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Standard Quantum Limit for Probing Mechanical Energy Quantization

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We derive a standard quantum limit for probing mechanical energy quantization in a class of systems with mechanical modes parametrically coupled to external degrees of freedom. To resolve a single mechanical quantum, it requires a strong-coupling regime—the decay rate of external degrees of freedom is smaller than the parametric coupling rate. In the case for cavity-assisted optomechanical systems, e.g., the one proposed by Thompson *et al.* [Nature (London) **452**, 72 (2008)], zero-point motion of the mechanical oscillator needs to be comparable to the linear dynamical range of the optical system which is characterized by the optical wavelength divided by the cavity finesse.

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Introduction.—Recently, significant cooling of mechanical modes of harmonic oscillators has been achieved by extracting heat through parametric damping or active feedback [1,2]. Theoretical calculations suggest that oscillators with a large thermal occupation number ($k_B T \gg \hbar\omega_m$) can be cooled to be close to their ground state, if they have high enough quality factors [3]. Once the ground state is approached, many interesting studies of macroscopic quantum mechanics can be performed, e.g., teleporting a quantum state onto mechanical degrees of freedom [4] and creating quantum entanglement between a cavity mode and an oscillator [5] and between two macroscopic test masses [6]. Most proposals involve the oscillator position linearly coupled to photons, in which case the quantum features of the oscillator, to a great extent, are attributable to the quantization of photons. In order to probe the intrinsic quantum nature of an oscillator, one of the most transparent approaches is to directly measure its energy quantization, and quantum jumps between discrete energy eigenstates. Since linear couplings alone will not project an oscillator onto its energy eigenstates, nonlinearities are generally required [7–9]. For cavity-assisted optomechanical systems, one experimental scheme, proposed in the pioneering work of Thompson *et al.* [1], is to place a dielectric membrane inside a high-finesse Fabry-Perot cavity, forming a pair of coupled cavities [10]. If the membrane is appropriately located, a dispersive coupling between the membrane position and the optical field is predominantly quadratic, allowing the detection of mechanical energy quantization.

In this Letter, we show that in the experimental setup of Thompson *et al.* the optical field also couples linearly to the membrane. Because of finiteness of cavity finesse

(either intentional for readout or due to optical losses), this linear coupling introduces quantum backaction. Interestingly, it sets forth a simple standard quantum limit, which dictates that only those systems whose cavity-mode decay rates are smaller than the optomechanical coupling rate can successfully resolve energy levels. We will further show that a similar constraint applies universally to all experiments that attempt to probe mechanical energy quantization via parametric coupling with external degrees of freedom (either optical or electrical).

Coupled cavities.—Optical configuration of coupled cavities is shown in Fig. 1. Given the specification in Ref. [1], transmissivities of the membrane and end mirrors are quite low, and thus a two-mode description is appropriate [11,12], with the corresponding Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}} = & \hbar\omega_m(\hat{q}^2 + \hat{p}^2)/2 + \hbar\omega_0(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \\ & - \hbar\omega_s(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) + \hbar G_0 \hat{q}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \\ & + \hat{\mathcal{H}}_{\text{ext}} + \hat{\mathcal{H}}_{\xi}. \end{aligned} \quad (1)$$

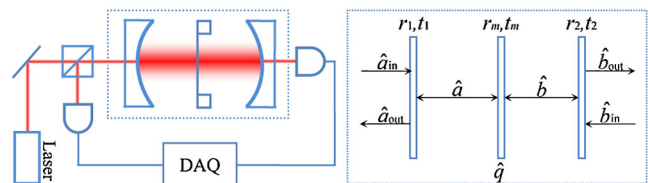


FIG. 1 (color online). The left panel presents the schematic configuration of coupled cavities in the proposed experiment [1]. The right panel shows optical modes, and we denote reflectivity and transmissivity of the optical elements by r_i and t_i ($i = 1, 2, m$). DAQ stands for data acquisition.

