 we know that any two points $x, y \in W$ are joined by a unique geodesic in $U_0$, namely $t \to \exp_x(tv)$ with $y = \exp_x(v)$. Unfortunately, for some $t_0 \in [0, 1]$ the image $\exp_x(t_0v)$ may lay outside the normal neighborhood $W$. So we consider a set

$$\mathcal{D}_W = \{(x, y) \in W \times W : \exp_x(tv) \in W \text{ for all } 0 \leq t \leq 1, \text{ where } \exp_x(v) = y\}.$$

For any pair $(x, y) \in \mathcal{D}_W$ we can naturally define the weighted mean of $x$ and $y$.

**Definition 3.70.** The $t$-weighted mean of a pair $(x, y) \in \mathcal{D}_W$ is given by

$$x \#_t y = \exp_x(tv),$$

where $0 \leq t \leq 1$.

**Lemma 3.71.** The set $\mathcal{D}_W$ is open.

**Proof.** For any pair $(x, y) \in \mathcal{D}_W$, we define a map $\alpha : [0, 1] \times W \times W \to U_0$ by

$$\alpha(t, x, y) = x \#_t y,$$

where $0 \leq t \leq 1$. From the setting of $\mathcal{D}_W$ we have that $\alpha ([0, 1] \times \{x\} \times \{y\}) \subseteq W$. Furthermore, the map $\alpha$ is continuous on a compact set $[0, 1] \times \{x\} \times \{y\}$ in $[0, 1] \times W \times W$ by Theorem 3.69. Then there are open sets $O_1$ in $W$ containing $\{x\}$ and $O_2$ in $W$ containing $\{y\}$ such that $\alpha ([0, 1] \times O_1 \times O_2) \subseteq W$. This means that $O_1 \times O_2$ is an open set in $\mathcal{D}_W$ containing $\{x\} \times \{y\}$. \qed
Now we define the midpoint of $x$ and $y$ as the geodesic image at $t = \frac{1}{2}$, where $(x, y) \in D_W$.

**Definition 3.72.** For any pair $(x, y) \in D_W$, we define the midpoint $x \# y$ by

$$x \# y := \exp_x \left( \frac{1}{2} v \right).$$

(3.9)

**Remark 3.73.** From the preceding definition we can see that $x \# \exp_x(tv) = \exp_x \left( \frac{t}{2} v \right)$ for any $t \in [0, 1]$. Indeed, for $u = tv$, $\exp_x(u) \in W$ and moreover, $(x, \exp_x(u)) \in D_W$. Then

$$x \# \exp_x(u) = \exp_x \left( \frac{1}{2} u \right) = \exp_x \left( \frac{t}{2} v \right).$$

Similarly we can define the midpoint $y \# x$ for a pair $(y, x) \in D_W$ by

$$y \# x := \exp_y \left( \frac{1}{2} w \right),$$

where $x = \exp_y(w)$.

**Proposition 3.74.** For any pair $(x, y) \in D_W$,

$$x \# y = y \# x.$$

**Proof.** Let $\alpha(t) = \exp_x(tv)$ be the unique geodesic joining two points $x$ and $y = \exp_x(v)$ for $t \in [0, 1]$. By Remark 3.68 we have seen that

$$\alpha(t) = \exp_x(tv) = \pi \Phi_F(t, v),$$

and so $\alpha'(t) = \Phi_F(t, v)$ by Proposition 3.65. Let $w = -\alpha'(1) = -\Phi_F(1, v)$. The semigroup property of the local flow $\Phi_F$ implies

$$\Phi_F(-t, -w) = \Phi_F(-t, \Phi_F(1, v)) = \Phi_F(1 - t, v) = \alpha'(1 - t).$$

Moreover, (2) of Proposition 3.67 gives us

$$\alpha(1 - t) = \pi \Phi_F(-t, -w) = \pi \Phi_F(t, w) = \exp_y(tw).$$
So the map \( t \to \alpha(1-t) = \pi \Phi_F(t, w) \) is the unique geodesic joining the two points \( y \) and \( x \) in \( W \). Thus, (2) of Proposition 3.67 and the definition of the local flow imply

\[
y\#x = \exp_y \left( \frac{1}{2} w \right) = \pi \Phi_F \left( \frac{1}{2}, w \right) \\
= \pi \Phi_F \left( -\frac{1}{2}, -w \right) = \pi \Phi_F \left( -\frac{1}{2}, \Phi_F(1, v) \right) \\
= \pi \Phi_F \left( \frac{1}{2}, v \right) = \exp_x \left( \frac{1}{2} v \right) = x\#y.
\]

\[\square\]

From Definition 3.72 and Remark 3.73 we have a nice reformulation in terms of the local flow \( \Phi_F \): for any \([0, 1] \times \{w\} \subseteq dom(\Phi_F)\) and \( r \in [0, 1] \),

\[
\pi \Phi_F(0, w) \# \pi \Phi_F(r, w) = \pi \Phi_F \left( \frac{r}{2}, w \right).
\]

(3.10)

Now we show that the \( t \)-weighted mean \( \alpha(t) = \exp_x(tv) \) is uniquely determined by its midpoint preservation.

**Lemma 3.75.** For any pair \((x, y) \in \mathcal{D}_W\), the map \( \alpha(t) = \exp_x(tv), 0 \leq t \leq 1 \), is the unique midpoint-preserving continuous map with \( \alpha(0) = x \) and \( \alpha(1) = y \).

**Proof.** By Theorem 3.69 we have seen that for any pair \((x, y)\) in \( W \times W \), there is a unique smooth map \( \exp_x(tv) \) in \( U_0 \) with \( \exp_x(0_x) = x \) and \( \exp_x(v) = y \). This implies that for any pair \((x, y)\) in \( \mathcal{D}_W \), \( \alpha(t) = \exp_x(tv) \) is the unique continuous map in a normal neighborhood \( W \) with

\[
\alpha(0) = \exp_x(0 \cdot v) = \exp_x(0_x) = x, \quad \alpha(1) = \exp_x(v) = y.
\]

We know that \( \exp_x(tv) = \pi \Phi_F(t, v) \). It remains to show that

\[
\pi \Phi_F(s, v) \# \pi \Phi_F(t, v) = \pi \Phi_F \left( \frac{s + t}{2}, v \right).
\]
Without loss of generality, we assume that $0 \leq s \leq t \leq 1$. From the equation (3.10) we have

$$
\pi \Phi_F(0, w) \# \pi \Phi_F(t - s, w) = \pi \Phi_F\left(\frac{t - s}{2}, w\right),
$$

where $(\pi \Phi_F(0, w), \pi \Phi_F(t - s, w)) \in D_W$. Taking $w = \Phi_F(s, v)$, we conclude

$$
\pi \Phi_F(s, v) \# \pi \Phi_F(t, v) = \pi \Phi_F(0, \Phi_F(s, v)) \# \pi \Phi_F(t - s, \Phi_F(s, v))
= \pi \Phi_F\left(\frac{t - s}{2}, \Phi_F(s, v)\right)
= \pi \Phi_F\left(\frac{s + t}{2}, v\right).
$$

We show some useful identities involving the weighted mean.

**Proposition 3.76.** Let $X$ be a manifold with spray $F$. For any pair $(x, y) \in D_W$, and $r, s, t \in [0, 1]$, we have

1. $x \#_0 y = x$, $x \#_1 y = y$, $x \#_t x = x$;

2. $x \#_t y = y \#_{1-t} x$;

3. $(x \#_r y) \#_t (x \#_s y) = x \#_{(1-t)r + ts} y$;

4. $x \#_t (x \#_s y) = x \#_{ts} y$.

**Proof.** We use that $x \#_t y = \exp_x(tv) = \pi \Phi_F(t, v)$ is a unique geodesic in Lemma 3.75.

(1) Immediate from Lemma 3.75.

(2) In the proof of Proposition 3.74 we have seen that $\pi \Phi_F(t, w)$ is locally
the unique geodesic with $\pi \Phi_F(0, w) = y$ and $\pi \Phi_F(1, w) = x$, where $w = -\Phi_F(1, v)$. This gives us that $\pi \Phi_F(t, w) = y \#_t x$, and therefore,

$$
x \#_t y = \pi \Phi_F(t, v) = \pi \Phi_F(t - 1, \Phi_F(1, v)) = \pi \Phi_F(1 - t, w) = y \#_{1-t} x.
$$
(3) For fixed $r$ and $s$, $\alpha(t) = x#(1-t)r+tsy = \pi \Phi_F((1-t)r + ts, v)$ is locally the unique geodesic such that $\alpha(0) = x#ry$ and $\alpha(1) = x#sy$. Furthermore, for any $t,u \in [0,1]$

$$\alpha\left(\frac{t + u}{2}\right) = \pi \Phi_F\left(\frac{1 - t + 1 - u}{2}r + \frac{t + u}{2}s, v\right)$$

$$= \pi \Phi_F\left(\frac{(1 - t)r + ts + (1 - u)r + us}{2}, v\right)$$

$$= \pi \Phi_F((1-t)r+ts,v) \neq \pi \Phi_F((1-u)r+us,v)$$

$$= \alpha(t)#\alpha(u).$$

By Lemma 3.75 we conclude (3).

(4) Using (1) and (3), we have

$$x#(x#sy) = (x#0y)#1(x#sy) = x#(1-t)0+tsy = x#isy.$$

The Lie-Trotter product formula

$$e^{A+B} = \lim_{n \to \infty} \left( e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n$$

for square matrices $A$ and $B$ is of great utility and generalizes to various settings such as semigroups of operators and Lie theory. Now we derive the Lie-Trotter formula with the midpoint operation.

In the following theorem and proof, we denote $\exp_x$ as simply $\exp$ and set $\log$ to be the inverse of $\exp$ on the $\epsilon$-ball $B(\epsilon)$ that appeared in Theorem 3.69.

**Theorem 3.77.** For $v, w \in T_x(X)$,

$$v + w = \lim_{t \to 0} \frac{1}{t} \log(\exp(2tv)#\exp(2tw))$$

$$= \lim_{n \to \infty} n \log \left( \exp \left( \frac{2v}{n} \right) \# \exp \left( \frac{2w}{n} \right) \right).$$

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Proof. Set $E = T_x(X)$. Since $x#x = x$ from (1) of Proposition 3.76, there is an open set $U$ containing $x \in X$ such that

$$U \times U \in \mathcal{D}_W.$$ 

Then the map $f : E \times E \to E$,

$$f(v, w) = \log(\exp(2v)\#\exp(2w)),$$

is well defined. By Theorem 3.69 we know that the map $f$ is smooth. By the definition of the directional derivative for $f$, we have

$$df_{(0,0)}(v, w) = \lim_{t \to 0} \frac{1}{t} \log(\exp(2tv)\#\exp(2tw)).$$

We compute $f(v, 0)$. The map $\beta(t) = \exp(2tv)$ is a geodesic in $W$ from $x$ to $\exp(2v)$, and so $x\#\exp(2v) = \exp(v)$. Thus

$$f(v, 0) = \log(\exp(2v)\#x) = \log(x\#\exp(2v)) = v,$$

and similarly $f(0, w) = w$. Thus the partial derivatives of $f$ are projections into the respective coordinates. Since the directional derivative is the sum of the partial derivatives, we have

$$df_{(0,0)}(v, w) = v + w.$$

Replacing $t$ by $\frac{1}{n}$, we obtain the second equality. \hfill \square

**Corollary 3.78.** Let $v, w \in T_x(X)$. Then

$$\exp(v + w) = \lim_{t \to 0} (\exp(2tv)\#\exp(2tw))^{1/t} = \lim_{n \to \infty} \left( \exp \left( \frac{2v}{n} \right) \# \exp \left( \frac{2w}{n} \right) \right)^n.$$

E. Ahn, S. Kim, and Y. Lim have shown the various extensions of the Lie-Trotter formula in [7]. We have also the extension in a smooth manifold with spray. The proof is similar to that of Theorem 3.77.
Proposition 3.79. Let \( v_1, \ldots, v_n \in T_x(X) \). Then
\[
\sum_{i=1}^{n} v_i = \lim_{t \to 0} \frac{1}{t} \log \left( \exp(nt v_n) \# \frac{1}{n} (\exp(nt v_{n-1}) \# \frac{1}{n-1} \cdots \# \frac{1}{1} (\exp(nt v_2) \# \exp(nt v_1))) \right).
\]

3.3 The Locally Midpoint-preserving Maps

From the preceding section we saw that we can define a midpoint operation in a manifold with spray as a geodesic image at \( t = \frac{1}{2} \). In this section we see that continuous maps locally preserving the midpoint operation \( \# \) are smooth in manifolds with spray. We first introduce locally midpoint-preserving maps and develop some fundamental properties associated with them.

Definition 3.80. For two manifolds \( X \) and \( Y \) with sprays, we say that \( f : X \to Y \) is a locally midpoint-preserving map if for each \( x_0 \in X \) there exists an open set \( U \) containing \( x_0 \) such that
\[
f(x \# y) = f(x) \# f(y)
\]
for all \( x, y \in U \).

Lemma 3.81. Let \( T \) be a subset of \([0,1]\) containing \( \{0,1\} \) and closed under the operation of taking midpoint. Then \( T \) contains all dyadic rational numbers and if it is closed, then it is equal to \([0,1]\).

Proof. By hypothesis clearly \( T \) contains all dyadic rational numbers between 0 and 1 with denominator \( 2^0 \) and contains all those with denominator \( 2^{n+1} \) if it contains those with denominator \( 2^n \). By induction \( T \) contains all dyadic rational numbers, and moreover, is dense. If it is closed, therefore, it is all of \([0,1]\). \( \square \)

Lemma 3.82. Let \( \gamma : [0,b] \to E, \ b > 0 \) be a continuous and midpoint-preserving map with \( \gamma(0) = 0 \) into a topological vector space \( E \). Then there exists \( v \in E \) such that \( \gamma(t) = tv \) is a linear extension of \( \gamma \) to all of \( \mathbb{R} \).
Proof. Suppose $b = 1$. Let $v = \gamma(1) \in E$. Set the set $T = \{t \in [0, 1] : \gamma(t) = tv\}$. Then clearly $0, 1 \in T$. If $s, t \in T$, then

$$\gamma\left(\frac{s + t}{2}\right) = \frac{1}{2}(\gamma(s) + \gamma(t)) = \frac{1}{2}(sv + tv) = \left(\frac{s + t}{2}\right)v.$$ 

Thus $T$ is closed under taking midpoints. By Lemma 3.81 $T = [0, 1]$ since $T$ is easily seen to be closed. If $b \neq 1$ then we first scale by $\frac{1}{b}$ from $[0, 1]$ to $[0, b]$ and apply the preceding paragraph to the composition of $\gamma$ with the scaling. \qed

**Theorem 3.83.** Let $X, Y$ be manifolds with sprays and let $f : X \to Y$ be a continuous and locally midpoint-preserving map. Then $f$ is smooth. Furthermore, if the exponential maps for $X, Y$ are analytic, then $f$ is analytic.

**Proof.** We choose a distinguished point $x_0 \in X$, take $f(x_0)$ for the distinguished point in $Y$, and define $\tilde{f}$ locally near $0_{x_0} \in T_{x_0}(X)$ by $\tilde{f} = \log_{f(x_0)} \circ f \circ \exp_{x_0}$. \hfill X \xrightarrow{f} Y

\hfill \exp_{x_0} \uparrow \quad \uparrow \exp_{f(x_0)}

\hfill T_{x_0}(X) \xrightarrow{\tilde{f}} T_{f(x_0)}(Y)

"Locally" means choosing a normal ball $B = \exp_{x_0}(B(0_{x_0}, r))$ such that $f(B)$ is contained in a normal ball around $f(x_0)$ and so that log can be defined as the inverse of $\exp_{f(x_0)}$.

In this proof we denote $\{tu : 0 \leq t \leq 1\} = [0, 1]u$ and $\{\exp_{x_0}(tu) : 0 \leq t \leq 1\} = \exp_{x_0}([0, 1]u)$ for convenience. We proceed in steps.

1. We show for $u \in B(0_{x_0}, r)$ there exists $w \in T_{f(x_0)}(Y)$ such that $\tilde{f}(tu) = tw$ for all $t \in [0, 1]$. From Lemma 3.75 we have seen the mapping $t \to \exp_{x_0}(tu)$ is midpoint-preserving, and so is $tu \to \exp_{x_0}(tu)$ from $[0, 1]u$ to $\exp_{x_0}([0, 1]u)$. Let $x = \exp_{x_0}(u)$, let $y = f(x)$, and let $v = \log_{f(x)}(y)$. Then $\exp_{f(x)}(tv)$
is also closed under the midpoint operation \#, and so \( \log_{f(x_0)} \) is midpoint-preserving from \( \exp_{f(x_0)}([0,1]) \) to \([0,1]\). The map
\[ t \to tu \to \exp_{x_0}(tu) \to f(\exp_{x_0}(tu)) \]
from \([0,1]\) into \(Y\) is a composition of midpoint-preserving maps, and hence is midpoint-preserving. Since it carries \(0_{x_0}\) to \(f(x_0)\) and \(1\) to \(f(x) = y = \exp_{f(x_0)}(v)\), one argues directly from Lemma 3.81 that all dyadic rational numbers are carried into \(\exp_{f(x_0)}([0,1])\), and hence all of \([0,1]\) are since \(\exp_{f(x_0)}([0,1])\) is compact and the composition is continuous. It follows that
\[ f(\exp_{x_0}([0,1]u)) \subseteq \exp_{f(x_0)}([0,1]v). \]
Combining all information together, we conclude that \(\tilde{f} = \log_{f(x_0)} \circ f \circ \exp_{x_0}\) restricted to \([0,1]u\) is midpoint-preserving. Thus \(t \to \tilde{f}(tu)\) is a continuous and midpoint-preserving map carrying \(0_{x_0}\) to \(0_{f(x_0)}\). By Lemma 3.82 there exists \(w \in T_{f(x_0)}(Y)\) such that \(\tilde{f}(tu) = tw\) for all \(t \in [0,1]\).

(2) We show that we can uniquely extend \(\tilde{f}\) so that \(\tilde{f}\) is globally defined, continuous, and positively homogeneous, namely \(\tilde{f}(u) = m\tilde{f}\left(\frac{u}{m}\right)\) for \(m > 0\).

Let \(u \in T_{x_0}(X)\) and choose \(0 < m < n\) such that \(\frac{u}{m}, \frac{u}{n} \in B(0_{x_0}, r)\). Then \(\frac{u}{n} = s\left(\frac{u}{m}\right)\), where \(s = \frac{m}{n} < 1\). By Step 1, there exists \(w \in T_{f(x_0)}(Y)\) such that \(\tilde{f}\left(t \frac{u}{m}\right) = tw\) for all \(t \in [0,1]\). It follows that
\[ n\tilde{f}\left(\frac{u}{n}\right) = \left(\frac{m}{s}\right) \tilde{f}\left(\frac{su}{m}\right) = \left(\frac{m}{s}\right) sw = mw = m\tilde{f}\left(\frac{u}{m}\right). \]
Thus the definition of \(\tilde{f}(u) = m\tilde{f}\left(\frac{u}{m}\right)\) is independent of the real number \(m > 0\), as long as \(m\) is sufficiently large so that \(\left\|\frac{u}{m}\right\| < r\).

