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Towards a splitter theorem for internally 4-connected binary matroids ☆

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ABSTRACT

We prove that if M is a 4-connected binary matroid and N is an internally 4-connected proper minor of M with at least 7 elements, then, unless M is a certain 16-element matroid, there is an element e of $E(M)$ such that either $M\backslash e$ or M/e is internally 4-connected having an N-minor. This strengthens a result of Zhou and is a first step towards obtaining a splitter theorem for internally 4-connected binary matroids.

1. Introduction

Our goal in this article is to make progress towards a splitter theorem for internally 4-connected binary matroids. Such a theorem would provide a guarantee that if M and N are internally 4-connected binary matroids, and M has a proper N-minor, then M has a minor M' such that M' is internally 4-connected with an N-minor, and M' can be produced from M by a bounded number of simple operations.

A chain theorem resembles a splitter theorem, except that the requirement that M' has an N-minor is dropped. In a previous article we proved a chain theorem for internally 4-connected binary matroids [1]. In particular, we showed that if M is an internally 4-connected binary matroid, then M has an internally 4-connected minor, M', such that $|E(M)| - |E(M')| \leqslant 6$. (In almost every case, this bound can be improved to 3.) In this paper, we take a necessary step towards a splitter theorem, by proving that, as long as M is 4-connected, we can produce a proper minor M' of M such that M' has an N-minor and $|E(M)| - |E(M')| \leqslant 2$. (In almost every case, this bound can be improved to 1.)

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We note here that there is no hope of extending our main theorem to the case where M, N, and M' are all required to be 4-connected. This is true even if we relax the bound on $|E(M)| - |E(M')|$ to be any fixed constant. To see this, consider the toroidal grid graph $G_{m\times n}$ with vertex set $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$, where (i, j) and (x, y) are adjacent if and only if $i = x$ and $j - y \equiv \pm 1 \bmod n$, or if $j = y$ and $i - x \equiv \pm 1 \bmod m$. If m is any positive integer, then $N = M(G_{m\times m})$ is a proper minor of $M = M(G_{(m+1)\times m})$, and both matroids are 4-connected. But there is no proper minor M' of M such that N is a proper minor of M', and M' is 4-connected. Further examples demonstrating the limits of possible splitter theorems can be found in [3].

We recall some key definitions before stating our main result. Let M be a matroid on the ground set E. If $X \subseteq E$, then $\lambda_M(X)$ is defined to be

$$r(X) + r^*(X) - |X| = r(X) + r(E - X) - r(M).$$

Note $\lambda_M(X) = \lambda_M(E - X)$. A partition (X, Y) of E is a *k-separation*, for a positive integer k, if $|X|, |Y| \geqslant k$ and $\lambda_M(X) < k$. If $\lambda_M(X) < k$, then X is said to be *k-separating*. If every k-separation of M satisfies $k \geqslant n$, for some value n, then M is n-connected. If M is 3-connected, and every 3-separation (X, Y) satisfies $\min\{|X|, |Y|\} = 3$, then M is *internally* 4-*connected*.

Theorem 1.1. *Let M be a* 4*-connected binary matroid and N be an internally* 4*-connected proper minor of M with at least* 7 *elements. Then, for some e in $E(M)$, either $M \setminus e$ or M/e is internally* 4*-connected having an N-minor unless $M \cong D_{16}$. In the exceptional case, there are elements $e, f \in E(M)$ such that $M' = M \setminus e/f$ is internally* 4*-connected with an N-minor.*

In the statement of Theorem 1.1, D_{16} refers to the 16-element rank-8 binary matroid represented over $GF(2)$ by the matrix $[I_8|A]$, where A is the following matrix.

$$\begin{bmatrix} 1&1&0&0&1&1&1&0\\ 1&0&1&0&1&1&0&1\\ 0&1&0&0&1&0&1&1\\ 0&0&0&1&0&1&1&1\\ 1&1&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 0&1&1&1&0&0&0&0 \end{bmatrix}$$

Evidently D_{16} is isomorphic to its dual. Moreover, D_{16} has two $\mathrm{AG}(3, 2)$-minors on disjoint ground sets.

Theorem 1.1 strengthens the following result by Zhou [5, Theorem 3.1], which plays a fundamental role in our proof. A matroid is *weakly* 4-*connected* if it is 3-connected, and, whenever (X, Y) is a 3-separation, $\min\{|X|, |Y|\} \leqslant 4$.

Theorem 1.2. *Let M be a* 4*-connected binary matroid and N be an internally* 4*-connected proper minor of M with at least* 7 *elements. Then, for some e in $E(M)$, either $M \setminus e$ or M/e is weakly* 4*-connected having an N-minor.*

We briefly describe the structure of the proof of Theorem 1.1. We assume that M and N are as in the statement of the theorem, and that there is no element $e \in E(M)$ such that $M \setminus e$ or M/e is internally 4-connected with an N-minor. By duality and Theorem 1.2, there is an element $e \in E(M)$ such that $M \setminus e$ is weakly 4-connected with an N-minor. We deduce that $M \setminus e$ contains a *quad* Q, that is, a 4-element circuit-cocircuit. Lemma 2.3 says that if 1 is an arbitrary element in Q, then either $M \setminus 1$ or $M/1$ is weakly 4-connected with an N-minor. The first case quickly leads to a contradiction, so $M/1$ is weakly 4-connected, and must contain a quad Q_1. In fact, if $Q = \{1, 2, 3, 4\}$, then M/i is weakly 4-connected, and contains a quad Q_i, for every element $i \in Q$. We show that $e \in Q_i$ for each i. Let Q_i be $\{e, x_i, y_i, z_i\}$. We gain additional structure by considering the minors $M \setminus x_1$, $M \setminus y_1$, $M \setminus z_1$, $M \setminus y_2$, $M \setminus z_2$, and $M \setminus x_3$. Each of these is weakly 4-connected with a quad. By repeatedly exploiting the fact that circuits meet cocircuits in an even number of elements in binary matroids, we find that $Q_1 \cup \cdots \cup Q_4 = \{e, x_1, y_1, z_1, y_2, z_2, x_3\}$. The entire ground set consists of these 7 elements together with $\{1, 2, 3, 4\}$ and 5 other elements found in various quads. At this point, we have learned enough about the structure of M to construct a representation for it and deduce that it is isomorphic to D_{16}.

We conclude the paper by showing that it really is necessary to make an exception for D_{16} in the statement of Theorem 1.1; that is, D_{16} really is 4-connected and has an internally 4-connected minor, N, such that no single-element deletion or contraction of D_{16} is internally 4-connected with an N-minor.

2. Some preliminaries

Recall that a *triangle* is a 3-element circuit, and a *triad* is a 3-element cocircuit. An n-connected matroid with at least $2(n-1)$ elements does not contain a circuit or cocircuit with fewer than n elements [2, Proposition 8.2.1]. Hence a 4-connected matroid with at least 6 elements does not contain a triangle or triad.

A circuit and a cocircuit cannot meet in a single element. We refer to this property as *orthogonality*. Let M be a binary matroid. Then a circuit and a cocircuit of M must intersect in an even number of elements [2, Theorem 9.1.2 (ii)]. If C_1 and C_2 are circuits of M, then $C_1 \triangle C_2$, the symmetric difference of C_1 and C_2, is a disjoint union of circuits [2, Theorem 9.1.2 (iv)].

Let (X, Y) be a k-separation of the matroid M. If $y \in Y$ is in $\mathrm{cl}(X)$, then $r(X \cup y) = r(X)$. As $r(Y - y) \leqslant r(Y)$, it follows that $(X \cup y, Y - y)$ is a k-separation of M (provided $|Y - y| \geqslant k$). Corollary 8.1.5 of [2] implies that (X, Y) is a k-separation of M if and only if it is a k-separation of M^*. Therefore, if y is in $Y \cap \mathrm{cl}^*(X)$ and $|Y - y| \geqslant k$, then $(X \cup y, Y - y)$ is a k-separation of M^*, and hence of M.