Here \hat{q} , \hat{p} are normalized position and momentum of the membrane; \hat{a} , \hat{b} are annihilation operators of cavity modes in the individual cavities (both resonate at ω_0); $\omega_s \equiv t_m c/L$ is the optical coupling constant for \hat{a} and \hat{b} , through transmission of the membrane [12]; $G_0 \equiv 2\sqrt{2}\omega_0 x_q/L$ is the optomechanical coupling constant with L denoting the cavity length and zero-point motion $x_q \equiv \sqrt{\hbar/(2m\omega_m)}$; $\hat{\mathcal{H}}_{\text{ext}}$ and $\hat{\mathcal{H}}_\xi$ correspond to the coupling of the system to the environment and quantify the fluctuation and dissipation mechanism. By introducing optical normal modes, namely, the common mode $\hat{c} \equiv (\hat{a} + \hat{b})/\sqrt{2}$ and differential mode $\hat{d} \equiv (\hat{a} - \hat{b})/\sqrt{2}$,

$$\hat{\mathcal{H}}/\hbar = \frac{\omega_m}{2}(\hat{q}^2 + \hat{p}^2) + \omega_- \hat{c}^\dagger \hat{c} + \omega_+ \hat{d}^\dagger \hat{d} + G_0 \hat{q}(\hat{c}^\dagger \hat{d} + \hat{d}^\dagger \hat{c}) + i(\sqrt{2}\gamma_c \hat{c}^\dagger \hat{c}_{\text{in}} + \sqrt{2}\gamma_d \hat{d}^\dagger \hat{d}_{\text{in}} - \text{H.c.}) + \hat{\mathcal{H}}_\xi/\hbar, \quad (2)$$

where $\omega_\pm \equiv \omega_0 \pm \omega_s$ and in the Markovian approximation $\hat{\mathcal{H}}_{\text{ext}}$ is written out explicitly in the second line (with $\gamma_{c,d}$ denoting decay rates and H.c. for Hermitian conjugate).

Before analyzing the detailed dynamics, here we follow Thompson *et al.* [1] and Bhattacharya and Meystre [11] by assuming $\omega_m \ll \omega_s$ and $G_0 \ll |\omega_+ - \omega_-| = 2\omega_s$, analogous to the dispersive regime in the photon-number counting experiment with a superconducting qubit [13,14]. This allows us to treat $\hbar G_0 \hat{q}(\hat{c}^\dagger \hat{d} + \hat{d}^\dagger \hat{c})$ as a perturbation and diagonalize the Hamiltonian formally. Up to $G_0^2/(2\omega_s)^2$, the optical and optomechanical coupling parts of the original Hamiltonian can be written as

$$\hat{\mathcal{H}}/\hbar = \left(\omega_- - \frac{G_0^2 \hat{q}^2}{2\omega_s}\right) \hat{c}^\dagger \hat{c} + \left(\omega_+ + \frac{G_0^2 \hat{q}^2}{2\omega_s}\right) \hat{d}^\dagger \hat{d}. \quad (3)$$

At first sight, frequency shift of the eigenmodes \hat{c} and \hat{d} is proportional to \hat{q}^2 . Since frequency separation of two normal modes is $2\omega_s \gg \gamma_{c,d}$, they can be independently driven and detected. Besides, with $\gamma_{c,d} < \omega_m$, only averaged membrane motion is registered, and $\hat{q}^2 = \hat{N} + 1/2$ with \hat{N} denoting the number of quanta. Therefore, previous authors had concluded that such a purely dispersive coupling allows quantum nondemolition measurements of the mechanical quanta.

However, the new eigenmodes \hat{o} and \hat{e} are given by

$$\hat{o} = \hat{c} - [(G_0 \hat{d})/(2\omega_s)] \hat{q}, \quad \hat{e} = \hat{d} + [(G_0 \hat{c})/(2\omega_s)] \hat{q}. \quad (4)$$

If we pump \hat{c} with classical amplitude \bar{c} and left \hat{d} in a vacuum state, the detected mode \hat{o} will have a negligible linear response. However, the idle mode $\hat{e} \approx [G_0 \bar{c}/(2\omega_s)] \hat{q}$, which is dominated by linear coupling. If we choose to drive \hat{d} , the role of \hat{o} and \hat{e} will simply swap. Such linear coupling can potentially demolish the energy eigenstates that we wish to probe. We can make an order-of-magnitude estimate. The optomechanical coupling term in Eq. (2), at the linear order, reads $G_0 \hat{q}(\bar{c} \hat{d} + \bar{c}^* \hat{d}^\dagger)$.