For \(u \in T_{x_0}(X)\), fix \(m > 0\) such that \(\left\|\frac{u}{m}\right\| < r\). Then there is an open neighborhood \(U\) of \(u\) such that \(\left\|\frac{v}{m}\right\| < r\) for \(v \in T_{x_0}(X)\). Then on \(U\) the extension of \(\tilde{f}\) is given by \(v \to m\tilde{f}\left(\frac{v}{m}\right)\), which is continuous.
For $u \in T_{x_0}(X)$ and $s > 0$, pick $m > s$ such that $\left\| \frac{su}{m} \right\| < r$. Then

$$\tilde{f}(su) = m \tilde{f} \left( \frac{su}{m} \right) = m \left( \frac{s}{m} \right) \tilde{f}(u) = s \tilde{f}(u).$$

Thus $\tilde{f}$ is positively homogeneous.

(3) We use the Lie-Trotter formula to show that $\tilde{f}$ is additive. We temporarily abbreviate the exponential functions $\exp_{x_0}$ and $\exp f_{(x_0)}$ by $\exp$, distinguishing them by context, and the corresponding log functions by $\log$. We first use the positive homogeneity and the local equality $\tilde{f} = \log \circ f \circ \exp$ in various equivalent forms to calculate for $u, v \in T_{x_0}(X)$ and $n$ large:

$$\tilde{f} \left( n \log \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) \right) = n(\tilde{f} \circ \log) \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) = n(\log \circ f) \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) = n \log \left( f \left( \exp \left( \frac{2u}{n} \right) \right) \# f \left( \exp \left( \frac{2v}{n} \right) \right) \right) = n \log \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) = n \log \left( \exp \left( \frac{2}{n} \tilde{f}(u) \right) \# \exp \left( \frac{2}{n} \tilde{f}(v) \right) \right).$$

We thus have by the Lie-Trotter formula:

$$\tilde{f}(u + v) = \tilde{f} \left( \lim_{n \to \infty} n \log \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) \right) = \lim_{n \to \infty} \tilde{f} \left( n \log \left( \exp \left( \frac{2u}{n} \right) \# \exp \left( \frac{2v}{n} \right) \right) \right) = \lim_{n \to \infty} n \log \left( \exp \left( \frac{2}{n} \tilde{f}(u) \right) \# \exp \left( \frac{2}{n} \tilde{f}(v) \right) \right) = \tilde{f}(u) + \tilde{f}(v).$$

(4) Since $\tilde{f}$ is continuous, additive, and positively homogeneous, it follows easily that it is a continuous linear mapping, hence smooth. Near $x_0$, $f = \tilde{f}$, etc.
\[ \exp_{f(x_0)} \circ f \circ \log_{x_0} \] and is thus smooth. Since \( x_0 \) was an arbitrary choice for the distinguished point, the map \( f \) is smooth in a neighborhood of every point, hence smooth. If each exponential map \( \exp_x \) is analytic, then the local inverse \( \log \) is also analytic since the derivative at \( 0_x \) is invertible (indeed, it is identity). Thus, the composition \( f = \exp_{f(x)} \circ f \circ \log_x \) is locally analytic, hence analytic.

\[ \square \]
4. Loos Symmetric Spaces

We introduce Loos symmetric spaces in Section 1. We refer an important result of K.-H. Neeb that we associate a spray to a symmetric space with the same symmetries and it is uniquely determined by this property. The concept of a spray is central to our discussion below because it encodes the exponential function of the underlying manifold.

Later we consider a smooth gyrocommutative gyrogroup with unique square roots, and see that it is a Loos symmetric space. Furthermore, we show that it becomes a gyrovector space with a scalar multiplication derived from the exponential map associated with the spray.

In Section 2 we develop the concept of parallel transport. A. Ungar defined a parallel transport of rooted gyrovectors via certain condition of their values, but we show the condition using the well-known properties obtained by K.-H. Neeb, J. Lawson and Y. Lim.

In Section 3 we define vertices parallelogram, analogous to the Euclidean parallelogram in vector spaces. Then we study its equivalent conditions with point reflections and midpoints, and moreover, investigate several properties.

4.1 Loos Symmetric Spaces

In this section we discuss symmetric spaces in the sense of Loos as spaces endowed with a binary operation satisfying certain axioms. An advantage of this approach is that it has excellent properties such as the tangent bundle of a symmetric space having a natural structure of a symmetric space.
Loos symmetric spaces have been studied by K.-H. Neeb in [15]. We first review some of his results.

**Definition 4.84.** Let $X$ be a smooth manifold. We say that $(X, \bullet)$ is a Loos symmetric space if $(x, y) \mapsto x \bullet y : X \times X \to X$ is a smooth map with the following properties for all $a, b, c \in X$:

1. $a \bullet a = a$;
2. $a \bullet (a \bullet b) = b$;
3. $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$;
4. Every $a \in X$ has a neighborhood $U$ such that $a \bullet x = x$ implies $a = x$ for $x \in U$.

K.-H. Neeb has shown the following property in Lemma 3.2, [15].

**Lemma 4.85.** Let $(X, \bullet)$ be a Loos symmetric space, and for $x \in X$ we put $S_x(y) = x \bullet y$. Then

$$dS_x(x) = -id_{T_x(X)}.$$  

The following are important results of K.-H. Neeb in Theorems 3.3, 3.4, and 3.6 of [15].

**Theorem 4.86.** Let $(X, \bullet)$ be a Loos symmetric space, and set $\mu(x, y) := x \bullet y$ for $x, y \in X$.

1. Identifying $T(X \times X)$ with $T(X) \times T(X)$, we define

$$v \bullet w := T(\mu)(v, w)$$

and obtain a Loos symmetric space on $T(X)$. In each tangent space $T_x(X)$,

$$v \bullet w = 2v - w.$$
(2) The function

\[ F : T(X) \to T(T(X)), \quad F(v) := -T(S_{\frac{v}{2}} \circ Z)(v) \]

defines a spray on \( X \), where \( Z : X \to T(X) \) is the zero section and \( S_{\frac{v}{2}} \) is the point symmetry for \( \frac{v}{2} \) from part (1).

(3) \( \text{Aut}(X, \bullet) = \text{Aut}(X, F) \), where the former consists of all diffeomorphisms that are automorphisms with respect to \( \bullet \) and the latter consists of all diffeomorphisms that preserve the spray \( F \).

(4) \( F \) is uniquely defined as the only spray invariant under all symmetries \( S_x, x \in X \).

(5) \((X, F)\) is geodesically complete (all geodesics extend to \( \mathbb{R} \)).

(6) Let \( \alpha : \mathbb{R} \to X \) be a geodesic and call the maps \( \tau_{\alpha, s} := S_{\alpha(\frac{t}{2})} \circ S_{\alpha(0)}, s \in \mathbb{R} \), translations along \( \alpha \). Then these are automorphisms of \((X, \bullet)\) with

\[ \tau_{\alpha, s}(\alpha(t)) = \alpha(t + s) \text{ and } d\tau_{\alpha, s}(\alpha(t)) = P_{t+s}^t(\alpha) \]

for all \( s, t \in \mathbb{R} \). We write

\[ P_{t+s}^t(\alpha) : T_{\alpha(t)}(X) \to T_{\alpha(t+s)}(X) \]

for the linear map given by parallel transport along \( \alpha \).

A basic result in the theory of Lie groups is that every one-parameter subgroup of a Lie group has the unique representation \( t \mapsto \exp(tA) \), where \( A \) is a member of the Lie algebra. In [11] J. Lawson and Y. Lim have shown that a similar result holds for Loos symmetric spaces.
Proposition 4.87. Let \((X, \bullet)\) be a Loos symmetric space and let \(x \in X\). For each \(v \in T_x(X)\), the map

\[ \alpha_v : \mathbb{R} \to X, \quad \alpha_v(t) = \exp_x(tv) \]

is a geodesic arising from the spray \(F\) of (2) in Theorem 4.86.

The correspondence \(v \leftrightarrow \alpha_v\) is a one-to-one correspondence between \(T_x(X)\) and all maximal geodesics taking on the value \(x\) at 0.

Furthermore, \(\alpha : \mathbb{R} \to X\) is a geodesic of symmetry if and only if it is a maximal geodesic for the associated spray \(F\) of \(X\).

We will derive some examples of sprays on Loos symmetric spaces.

Example 4.88. Let \(E\) be a Banach space with trivial tangent bundle \(E \times E\). Consider that \((E, \bullet)\) is a symmetric space with \(x \bullet y = 2x - y\). Then the tangent bundle \(T(E)\) becomes a Loos symmetric space by (1) of Theorem 4.86.

If we denote a point in the tangent bundle by a pair \((x, v)\) in \(E \times E\), then we have

\[ \left( S_{(x, \frac{v}{2})} \circ Z \right)(y) = S_{(x, \frac{v}{2})}(y, 0) = (2x - y, v), \]

and so,

\[ T(S_{(x, \frac{v}{2})} \circ Z)(y, w) = (2x - y, v, -w, 0) \in (X \times E) \times (E \times E). \]

Therefore,

\[ F(x, v) = -T(S_{(x, \frac{v}{2})} \circ Z)(x, v) = (x, v, v, 0). \]

Example 4.89. For an open convex cone \(\Omega\) of all positive definite matrices, \((\Omega, \mu)\) is a Loos symmetric space, where

\[ \mu(A, B) := A \bullet B = AB^{-1}A. \]

Then the tangent bundle \(T\Omega\) can be denoted by
\[ T\Omega = \Omega \times \text{Sym}, \]

where \text{Sym} is the space of symmetric matrices. Moreover, \((T(\Omega), T\mu)\) becomes a Loos symmetric space, where

\[ T\mu((A, X), (B, Y)) = (A \bullet B, XB^{-1}A + AB^{-1}X - AB^{-1}YB^{-1}A). \]

This implies that

\[ \left( S_{(A, \frac{X}{2})} \circ Z \right)(B) = S_{(A, \frac{X}{2})}(B, 0) = \left( A \bullet B, \frac{1}{2}(XB^{-1}A + AB^{-1}X) \right). \]

By Lemma 4.85 the first coordinate of the differential \(d(S_{(A, \frac{X}{2})} \circ Z)(A)\) is \(-id_{T\mu(\Omega)}\), and the second coordinate at \(X\) is

\[ \frac{1}{2}(-XB^{-1}XB^{-1}A - AB^{-1}XB^{-1}X). \]

So

\[ T(S_{(A, \frac{X}{2})} \circ Z)(A, X) = \left( A \bullet A, X, -X, \frac{1}{2}(-XA^{-1}X - XA^{-1}X) \right) \]

\[ = (A, X, -X, -XA^{-1}X). \]

Therefore,

\[ F(A, X) = -T(S_{(A, \frac{X}{2})} \circ Z)(A, X) = (A, X, XA^{-1}X). \]

### 4.2 Parallel Transport

In this section we consider \((X, \oplus)\) as a smooth gyrocommutative gyrogroup with unique square roots, where the gyroaddition \(\oplus : X \times X \rightarrow X\) is smooth.

In Theorem 2.51 we have seen that \((X, \oplus)\) is equivalent to a symmetric mean \((X, \bullet)\) via the following formulas:

\[ a \bullet b = a \ominus (\ominus a \oplus b) = a \oplus (a \ominus b), \]

\[ a \oplus b = S_{a^{1/2}}(S_0b) = Q(a^{1/2})b, \]  

(4.11)
where $S_x$ represents the point reflection at $x$.

In order to show that the binary operation $\bullet$ is smooth, we first apply the Implicit Mapping Theorem (Theorem 5.9 of Chapter I, [20]) with the function

$$f(a, b) = a \oplus b : X \times X \to X.$$  

Then $L_a$, the left translation by $a$, is a diffeomorphism, whose inverse map is the left translation by $\ominus a$. It follows that $dL_a(b) = D_2f(a, b)$ is an isomorphism for all $b$. Thus, the function

$$g(x) = \ominus x : U_0 \to V$$

satisfying $f(x, g(x)) = 0$ is also smooth on a sufficiently small open neighborhood $U_0$ of $a$. So the binary operation $\bullet$ obtained from the following compositions is smooth:

$$(a, b) \mapsto (a, \ominus b) \mapsto (a, a \ominus b) \mapsto a \oplus (a \ominus b).$$

From Axiom (4) of a symmetric mean, we know that there is only one fixed point. Thus, $(X, \bullet)$ becomes a Loos symmetric space.

Therefore, we can use Theorem 4.86 and Proposition 4.87 for a smooth gyrocommutative gyrogroup $(X, \oplus)$ with unique square roots.

In the next theorem we show that the smooth gyrocommutative gyrogroup $(X, \oplus)$ with unique square roots becomes a gyrovector space. In the following we abbreviate the exponential function $\exp_0$ at the identity $0 \in X$ by $\exp$, and the corresponding log function by $\log$.

**Proposition 4.90.** Let $(X, \oplus)$ be the smooth gyrocommutative gyrogroup with unique square roots. Then $X$ becomes a gyrovector space with a scalar multiplication defined by

$$t \otimes x := \exp(t \log(x))$$
for any $t \in \mathbb{R}$ and $x \in X$. Furthermore, the scalar multiplication $\otimes : \mathbb{R} \times X \to X$ is smooth.

Proof. By Proposition 4.87 we have that there is a geodesic, or a $\bullet$ homomorphism,

$$\alpha_x : \mathbb{R} \to X, \alpha(t) = \exp(tv)$$

arising from the canonical spray, where $v = \log(x) \in T_0(X)$ for any $x \in X$. Then we define the scalar multiplication by

$$t \otimes x := \alpha_x(t) = \exp(t \log(x)).$$

It is derived from the composition of the following smooth maps:

$$(t, x) \mapsto (t, \log(x)) \mapsto t \log(x) \mapsto \exp(t \log(x)).$$

So the scalar multiplication $\otimes : \mathbb{R} \times X \to X$ is smooth. It is now enough to verify the axioms in Definition 2.56.

(1) $1 \otimes x = \exp(\log(x)) = x$, and $0 \otimes x = \exp(0) = 0$.

(2) For any $s, t \in \mathbb{R}$,

$$s \otimes (t \otimes x) = \exp(s \log(t \otimes x)) = \exp(st \log(x)) = (st) \otimes x.$$ 

(3) Since $\alpha_x(t) = \exp(tv)$ is a $\bullet$ homomorphism, it is also an $\oplus$ homomorphism by Lemma 2.54. So

$$s \otimes x \oplus t \otimes x = \alpha_x(s) \oplus \alpha_x(t) = \alpha_x(s \oplus t) = \alpha_x(s + t) = (s + t) \otimes x.$$ 

In particular, when $s = -1$ and $t = 1$, we have

$$(-1) \otimes x \oplus x = 0 \otimes x = 0.$$ 

We thus obtain $(-1) \otimes x = \ominus x$ because of the uniqueness of inverse.
We consider a map

\[ \beta : \mathbb{R} \to X, \quad \beta(t) = \text{gyr}[a, b]\alpha_x(t) = \text{gyr}[a, b](t \otimes x) \]

for any \(a, b \in X\). Since \(\text{gyr}[a, b]\) is a \(\oplus\) homomorphism, the map \(\beta\) is a \(\bullet\) homomorphism with \(\beta(0) = 0\) by Lemma 2.54. Thus by Proposition 4.87 we obtain \(\beta(t) = \exp(tw)\) for some \(w \in T_0(X)\). Therefore,

\[ \text{gyr}[a, b](t \otimes x) = \exp(tw) = t \otimes \exp(w) = t \otimes \beta(1) = t \otimes \text{gyr}[a, b]x. \]

\begin{flushright}
\(\square\)
\end{flushright}

**Remark 4.91.** In a smooth gyrocommutative gyrogroup \(G\) with unique square roots, we have that both the gyroaddition and the scalar multiplication are smooth. So the map

\[ (a, b, c) \mapsto \text{gyr}[a, b]c : G^3 \to G, \quad \text{gyr}[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)), \]

is also smooth. Moreover, we have the same property according to Lemma 2.59: for any \(a \in G\) and \(s, t \in \mathbb{R}\)

\[ \text{gyr}[s \otimes a, t \otimes a] = I, \]

where \(I\) is the identity map.

We now compute parallel transport in a smooth gyrocommutative gyrogroup \(X\) with unique square roots.

**Theorem 4.92.** Let \((X, \oplus)\) be a smooth gyrocommutative gyrogroup with unique square roots. Let

\[ \alpha : \mathbb{R} \to X, \quad \alpha(t) = \exp(tw), \]

and \(a = \alpha(1)\). Then parallel transport from \(T_0(X)\) along \(\alpha\) to \(T_a(X)\) is given by

\[ P^1_0(\alpha) = dL_a(0). \]
Furthermore, the following diagram commutes:

\[
\begin{array}{c}
T_0(X) \xrightarrow{dL_a(0) = P_0^1(\alpha)} T_a(X) \\
\downarrow \exp \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
We introduce rooted gyrovectors and free gyrovectors in a gyrocommutative gyrogroup as A. Ungar has done in Chapter 5 of [1].

**Definition 4.93.** Let $a$ and $b$ be elements, or points, in a gyrocommutative gyrogroup $(G, \oplus)$. A rooted gyrovector $ab$ is an ordered pair of points $a, b \in G$. The points $a$ and $b$ of the rooted gyrovector $ab$ are called, respectively, the tail and the head of the rooted gyrovector.

The value in $G$ of the rooted gyrovector $ab$ is $\ominus a \oplus b$. Accordingly, we write

$$v = \ominus a \oplus b.$$ 

Furthermore, any point $a \in G$ is identified with the rooted gyrovector $0a$ with head $a$, rooted at the origin $0$.

Given $v \in T_a(X)$, the rooted gyrovector $a \exp_a(v)$ is called the internal representative of $v$.

Two rooted gyrovectors $ab$ and $a'b'$ are said to be equivalent,

$$ab \sim a'b',$$

if they have the same value in $G$, that is, if

$$\ominus a \oplus b = \ominus a' \oplus b'.$$
Then the relation $\sim$ is given in terms of an equality so that, being reflexive, symmetric, and transitive, it is an equivalence relation. The resulting equivalence classes are called free gyro vectors.

We have another version of Theorem 4.92 in terms of rooted gyro vectors.

**Corollary 4.94.** Let $(X, \oplus)$ be a gyro vector space. Then the parallel transport of a vector $v$ in $T_0(X)$ along a geodesic from 0 to a has the internal representative $ab$, where

$$b = a \oplus \exp(v).$$

We next consider a general version of parallel transport for rooted gyro vectors using the equivalence of tangent vectors and rooted gyro vectors via internal representatives.

The figure illustrates a rooted gyro vector $a_1b_1 = a_1 \exp_{a_1}(v_1)$ that is a parallel transport of a rooted gyro vector $a_0b_0 = a_0 \exp_{a_0}(v_0)$. By definition it means that the tangent vector $v_1 \in T_{a_1}(X)$ is the parallel transport of the tangent vector

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\( v_0 \in T_{a_0}(X) \) along the geodesic

\[
\alpha(t) = a_0 \oplus t \otimes (\ominus a_0 \oplus a_1),
\]
where \( a_0 \neq a_1 \).