Lemma 2.1. *Let M be a 3-connected binary matroid and (X, Y) be a 3-separation of M. If $|X| = 5$ and $r(X) = 3$, then X is not a cocircuit of M.*

Proof. Assume that X is a cocircuit. We may view M as a restriction of $\mathrm{PG}(r-1, 2)$ where $r = r(M)$. As (X, Y) is a 3-separation of M, the subspaces of $\mathrm{PG}(r-1, 2)$ spanned by X and Y meet in a rank-2 flat of $\mathrm{PG}(r-1, 2)$. Since X is a cocircuit of M, it follows that $X \cap \mathrm{cl}(Y) = \emptyset$, so this rank-2 flat avoids X. Thus X is a subset of the 4-element set that is obtained from the binary projective plane, $\mathrm{PG}(2, 2)$, by deleting a line. As $|X| = 5$, this is impossible. □

Lemma 2.2. *Let Q be a quad of the binary matroid M. If x and y are elements of Q, then $M \backslash x$ is isomorphic to $M \backslash y$.*

Proof. We may as well assume $x \neq y$. Let E be the ground set of M and let $Q = \{x, y, a, b\}$. Let $\phi : (E - x) \to (E - y)$ be defined so that $\phi(y) = x$, $\phi(a) = b$, $\phi(b) = a$, and $\phi(e) = e$ for every element $e \in E - Q$.

Let C be a circuit of $M \backslash x$. If $C \subseteq E - Q$, then clearly $\phi(C) = C$ is a circuit of $M \backslash y$. Assume that C meets $Q - x$. Since $Q - x$ is a cocircuit of $M \backslash x$, it follows that $|C \cap (Q - x)| = 2$. If $y \notin C$, then $\phi(C) = C$ is a circuit of $M \backslash y$, so we assume $y \in C$. Then $\phi(C) = C \triangle Q$ is a disjoint union of circuits of M. No circuit of M can meet Q in a single element, and no circuit can be properly contained in C. Therefore $\phi(C)$ is a circuit of M that does not contain y. Hence $\phi(C)$ is a circuit of $M \backslash y$. A similar argument shows that if C is a circuit of $M \backslash y$ that meets $Q - y$, then $\phi^{-1}(C)$ is a circuit of $M \backslash x$. Hence ϕ is the desired isomorphism. □

Lemma 2.3. *Let M be a 4-connected binary matroid. Let e be an element such that $M \backslash e$ is weakly 4-connected. Suppose $M \backslash e$ has a quad Q. Let 1 be an element of Q. Then the following statements hold.*

(i) *$M \backslash e \backslash 1$ is 3-connected and $M \backslash 1$ is weakly 4-connected.*
(ii) *$M \backslash e / 1$ is 3-connected and $M / 1$ is weakly 4-connected.*

Proof. Assume $|E(M)| < 6$. It is trivial to check that there are no 3-connected binary matroids with 4 or 5 elements. Therefore $|E(M)| \leqslant 3$, which contradicts the fact that $M \setminus e$ has a quad. Therefore $|E(M)| \geqslant 6$, so M has no triangles or triads.

We first establish (i).

2.3.1. *$M \setminus e \setminus 1$ is 3-connected.*

If not, then $M \setminus e \setminus 1$ has a 2-separation (U, V). Without loss of generality, $|U \cap (Q - 1)| \geqslant 2$. If $|U \cap (Q - 1)| = 3$, then $1 \in \mathrm{cl}_{M \setminus e}(U)$, so $(U \cup 1, V)$ is a 2-separation of $M \setminus e$; a contradiction. Thus we may assume that $|U \cap (Q - 1)| = 2$, so $V \cap (Q - 1) = \{g\}$, say. Since $Q - 1$ is a cocircuit of $M \setminus e \setminus 1$, $g \in \mathrm{cl}^*_{M \setminus e \setminus 1}(U)$. Therefore $(U \cup g, V - g)$ is a 2-separation of $M \setminus e \setminus 1$ unless $|V| = 2$. If $(U \cup g, V - g)$ is a 2-separation of $M \setminus e \setminus 1$, then, as $U \cup g \supseteq Q - 1$, we obtain a contradiction as above. Thus we may assume that $|V| = 2$.

Since $M \setminus e \setminus 1$ is certainly 2-connected, it follows from [2, Corollary 8.2.2] that V is a circuit or cocircuit of $M \setminus e \setminus 1$. As $Q - 1$ is a cocircuit meeting V in $\{g\}$, orthogonality implies V is a cocircuit. Since M has no cocircuits with fewer than 4 elements, $V \cup \{e, 1\}$ is a cocircuit of M. Now $Q \cap (V \cup \{e, 1\}) = \{g, 1\}$. As Q is a quad in $M \setminus e$, but not in M, $Q \cup e$ is a cocircuit of M. Therefore $(Q \cup e) \triangle (V \cup \{e, 1\})$ is a disjoint union of cocircuits of M. But the last set has only 3 elements, contradicting the fact that M is 4-connected. We conclude that (2.3.1) holds.

Suppose $M \setminus 1$ is not weakly 4-connected. Then it has a 3-separation (X, Y) with $|X|, |Y| \geqslant 5$. Without loss of generality, $e \in X$. Since neither $(X \cup 1, Y)$ nor $(X, Y \cup 1)$ is a 3-separation of M, neither $\mathrm{cl}_M(X)$ nor $\mathrm{cl}_M(Y)$ contains 1. Therefore $Q - 1$ is contained in neither X nor Y.

We first assume that $|(Q - 1) \cap X| = 2$ and let $(Q - 1) \cap Y = \{f\}$. Then $f \in \mathrm{cl}^*_{M \setminus 1}(X)$, since $(Q \cup e) - 1$ is a cocircuit of $M \setminus 1$, so $(X \cup f, Y - f)$ is a 3-separation of $M \setminus 1$. However, $1 \in \mathrm{cl}_M(X \cup f)$, so this implies that $(X \cup \{f, 1\}, Y - f)$ is a 3-separation of M, which is impossible.

We deduce that $|(Q - 1) \cap Y| = 2$. Let g be the single element in $(Q - 1) \cap X$. Now $(X - e, Y)$ is a 3-separation in $M \setminus 1 \setminus e$. As $Q - 1$ is a cocircuit of $M \setminus 1 \setminus e$, it follows that $g \in \mathrm{cl}^*_{M \setminus 1 \setminus e}(Y)$, so $(X - \{e, g\}, Y \cup g)$ is a 3-separation in $M \setminus 1 \setminus e$. But $Q \subseteq Y \cup \{g, 1\}$, so $1 \in \mathrm{cl}_{M \setminus e}(Y \cup g)$. Therefore $(X - \{e, g\}, Y \cup \{g, 1\})$ is a 3-separation in $M \setminus e$. As $M \setminus e$ is weakly 4-connected, it follows that $|X - \{e, g\}| \leqslant 4$, so $|X|$ is 5 or 6.

Now e must be in $\mathrm{cl}_{M \setminus 1}(X - e)$, for otherwise $(X - e, Y)$ is a 2-separation in $M \setminus 1 \setminus e$, contradicting (2.3.1). On the other hand, $e \notin \mathrm{cl}_M(X - \{e, g\})$, or else $(X - g, Y \cup \{g, 1\})$ is a 3-separation in M, which contradicts the fact that M is 4-connected. We deduce from this that there is a circuit C contained in X that contains both e and g.

Assume that $|X| = 5$. Then $X - \{e, g\}$ is a 3-element 3-separating set in $M \setminus e$. As M has no triangles, $X - \{e, g\}$ is a triad of $M \setminus e$, so $X - g$ is a cocircuit of M. Furthermore, $|C| > 3$, and $|C \cap (X - g)|$ is even, so C must be equal to X. Therefore $r_{M \setminus 1}(X) = 4$. As

$$\lambda_{M \setminus 1}(X) = r_{M \setminus 1}(X) + r^*_{M \setminus 1}(X) - |X| = 2,$$

it follows that $r^*_{M \setminus 1}(X) = 3$. Now $M^*/1 = (M \setminus 1)^*$ is 3-connected, (X, Y) is a 3-separation in $M^*/1$, $r_{M^*/1}(X) = 3$, and X is a cocircuit in $M^*/1$. This contradiction to Lemma 2.1 shows that $|X| = 6$.

Since $X - \{e, g\}$ is a 4-element 3-separating set in $M \setminus e$ that contains no triangles, it is a quad of $M \setminus e$. Therefore $X - \{e, g\}$ and $X - g$ are a circuit and a cocircuit in M, respectively. Thus $|C \cap (X - g)|$ is even. As $|C| > 3$, this means that $|C \cap (X - g)| = 4$. Now $C \triangle (X - \{e, g\})$ has cardinality 3 and is a disjoint union of circuits. This contradiction completes the proof of statement (i).