According to the Fermi's golden rule, it causes decoherence of the energy eigenstate near the ground level at a rate of

$$\tau_{\text{dec}}^{-1} = G_0^2 |\bar{c}|^2 \tilde{S}_{\hat{d}}(-\omega_m) \approx G_0^2 |\bar{c}|^2 \gamma_d / (2\omega_s^2), \quad (5)$$

where we have assumed that \hat{c} is on resonance, and

$$\tilde{S}_{\hat{d}} \equiv \int dt e^{i\omega t} \langle \hat{d}(t) \hat{d}^\dagger(0) \rangle = 2\gamma_d / [(\omega - 2\omega_s)^2 + \gamma_d^2]. \quad (6)$$

On the other hand, from Eq. (3) and linear response theory [15], the measurement time scale to resolve the energy eigenstate (i.e., measuring \hat{N} with a unit error) with a shot-noise limited sensitivity is approximately given by

$$\tau_{\text{mea}} \approx [\gamma_c^2 \omega_s^2 / (G_0^4 |\bar{c}|^2)] \tilde{S}_{\hat{c}}(0) = 2\omega_s^2 \gamma_c / (G_0^4 |\bar{c}|^2), \quad (7)$$

where $\tilde{S}_{\hat{c}}(0)$ is the spectral density of \hat{c} at zero frequency. Requiring $\tau_{\text{mea}} \leq \tau_{\text{dec}}$ yields

$$(\gamma_c \gamma_d / G_0^2) \leq 1. \quad (8)$$

In the case when transmissivity of end mirrors $t_1 = t_2 \equiv t_0$, we have $\gamma_c = \gamma_d = ct_0^2/(2L)$. Defining the cavity finesse as $\mathcal{F} \equiv \pi/t_0^2$, the above inequality reduces to $\lambda/(\mathcal{F}x_q) \lesssim 8\sqrt{2}$. Therefore, to probe mechanical energy quantization, it requires a strong-coupling regime [cf. Eq. (8)], or equivalently, for such an optomechanical system, zero-point mechanical motion x_q to be comparable to linear dynamical range λ/\mathcal{F} of the cavity.

We now carry out a detailed analysis of the dynamics according to the standard input-output formalism [16]. In the rotating frame at the laser frequency ω_+ , the nonlinear quantum Langevin equations are given by

$$\dot{\hat{q}} = \omega_m \hat{p}, \quad (9)$$

$$\dot{\hat{p}} = -\omega_m \hat{q} - \gamma_m \hat{p} - G_0(\hat{c}^\dagger \hat{d} + \hat{d}^\dagger \hat{c}) + \xi_{\text{th}}, \quad (10)$$

$$\dot{\hat{c}} = -\gamma_c \hat{c} - iG_0 \hat{q} \hat{d} + \sqrt{2}\gamma_c \hat{c}_{\text{in}}, \quad (11)$$

$$\dot{\hat{d}} = -(\gamma_d + 2i\omega_s) \hat{d} - iG_0 \hat{q} \hat{c} + \sqrt{2}\gamma_d \hat{d}_{\text{in}}. \quad (12)$$

Here the mechanical damping and associated Brownian thermal force ξ_{th} originate from $\hat{\mathcal{H}}_\xi$ under the Markovian approximation. These equations can be solved perturbatively by decomposing every Heisenberg operator $\hat{\alpha}$ into different orders such that $\hat{\alpha} = \bar{\alpha} + \epsilon \hat{\alpha}^{(1)} + \epsilon^2 \hat{\alpha}^{(2)} + \mathcal{O}[\epsilon^3]$. We treat $G_0/(2\omega_s)$ and vacuum fluctuations $\sqrt{2}\gamma_c \hat{c}_{\text{in}}^{(1)}$ and $\sqrt{2}\gamma_d \hat{d}_{\text{in}}^{(1)}$ (simply denoted by $\sqrt{2}\gamma_c \hat{c}_{\text{in}}$ and $\sqrt{2}\gamma_d \hat{d}_{\text{in}}$ in later discussion) as being of the order of ϵ ($\ll 1$).