**Theorem 4.95.** Let \((X, \oplus)\) be a gyrovector space. A rooted gyrovector \(a_1 b_1\) is a parallel transport of a rooted gyrovector \(a_0 b_0\), where \(a_0 \neq a_1\) if their values satisfy

\[
\ominus a_1 \oplus b_1 = \text{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0). \tag{4.12}
\]

**Proof.** By definition of parallel transport of rooted gyrovectors, we have that

\[
b_0 = \exp_{a_0}(v_0), \quad b_1 = \exp_{a_1}(v_1),
\]
where

\[
v_1 = P^1_0(\alpha)(v_0) \tag{4.13}
\]
for the geodesic \(\alpha\) with \(\alpha(0) = a_0\) and \(\alpha(1) = a_1\). From (6) of Theorem 4.86 generally we get the following diagram commutes:

\[
\begin{array}{ccc}
T_{a_0}(X) & \xrightarrow{d\tau_{\alpha,1}(\alpha(0))=P^1_0(\alpha)} & T_{a_1}(X) \\
\exp_{a_0} & & \exp_{a_1} \\
X & \xrightarrow{\tau_{\alpha,1}=S_{\frac{1}{2}}S_{\alpha(0)}} & X
\end{array}
\]

Taking \(\exp_{a_1}\) on both sides in the equation (4.13) and using the above diagram, we have

\[
\exp_{a_1}(v_1) = \exp_{a_1}(d\tau_{\alpha,1}(\alpha(0))v_0) \\
= \tau_{\alpha,1}(\exp_{a_0}(v_0)) \\
= S_{\frac{1}{2}}S_{\alpha(0)}(\exp_{a_0}(v_0)).
\]

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Equivalently,

\[
b_1 = S_{\alpha(\frac{1}{2})}S_{\alpha(0)}(b_0)
\]

\[
= \alpha \left(\frac{1}{2}\right) \bullet (\alpha(0) \bullet b_0)
\]= \left( \alpha \left(\frac{1}{2}\right) \bullet \alpha(0) \right) \bullet \left( \alpha \left(\frac{1}{2}\right) \bullet b_0 \right)
\]= \alpha(1) \bullet \left( \alpha \left(\frac{1}{2}\right) \bullet b_0 \right)
\]

\[
= a_1 \bullet ((a_0\#a_1) \bullet b_0).
\]

The third equality follows from (3) of Definition 1.2, and the fourth follows from the fact that \(\alpha \left(\frac{1}{2}\right)\) is a midpoint of \(a_0 = \alpha(0)\) and \(a_1 = \alpha(1)\).

Since the point symmetry \(S_{a_1}\) is involutive, we have

\[
a_1 \bullet b_1 = a_1 \bullet [a_1 \bullet ((a_0\#a_1) \bullet b_0)] = (a_0\#a_1) \bullet b_0.
\]

Using Proposition 2.61, Proposition 4.90, (4) of Lemma 2.47, and Lemma 2.43, we get equivalently

\[
a_1 \ominus (\ominus a_1 \oplus b_1) = 2 \otimes \left\{ \frac{1}{2} \otimes (a_0 \boxplus a_1) \right\} \ominus b_0
\]

\[
= (a_0 \boxplus a_1) \ominus b_0
\]

\[
= (a_1 \boxplus a_0) \ominus b_0
\]

\[
= a_1 \ominus \text{gyr}[a_1, \ominus a_0](a_0 \ominus b_0).
\]

The left cancellation gives us

\[
\ominus (\ominus a_1 \oplus b_1) = \text{gyr}[a_1, \ominus a_0](a_0 \ominus b_0).
\]

By the automorphic inverse property we conclude

\[
\ominus a_1 \oplus b_1 = \ominus \text{gyr}[a_1, \ominus a_0](a_0 \ominus b_0) = \text{gyr}[a_1, \ominus a_0](\ominus a_0 \oplus b_0).
\]

\[\square\]
Remark 4.96. In the special case with $a_0 = 0$, the equation (4.12) becomes

$$\Theta a_1 \oplus b_1 = \text{gyr}[a_1, 0]b_0 = b_0.$$  

Equivalently, we have

$$b_1 = a_1 \oplus b_0.$$  

This gives us the result of Theorem 4.92.

4.3 Parallelogram

The parallelogram is one of the interesting topics in Euclidean geometry. Especially, the common parallelogram vector addition law in Euclidean geometry is an alternative statement of triangle vector addition law.

![Parallelogram Diagram](image)

The figure shows us a parallelogram $abdc$ in Euclidean vector space, where $a, b, c,$ and $d$ are points. By the parallelogram vector addition law we have two equivalent conditions

$$d = c + (b - a),$$

$$d = b + (c - a).$$

In the gyrovector space where there is no parallel postulate, we still have an analogous definition.
Definition 4.97. Let $a$, $b$, and $c$ be any points in a gyrovector space $G$. Then the points $a$, $b$, $c$, and $d$ are the vertices of parallelogram $abdc$, ordered either clockwise or counterclockwise, if

$$d = (b \boxplus c) \ominus a.$$ 

We call $abdc$ the vertices of the parallelogram.

We have equivalent conditions for the vertices of a parallelogram.

Proposition 4.98. Let $a$, $b$, $c$, and $d$ be any points in a gyrovector space $G$. Then the following are equivalent:

1. $abdc$ are the vertices of a parallelogram.

2. The midpoint of the two points $a$ and $d$ coincides with the midpoint of the two points $b$ and $c$.

3. The following conditions hold:

$$a = (b \boxplus c) \ominus d,$$
$$b = (a \oplus d) \ominus c,$$
$$c = (a \oplus d) \ominus b,$$
$$d = (b \boxplus c) \ominus a.$$ 

4. The rooted gyrovectors satisfy the following conditions:

$$\ominus b \oplus d = \text{gyr}[b, \ominus c] \text{gyr}[c, \ominus a](\ominus a \oplus c),$$
$$\ominus c \oplus d = \text{gyr}[c, \ominus b] \text{gyr}[b, \ominus a](\ominus a \oplus b).$$

Proof. By Proposition 2.41 and (4) of Lemma 2.47 the condition

$$d = (b \boxplus c) \ominus a$$ 

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is equivalent to

\[ a \boxplus d = d \boxplus a = b \boxplus c. \]

Multiplying \( \frac{1}{2} \) on both sides we have equivalently

\[ \frac{1}{2} \otimes (a \boxplus d) = \frac{1}{2} \otimes (b \boxplus c). \]

This gives us the equivalence among (1), (2), and (3).

We now prove the equivalence between (1) and (4).

(1) \( \Rightarrow \) (4) Assume that \( abdc \) are the vertices of a parallelogram. Then we have

\[ d = (b \boxplus c) \ominus a. \]

Applying Lemma 2.43 and the gyrocommutativity to this condition, we have

\[

d = (b \boxplus c) \ominus a \\
= b \oplus \text{gyr}[b, \ominus c](c \ominus a) \\
= b \oplus \text{gyr}[b, \ominus c] \text{gyr}[c, \ominus a](\ominus a \oplus c).
\]

Thus, by the left cancellation

\[ \ominus b \oplus d = \text{gyr}[b, \ominus c] \text{gyr}[c, \ominus a](\ominus a \oplus c). \]

Using (4) of Lemma 2.47 and following the above steps, we can obtain the other condition.

(4) \( \Rightarrow \) (1) Reversing the preceding steps yields (4) \( \Rightarrow \) (1).

Remark 4.99. In Section 2.2 we have seen that a uniquely 2-divisible gyrocommutative gyrogroup \( G \) gives rise to a symmetric mean by setting

\[ S_{xy} = x \bullet y = x \oplus (x \ominus y) = 2 \otimes x \ominus y. \]
By (2) of Proposition 4.98 we have that \( abdc \) are the vertices of a parallelogram if and only if there is a unique point \( x \in G \) such that

\[
S_x a = d \quad \text{and} \quad S_x b = c.
\]

By Lemma 1.22, equivalently, we know that \( abdc \) are the vertices of a parallelogram if and only if there is a unique point \( x \in G \) such that

\[
S(x)(a\#_t b) = d\#_t c,
\]

where \( a\#_t b \) is the \( t \)-weighted mean for all \( t \in [0, 1] \). Furthermore, we know that \( x \) must be the midpoint of \( a \) and \( d \) as well as the midpoint of \( b \) and \( c \).

We show a gyration property on the vertices of a parallelogram \( abdc \).

**Proposition 4.100.** Let \( abdc \) be the vertices of a parallelogram. Then

\[
gyr[a, \ominus d] = gyr[a, \ominus b] gyr[b, \ominus c] gyr[c, \ominus d].
\]

**Proof.** Applying gyrocommutativity and Lemma 2.30 to equation (2.4) yields

\[
gyr[a', b' \oplus c'] gyr[b', c'] = gyr[a' \oplus b', gyr[a', b'] c'] gyr[a', b']
\]

\[
= gyr[gyr[a', b'](b' \oplus a'), gyr[a', b'] c'] gyr[a', b']
\]

\[
= gyr[a', b'] gyr[b' \oplus a', c'].
\]

By the loop properties, we have

\[
gyr[a', b' \oplus c'] gyr[b' \oplus c', c'] = gyr[a', b' \oplus a'] gyr[b' \oplus a', c'].
\]

Let us take \( a = \ominus c' \), \( c = \ominus a' \), and \( b = b' \oplus a' \). Then by Proposition 2.41

\[
b' \oplus c' = (b \boxdot a') \oplus c' = (b \boxplus c) \ominus a = d.
\]

Thus,

\[
gyr[\ominus c, d] gyr[d, \ominus a] = gyr[\ominus c, b] gyr[b, \ominus a].
\]
By taking the inverse on both sides and using Lemma 2.33 we obtain

\[ \text{gyr}[\ominus a, d]\text{gyr}[d, \ominus c] = \text{gyr}[\ominus a, b]\text{gyr}[b, \ominus c]. \]
\[ \text{gyr}[\ominus a, d] = \text{gyr}[\ominus a, b]\text{gyr}[b, \ominus c]\text{gyr}[\ominus c, d]. \]
\[ \text{gyr}[a, \ominus d] = \text{gyr}[a, \ominus b]\text{gyr}[b, \ominus c]\text{gyr}[c, \ominus d]. \]

We have an interesting property that the vertices of a parallelogram are invariant under a left translation.

**Proposition 4.101.** The vertices of a parallelogram are invariant under a left translation.

**Proof.** We assume that abdc are the vertices of a parallelogram. We need to show that the left translation of abdc by x is again the vertices of a parallelogram. In other words,

\[ x \oplus ((b \boxplus c) \ominus a) = \{(x \oplus b) \boxplus (x \oplus c)\} \ominus (x \oplus a). \] (4.14)

By Lemma 2.43, the right gyroassociativity, and Lemma 2.37, we have

\[ \{(x \oplus b) \boxplus (x \oplus c)\} \ominus (x \oplus a) \]
\[ = (x \oplus b) \oplus \text{gyr}[x \oplus b, \ominus(x \oplus c)]\{(x \oplus c) \ominus (x \oplus a)\} \]
\[ = x \oplus \{b \oplus \text{gyr}[b, x]\text{gyr}[x \oplus b, \ominus(x \oplus c)]\text{gyr}[x, c](c \ominus a)\}. \]

In the proof of Proposition 4.100 we have seen

\[ \text{gyr}[a', b']\text{gyr}[b' \oplus a', c'] = \text{gyr}[a', b' \oplus c']\text{gyr}[b', c']. \]
Let us take $a' = b$, $b' = x$, and $c' = \ominus(x \oplus c)$. Then by the left cancellation, the loop properties, and (2) of Lemma 2.33, we have

$$\text{gyr}[b, x]\text{gyr}[x \ominus b, \ominus(x \ominus c)] = \text{gyr}[b, x \ominus (x \ominus c)]\text{gyr}[x, \ominus(x \ominus c)]$$

$$= \text{gyr}[b, \ominus c]\text{gyr}[\ominus c, \ominus x]$$

$$= \text{gyr}[b, \ominus c]\text{gyr}[c, x].$$

From (1) of Lemma 2.33 and Lemma 2.43, this implies

$$\{(x \oplus b) \boxplus (x \ominus c)\} \ominus (x \ominus a)$$

$$= x \ominus \{b \oplus \text{gyr}[b, x]\text{gyr}[x \ominus b, \ominus(x \ominus c)]\text{gyr}[x, c]\ominus a\}$$

$$= x \ominus \{b \oplus \text{gyr}[b, \ominus c]\text{gyr}[c, x]\text{gyr}[x, c]\ominus a\}$$

$$= x \ominus \{b \oplus \text{gyr}[b, \ominus c]\}(c \ominus a)\}$$

$$= x \ominus \{(b \boxplus c) \ominus a\}.$$

We show the parallelogram addition law of rooted gyro vectors.

**Corollary 4.102.** Let $abdc$ be the vertices of a parallelogram. Then

$$(\ominus a \oplus b) \boxplus (\ominus a \oplus c) = (\ominus a \oplus d).$$

**Proof.** Replacing $x$ by $\ominus a$ in the equation (4.14), we obtain

$$(\ominus a \oplus b) \boxplus (\ominus a \oplus c) = \ominus a \oplus \{(b \boxplus c) \ominus a\} = \ominus a \oplus d.$$
5. Geodesics

The shortest path between two points in a differentiable manifold can be found by writing the equation for the length of a curve, and then minimizing this length using the calculus of variations. This has some minor technical problems because there is an infinite dimensional space of different ways to parameterize the shortest path.

Geodesics are commonly seen in the study of Riemannian geometry and more generally metric geometry. In Euclidean geometry geodesics are straight lines, also the shortest curves. In Riemannian geometry geodesics are not the same as shortest curves between two points, though two concepts are closely related. Going the long way round on a great circle between two points on a sphere is a geodesic, but not the shortest curve between two points. If two points are antipodal, then there are infinitely many geodesics between them.

In Section 1 we recall the length of curves and the length metric in a metric space. We see the well-known result that the length metric between two points is the same as the usual distance induced by the norm in a Banach space.

In Section 2 we introduce a Finsler manifold with spray which has seminegative curvature, and define the Finsler gyrovector space. Applying the result that a smooth gyrocommutative gyrogroup with unique square roots is a gyrovector space, we see that the exponential map associated with spray in the Finsler gyrovector space with seminegative curvature gives us a length minimizing geodesic.
### 5.1 Length of Curves

Let us consider $X$ as a metric space with metric $d$. Let $\alpha : [a, b] \to X$ be a continuous function. We simply call it a curve, and consider a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $[a, b]$. Then the sum associated with this partition is

$$\sum_{j=1}^{n} d(\alpha(t_j), \alpha(t_{j-1})).$$

If we further subdivide a given partition, then the associated sum of the new partition is at least as large as that of the old partition. So we define a length of the curve $\alpha$, denoted by $L(\alpha)$, to be the supremum of all sums of partitions:

$$L(\alpha) = \sup \left\{ \sum_{j=1}^{n} d(\alpha(t_j), \alpha(t_{j-1})) : a = t_0 < t_1 < \cdots < t_n = b, n \in \mathbb{N} \right\}.$$

The curve is said to be rectifiable if its length is finite.

**Remark 5.103.** By the triangle inequality of the metric $d$, we have

$$d(x, y) \leq L(\alpha),$$

where $\alpha$ is a curve passing through the point $x$ at $t = a$ and the point $y$ at $t = b$.

We now define a length metric $\delta$ on a path-connected metric space $(X, d)$ using rectifiable curves.

**Definition 5.104.** Let $x$ and $y$ be any two points in a path-connected metric space $X$. Then the length metric $\delta : X \times X \to [0, \infty)$ is defined by

$$\delta(x, y) = \inf L(\alpha),$$

where the infimum is taken over the set of all rectifiable curves $\alpha$ connecting $x$ to $y$. 

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Remark 5.105. From Proposition 2.1.5 in [19] we know that the map $\delta$ satisfies all metric axioms. So we usually call $(X, \delta)$ a length metric space.

We now consider $X$ a Banach space with norm $\| \cdot \|$. The distance in the Banach space $X$ is defined by

$$d(x, y) = \| x - y \|$$

for any $x, y \in X$. In this setting we see that

$$d(x, y) = \delta(x, y).$$

Let $k$ be a non-negative integer. The class $C^0$ consists of all continuous functions. The function $f$ is said to be of class $C^k$ if the derivatives $f'$, $f''$, $\ldots$, $f^{(k)}$ exist and are continuous. The function $f$ is said to be of class $C^\infty$ or smooth if it has derivatives of all orders.

Lemma 5.106. Suppose that $\alpha : [a, b] \to X$ is a $C^1$ curve into a Banach space $X$. Then

$$L(\alpha) = \int_a^b \| \alpha'(t) \| \, dt.$$ 

Proof. Let us consider any partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $[a, b]$. Then

$$\sum_{j=1}^{n} d(\alpha(t_j), \alpha(t_{j-1})) = \sum_{j=1}^{n} \left\| \int_{t_{j-1}}^{t_j} \alpha'(t) \, dt \right\|$$

$$\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \| \alpha'(t) \| \, dt$$

$$= \int_a^b \| \alpha'(t) \| \, dt.$$ 

Hence, $L(\alpha) \leq \int_a^b \| \alpha'(t) \| \, dt$. 

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The reverse inequality is a little tricky. For given $\epsilon > 0$, we know that $\int_a^b \|\alpha'(t)\| \, dt$ can be approximated up to $\epsilon$ by a Riemann sum of the form

$$\sum_{j=1}^n \|\alpha'(t_{j-1})\| (t_j - t_{j-1})$$

provided the partition $\{t_j\}$ is fine enough, that is,

$$\Delta := \max\{t_j - t_{j-1} : j = 1, 2, \ldots, n\} \leq \rho_1$$

for some small $\rho_1 > 0$.

We now use the uniform continuity of $\alpha'$ on $[a, b]$ to obtain a $\rho_2 > 0$ such that

$$\|\alpha'(s) - \alpha'(t)\| \leq \epsilon \text{ whenever } |s - t| \leq \rho_2.$$

Then for all $0 < h \leq \rho_2$ and $t \in [a, b]$,

$$\left\| \frac{\alpha(t+h) - \alpha(t)}{h} - \alpha'(t) \right\| \leq \frac{1}{h} \int_t^{t+h} \|\alpha'(s) - \alpha'(t)\| \, ds \leq \frac{1}{h}(h\epsilon) = \epsilon.$$

Let $\{t_j\}$ be a partition with $\Delta \leq \min\{\rho_1, \rho_2\}$. Setting $h = t_j - t_{j-1}$ we have

$$\int_a^b \|\alpha'(t)\| \, dt \leq \sum_{j=1}^n \|\alpha'(t_{j-1})\| (t_j - t_{j-1}) + \epsilon$$

$$\leq \sum_{j=1}^n \frac{||\alpha(t_j) - \alpha(t_{j-1})||}{t_j - t_{j-1}} (t_j - t_{j-1}) + \sum_{j=1}^n \epsilon(t_j - t_{j-1}) + \epsilon$$

$$= \sum_{j=1}^n \|\alpha(t_j) - \alpha(t_{j-1})\| + \epsilon(b - a + 1)$$

$$\leq L(\alpha) + \epsilon(b - a + 1).$$

Taking the limit as $\epsilon$ goes to 0 yields $\int_a^b \|\alpha'(t)\| \, dt \leq L(\alpha)$. \hfill \QED

**Remark 5.107.** Lemma 5.106 is true for any piecewise $C^1$ curve $\alpha$ also, simply by adding together the results for each smooth segment of $\alpha$.

**Lemma 5.108.** For any $x, y \in X$, a Banach space,

$$d(x, y) = \|x - y\| = \delta(x, y).$$
Proof. From Remark 5.103 we know

\[ d(x, y) = \|x - y\| \leq \delta(x, y) = \inf L(\alpha) \]

for any curve \( \alpha \) passing through the point \( x \) at \( t = 0 \) and the point \( y \) at \( t = 1 \).