To prove (ii), we first show that

2.3.2. *$M \setminus e/1$ is 3-connected.*

Suppose $M \setminus e/1$ has (U, V) as a 2-separation. We can assume $|(Q - 1) \cap U| \geqslant 2$. Now $Q - 1$ is a circuit of $M \setminus e/1$. If $Q - 1 \subseteq U$, then, as Q is a cocircuit of $M \setminus e$, we deduce that $(U \cup 1, V)$ is a 2-separation of $M \setminus e$; a contradiction. If $|(Q - 1) \cap U| = 2$ and $(Q - 1) \cap V = \{f\}$, then either $(U \cup f, V - f)$ is a 2-separation of $M \setminus e/1$ with $Q - 1 \subseteq U \cup f$, or $|V| = 2$. In the former case, we argue as above. In the latter case, V is a circuit or a cocircuit of M, contradicting the fact that M has no triangles and no triads. Hence (2.3.2) holds.

Suppose $M/1$ is not weakly 4-connected. Then it has a 3-separation (X, Y) with $|X|, |Y| \geqslant 5$. Without loss of generality, $e \in X$. Therefore $(X - e, Y)$ is a 3-separation of $M/1 \setminus e$. Suppose $Q - 1 \subseteq X$. Then $1 \in \mathrm{cl}^*_{M\setminus e}(X)$, as Q is a cocircuit of $M \setminus e$. Hence $((X - e) \cup 1, Y)$ is a 3-separation of $M \setminus e$. This contradicts the fact that this matroid is weakly 4-connected.

Next suppose $Q - 1 \subseteq Y$. Then $(X - e, Y \cup 1)$ is a 3-separation of $M \setminus e$. Thus $|X - e| \leqslant 4$, and $X - e$ is a quad of $M \setminus e$, since otherwise $X - e$ contains a triangle of $M \setminus e$, and hence of M. Therefore X is a cocircuit of M, and of $M/1$. Hence $r^*_{M/1}(X) = 4$, and it follows that $r_{M/1}(X) = 3$. Thus we have a contradiction to Lemma 2.1.

Suppose next that $|(Q - 1) \cap X| = 2$ and let $(Q - 1) \cap Y = \{f\}$. Then $((X - e) \cup f, Y - f)$ is a 3-separation of $M/1 \setminus e$, so $((X - e) \cup \{f, 1\}, Y - f)$ is a 3-separation of $M \setminus e$. But $e \in \mathrm{cl}_{M/1}(X - e)$, for otherwise $(X - e, Y)$ is a 2-separation of $M/1 \setminus e$, contradicting (2.3.2). Therefore $e \in \mathrm{cl}_M((X - e) \cup 1)$, and it follows that $(X \cup \{f, 1\}, Y - f)$ is a 3-separation of M. As M is 4-connected, this is a contradiction.

Finally, suppose $|(Q - 1) \cap Y| = 2$ and let $(Q - 1) \cap X = \{g\}$. As $Q - 1$ is a circuit of $M/1$, it follows that $(X - g, Y \cup g)$ is a 3-separation of $M/1$ with $Q - 1 \subseteq Y \cup g$. If $|X - g| \geqslant 5$, then we have reduced to an earlier case. Thus we assume that $|X| = 5$. Then $(X - \{g, e\}, Y \cup g)$ is a 3-separation of $M/1 \setminus e$ and $Q - 1 \subseteq Y \cup g$. Hence $(X - \{g, e\}, Y \cup \{g, 1\})$ is a 3-separation of $M \setminus e$. Thus $X - \{g, e\}$ is a triad of $M \setminus e$, so $X - g$ is a cocircuit of M and hence of $M/1$.

We have $r_{M/1}(X) + r^*_{M/1}(X) = 7$. Suppose $r_{M/1}(X) = 3$. Then, as $X - g$ is a cocircuit of $M/1$, we deduce that $(M/1)|X$ is the union of two triangles, T_1 and T_2, that meet in g. Thus $T_1 \cup 1$ and $T_2 \cup 1$ are circuits of M, so $T_1 \triangle T_2 = X - g$ is a circuit of M. Since it is also a cocircuit, M has a quad, which is impossible.

We may now assume that $r^*_{M/1}(X) = r^*_M(X) = 3$. As X is a 5-element rank-3 set in M^*, it contains a triangle of M^*, and hence M contains a triad. This contradiction completes the proof of (ii). □

3. The main result

Proof of Theorem 1.1. First assume that $|E(N)| = 7$. By duality, we can assume that $r(N) \leqslant 3$. Then N is a 3-connected binary matroid with rank 3 and 7 elements. Since PG(2, 2) contains only 7 elements, this shows that $N \cong F_7$ or F_7^*. Since $M \neq N$, a result by Zhou [4, Corollary 1.2], shows that M has an N_1-minor, where N_1 is one of 5 possible 10- or 11-element matroids. It is easily confirmed that N_1 is non-regular, and internally 4-connected, but not 4-connected. Thus N_1 has an N-minor and $N_1 \neq M$. By relabeling N_1 as N, we can assume that $|E(N)| \geqslant 8$.

We will assume that M has no element e such that $M \setminus e$ or M/e is internally 4-connected having an N-minor. This implies the following fact.

1.1.1. *Let x be an element of M.*

(i) *If $M \setminus x$ is weakly 4-connected, and has an N-minor, then $M \setminus x$ has a quad.*
(ii) *If M/x is weakly 4-connected, and has an N-minor, then M/x has a quad.*

To prove (1.1.1), we assume that $M \setminus x$ has an N-minor, and is weakly 4-connected. Our assumption means that $M \setminus x$ is not internally 4-connected. Therefore $M \setminus x$ has a 3-separation (X, Y) such that $|X| = 4$ or $|Y| = 4$. We will assume the former, without loss of generality. If X is not a quad, then it contains both a triangle and a triad. Therefore M contains a triangle, which is impossible. Thus $M \setminus x$ contains a quad. The proof of the second statement is identical.

By Theorem 1.2 and duality, for some e in $E(M)$, the matroid $M \setminus e$ is weakly 4-connected and has an N-minor. Then (1.1.1) implies $M \setminus e$ has a quad $Q = \{1, 2, 3, 4\}$. If $Q \subseteq E(N)$, then Q is a 4-element 3-separating set in N. Since $|E(N)| \geqslant 8$, this contradicts the fact that N is internally 4-connected. Thus, we can assume that the element $1 \in Q$ is not in $E(N)$, and that therefore N is a minor of $M \setminus e \setminus 1$ or of $M \setminus e/1$. Then, by Lemma 2.3, either

(i) $M \setminus e \setminus 1$ has an N-minor and $M \setminus 1$ is weakly 4-connected; or
(ii) $M \setminus e/1$ has an N-minor and $M/1$ is weakly 4-connected.

For all i in Q, the matroid $M \setminus e \setminus i$ is isomorphic to $M \setminus e \setminus 1$ by Lemma 2.2. Therefore, if (i) holds, then $M \setminus e \setminus i$ has an N-minor and is weakly 4-connected, for all $i \in Q$. By duality and Lemma 2.2, $M \setminus e/i$ is isomorphic to $M \setminus e/1$ for all i in Q. Therefore, if (ii) holds, then $M \setminus e/i$ has an N-minor and is weakly 4-connected for all $i \in Q$.

Suppose first that (i) holds. As $M \setminus 1$ is weakly 4-connected, it has a quad Q_1 by (1.1.1). Now Q and Q_1 are circuits of M, while $Q \cup e$ and $Q_1 \cup 1$ are cocircuits. Since $1 \in Q$, it follows that $|Q_1 \cap (Q-1)|$ is odd. As $|Q_1 \cap ((Q-1) \cup e)|$ is even, we deduce that $e \in Q_1$. If

$$\left|Q_1 \cap (Q-1)\right| = 3,$$

then $|Q_1 \triangle Q| = 2$, meaning that M has a circuit of size at most 2. This is impossible, so $|(Q_1 - e) \cap (Q-1)| = 1$. We may assume that $Q_1 = \{e, 2, x_1, y_1\}$ where $|\{1,2,3,4,e,x_1,y_1\}| = 7$. By symmetry, $M \setminus 2$ has a quad Q_2 and $e \in Q_2$. Thus Q_2 is a circuit of M and $Q_2 \cup 2$ is a cocircuit of M. As above, $|(Q_2 - e) \cap (Q-2)| = 1$. Note that $M \setminus e \setminus 1 = M \setminus 1 \setminus e \cong M \setminus 1 \setminus 2$ by Lemma 2.2, because $\{2, e\} \subseteq Q_1$. Thus, by symmetry, $1 \in Q_2$ and $|(Q_2 - 1) \cap (Q_1 - 2)| = 1$. Hence $Q_2 = \{e, 1, x_2, y_2\}$ where $|\{1,2,3,4,e,x_1,y_1,x_2,y_2\}| = 9$.