To the zeroth order, $\bar{c} = \sqrt{2I_0}/(\gamma_c \hbar \omega_0)$ with I_0 denoting the input optical power and $\bar{d} = 0$. Up to the first order, the radiation pressure term reads $G_0 \bar{c} [\hat{d}^{(1)} + \hat{d}^{(1)\dagger}]$ (\bar{c} is set to be real by choosing an appropriate phase reference). In the frequency domain, it can be written as

$$\tilde{F}_{\text{rp}} = \frac{2\sqrt{\gamma_d}G_0\tilde{c}[(\gamma_d - i\omega)\tilde{v}_1 - 2\omega_s\tilde{v}_2] + 4G_0^2\tilde{c}^2\omega_s\tilde{q}}{(\omega + 2\omega_s + i\gamma_d)(\omega - 2\omega_s + i\gamma_d)}, \quad (13)$$

where \tilde{v}_1 , \tilde{v}_2 , and \tilde{q} are Fourier transformations of $\hat{v}_1(t) \equiv (\hat{d}_{\text{in}} + \hat{d}_{\text{in}}^\dagger)/\sqrt{2}$, $\hat{v}_2(t) \equiv (\hat{d}_{\text{in}} - \hat{d}_{\text{in}}^\dagger)/(i\sqrt{2})$, and $\hat{q}(t)$, respectively. The part, containing vacuum fluctuations, is the backaction \hat{F}_{BA} , which induces the quantum limit. The other part proportional to \tilde{q} is the optical-spring effect. Within the time scale for measuring energy quantization, of the order of $\gamma_c^{-1} (\ll \gamma_m^{-1})$, the positive damping can be neglected, but the negative rigidity has an interesting consequence—it modifies ω_m to an effective $\omega_{\text{eff}} (< \omega_m)$. Correspondingly, the position of the high- Q membrane is

$$\hat{q}(t) = \hat{q}_m + \Lambda^2 \int_0^t dt' \sin\omega_{\text{eff}}(t-t')[\hat{F}_{\text{BA}}(t') + \xi_{\text{th}}(t')] \quad (14)$$

with $\Lambda \equiv \sqrt{\omega_m/\omega_{\text{eff}}}$. The free quantum oscillation $\hat{q}_m = \Lambda(\hat{q}_0 \cos\omega_{\text{eff}}t + \hat{p}_0 \sin\omega_{\text{eff}}t)$ and \hat{q}_0 and \hat{p}_0 are the initial position and momentum normalized with respect to $\sqrt{\hbar/(m\omega_{\text{eff}})}$ and $\sqrt{\hbar m\omega_{\text{eff}}}$.

The dispersive response is given by the second-order perturbation $\mathcal{O}[\epsilon^2]$. Adiabatically eliminating rapidly oscillating components and assuming $\omega_m \ll \omega_s$, which can be shown to maximize the signal-to-noise ratio, we obtain

$$\begin{aligned} \hat{c}^{(2)}(t) &= -iG_0 \int_0^t dt' e^{-\gamma_c(t-t')} \hat{q}(t') \hat{d}^{(1)}(t') \\ &\approx G_{\text{eff}}^2 \tilde{c} \hat{N}(t) / (2i\gamma_c \omega_s). \end{aligned} \quad (15)$$

Here $G_{\text{eff}} \equiv \Lambda G_0$, and $\hat{N}(t) \equiv \hat{N}_0 + \Delta\hat{N}(t)$ contains the number of mechanical quanta $\hat{N}_0 \equiv (\hat{q}_0^2 + \hat{p}_0^2)/2$ and the noise term $\Delta\hat{N}(t)$ due to the backaction and thermal noise. To read out $\hat{N}(t)$, we integrate output phase quadrature for a duration τ . According to the input-output relation $\hat{c}_{\text{out}} + \hat{c}_{\text{in}} = \sqrt{2\gamma_c}\hat{c}$, the estimator reads

$$\hat{Y}(\tau) = \int_0^\tau dt [\hat{u}_2(t) - G_{\text{eff}}^2 \tilde{c} \hat{N}(t) / (\sqrt{\gamma_c} \omega_s)], \quad (16)$$