Taking a straight line segment

\[ \gamma(t) = x + t(-x + y) \]

in a Banach space \( X \), we have by Lemma 5.106

\[ L(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt = \|x - y\|, \]

and so,

\[ d(x, y) \leq \delta(x, y) = \inf L(\alpha) \leq L(\gamma) = d(x, y). \]

Therefore, the distance between two points \( x \) and \( y \) in a Banach space \( X \) is

\[ d(x, y) = \|x - y\| = \delta(x, y). \]

\[ \square \]

**Definition 5.109.** Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A map \( f : X \to Y \) is called an isometry if for any \( a, b \in X \),

\[ d_Y(f(a), f(b)) = d_X(a, b). \]

We know that an isometry \( f \) preserves the length of a curve \( \alpha \) due to the definitions.

**Lemma 5.110.** Let \( f : X \to Y \) be an isometry of metric spaces. For any curve \( \alpha \),

\[ L(f \circ \alpha) = L(\alpha). \]
5.2 Length Minimizing Geodesic

In the previous section we have seen that a line is a length minimizing geodesic in a Banach space. In this section we see that an exponential map associated with spray, which we have seen in Chapter 3, yields a length minimizing geodesic in the setting of a symmetric space equipped with Finsler geometry.

We recall the following facts from [15].

**Definition 5.111.** Let $X$ be a Banach manifold.

(1) A tangent norm on $X$ is a function $b : T(X) \to [0, \infty)$ whose restriction to every tangent space $T_x(X)$ is a norm.

(2) A continuous tangent norm $b$ is called compatible if for each $p \in X$ there exist a chart $\phi$ on some neighborhood $U$ of $p$ such that

$$m \cdot b(v) \leq \|d\phi(x)(v)\| \leq M \cdot b(v)$$

for all $v \in T_x(X)$ and $x \in U$, where $m, M > 0$.

We usually write $\|v\|_x := b(v)$ for $v \in T_x(X)$ and $x \in X$. We define a Finsler manifold with spray which has seminegative curvature.

**Definition 5.112.** Let $F$ be the spray naturally associated with the symmetric space $X$.

(1) The triple $(X, b, F)$ is called a Finsler manifold with spray if the tangent norm $b : T(X) \to [0, \infty)$ is invariant under parallel transport along geodesics.

(2) A Finsler manifold with spray $(X, b, F)$ has seminegative curvature if for all $p \in X$ and $x, v \in T_p(X) \cap D_{\exp}$ we have that $d\exp_p(x)$ is invertible and

$$\|d\exp_p(x)(v)\|_p \geq \|v\|_p$$
where $D_{\exp}$ is the domain of the exponential function defined by the spray (see Section 3.1).

We see that a Finsler manifold with spray which has seminegative curvature gives us a length metric increasing property associated with an exponential map. In the following we abbreviate the exponential functions by exp, distinguishing them by context, and the corresponding log functions by log.

**Proposition 5.113.** Let $X$ be a Finsler manifold with spray which has seminegative curvature. Then for any $x, y \in X$,

$$\| \log x - \log y \| \leq \delta(x, y).$$

**Proof.** Let us consider $\gamma$ arbitrary rectifiable piecewise $C^1$ curve such that $\gamma(0) = x$ and $\gamma(1) = y$. Then $\log \gamma$ is a piecewise $C^1$ curve in $T(X)$ such that

$$\log \gamma(0) = \log x \text{ and } \log \gamma(1) = \log y.$$

Since $X$ has seminegative curvature, we have

$$L(\gamma) = \int_0^1 \| \frac{d}{dt} \gamma(t) \| dt = \int_0^1 \| \frac{d}{dt} \exp(\log \gamma(t)) \| dt$$

$$= \int_0^1 \| d \exp_{\gamma(t)}(\log \gamma(t)) \left( \frac{d}{dt} \log(\gamma(t)) \right) \| dt$$

$$\geq \int_0^1 \| \frac{d}{dt} \log(\gamma(t)) \| dt = L(\log(\gamma)).$$

So

$$\| \log x - \log y \| \leq L(\log(\gamma)) \leq L(\gamma),$$

and hence,

$$\| \log x - \log y \| \leq \inf_{\gamma} L(\gamma) = \delta(x, y).$$

$\square$
Remark 5.114. By Lemma 5.108 we have alternatively

\[ \delta(\log(x), \log(y)) = \| \log x - \log y \| \leq \delta(x, y) \]

for any \( x, y \in X \).

Definition 5.115. A Finsler gyrovector space \( X \) is a gyrovector space modeled on a Banach space \( E \) satisfying the tangent norm \( \| \cdot \| : T(X) \to [0, \infty) \) is invariant under parallel transport along geodesics.

We assume that \( X \) is a smooth gyrocommutative gyrogroup with unique square roots, modeled on a Banach space \( E \), satisfying

(F1) the derivative of gyration at 0, \( d\text{gyr}[x, y](0) \) is an isometry for any \( x, y \in X \), and

(F2) for any \( v \in T_a(X) \),

\[ \|v\|_a := \|v_0\|_0 \]

for some \( v_0 \in T_0(X) \) such that \( v = P_0^1(\alpha)v_0 \), where \( \alpha : \mathbb{R} \to X, \alpha(t) := \exp(tw) \) is a geodesic and \( a = \alpha(1) \).

Remark 5.116. From (F2) we have by Theorem 4.92

\[ v = P_0^1(\alpha)v_0 = dL_a(0)v_0. \]

In Section 4.2 we have seen that a smooth gyrocommutative gyrogroup with unique square roots is a Loos symmetric space with the canonical spray, and a gyrovector space with a smooth scalar multiplication. In order to have that \( X \) is a Finsler gyrovector space, we need to show that the tangent norm \( \| \cdot \| : T(X) \to [0, \infty) \) is invariant under parallel transport along geodesics.

For convenience, we denote \( d_bL_a = dL_a(b) \) and \( d_0 \text{gyr}[x, y] = d\text{gyr}[x, y](0) \).
Lemma 5.117. Let $X$ be a gyrovector space satisfying (F1) and (F2). For any $w \in T_{a \oplus b}(X)$,

$$\|w\|_{a \oplus b} = \|v\|_b$$

for some $v \in T_b(X)$ such that $w = d_bL_a(v)$.

Proof. From (F2) we have

$$\|w\|_{a \oplus b} = \|w_0\|_0$$

for some $w_0 \in T_0(X)$ such that $w = d_bL_{a \oplus b}(w_0)$. We set

$$v := (d_bL_b \circ d_0 \text{ gyr}[b, a])(w_0).$$

Then $v \in T_b(X)$, and moreover, $\|v\|_b = \|w_0\|_0$ because of (F1). Since $L_{a \oplus b} = L_aL_b \text{ gyr}[b, a]$ from the right gyroassociativity, we obtain

$$d_0L_{a \oplus b} = d_bL_a \circ d_0L_b \circ d_0 \text{ gyr}[b, a],$$

and therefore,

$$w = d_0L_{a \oplus b}(w_0)
= (d_bL_a \circ d_0L_b \circ d_0 \text{ gyr}[b, a])(w_0)
= d_bL_a(v).$$

We show that a left translation is an isometry with respect to a length metric.

Lemma 5.118. Let $X$ be a Finsler gyrovector space. For any $a, x, y \in X$,

$$\delta(a \oplus x, a \oplus y) = \delta(x, y).$$

Proof. Let $\gamma$ be any piecewise $C^1$ curve passing through the point $x$ at $t = 0$ and $y$ at $t = 1$. By Lemma 5.117

$$L(L_a \circ \gamma) = \int_0^1 \|d_{\gamma(t)}L_a(\gamma'(t))\|_{a \oplus \gamma(t)}dt = \int_0^1 \|\gamma'(t)\|_{\gamma(t)}dt = L(\gamma).$$
Thus, \( \delta(L_a(x), L_a(y)) = \delta(x, y) \).

We now show that the exponential map associated with spray gives us a length minimizing geodesic.

**Theorem 5.119.** Let \( X \) be a Finsler gyrovector space with seminegative curvature. Then for any \( x, y \in X \),

\[
\delta(x, y) = L(\alpha),
\]

where \( \alpha : [0, 1] \to X, \alpha(t) = \exp(tv) \)

and \( \ominus x \oplus y = \exp(v) \).

**Proof.** For \( \ominus x \oplus y \in X \) we have by Proposition 4.87 that there exists a \( v \in T_0(X) \) such that \( \exp(v) = \ominus x \oplus y \). We set

\[
\alpha : [0, 1] \to X, \alpha(t) = \exp(tv).
\]

Then \( \alpha \) is a geodesic, or a \( \bullet \) homomorphism with \( \alpha(0) = 0 \in X \). By Lemma 2.54 we know that \( \alpha \) is a \( \oplus \) homomorphism. So

\[
\alpha(t) = \alpha(t \oplus 0) = \alpha(t) \oplus \alpha(0) = L_{\alpha(t)}(\alpha(0)) = \tau_{\alpha,t}(\alpha(0)),
\]

and then,

\[
\alpha'(t) = d_{\alpha(0)} \tau_{\alpha,t}(\alpha'(0)) = P^t_{\alpha}(\alpha)v.
\]

Thus, \( \|\alpha'(t)\|_{\alpha(t)} = \|P^t_{\alpha}(\alpha)v\| = \|v\|_0 \), and so, \( L(\alpha) = \|v\|_0 \). Lemma 5.108 and Proposition 5.113 give us

\[
\|v\|_0 = d(0, v) = \delta(0, v) \leq \delta(\alpha(0), \alpha(1)) \leq L(\alpha) = \|v\|_0.
\]

This implies that \( \delta(0, \ominus x \oplus y) = \|v\|_0 \). Hence, by Lemma 5.118

\[
\delta(x, y) = \delta(0, \ominus x \oplus y) = \|v\|_0 = L(\alpha).
\]
Theorem 5.119 says alternatively that the map

\[(L_x \circ \alpha)(t) = x \oplus \exp(tv)\]

is a length minimizing geodesic connecting \(x\) to \(y\). In Proposition 4.90 we have seen that

\[t \otimes (\ominus x \oplus y) = \exp(t \log(\ominus x \oplus y)) = \exp(tv)\]

for any \(t \in [0, 1]\). So we have the following corollary.

**Corollary 5.120.** Let \(X\) be a Finsler gyrovector space with seminegative curvature. Then for any \(x, y \in X\),

\[L : [0, 1] \times X \times X \rightarrow X, \quad L(t, x, y) = x \oplus t \otimes (\ominus x \oplus y)\]

is a length minimizing geodesic.

Later we call the set of \(L(t, x, y)\) for all \(t \in \mathbb{R}\) a gyroline passing through the point \(x\) at \(t = 0\) and the point \(y\) at \(t = 1\).
6. Ordered Gyrovector Spaces

Given a vector space $V$ it is well-known to define a partial order on $V$ via a proper convex cone. A subset $C$ of the vector space $V$ is a convex cone if and only if

$$\alpha x + \beta y \in C$$

for any positive scalars $\alpha, \beta$ and any $x, y \in C$. We further assume that $C$ is a proper convex cone: that is,

$$C \cap -C = \{0\}.$$

For the proper convex cone $C$ the relation

$$x \leq y \text{ if and only if } y - x \in C$$

is a partial order. Based on this idea we define a partial order on the gyrovector space in Section 2. Moreover, we develop its interesting properties related with gyrolines and cogyrolines that we study in Section 1.

In this chapter we consider $G$ as the gyrovector space derived from a smooth gyrocommutative gyrogroup with unique square roots. We review the properties of scalar multiplication. For any $a, b, x \in G$ and $s, t \in \mathbb{R}$,

(A1) $1 \otimes x = x$, $0 \otimes x = 0$, and $(-1) \otimes x = \ominus x$,

(A2) $s \otimes x \oplus t \otimes x = (s + t) \otimes x$,

(A3) $s \otimes (t \otimes x) = (st) \otimes x$,

(A4) $t \otimes \text{gyr}[a, b]x = \text{gyr}[a, b](t \otimes x)$,

(A5) $\text{gyr}[s \otimes x, t \otimes x] = I$,

where $I$ is the identity map.
6.1 Gyrolines and Cogyrolines

The Euclidean line is uniquely determined by any two distinct points. One can express it as

\[ a + t(-a + b) \quad \text{Euclidean Line} \]
\[ t(b - a) + a \quad \text{Euclidean Line} \]
calling each of them the line representation by two points \( a \) and \( b \). In the Euclidean geometry they represent the same line via the associative algebra of vector spaces.

In full analogy with the Euclidean geometry, gyrolines are also uniquely determined by any two distinct points. One can replace the above two expressions by

\[ a \oplus t \otimes (\ominus a \oplus b) \quad \text{Gyroline} \]
\[ t \otimes (b \boxplus a) \oplus a \quad \text{Cogyroline} \]
calling them, respectively, the gyroline and the cogyroline determined by two points \( a \) and \( b \). They are different due to the nonassociativity of a gyrovector space.

We may also consider the midpoints of gyrolines and cogyrolines at \( t = \frac{1}{2} \). We have seen the gyromidpoint in Lemma 2.61; for any \( a, b \in G \)

\[ a \oplus \frac{1}{2} \otimes (\ominus a \oplus b) = \frac{1}{2} \otimes (a \boxplus b). \]

Similarly, we can define the cogyromidpoint of \( a \) and \( b \) as

\[ \frac{1}{2} \otimes (b \boxplus a) \oplus a. \]

In Theorem 6.15 of [1] A. Ungar has found a very interesting relation between the gyromidpoint and the cogyromidpoint.

**Theorem 6.121.** Let \( G \) be a gyrovector space, and let \( a, b \in G \). Then the unique solution of the system of two equations

\[ x \oplus y = a \]
\[ \ominus x \oplus y = b \]

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for the unknowns $x$ and $y$ is

$$x = \frac{1}{2} \otimes (a \triangledown b),$$

$$y = \frac{1}{2} \otimes (a \triangledown b) \oplus b.$$

The solution $x$ is the gyromidpoint of $a$ and $\triangledown b$, and the solution $y$ is the cogyromidpoint of $a$ and $b$. We investigate more these midpoints in this section.

### 6-1.1 Gyrolines and Gyromidpoints

We define a gyroline and a gyromidpoint.

**Definition 6.122.** Let $a$ and $b$ be any two distinct points in a gyrovector space $G$. The gyroline in $G$ that passes through $a$ and $b$ is the set of all points

$$L(t, a, b) = a \oplus t \otimes (\triangledown a \oplus b)$$

with $t \in \mathbb{R}$.

Considering the real parameter $t$ as time, the gyroline $L$ satisfies

$$L(0, a, b) = a \oplus 0 = a,$n$$

$$L(1, a, b) = a \oplus (\triangledown a \oplus b) = b.$$

Naturally we define the gyromidpoint of $a$ and $b$ as the image of the gyroline $L(t, a, b)$ at $t = \frac{1}{2}$.

**Definition 6.123.** The gyromidpoint of any two distinct points $a$ and $b$ in a gyrovector space $G$ is given by

$$L\left(\frac{1}{2}, a, b\right) = a \oplus \frac{1}{2} \otimes (\triangledown a \oplus b).$$

**Remark 6.124.** As we have seen in Section 2.3, for given $a, b \in G$ we can consider a gyroline $L$ as a geodesic $\alpha_{a,b}$ in a symmetric mean such that

$$\alpha_{a,b} : \mathbb{R} \to G, \quad \alpha_{a,b}(t) = a \#_{t} b = L(t, a, b).$$
We now see the basic properties of gyrolines.

**Lemma 6.125.** Let $a$ and $b$ be any two distinct points in a gyrovector space $G$.

Let $s, t, u \in \mathbb{R}$.

(1) $L(t, \ominus a, \ominus b) = \ominus L(t, a, b)$.

(2) $L(t, a, b) = L(1 - t, b, a)$.

(3) $L(u, L(s, a, b), L(t, a, b)) = L((1 - u)s + ut, a, b)$.

**Proof.** Based on Remark 6.124 we use Remark 1.21 and Lemma 1.22. Particularly (3) follows from Remark 1.21.

(1) By Lemma 1.22 we have

\[
\ominus L(t, a, b) = S_0(a \#_t b) = S_0 a \#_t S_0 b = (\ominus a) \#_t (\ominus b) = L(t, \ominus a, \ominus b).
\]

(2) By Remark 1.21 we have

\[
L(t, a, b) = \alpha_{a,b}(t) = \alpha_{a,b}(0 \#_t 1) = \alpha_{a,b}(1 \#_t 1 - 0) = \alpha_{a,b}(1) \#_{1-t} \alpha_{a,b}(0) = b \#_{1-t} a = L(1 - t, b, a).
\]

\[\square\]

**Corollary 6.126.** Let $a$ and $b$ be any two distinct points in a gyrovector space $G$.

Then the equation $L(t, a, x) = b$ for the unknown $x$ and nonzero $t$ has a unique solution

\[x = L\left(\frac{1}{t}, a, b\right).\]

**Proof.** By (3) of Lemma 6.125 we have

\[
L(t, a, x) = L\left(t, L(0, a, b), L\left(\frac{1}{t}, a, b\right)\right)
\]

\[= L(1, a, b) = b.
\]
Suppose that $y$ is another solution of the equation $L(t, a, y) = b$. So we have $L(t, a, x) = L(t, a, y)$, that is,

$$a \oplus t \otimes (\ominus a \oplus x) = a \oplus t \otimes (\ominus a \oplus y).$$

By the left cancellation, multiplying $\frac{1}{t}$ on both sides, again the left cancellation, we obtain $x = y$. \qed

We now show that a gyroline is invariant under a left translation.

**Proposition 6.127.** A left translation of a gyroline is again a gyroline.

**Proof.** The left translation of the gyroline $L(t, a, b)$ by $x$ is given by

$$x \oplus L(t, a, b) = x \oplus \{a \oplus t \otimes (\ominus a \oplus b)\}.$$

By the left gyroassociativity, (A4), and Lemma 2.37, we have

$$x \oplus L(t, a, b) = x \oplus \{a \oplus t \otimes (\ominus a \oplus b)\} = (x \oplus a) \oplus \text{gyr}[x, a] \{t \otimes (\ominus a \oplus b)\} = (x \oplus a) \oplus t \otimes \text{gyr}[x, a] (\ominus a \oplus b) = (x \oplus a) \oplus t \otimes \{\ominus (x \oplus a) \oplus (x \oplus b)\} = L(t, x \oplus a, x \oplus b).$$ \qed

**Example 6.128.** We know that an open convex cone $\Omega$ of positive definite matrices is a gyrovector space with the gyroaddition and scalar multiplication: for any $A, B \in \Omega$ and $t \in \mathbb{R}$

$$A \oplus B = A^{\frac{1}{2}}BA^{\frac{1}{2}}, \ t \odot A = A^t.$$

The gyroline passing through $A$ at $t = 0$ and $B$ at $t = 1$ can be expressed as

$$L(t, A, B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}.$$
We usually call it a \( t \)-weighted geometric mean of \( A \) and \( B \), and denote \( A\#_t B \).
Many scholars have studied the properties of the \( t \)-weighted geometric means and have tried to find the appropriate extension method of two variable geometric means on \( \Omega \). See [5], [12], and [14].

6-1.2 Cogyrolines and Cogyromidpoints

We define a cogyroline and a cogyromidpoint.

**Definition 6.129.** Let \( a \) and \( b \) be any two distinct points in a gyrovector space \( G \). The cogyroline in \( G \) that passes through \( a \) and \( b \) is the set of all points

\[
S(t, a, b) = t \otimes (b \boxdot a) \oplus a
\]

with \( t \in \mathbb{R} \).

