Some Classes of Continua Related to Clan Structures.

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by

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ABSTRACT

Let $X$ be a continuum (a compact, connected Hausdorff space). A clan structure on $X$ is a continuous, associative operation $m$ on $X$ with an identity element $1$; $X$ is said to admit a clan structure if there is a clan structure on $X$. Our concern here is the investigation of certain classes of continua which are related to continua admitting clan structures.

In the study of clan structures on $X$, it often happens that, with suitable restrictions on the multiplication and possibly some topological restrictions on $X$, one finds that $X$ possesses a good deal of additional topological structure. The question arises: Does this topological structure allow one to define a clan multiplication on $X$? To rephrase the question: Suppose $m$ is a clan structure on $X$, where some additional properties $P$ are assumed. Let $\mathcal{I}$ be the topological properties of $X$ which are known to be implied, and let $\mathcal{C}$ denote the class of continua which possesses these properties $\mathcal{I}$. Does every member of $\mathcal{C}$ admit a clan structure? Much of the work here was done with this question in mind.

In Chapter II, we define the class of ruled spaces. It is shown that a large class of clans are ruled spaces, and a counterexample to the converse is given. The notion of a
partially ordered space is introduced and ruled spaces are characterized as a certain type of partially ordered space. It is shown that all ruled spaces are acyclic and may be embedded by an order preserving homeomorphism into a compact connected semilattice with identity.

At this point we restrict our attention to metric ruled spaces. It is found that trees can be characterized as ruled spaces. The notion of a radially convex metric is introduced, and we show that every metric ruled space admits a radially convex metric. This proves to be a useful tool in defining multiplications on metric ruled spaces. For example, we define a clan structure on a ruled space which has a 'free arc'. The chapter concludes with some theorems about ruled subsets of the plane. It is shown that many such spaces admit semilattice structures.

In Chapter III, we define a dendritic extension of a clan and prove that every dendritic extension of a clan admits a clan structure. As a corollary, we obtain that every locally connected metric continuum containing at most one circle admits a clan structure.

In Chapter IV, the notion of an inessential space is introduced and inessential continua are studied. We prove that every clan which is not a group is inessential. The main result of this chapter is a type of unique factorization theorem for the product of circles; namely, we prove that
the only one-dimensional factor space of the countable product of circles is a circle.
By a continuum we mean a compact connected Hausdorff space.

Since we will be working with spaces which are not necessarily metric, we employ nets. For a complete account, see Chapter 2 of Kelley [9]. Briefly, a set $D$ is directed by a reflexive, transitive relation $\leq$ if and only if for each $\alpha, \beta \in D$, there exists $\gamma \in D$ such that $\alpha, \beta \leq \gamma$. A net in a space $X$ is a function $f$ from a directed set $D$ to $X$. $f$ clusters at $x \in X$ if and only if for each open $U$ containing $x$ and each $\alpha \in D$ there is $\beta \in D$, $\beta \geq \alpha$, such that $f(\beta) \in U$; $f$ converges to $x$ if and only if for each open $U$ containing $x$ there is $\beta \in D$ such that $f(\alpha) \in U$ for all $\alpha \geq \beta$. For $\alpha \in D$, we write $f(\alpha) = x_\alpha$ and denote the net $f$ by $x_\alpha$ or sometimes $\{x_\alpha\}_{\alpha \in D}$. $x_\alpha \rightarrow x$ means the net $x_\alpha$ converges to $x$. The main facts we use about nets are the following: A function $g$ from a space $X$ to a space $Y$ is continuous if and only if $x_\alpha \rightarrow x$ in $X$ implies $g(X_\alpha) \rightarrow g(x)$ in $Y$. A subset $A \subseteq X$ is closed in $X$ if and only if $x_\alpha \rightarrow x$ and $x_\alpha \in A$ implies $x \in A$. $X$ is compact if and only if each net in $X$ clusters at some point of $X$.

Let $X$ be a topological space and let $C(X)$ be the collection of all closed subsets of $X$. The finite topology
on $C(X)$ is the topology generated by the subbasis consisting of all sets

$$N(U,V) = \{ A \in C(X) : A \subseteq U, A \cap V \neq \emptyset \}$$

where $U, V$ are open in $X$ and $\emptyset$ denotes the empty set. For details, see Michael [15]. There it is shown that if $X$ is compact Hausdorff, then $C(X)$ is compact Hausdorff.

1.1. Lemma. Suppose $f : Y \times X \to X$ is continuous and $X$ is compact. Then

(i) if $y_\alpha \to y$ in $Y$, then $f(y_\alpha, A) \to f(y, A)$ in $C(X)$ for all $A \in C(X)$.

(ii) if $y_\alpha$ clusters at $y$ and if $f(y_\alpha, A)$ is a tower such that $f(y_\alpha, A) \subseteq f(y_\beta, A)$ for $\alpha \leq \beta$, then $f(y, A) = \left( \bigcup f(y_\alpha, A) \right)^*$. 

Proof.

(i) Let $N(U; V_1, \cdots V_n)$ contain $f(y, A)$. Suppose that for each $\alpha$ there is $\beta \geq \alpha$ such that $f(y_\beta, A)$ is not a subset of $U$. Then we can choose a net $f(y_\beta, a_\beta)$ in $X \setminus U$. However $a_\beta$ clusters at some point $a \in A$ and hence $f(y_\beta, a_\beta)$ clusters at $f(y, a) \in U$, a contradiction. So there is a $\beta$ such that $f(y_\alpha, A) \subseteq U$ for all $\alpha \geq \beta$. Pick $f(y, a_1) \in f(y, A) \cap V_i \neq \emptyset$ for $i = 1, \cdots, n$. There is $\beta' \geq \beta$ such that $f(y_\alpha, a_1) \in V_i$ for $\alpha \geq \beta'$. Hence $f(y_\alpha, A) \in N(U; V_1, \cdots V_n)$ for $\alpha \geq \beta'$ and so $f(y_\alpha, A) \to f(y, A)$.

(ii) This is a lemma of Koch, [11], p. 398.

Let $X$ be a Hausdorff space. A continuous operation $m : X \times X \to X$ is a semigroup structure on $X$ if $m$ is
associative \((m(x, m(y, z)) = m(m(x, y), z)\) for all \(x, y, z \in X\).

If, in addition, \(X\) is a continuum and there is an identity 
\(1 \in X\) \((m(x, l) = m(l, x) = x\) for all \(x \in X\),\) then \(m\) is a 
clan structure on \(X\). Whenever possible, operations will be 
denoted by juxtaposition; thus \(m(x, y) = xy\).

Suppose \(S\) is a semigroup; that is, \(S\) is a Hausdorff 
space and we are given a semigroup structure on \(S\). A 
nonvoid subset \(A \subseteq S\) is an ideal of \(S\) provided 
x\(A \cup Ax \subseteq A\) for all \(x \in S\). It is easily shown that the 
intersection of a collection of ideals of \(S\) is again an 
ideal of \(S\), provided it is nonvoid. The kernel of \(S\), 
\(K(S)\), is defined as the intersection of all ideals of \(S\), 
provided this intersection is nonvoid. If \(S\) is compact, 
then \(K(S)\) exists.

A semilattice is a semigroup \(S\) such that \(x^2 = x\) and 
\(xy = yx\) for all \(x, y \in S\). The kernel of a semilattice is a 
point, if it exists. Thus compact semilattices have a zero; 
that is, an element \(0\) such that \(x0 = 0x = 0\) for all \(x \in S\).

Let \(S\) be a semilattice. The relation \(\leq\) defined on 
\(S\) by \(x \leq y\) iff \(xy = x\) is reflexive, transitive, and anti-
symmetric. In addition, the graph of \(\leq\) \(((x, y) : x \leq y)\) 
is a closed subset of \(S \times S\). In general, if \(X\) is a 
Hausdorff space and \(\leq\) is a partial ordering of the set \(X\) 
such that the graph of \(\leq\) is closed in \(X \times X\), then we say 
\(X\) is a partially ordered space. So a compact semilattice 
is a partially ordered space with minimum element \(0\).
An arc is a continuum with exactly two non-cutpoints. Equivalently, an arc is a linearly ordered continuum. Let $A$ be an arc with non-cutpoints $p$ and $q$. Define a multiplication on $A$ by $xy = x$ if $x \in [p,y]$ and $xy = y$ otherwise. Here $[p,y]$ denotes the subarc of $A$ with endpoints $p$ and $y$ if $p \neq y$ and $[p,y] = p$ if $p = y$. With this multiplication, $A$ is a compact connected semilattice with zero $p$ and identity $q$. The partial order induced by the operation is linear and $xy = \min\{x,y\}$ with respect to this order. $A$ is then called a min thread. A standard thread is an arc $A$ with a semigroup structure so that one endpoint is a zero and the other endpoint is an identity.

We list two results on standard threads which will be used in the next chapter.

1.2. Theorem. Phillips [16]. Let $T$ be a standard thread in a semigroup $S$. For $x \in S$, $xT$ is either a point or an arc with endpoints $x \cdot 1$ and $x \cdot 0$.

1.3. Theorem. Koch [12]. Let $S$ be a clan. If $K(S) \neq S$ and each subgroup of $S$ is totally disconnected, then there is a standard thread in $S$ containing the identity of $S$ and meeting $K(S)$.

It should be noted that Phillips states his theorem only for metric standard threads; however, no use is made of metricity in his proof.
CHAPTER II
RULED SPACES

In [14], Koch and McAuley define a class of metric continua, called ruled continua, and show that each ruled continuum admits a clan structure with zero. There are eight conditions which define the class. The first four are of interest in themselves because they define a class closely related with clans having a zero.

2.1. Definition. Let X be a continuum. Suppose there is given $0 \in X$, $I \subset X$, and $\mathcal{C} = \{[0,e] : e \in I\}$, a family of arcs in X. We say that X is a ruled space with ruling $\mathcal{C}$ provided:

(i) $X = \bigcup \mathcal{C}$

(ii) For each $e \in I$, there is a unique arc $[0,e] \in \mathcal{C}$

(iii) If $e,f \in I$ with $e \neq f$, then $[0,e] \cap [0,f]$ is a proper subarc of each of $[0,e]$ and $[0,f]$.

Notation. For $x \in X$, $[0,x]$ denotes the subarc with endpoints $0$ and $x$ of any member of $\mathcal{C}$ containing $x$.

(iv) If $x_\alpha \to x$ in $X$, then $[0,x_\alpha] \to [0,x]$ in $C(X)$.

2.2. Theorem. Suppose $S$ is a clan with $0$ containing a standard thread $T$ from $0$ to $1$. Then $S$ is ruled by a set of left translates of $T$.

Proof. Take $\mathcal{C} = \{eT : eT \subset yT \text{ implies } eT = yT\}$. Let $0$ be the zero of $S$ and $I = \{e : eT \in \mathcal{C}\}$. By 1.2, we have
that \( xT \) is an arc with endpoints \( x \) and \( 0 \) for all \( x \neq 0 \).

2.2(i). Let \( x \in S \). Pick a maximal tower \( \{x_\alpha T\} \) of left translates of \( T \) which contains \( xT \). Let \( e \) be a cluster point of the net \( x_\alpha \). By 1.1(ii), \( eT = (Ux_\alpha T)^* \). Suppose \( eT \subseteq yT \). Then \( x_\alpha T \subseteq yT \) for all \( x_\alpha T \). Hence \( yT \in \{x_\alpha T\} \), from which we get that \( eT = yT \). Thus \( x \in eT \in \mathcal{C} \).

2.1(ii) is clear.