By symmetry again, $M \setminus 3$ has a quad Q_3 and $e \in Q_3$. Moreover, $|Q_3 \cap (Q-3)| = 1$. Assume that $2 \in Q_3$. Then the cocircuit $Q_3 \cup 3$ meets the circuit Q_2 in at least one element, e. It follows that $|Q_3 \cap Q_2| = 2$. But as $2 \in Q_3$, this means that the circuit Q_3 meets the cocircuit $Q_2 \cup 2$ in 3 elements, which is impossible. Therefore either $4 \in Q_3$ or $1 \in Q_3$.

Assume that $4 \in Q_3$, so $Q_3 = \{e, 4, x_3, y_3\}$. We also know that $M \setminus 4$ has a quad Q_4 and $e \in Q_4$. By symmetry with the previous arguments, $Q_4 = \{e, 3, x_4, y_4\}$ and $|\{1,2,3,4,e,x_3,y_3,x_4,y_4\}| = 9$. Since M is binary, $|(Q_4 - e) \cap (Q_1 - e)| = 1$ and $|(Q_4 - e) \cap (Q_2 - e)| = 1$ so, without loss of generality, $x_4 = x_1$ and $y_4 = y_2$. By symmetry, $x_3 = y_1$ and $y_3 = x_2$. Now let $Z = \{1,2,3,4,e,x_1,y_1,x_2,y_2\}$. Then Z is spanned by $\{1,2,3,x_1,y_1\}$ in M. Since $\{1,2,3,4,e\}$ and $\{1,2,x_1,y_1,e\}$ are cocircuits of M, so is $\{3,4,x_1,y_1\}$. Hence Z is spanned by $\{1,2,3,x_1,e\}$ in M^*. Thus $r(Z) + r^*(Z) - |Z| \leqslant 1$. Since M is 4-connected, we deduce that $|E(M) - Z| \leqslant 1$. Hence we obtain a contradiction unless $|E(M)| \in \{9, 10\}$. In the exceptional case, as $M \setminus e$ has a quad and an N-minor, and $|E(N)| \geqslant 8$, we have $|E(M)| = 10$. Recall that $M \setminus e \setminus 1$ has an N-minor. But $(M \setminus e \setminus 1)^*$ has $\{2, x_1, y_1\}$ and $\{2,3,4\}$ as circuits. Now let $E(M) - Z = \{f\}$. Then, as $r(Z) = 5 = r^*(Z)$ and $\{1,2,3,4,e\}$ is a cocircuit of M, we deduce that $r(\{x_1,y_1,x_2,y_2,f\}) = 4$. Thus this set contains a circuit C, and C contains at least 4 elements. Note that $\{1,2,x_1,y_1,x_2,y_2\}$ is the symmetric difference of Q, Q_3, and Q_4. Since M has no circuits with fewer than 4 elements, it follows that $\{1,2,x_1,y_1,x_2,y_2\}$ is a circuit. Therefore $C \neq \{x_1,y_1,x_2,y_2\}$. But, by orthogonality with each of the sets $Q_i \cup i$, we deduce that C contains $\{x_1,y_1,x_2,y_2\}$. Hence $C = \{x_1,y_1,x_2,y_2,f\}$. But the symmetric difference of this with $\{1,2,x_1,y_1,x_2,y_2\}$ is $\{1,2,f\}$; which contradicts the fact that M has no triangles. We conclude that $4 \notin Q_3$.

We now know that $1 \in Q_3$. Then $Q_3 = \{e, 1, x_3, y_3\}$ for some x_3 and y_3. Thus $\{3, e, 1, x_3, y_3\}$ is a cocircuit. But $\{e, 2, x_1, y_1\}$ is a circuit so $|\{x_1, y_1\} \cap \{x_3, y_3\}|$ is odd. On the other hand, $\{1, 2, x_1, y_1, e\}$ is a cocircuit and $\{e, 1, x_3, y_3\}$ is a circuit, so $|\{x_1, y_1\} \cap \{x_3, y_3\}|$ is even. This contradiction completes the proof that $M \setminus e \setminus 1$ does not have an N-minor.

We now assume that case (ii) holds, so that $M \setminus e/1$ has an N-minor and is weakly 4-connected. Then, by Lemma 2.2 and (1.1.1), for all i in Q, the matroid M/i has an N-minor and is weakly 4-connected having a quad Q_i. Moreover, for any i and f in Q_i, it follows that $M/i/f$ or $M/i \setminus f$ has an N-minor. The first case is dual to the case above, which was eliminated. Thus we may assume that $M/i \setminus f$ has an N-minor. By the dual of Lemma 2.3(ii), $M \setminus f$ is weakly 4-connected, thus each $M \setminus f$ has a quad by (1.1.1).

Since $Q \cup e$ is cocircuit in M, and $Q \cup i$ is a circuit, for each i in $\{1,2,3,4\}$, the intersection $(Q \cup e) \cap (Q_i \cup i)$ has even cardinality. Therefore $|(Q \cup e) \cap Q_i|$ is odd. Since Q is a circuit and Q_i is a cocircuit, $|Q \cap Q_i|$ is even, so we conclude that $e \in Q_i$ and we let $Q_i = \{e, x_i, y_i, z_i\}$.

1.1.2. $(Q_i - e) \cap Q = \emptyset$ *for all i in* $\{1,2,3,4\}$.

As $Q_i \cup i$ is a circuit and $Q \cup e$ is a cocircuit, $|(Q_i - e) \cap (Q - i)|$ is even. Assume $|(Q_i - e) \cap (Q - i)| = 2$. Then, as Q is a circuit, $(Q_i \cup i) \triangle Q$ is a disjoint union of circuits. But $|(Q_i \cup i) \cap Q| = 3$, so $|(Q_i \cup i) \triangle Q| = 3$. This contradicts the fact that M is 4-connected.

We may assume that

1.1.3. $x_1 = x_2$ *and* $\{x_1, y_1, z_1\} \cap \{y_2, z_2\} = \emptyset$.

To see this, observe that $\{e, x_1, y_1, z_1, 1\}$ is a circuit and $\{e, x_2, y_2, z_2\}$ is a cocircuit. Hence $|\{x_1, y_1, z_1\} \cap \{x_2, y_2, z_2\}| = 1$ by (1.1.2), and (1.1.3) holds.

Let $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ be a quad of $M \setminus x_1$. The circuit $\{e, x_1, y_1, z_1, 1\}$ and the cocircuit $\{\alpha_1, \beta_1, \gamma_1, \delta_1, x_1\}$ imply that $|\{e, y_1, z_1, 1\} \cap \{\alpha_1, \beta_1, \gamma_1, \delta_1\}|$ is odd. The circuit $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ and cocircuit $\{e, x_1, y_1, z_1\}$ imply that $|\{e, y_1, z_1\} \cap \{\alpha_1, \beta_1, \gamma_1, \delta_1\}|$ is even. Thus $1 \in \{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ so, without loss of generality,

1.1.4. $1 = \alpha_1$.

1.1.5. *We may assume that* $2 = \beta_1$ *and*

$$\{\gamma_1, \delta_1\} \cap \{e, x_1, y_1, z_1, y_2, z_2, 1, 2, 3, 4\} = \emptyset.$$

Since $x_2 = x_1$, the set $\{e, x_1, y_2, z_2, 2\}$ is a circuit of M and $\{1, \beta_1, \gamma_1, \delta_1, x_1\}$ is a cocircuit of M by (1.1.4). Thus $|\{e, y_2, z_2, 2\} \cap \{1, \beta_1, \gamma_1, \delta_1\}|$ is odd. In addition, $\{e, x_1, y_2, z_2\}$ is a cocircuit of M by (1.1.3), and $\{1, \beta_1, \gamma_1, \delta_1\}$ is a circuit, so $|\{e, y_2, z_2\} \cap \{1, \beta_1, \gamma_1, \delta_1\}|$ is even. Hence $2 \in \{\beta_1, \gamma_1, \delta_1\}$ and we may assume that $2 = \beta_1$. Then $\{1, 2, \gamma_1, \delta_1, x_1\}$ and $\{e, x_1, y_2, z_2\}$ are cocircuits. If $|\{e, y_2, z_2\} \cap \{1, 2, \gamma_1, \delta_1\}| = 2$, then $|\{1, 2, \gamma_1, \delta_1, x_1\} \triangle \{e, x_1, y_2, z_2\}| = 3$, and this leads to a contradiction. Thus $|\{e, y_2, z_2\} \cap \{1, 2, \gamma_1, \delta_1\}| = 0$. Similarly, $|\{e, y_1, z_1\} \cap \{1, 2, \gamma_1, \delta_1\}| = 0$. Finally, it is clear that $\{1, 2\} \cap \{\gamma_1, \delta_1\} = \emptyset$. If $\{3, 4\} \cap \{\gamma_1, \delta_1\} \neq \emptyset$, then we must have $\{1, 2, 3, 4\} = \{1, 2, \gamma_1, \delta_1\}$ so $\{1, 2, 3, 4, x_1\}$ and $\{1, 2, 3, 4, e\}$ are cocircuits of M, and $e = x_1$; a contradiction. We conclude that (1.1.5) holds.