**Remark 6.130.** By Lemma 2.61 we have alternatively

\[
S(t, a, b) = 2t \otimes \left( \frac{1}{2} \otimes (b \boxdot a) \right) \oplus a
= 2t \otimes \left( \frac{1}{2} \otimes (\oplus a \boxplus b) \right) \oplus a
= 2t \otimes L \left( \frac{1}{2}, \ominus a, b \right) \oplus a.
\]

 Considering the real parameter \( t \) as time, the cogyroline \( S \) satisfies

\[
S(0, a, b) = 0 \oplus a = a,
S(1, a, b) = (b \boxdot a) \oplus a = b
\]

by Corollary 2.42. Naturally we define the cogyromidpoint of \( a \) and \( b \) as the image of the cogyroline \( S(t, a, b) \) at \( t = \frac{1}{2} \).

**Definition 6.131.** The cogyromidpoint of any two distinct points \( a \) and \( b \) in a gyrovector space \( G \) is given by

\[
S \left( \frac{1}{2}, a, b \right) = \frac{1}{2} \otimes (b \boxdot a) \oplus a.
\]
Lemma 6.132. Let $a$ and $b$ be any two distinct points in a gyrovector space $G$.

Then

$$S\left(\frac{1}{2}, a, b\right) = \frac{1}{2} \otimes \text{gyr} \left[ S\left(\frac{1}{2}, a, b\right), a \right] (a \oplus b).$$

Proof. By the gyrocommutativity, (A4), the Bruck identity of Lemma 2.47, (A3), the left loop property, and Corollary 2.42, we have

$$2 \otimes S\left(\frac{1}{2}, a, b\right) = 2 \otimes \left\{ \frac{1}{2} \otimes (b \boxplus a) \oplus a \right\}$$

$$= 2 \otimes \text{gyr} \left[ \frac{1}{2} \otimes (b \boxplus a), a \right] \left\{ a \oplus \frac{1}{2} \otimes (b \boxplus a) \right\}$$

$$= \text{gyr} \left[ \frac{1}{2} \otimes (b \boxplus a), a \right] 2 \otimes \left\{ a \oplus \frac{1}{2} \otimes (b \boxplus a) \right\}$$

$$= \text{gyr} \left[ \frac{1}{2} \otimes (b \boxplus a) \oplus a, a \right] \left\{ a \oplus ((b \boxplus a) \oplus a) \right\}$$

$$= \text{gyr} \left[ S\left(\frac{1}{2}, a, b\right), a \right] (a \oplus b).$$

Thus, we proved. \qed

We now derive the basic properties of cogyrolines similar those of gyrolines.

Lemma 6.133. Let $a$ and $b$ be any two distinct points in a gyrovector space $G$.

Let $s, t, u \in \mathbb{R}$.

(1) $S(t, \ominus a, \ominus b) = \ominus S(t, a, b)$.

(2) $S(t, a, b) = S(1 - t, b, a)$.

(3) $S(u, S(s, a, b), S(t, a, b)) = S((1 - u)s + ut, a, b)$. 

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Proof. First of all, we show a useful property of coaddition. By the automorphic inverse property and (2) of Lemma 2.33

\[ \ominus(a \ominus b) = \ominus(a \ominus \text{gyr}[a, \ominus b] b) \]

\[ = \ominus a \ominus \text{gyr}[a, \ominus b] b \]

\[ = \ominus a \ominus \text{gyr}[\ominus a, b] b \]

\[ = (\ominus a) \ominus (\ominus b). \]

(1) By the above property and the automorphic inverse property, we have

\[ S(t, \ominus a, \ominus b) = t \otimes \{ (\ominus b) \ominus (\ominus a) \} \ominus a \]

\[ = \ominus t \otimes (b \ominus a) \ominus a \]

\[ = \ominus S(t, a, b). \]

(2) By (A2), the right gyroassociativity, Lemma 2.59, Corollary 2.42, and (4) of Lemma 2.47, we have

\[ S(1-t, b, a) = (1-t) \otimes \{ a \ominus b \} \ominus b \]

\[ = \{ (-t) \otimes (a \ominus b) \ominus (a \ominus b) \} \ominus b \]

\[ = (-t) \otimes (a \ominus b) \ominus \text{gyr}[a \ominus b, (-t) \otimes (a \ominus b)] \{ (a \ominus b) \ominus b \} \]

\[ = (-t) \otimes (a \ominus b) \ominus a \]

\[ = t \otimes ((\ominus a) \ominus b) \ominus a \]

\[ = t \otimes (b \ominus a) \ominus a. \]

(3) By Lemma 2.44, Lemma 2.59, and (A2), we have

\[ S(t, a, b) \ominus S(s, a, b) = \{ t \otimes (b \ominus a) \ominus a \} \ominus \{ s \otimes (b \ominus a) \ominus a \} \]

\[ = t \otimes (b \ominus a) \ominus s \otimes (b \ominus a) \]

\[ = (t-s) \otimes (b \ominus a). \]
Then by (A3), the left gyroassociativity, Lemma 2.59, and (A2), we obtain

\[
S(u, S(s, a, b), S(t, a, b)) = \]

\[
= u \otimes \{S(t, a, b) \boxdot S(s, a, b)\} \oplus S(s, a, b)
\]

\[
= u \otimes \{(t - s) \otimes (b \ominus a)\} \oplus \{s \otimes (b \ominus a) \oplus a\}
\]

\[
= u(t - s) \otimes (b \ominus a) \oplus \{s \otimes (b \ominus a) \oplus a\}
\]

\[
= \{u(t - s) \otimes (b \ominus a) \oplus s \otimes (b \ominus a)\} \oplus \text{gyr}[u(t - s) \otimes (b \ominus a), s \otimes (b \ominus a)]a
\]

\[
= ((1 - u)s + ut) \otimes (b \ominus a) \oplus a.
\]

\[\square\]

**Corollary 6.134.** Let $a$ and $b$ be any two distinct points in a gyrovector space $G$. Then the equation $S(t, a, x) = b$ for the unknown $x$ and nonzero $t$ has a unique solution

\[
x = S\left(\frac{1}{t}, a, b\right).
\]

**Proof.** By (3) of Lemma 6.133 we know that $x = S\left(\frac{1}{t}, a, b\right)$ is the solution of the equation $S(t, a, x) = b$.

Suppose that $y$ is another solution of the equation $S(t, a, y) = b$. Then we have $S(t, a, x) = S(t, a, y)$. By Proposition 2.41, multiplying $\frac{1}{t}$ on both sides, again Proposition 2.41, we obtain $x = y$. \[\square\]

We have seen that a left translation of a gyroline is again a gyroline. On the other hand, this does not hold for a cogyroline. However, it satisfies the following property.

**Proposition 6.135.** Let $a$ and $b$ be any two distinct points in a gyrovector space $G$. For any $x \in G$,

\[
S(t, a \oplus x, b \oplus \text{gyr}[b, a]x) = S(t, a, b) \oplus \text{gyr}[S(t, a, b), x]x.
\]
Proof. By Lemma 2.44, the left gyroassociativity, the left loop property, we have

\[ S(t, a \oplus x, b \oplus \text{gyr}[b, a]x) = t \otimes ( (b \oplus \text{gyr}[b, a]x) \boxdot (a \oplus x) ) \oplus (a \oplus x) \]
\[ = t \otimes (b \boxdot a) \oplus (a \oplus x) \]
\[ = \{t \otimes (b \boxdot a) \oplus a\} \oplus \text{gyr}[t \otimes (b \boxdot a), a]x \]
\[ = \{t \otimes (b \boxdot a) \oplus a\} \oplus \text{gyr}[t \otimes (b \boxdot a) \oplus a, a]x. \]

\[ \square \]

Remark 6.136. Taking \( x = \ominus a \) gives us

\[ S(t, 0, b \boxdot a) = S(t, a, b) \boxdot a. \]

We now obtain the equality between gyrolines and cogyrolines under a certain condition.

**Proposition 6.137.** Let \( a \) and \( b \) be any two points in a gyrovector space \( G \) such that

\[ \text{gyr}[a, b] = I. \]

Then for any \( t \in \mathbb{R} \),

\[ S(t, a, b) = \text{gyr}[t \otimes (\ominus a \oplus b), a]L(t, a, b). \]

Proof. From (2) of Lemma 2.33 we also have \( \text{gyr}[\ominus a, \ominus b] = I \). So

\[ b \boxdot a = (\ominus a) \boxdot b = (\ominus a) \oplus \text{gyr}[\ominus a, \ominus b]b = \ominus a \oplus b. \]

Thus, the gyrocommutativity implies

\[ S(t, a, b) = t \otimes (\ominus a \oplus b) \oplus a \]
\[ = \text{gyr}[t \otimes (\ominus a \oplus b), a](a \oplus t \otimes (\ominus a \oplus b)). \]

\[ \square \]
Example 6.138. On the open convex cone $\Omega$ of positive definite matrices, the cogyroline passing through $A$ at $t = 0$ and $B$ at $t = 1$ can be expressed as

$$S(t, A, B) = 2t \otimes L\left(\frac{1}{2}, \ominus A, B\right) \oplus A = (A^{-1} \# B)^{t} A (A^{-1} \# B)^{t}.$$ 

We usually call it a $t$-weighted spectral geometric mean of $A$ and $B$. If you are interested in more properties of two variable spectral geometric means with usual geometric means on $\Omega$, then see [8].

### 6.2 The Gyro-Order

Let $C$ be a set satisfying for any $v, w \in C$ and $a, b \in G$

(C1) $t \otimes v \in C$ for any $t \geq 0$,

(C2) $v \oplus w \in C$,

(C3) $\text{gyr}[a, b](C) \subseteq C$,

(C4) $(\ominus C) \cap C = \{0\}$,

where $\ominus C = \{\ominus v : v \in C\}$. For any $a, b \in G$, we define a relation $\leq$ such as $a \leq b$ if and only if

$$\ominus a \oplus b \in C.$$ 

Alternatively, $a \ominus b \in \ominus C$.

**Proposition 6.139.** The relation $\leq$ on a gyrovector space $G$ is a partial order.

**Proof.** We show that the relation $\leq$ is reflexive, anti-symmetric, and transitive.

Let $a, b, c \in G$.

(a) Since $\ominus a$ is the inverse of $a$, we have $\ominus a \oplus a = 0 \in C$. Thus, $a \leq a$. 

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(b) Suppose that \( a \leq b \) and \( b \leq a \). Then \( \ominus a \oplus b \in C \) and \( \ominus b \oplus a \in C \). By the gyration invariance (C3), we have

\[
a \oplus (\ominus b) = \text{gyr}[a, \ominus b](\ominus b \oplus a) \in \text{gyr}[a, \ominus b](C) \subseteq C.
\]

By the automorphic inverse property we have \( a \oplus (\ominus b) = \ominus (\ominus a \oplus b) \in \ominus C \).
By (C4) we have \( \ominus a \oplus b = 0 \), and conclude \( a = b \).

(c) Suppose that \( a \leq b \) and \( b \leq c \). Then \( \ominus a \oplus b \in C \) and \( \ominus b \oplus c \in C \). By the gyration invariance (C3), we have \( \text{gyr}[\ominus a, b](\ominus b \oplus c) \in C \). From Lemma 2.37 and (C2), we have

\[
(\ominus a \oplus b) \oplus \text{gyr}[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c \in C.
\]

Thus, \( a \leq c \).

\[\square\]

**Example 6.140.** We remind that an open convex cone \( \Omega \) of positive definite matrices is a gyrovector space with the gyroaddition and scalar multiplication: for any \( A, B \in \Omega \) and \( t \in \mathbb{R} \)

\[
A \oplus B = A^{\frac{1}{2}}BA^{\frac{1}{2}}, \quad t \otimes A = A^t.
\]

Let us consider a set \( C = \{ A \in \Omega : A \geq I \} \), where \( I \) is the identity matrix. Moreover, the identity matrix is the zero gyrovector in the open convex cone. Then the set \( C \) satisfies all conditions (C1) through (C4). Assume that \( A, B \in C \) and \( X, Y \in \Omega \).

(1) Since \( t \otimes A = A^t \geq I \) for any \( t \geq 0 \), we have \( t \otimes A \in C \).

(2) Since \( A \oplus B = A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^{\frac{1}{2}}IA^{\frac{1}{2}} = A \geq I \), we have \( A \oplus B \in C \).
(3) The gyration $\text{gyr}[X,Y]$ is defined by

$$\text{gyr}[X,Y] = Q(Q(X^\frac{1}{2})Y)^{-\frac{1}{2}}Q(X^\frac{1}{2})Q(Y^\frac{1}{2})$$

as we have seen in Theorem 2.52. So

$$\text{gyr}[X,Y]A = Q(Q(X^\frac{1}{2})Y)^{-\frac{1}{2}}Q(X^\frac{1}{2})Q(Y^\frac{1}{2})A$$

$$= Q(Q(X^\frac{1}{2})Y)^{-\frac{1}{2}}(X^\frac{1}{2}Y^\frac{1}{2}AY^\frac{1}{2}X^\frac{1}{2})$$

$$\geq Q(Q(X^\frac{1}{2})Y)^{-\frac{1}{2}}(X^\frac{1}{2}Y^\frac{1}{2}IY^\frac{1}{2}X^\frac{1}{2})$$

$$= Q(Q(X^\frac{1}{2})Y)^{-\frac{1}{2}}(X^\frac{1}{2}YX^\frac{1}{2}) = I.$$

Thus, $\text{gyr}[X,Y]A \in \mathcal{C}$.

(4) Suppose that $X \in \mathcal{C}$ and $X \in \Theta \mathcal{C}$. This gives us that $X \geq I$ and $X^{-1} \geq I$.

Thus, $X = I$.

Via Proposition 6.139 we can construct the partial order $\leq$ defined by $A \leq B$ if and only if

$$I \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.$$

So it is a Loewner order.

**Definition 6.141.** The partial order $\leq$ on the smooth gyrocommutative gyrogroup $G$ with unique square roots is called a gyro-order. Moreover, $(G, \leq)$ is called an ordered gyrovector space.

We now investigate the properties of the gyro-order on the ordered gyrovector space $(G, \leq)$.

**Proposition 6.142.** If $a \leq b$ for any $a, b \in G$, then $\Theta b \leq \Theta a$. 

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Proof. By the gyrocommutativity and the gyration invariance (C3), we have
\[
\ominus (\ominus b) \oplus (\ominus a) = b \oplus (\ominus a)
= \text{gyr}[b, \ominus a](\ominus a \oplus b)
\in \text{gyr}[b, \ominus a]C \subseteq C.
\]
Therefore, \( \ominus b \leq \ominus a \) whenever \( a \leq b \).

Lemma 6.143. Let \( a, b \in G \) and \( s, t \in \mathbb{R} \). Then

1. If \( s \leq t \), then \( s \otimes x \leq t \otimes x \) for \( x \in C \),
2. If \( a \leq b \), then \( x \oplus a \leq x \oplus b \) for any \( x \in G \).

Proof. (1) Assume that \( s \leq t \). In other words, \( -s + t \geq 0 \). By (A2) and (C1) we have
\[
\ominus (s \otimes x) \oplus (t \otimes x) = (-s + t) \otimes x \in C,
\]
where \( x \in C \). Thus, \( s \otimes x \leq t \otimes x \).

(2) Assume that \( a \leq b \). This means that \( \ominus a \oplus b \in C \). By Lemma 2.37 and the gyration invariance (C3), we have
\[
\ominus (x \oplus a) \oplus (x \oplus b) = \text{gyr}[x, a](\ominus a \oplus b) \in \text{gyr}[x, a](C) \subseteq C.
\]
Therefore, \( x \oplus a \leq x \oplus b \) for any \( x \in G \).

We now show that the gyroline \( L \) is monotone with respect to parameters.

Theorem 6.144. We assume that \( a \leq b \) for \( a, b \in G \). Then
\[
L(s, a, b) \leq L(t, a, b)
\]
for all \( s, t \in \mathbb{R} \) such that \( s \leq t \).
Proof. Assume that \( a \leq b \) for \( a, b \in G \). Then \( \ominus a \oplus b \in C \). By (1) of Lemma 6.143 we have

\[
 s \otimes (\ominus a \oplus b) \leq t \otimes (\ominus a \oplus b).
\]

By (2) of Lemma 6.143 we conclude

\[
 L(s, a, b) = a \oplus s \otimes (\ominus a \oplus b) \leq a \oplus t \otimes (\ominus a \oplus b) = L(t, a, b).
\]

For \( t \in [0, 1] \) we call \( L(t, a, b) \) a \( t \)-weighted mean of \( a \) and \( b \). Then by Theorem 6.144 we have the following corollary.

**Corollary 6.145.** We assume that \( a \leq b \) for \( a, b \in G \), and \( t \in [0, 1] \). Then

\[
 a \leq L(t, a, b) \leq b.
\]

**Definition 6.146.** The gyro-order on the smooth gyrocommutative gyrogroup \( G \) with unique square roots is called a Loewner-Heinz order if

\[
 \frac{1}{2} \otimes a \leq \frac{1}{2} \otimes b
\]

whenever \( a \leq b \). In this case, we call \((G, \leq)\) a Loewner-Heinz ordered gyrovector space.

We show that the midpoint is monotone with respect to variables on the Loewner-Heinz ordered gyrovector space.

**Proposition 6.147.** Let \((G, \leq)\) be the Loewner-Heinz ordered gyrovector space. If \( a \leq b \) for any \( a, b \in G \), then for any \( x \in G \)

\[
 x \oplus \frac{1}{2} \otimes (\ominus x \oplus a) \leq x \oplus \frac{1}{2} \otimes (\ominus x \oplus b).
\]
Proof. Assume that $a \leq b$ for any $a, b \in G$. By (2) of Lemma 6.143 we have
\[ \ominus x \oplus a \leq \ominus x \oplus b. \]

Since $\leq$ is the Loewner-Heinz order,
\[ \frac{1}{2} \otimes (\ominus x \oplus a) \leq \frac{1}{2} \otimes (\ominus x \oplus b). \]

Again by (2) of Lemma 6.143 we proved. \[\square\]

Via the previous property we show the Loewner-Heinz inequality.

**Theorem 6.148.** Let $(G, \leq)$ be the Loewner-Heinz ordered gyrovector space. If $a \leq b$ for any $a, b \in G$, then $t \otimes a \leq t \otimes b$ where $t \in [0, 1]$.

**Proof.** Assume that $a \leq b$ for any $a, b \in G$. We set $T = \{ t \in [0, 1] : t \otimes a \leq t \otimes b \}$.

Then clearly $0, 1 \in T$, and $\frac{1}{2} \in T$ because $\leq$ is the Loewner-Heinz order.

Let $s, t \in T$. By (2) of Lemma 6.125 and Proposition 6.147, we have
\[
\left( \frac{s + t}{2} \right) \otimes a = s \otimes a \oplus \frac{1}{2} \otimes (\ominus s \otimes a \oplus t \otimes a) \\
\leq s \otimes a \oplus \frac{1}{2} \otimes (\ominus s \otimes a \oplus t \otimes b) = t \otimes b \oplus \frac{1}{2} \otimes (\ominus t \otimes b \oplus s \otimes a) \\
\leq t \otimes b \oplus \frac{1}{2} \otimes (\ominus t \otimes b \oplus s \otimes b) = \left( \frac{s + t}{2} \right) \otimes b.
\]

So $T$ contains all dyadic rational numbers in $[0, 1]$. By the density of the dyadic rational numbers and the continuity of operations, $T = [0, 1]$. \[\square\]

We now show that the $t$-weighted mean $L(t, \cdot, \cdot)$ for any $t \in [0, 1]$ is monotone with respect to variables.