2.1(iii) Let \( e, f \in I \) with \( e \neq f \). Give \( eT \) and \( fT \) the cutpoint orderings which make \( e \) and \( f \) the maximum elements of \( eT \) and \( fT \) respectively. \( eT \cap fT \) is a closed subset of \( eT \) (and of \( fT \)). Let \( z = \max\{eT \cap fT\} = \max\{eT \cap fT\} \). We show \( zT = eT \cap fT \). Since \( z \in eT \cap fT \), we have \( z = eT_1 = fT_2 \) for some \( t_1, t_2 \in T \). Therefore \( zT = eT_1 T = fT_2 T \subseteq eT \cap fT \). Now let \( w \in eT \cap fT \). Then \( w \leq z \) in the cutpoint order on \( eT \). Since \( zT \) is an arc contained in \( eT \) having \( z \) and \( 0 \) as endpoints, \( w \in zT \). Clearly, \( z \neq e, f \); otherwise \( e = f \). Therefore \( eT \cap fT \) is a proper subarc of each.

2.1(iv) Let \( x_\alpha \to x \) in \( X \). By 1.1(i) \( x_\alpha T \to xT \). This completes the proof.

2.3. Corollary. If \( S \) is a clan with \( 0 \) and totally disconnected subgroups, then \( S \) is a ruled space. In
particular, every compact connected semilattice with identity is a ruled space.

**Proof.** By 1.3, there is a standard thread from 0 to 1 in $S$.

2.4. Example. A ruled space which admits no clan structure with 0.

Let $S = [0,1) \times [0,1] \cup (1,0)$ and let

$L = [0,\Omega) \times [0,1] \cup (\Omega,0)$, where $[0,1]$ is the unit interval and $[0,\Omega)$ is the set of ordinals out to $\Omega$, the first uncountable ordinal. With the dictionary order ($(a,b) < (c,d)$ if and only if $a < c$ or $a = c$ and $b < d$), $S$ and $L$ are arcs. We establish two facts.

(a) Suppose $f: L \to S$ is continuous. Then $f(L) \subseteq [x] \times [0,1]$ for some $x \in [0,1]$.

**Proof.** Suppose not. Then we can find a closed subarc $[(a,b),(c,d)] \subset f(L)$ with $a < c$ which does not contain $f(\Omega,0)$. Since $f$ is continuous $f(A) \cap [(a,b),(c,d)] = \emptyset$ for some subarc $A$ of $L$ containing $(\Omega,0)$. But there exist uncountably many disjoint open sets in $[(a,b),(c,d)]$ and there do not exist uncountably many disjoint open sets in $f^{-1}[(a,b),(c,d)] \subseteq L \setminus A$. This is a contradiction.

(b) Suppose $f: S \to L$ is continuous. Then $(\Omega,0) \not\in f(S)$ or $f(S) = [(\Omega,0)]$. 
Proof. Suppose $S \backslash f^{-1}(\mathcal{N},0) \neq \emptyset$. Then there is a sequence $\{x_i\}$ in $S \backslash f^{-1}(\mathcal{N},0)$ which converges to a point $x$ in $f^{-1}(\mathcal{N},0)$. Then $f(x_i)$ converges to $(\mathcal{N},0)$ in $L$, an impossibility since $\{f(x_i)\}$ is a countable subset of $L \backslash (\mathcal{N},0)$.

To construct the example, take two disjoint copies of $L$, $[m_1,M_1]$ and $[m_2,M_2]$, where $M = (\mathcal{N},0)$ and $m = (0,0)$ in $L$. Let $S = [m',M']$, where $m' = (0,0)$ and $M' = (1,0)$, and let $R$ be the space obtained from these three arcs by identifying $M_1,m_2,m'$. Suppose $R$ admits a clan structure with zero. By Cor. 2, p. 613, of [10], the identity $1$ of $R$ must be one of the endpoints $m_1,M_2,M'$ of $R$, and the arc $T$ from $0$ to $1$ is a standard thread. We distinguish four cases.

Case 1. $0 = M_1$. Then $T = L$ or $T = S$. If $T = L$, then $M'T = S$, a contradiction of (a) above. If $T = S$, then $m_1T = L$, a contradiction of (b).

Case 2. $1 = M_1$.

(i) If $0 \in (m_1,M_1)$, then $T \subset [m_1,M_1]$ is metric. Since $S \subset M'T$, this implies that $S$ is metric, which is not so.

(ii) If $0 \in [m_2,M_2]$, there is $x \in T$ such that $M'[1,x] = S$. This is a contradiction of (a).

(iii) If $0 \in (m',M')$, then $M_2[m_1,M_1] = M_2$ and there exists $x \in (M_1,0)$ such that $M_2[x,0] = [m_2',M_2]$, a contradiction.

Case 3. $1 = M_2$. This is handled much like Case 2.
Case 4. \( l = M' \).

(i) If \( 0 \in [m_1, M_1] \), then \( M_2 S = M_2 \). Hence \( M_2 [0, M_1] = [0, M_2] \), a contradiction, since \([0, M_1]\) has only one point of non-first countability and \([0, M_2]\) has two such points.

(ii) If \( 0 \in [M_2, m_2] \), then \( m_1 S = [m_1, x] \) for some \( x \in [m_2, M_2] \). Hence \( x[M_1, 0] = [x, 0] \), which cannot be since \([x, 0]\) has more points of non-first countability than \([M_1, 0]\).

(iii) If \( 0 \in (m', M') \), then \( M_2 [x, M'] = [M_2, m_2] \) for some \( x \in (0, M') \), a contradiction to (b) above. This concludes the example.

Ruled Spaces as Partially Ordered Spaces.

2.5. Definition. A partially ordered space is a Hausdorff space \( X \) together with a partial ordering relation \( \leq \) on the set \( X \) so that the graph of \( \leq \) is closed in \( X \times X \). An element \( x \in X \) is maximal (minimal) with respect to \( \leq \) iff \( x \leq y (y \leq x) \) implies \( x = y \). For \( A \subseteq X \), \( L(A) = \{ y \in X : y \leq a \text{ for some } a \in A \} \) and \( M(A) = \{ y \in X : a \leq y \text{ for some } a \in A \} \). Clearly if \( A \) is closed, then \( L(A) \) and \( M(A) \) are closed. We refer to sets of the form \( L(x) \) and \( M(x) \) as lower sets and upper sets respectively.

2.6. Theorem. Let \( X \) be a ruled space. Define \( \leq \) by \( x \leq y \iff x \in [0, y] \). Then

(i) \( X \) is a partially ordered space with respect to \( \leq \).

(ii) \( 0 \) is the minimum element of \( X \) and \( 1 \) is the set of maximum elements of \( X \).
(iii) For $x \in X$, $L(x) = [0,x]$ and $M(x) = U[[x,e] : e \in I(x)]$ where $I(x)$ denotes the set of all $e \in I$ such that $x \in [0,e]$, and $[x,e] = [0,e] \setminus [0,x]$.

Proof. (i) $\leq$ is a partial ordering of the set $X$. That the graph of $\leq$ is closed is seen as follows. Let $x_\alpha \to x$ and $y_\alpha \to y$ in $X$ with $x_\alpha \leq y_\alpha$ for each $\alpha$. Then $x_\alpha \in [0,y_\alpha]$. Since $[0,x_\alpha] \subseteq [0,y_\alpha] \to [0,y]$, we have $[0,x] \subseteq [0,y]$. So $x \leq y$. (ii) and (iii) are clear.

We shall refer to the order defined in 2.6 as the natural partial order on the ruled space $X$. Note that it has the property that lower sets are connected and upper sets of unrelated points are disjoint. This property is characteristic of ruled spaces, as the next theorem shows.

2.7. Lemma. If $X$ is a compact partially ordered space, then for each $x \in X$, there is a maximal element $M$ of $X$ such that $x \leq M$.

Proof. Let $C$ be a maximal chain in $X$ which contains $x$. Let $y,z \in C^*$, the closure of $C$, and let $y_\alpha \to y$, $z_\alpha \to z$ where $y_\alpha,z_\alpha \in C$ for each $\alpha$. Then $y_\alpha \leq z_\alpha$ or $z_\alpha \leq y_\alpha$ for each $\alpha$, and one of these must be the case cofinally. Hence we may suppose $y_\alpha \leq z_\alpha$ cofinally. This implies that $y \leq z$. Thus $C^* = C$ since $C^*$ is a chain in $X$ and $C$ is maximal. From the compactness of $C$, we get that $C$ has a maximum element $M$.

2.8. Theorem. Suppose $X$ is a compact partially ordered space with a unique minimal element $0$. If (1) $L(x)$ is
connected for each \( x \in X \), and (ii) for \( x,y \in X \) such that \( y \preceq x \) and \( x \preceq y \), we have \( M(x) \cap M(y) = \emptyset \), then \( X \) is ruled by the maximal chains in \( X \).

**Proof.** Let \( x \neq 0 \), and let \( y,z \in L(x) \). Then \( z \preceq y \) or \( y \preceq z \), since \( M(y) \cap M(z) \neq \emptyset \). Thus \( L(x) \) is a compact connected linearly ordered subset of \( X \) and thus is an arc. Now let \( I = \{ e \in X : e \text{ is maximal} \} \) and let \( \mathcal{M} = \{ L(e) : e \in I \} \). For \( e \in I \), \( L(e) \) is a maximal chain of \( X \). Suppose \( C \) is a maximal chain of \( X \). Then by 2.8, \( C \) has a maximum element \( e \) and \( e \) is maximal in \( X \). Thus \( C = L(e) \).

2.1(i) Lemma 2.8 implies that \( X = \bigcup \mathcal{M} \).

2.1(ii) and 2.1(iii) are clear.

2.1(iv) Let \( x_\alpha \to x \) in \( X \). Pick a convergent subnet \( L(x_\beta) \) of \( L(x_\alpha) \) in \( C(X) \). Then \( L(x_\beta) \to L \), a compact connected subset of \( X \) containing 0 and \( x \) and contained in \( L(x) \), an arc. Therefore \( L = L(x) \). Consequently \( L(x_\alpha) \to L(x) \).

In the next definition and in Chapter IV at one point, we make use of cohomology groups. For a description of these, see [18] and [4].

2.9. **Definition.** A connected space \( X \) is acyclic iff it has the cohomology groups of a point; that is, if \( H^p(X) = 0 \) for \( p \geq 1 \). (Iff means if and only if)

2.10. **Theorem.** Every ruled space is acyclic.

**Proof.** This follows from theorem 1 of [19], which we state in the form we use it: Let \( X \) be a compact Hausdorff space and \( \preceq \) a partial ordering of \( X \). If for each pair of closed subsets \( A \) and \( B \) of \( X \),
L(A) ∩ L(B) = L(C) for some closed subset C of X and
L(A) is connected and if L(x) is acyclic for each x ∈ X,
then X is acyclic. In our case L(x) = [0, x] is acyclic,
L(A) = U([0, a] : a ∈ A] is connected and L(A) ∩ L(B) =
L(L(A) ∩ L(B)). Thus the theorem applies.

2.11. Remark. Let S be a compact connected semilattice
with identity and let R be the partial order of S
induced by the operation. S is ruled by a set of transla­
tes of a standard thread from 0 to 1. Let R_o denote the
natural partial order obtained from this ruling. Then
R_o ⊆ R. Does a compact connected semilattice have the same
property? In general, when is a partially ordered space a
ruled space? The circle can be partially ordered, but it
is not acyclic.

2.12. Definition. A closed subset A of a ruled space X
is a ruled subset of X iff when x ∈ A, [0, x] ⊆ A. Let
R be the collection of ruled subsets of X.

2.13. Theorem. As a subspace of C(X), R is compact and
connected. The union and intersection of two ruled subsets
is a ruled subset. The operation of union in R is con­
tinuous. Consequently, (R, U) is a compact connected semi­
lattice with identity {0} and zero X.

Proof. Since C(X) is compact, we need only show that R
is closed in C(X). Suppose A_α → A, where A_α ∈ R for each
α ∈ D, a directed set. Then A ∈ C(X). Let x ∈ A. We
will show \([0, x] \subseteq A\). Let \(D' = \{ V : V \text{ is open and contains } x \}\).