1.1.6. $x_1 \notin \{x_3, y_3, z_3\}$.

Recall that $\{1, 2, \gamma_1, \delta_1\}$ is a circuit and $\{e, x_3, y_3, z_3\}$ is a cocircuit, hence $|\{1, 2, \gamma_1, \delta_1\} \cap \{x_3, y_3, z_3\}|$ is even. As $|\{1, 2, \gamma_1, \delta_1, x_1\} \cap \{e, x_3, y_3, z_3, 3\}|$ is even and $3 \notin \{1, 2, \gamma_1, \delta_1, x_1\}$, by (1.1.2) and (1.1.5), it follows that $|\{1, 2, \gamma_1, \delta_1, x_1\} \cap \{e, x_3, y_3, z_3\}|$ is even. Since $e \in \{\gamma_1, \delta_1\}$ by (1.1.5) and $e \notin Q$, we conclude that $e \in \{1, 2, \gamma_1, \delta_1\}$ and therefore (1.1.6) holds.

1.1.7. *We may assume that* $Q_3 = \{e, x_3, y_1, z_2\}$. *Moreover,* $x_3 \notin \{\gamma_1, \delta_1\}$.

To see this, note that the cocircuits $\{e, x_1, y_1, z_1\}$ and $\{e, x_1, y_2, z_2\}$ and the circuit $\{e, x_3, y_3, z_3, 3\}$ of M imply using (1.1.2) and (1.1.6) that each of $\{y_1, z_1\}$ and $\{y_2, z_2\}$ meets $\{x_3, y_3, z_3\}$ in a single element. By (1.1.3), $\{y_1, z_1\} \cap \{y_2, z_2\} = \emptyset$, and the first part of (1.1.7) follows. If $x_3 \in \{\gamma_1, \delta_1\}$, then it follows from (1.1.2) and (1.1.5) that the circuit $\{1, 2, \gamma_1, \delta_1\}$ meets the cocircuit $\{e, x_3, y_1, z_2\}$ in a single element; a contradiction.

Next we consider Q_4. The arguments of (1.1.6) also show that $x_1 \notin \{x_4, y_4, z_4\}$. Since $\{e, x_4, y_4, z_4, 4\}$ is a circuit, and $\{e, x_1, x_2, x_3\}$, $\{e, x_1, y_2, z_2\}$, and $\{e, x_3, y_1, z_2\}$ are cocircuits, it follows that $\{x_4, y_4, z_4\}$ meets each of $\{x_3, y_1, z_2\}$, $\{y_1, z_1\}$, and $\{y_2, z_2\}$ in a single element.

1.1.8. $\{x_3, y_1\} \cap \{x_1, y_2\} = \emptyset = \{x_3, z_2\} \cap \{x_1, z_1\}$.

This follows by considering the intersection of the circuit $\{e, x_3, y_1, z_2, 3\}$ with the cocircuits $\{e, x_1, y_2, z_2\}$ and $\{e, x_1, y_1, z_1\}$.

By using (1.1.3) and the fact that $x_1 \notin \{x_4, y_4, z_4\}$, we deduce that there are the following three possibilities for $\{x_4, y_4, z_4\}$:

(A) $\{y_1, y_2, y'\}$ for some $y' \notin \{y_1, y_2, z_1, z_2, x_1, x_3\}$;
(B) $\{z_1, y_2, x_3\}$;
(C) $\{z_1, z_2, z'\}$ for some $z' \notin \{y_1, y_2, z_1, z_2, x_1, x_3\}$.

Cases (A) and (C) are symmetric, so we may assume that (A) or (B) holds.

Now $M \setminus y_1$ has a quad. By (1.1.4) and symmetry, this quad is $\{1, \beta_2, \gamma_2, \delta_2\}$. Thus $\{1, \beta_2, \gamma_2, \delta_2, y_1\}$ is a cocircuit of M. As $\{e, x_3, y_1, z_2, 3\}$ is a circuit, we deduce that $|\{1, \beta_2, \gamma_2, \delta_2\} \cap \{e, x_3, z_2, 3\}|$ is odd. Also, since $\{1, \beta_2, \gamma_2, \delta_2\}$ is a circuit and $\{e, x_3, y_1, z_2\}$ is a cocircuit, $|\{1, \beta_2, \gamma_2, \delta_2\} \cap \{e, x_3, z_2\}|$ is even. Thus, without loss of generality, and arguing as for (1.1.5), we get that

1.1.9. $3 = \beta_2$ *and* $\{\gamma_2, \delta_2\} \cap \{e, x_3, z_2\} = \emptyset$.

We now have that $\{1, 3, \gamma_2, \delta_2, y_1\}$ is a cocircuit and $\{1, 3, \gamma_2, \delta_2\}$ is a circuit of M. Assume that (A) holds. Then $\{e, y_1, y_2, y', 4\}$ is a circuit of M. Since $|\{1, 3, \gamma_2, \delta_2\} \cap \{e, y_2, y', 4\}|$ is odd and $|\{1, 3, \gamma_2, \delta_2\} \cap \{e, y_2, y'\}|$ is even, it follows that $4 \in \{\gamma_2, \delta_2\}$. Hence $\{1, 3, \gamma_2, \delta_2\} = \{1, 3, 4, 2\}$. But this means that $\{1, 3, 4, 2, y_1\}$ and $\{1, 2, 3, 4, e\}$ are cocircuits of M, so $y_1 = e$; a contradiction. We conclude that (A) does not hold. Thus (B) holds and

1.1.10. *M has $\{e, x_3, y_2, z_1, 4\}$ as a circuit and has $\{e, x_3, y_2, z_1\}$ as a cocircuit.*

The matroid $M \setminus z_1$ has a quad and it must contain 1, by the same argument as (1.1.4). Let $\{1, \beta_3, \gamma_3, \delta_3\}$ be this quad. Then $|\{1, \beta_3, \gamma_3, \delta_3\} \cap \{e, x_3, y_2, z_1\}|$ and $|\{1, \beta_3, \gamma_3, \delta_3, z_1\} \cap \{e, x_3, y_2, z_1, 4\}|$ are both even. Therefore $|\{1, \beta_3, \gamma_3, \delta_3\} \cap \{e, x_3, y_2, 4\}|$ is odd. It follows that $4 \in \{1, \beta_3, \gamma_3, \delta_3\}$. Without loss of generality we assume that $\beta_3 = 4$. Thus we have the following, where the assertion in the last sentence follows by a similar argument used for (1.1.5).

1.1.11. *M has $\{1, 4, \gamma_3, \delta_3\}$ as a circuit and has $\{1, 4, \gamma_3, \delta_3, z_1\}$ as a cocircuit. Moreover, $\{\gamma_3, \delta_3\} \cap \{e, x_3, y_2\} = \emptyset$.*

From (1.1.5) we see that $4 \notin \{\gamma_1, \delta_1\}$. Assume that $2 \in \{\gamma_3, \delta_3\}$. Then $\{1, 2, 3, 4\}$ and $\{1, 4, \gamma_3, \delta_3\}$ are circuits of M intersecting in 3 elements, so $\{1, 2, 3, 4\} = \{1, 4, \gamma_3, \delta_3\}$. Then $\{1, 2, 3, 4, e\}$ and $\{1, 2, 3, 4, z_1\}$ are cocircuits, and this leads to a contradiction. Therefore $2 \notin \{\gamma_3, \delta_3\}$. Since $\{1, 2, \gamma_1, \delta_1\}$ is a circuit and $\{1, 2, \gamma_1, \delta_1, x_1\}$ is a cocircuit, $|\{\gamma_3, \delta_3\} \cap \{\gamma_1, \delta_1, x_1\}|$ and $|\{\gamma_1, \delta_1\} \cap \{\gamma_3, \delta_3, z_1\}|$ are both odd. Thus $x_1 \in \{\gamma_3, \delta_3\}$ if and only if $z_1 \in \{\gamma_1, \delta_1\}$.