where $\hat{u}_2 \equiv (\hat{c}_{\text{in}} - \hat{c}_{\text{in}}^\dagger)/(i\sqrt{2})$. For the Gaussian and Markovian process, the correlation function $\langle \hat{c}_2(t) \hat{c}_2^\dagger(t') \rangle = \delta(t-t')/2$. For typical experiments, the thermal occupation number $\bar{n}_{\text{th}} \equiv k_B T / (\hbar\omega_m)$ is much larger than unity, and $\langle \xi_{\text{th}}(t) \xi_{\text{th}}^\dagger(t') \rangle \approx 2\gamma_m \bar{n}_{\text{th}} \delta(t-t')$. Through evaluating the four-point correlation function of backaction noise and $\xi_{\text{th}}(t)$ in $\langle \Delta\hat{N}(t) \Delta\hat{N}(t') \rangle$, we obtain the resolution ΔN as a function of τ

$$\Delta N^2 = \left(\frac{\gamma_c \omega_s^2}{G_{\text{eff}}^4 \tilde{c}^2 \tau} \right) + \frac{5}{6} \left(\frac{\gamma_d G_{\text{eff}}^2 \tilde{c}^2 \tau}{2\sqrt{2}\omega_s^2} \right)^2 + \frac{5}{6} \left(\frac{\gamma_m k_B T \tau}{\sqrt{2}\hbar\omega_{\text{eff}}} \right)^2. \quad (17)$$

In order to successfully observe energy quantization, the following conditions are simultaneously required: (i) the resolution ΔN^2 should have a minimum equal or less than

unity; (ii) this minimum should be reachable within τ that is longer than the cavity storage time $1/\gamma_c$ (which in turn must be longer than the oscillation period $1/\omega_{\text{eff}}$ of the membrane); (iii) the system dynamics should be stable when taking into account optical rigidity which is approximately equal to $G_0^2 \tilde{c}^2 / \omega_s$ for $\omega_m \ll \omega_s$.

Specifically, the standard quantum limit in condition (i), set by the first two terms in ΔN^2 , gives $\gamma_c \gamma_d / G_{\text{eff}}^2 \lesssim 1$, or equivalently $(\gamma_c \gamma_d / G_0^2) \lesssim \Lambda^2$. If we neglect the optical-spring effect ($\Lambda = 1$), we simply recover Eq. (8). A strong negative optical rigidity ($\omega_{\text{eff}} \ll \omega_m$, i. e., $\Lambda \gg 1$) can significantly enhance the effective coupling strength and ease the requirements on optomechanical properties. However, a small ω_{eff} also makes the system susceptible to the thermal noise. Taking into account all the above conditions, the optimal $\omega_{\text{eff}} = \omega_m \sqrt{\bar{n}_{\text{th}} / Q_m}$ with mechanical quality factor $Q_m \equiv \omega_m / \gamma_m$, and there is a nontrivial constraint on the thermal occupation number, which reads $(\bar{n}_{\text{th}} / Q_m) \leq [G_0^2 / (\omega_s \gamma_c)]^{2/3}$.

For a numerical estimate, we use a similar specification as given in Ref. [1] but assume a slightly higher mechanical quality factor Q_m , lower environmental temperature T , and lower input optical power I_0 such that all mentioned conditions are satisfied. The parameters are the following: $m = 50$ pg, $\omega_m / (2\pi) = 10^5$ Hz, $Q_m = 3.2 \times 10^7$, $\lambda = 532$ nm, $L = 3$ cm, $r_m = 0.9999$, $\mathcal{F} = 6 \times 10^5$, $T = 0.1$ K, and $I_0 \approx 5$ nW. The resulting resolution ΔN is shown in Fig. 2, and we are able to resolve single mechanical quantum when $\tau \approx 0.1$ ms.