For each \(V \in D'\), we have \(A \in N(X, V)\). Therefore we can choose \(a(V) \in D\) so that \(A_\beta \cap V \neq \emptyset\) for \(a(V) \leq \beta\). Well-order \(D'\). We will construct a subset \(A\) of \(A_\alpha\) such that \(A_\alpha(V) \cap \neq \emptyset\) for all \(V \in D'\). Choose \(a(V_1) \in D\) so that \(A_\alpha(V_1) \cap V_1 \neq \emptyset\). Suppose \(a(V_i)\) has been defined for \(i < n\) so that \(a(V_i) < a(V_j)\) for \(i < j < n\) and \(A_\alpha(V_i) \cap V_i \neq \emptyset\) for \(i < n\). If the set \(\{a(V_i) : i < n\}\) is not cofinal in \(D\), there is a \(\beta \in D\) such that \(a(V_i) < \beta\) for all \(i < n\). Since the set \(\gamma \in D : A_\gamma \cap V_n \neq \emptyset\) is residual in \(D\), we can choose \(a(V_n) \geq \beta\) for which \(A_\alpha(V_n) \cap V_n \neq \emptyset\). Hence by the principal of transfinite induction, the existence of the subnet \(A_\alpha(V)\) is established. Then \(A_\alpha(V) \rightarrow A\). Choose \(x_\alpha(V) \in A_\alpha(V) \cap V\) for each \(V \in D'\). Then \(x_\alpha(V) \rightarrow x\) and \([0, x_\alpha(V)] \subseteq [0, x]\). But \([0, x_\alpha(V)] \subseteq A_\alpha(V)\). Hence \([0, x] \subseteq A\) and \(A \in R\).

Union is continuous. It is evident that the intersection and union of ruled subsets is a ruled subset. Let \(A, B \in R\).

Let \(N(U; V_1', \ldots, V_n')\) be a neighborhood of \(A \cup B\). Let \(V_1', V_2', \ldots, V_m'\) denote the \(V_i\)'s which meet \(A\) and \(V_1'', V_2'', \ldots, V_k''\) denote the \(V_i\)'s which meet \(B\). Note that every \(V_i\) is a \(V'\) or a \(V''\). Suppose \(A' \in N(U; V_1', \ldots, V_m')\) and \(B' \in N(U; V_1'', \ldots, V_k'')\). Then \((A' \cup B') \cap V_i \neq \emptyset\) for \(i = 1, \ldots, n\) and \(A' \cup B' \subseteq U\). We conclude that union is continuous.
R is connected. Suppose $R = U \cup Z$, where $Y \cap Z = \emptyset$ and $Y$ is open and closed in $R$. Assume $\{0\} \in Y$. We will show $Z = \emptyset$. Suppose $A \in Z$. Choose a basic neighborhood $N(U; V_1, \ldots, V_n)$ of $A$ lying in $Z$. Let $x_i \in A \cap V_i$ for $i = 1, \ldots, n$. Then $B = \bigcup_{i=1}^{n} [0, x_i] \in N(U; V_1, \ldots, V_n) \subset Z$.

Consider the function $f : \prod_{i=1}^{n} [0, x_i] \to K = \bigcup_{i=1}^{n} [0, y_i] : y_i \in [0, x_i]$ given by $f(y_1, \ldots, y_n) = \bigcup_{i=1}^{n} [0, y_i]$. From 2.1(iv) and the continuity of union we get that $f$ is continuous. Since $f$ is onto, $K$ is connected. But $\{0\} \in K$ and $B \in K$. This contradicts $Y \cap Z = \emptyset$.

2.15. Theorem. The function $\psi : X \to R$ defined by $\psi(x) = [0, x]$ is a homeomorphism into and preserves the natural ordering on $X$.

Proof. $\psi$ is 1-1. If $x \leq y$, then $\psi(x) = [0, x] \subseteq [0, y] = \psi(y)$; hence $\psi$ is order preserving. If $x_\alpha \to x$ then $\psi(x_\alpha) = [0, x_\alpha] \to [0, x] = \psi(x)$; hence $\psi$ is continuous. Since $X$ is compact, $\psi$ is an embedding.

The Branch Point Multiplication

Let $X$ be a ruled space. For $x, y \in X$, $[0, x] \cap [0, y]$ is a subarc of each. Furthermore, this arc is a chain with respect to the natural partial ordering on $X$. The function $\varphi : X \times X \to X$ defined by $\varphi(x, y) = \sup([0, x] \cap [0, y])$ will be called the branch point.
multiplication on $X$. In most cases $\varphi$ is not continuous; in fact, we will show later that if $X$ is metric, the class of spaces for which $\varphi$ is continuous is precisely the class of metric trees.

2.14. Proposition. Let $X$ be a ruled space. Then for $x,y \in X$,

(i) If $\varphi(x,y) = x$, then $I(y) \subseteq I(x)$.

(ii) If $I(y) \subseteq I(x)$ and $I(y) \neq I(x)$, then $\varphi(x,y) = x$.

(iii) $\varphi(x,y) \neq x,y$ iff $I(x) \cap I(y) = \emptyset$.

Proof. (i) If $\varphi(x,y) = x$, then $x \in [0,y]$. Hence if $y \in [0,e]$, then $x \in [0,e]$; that is, $I(y) \subseteq I(x)$.

(ii) Let $f \in I(x)$ such that $y \in [0,f]$. Let $e \in I(y)$; then $x,y \in [0,e]$. Therefore $\varphi(x,y) = x$ or $y$. If $\varphi(x,y) = y$, then (i) implies that $y \in [0,x] \subseteq [0,f]$, a contradiction. Therefore $\varphi(x,y) = x$.

(iii) If $I(x) \cap I(y) = \emptyset$ and $\varphi(x,y) = x$, then (i) implies that $\emptyset \neq I(y) \subseteq I(x)$, a contradiction.

2.15. Proposition. If $\varphi(x,y) \neq x,y$, $x_\alpha \to x$, and $y_\alpha \to y$, then $\varphi(x_\alpha,y_\alpha) \neq x_\alpha,y_\alpha$ residually.

Proof. If not, then $\varphi(x_\alpha,y_\alpha) = x_\alpha$ cofinally or $\varphi(x_\alpha,y_\alpha) = y_\alpha$ cofinally. Suppose $\varphi(x_\alpha,y_\alpha) = x_\alpha$ cofinally.

Then $x_\alpha \in [0,y_\alpha]$ cofinally, and hence $x \in [0,y]$, a contradiction.
Metric Ruled Spaces

2.16. Definition. A tree is a locally connected metric continuum containing no simple closed curve.

2.17. Definition. Let $X$ be a space and $x \in X$. $x$ is an endpoint of $X$ iff for each open $U$ containing $x$ there is an open $V \subset U$ containing $x$ such that $\text{Bdry } V$ is a point. $x$ is a cutpoint of $X$ if $X \setminus x$ is not connected.

2.18. Theorem. Whyburn, p. 88, [21]. In order that a metric continuum $X$ be a tree it is necessary and sufficient that each point of $X$ be either a cutpoint of $X$ or an endpoint of $X$.

2.19. Theorem. Koch and McAuley [13] and [14]. Let $X$ be a tree and let $0 \in X$. For $x \in X$, let $[0,x]$ denote the arc from $0$ to $x$. (It follows from the definition that a tree is uniquely arcwise connected.) Let $I = \{\text{endpoints of } X\} \setminus \{0\}$. Then $\mathcal{O} = \{[0,e] : e \in I\}$ is a ruling of $X$. Furthermore, the branch point multiplication associated with this ruling is continuous.

Proof. The proof that $\mathcal{O}$ is a ruling is on pp. 1-2 of [14] and the proof the $\varphi$ is continuous is Lemma 5 of [13]. The assumption that $I$ is compact is made in Lemma 5, but is not used.

2.20. Theorem. Suppose $X$ is a metric continuum with a family $\mathcal{O}$ of arcs in $X$ satisfying properties (i)-(iii) of 2.1. Then if $\varphi$ is continuous in one variable, satisfies
2.1(iv) and $X$ is a tree.

**Proof.** Let $x_n \to x$. Let $N(U;V)$ be a subbasic neighborhood of $[0,x]$. Pick $p \in [0,x]$ so that $[0,p] \cap V \neq \emptyset$. Since $\phi(x_n,x) \to x$, there is an $N$ such that for all $n > N$, $x_n \in V$ and $\phi(x_n,x) \in [p,x]$. Hence for $n > N$, $[0,x_n] \in N(U,V)$, and 2.1(iv) is satisfied.

Now suppose $x \in (0,e)$. We show that $x$ is a cutpoint of $X$. Under the natural partial ordering of $X$, $M(x)$ is a closed set. Let $K = X \setminus (M(x) \setminus \{x\})$. Suppose $y_n \to y$, where $y \in M(x) \setminus \{x\}$. Since $\phi(y_n,y) \to y$, $\phi(y_n,y) \in M(x) \setminus \{x\}$ for sufficiently large $n$. Hence $y_n \in M(x) \setminus \{x\}$ for sufficiently large $n$. Thus $K$ is closed, and $x$ is a cutpoint of $X$. Now we show that $e \in I$ is an endpoint of $X$. Let $U$ be a neighborhood of $e$. Suppose for each $x \in [0,e)$ there is $y \not\in U$ such that $x \in [0,y]$. Pick a sequence $x_n \to e$ on $[0,e]$, and a corresponding sequence $y_n \not\in U$ such that $x_n \in [0,y_n]$. We may assume $y_n \to y \not\in U$. Then $\phi(y_n,e) = x_n \to e$ and $\phi(y_n,e) = \phi(y,e)$. We conclude that $e \not\in I$, a contradiction. Thus $M(x) \subseteq U$ for some $x \in [0,e)$, and so $e$ is an endpoint of $X$. Suppose $\phi(x,y) = 0$ for some $x,y \neq 0$. Let $K = \{z \in X : \phi(x,z) \neq 0\}$. By the continuity of the function $\phi(x,z)$, $K$ is open. If $w \not\in K \cup \{0\}$, then the continuity of $\phi(w,z)$ gives us that $w \not\in K^*$. Therefore $K^* = K \cup \{0\}$ and so $0$ is a cutpoint of $X$. Suppose $\phi(x,y) = 0$ implies $x = 0$ or $y = 0$. Let $U$ be an open set containing $0$. Pick $e \in I$. 

An argument similar to the one showing $e$ is an endpoint of $X$ shows that $0$ is an endpoint of $X$. Since every point of $X$ is an endpoint of $X$ or a cutpoint of $X$, $X$ is a tree by 2.18.

Koch and McAuley's fifth condition will now be introduced.

2.21. Definition. Let $X$ be a metric ruled space. A metric $d$ on $X$ is radially convex iff for all $x,y \in X$ such that $x \in [0,y)$, $d(0,x) < d(0,y)$.

2.23. Theorem. Carruth, p.11, [1]. If $X$ is a partially ordered compact metric space, then there is an order-preserving embedding of $X$ into the Hilbert cube. (The Hilbert cube is the product $\prod_{i=1}^{\infty} I_i$ of unit intervals. The order is given by $(x_i) < (y_i)$ iff $x_i \leq y_i$ for $i = 1,2,\cdots$. The metric is given by $d((x_i),(y_i)) = \sum_{i=0}^{\infty} \frac{|x_i-y_i|}{2^i}$).

2.23. Corollary. If $X$ is a metric ruled space, then $X$ admits a radially convex metric.

Proof. $X$ is partially ordered by the natural partial ordering. Let $h$ be an order-preserving embedding of $X$ into the Hilbert cube. Then the metric $d$ induced by $h$ is radially convex.

2.24. Definition. Let $X$ be a metric ruled space with radially convex metric $d$. Denote $d(0,x)$ by $d_x$ for $x \in X$. 
Let $t$ be a nonnegative real number. Then $(x, t)$ denotes the point $z \in [0, x]$ such that $d_z = t$ for $t \leq d_x$ and $(x, t) = x$ for $t \geq d_x$.