Suppose $x_1 \in \{\gamma_3, \delta_3\}$, say $x_1 = \gamma_3$. Then $z_1 = \gamma_1$, without loss of generality. Thus $\{1, 2, z_1, \delta_1\}$ is a circuit and $\{e, x_1, y_1, z_1\}$ is a cocircuit, so $|\{1, 2, \delta_1\} \cap \{e, x_1, y_1\}| = 1$. By (1.1.2), neither 1 nor 2 is in $\{e, x_1, y_1, z_1\}$, so $\delta_1 \in \{e, x_1, y_1\}$. But $\delta_1 \neq x_1$ by (1.1.5). If $\delta_1 = e$, then $\{1, 2, e, z_1\}$ is a circuit and $\{e, x_1, y_2, z_2\}$ is a cocircuit. Note $z_1 \neq y_2$ by (1.1.10) and $z_1 \neq z_1$ by (1.1.8). Hence $1 \in \{y_2, z_2\}$. But $\{x_1, y_2, z_2\} \cap \{1, 2, 3, 4\} = \emptyset$ by (1.1.2). Hence $\delta_1 \neq e$. Thus $\delta_1 = y_1$. Then $\{1, 2, z_1, y_1, x_1\}$ and $\{e, x_1, y_1, z_1\}$ are both cocircuits. Their symmetric difference has exactly 3 elements; a contradiction. We deduce that $x_1 \notin \{\gamma_3, \delta_3\}$ and $z_1 \notin \{\gamma_1, \delta_1\}$ so

1.1.12. $|\{\gamma_1, \delta_1\} \cap \{\gamma_3, \delta_3\}| = 1$.

Now $M \setminus y_2$ has a quad Y_2, so $Y_2 \cup y_2$ is a cocircuit of M. By considering the circuit $\{e, x_1, y_2, z_2, 2\}$ and the cocircuit $\{e, x_1, y_2, z_2\}$, we deduce that $|Y_2 \cap \{e, x_1, z_2\}|$ is even and $|Y_2 \cap \{e, x_1, z_2, 2\}|$ is odd, so $2 \in Y_2$. Similarly, using the circuit $\{e, x_3, y_2, z_1, 4\}$ and the cocircuit $\{e, x_3, y_2, z_1\}$, we deduce that $4 \in Y_2$. Thus $Y_2 = \{2, 4, \gamma_5, \delta_5\}$, say.

The matroid $M \setminus z_2$ has a quad Z_2. Since $M/2$ and $M/3$ have $\{e, x_1, y_2, z_2\}$ and $\{e, x_3, y_1, z_2\}$ as quads, it follows that $\{2, 3\} \subseteq Z_2$. Thus $Z_2 = \{2, 3, \gamma_4, \delta_4\}$, say. Similarly, $M \setminus x_3$ has a quad X_3 and $X_3 = \{3, 4, \gamma_6, \delta_6\}$.

To keep track of the argument to follow, we list in Table 1 the circuits and cocircuits that have arisen from the various quads we have identified. In each of the circuits and cocircuits listed, the elements are distinct.

Table 1
Some known circuits and cocircuits.

Circuits	Cocircuits
$\{1, 2, 3, 4\}$	$\{1, 2, 3, 4, e\}$
$\{e, x_1, y_1, z_1, 1\}$	$\{e, x_1, y_1, z_1\}$
$\{e, x_1, y_2, z_2, 2\}$	$\{e, x_1, y_2, z_2\}$
$\{e, x_3, y_1, z_2, 3\}$	$\{e, x_3, y_1, z_2\}$
$\{e, x_3, y_2, z_1, 4\}$	$\{e, x_3, y_2, z_1\}$
$\{1, 2, \gamma_1, \delta_1\}$	$\{1, 2, \gamma_1, \delta_1, x_1\}$
$\{1, 3, \gamma_2, \delta_2\}$	$\{1, 3, \gamma_2, \delta_2, y_1\}$
$\{1, 4, \gamma_3, \delta_3\}$	$\{1, 4, \gamma_3, \delta_3, z_1\}$
$\{2, 3, \gamma_4, \delta_4\}$	$\{2, 3, \gamma_4, \delta_4, z_2\}$
$\{2, 4, \gamma_5, \delta_5\}$	$\{2, 4, \gamma_5, \delta_5, y_2\}$
$\{3, 4, \gamma_6, \delta_6\}$	$\{3, 4, \gamma_6, \delta_6, x_3\}$

Next we prove the following sublemma.

1.1.13. *Suppose that* $1 \leqslant i < j \leqslant 6$*. Then* $\{\gamma_i, \delta_i\} \neq \{\gamma_j, \delta_j\}$*. Moreover, if* $\{i, j\}$ *is* $\{1, 6\}$, $\{2, 5\}$*, or* $\{3, 4\}$*, then* $\{\gamma_i, \delta_i\} \cap \{\gamma_j, \delta_j\} = \emptyset$*.*

To prove this, we may assume that $i = 1$, as the other cases follow by an identical argument. If $j \in \{2, 3, 4, 5\}$, then $\{\gamma_i, \delta_i\}$ cannot be equal to $\{\gamma_j, \delta_j\}$, for otherwise we can take the symmetric difference of two of the circuits in Table 1 and find a circuit of size at most 2. If $j = 6$ and $\{\gamma_i, \delta_i\} \cap \{\gamma_j, \delta_j\}$ is non-empty, then $\{\gamma_i, \delta_i\}$ and $\{\gamma_j, \delta_j\}$ must be equal, for otherwise the symmetric difference of $\{1, 2, 3, 4\}$, $\{1, 2, \gamma_1, \delta_1\}$, and $\{3, 4, \gamma_6, \delta_6\}$ contains a circuit of size at most 2. Now taking the symmetric difference of $\{1, 2, \gamma_1, \delta_1, x_1\}$ and $\{3, 4, \gamma_6, \delta_6, x_3\}$ shows that $\{1, 2, 3, 4, x_1, x_3\}$ is a cocircuit of M. This is a contradiction, as the cocircuit $\{1, 2, 3, 4, e\}$ leads to a cocircuit of size at most 3. Thus (1.1.13) holds.

We now consider the 6 elements $x_1, y_1, z_1, y_2, z_2, x_3$. From (1.1.3), (1.1.6), and Table 1, these elements are distinct. The 3-element subsets of this set that lie in a known 4-cocircuit with e match up with the 3-point lines in a copy of $M(K_4)$. Moreover, for each 2-element subset $\{i, j\}$ of $\{1, 2, 3, 4\}$, the listed 5-cocircuit containing $\{i, j\}$ contains the unique element of $\{x_1, y_1, z_1, y_2, z_2, x_3\}$ that is common to the indicated 5-circuits containing $\{e, i\}$ and $\{e, j\}$. This reveals more symmetry than may have been immediately apparent.

For example, by repeating the arguments of (1.1.5) with the circuit $\{1, 3, \gamma_2, \delta_2\}$ and the two cocircuits of the form Q_i containing y_1, namely $\{e, x_1, y_1, z_1\}$ and $\{e, x_3, y_1, z_2\}$, we show that $\{\gamma_2, \delta_2\} \cap \{e, x_1, y_1, z_1, z_2, x_3\} = \emptyset$. The orthogonality of the circuit $\{1, 3, \gamma_2, \delta_2\}$ and the cocircuit $\{e, x_1, y_2, z_2\}$ implies that $y_2 \notin \{\gamma_2, \delta_2\}$. Moreover, if $2 \in \{\gamma_2, \delta_2\}$, then $\{1, 2, 3, 4\}$ and $\{1, 3, \gamma_2, \delta_2\}$ must be equal, implying that $\{1, 2, 3, 4, e\}$ and $\{1, 2, 3, 4, y_1\}$ are both cocircuits, which is impossible. Similarly, $4 \notin \{\gamma_2, \delta_2\}$. By applying these arguments in the other symmetric cases we arrive at the following conclusion.