Even though we have been focusing on the double-sided setup where $t_1 \approx t_2$, the quantum limit also exists in the single-sided case originally proposed in Ref. [1]. Ideally, a single-sided setup consists of a totally reflected end mirror, and the vacuum fluctuations only enter from the front mirror. Therefore, the quantum noises inside two subcavities have the same origin but different optical paths. Through similar input-output calculations, we find that if laser detuning is equal to $\pm\omega_s$, the quantum noises destructively interfere with each other at low frequencies, due to the same mechanism studied in great detail in Ref. [17], achieving an ideal quantum nondemolition measurement. However, in reality, the end mirror always has some finite transmission or optical loss which introduces uncorrelated

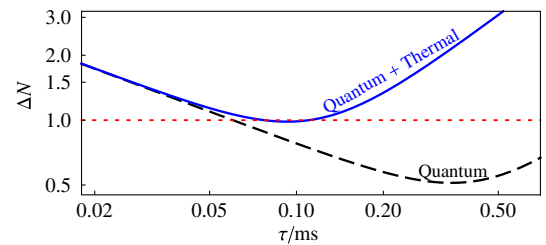


FIG. 2 (color online). The resolution ΔN for measuring mechanical energy quantization depending on the integration duration τ with total noise (solid line) and quantum noise only (dashed line).

vacuum fluctuations. As it turns out, the quantum limit is similar to Eq. (8), only with $\gamma_{c,d}$ replaced by the damping rate of two subcavities.

General systems.—Actually, the standard quantum limit obtained above applies to all schemes that attempt to probe mechanical energy quantization via parametric coupling. Let us consider n mechanical modes parametrically coupled with n' normal external modes, describable by the following Hamiltonian:

$$\hat{\mathcal{H}} = \sum_{\nu=1}^n \hbar \Omega_{\nu} (\hat{q}_{\nu}^2 + \hat{p}_{\nu}^2)/2 + \sum_{i=1}^{n'} \hbar \omega_i \hat{a}_i^{\dagger} \hat{a}_i + \sum_{i,j=1}^{n'} \sum_{\nu=1}^n \hbar \chi_{ij\nu} \hat{q}_{\nu} (\hat{a}_i^{\dagger} \hat{a}_j + \hat{a}_j^{\dagger} \hat{a}_i) + \hat{\mathcal{H}}_{\text{ext}} + \hat{\mathcal{H}}_{\xi}. \quad (18)$$

Here Greek indices identify mechanical modes and Latin indices identify external modes; Ω_{ν} and ω_i are eigenfrequencies; \hat{q}_{ν} , \hat{p}_{ν} are normalized positions and momenta; \hat{a}_i are annihilation operators of the external degrees of freedom; and $\chi_{ij\nu} = \chi_{jiv}$ are coupling constants. Similarly, we focus on the regime where $|\chi_{ij\nu}| \ll |\omega_i - \omega_j|$ (dispersive) and $\Omega_{\nu} \ll |\omega_i - \omega_j|$ (adiabatic) and obtain

$$\hat{\mathcal{H}} = \sum_{\nu=1}^n \hbar \Omega_{\nu} (\hat{q}_{\nu}^2 + \hat{p}_{\nu}^2)/2 + \sum_{i=1}^{n'} \hbar \omega'_i \hat{\delta}_i^{\dagger} \hat{\delta}_i + \hat{\mathcal{H}}_{\text{ext}} + \hat{\mathcal{H}}_{\xi}, \quad (19)$$

where, up to $\chi_{ij\nu}^2/|\omega_i - \omega_j|^2$,

$$\omega'_i = \omega_i + \sum_{\nu} \chi_{iiv} \hat{q}_{\nu} + \sum_{j \neq i} \sum_{\nu} \frac{(\chi_{ij\nu} \hat{q}_{\nu})^2}{\omega_i - \omega_j}. \quad (20)$$

In order to have quadratic couplings between a pair of external and mechanical modes, $\hat{\delta}_1$ and \hat{q}_1 , for instance, we require that $\chi_{11\nu} = 0$ and $\chi_{1i\nu} = \chi_{i1} \delta_{1\nu}$, and then

$$\omega'_1 = \omega_1 + \sum_{i \neq 1} \frac{\chi_{i1}^2}{\omega_1 - \omega_i} \hat{q}_1^2. \quad (21)$$