**2.25. Lemma.** If $x_n \to x$, and $t_n \to t$ then $(x_n, t_n) \to (x, t)$.

**Proof.** Let $y$ be a cluster point of $(x_n, t_n)$. Then $y \in [0, x]$ and $d_y = t$. Hence $y = (x, t)$.

**2.26. Theorem.** Let $X$ be a metric ruled space. Then there is an arc $[0, e] \in \mathcal{A}$ and an order-preserving retraction of $X$ onto $[0, e]$.

**Proof.** Let $d$ be a radially convex metric on $X$. Let $[0, e] \in \mathcal{A}$ be an arc of maximum diameter. Define $f(x) = (e, d_x)$ for all $x \in X$. Then $f(x) = x$ for $x \in [0, e]$. Suppose $x_n \to x$. Then by the continuity of the metric $d_{x_n} \to d_x$.

Hence by 2.25 $(e, d_{x_n}) \to (e, d_x)$. Thus $f$ is a retraction.

If $x \in [0, y]$, then $d_x \leq d_y$, so $f(x) = (e, d_x) \in [e, f(y)]$.

**2.27. Definition.** A space $X$ is contractible if there is a continuous function $H : X \times I \to X$ such that $H(x, 1) = x$ for all $x \in X$ and $H(x, 0) = H(y, 0)$ for all $x, y \in X$.

**2.28. Theorem.** Every metric ruled space is contractible.

**Proof.** Let $X$ be a metric ruled space with radially convex metric $d$. Choose $[0, e] \in \mathcal{A}$ so that the diameter of $[0, e]$ is maximum. Define $H : X \times [0, e] \to X$ by $H(x, t) = (x, d_t)$. Then $H(x, e) = (x, d_e) = x$ and $H(x, 0) = (x, d_0) = 0$ for all $x \in X$. For the continuity of $H$, let $x_n \to x$ in $X$
and \( t_n \to t \) in \([0,e]\). Then \( H(x_n,t_n) = (x_n,d_{t_n}) \to (x,d_t) = H(x,t) \), by 2.25.

2.29. Theorem. Suppose \( X \) is a metric ruled space with a free arc; that is, there is an \( e \in I \) and \( p \in [0,e) \) so that \([p,e]\) is open in \( X \). Then \( X \) admits a clan structure with 0.

**Proof.** Let \( d' \) be a radially convex metric on \( X \). By scaling down \( d' \), we can assume \( d'(0,x) \leq \frac{1}{2} \) for all \( x \in X \).

Now define a new metric on \( X \) as follows: Let \( r = d' \) and let \( h : [p,e] \to [0,1-r] \) be an onto homeomorphism.

\[
d(x,y) = \begin{cases} 
   d'(x,y) & \text{if } x,y \notin (p,e] \\
   |h(x) - h(y)| & \text{if } x,y \in [p,e] \\
   d'(x,p) + |h(p) - h(y)| & \text{if } x \notin (p,e], y \in [p,e].
\end{cases}
\]

\( d \) is a radially convex metric on \( X \) with the property that \( d(0,e) = 1 \) and \( d(0,x) \leq 1 \) all \( x \in X \). Let \( c \) be the point in \([p,e]\) such that \( d(0,c) = \frac{1}{2} \). Define multiplication on \( X \) as follows:

\[
xy = yx = \begin{cases} 
   0 & \text{if } x,y \notin (c,e] \\
   (y, \max(0,d_x + d_y - 1)) & \text{if } x \in [c,e].
\end{cases}
\]

We must show this is well defined, associative, continuous with zero 0 and identity e.

**Well-defined.** Note that \((X \setminus (c,e)] \times X \setminus (c,e]) \cap [c,e] \times X = [c] \times X \setminus (c,e] \). Hence we need only show that \( cy = 0 \) for
y \notin (c,e] by either rule. But if \( y \notin (c,e] \) then \( d_y \leq \frac{1}{2} \). So 
\[ \max(0, d_c + d_y - 1) = 0, \text{ and } cy = 0. \]

**Associative.** Let \( x, y, z \in X \).

**Case 1.** At least two of \( x, y, z \) are not in \( (c,e] \). By the 
commutativity of the operation, we may assume \( x, y \notin (c,e] \).
Then \( yz \notin (c,e] \). Hence \( x(yz) = 0 = (xy)z \).

**Case 2.** At least two of \( x, y, z \) are in \([c,e]\), say \( x, y \). Then 
\[ x(yz) = x(z, \max(0, d_y + d_z - 1)) \]
\[ = (z, \max(0, d_x + \max(0, d_y + d_z - 1) - 1)) \]
\[ = (z, \max(0, d_x - 1, d_x + d_y + d_z - 2)), \]
and \( (xy)z = (e, \max(d_x + d_y - 1, 0))z \)
\[ = \begin{cases} 
0 & \text{if } xy \notin (c,e), \ z \notin (c,e) \\
(z, \max(d_x + d_y + d_z - 2, 0)) & \text{if } xy \in [c,e] \text{ or } z \in [c,e] 
\end{cases} \]

But if \( xy, z \notin (c,e] \), then \( d_x + d_y - 1 \leq \frac{1}{2} \) and \( d_z \leq \frac{1}{2} \). So 
\( d_x + d_y + d_z - z \leq 0 \), and \( (xy)z = 0 = x(yz) \).

**Continuous.** Note that \( X \times X = [x, e] \times X \cup X \times [c,e] \cup (X \setminus (c,e))^2 \), 
a finite union of closed sets. We show continuity of the 
multiplication on each of these. Since \( xy = 0 \) on the last 
set, it is continuous there. Now suppose \( x_n \rightarrow x \) in \([c,e]\) 
and \( y_n \rightarrow y \) in \( X \). By 2.25, 
\[ x_n y_n = (y_n, \max(d_{x_n} + d_{y_n} - 1, 0) \]
\[ \rightarrow (y, \max(d_x + d_y - 1, 0)) \]. Finally note that \( xe = 
(x, \max(d_e + d_x - 1, 0)) = (x, d_x) = x \) and \( Ox = 0 \) for all 
\( x \in X \). This completes the proof of 2.29.
2.30. Corollary. The class of metric continua which admit a clan structure with 0 and a standard thread $T$ from 0 to 1 such that some subarc of $T$ is a neighborhood of 1 is topologically characterized as the class of metric ruled spaces which possess a free arc.

Ruled Subsets of the Plane

Let $S$ be a compact connected semilattice with 1 contained in the plane $\pi$. Then $S$ is a ruled subset of the plane. We now investigate the problem of making a semilattice with 1 out of a ruled subset of the plane. Some terminology from Whyburn, Chapter IV, [21], will be adopted. A two-cell is the homeomorphic image of the closed unit disk. A spanning arc of a two-cell $D$ is an arc $A \subseteq D$ such that $A$ meets the boundary of $D$ in its endpoints. It is characteristic of a two-cell that a spanning arc separates it into exactly two components.

In the following lemmas, $X$ is a ruled subset of the plane $\pi$ and $U = \pi \setminus X$.

2.31. Lemma. $I \subseteq \mathrm{Bdry} \ U$.

Proof. Let $e \neq f \in I$. Let $D$ be a two-cell containing $[0,e] \cup [0,f]$ so that $[e,f]$ is a spanning arc of $D$. Denote the two components of $D \setminus [e,f]$ by $N_1$ and $N_2$. Define $R = \{ x \in X : ([0,x] \setminus [0,e] \cup [0,f]) \cap N_1 \}$ meets $[0,e] \cup [0,f]$ and $L = \{ x \in X : ([0,x] \setminus [0,e] \cup [0,f]) \cap N_1 \}$ meets $[0,e] \cup [0,f]$. Clearly $X \setminus [e,f] = R \cup L$. Suppose
e ∈ int X. Let N be a two-cell containing e in its interior so that N ⊂ X and N ∩ [e, f] = {b}. Then [e, f] does not separate N. Hence N\[e, f] ⊂ L or N\[e, f] ⊂ R. Suppose N\[e, f] ⊂ L. Let b₁ → b, where b₁ ∈ N₁ for all i. Then [0, b₁] → [0, b] and [0, b] ⊂ int D. But [b₁, 0] ∩ (π\D) ≠ ∅ for all i. This is a contradiction to 2.1(iv).

2.32. Definition. Let x ∈ Bdry U. Then x is said to be accessible from U iff there is an arc A ⊂ U* such that A ∩ Bdry U = {x}.

2.33. Lemma. If 0 is accessible from U, then each element of I is accessible from U.

Proof. By theorem 4.1, page 111 of Whyburn [21], it suffices to show that U ∪ {e} is locally connected. Let D be a two-cell containing X in its interior and let A be an arc in D from 0 to boundary D so that A ∩ X = {0}. D exists since X is compact and A exists since X does not separate D (X is acyclic) and 0 is accessible. Let N be a two-cell and b ∈ [0, e) so that (b, e] ⊂ int N ⊂ N ⊂ int D and b ∈ Bdry N. We will show that the component C of N' = (int N) ∩ (U U {e}) which contains e is a neighborhood of e intU U {e}. Since such two-cells N form a neighborhood basis at e, it then will follow that U U {e} is locally connected.

We establish the following: If p, q lie in distinct components of N' such that p, q ∉ C, then there is an x ∈ X
so that some subarc of \([0,x]\) spans \(N\), separating \(p\) from \(q\).

Let \(B\) be an arc connecting \(p\) and \(q\) lying in \(\text{int}(D \setminus (A \cup X))\), a connected open subset of \(D\). Let \(B' = B \setminus ([p,p') \cup [q,q'])\), where \([p,p']\) is the subarc of \(B\) such that \(p' \in \text{bdry} \ N \) and \([p,p'] \subseteq \text{int} \ N\), and similarly for \([q,q']\). We consider two cases:

**Case 1.** (See Figure 1). Suppose \(p\) and \(q\) lie in the same component \(K\) of \(\text{int} \ N \setminus B'\). Then we can join \(p\) and \(q\) by an arc \(E\) lying in \(K\) so that \(E \cap [e,b] = \emptyset\) and \([p,p'] \cap E = \{p\}, [q,q'] \cap E = \{q\}\). Note that \([p,p'] \cup E \cup [q,q']\) spans \(N\). Let \(N_1\) and \(N_2\) denote the two components of \(\text{int} \ N \setminus ([p,p'] \cup E \cup [q,q'])\), where \(e \in N_1\).

Let \(N''\) be the two-cell whose boundary is \(E \cup B\). If \(X \cap \text{int} \ N'' \subset N_2 \cap K\), then an arc connecting \(p\) and \(q\) can be constructed so as to lie in \((\text{int} \ N) \cap U\), contradicting the fact that \(p\) and \(q\) lie in different components of this set. Hence some point \(x\) in \(X \cap \text{int} \ N''\) must fail to lie in \(K\). We conclude that some subarc of \([0,x]\) spans \(N\), separating \(p\) from \(q\).

![Figure 1](image1.png)

![Figure 2](image2.png)
Case 2. (See Figure 2). Suppose $p$ and $q$ lie in different components of $(\text{int } N) \setminus B'$. Then some subarc $[a,c]$ of $B$ spans $N$, separating $p$ from $q$. Since at most a finite number of these subarc can span $N$, separating $p$ from $q$, we can choose $[a,c]$ so that the arc $[c,q] \subset B$ does not contain a subarc which spans $N$, separating $p$ from $q$. Pick $p_o \in (a,x)$. If $p_o$ and $q$ lie in different components of $(U \cup \{e\}) \cap \text{int } N$, then we apply Case 1 to get $x \in X$ so that some subarc of $[0, x]$ spans $N$, separating $p_o$ from $q$, and hence $p$ from $q$. If $p_o$ and $q$ lie in the same component, then we could have chosen $B$ to have one less spanning arc separating $p$ from $q$. Hence if we choose $B$ to have a minimum number of such spanning arcs, Case 1 applies. This establishes the statement.

Now we will use this to show that $C$ is a neighborhood of $e$ in $U \cup \{e\}$.