**Proposition 6.149.** Let $(G, \leq)$ be the Loewner-Heinz ordered gyrovector space.

We assume that $a_1 \leq b_1$ and $a_2 \leq b_2$ for any $a_1, a_2, b_1, b_2 \in G$. Then
\[ L(t, a_1, a_2) \leq L(t, b_1, b_2), \]

where $t \in [0, 1]$. \[109\]
Proof. From (2) of Lemma 6.143, Theorem 6.148, and (2) of Lemma 6.125, we have

\[ a_1 \oplus t \otimes (\ominus a_1 \oplus a_2) \leq a_1 \oplus t \otimes (\ominus a_1 \oplus b_2) = b_2 \oplus (1 - t) \otimes (\ominus b_2 \oplus a_1) \]

\[ \leq b_2 \oplus (1 - t) \otimes (\ominus b_2 \oplus b_1) = b_1 \oplus t \otimes (\ominus b_1 \oplus b_2). \]

We apply the Loewner-Heinz inequality to some inequalities.

**Proposition 6.150.** Let \((G, \leq)\) be the Loewner-Heinz ordered gyrovector space.

For any \(t \in [0, 1]\), \(L(t,a,b) \leq 0\) implies

\[ L(t, r \otimes a, r \otimes b) \leq 0 \]

for all \(r \geq 1\).

Proof. Assume that \(L(t,a,b) \leq 0\) for any \(t \in [0, 1]\). Setting \(c = \ominus a \oplus b\), we have \(b = a \oplus c\) and \(L(t,a,b) = a \oplus t \otimes c \leq 0\). By (2) of Lemma 6.143 and Proposition 6.142, we get

\[ t \otimes c \leq \ominus a, \text{ or } a \leq (-t) \otimes c. \]

Suppose that \(1 \leq r \leq 2\). Pick \(\tau \in [0, 1]\) such that \(r = 2 - \tau\). Then by Theorem 6.148, we have

\[ (1 - \tau) \otimes a \leq -t(1 - \tau) \otimes c. \]
Note that
\[
(2 - \tau) \otimes b = 2 \otimes (a \oplus c) \oplus (a \oplus c)
\]
\[
= \{a \oplus (2 \otimes c \oplus a) \}\oplus (a \oplus c)
\]
\[
= a \oplus \{(2 \otimes c \oplus a) \oplus \text{gyr}[2 \otimes c \oplus a, (-\tau)] (a \oplus c)\}
\]
\[
= a \oplus \{(2 \otimes c \oplus a) \oplus \text{gyr}[2 \otimes c, a] (-\tau) (a \oplus c)\}
\]
\[
= a \oplus \{2 \otimes c \oplus (a \oplus (-\tau) (a \oplus c))\}
\]
\[
= a \oplus \{2 \otimes c \oplus L(\tau, a, \ominus c)\}
\]
\[
= a \oplus L(\tau, 2 \otimes c \ominus a, c).
\]

The second equality follows from the Bruck identity ((3) of Lemma 2.47), the third follows from the right gyroassociativity, the fourth follows from the left loop property, the fifth follows from the left gyroassociativity, the sixth follows from the automorphic inverse property, and the last follows from Proposition 6.127.

Then
\[
L(t, (2 - \tau) \otimes a, (2 - \tau) \otimes b)
\]
\[
= (2 - \tau) \otimes a \oplus t \otimes ((2 - \tau) \otimes a \oplus (2 - \tau) \otimes b)
\]
\[
= (2 - \tau) \otimes a \oplus t \otimes \{(2 - \tau) \otimes a \oplus (a \oplus L(\tau, 2 \otimes c \ominus a, c))\}
\]
\[
= (2 - \tau) \otimes a \oplus t \otimes \{(1 - \tau) \otimes a \oplus L(\tau, 2 \otimes c \ominus a, c)\}
\]
\[
= a \oplus \{(1 - t) \otimes a \oplus t \otimes \{(1 - \tau) \otimes a \oplus L(\tau, 2 \otimes c \ominus a, c)\}\}
\]
\[
= a \oplus L(t, (1 - \tau) \otimes a, L(\tau, 2 \otimes c \ominus a, c))
\]
\[
\leq a \oplus L(t, -t(1 - \tau) \otimes c, L(\tau, (2 - t) \otimes c, c))
\]
\[
= a \oplus \{(1 - t)(-t(1 - \tau)) + t((1 - \tau)(2 - t) + \tau)\} \otimes c
\]
\[
= a \oplus t \otimes c \leq 0.
\]

The second equality follows from the above result, the third and the fourth follow from the left gyroassociativity, and the inequality follows from Proposition 6.149.
Applying the preceding result, we have that

\[ L(t, r \otimes a, r \otimes b) \leq 0 \text{ implies } L(t, 2r \otimes a, 2r \otimes b) \leq 0. \]

Using induction the statement is true for \( r = 2^k(2-\tau) \), where \( k \in \mathbb{N} \) and \( 0 \leq \tau \leq 1 \), and therefore, for all real numbers. \( \square \)

**Proposition 6.151.** Let \((G, \leq)\) be the Loewner-Heinz ordered gyrovector space. If \( a \leq b \) for any \( a, b \in G \), then

\[ L\left( \frac{p}{p + r}, p \otimes a, (-r) \otimes b \right) \leq 0 \quad (6.15) \]

for all \( r \geq 0 \) and \( p \geq 0 \).

**Proof.** First we prove the inequality (6.15) by induction for positive integer \( r \). Since

\[ L\left( \frac{p}{p + r}, p \otimes a, (-1) \otimes b \right) \leq L\left( \frac{p}{p + r}, p \otimes a, (-1) \otimes a \right) = 0, \]

the inequality (6.15) holds for \( r = 1 \). Suppose that

\[ L\left( \frac{p}{p + n}, p \otimes a, (-n) \otimes b \right) \leq 0 \]

for some positive integer \( r = n \). By (2) of Lemma 6.125, we have equivalently

\[ L\left( \frac{n}{p + n}, (-n) \otimes b, p \otimes a \right) \leq 0. \]

Using (2) of Lemma 6.143 and Proposition 6.127 in the above inequality implies

\[ \left( \frac{n}{p + n} \right) \otimes (n \otimes b \oplus p \otimes a) = n \otimes b \oplus L\left( \frac{n}{p + n}, (-n) \otimes b, p \otimes a \right) \leq n \otimes b \oplus 0 = n \otimes b. \]

By Theorem 6.148, we have

\[ \left( \frac{1}{p + n} \right) \otimes (n \otimes b \oplus p \otimes a) \leq b. \]
Then

$$L\left(\frac{p}{p+n+1}, p \otimes a, -(n+1) \otimes b\right)$$

$$= (-n) \otimes b \oplus L\left(\frac{p}{p+n+1}, n \otimes b \oplus p \otimes a, (-1) \otimes b\right)$$

$$\leq (-n) \otimes b \oplus L\left(\frac{p}{p+n+1}, n \otimes b \oplus p \otimes a, \left(\frac{-1}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a)\right)$$

$$= (-n) \otimes b \oplus \left(\frac{n}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a)$$

$$= L\left(\frac{n}{p+n}, (-n) \otimes b, p \otimes a\right) \leq 0.$$ 

So the inequality (6.15) holds for all positive integer $r$.

If $0 \leq r \leq 1$, then $(-r) \otimes b \leq (-r) \otimes a$ by Theorem 6.148 and Proposition 6.142, and hence,

$$L\left(\frac{p}{p+r}, p \otimes a, (-r) \otimes b\right) \leq L\left(\frac{p}{p+r}, p \otimes a, (-r) \otimes a\right) = 0.$$ 

If $r = n + \tau$ for positive integer $n$ and $0 \leq \tau \leq 1$, then

$$L\left(\frac{p}{p+n}, p \otimes a, (-n) \otimes b\right) \leq 0

implies that

$$\left(\frac{n}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a) \leq n \otimes b.$$ 

By Theorem 6.148 for $t = \frac{\tau}{n}$, we have

$$\left(\frac{\tau}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a) \leq \tau \otimes b.$$
Hence,

\[
L\left(\frac{p}{p+r}, p \otimes a, (-r) \otimes b \right) \\
= L\left(\frac{p}{p+n+\tau}, p \otimes a, -(n+\tau) \otimes b \right) \\
= (-n) \otimes b \oplus L\left(\frac{p}{p+n+\tau}, p \otimes a, (-\tau) \otimes b \right) \\
\leq (-n) \otimes b \oplus L\left(\frac{p}{p+n+1}, n \otimes b \oplus p \otimes a, \left(-\frac{\tau}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a) \right) \\
= (-n) \otimes b \oplus \left(\frac{n}{p+n}\right) \otimes (n \otimes b \oplus p \otimes a) \\
= L\left(\frac{n}{p+n}, (-n) \otimes b, p \otimes a \right) \leq 0.
\]

We have seen that for any \(a, b \in G\) with \(a \leq b\) and any \(t \in [0, 1]\),

\[a \leq L(t, a, b) \leq b.\]

On the other hand, the cogyroline \(S(t, a, b)\) does not hold, but satisfies the following property.

**Proposition 6.152.** For any \(a, b \in G\) and \(t \in [0, 1]\),

\[a \leq S(t, a, b) \text{ and } S(t, b, a) \leq b\]

are equivalent.

**Proof.** Assume that \(a \leq S(t, a, b)\). By (2) of Lemma 6.133 we have

\[S(t, a, b) = S(1-t, b, a) = (1-t) \otimes (a \boxdot b) \oplus b.\]

So (2) of Lemma 6.143 and the left cancellation imply

\[\ominus(1-t) \otimes (a \boxdot b) \oplus a \leq \ominus(1-t) \otimes (a \boxdot b) \oplus \{(1-t) \otimes (a \boxdot b) \oplus b\} = b.\]
Since $\Theta(1-t) \otimes (a \boxtimes b) \oplus a = (1-t) \otimes (b \boxtimes a) \oplus a = S(1-t, a, b) = S(t, b, a)$, we have

$$S(t, b, a) \leq b.$$ 

Similarly we can prove the reverse implication. $\square$
7. Quantum System of Qubits

Bloch vectors and their generated density matrices are important concepts in quantum computation and quantum information. We see that Bloch vectors are the hyperbolic vectors, called gyrovectors naturally linked with the Einstein’s vector addition of relativistically admissible vectors.

In Section 1 we give a gyrogroup structure to the setting of density matrices that can provide useful algebraic tools in their study.

In Section 2 we introduce the Bloch vector and its generated density matrix, called a qubit. Via the diagonalization of the qubits, we see that the set of Bloch vectors with the Einstein vector addition is isomorphic to the set of their generated density matrices with certain operation.

In Section 3 we study the Minkowski space and the Lorentz transformation. We see that a positive definite Lorentz transformation is a Lorentz boost. Via a polar decomposition we show an isomorphism between the set of Bloch vectors with the Einstein vector addition and the set of Lorentz boosts with certain operation.

In Section 4 we introduce the Bures fidelity, the most important distance measure in quantum information and computation. We see the several equivalent formulas of the Bures fidelity generated by two qubits.

7.1 Density Matrices

A density matrix is a positive semidefinite Hermitian matrix of trace 1 on a complex Hilbert space. In one of the standard approaches to quantum theory, the states of an isolated quantum system are taken to be the set of density matrices on some Hilbert space, each matrix representing some state of the system.
Remark 7.153. For a density matrix $A$ the following are equivalent:

(1) $A$ is an orthogonal projection onto a 1-dimensional subspace.

(2) $\text{tr}(A^2) = 1$.

Such density matrices are called pure states.

The other density matrices are mixed states. The positive definite density matrices sit inside the mixed state density matrices as a dense open subset of the density matrices.

Let us consider $\mathbb{D} := \mathbb{D}_n$ as a set of all $n \times n$ invertible density matrices, or all $n \times n$ positive definite Hermitian matrices with trace 1. We have seen that the open cone $\Omega$ of positive definite matrices with the gyroaddition and the scalar multiplication

$$
\oplus : \Omega \times \Omega \to \Omega, \ A \oplus B = A^{\frac{1}{2}}BA^{\frac{1}{2}},
$$

$$
\otimes : \mathbb{R} \times \Omega \to \Omega, \ t \oplus A = A^t.
$$

is a gyrovector space. On the set $\mathbb{D}$ we naturally define the binary operation $\odot : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ and the scalar multiplication $\circ : \mathbb{R} \times \mathbb{D} \to \mathbb{D}$ by

$$
A \odot B = \frac{1}{\text{tr}(A \oplus B)} A \oplus B,
$$

$$
t \circ A = \frac{1}{\text{tr}(A^t)} A^t.
$$

In order to show that $(\mathbb{D}, \odot, \circ)$ is a gyrovector space, we need the following lemma. The proof is straightforward.

Lemma 7.154. For any $A, B \in \Omega$ and $c > 0$,

$$(cA) \oplus B = c(A \oplus B) = A \oplus (cB).$$
Remark 7.155. Lemma 7.154 does not hold generally for scalar multiplication $\otimes$.

That is,

$$(c \otimes A) \oplus B \neq c \otimes (A \oplus B), \quad A \oplus (c \otimes B) \neq c \otimes (A \oplus B).$$

We remind of the polar decomposition of an $n \times n$ matrix $A$ in Theorem 7.3.2 of [4]. It may be written in the form

$$A = P \cdot U,$$

where $P$ is positive semidefinite and $U$ is unitary. The matrix $P$ is uniquely determined as $P := (AA^*)^{\frac{1}{2}}$, where $A^*$ is a conjugate transpose of $A$. If $A$ is invertible, then the matrix $P$ is positive definite and the matrix $U$ is uniquely determined as $U := P^{-1}A$.

Remark 7.156. We have seen in Theorem 2.52 that the map

$$\text{gyr}[A, B] = Q(Q(A^\frac{1}{2})B)^{-\frac{1}{2}}Q(A^\frac{1}{2})Q(B^\frac{1}{2})$$

is a gyroautomorphism on the open cone $(\Omega, \oplus)$. Alternatively, the gyroautomorphism $\text{gyr}[A, B]$ can be defined by a unitary conjugation. Indeed,

$$\text{gyr}[A, B]X = U(A^\frac{1}{2}, B^\frac{1}{2})XU(A^\frac{1}{2}, B^\frac{1}{2})^{-1},$$

where $U(A^\frac{1}{2}, B^\frac{1}{2})$ is a unitary part of the polar decomposition for $A^\frac{1}{2}B^\frac{1}{2}$ such that

$$A^\frac{1}{2}B^\frac{1}{2} = (A \oplus B)U(A^\frac{1}{2}, B^\frac{1}{2}).$$

So the gyroautomorphism preserves the trace.

Proposition 7.157. The system $(\mathbb{D}, \odot, \circ)$ is a gyrovector space.

Proof. We first show the weakened axioms of gyrocommutative gyrogroups.
(1) By Lemma 7.154 it is easy to see that $\frac{1}{n}I$ is the left identity in $\mathbb{D}$.

(2) For each $A \in \mathbb{D}$, $\frac{1}{\text{tr}(A^{-1})}A^{-1}$ is the left inverse of $A$ by Lemma 7.154.

(3) From Remark 7.156 we have that the gyroautomorphisms are well defined on the set $\mathbb{D}$. Moreover, for any $X, Y \in \mathbb{D}$

\[ \text{gyr}[A, B]X \odot \text{gyr}[A, B]Y = \frac{1}{\text{tr}(\text{gyr}[A, B](X \oplus Y))} \text{gyr}[A, B](X \oplus Y) \]

\[ = \text{gyr}[A, B] \frac{1}{\text{tr}(X \oplus Y)} (X \oplus Y) \]

\[ = \text{gyr}[A, B](X \odot Y). \]

Thus, $\text{gyr}[A, B] \in \text{Aut}(\mathbb{D}, \odot)$ for all $A, B \in \mathbb{D}$.

(4) For any $A, B, C \in \mathbb{D}$, we have by Lemma 7.154

\[ A \odot (B \odot C) \]

\[ = \frac{1}{\text{tr} \left( A \oplus \frac{1}{\text{tr}(B \oplus C)} (B \oplus C) \right)} \left( A \oplus \frac{1}{\text{tr}(B \oplus C)} (B \oplus C) \right) \]

\[ = \frac{1}{\text{tr}(A \oplus (B \oplus C))} (A \oplus (B \oplus C)), \]

and

\[ (A \odot B) \odot \text{gyr}[A, B]C \]

\[ = \frac{1}{\text{tr} \left( \frac{1}{\text{tr}(A \oplus B)} (A \oplus B) \oplus \text{gyr}[A, B]C \right)} \left( \frac{1}{\text{tr}(A \oplus B)} (A \oplus B) \oplus \text{gyr}[A, B]C \right) \]

\[ = \frac{1}{\text{tr}(A \oplus (B \oplus \text{gyr}[A, B]C))} ((A \oplus B) \oplus \text{gyr}[A, B]C). \]

The left gyroassociativity under the operation $\oplus$ implies

\[ A \odot (B \odot C) = (A \odot B) \odot \text{gyr}[A, B]C. \]

(5) We show the left loop property of a gyroautomorphism under the operation $\odot$. From the polar decomposition of $X^{\frac{1}{2}}Y^{\frac{1}{2}}$, we have the explicit formula of
the unitary part

\[ U(X^{\frac{1}{2}}, Y^{\frac{1}{2}}) = (X^{\frac{1}{2}}YX^{\frac{1}{2}})^{-1}X^{\frac{1}{2}}Y^{\frac{1}{2}}. \]

So for any \( c > 0 \),

\[ U((cX)^{\frac{1}{2}}, Y^{\frac{1}{2}}) = c^{-\frac{1}{2}}U(X^{\frac{1}{2}}, Y^{\frac{1}{2}}) = U(X^{\frac{1}{2}}, (cY)^{\frac{1}{2}}). \]

This gives us \( \text{gyr}[cX, Y] = \text{gyr}[X, Y] = \text{gyr}[X, cY] \). Hence, by the left loop property under the operation \( \oplus \),

\[ \text{gyr}[A \odot B, B] = \text{gyr}\left[ \frac{1}{\text{tr}(A \oplus B)}(A \oplus B), B \right] = \text{gyr}[A \oplus B, B] = \text{gyr}[A, B]. \]

We note that the property \( \text{gyr}[cX, Y] = \text{gyr}[X, Y] = \text{gyr}[X, cY] \) gives us

\[ \text{gyr}[s \circ X, t \circ Y] = \text{gyr}[X, Y]. \]

So we can prove (A5) for the scalar multiplication \( \circ \).

(6) For any \( A, B \in \mathbb{D} \)

\[ \text{gyr}[A, B](B \odot A) = \text{gyr}[A, B]B \odot \text{gyr}[A, B]A \]

\[ = \frac{1}{\text{tr}(\text{gyr}[A, B]B \oplus \text{gyr}[A, B]A)}(\text{gyr}[A, B]B \oplus \text{gyr}[A, B]A) \]

\[ = \frac{1}{\text{tr}(\text{gyr}[A, B](B \oplus A))}\text{gyr}[A, B](B \oplus A) \]

\[ = \frac{1}{\text{tr}(A \oplus B)}(A \oplus B) \]

\[ = A \odot B. \]

The first equality follows from the fact that \( \text{gyr}[A, B] \in \text{Aut}(\mathbb{D}, \odot) \), the third follows from the fact that \( \text{gyr}[A, B] \in \text{Aut}(\Omega, \oplus) \), and the fourth follows from the gyrocommutativity under the operation \( \oplus \).