However, there still are linear couplings which originate from idle modes. This is because, up to $\chi_{ij\nu}/|\omega_i - \omega_j|$,

$$\hat{\delta}_i = \hat{a}_i + \sum_{j \neq i} \frac{\chi_{ij1} \hat{a}_j}{\omega_i - \omega_j} \hat{q}_1 \approx \hat{a}_i + \frac{\chi_{i1} \bar{a}_1}{\omega_i - \omega_1} \hat{q}_1 \quad (i \neq 1), \quad (22)$$

where \hat{a}_1 is replaced with its classical amplitude \bar{a}_1 , for $\bar{a}_1 \gg \hat{a}_1$. From Eqs. (21) and (22), both linear and dispersive couplings are inversely proportional to $|\omega_i - \omega_1|$. Therefore, we only need to consider a tripartite system formed by \hat{q}_1 , $\hat{\delta}_1$, and $\hat{\delta}_2$ which is the closest to $\hat{\delta}_1$ in frequency. The resulting Hamiltonian is identical to Eq. (2), and thus the same standard quantum limit applies.

Conclusion.—We have demonstrated the existence of a quantum limit for probing mechanical energy quantization in general systems where mechanical modes parametri-

cally interact with optical or electrical degrees of freedom. This work will shed light on choosing the appropriate parameters for experimental realizations.

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- [1] J. D. Thompson *et al.*, Nature (London) **452**, 72 (2008).
 - [2] A. Naik *et al.*, Nature (London) **443**, 193 (2006); S. Gigan *et al.*, Nature (London) **444**, 67 (2006); O. Arcizet *et al.*, Nature (London) **444**, 71 (2006); D. Kleckner and D. Bouwmeester, Nature (London) **444**, 75 (2006); A. Schliesser *et al.*, Phys. Rev. Lett. **97**, 243905 (2006); M. Poggio *et al.*, Phys. Rev. Lett. **99**, 017201 (2007); T. Corbitt *et al.*, Phys. Rev. Lett. **98**, 150802 (2007); **99**, 160801 (2007); P. F. Cohadon *et al.*, Phys. Rev. Lett. **83**, 3174 (1999); B. Abbott *et al.* (LIGO Scientific Collaboration), New J. Phys. **11**, 073 032 (2009).
 - [3] S. Mancini *et al.*, Phys. Rev. Lett. **80**, 688 (1998); F. Marquardt *et al.*, Phys. Rev. Lett. **99**, 093902 (2007); I. Wilson-Rae *et al.*, Phys. Rev. Lett. **99**, 093901 (2007); C. Genes *et al.*, Phys. Rev. A **77**, 033804 (2008); S. Danilishin *et al.*, arXiv:0809.2024.
 - [4] S. Mancini *et al.*, Phys. Rev. Lett. **90**, 137901 (2003).
 - [5] D. Vitali *et al.*, Phys. Rev. Lett. **98**, 030405 (2007).
 - [6] H. Müller-Ebhardt *et al.*, Phys. Rev. Lett. **100**, 013601 (2008); M. Hartmann and M. Plenio, Phys. Rev. Lett. **101**, 200503 (2008).
 - [7] D. H. Santamore *et al.*, Phys. Rev. B **70**, 144301 (2004).
 - [8] I. Martin and W. H. Zurek, Phys. Rev. Lett. **98**, 120401 (2007).
 - [9] K. Jacobs *et al.*, Phys. Rev. Lett. **98**, 147201 (2007).
 - [10] A similar configuration has been proposed by V. B. Braginsky *et al.* for detecting gravitational waves, Phys. Lett. A **232**, 340 (1997); **246**, 485 (1998).
 - [11] M. Bhattacharya and P. Meystre, Phys. Rev. Lett. **99**, 073601 (2007); Phys. Rev. A **77**, 033819 (2008).
 - [12] A. M. Jayich *et al.*, New J. Phys. **10**, 095 008 (2008).
 - [13] D. I. Schuster *et al.*, Nature (London) **445**, 515 (2007).
 - [14] A. A. Clerk and D. W. Utami, Phys. Rev. A **75**, 042302 (2007).
 - [15] A. A. Clerk *et al.*, arXiv:0810.4729.
 - [16] C. Gardiner, *Quantum Noise* (Springer, Berlin, 1991).
 - [17] F. Elste *et al.*, Phys. Rev. Lett. **102**, 207209 (2009).