Suppose it is not a neighborhood. Then there exist distinct components $C_1, C_2, \ldots$ of $(\pi \setminus X) \cup \{e\} \cap \text{int } N$ and points $p_1, p_2, \ldots$ such that $p_i \rightarrow e$ and $p_i \in C_i$. By the above argument, there exist points $x_i, y_i \in X$ so that $[y_i, x_i] \subseteq [0, x_i] \in \Omega$ and $[y_i, x_i]$ spans $N$ separating $p_i$ from $p_{i+1}$. We may assume $x_i \rightarrow x$ and $y_i \rightarrow y$ in $\text{Bdry } N$. Hence $[y_i, x_i] \rightarrow [y, x] \subseteq [0, x]$. Let $N_\varepsilon(e)$ be given. Then for sufficiently large $i$, $p_i, p_{i+1} \in N_\varepsilon(e)$ and so $[y_i, x_i] \cap N_\varepsilon(e) \neq \emptyset$ for sufficiently large $i$. We conclude that $e \in [x, y]$, a contradiction, since $e \in I$. This completes the proof 2.33.
Now let \( d \) be a radially convex metric on \( X \). Choose \( u \in I \) so that \([0,u]\) has maximum diameter. Since \( X \) is acyclic, it does not separate the plane. Hence, assuming \( 0 \) is accessible, there is an arc \( A \) from 0 to \( u \) lying except for 0 and \( u \) in \( \pi \setminus X \). Let \( N \) be the two-cell whose boundary is \( A \cup [0,u] \) and let \( X_1 = N \cap X \).

2.35. Theorem. \( X_1 \) admits the structure of a semilattice with identity.

Proof. Let \( I_1 = I \cap X \). For each \( e \in I_1 \setminus \{u\} \), let \( A_e \) be an arc in \( N \) from \( e \) to \( A \setminus [0,u] \) lying except for \( e \) and its endpoint on \( A \) in \( \text{int } N \setminus X_1 \). Define a linear order on \( I_1 \) as follows:

- \( e < u \) for \( e \in I_1 \setminus \{u\} \)
- \( e < f \) for \( e,f \in I_1 \setminus \{u\} \) iff the spanning arc \( A_f \cup [f,\phi(f,u)] \) of \( N \) separates \( e \) from \( u \). Since spanning arcs of 2-cells separate them into exactly two components, this is a linear order on \( I_1 \). In addition, it has the following properties:

1. For \( x \in X_1 \), \( I_1(x) \) is convex.
2. If \( I_1(x) < I_1(y) \), \( x_n \to x \) and \( y_n \to y \), then \( I_1(x_n) < I_1(y_n) \) for sufficiently large \( n \).

Proof. (1) Suppose \( f < g < e \) and \( e,f \in I_1(x) \). Let \( B \) be an arc in \( N \) from \( f \) to \( e \) so that \( B \cap X_1 = \{e,f\} \). Then \( g \) is in the 2-cell bounded by \( B \cup [e,\phi(e,f)] \cup [f,\phi(e,f)] \). We conclude \([0,g] \) meets \([e,\phi(e,f)] \cup [f,\phi(e,f)] \), and hence \( x \in [0,g] \).
By 2.15, \( I_1(x) \cap I_1(y) = \emptyset \) or one is a subset of the other. Thus the notation \( I_1(x) < I_1(y) \) means \( I_1(x) \cap I_1(y) = \emptyset \) and \( e < f \) for \( e \in I_1(x), f \in I_1(y) \). Also by 2.15, \( \phi(x,y) \neq x,y \). Since \( I_1(x) < I_1(y), x \notin [0,u] \). Choose \( e \in I_1(x), f \in I_1(y) \).

**Case 1.** There is a \( g \in I_1 \) such that \( e < g < f \) and \( g \notin I_1(x) \) and \( g \notin I_1(y) \). Then the spanning arc \( A_g \cup [g, \phi(g,n)] \) of \( N \) separates \( x \) from \( y \); hence \( x_n \) from \( y_n \) for sufficiently large \( n \). From this we conclude that \( I_1(x_n) < I_1(y_n) \) for sufficiently large \( n \).

**Case 2.** For each \( g \in I_1 \) such that \( c < g < f \) and \( g \notin I_1(x) \) or \( g \notin I_1(y) \). Choose \( e_n \in I_1(x_n) \) and \( f_n \in I_1(y_n) \). Using property 2.1(iv), it follows that for sufficiently large \( n \) \( e_n < e \) or \( e_n \in I_1(x) \) and \( f_n > f \) or \( f_n \in I_1(y) \). Therefore for sufficiently large \( n \), \( I_1(x_n) < I_1(y_n) \). Define a linear order on the set \( X_1 \) as follows:

\[
x \leq y \text{ iff } I_1(x) < I_1(y) \text{ or } x \in [0,y].
\]

This is clearly reflexive and antisymmetric. For the transitivity, suppose \( s \leq y \) and \( y \leq z \). If \( I_1(x) < I_1(y) \) and \( I_1(y) < I_1(z) \), then \( I_1(x) < I_1(z) \). If \( x \in [0,y] \) and \( y \in [0,z] \), then \( x \in [0,z] \). If \( I_1(x) < I_1(y) \) and \( y \in [0,z] \), then \( I_1(z) \leq I_1(y) \) and so \( I_1(x) < I_1(z) \). If \( x \in [0,y] \) and \( I_1(y) < I_1(z) \), then \( I_1(y) \subset I_1(x) \) and \( I_1(y) \cap I_1(z) = \emptyset \). Suppose \( I_1(x) \cap I_1(z) = \emptyset \). Then by 2.15, \( x \in [0,z] \). Then \( \leq \) is
transitive. 2.15 implies \( \leq \) is linear. Now define multiplication on \( X_1 \) as follows:

\[
xy = (\min\{x,y\}, \min\{d_x, d_y\})
\]

This is clearly an algebraic semilattice with identity \( u \) and zero \( 0 \).

**Continuity.** Let \( x_n \to x \) and \( y_n \to y \) in \( X_1 \).

**Case 1.** \( \varphi(x,y) = x \) or \( y \). Suppose \( x \). Then \( x \in [0,y] \), and

\[
x_ny_n = (\min\{x_n, y_n\}, \min\{d_{x_n}, d_{y_n}\})
\]

We break the sequence \( n \) into the two subsequences \( n', n'' \) such that \( x_n' = \min\{x_n', y_n'\} \) and \( y_n'' = \min\{x_n'', y_n''\} \). Assume both are cofinal. Since \( [0, x_n'] \to [0, x] \subseteq [0, y] \) and

\[
\min\{d_{x_n'}, d_{y_n'}\} \to \min\{d_x, d_y\} = d_x,
\]

we have that

\[
(x_n', \min\{d_{x_n'}, d_{y_n'}\}) \in [0, x_{n'}] \text{ converges to } (x, d_x).
\]
The other subsequence is handled similarly.

**Case 2.** \( \varphi(x,y) \neq x, y \). Then \( I(x) \cap I(y) = \emptyset \). Suppose \( I(x) < I(y) \). Then by (2), \( I(x_n) < I(y_n) \) for sufficiently large \( n \). Hence

\[
x_ny_n = (x_n, \min\{d_{x_n}, d_{y_n}\}) \to (x, \min\{d_x, d_y\}) = xy.
\]

This completes the proof of 2.35.

2.36. **Corollary.** If \( X \) is a ruled subset of the plane and \( 0 \) is accessible, then \( X \) admits a semilattice structure.

2.37. **Corollary.** If \( X \) is a ruled subset of a 2-cell \( N \) such that \( X \cap \text{Bdry } N \in \mathcal{A} \), then \( X \) admits the structure of a semilattice with identity.
CHAPTER III
DENDRITIC EXTENSIONS OF CLANS

In [20], Wallace states that every 1-dimensional locally connected metric clan is topologically either a tree or contains exactly one simple closed curve. In this chapter, we present a sequence of theorems from which will follow the converse of Wallace's statement, thus giving a topological characterization of this clan structure.

3.1. Definition. An arc $A$ in a semigroup $S$ is a local usual thread in $S$ if there is an interval $[0,a]$ of non-negative reals and a homeomorphism $h : [0,a] \to A$ such that if $x,y, x+y \in [0,a]$, then $h(x+y) = h(x)h(y)$.

3.2. Theorem. Let $f : [0,a] \to S$, a semigroup, be a continuous function satisfying (i) or (ii):

(i) if $x,y, x+y \in [0,a]$, then $f(x+y) = f(x)f(y)$.

(ii) if $x,y \in [0,a]$, then $f(\min(x,y)) = f(x)f(y)$, where $\min[0,a] = a$.

Let $S' = \{a\} \times S \cup \{(x,f(x)) : x \in [0,a]\}$. Then (a) $S'$ is a subsemigroup of $[0,a] \times S$, where $[0,a]$ is regarded as the Calabi interval if $f$ satisfies (i) and the min interval if $f$ satisfies (ii); (b) $\{(x,f(x)) : x \in [0,a]\}$ is a local usual thread in $S'$ or is a min thread in $S'$ if $f$ satisfies (i) or (ii) respectively; (c) $\{a\} \times S$ is a closed ideal of $S'$.
Proof. Denote the Calabi addition on \([0,a]\) by \(\circ\). Thus \(x \circ y = x + y\) if \(x + y \in [0,a]\) and \(x \circ y = a\) otherwise. We prove (a) - (c) assuming (i) holds. The proof for (ii) is similar.

(a) Let \((t_1,x), (t_2,y) \in S'\). Then their product is \((t_1 \circ t_2, xy)\). If \(t_1 \circ t_2 \in [0,a]\), then \(t_1, t_2, t_1 + t_2 \in [0,a]\) and hence \(f(t_1) = x, f(t_2) = y\), and \(f(t_1 + t_2) = f(t_1)f(t_2) = xy\). Therefore \((t_1 \circ t_2, xy) \in S'\). If \(t_1 \circ t_2 = a\), then \((t_1 \circ t_2, xy) = (a, xy) \in S'\). Hence \(S'\) is a subsemigroup of \([0,a] \times S\).

(b) The function \(h : [0,a] \to S', \) defined by \(h(x) = (x, f(x))\) is 1-1 and continuous. So \(\{(x, f(x)) : x \in [0,a]\}\) is a local usual thread in \(S'\).

(c) is clear.

3.3. Corollary. A translate of a local usual thread need not be an arc or a point.

Proof. Let \(S\) be the unit circle in the complex plane.
Let \(f : [0,2\pi] \to S\) be given by \(f(t) = e^t\); then \(f\) satisfies (i) of 3.2. Thus \(L = \{(t, e^t) : t \in [0,2\pi]\}\) is a local usual thread in \([0,2\pi] \times S\). But \((2\pi, e^0)T = \{2\pi\} \times S\), a circle.

3.4. Corollary. Let \(S\) be a semigroup with \(1\). Let \(f : [0,1] \to S\) be given by \(f(x) = 1\). Then \(f\) satisfies (ii) of 3.2. \(S'\) is then a semigroup containing \(S\) as a closed ideal. Further \((S' \setminus S)^*\) is a min thread in \(S'\).

3.5. Definition. Let \(S\) be a semigroup with \(1\) and let \(b \in S\). \(S\) is said to have an arc extension at \(b\) if there
is a semigroup $S'$ containing $S$ such that $1$ is the
identity of $S'$ and $(S' \setminus S) \cup \{b\}$ is an arc.

3.6. Theorem. Let $S$ be a semigroup with $1$. Suppose
$T = [d,1]$ is a local usual thread (or a min thread) in $S$
such that $S \setminus (d,1]$ is a closed ideal of $S$. Let $b \in S$. If
there is a $c \in S$ such that $cd = b$ and $cx = xc \not\in T$ for all
$x \in T$, then $S$ has an arc extension at $b$.