We next show the axioms (A1) - (A4) for the scalar multiplication \( \circ \). Let \( s, t \in \mathbb{R} \) and \( A, B, X \in \mathbb{D} \).
(A1) One can see easily that
\[ 1 \circ A = A, \quad 0 \circ A = \frac{1}{n} I, \quad \text{and} \quad (-1) \circ A = \frac{1}{\text{tr}(A^{-1})} A^{-1}. \]

(A2) By Lemma 7.154
\[ s \circ A \oplus t \circ A = \frac{1}{\text{tr}(A^s)} A^s \oplus \frac{1}{\text{tr}(A^t)} A^t = \frac{1}{\text{tr}(A^s)\text{tr}(A^t)} A^{s+t}. \]
So we have
\[ s \circ A \circ t \circ A = \frac{1}{\text{tr}(s \circ A \oplus t \circ A)} (s \circ A \oplus t \circ A) \]
\[ = \frac{1}{\text{tr}(A^s)} \text{tr}(A^s+t) \frac{1}{\text{tr}(A^t)} \text{tr}(A^s+t) \]
\[ = \frac{1}{\text{tr}(A^{s+t})} A^{s+t} = (s + t) \circ A. \]

(A3) By the direct computation we have
\[ s \circ (t \circ A) = s \circ \frac{1}{\text{tr}(A^t)} A^t \]
\[ = \frac{1}{\text{tr}(A^t)} \left( \frac{1}{\text{tr}(A^s)} A^s \right)^t \]
\[ = \frac{1}{\text{tr}(A^{st})} A^{st} = (st) \circ A. \]

(A4) In the gyrovector space \((\Omega, \oplus, \otimes)\), we have
\[ \text{gyr}[A, B]X^t = \text{gyr}[A, B](t \otimes X) = t \otimes \text{gyr}[A, B]X = (\text{gyr}[A, B]X)^t. \]

By Remark 7.156 we obtain
\[ t \circ \text{gyr}[A, B]X = \frac{1}{\text{tr}(\text{gyr}[A, B]X)^t} (\text{gyr}[A, B]X)^t \]
\[ = \frac{1}{\text{tr}(\text{gyr}[A, B]X)^t} \text{gyr}[A, B]X^t \]
\[ = \text{gyr}[A, B] \frac{1}{\text{tr}(X^t)} X^t = \text{gyr}[A, B](t \circ X). \]
7.2 Bloch Gyrovectors and Qubits

A qubit is the density matrix on a complex Hilbert space with dimension 2. In other words, the qubit is the two-by-two positive semidefinite Hermitian matrix with trace 1. Indeed, any two-by-two Hermitian matrix of trace 1 must have a parametrization of the form

$$\rho_v = \frac{1}{2} \begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix},$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. So the qubit can be described as $\rho_v$ for some $v \in \mathbb{R}^3$ such that $\|v\| \leq 1$. In this case the vector $v$ is known as the Bloch vector or Bloch vector representation of the state $\rho_v$.

**Remark 7.158.** Interestingly, we have

$$\det(\rho_v) = \left(\frac{1}{2\gamma_v}\right)^2 = \frac{1}{4} (1 - \|v\|^2),$$

where $\gamma_v := \frac{1}{\sqrt{1 - \|v\|^2}}$ is the Lorentz factor of the Bloch vector $v$. So the mixed states are parameterized by the open unit ball $\mathbb{B}$ in $\mathbb{R}^3$.

**Remark 7.159.** The Pauli matrices are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$.

The parameterization of qubits can be written in an alternative method using the Pauli matrices:

$$\rho_v = \frac{1}{2}(I + v \cdot \sigma),$$

where $v \cdot \sigma = v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3$ for $v = (v_1, v_2, v_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. 

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Lemma 7.160. For any $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{B}$,

$$\text{tr}(\rho_u \oplus \rho_v) = \frac{1}{2}(1 + u \cdot v).$$

Proof. By the direct computation we have

$$\text{tr}(\rho_u \oplus \rho_v) = \text{tr}(\rho_u \rho_v)$$

$$= \frac{1}{4}\{(1 + u_3)(1 + v_3) + (u_1 - iu_2)(v_1 + iv_2)$$

$$+ (u_1 + iu_2)(v_1 - iv_2) + (1 - u_3)(1 - v_3)\}$$

$$= \frac{1}{2}(1 + u_1v_1 + u_2v_2 + u_3v_3).$$

We now investigate the diagonalization of the invertible qubit.

Proposition 7.161. For any $v = (v_1, v_2, v_3) \in \mathbb{B}$, the qubit $\rho_v$ can be diagonalized such as

(a) If $v_1 = v_2 = 0$, then

$$\rho_v = \frac{1}{2} \begin{pmatrix} 1 + v_3 & 0 \\ 0 & 1 - v_3 \end{pmatrix}.$$

(b) If $v_1 \neq 0$ or $v_2 \neq 0$, then $\rho_v = \frac{1}{2} U_v D_v U_v^{-1}$, where

$$U_v = \begin{pmatrix} v_1 - iv_2 & -v_1 + iv_2 \\ -v_3 + \|v\| & v_3 + \|v\| \end{pmatrix} \text{ and } D_v = \begin{pmatrix} 1 + \|v\| & 0 \\ 0 & 1 - \|v\| \end{pmatrix}.$$

Proof. There is nothing to prove for the case (a). So let us assume that $v_1 \neq 0$ or $v_2 \neq 0$. The eigenvalues of the invertible qubit $\rho_v$ are $\lambda_+ = 1 + \|v\|$ and $\lambda_- = 1 - \|v\|$. 

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The eigenvector for $\lambda_+$ can be
\[
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix} = \begin{pmatrix}
v_1 - iv_2 \\
-v_3 + \|v\| \\
\end{pmatrix}.
\]
Similarly, the eigenvector for $\lambda_-$ can be
\[
\begin{pmatrix}
y_1 \\
y_2 \\
\end{pmatrix} = \begin{pmatrix}
-v_1 + iv_2 \\
v_3 + \|v\| \\
\end{pmatrix}.
\]
Then the matrix
\[
U_v = \begin{pmatrix}
v_1 - iv_2 & -v_1 + iv_2 \\
-v_3 + \|v\| & v_3 + \|v\| \\
\end{pmatrix}
\]
is invertible, and $\rho_v = \frac{1}{2} U_v D_v U_v^{-1}$. \qed

We now compute the square root of the invertible qubit.

**Corollary 7.162.** For any $v = (v_1, v_2, v_3) \in \mathbb{B}$, the square root of the qubit $\rho_v$ is of the form
\[
\rho_v^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{\frac{\gamma_v}{1 + \gamma_v}} \begin{pmatrix}
1 + v_3 + \frac{1}{\gamma_v} & v_1 - iv_2 \\
v_1 + iv_2 & 1 - v_3 + \frac{1}{\gamma_v} \\
\end{pmatrix}.
\]

**Proof.** If $v_1 = v_2 = 0$, the equation (7.16) reduces to
\[
\rho_v^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{1 + v_3} & 0 \\
0 & \sqrt{1 - v_3} \\
\end{pmatrix}.
\]
The proof for the case that $v_1 = v_2 = 0$ is done from Proposition 7.161. So let us assume that $v_1 \neq 0$ or $v_2 \neq 0$. The eigenvalues of the square root $\rho_v^{\frac{1}{2}}$ are respectively
\[
\sqrt{1 \pm \|v\|} = \frac{\sqrt{\gamma_v + 1} \pm \sqrt{\gamma_v - 1}}{\sqrt{2\gamma_v}}.
\]
Then by the matrix multiplication

\[
\frac{1}{2} \mathbf{D}^\frac{1}{2} \mathbf{U} \mathbf{v} \mathbf{D}^\frac{1}{2} \mathbf{U}^{-1} = \frac{1}{2} \begin{pmatrix}
\sqrt{\gamma} v_3 + \sqrt{\gamma} \|\mathbf{v}\| & \sqrt{\gamma} (v_1 - iv_2) \\
\sqrt{\gamma} (v_1 + iv_2) & -\sqrt{\gamma} v_3 + \sqrt{\gamma} \|\mathbf{v}\|
\end{pmatrix}
\]

\[
= \frac{1}{2} \sqrt{\gamma} \left( \begin{array}{cc}
1 + v_3 + \frac{1}{\gamma} & v_1 - iv_2 \\
v_1 + iv_2 & 1 - v_3 + \frac{1}{\gamma}
\end{array} \right).
\]

\[\square\]

Lemma 7.163. For any \(\mathbf{v} \in \mathbb{B}\),

\[
\text{tr}(\rho_{\mathbf{v}}^\frac{1}{2}) = \sqrt{\frac{\gamma_v + 1}{\gamma_v}}.
\]

Proof. Since the eigenvalues of \(\rho_{\mathbf{v}}^\frac{1}{2}\) are respectively

\[
\sqrt{1 \pm \|\mathbf{v}\|} = \frac{\sqrt{\gamma_v + 1} \pm \sqrt{\gamma_v - 1}}{\sqrt{2\gamma_v}},
\]

we have

\[
\text{tr}(\rho_{\mathbf{v}}^\frac{1}{2}) = \frac{1}{\sqrt{2}} \left( \frac{2\sqrt{\gamma_v + 1}}{\sqrt{2\gamma_v}} \right) = \sqrt{\frac{\gamma_v + 1}{\gamma_v}}.
\]

\[\square\]

Remark 7.164. Let us recall an Einstein vector addition \(\oplus_E\) and a scalar multiplication \(\otimes\) on the open unit ball \(\mathbb{B}\) from Example 2.48 and Example 2.57. For any \(\mathbf{u}, \mathbf{v} \in \mathbb{B}\) and \(t \in \mathbb{R}\),

\[
\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \mathbf{u} \cdot \mathbf{v}} \left\{ \mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} + \frac{\gamma_u}{1 + \gamma_u} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right\}.
\]

\[
t \otimes \mathbf{v} = \tanh(t \tanh^{-1} \|\mathbf{v}\|) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\{ \frac{(1 + \|\mathbf{v}\|^t - (1 - \|\mathbf{v}\|)^t}{(1 + \|\mathbf{v}\|^t + (1 - \|\mathbf{v}\|)^t} \right\} \frac{\mathbf{v}}{\|\mathbf{v}\|}.
\]

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For those familiar with special relativity, the Einstein vector addition can also be defined by applying the Lorentz boost

\[ B(\mathbf{u}) = \begin{pmatrix} \gamma_{\mathbf{u}} & \gamma_{\mathbf{u}} \mathbf{u} \\ \gamma_{\mathbf{u}} \mathbf{u}^T & I + \frac{\gamma_{\mathbf{u}}^2}{1 + \gamma_{\mathbf{u}}} \mathbf{u}^T \mathbf{u} \end{pmatrix} \]

to \( (1 \mathbf{v})^T \) and obtaining

\[ B(\mathbf{u}) \begin{pmatrix} 1 \\ \mathbf{v}^T \end{pmatrix} = \begin{pmatrix} t \\ t(\mathbf{u} \oplus_E \mathbf{v})^T \end{pmatrix}, \]

where \( t = \gamma_{\mathbf{u}}(1 + \mathbf{u} \cdot \mathbf{v}) \). We will see more detail in the next section.

As we have seen Proposition 7.157, the system \((\mathbb{D}, \odot, \circ)\) is a gyrovector space. Especially the set \(\mathbb{D}_2\) of all invertible qubits is also a gyrovector space. We now show the main result in this section, an isomorphism between the open unit ball \(\mathbb{B}\) with Einstein vector addition and scalar multiplication and the set \(\mathbb{D}_2\) with certain operations.

**Theorem 7.165.** The map \( \rho : (\mathbb{B}, \oplus_E, \otimes) \to (\mathbb{D}_2, \odot, \circ) \) defined by

\[ \mathbf{v} \mapsto \rho_\mathbf{v} \]

is an isomorphism.

**Proof.** It is easy to see that the map \( \rho \) is a bijection. It needs to be shown that

\[ \rho_{\mathbf{u} \oplus \mathbf{v}} = \rho_\mathbf{u} \odot \rho_\mathbf{v} = \frac{1}{\text{tr}(\rho_\mathbf{u} \oplus \rho_\mathbf{v})} (\rho_\mathbf{u} \oplus \rho_\mathbf{v}) \]

for any \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) in the open unit ball \(\mathbb{B}\).

Set \( x := \frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}} + 1} \). By Corollary 7.162 we have

\[ \rho_{\mathbf{u}}^{1/2} = \frac{1}{2 \sqrt{x}} \begin{pmatrix} u_3 + \frac{1}{x} & u_1 - iu_2 \\ u_1 + iu_2 & -u_3 + \frac{1}{x} \end{pmatrix}. \]
It is enough to check the (1,1) and (1,2) entries of $\rho_u \odot \rho_v$ since $\rho_u \odot \rho_v$ is Hermitian and positive definite. When we compute with $\|u\|$ in the following, we need the property of the Lorentz gamma factor:

$$\|u\|^2 = \frac{\gamma_u^2 - 1}{\gamma_u^2}. \quad (7.17)$$

Let us compute the (1,1)-component of $\rho_u \odot \rho_v$. Then we have

$$\begin{align*}
\frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (1 + v_3) + (u_1 - iu_2)(v_1 + iv_2) \right\} \left( u_3 + \frac{1}{x} \right) \\
+ \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (v_1 - iv_2) + (u_1 - iu_2)(1 - v_3) \right\} (u_1 + iu_2) \\
= \frac{x}{8} \left[ \left( u_3 + \frac{1}{x} \right)^2 (1 + v_3) + 2 \left( u_3 + \frac{1}{x} \right) (u_1v_1 + u_2v_2) + (u_1^2 + u_2^2)(1 - v_3) \right] \\
= \frac{x}{8} \left[ \|u\|^2 + 2(u \cdot v)u_3 - \|u\|^2v_3 + \frac{2}{x}u_3 + \frac{2}{x}(u \cdot v) + \frac{1}{x^2}(1 + v_3) \right].
\end{align*}$$

Let us compute the (1,2)-component of $\rho_u \odot \rho_v$. Then we have

$$\begin{align*}
\frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (1 + v_3) + (u_1 - iu_2)(v_1 + iv_2) \right\} \left( v_1 + iv_2 \right) \\
+ \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (v_1 - iv_2) + (u_1 - iu_2)(1 - v_3) \right\} \left( -u_3 + \frac{1}{x} \right)
\end{align*}$$

So from Lemma 7.160, the (1,1)-component of $\rho_u \odot \rho_v$ is

$$\begin{align*}
\left( \frac{2}{1 + u \cdot v} \right) \frac{1}{4} \left\{ 1 + u \cdot v + u_3 + \frac{1}{\gamma_u}v_3 + \frac{\gamma_u}{\gamma_u + 1}(u \cdot v)u_3 \right\} \\
= \frac{1}{2} \left\{ 1 + \frac{1}{1 + u \cdot v} \left( u_3 + \frac{1}{\gamma_u}v_3 + \frac{\gamma_u}{\gamma_u + 1}(u \cdot v)u_3 \right) \right\} \\
= \frac{1}{2} (1 + (u \oplus_E v)_3),
\end{align*}$$

where $(u \oplus_E v)_j$ represents the $j$ th coordinate of Einstein vector addition $u \oplus_E v$.

Let us compute the (1,2)-component of $\rho_u \odot \rho_v$. Then we have

$$\begin{align*}
\frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (1 + v_3) + (u_1 - iu_2)(v_1 + iv_2) \right\} (u_1 - iu_2) \\
+ \frac{x}{8} \left\{ \left( u_3 + \frac{1}{x} \right) (v_1 - iv_2) + (u_1 - iu_2)(1 - v_3) \right\} \left( -u_3 + \frac{1}{x} \right)
\end{align*}$$

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The real part of the (1,2)-component for $\rho_u \oplus \rho_v$ is
\[
\frac{x}{8} \left[ \left( u_3 + \frac{1}{x} \right) (1 + v_3) u_1 + (u_1^2 - u_2^2) v_1 + 2u_1 u_2 v_2 \right] \\
+ \frac{x}{8} \left[ \left( u_3 + \frac{1}{x} \right) v_1 \left( -u_3 + \frac{1}{x} \right) + u_1(1 - v_3) \left( -u_3 + \frac{1}{x} \right) \right] \\
= \frac{x}{8} \left\{ 2u_1 \left( u_3 v_3 + \frac{1}{x} \right) + (u_1^2 - u_2^2 - u_3^2)v_1 + 2u_1 u_2 v_2 + \frac{1}{x^2} v_1 \right\} \\
= \frac{x}{8} \left\{ \frac{2}{x} u_1 + 2(u \cdot v) u_1 - \|u\|^2 v_1 + \frac{1}{x^2} v_1 \right\} \\
= \frac{1}{4} \left( u_1 + \frac{\gamma_u}{\gamma_u + 1} (u \cdot v) u_1 \right).
\]
So from Lemma 7.160, the real part of the (1,2)-component for $\rho_u \circ \rho_v$ is
\[
\left( \frac{2}{1 + u \cdot v} \right) \frac{1}{4} \left( u_1 + \frac{\gamma_u}{\gamma_u + 1} (u \cdot v) u_1 \right) = \frac{1}{2} (u \oplus_E v)_1.
\]

The imaginary part of the (1,2)-component for $\rho_u \oplus \rho_v$ is
\[
\frac{x}{8} \left[ - \left( u_3 + \frac{1}{x} \right) (1 + v_3) u_2 + (u_1^2 - u_2^2) v_2 - 2u_1 u_2 v_1 \right] \\
+ \frac{x}{8} \left[ - \left( u_3 + \frac{1}{x} \right) v_2 \left( -u_3 + \frac{1}{x} \right) - u_2(1 - v_3) \left( -u_3 + \frac{1}{x} \right) \right] \\
= \frac{x}{8} \left\{ -2u_2 \left( u_3 v_3 + \frac{1}{x} \right) + (u_1^2 - u_2^2 + u_3^2)v_2 - 2u_1 u_2 v_2 - \frac{1}{x^2} v_2 \right\} \\
= \frac{x}{8} \left\{ -\frac{2}{x} u_2 - 2(u \cdot v) u_2 + \|u\|^2 v_2 - \frac{1}{x^2} v_2 \right\} \\
= -\frac{1}{4} \left( u_2 + \frac{\gamma_u}{\gamma_u} v_2 + \frac{\gamma_u}{\gamma_u + 1} (u \cdot v) u_2 \right).
\]
So from Lemma 7.160, the imaginary part of the (1,2)-component for $\rho_u \circ \rho_v$ is
\[
- \left( \frac{2}{1 + u \cdot v} \right) \frac{1}{4} \left( u_2 + \frac{\gamma_u}{\gamma_u + 1} (u \cdot v) u_2 \right) = -\frac{1}{2} (u \oplus_E v)_2.
\]

Therefore, we conclude
\[
\rho_u \circ \rho_v = \frac{1}{2} \begin{pmatrix}
1 + (u \oplus_E v)_3 & (u \oplus_E v)_1 - i(u \oplus_E v)_2 \\
(u \oplus_E v)_1 + i(u \oplus_E v)_2 & 1 - (u \oplus_E v)_3
\end{pmatrix} = \rho_{(u \circledast_E v)}.
\]

Now it remains to show that
\[
\rho_{u \circledast v} = t \circ \rho_v = \frac{1}{\text{tr}(\rho_v)} \rho_v.
\]

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for any \( t \in \mathbb{R} \). Set
\[
T = \{ t \in \mathbb{R} : \rho_{t \otimes v} = \frac{1}{\text{tr}(\rho_v^t)} \rho_v^t \text{ for any } v \in \mathbb{B} \}
\]

We show that the set \( T \) contains all dyadic rational numbers.