1.1.14. $\{e, x_1, y_1, z_1, y_2, z_2, x_3, 1, 2, 3, 4\}$ *avoids* $\{\gamma_i, \delta_i : 1 \leqslant i \leqslant 6\}$*.*

Moreover, by (1.1.2):

1.1.15. $\{e, x_1, y_1, z_1, y_2, z_2, x_3\}$ *avoids* $\{1, 2, 3, 4\}$*.*

By using (1.1.14) and comparing circuits and cocircuits in Table 1, we see that $\{\gamma_1, \delta_1\}$ meets each of $\{\gamma_2, \delta_2\}$, $\{\gamma_3, \delta_3\}$, $\{\gamma_4, \delta_4\}$, and $\{\gamma_5, \delta_5\}$ in a single element. From (1.1.13) we know that $\{\gamma_1, \delta_1\}$ avoids $\{\gamma_6, \delta_6\}$.

Without loss of generality, we may assume that $\gamma_1 = \gamma_2$. Then one of the following two cases occurs.

1.1.16. $\{(\gamma_1,\delta_1),(\gamma_2,\delta_2),(\gamma_3,\delta_3),(\gamma_4,\delta_4),(\gamma_5,\delta_5),(\gamma_6,\delta_6)\}$ *is*

(I) $\{(\gamma_1,\delta_1),(\gamma_1,\delta_2),(\delta_1,\delta_2),(\gamma_1,\delta_4),(\delta_1,\delta_4),(\delta_2,\delta_4)\}$; *or*
(II) $\{(\gamma_1,\delta_1),(\gamma_1,\delta_2),(\gamma_1,\delta_3),(\delta_1,\delta_2),(\delta_1,\delta_3),(\delta_2,\delta_3)\}$.

To see that this is true, we consider whether or not γ_1 is in $\{\gamma_3,\delta_3\}$. First assume that it is. Then by relabeling we can assume that $\gamma_3=\gamma_1$. From (1.1.13) we see that $\delta_2\notin\{\gamma_1,\delta_1\}$ and $\delta_3\notin\{\gamma_1,\delta_1,\delta_2\}$. By orthogonality between $\{2,3,\gamma_4,\delta_4\}$ and $\{1,2,\gamma_1,\delta_1,x_1\}$, and between $\{2,3,\gamma_4,\delta_4\}$ and $\{1,3,\gamma_1,\delta_2,y_1\}$, we see that

$$\big|\{\gamma_4,\delta_4\}\cap\{\gamma_1,\delta_1\}\big|=\big|\{\gamma_4,\delta_4\}\cap\{\gamma_1,\delta_2\}\big|=1.$$

But neither γ_4 nor δ_4 can be equal to γ_1, for then $\{\gamma_4,\delta_4\}$ and $\{\gamma_3,\delta_3\}$ would not be disjoint, as is demanded by (1.1.13). Thus $\{\gamma_4,\delta_4\}=\{\delta_1,\delta_2\}$. We can assume that $(\gamma_4,\delta_4)=(\delta_1,\delta_2)$. Orthogonality between $\{2,4,\gamma_5,\delta_5\}$ and the cocircuits $\{1,2,\gamma_1,\delta_1,x_1\}$ and $\{1,4,\gamma_1,\delta_3,z_1\}$ shows that

$$\big|\{\gamma_5,\delta_5\}\cap\{\gamma_1,\delta_1\}\big|=\big|\{\gamma_5,\delta_5\}\cap\{\gamma_1,\delta_3\}\big|=1.$$

By using (1.1.13), we can assume that $(\gamma_5,\delta_5)=(\delta_1,\delta_3)$. A similar argument shows that we can assume that $(\gamma_6,\delta_6)=(\delta_2,\delta_3)$. Thus we have verified that (II) holds, assuming that $\gamma_1\in\{\gamma_3,\delta_3\}$.

Next we assume that $\gamma_1\notin\{\gamma_3,\delta_3\}$. Then $\delta_1\in\{\gamma_3,\delta_3\}$. Note that $\delta_2\notin\{\gamma_1,\delta_1\}$. Orthogonality between $\{1,4,\gamma_3,\delta_3\}$ and $\{1,3,\gamma_1,\delta_2,y_1\}$ shows that $\delta_2\in\{\gamma_3,\delta_3\}$, so we may assume that $(\gamma_3,\delta_3)=(\delta_1,\delta_2)$. We know that $|\{\gamma_4,\delta_4\}\cap\{\gamma_1,\delta_1\}|=1$. But $\delta_1\notin\{\gamma_4,\delta_4\}$, for $\{\gamma_4,\delta_4\}$ is disjoint with $\{\gamma_3,\delta_3\}$. Thus $\gamma_1\in\{\gamma_4,\delta_4\}$. We can assume that $\gamma_4=\gamma_1$. We deduce from (1.1.13) that $\delta_4\notin\{\gamma_1,\delta_1,\delta_2\}$. By (1.1.13) and orthogonality between $\{2,4,\gamma_5,\delta_5\}$ and $\{1,2,\gamma_1,\delta_1,x_1\}$, we see that $\delta_1\in\{\gamma_5,\delta_5\}$. Applying the same argument to the cocircuit $\{2,3,\gamma_1,\delta_4,z_2\}$ shows that $\delta_4\in\{\gamma_5,\delta_5\}$. A similar argument shows that $\{\gamma_6,\delta_6\}=\{\delta_2,\delta_4\}$, so we have completed the proof of (1.1.16).

Now $\{1,2,\gamma_1,\delta_1\}$ is a quad of $M\setminus x_1$, and $M\setminus x_1/1$ has an N-minor. Thus $M\setminus x_1/\gamma_1$ has an N-minor by Lemma 2.2. Since $M\setminus e$ is weakly 4-connected, Lemma 2.3 implies that $M/1$ is weakly 4-connected. As $\{e,x_1,y_1,z_1\}$ is a quad of $M/1$, this in turn implies that $M\setminus x_1$ is weakly 4-connected, and hence, so is M/γ_1. Thus M/γ_1 has a quad G by (1.1.1). Then G is a cocircuit of M and $G\cup\gamma_1$ is a circuit of M. Since $|G\cap\{1,2,\delta_1,x_1\}|$ is odd and $|G\cap\{1,2,\delta_1\}|$ is even, it follows that $x_1\in G$. Similarly, $\{1,3,\gamma_2,\delta_2,y_1\}=\{1,3,\gamma_1,\delta_2,y_1\}$ is a cocircuit, and $|G\cap\{1,3,\delta_2,y_1\}|$ is odd while $|G\cap\{1,3,\delta_2\}|$ is even. Hence $y_1\in G$.

In case (II), $\{1,4,\gamma_1,\delta_3,z_1\}$ is a cocircuit, and we can argue that z_1 is in G. As $\{x_1,y_1,z_1\}\subseteq G$, and both G and $\{e,x_1,y_1,z_1\}$ are cocircuits, it follows that $G=\{e,x_1,y_1,z_1\}$. Thus $\{e,x_1,y_1,z_1,1\}$ and $\{e,x_1,y_1,z_1,\gamma_1\}$ are circuits, which leads to a contradiction.

Therefore case (I) holds. Since $\{2,3,\gamma_1,\delta_4,z_2\}$ is a cocircuit, we can deduce that $z_2\in G$. Let t be the element of $G-\{x_1,y_1,z_2\}$. By orthogonality, $\{t\}$ is disjoint from the set $J'=\{e,1,2,3,4,x_1,y_1,z_1,y_2,z_2,x_3,\gamma_1,\delta_1,\delta_2,\delta_4\}$. Let $J=J'\cup t$. Then J is spanned by $\{e,1,2,3,x_1,y_1,y_2,\gamma_1\}$ in M and in M^*. Thus

$$\lambda(J)=r(J)+r^*(J)-|J|\leqslant 8+8-16=0.$$

Hence $E(M)=J$.

It is easy to show that $\{e,1,2,3,x_1,y_1,y_2,\gamma_1\}$ must be both a basis and cobasis of M, and it is then straightforward to check that M is represented by the matrix $[I_8|A]$, where A is shown in Table 2. Thus $M\cong D_{16}$.

As $M/2\setminus e$ has an N-minor, we can complete the proof of Theorem 1.1 by proving the following sublemma.