Proof. We prove the theorem for $T$ a local usual thread.
The proof for $T$ a min thread is analogous. Let
$h : [0,a] \to [1,d] \times [0,a]$ be the given homeomorphism. Let
$S_0 = \{a\} \times (S \setminus (d,1])$, $T_0 = \{(x,h(x)) : x \in [0,a]\}$, and
$A = \{(x,ch(x)) : x \in [0,a]\}$. Since $S \setminus (d,1]$ is a closed
ideal of $S$, $S_0 \cup T_0 \subseteq [0,a] \times S$ is an isomorphic copy of $S$.
Since $cd = b$ and $cx \not\in T$ for all $x \in T$, we have that $A$ is
an arc in $[0,a] \times S$ such that $A \cap T_0 = \emptyset$ and $A \cap S_0 = (a,b)$.
Let $S' = S_0 \cup T_0 \cup A$ and define a multiplication on $S'$ as
follows:

$$xy = \begin{cases} (a,\pi_2(x)\pi_2(y)) & \text{if } \{x,y\} \cap T_0 = \emptyset \\ (\pi_1(x)\pi_1(y), \pi_2(x)\pi_2(y)) & \text{otherwise} \end{cases}$$

$\pi_1$ and $\pi_2$ denote the first and second projections of
$[0,a] \times S$. We will show $S'$ is a semigroup and is the
required arc extension of $S$ at $b$.

1. If $x,y \in S'$, then $xy \in S'$. 
Case 1. \( (x,y) \cap T_o = \emptyset \). Then \( xy = (a, \pi_2(x) \pi_2(y)) \in S_o \), since for any \( z \in S' \setminus T_o \), \( \pi_2(z) \notin T \) and \( S \setminus (d,1) \) is a subsemigroup of \( S \).

Case 2. \( x \in T_o \). Then \( x = (t, h(t)) \) for some \( t \in [0,a] \). If \( y = (a, p) \), \( p \in S \setminus (d,1) \), then \( xy = (a, h(t)p) \in S_o \) since \( S \setminus (d,1) \) is an ideal of \( S \). If \( y = (t', h(t')) \), then \( xy = (t \cdot t', h(t + t')) \in T_o \) if \( t + t' \in [0,a) \) and \( xy = (a, h(t)h(t')) \) otherwise. But when \( t + t' \geq a \), \( h(t)h(t') = h(t-(a-t') + (a-t'))h(t') = h(t-(a-t'))h(a-t')h(t') = h(t-(a-t'))h(a) = h(t-(a-t'))d \in S \setminus (d,1) \) since \( S \setminus (d,1) \) is an ideal of \( S \). Hence \( xy \in S_o \). If \( y = (t', ch(t')) \), we consider two cases.

(i) \( t + t' \in [0,a) \). Then \( h(t)ch(t') = ch(t)h(t') = ch(t+t') \) and thus \( xy = (t \cdot t', h(t)ch(t')) = (t \cdot t', ch(t \cdot t')) \in A \).

(ii) \( t + t' \geq a \). Then \( xy = (a, h(t)ch(t')) \in S_o \), since \( ch(t') \notin T \) and \( S \setminus (d,1) \) is an ideal of \( S \).

2. Associativity. Since the multiplication is coordinate-wise in the second factor, it is associative there. Hence we need only show that \( \pi_1(xy)z) = \pi_1((xy)z) \). Note that \( \pi_1(xy) \geq \pi_1(x) \circ \pi_1(y) \) always.

Case 1. \( \pi_1((xy)z) = a \). Then \( (xy), z) \cap T_o = \emptyset \) or \( \pi_1(xy) \circ \pi_1(z) = a \). Suppose the first. Then \( \pi_1(z) = a \) or \( z \in A \). If \( \pi_1(z) = a \), then \( \pi_1(yz) = a \); hence \( \pi_1(x(yz)) = a \). If \( z \in A \), we consider the possibilities for \( x \) and \( y \).
Suppose $y \not\in T_0$. Then $\pi_1(yz) = a$, since $A \cap T_0 = \emptyset$. Hence $\pi_1(x \cdot (yz)) = a$. Suppose $y \in T_0$. Then $yz \not\in T_0$, since $S \setminus (d,1]$ is an ideal. Hence if $x \not\in T_0$, then $\pi_1(x(yz)) = a$ and if $x \in T_0$ then since $xy \not\in T_0$, $\pi_1(xy) = \pi_1(x) \circ \pi_1(y) = a$. This implies that $\pi_1(x(yz)) = a$.

Now suppose $\pi_1(xy) \circ \pi_1(z) = a$. Then $\pi_1(xy) = \pi_1(x) \circ \pi_1(y)$ or $\pi_1(xy) = a$. If $\pi_1(xy) = \pi_1(x) \circ \pi_1(y)$, then $a = \pi_1(xy) \circ \pi_1(z) = \pi_1(x) \circ \pi_1(y) \circ \pi_1(z) \leq \pi_1(x(yz))$. Hence $\pi_1(x(yz)) = a$. If $\pi_1(xy) = a$, then $[x,y] \cap T_0 = \emptyset$ or $\pi_1(x) \circ \pi_1(y) = a$. Suppose $[x,y] \cap T_0 = \emptyset$. Then $x,y \in A$ or one of $\pi_1(x), \pi_1(y)$ is $a$. If $\pi_1(x) = a$, then $\pi_1(xy) = a$. If $x,y \in A$, then $yz \not\in T_0$. Hence $\{x,yz\} \cap T_0 = \emptyset$ and hence $\pi_1(x(yz)) = a$. Now suppose that $\pi_1(x) \circ \pi_1(y) = a$. Then $\pi_1(x(yz)) = a$. This concludes Case 1. Thus if $\pi_1((xy)z) = a$, then $\pi_1(x(yz)) = a$. The proof of the converse is analogous.

Case 2. $\pi_1(xyz) \neq a$. Then $\pi_1(x(yz)) = \pi_1(x) \circ \pi_1(y) \circ \pi_1(z) = \pi_1((xy)z)$ by Case 1.

3. Continuity. As with associativity, we need only show continuity in the first coordinate. Let $x_\alpha \to x$ and $y_\alpha \to y$ in $S'$.

Case 1. $\pi_1(xy) = \pi_1(x) \circ \pi_1(y) < a$. Then $[x,y] \cap T_0 \neq \emptyset$.

Suppose $x \in T_0$. Then there is a $\beta$ so that for $\alpha \geq \beta$, $x_\alpha \in T_0$, since $T_0$ is open in $S'$. Hence $[x_\alpha, y_\alpha] \cap T_0 \neq \emptyset$ for $\alpha \geq \beta$. This implies that $\pi_1(x_\alpha y_\alpha) = \pi_1(x_\alpha) \circ \pi_1(y_\alpha) \to \pi_1(x) \circ \pi_1(y) = \pi_1(xy)$. 
Case 2. Suppose \( \pi_1(xy) = a \). If \( \pi_1(x) \circ \pi_1(y) < a \), then 
\[ \{x,y\} \subset A \subset S \setminus T_0 * \] and hence \( \{x, y\} \cap T_0 = \emptyset \) for \( \alpha \geq \beta \), 
some \( \beta \). Thus \( \pi_1(x_\alpha y_\alpha) = a \) for \( \alpha \geq \beta \). We conclude that 
\( \pi_1(x_\alpha y_\alpha) \rightarrow \pi_1(xy) \). If \( \pi_1(x) \circ \pi_1(y) = a \), then \( \pi_1(x_\alpha) \circ \pi_1(y_\alpha) \rightarrow a \). 
Since \( \pi_1(x_\alpha) \circ \pi_1(y_\alpha) \leq \pi_1(x_\alpha y_\alpha) \), we have \( \pi_1(x_\alpha y_\alpha) \rightarrow \pi_1(xy) = a \). 
This completes the proof.

3.7. Example. Let \( S = \{(x,y) : x^2 + y^2 = 1\} \), 
\( A = \{(x,0) : x \in [1,2]\} \), \( B = \{(0,y) : y \in [1,2]\} \), and 
\( X = S \cup A \cup B \). We assert that \( X \) admits a clan structure.
For by using 3.2(ii) a clan structure can be defined on a 
space homeomorphic to \( S \cup A \). By taking an appropriate 
homeomorphism, this multiplication is realized on \( S \cup A \) as 
follows:
\[
xy = yx = \begin{cases} 
\text{circle product if } x, y \in S \\
\min\{x, y\} & \text{if } x, y \in A \\
x & \text{if } x \in S, y \in A.
\end{cases}
\]
Note that each \( y \in A \) is an identity for \( S \). Let \( l = (2,0) \), 
the identity of \( S \cup A \), and let \( b = (0,1) \). Choose any 
d \( \in A \setminus \{l\} \) and note that \( b[d,1] = b \notin [d,1] \). In addition, 
\( (S \cup A) \setminus (d,1) \) is a closed ideal of \( S \cup A \). Hence, using 3.6, 
an arc-extension of \( S \cup A \) at \( b \) can be made. The 
resulting clan is homeomorphic to \( X \), and the multiplication 
can be realized on \( X \) as follows: Let \( h : [d,1] \rightarrow B \) be a 
homeomorphism with \( h(d) = b \). Let \( B \) have the order which 
makes \( b \) minimum. Then
This example provides the motivation for the following definition and theorem.

3.8. Definition. Let $K$ be a clan contained in a metric continuum $X$. $X$ is a dendritic extension of $K$ provided:

(i) For each component $C$ of $X \setminus K$, $C$ is open, $\text{Bdry } C$ is a point, and $C^*$ is a tree.

Note. Since $X$ is compact metric and each component of $X \setminus K$ is open, there are countably many of them. Label them $\{C_i\}_{i=1}^\infty$. Let $k(x)$ denote the boundary point of the $C_i$ which contains $x$, if $x \in X \setminus K$, and let $[x,k(x)]$ denote the unique arc from $x$ to $k(x)$.

(ii) If $x_n \in X \setminus K$ and $x_n \to x \in K$, then $[x_n,k(x_n)] \to [x]$.

(iii) For some $x \in X \setminus K$, $k(x) = l$.

We now prove a sequence of lemmas leading to a proof that a dendritic extension $X$ of a clan $K$ admits a clan structure.

3.9. Lemma. The function $f : X \to K$ defined by $f(x) = x$ for $x \in K$ and $f(x) = k(x)$ for $x \not\in K$ is a retraction.
Proof. Let \( x_n \to x \). If \( x \not\in K \), then \( x \in C_i \) for some \( i \) and \( x_n \in C_i \) for sufficiently large \( n \). Hence \( f(x_n) = f(x) \) for sufficiently large \( n \) and so \( f(x_n) \to f(x) \). If \( x \in K \), break \( x_n \) into two subsequences \( x_n' \in K \) and \( x_n'', \not\in K \). Then \( f(x_n') = x_n \to x = f(x) \) and \( [x_n'', k(x_n'')] \to x \). Hence \( f(x_n'') \to f(x) \).

3.10. Lemma. Let \( E \) be the set of endpoints of \( X \) which are not in \( K \). Let \( \{e_i\}_{i=1}^{\infty} \) be a dense subset of \( E \). Then \( X \setminus (K \cup \bigcup_{i=1}^{\infty} [e_i, k(e_i)]) \) is contained in \( E \).

Proof. Let \( x \in X \setminus (K \cup \bigcup_{i=1}^{\infty} [e_i, k(e_i)]) \), and let \( C \) be the component of \( X \setminus K \) containing \( x \). If \( x \) is not an endpoint of \( C^* \), then \( C^* \setminus x \) has at least two components. Let \( C' \) be a component not containing \( k(x) \). Then \( C' \) is open in \( X \) and \( e_i \in C' \) for some \( i \). Therefore \( x \in [e_i, k(x)] \), a contradiction. Thus \( x \) is an endpoint of \( C^* \) different from \( k(x) \) and hence is an endpoint of \( X \).

Let \( K_0 = K \cup [e_0, b_0] \), where \( b_0 = k(e_0) = 1 \), and inductively define \( K_n = K_{n-1} \cup [e_n, b_n] \), where \( b_n \) is the point in \( [e_n, k(e_n)] \) such that \( K_{n-1} \cap [e_n, b_n] = b_n \). Then \( X = (\bigcup_{i=0}^{\infty} K_i)^* = \bigcup_{i=1}^{\infty} K_i \cup E \).