Easily \( 0, 1 \in T \). Moreover, \( \frac{1}{2} \in T \). Indeed,
\[
\frac{1}{2} \otimes v = \frac{\gamma v}{\gamma + 1} v.
\]

So we obtain from Corollary 7.162 and Lemma 7.163
\[
\rho_{\frac{1}{2} \otimes v} = \frac{1}{2} \left( \begin{array}{cc}
1 + \frac{\gamma v}{\gamma + 1} v_3 & \frac{\gamma v}{\gamma + 1} (v_1 - iv_2) \\
\frac{\gamma v}{\gamma + 1} (v_1 + iv_2) & 1 - \frac{\gamma v}{\gamma + 1} v_3
\end{array} \right)
\]
\[
= \frac{1}{2} \frac{\gamma v}{\gamma + 1} \left( \begin{array}{cc}
1 + v_3 + \frac{1}{\gamma v} & v_1 - iv_2 \\
v_1 + iv_2 & 1 - v_3 + \frac{1}{\gamma v}
\end{array} \right)
\]
\[
= \frac{1}{\text{tr}(\rho_v^{1/2})} \rho_v^{1/2}.
\]

This gives us that \( \frac{t}{2} \in T \) whenever \( t \in T \).

From \( \rho_{u \otimes v} = \rho_u \otimes \rho_v \) we have
\[
\rho_{2 \otimes v} = \frac{1}{\text{tr}(\rho_v^2)} \rho_v^2 \quad \text{and} \quad \rho_{(-1) \otimes v} = \frac{1}{\text{tr}(\rho_v^{-1})} \rho_v^{-1}.
\]

That is, \( 2t \in T \) and \( -t \in T \) whenever \( t \in T \). Then for \( s, t \in T \)
\[
\rho_{(s \cdot t) \otimes v} = \rho_{(2s - t) \otimes v}
\]
\[
= \rho_{(2s) \otimes v \oplus_E (-t) \otimes v}
\]
\[
= \rho_{(2s) \otimes v} \odot \rho_{(-t) \otimes v}
\]
\[
= (2s) \odot \rho_v \odot (-t) \odot \rho_v
\]
\[
= (2s - t) \odot \rho_v = (s \cdot t) \odot \rho_v.
\]

In other words, \( s \cdot t = 2s - t \in T \) whenever \( s, t \in T \).

So the set \( T \) contains all dyadic rational numbers in \( \mathbb{R} \). This implies by the density of dyadic rational numbers that \( T = \mathbb{R} \). \( \square \)
7.3 Lorentz Boosts and Qubits

In Remark 7.164 we have seen the Lorentz boost generated by a gyrovector. We show in this section an isomorphism between the unit ball $\mathbb{B}$ of Bloch vectors with Einstein vector addition and a linear group of Lorentz boosts with certain operation.

In physics and mathematics, Minkowski space (or Minkowski spacetime) is the standard mathematical setting for Einstein’s theory of special relativity. In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional space for representing a spacetime. Minkowski space is introduced in 1908.

Minkowski spacetime is defined to be the vector space $\mathbb{R}^4$ equipped with the symmetric bilinear form, called the Lorentzian form of the spacetime,

$$
\eta([t \, \mathbf{x}],[t' \, \mathbf{x}']) = -tt' + \mathbf{x} \cdot \mathbf{x}'
$$

for $t, t' \in \mathbb{R}$ and $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$. A linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ is said to preserve the form $\eta$ if and only if

$$
\eta(A\mathbf{x},A\mathbf{y}) = \eta(\mathbf{x},\mathbf{y})
$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$.

**Remark 7.166.** A linear transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ preserves the form $\eta$ if and only if

$$
I_{1,3} = A^T I_{1,3} A,
$$

where

$$
I_{1,3} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$
The invertible linear transformations on $\mathbb{R}^4$ that preserve $\eta$ form a group under composition, usually referred to as the generalized orthogonal group $O(1,3)$. The Lorentz group is defined to be the subgroup of $O(1,3)$ of time-preserving or orthochronous $\eta$ preserving invertible linear transformations. One characterization of time preservation is that the vector $(1,0,0,0) \in \mathbb{R}^4$ is carried into a vector with a positive time component. We shall call such transformations Lorentz transformations or linear isometries of Minkowski space. We denote the Lorentzian group by $O^+(1,3)$.

**Lemma 7.167.** Let $P$ be a positive definite matrix preserving the Lorentzian form. Then $P$ and all of its powers $P^t$, $t \in \mathbb{R}$, are Lorentz transformations.

**Proof.** By the spectral theorem there exists an orthogonal matrix $U$ such that $P = U^T D U$, where $D = (d_{ij})$ is a diagonal matrix with positive diagonal entries. By Remark 7.166 we have $P I_{1,3} P = I_{1,3}$. Thus

$$D(U I_{1,3} U^T) D = U I_{1,3} U^T.$$ 

Setting $B = U I_{1,3} U^T$ gives us $DBD = B$. The only way this can happen for a diagonal matrix $D$ is that $d_{ii}d_{jj} = 1$ whenever $b_{ij} \neq 0$. It then follows for $t \in \mathbb{R}$ that $d_{ii}^t d_{jj}^t = 1$ whenever $b_{ij} \neq 0$, and thus $D^t B D^t = B$. Reversing our earlier argument, we conclude that $P^t = U^T D^t U$ preserves the Lorentzian form.

Let $t$ denote the unit vector in the time direction in Minkowski space. Since all of $P^t$ preserve the Lorentzian form, they must carry the vector $t$ into the open positive cone of time-like or its negative cone. Since the map from $\mathbb{R}$ to $\mathbb{R}$ given by $t \mapsto \pi_1(P^t(t))$, where $\pi_1$ is the projection into the time coordinate, is continuous and takes the value 1 at $t = 0$, we conclude by the Intermediate Value Theorem that it takes on only positive values. Thus, all of $P^t$ are Lorentz transformations. \qed
Proposition 7.168. The polar factors $P$ and $U$ of the polar decomposition $A = P \cdot U$ of a Lorentz transformation $A$ are also Lorentz transformations.

Proof. If $A \in O(1,3)$, then $A^T \in O(1,3)$, and so, $AA^T \in O(1,3)$. Since $AA^T$ is positive definite, it follows from Lemma 7.167 that $P = (AA^T)^{1/2}$ is a Lorentz transformation. Thus, $U = P^{-1}A$ is also a Lorentz transformation.

We see the characterization of an orthogonal matrix $U$ that is a Lorentz transformation. The proof follows directly from the equation

$$U^T I_{1,3} U = I_{1,3},$$

or equivalently, $I_{1,3} U = U I_{1,3}$.

Proposition 7.169. An orthogonal matrix $U$ is a Lorentz transformation if and only if it has the block form

$$U = \begin{pmatrix} 1 & 0 \\ 0^T & S \end{pmatrix},$$

where $S \in O(3)$, an orthogonal matrix on $\mathbb{R}^3$.

We now show the interesting result that the positive definite Lorentz transformations are precisely the Lorentz boosts. Normalizing the speed of light $s$, we consider an open unit ball in $\mathbb{R}^3$,

$$\mathbb{B} = \{ v \in \mathbb{R}^3 : \|v\| < 1 \}.$$

Let $A$ be the Lorentz boost generated by a vector $v \in \mathbb{B}$. Then

$$A = B(v) = \begin{pmatrix} \gamma_v & \gamma_v v \\ \gamma_v v^T & I + \frac{s^2}{1 + \gamma_v} v^T v \end{pmatrix}.$$

It is easy to see that $A$ is a symmetric Lorentz transformation. Moreover, its eigenvalues are
1, $\sqrt{\frac{1 + \|v\|}{1 - \|v\|}}$ and $\sqrt{\frac{1 - \|v\|}{1 + \|v\|}}$ which are all positive. So it is a positive definite Lorentz transformation.

Let us write an arbitrary positive definite Lorentz transformation $A$ in the form

$$A = \begin{pmatrix} r & x \\ x^T & S \end{pmatrix},$$

where $r$ is a positive scalar since $A$ is a Lorentz transformation, $S$ is a $3 \times 3$ positive definite matrix since $A$ is positive definite, and $x \in \mathbb{R}^3$ is a row vector.

From Remark 7.166 we have that

$$\begin{pmatrix} r & x \\ x^T & S \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0^T & I \end{pmatrix} \begin{pmatrix} r & x \\ x^T & S \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0^T & I \end{pmatrix}.$$

If we multiply out the left-hand side and set the $(1,1)$ entries equal on both sides of the equation, we obtain $-r^2 + xx^T = -1$, which implies $r = \sqrt{1 + \|x\|^2}$. We set

$$v := \frac{1}{\sqrt{1 + \|x\|^2}} x.$$

Then

$$\|v\|^2 = \frac{\|x\|^2}{1 + \|x\|^2} < 1,$$

and hence, $v \in \mathbb{B}$. We next compute

$$\gamma_v = \frac{1}{\sqrt{1 - \|v\|^2}} = \frac{1}{\sqrt{1 - \frac{\|x\|^2}{1 + \|x\|^2}}} = \sqrt{1 + \|x\|^2} = r.$$

Thus we obtain

$$A = \begin{pmatrix} \gamma_v & \gamma_v v \\ \gamma_v v^T & S \end{pmatrix}.$$
By equating the (2,2) entries in the equation $AI_{1,3}A = I_{1,3}$, we obtain the following string of equivalent equalities:

$$-\gamma^2_v v^T v + S^2 = I$$
$$S^2 = I + \gamma^2_v v^T v$$
$$S = \left( I + \gamma^2_v v^T v \right)^{\frac{1}{2}}.$$

The last equality follows from the fact that $S$ is positive definite, and so is $S^2$. Thus, it has unique square root $S$.

We now calculate

$$\left( I + \frac{\gamma^2_v}{1 + \gamma_v} v^T v \right)^2 = \left( I + \gamma_v - \frac{1}{\|v\|^2} v^T v \right)^2$$
$$= I + 2\gamma_v - \frac{1}{\|v\|^2} v^T v + \frac{(\gamma_v - 1)^2}{\|v\|^4} v^T v v^T v$$
$$= I + \frac{2\gamma_v - 2}{\|v\|^2} v^T v + \frac{\gamma_v^2 - 2\gamma_v + 1}{\|v\|^2} v^T v$$
$$= I + \frac{\gamma_v^2 - 1}{\|v\|^2} v^T v$$
$$= I + \gamma^2_v v^T v.$$

The first and last equalities follow from the Lorentz gamma identity (7.17). Thus,

$$A = \begin{pmatrix} 
\gamma_v & \gamma_v v \\
\gamma_v v^T & I + \frac{\gamma^2_v}{1 + \gamma_v} v^T v 
\end{pmatrix} = B(v).$$

We have established

**Proposition 7.170.** A positive definite Lorentz transformation is a Lorentz boost.

This is a very close and useful connection between the Einstein vector addition, Lorentz boosts, and polar decompositions, which we develop in this section.

**Proposition 7.171.** For any $u, v \in \mathbb{B}$,

$$B(u)B(v) = B(u \oplus_E v)U(u, v),$$

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where the right hand side is the polar decomposition of the left hand side in the Lorentz group $O^+(1,3)$.

**Proof.** We note from Proposition 7.169 that the first column of the product

$$B(u)B(v) = \begin{pmatrix} \gamma_u & \gamma_u u \\ \gamma_u u^T & I + \frac{\gamma_u^2}{1+\gamma_u} uu^T \end{pmatrix} \begin{pmatrix} \gamma_v & \gamma_v v \\ \gamma_v v^T & I + \frac{\gamma_v^2}{1+\gamma_v} vv^T \end{pmatrix}$$

must be the first column of the Lorentz boost $P$ in the polar decomposition $P \cdot U$ of $B(u)B(v)$. Since a matrix

$$P = \begin{pmatrix} r & x \\ x^T & S \end{pmatrix}$$

is the Lorentz boost generated by $w = \frac{1}{r}x$, we conclude that the matrix $P$ in the polar decomposition of $B(u)B(v)$ is the Lorentz boost for the vector

$$\frac{1}{\gamma_u \gamma_v (1+u \cdot v)} \left( \gamma_u \gamma_v v + \frac{\gamma_u^2 \gamma_v}{1+\gamma_u} (u \cdot v) u \right)$$

$$= \frac{1}{1+u \cdot v} \left( u + \frac{1}{\gamma_u} v + \frac{\gamma_u}{1+\gamma_u} (u \cdot v) u \right)$$

$$= u \oplus_E v.$$ 

Thus, $B(u)B(v) = B(u \oplus_E v)U(u,v)$, where we define $U(u,v)$ to be a unitary part $U$. \hfill \square

We now show the main result, an isomorphism between the open unit ball $\mathbb{B}$ with Einstein vector addition and the set $L$ of Lorentz boosts with the operation

$$B(u) \otimes B(v) := (B(u)B(v)^2B(u))^\frac{1}{2}.$$ 

**Theorem 7.172.** The map from $(\mathbb{B}, \oplus_E)$ to $(L, \otimes)$ defined by

$$v \mapsto B(v)$$

is an isomorphism.
Proof. By definition it is easy to see that the map \( v \mapsto B(v) \) is a bijection. By Proposition 7.171 and the uniqueness of the polar decomposition of \( B(u)B(v) \) in \( O^+(1,3) \), we have
\[
B(u \oplus_E v) = (B(u)B(v)^2B(u))^{\frac{1}{2}} = B(u) \oplus B(v).
\]
Thus the map \( v \mapsto B(v) \) is an isomorphism. \( \square \)

Remark 7.173. It is straightforward to verify that an isomorphism of groupoids preserves all properties of Definition 2.26, and hence, \((L, \otimes)\) is a gyrocommutative gyrogroup.

From Theorem 7.165 we have seen that the map \( v \mapsto \rho_v \) is an isomorphism. Since the inverse of isomorphism is also an isomorphism, we have the following.

Corollary 7.174. Three gyrocommutative gyrogroups \((\mathbb{B}, \oplus_E)\), \((\mathbb{D}_2, \odot)\), and \((L, \otimes)\) are isomorphic each other.

### 7.4 Bures Fidelity

In this section we study two important distance measures in quantum information and computation theory, the trace distance and the Bures fidelity.

The trace distance between two density matrices \( A \) and \( B \) is given by
\[
D(A, B) := \frac{1}{2} \text{tr}|A - B|,
\]
where \(|X| := (XX^*)^{\frac{1}{2}}\) means a positive part of the polar decomposition for any matrix \( X \). One can easily obtain
\[
D(\rho_u, \rho_v) = \frac{1}{2} \|u - v\|
\]
for any Bloch vectors \( u, v \in \mathbb{B} \). It has a simple geometric interpretation as a half of the Euclidean distance between two Bloch vectors.
We now introduce the Bures fidelity between two density matrices and provide its geometric observation.

**Definition 7.175.** The Bures fidelity between two invertible density matrices \(A\) and \(B\) is defined by

\[
F(A, B) = \left( \text{tr}(A \oplus B)^{1/2} \right)^2.
\]

We note that this definition extends to all of density matrices.

**Remark 7.176.** In the proofs of Theorem 2.52 and Proposition 7.157 we have seen that the gyroautomorphism \(\text{gyr}[A, B]\) preserves square roots and also preserves the trace. So we have

\[
\text{tr}(A \oplus B)^{1/2} = \text{tr}(\text{gyr}[A, B](B \oplus A))^{1/2} = \text{tr}(\text{gyr}[A, B](B \oplus A)^{1/2}) = \text{tr}(B \oplus A)^{1/2}.
\]

Thus, \(F(A, B) = F(B, A)\) for any \(A, B \in \mathcal{D}\). By continuity it is also symmetric for any two density matrices.

We show the equivalent formulas of the Bures Fidelity between two qubits.

**Proposition 7.177.** For any \(u, v \in (\mathbb{B}, \oplus_E)\), the Bures fidelity between two qubits \(\rho_u\) and \(\rho_v\) is

\[
F(\rho_u, \rho_v) = \frac{\gamma_{u \oplus_E v} + 1}{2\gamma_u\gamma_v}.
\]

**Proof.** From Theorem 7.165 we have

\[
\rho_u \oplus \rho_v = \text{tr}(\rho_u \oplus \rho_v)\rho_u \oplus \rho_v
\]

\[
= \text{tr}(\rho_u \oplus \rho_v)\rho_{u \oplus_E v}
\]

\[
= 1 + \frac{u \cdot v}{2} \rho_{u \oplus_E v}.
\]
Then by Lemma 7.163

$$\sqrt{\frac{1 + \mathbf{u} \cdot \mathbf{v}}{2}} \frac{\text{tr}(\rho_{u \oplus \rho_v})}{2} = \sqrt{1 + \mathbf{u} \cdot \mathbf{v}} \frac{\text{tr}(\rho_{u \oplus E v})}{2},$$

Since the Einstein vector addition satisfies the mutually equivalent Lorentz factor identity

$$\gamma_{u \oplus v} = \gamma_u \gamma_v (1 + \mathbf{u} \cdot \mathbf{v}),$$

we conclude

$$\left(\text{tr}(\rho_{u \oplus \rho_v})\right)^2 = \frac{1 + \mathbf{u} \cdot \mathbf{v}}{2} \left(\frac{\gamma_{u \oplus v} + 1}{\gamma_{u \oplus v}}\right)$$

$$= \frac{1 + \mathbf{u} \cdot \mathbf{v}}{2} \left(\frac{\gamma_{u \oplus v} + 1}{\gamma_u \gamma_v (1 + \mathbf{u} \cdot \mathbf{v})}\right)$$

$$= \frac{\gamma_{u \oplus v} + 1}{2 \gamma_u \gamma_v}. \quad \square$$

Expressing the magnitude of the Bloch vector $\mathbf{v}$ by the hyperbolic parameter $\phi_v$, called a rapidity,

$$\phi_v = \tanh^{-1} ||\mathbf{v}||$$

we have

$$\cosh \phi_v = \gamma_v.$$

By the half-angle formula of hyperbolic cosine functions

$$\cosh^2 \alpha = \frac{1 + \cosh 2 \alpha}{2},$$

we have

$$\cosh^2 \phi_w = \frac{1 + \cosh \phi_w}{2} = \frac{1 + \gamma_{u \oplus v}}{2},$$

where $\mathbf{w} = u \oplus_E \mathbf{v}$. Hence, from Proposition 7.177 we obtain the following.
Proposition 7.178. For any \( u, v \in (\mathbb{B}, \oplus_E) \), the Bures fidelity between two qubits \( \rho_u \) and \( \rho_v \) is

\[
F(\rho_u, \rho_v) = \frac{1 + \cosh \phi_w}{2 \cosh \phi_u \cosh \phi_v} = \frac{\cosh^2 \frac{\phi_w}{2}}{\cosh \phi_u \cosh \phi_v},
\]

where \( w = u \oplus_E v \in \mathbb{B} \).

The Bures fidelity between two qubits \( \rho_u \) and \( \rho_v \) is related to a hyperbolic triangle \( \triangle ABC \) in the above figure formed by three hyperbolic sides \( \phi_u, \phi_v, \) and \( \phi_w \), where \( M \) is the midpoint of the side \( BC \). It is a multiplication of the ratio \( \frac{\cosh \frac{\phi_w}{2}}{\cosh \phi_u} \) and \( \frac{\cosh \frac{\phi_w}{2}}{\cosh \phi_v} \).
References


Vita

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