1.1.17. *$M/2\setminus e$ is internally 4-connected.*

Certainly $M/2\setminus e$ is 3-connected by Lemma 2.3. Assume it is not internally 4-connected and let (X,Y) be a 3-separation of it with $|X|,|Y|\geqslant 4$. Let $S=\{1,3,4,\gamma_1,\delta_1,\delta_2,\delta_4\}$ and $T=\{t,x_1,y_1,z_1,y_2,z_2,x_3\}$. Then (S,T) is a 4-separation of $M/2\setminus e$. Evidently every 4-element subset

Table 2
A representation of D_{16}.

	x_3	z_1	t	e	δ_1	3	1	4
δ_4	1	1	0	0	1	1	1	0
γ_1	1	0	1	0	1	1	0	1
δ_2	0	1	0	0	1	0	1	1
2	0	0	0	1	0	1	1	1
y_1	1	1	1	0	0	0	0	0
y_2	1	1	0	1	0	0	0	0
x_1	1	0	1	1	0	0	0	0
z_2	0	1	1	1	0	0	0	0

of S spans S in $M/2 \setminus e$. By duality, every 4-element subset of T spans T in $(M/2 \setminus e)^*$. Clearly $|S \cap X| \geqslant 4$ or $|S \cap Y| \geqslant 4$. Assume the former. If $|Y \cap T| \geqslant 4$, then, via closure, we can move the elements of $Y \cap S$ into X and, via coclosure, we can move the elements of $X \cap T$ into Y, where each of these moves maintains a 3-separation. It follows that (S, T) is a 3-separation of $M/2 \setminus e$; a contradiction. Thus $|Y \cap T| \leqslant 3$. Now if $|Y| > 4$, we can move elements of $Y \cap S$ into X via closure one at a time until we have a 3-separation (X', Y') with $|Y'| = 4$ and $|Y' \cap T| \leqslant 3$. If x is an element in $Y' \cap S$, then both Y' and $Y' - x$ are 3-separating. Thus Y' is a 4-element fan of $M/2 \setminus e$ so at most one element of Y' is in the closure of X' and at most one element of Y' is in the coclosure of X'. Thus each of $Y' \cap S$ and $Y' \cap T$ has at most one element; a contradiction. We deduce that (1.1.17) holds, and this completes the proof of Theorem 1.1. □

We conclude by demonstrating that it really is necessary to make an exception for D_{16} in the statement of Theorem 1.1. Let $M = [I|A]$, where A is the labeled matrix in Table 2.

We start by showing that M is 4-connected. Assume that this is not the case. When we constructed A during the proof of Theorem 1.1, the element e was chosen so that $M \setminus e$ is weakly 4-connected. Thus $M \setminus e$ is 3-connected, and clearly so is M. Therefore there is a 3-separation (X, Y) of M. It is very easy to confirm that M does not contain any triangles, nor any triads (since it is self-dual). Therefore $|X|, |Y| \geqslant 4$.

Assume that $|X|, |Y| \geqslant 5$. Then $(X - \{2, e\}, Y - \{2, e\})$ is a 3-separation of $M/2 \setminus e$. Since this matroid is internally 4-connected, by (1.1.17), we can assume that $2, e \in Y$, and that $|Y| = 5$. But it is routine to verify that any 5-element 3-separating set in a 3-connected binary matroid contains a triangle or a triad, so this is impossible. Therefore we can assume that $|Y| = 4$. Moreover, Y is a quad, since otherwise it would contain a triangle or triad.

Let $S_1 = \{\delta_4, \gamma_1, \delta_2, 2\}$, and let $S_2 = \{\delta_1, 3, 1, 4\}$. Moreover, let $T_1 = \{y_1, y_2, x_1, z_2\}$ and let $T_2 = \{x_3, z_1, t, e\}$. Then $M/S_1 \setminus S_2$ and $M/T_1 \setminus T_2$ are both isomorphic to AG(3, 2). Assume that $Y \subseteq S_1 \cup S_2$. Since AG(3, 2) has no circuits or cocircuits with fewer than 4 elements, Y is one of the 14 quads in $M/T_1 \setminus T_2$. But it is easy to verify that none of these is a quad of M. For example, $\{\delta_4, \gamma_1, \delta_2, \delta_1\}$ is a quad in $M/T_1 \setminus T_2$. If it were a cocircuit in M, then the rows δ_4, γ_1, δ_2 would sum to the row that is everywhere zero, except in the column labeled δ_1. This is not the case, so $\{\delta_4, \gamma_1, \delta_2, \delta_1\}$ is not a quad of M. In this way we verify that no quad of $M/T_1 \setminus T_2$ is a quad of M, and therefore $Y \nsubseteq S_1 \cup S_2$. An identical argument shows that $Y \nsubseteq T_1 \cup T_2$.

It is easy to see that $S_1 \cup S_2$ and $T_1 \cup T_2$ are flats of M, so $|Y \cap (S_1 \cup S_2)| = |Y \cap (T_1 \cup T_2)| = 2$. If $|Y \cap S_1| = 2$ or $|Y \cap S_2| = 2$, then $M/S_1 \setminus S_2$ contains a circuit or cocircuit of size 2. Therefore $|Y \cap S_1| = |Y \cap S_2| = 1$. The same argument shows that $|Y \cap T_1| = |Y \cap T_2| = 1$. But it is obvious that no 4-element circuit of M meets S_1, S_2, T_1, and T_2 in a single element each. This contradiction completes the demonstration that M is 4-connected.

By considering the row and column labels of the matrix in Table 2, we see that the permutation that swaps the following pairs is an isomorphism, ϕ, from M to M^*.

$$\{\delta_4, x_3\},\ \{\gamma_1, z_1\},\ \{\delta_2, t\},\ \{2, e\},\ \{y_1, \delta_1\},\ \{y_2, 3\},\ \{x_1, 1\},\ \{z_2, 4\}.$$

Let $N = M/2 \setminus e$. Then N is an internally 4-connected minor of M by (1.1.17). We will now show that no single-element deletion or contraction of M is internally 4-connected with an N-minor.

The matrix produced from A by:

(i) pivoting on the entry in the δ_4 row and the δ_1 column;
(ii) swapping the 1 column and the 3 column;
(iii) swapping the x_3 column and the z_1 column;
(iv) swapping the x_1 row and the z_2 row

is identical to A. This shows that there is an automorphism Ω_1 of M swapping the pairs

$$\{\delta_4, \delta_1\},\ \{1, 3\},\ \{x_3, z_1\},\ \{x_1, z_2\}$$

and acting as the identity on the rest of the matroid. Similarly, if we act on A by:

(i) pivoting on the entry in the γ_1 row and the 3 column;
(ii) pivoting on the entry in the x_1 row and the e column;
(iii) swapping the δ_1 column and the 4 column;
(iv) swapping the t column and the x_3 column

then we produce an identical copy of A. Thus there is an automorphism Ω_2 of M that swaps

$$\{\gamma_1, 3\},\ \{x_1, e\},\ \{\delta_1, 4\},\ \{t, x_3\}$$

and acts as the identity on other elements.

Since Ω_1 and Ω_2 are also automorphisms of M^*, we see that $\phi^{-1} \circ \Omega_1 \circ \phi$ and $\phi^{-1} \circ \Omega_2 \circ \phi$ are automorphisms of M that swap, respectively, the pairs

$$\{\delta_4, \gamma_1\},\ \{1, 4\},\ \{x_3, y_1\},\ \{x_1, y_2\} \text{ and}$$

$$\{\delta_4, \delta_2\},\ \{1, 2\},\ \{z_2, y_1\},\ \{z_1, y_2\}$$

while leaving all other elements unchanged. By studying these four automorphisms, we see that

$$O_1 = \{e, t, x_1, y_1, z_1, y_2, z_2, x_3\} \quad \text{and} \quad O_2 = \{1, 2, 3, 4, \gamma_1, \delta_1, \delta_2, \delta_4\}$$

are contained in orbits of the automorphism group of M.

Consider M/e. It is represented by the matrix $[I_7|A']$ where A' is

$$\begin{array}{c} \\ \delta_4 \\ \gamma_1 \\ \delta_2 \\ y_1 \\ y_2 \\ x_1 \\ z_2 \end{array} \begin{array}{c} \begin{array}{cccccccc} x_3 & z_1 & t & 2 & \delta_1 & 3 & 1 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

It is easily checked that M/e has no triangles. Since $\{1, 3, 4\}$ is a triangle of N, we deduce that M/e cannot have an N-minor. (This also shows that O_1 and O_2 are in fact orbits.) Certainly $M \setminus e$ is not internally 4-connected, since it contains the quad $\{1, 2, 3, 4\}$. Consequently, we cannot delete or contract an element from O_1 to produces an internally 4-connected matroid with an N-minor.

Since $\phi(e) = 2$, we see that $M^*/2$ does not have an N-minor. As N is self-dual, this means that $M^*/2$ does not have an N^*-minor, so $M \setminus 2$ does not have an N-minor. Moreover, $M/2$ has a quad, so it is not internally 4-connected. Thus we cannot delete or contract any element from O_2 to produce an internally 4-connected matroid with an N-minor, and we have completed the proof of our claim.

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