3.11. Lemma. The multiplication on \( K \) can be extended to a multiplication on \( \bigcup_{i=1}^{\infty} K_i \).
Proof. By 3.2, we may regard \( K_0 \) as the clan \( K \) with the min interval adjoined. From this point, we proceed by arc-extension. Note that \( b_1 \neq e_0 \), the identity of \( K_0 \).

Hence we may choose \( d_1 \in [b_0, e_0) \) so that \( b_1[d_1, e_0] = b_1 \notin [d_1, e_0] \). Note also that \( K_0 \setminus (d_1, e_0] \) is a closed ideal of \( K_0 \). Hence we can apply 3.6 to extend the multiplication. Suppose the multiplication has been extended to \( K_{n-1} \) by \( n-2 \) applications of 3.6 so that the points \( d_i \in [b_0, e_0] \) satisfy:

(i) \( d_i < d_{i+1} < e_0 \) for \( i = 1, 2, \ldots, n-2 \).
(ii) \( b_i \notin [d_j, e_0] \) for \( i \leq j \leq n-2 \).
(iii) \( b_i[d_i, e_0] = b_i \) for \( i \leq n-1 \).
(iv) the distance from \( d_i \) to \( e_0 \) is less than \( \frac{1}{n} \).

Now \( b_n \neq e_1 \) for \( i = 0, \ldots, n-1 \) and \( b_n \in K_{n-1} \). Hence we can choose \( d_n \in (d_{n-1}, e_0) \) so that \( \text{dist}(d_n, e_0) < \frac{1}{n} \) and \( b_n[d_n, e_0] = b_n \notin [d_n, e_0] \). By the particular way multiplication is defined in 3.6, \( K_{n-1} \setminus (d_n, e_0] \) is a closed ideal of \( K_{n-1} \). Hence 3.6 may be applied again to extend to \( K_n \) and (i)-(iv) remain satisfied. Hence by induction we can extend to \( \bigcup_{i=1}^{\infty} K_i \).

Define for each natural number \( n \) a map \( n : X \to K_n \) by \( n(x) = x \) for \( x \in K_n \) and \( n(x) = \text{the point on } [x, k(x)] \) such that \( [x, k(x)] \cap K_n = n(x) \) for \( x \notin K_n \).

3.12. Lemma. \( n \) is a retraction and the restriction of \( n \) to \( \bigcup_{i=1}^{\infty} K_i \) is a homomorphism.
Proof. $X$ is a dendritic extension of $K_n$ (except (iii) is not satisfied) and the map $f$ defined in 3.9 is defined precisely as $n$. Hence $n$ is a retraction. For the second half of the assertion: Let $m \geq n$. Note that $n \circ m = n$. From the way the multiplication was extended to $UK_i$, it is clear that $n|K_{n+1}$ (the restriction of $n$ to $K_{n+1}$) is a homomorphism. Suppose that $n|K_m$ is a homomorphism. Since $m|K_{m+1}$ is a homomorphism and $n|K_{m+1} = n|K_m \circ n|K_{m+1}$, it follows that $n|K_{m+1}$ is a homomorphism. Hence by induction $n|UK_i$ is a homomorphism.

Now we state the lemma which allows the extension of the multiplication on $UK_i$ to $X$.

3.13. Lemma. Let $x, y \in X$. Suppose $x_n, x'_n \to x$ and $y_n, y'_n \to y$, where $x_n, x'_n, y_n, y'_n \in UK_i$ for each $n$. Then there is a $z \in X$ such that $x_n y_n, x'_n y'_n \to z$.

Proof. We distinguish three cases.

Case 1. $x, y \neq c_o$. Then choose $M$ so large that $o(x), o(y) \notin [d_M, c_o]$. We may assume that $o(x_n), o(y_n), o(x'_n), o(y'_n) \notin [d_M, e_o]$ for all $n$. We will show the following: If $z, w \in UK_i$ such that $o(z), o(w) \notin [d_M, e_o]$, then $zw = M(z)M(w)$. This is clear if $z, w \in K_M$. Suppose it has been shown for $z, w \in K_{M+m'}$, and let $z, w \in K_{M+m+1}$ such that $o(z), o(w) \notin [d_M, e_o]$. If $z \in (b_{M+m+1}, e_{M+m+1})$ and $w \in K_{M+m}$, then $o(z) = o(b_{M+m+1})$. Hence $o(b_{M+m+1}) \notin [d_M, e_o]$. Therefore $zw = b_{M+m+1}w = M(b_{M+m+1})M(w) = M(z)M(w)$. A
similar argument handles the case where \( z, w \in \{b_{M+m+1}, e_{M+m+1}\} \). Therefore the claim is established by induction. We conclude that \( x_n y_n = M(x_n)M(y_n) \) and \( x'_n y'_n = M(x'_n)M(y'_n) \) for all \( n \). But \( M(x_n)M(y_n) \to M(x)M(y) \) and \( M(x'_n)M(y'_n) \to M(x)M(y) \) by the continuity of \( M \) and the continuity of the multiplication on \( K_M \). Hence the choice \( z = M(x)M(y) \) satisfies the lemma.

**Case 2.** \( x = y = e_o \). We will show \( z = e_o \). Let \( V_1 = \{ z \in X : o(z) \in (d_1, e_o) \} \). Then these \( V_i \)'s form a neighborhood basis at \( e_o \). We will show the following: If \( z, w \in V_i \cap K_m \), then \( zw \in V_i \cap K_m \). If \( z, w \in V_i \cap K_o = (d_1, e_o) \), this is clear. Suppose it has been shown true for \( z, w \in V_i \cap K_m \) and suppose \( z, w \in K_m \cap V_i \). If \( z \in (b_m, e_m] \) and \( w \in K_{m-1} \), then \( zw \in [b_m, e_m] \) or \( zw = b_m w \) from the manner in which multiplication is defined. Since \( o(z) \in (d_1, e_o) \), we have \( o(b_m) = O(z) \in (d_1, e_o) \) and in either case \( zw \in V_i \cap K_m \). If \( z, w \in (b_m, e_m] \), then \( zw = b_m b_m \) and \( b_m \in V_i \cap K_{m-1} \). Hence \( zw \in V_i \cap K_{m-1} \subset V_i \cap K_m \). We conclude that for each \( i \), \( x_n y_n \in V_i \) for sufficiently large \( i \).

Hence \( x_n y_n \to e_o \). Similarly, \( x'_n y'_n \to e_o \).

**Case 3.** \( x = e_o \), \( y \neq e_o \). We will show \( z = y \). First we establish the following: (A) If \( z, w \in U_{K_1} \) such that \( o(z) \in (d_m, e_o) \) and \( o(w) \notin [d_m, e_o] \), then \( zw = o(z)w \). The proof is by induction on \( K_1 \). For \( z = o(z) \), it is clear. Suppose the statement is true for \( z \in K_1 \), \( i \geq 0 \). Let \( z \in (b_{i+1}, e_{i+1}] \). Then we have that \( w \notin [d_{i+1}, e_{i+1}] \); hence
zw = b_{i+1}w = o(b_{i+1})w = o(z)w. Thus (A) is established.

(B) If x ∈ (d_m,e_o], o(w) ⊈ (d_m,e_o], then

\[ xw = xm(w) \text{ or } xw ∈ [w,m(w)]. \]

If w = m(w), then (B) is clear. If w ∈ (b_{m+1},e_{m+1}], then

\[ xw = xb_{m+1} = xm(w) \text{ or } xw ∈ [b_{m+1},w], \]

depending on whether \( x ≤ d_{m+1} \) or \( d_{m+1} < x \). Suppose true for \( w ∈ K_{m+n+1} \) and let \( w ∈ (b_{m+n+1},e_{m+n+1}] \). If \( x > d_{m+n+1} \), then \( xw ∈ (b_{m+n+1},w] ⊂ (m(w),w] \).

If \( x ≤ d_{m+n+1} \), then \( xw = xb_{m+n} = xm(b_{m+n}) = xm(w) \). Thus (B) is established. Now we distinguish two subcases of Case 3.

**Subcase 1.** \( y ∈ K_M, \) some \( M \). Choose \( M \) so large that \( o(y) \not∈ [d_M,e_o] \). We may assume that \( o(y_n) \not∈ [d_M,e_o] \) and \( o(x_n) ∈ (d_M,e_o) \) for all \( n \). Then by (A), \( x_ny_n = o(x_n)y \) and by (B) \( o(x_n)y_n = o(x_n)M(y_n) \) or \( o(x_n)y_n ∈ [y_n,M(y_n)] \). Therefore since \( o(x_n)M(y_n) → e_oM(y) = e_oy = y \) and \( [y_n,M(y_n)] → \{y\} \), we conclude \( x_ny_n → y \); similarly \( x_ny_n → y \).

**Subcase 2.** \( y \not∈ K_i \) for all \( i \). Then \( y \) is an endpoint of \( X \). Let \( U \) be a connected neighborhood of \( y \) with a one point boundary \( b_k \). We will show the following: If \( x ∈ (d_k,e_o], y ∈ UK_1 \cap U \), then \( xy ∈ U \). Now if \( y ∈ (b_k,e_k] \) then \( xy ∈ (b_k,e_k] \) from the manner in which multiplication is defined. Suppose \( y ∈ (b_j,e_j] \), where \( b_j ∈ (b_k,e_k] \). Then \( d_j ∈ (d_k,e_o] \) and \( xy ∈ (b_j,e_j] \) or \( xy = xb_j ∈ (b_j,y] \). But for each \( y ∈ UK_1 \cap U \), there is a finite sequence \( b_{j_1},b_{j_2},...,b_{j_λ} \) such that \( y ∈ (b_{j_1},e_{j_1}], \)

\( b_{j_1} ∈ (b_{j_2},e_{j_2}],...b_{j_λ} ∈ (b_k,e_k] \) and so the statement is
established by induction. Now \( o(x_n) \in (d_k, e_o] \) for sufficiently large \( n \), and \( y_n \in U \) for sufficiently large \( n \). Hence \( x_n y_n = o(x_n)y_n \in U \) for sufficiently large \( n \). We conclude that \( x_n y_n \to y \). Similarly, \( y'_n \to y \). This completes the proof of 3.13.


**Proof.** Define multiplication on \( X \) as follows: For \( x, y \in X \) let \( x_n \to x \), \( y_n \to y \), where \( x_n, y_n \in UK_1 \). Define \( x \cdot y = \lim_{n} x_n y_n \). By 3.13 this limit exists and is independent of the choice of the sequences \( x_n \) and \( y_n \). Hence \( \cdot \) is well-defined. Obviously the restriction of this multiplication \( UK_1 \) is continuous and agrees with the multiplication obtained by construction (3.11). It follows that \( \cdot \) is continuous on \( X \), is associative and has identity \( e_o \).

3.15. Corollary. The class of 1-dimensional locally connected metric clans is topologically characterized as the class of 1-dimensional locally connected metric continua which contain at most one circle.

**Proof.** Every such continuum is either a dendritic extension of a point if it contains no circle or is a dendritic extension of a circle if it contains one. Thus, by 3.14, each such continuum admits a clan structure. For the converse, see Wallace [20].
3.16. Remark. The class of continua in 3.15 contains the class of trees. It is apparent that the clan structure which is put on a tree using 3.14 is not idempotent; hence is not a semilattice structure. Koch and McAuley showed in [14], however, that all trees admit a semilattice structure with 1.
CHAPTER IV
INESSENTIAL SPACES

4.1. Definition. A space $X$ is inessential iff there is a connected space $T$ and a continuous function $m : T \times X \to X$ such that for some points $p$ and $q$ in $T$, $m(p,x) = x$ for each $x \in X$ and $m(q,X) \neq X$. Otherwise, $X$ is essential.

4.2. Theorem. If a continuum $X$ admits a clan structure which is not a group structure, then $X$ is inessential.
Proof. Let $m$ be a clan structure on $X$ which is not a group structure. Then there must exist some $q \in X$ such that $m(q,X) \neq X$ or $m(X,q) \neq X$. Otherwise, $m$ would be a group structure on $X$. The theorem follows, taking $p = 1$.

4.3. Remark. We note that if $X$ is an essential continuum, then every clan on $X$ is a group. The question of when a clan is a group has been investigated by Madison and Selden [17], among others. They have proved that every clan structure on a space which admits a group structure is a group structure. We conjecture here that every continuum which admits a group structure is essential. This is known to be true in the finite dimensional case.

4.4. Corollary. Let $X$ be a locally connected metric continuum containing exactly one circle $C$. Then $X$ is inessential if $C \neq X$.
Proof. Each such continuum admits a clan structure which is not a group structure by 3.15.
4.5. Theorem. If \( X \) is inessential, then \( X \times Y \) is inessential for any space \( Y \).

Proof. Let \( T, m, p, q \) be given for \( X \). Define
\[
M : T \times (X \times Y) \to X \times Y \text{ by } M(t, x, y) = (m(t, x), y) \text{ for } t \in T, \]
x \in X, and \( y \in Y \). Then \( M(p, (x, y)) = (m(p, x), y) = (x, y) \) and
\[
M(q, X \times Y) = m(q, X) \times Y \neq X \times Y.
\]

4.6. Corollary. Every factor space of an essential space is essential. (\( X \) is a factor space of \( Z \) if \( X \times Y = Z \) for some space \( Y \)).

It is an open question whether the product of two essential spaces is essential. However, a partial converse to 4.5 was proved by Ganea [5]. Ganea has a different definition of inessential space, but his proof carries over to our definition.

4.7. Theorem. Let \( \{X_\alpha\}_{\alpha \in \Lambda} \) be a family of compact Hausdorff spaces such that for any finite \( F \subset \Lambda \), we have that \( \tau \) is essential. Then \( \prod_{\alpha \in \Lambda} X_\alpha \) is essential.

Proof. Suppose \( X = \prod_{\alpha \in \Lambda} X_\alpha \) is inessential. Let \( T, m, p, q \) be given for \( X \). Then \( m(q, X) \) is a proper closed subset of \( X \).

Let \( U = X \setminus m(q, X) \). There is a finite \( F \subset \Lambda \) and points \( a_\alpha \in X_\alpha \) for each \( \alpha \in F \) such that \( (a_\alpha)_{\alpha \in F} \notin U \). We will show that \( Y = \prod_{\alpha \in \Lambda} X_\alpha \) is inessential, contrary to hypothesis. Let \( a_\alpha \in X_\alpha \) for \( \alpha \notin F \) be arbitrarily chosen.
Let $j : y \to X$ be the injection given by $j(x_\alpha)_{\alpha \in F} = (x_\alpha)_{\alpha \in F} \times (a_\alpha)_{\alpha \in \Lambda \setminus F}$.

Let $\pi : X \to Y$ be the natural projection. Define $M : T \times Y \to Y$ by the following diagram:

$$
\begin{array}{c}
T \times Y \\
\downarrow i \times j \\
T \times X \\
\downarrow m \\
X \\
\downarrow \pi \\
Y \\
\end{array}
$$

\[ \pi m(i \times j) = M \]

Here $i$ denotes the identity map on $T$. Now

\[ M(p, (x_\alpha)_{\alpha \in F}) = \pi m(i \times j)(p, (x_\alpha)_{\alpha \in F}) = \pi m(p, (x_\alpha)_{\alpha \in F} \times (a_\alpha)_{\alpha \in \Lambda \setminus F}) = (x_\alpha)_{\alpha \in F} \]

Also $M(q, X) = \pi m(i \times j)(q, y) = \pi m(q, y \times (a_\alpha)_{\alpha \in \Lambda \setminus F})$. Since the point $(a_\alpha)_{\alpha \in \Lambda}$ is not in $m(q, X)$, it is not in $m(q, y \times (a_\alpha)_{\alpha \in \Lambda \setminus F})$. Hence $(a_\alpha)_{\alpha \in F} \notin M(q, Y)$. This completes the proof.

### 4.7 Theorem
Every compact connected manifold (locally Euclidean space) is essential.

**Proof.** We use the following fact about the cohomology groups of such spaces: if $M$ is a compact connected manifold of dimension $n$, then $H^n(M, \text{integers}) \neq 0$ and $H^n(p, \text{integers}) = 0$ for any proper closed subset $P$ of $M$. This follows from Theorem 6.8 of Eilenberg and Steenrod [4] and Theorem VIII3 of Hurewicz and Wallman [8].
Now suppose $M$ is an inessential compact connected manifold of dimension $n$. Let $T, m, p, q$ be given for $M$. Let $P = m(q, M)$. Let $f : M \to P$ be defined by $f(x) = m(q, x)$ for $x \in M$ and let $i : P \to M$ be the inclusion map. By the generalized homotopy theorem for AWS cohomology, $(if)^* : H^n(M) \to H^n(M)$, the homomorphism induced by $if$, is the identity homomorphism on $H^n(M)$. Since $H^n(M) \neq 0$ (with integer coefficients) $(if)^*$ is a non trivial homomorphism. On the other hand, $(if)^* = f^*i^*$, and since $H^n(P) = 0$, $f^*$ is the trivial homomorphism. Thus we have a contradiction.

4.9. Corollary. Let $\{M_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact connected manifolds. Then $M = \prod_{\alpha \in \Lambda} M_\alpha$ is an essential space.

Proof. Follows from 4.7 and 4.8 and the fact that the finite product of compact connected manifolds is a compact connected manifold.

4.10. Theorem. Every one-dimensional factor space of a countable product of circles is a circle. Thus the factorization of a countable product of circles into one dimensional factor spaces is unique.

Proof. Let $T = \prod_{i \in \Lambda} C_i$, where $\Lambda$ is a countable set and $C_i$ is a circle for $i \in \Lambda$. Let $X$ be a one dimensional factor space of $T$. Since $X$ is the continuous image of a locally connected metric continuum, it is a locally connected metric continuum. We will show that $X$ contains at most one
circle. The theorem then follows from 4.4, 4.6, and 4.8.
Suppose $X$ contains more than one circle. We consider three cases.

Case 1. $X$ contains two circles $A$ and $B$ such that $A \cap B$ is a point $p$. For each $x \in B \setminus p$, let $u_x = \{ y \in X : d(y,x) < \frac{d(x,A)}{2} \}$. Let $u = \bigcup (u_x : x \in B \setminus p)$. Then $u^* \supset B$ and $u^* \cap A = p$. Note $B \cup (u^* \setminus u)$ is a closed set in $u^*$. By Theorem VI4, p.83, of [8], there is a continuous function $g : u^* \rightarrow B$ such that $g(x) = x$ for $x \in B$ and $g(x) = p$ for $x \in u^* \setminus u$. Now let $V = X \setminus u^*$. Then $V^* \supset A$ and $V^* \cap B = p$ and $V^* \cap V = u^* \setminus u$. Furthermore $(V^* \setminus V) \cup A$ is a closed subset of $V^*$. Hence there exists a continuous function $h : V^* \rightarrow A$ such that $h|A$ is the identity on $A$ and $h(x) = p$ for $x \in V^* \setminus V$. The map $f : X \rightarrow A \cup B$ defined by $f(x) = h(x)$ for $x \in V^*$ and $f(x) = g(x)$ for $x \in u^*$ is a retraction of $X$ onto $A \cup B$. Let $\pi_1(X)$ denote the first homotopy group of $X$, based at $p$. Since $A \cup B$ is a retract of $X$, $\pi_1(A \cup B)$ is a subgroup of $\pi_1(X)$. Thus $\pi_1(X)$ is nonabelian since $\pi_1(A \cup B)$ is nonabelian. Let $Y$ be a space such that $X \setminus Y = T$. By a theorem in Hilton [6], $\pi_1(X) + \pi_1(Y) \simeq \pi_1(T)$. Also, $T$ is an arcwise connected topological group. Hence $\pi_1(T)$ is abelian.[6] But $\pi_1(X)$ is a subgroup of $\pi_1(T)$ and so it is abelian. This is a contradiction.
**Case 2.** X contains two disjoint circles A and B. Let C be an arc in X such that \( A \cap C = a \) and \( B \cap C = b \) are the endpoints of C. For \( x \in A \setminus a \), let \( u_x = \{ y \in X : d(x,y) < \frac{d(x,B \cup C)}{2} \} \), and let \( u = U(u_x : x \in A \setminus a) \). Then \( u^* \supset A \) and \( u^* \cap (B \cup C) = a \). Furthermore, \( (u^* \setminus u) \cup A \) is a closed set in \( u^* \). Hence there is a map \( g : u^* \to A \) such that \( g(x) = x \) for \( x \in A \) and \( g(x) = a \) for \( x \in u^* \setminus u \). For \( x \in B \setminus b \) let 
\[ V_x = \{ y \in X : d(x,y) < \frac{d(x,u^* \cup C)}{2} \} , \]
and let \( V = U(V_x : x \in B \setminus b) \). Then \( V^* \supset B \), and \( V^* \cap (u^* \cup C) = b \).
Furthermore \( (V^* \setminus V) \cup B \) is a closed subset of \( V^* \). Hence there is a map \( h : V^* \to B \) such that \( h(x) = x \) for \( x \in B \) and \( h(x) = b \) for \( x \in V^* \setminus V \). Let \( K = X \setminus (u \cup V) \). Then 
\( C \cup (V^* \setminus V) \cup (u^* \setminus u) \) is a closed subset of \( K \). Hence by the Tietze extension theorem, there is a continuous function \( k : K \to C \) such that \( k(x) = x \) for \( x \in C \), \( k(x) = a \) for \( x \in u^* \setminus u \) and \( k(x) = b \) for \( x \in V^* \setminus V \). Hence the map 
\[ f : X \to A \cup C \cup B \] defined by \( f(x) = g(x) \) for \( x \in u^* \), \( f(x) = h(x) \) for \( x \in V^* \), and \( f(x) = k(x) \) for \( x \in K \) is a retraction of \( X \) onto \( A \cup C \cup B \). As in Case 1, this contradicts the fact that \( \pi_1(X) \) is abelian.

**Case 3.** Neither Case 1 nor Case 2 occurs. Let A and B be distinct circles in X. We may assume \( A \cap B \) is an arc C. Call the endpoints of C p and q. What we now have are three arcs C, \( A_1 \) and \( B_1 \) with endpoints p and q which
meet pairwise in \( p \) and \( q \). Let \( C' = C \setminus \{p, q\} \), \( A'_1 = A_1 \setminus \{p, q\} \), and \( B'_1 = B_1 \setminus \{p, q\} \). If \( C' \), \( A'_1 \), and \( B'_1 \) all lie in the same component of \( X \setminus \{p, q\} \), then Case 1 or Case 2 must occur, contrary to our assumption. Thus there is a component \( K \) of \( X \setminus \{p, q\} \) which contains one and only one of \( A'_1 \), \( B'_1 \), and \( C' \). Suppose \( K \) contains \( C' \). Let \( K_1 = K \cup \{p, q\} \), and \( K_2 = X \setminus K \). \( K_1 \) and \( K_2 \) are closed subsets of \( X \), \( K_1 \cap K_2 = \{p, q\} \), and \( K_1 \cup K_2 = X \). There is a map \( g : K_2 \to A_1 \cup B_1 \) such that \( g(x) = x \) for \( x \in A_1 \cup B_1 \). Also there is a map \( h : K_1 \to C \) such that \( h(x) = x \) for \( x \in C \).

Hence the map \( k : X \to A_1 \cup C \cup B_1 \) defined by \( k(x) = g(x) \) for \( x \in K_2 \) and \( k(x) = h(x) \) for \( x \in K_1 \) is a retraction of \( X \) onto \( A_1 \cup C \cup B_1 \). As in Case 1, this contradicts the fact that \( \pi_1(X) \) is abelian. This completes the proof.
BIBLIOGRAPHY


BIOGRAPHY

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