Topics in Quasi-Ordered Spaces.

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by

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ABSTRACT

The object of this paper is to study certain aspects of the theory of quasi-ordered topological spaces. Primarily, the results concern closed quasi-orders on compact Hausdorff spaces.

In the initial part of Chapter I, results due to Nachbin are used to prove that every compact metric partially ordered space is imbeddable in the Hilbert cube by an order preserving homeomorphism. As a corollary, it is concluded that every compact metric partially ordered space admits a radially convex metric. The remainder of Chapter I is devoted to the study of compact quasi-ordered spaces which are irreducible between two subsets. If one of the subsets is the set of minimal elements and if the quasi-order is assumed to be continuous, closed, and monotone, it is shown that the space is a chain and that each maximal set of equivalent elements is connected.

In Chapter II, the notion of order $\pi$-connectivity is introduced, where $\pi$ is an open set containing the diagonal of the space in question. Then, an investigation of the interrelationships between order density, monotonicity, order $\pi$-connectivity, and bi-connectivity is presented. A
fundamental theorem is proved which states that a continuum A in X is order $\tau$-connected for each open set $\tau$ containing the diagonal of X if and only if A is order dense and bi-connected. The principal result of this chapter is the following very useful characterization of compact bi-connected chains. A subset A of the compact quasi-ordered space X is a compact bi-connected chain if and only if A is the limit of a net $\{A_\alpha : \alpha \in D\}$ of chains such that, for each open set $\tau$ containing the diagonal of X, the set of all $\alpha$ such that $A_\alpha$ is order $\tau$-connected is residual in D.

The first concern of Chapter III is a proof of the existence of compact bi-connected chains in quasi-ordered spaces which have 'no gaps' in an appropriate sense. An alternate solution is given to a conjecture of Koch, recently settled by Ward, on the existence of arcs in quasi-ordered spaces. Next, the ability to cover compact bi-connected chains by compact bi-connected chains under dense from below maps is obtained. In particular, such chains are coverable under order preserving open maps. A proof is given that each tree is liftable under a light open map, yielding the non-metric analog of a theorem due to Whyburn on the liftability of dendrites. Finally, possible extensions of the theory of liftability are discussed.

In Chapter IV, the results on bi-connectivity are applied to the theory of compact topological semigroups. First, an
alternate proof of a theorem due to Koch on the existence of threads in compact semigroups is given. Secondly, an outline of a proof of the recent result of Hofmann and Mostert, that each algebraically irreducible semigroup is abelian, is presented. Finally, it is shown that an algebraically irreducible semigroup $A$ in $S$ is a bi-connected chain with respect to the right ideal quasi-order of $S$, as is each left translate of $A$. 
The relationship between topology and order has been of interest to many mathematicians for a number of years. In 1945, Wallace [52] proved that each non-void closed subset of a compact quasi-ordered space has a maximal element. Earlier references are Moore's book *Point Set Theory* [39;33 and 97] and *Analytic Topology* by Whyburn [68;41-57].

Later, in 1950, Nachbin wrote a monograph entitled *Topologia E Ordem* which was published by the University of Chicago Press. This monograph was the first major work on the relationship between topological and order structures. A translation of this work into English was published in 1965 [42].

Since 1950, many authors closely associated with the theory of topological semigroups have studied ordered topological spaces. Notable among these are Koch [21] and [24], Koch and Krule [26], Krule [31] and [32], Strother [51], Wallace [55] and [57], and Ward [58], [59], [60], [61], [62], [63], and [64]. This transition from the study of topological semigroups to the study of orders on topological spaces is a natural one, since the ideal ordering on a compact semigroup is a closed quasi-order.
Recently, Koch and Mc Auley [27] and [28] have studied semi-group structures on continua ruled by arcs. A natural question arises from their work concerning the existence of radially convex metrics. In the first part of Chapter I, results due to Nachbin are used to answer the question in the affirmative. That is, we prove that every compact metric partially ordered space admits a radially convex metric. The second part of Chapter I is devoted to the study of quasi-ordered spaces which are continua irreducible between two subsets. Our results parallel those of Krule [31] and Koch and Wallace [30].

The concept of bi-connectivity was studied by Whyburn [68], although the term, in connection with quasi-ordered spaces, is due to Koch [24]. Ward [64] has also considered bi-connected chains in quasi-ordered spaces. Bi-connectivity has appeared in the study of algebraically irreducible semi-groups, although not by that name, in [17] and [11]. In Chapter II we study interrelationships between bi-connectivity, monotonicity, order-density, and order \( \pi \)-connectivity in quasi-ordered spaces.

Chapters III and IV present a justification of the work done in Chapter II. In Chapter III we extend results of Koch and Ward regarding the existence of bi-connected chains. The results are actually obtainable by methods of Koch and Ward. However, our method gives directly the existence of
bi-connected chains and the existence of arcs follows as a special case. In particular, an alternate solution to that of Ward's [64] is given of a conjecture due to Koch [24]. A corollary, as pointed out by Koch, is an order theoretic extension of an arc-lifting theorem due to Stoilow [47], [48], and [49], Whyburn [65] and [68], and Zippin (see [37;328]). Using this corollary we prove that any tree is liftable under a light open map. Williams [69] and Cornette [8] have considered possible extensions of the theory of liftability. These are also discussed in Chapter III.

Chapter IV deals with applications of the results of Chapter II to topological semigroups. We first prove a theorem due to Koch [23] on the existence of threads in compact semigroups. The proof is designed to give also the commutativity of standard threads, a well known result due to Faucett [9]. Very recently, Hofmann and Mostert [11] proved a very strong theorem in the theory of topological transformation groups which yielded the fundamental result that each algebraically irreducible semigroup is abelian. We sketch a proof of this theorem using the methods of Chapter II and the transformation group theorem. The proof includes the fact that each \( H \)-class is connected in such semigroups. Koch's theorem clearly follows from the theorem of Hofmann and Mostert, but its proof is presented without the use of the transformation group theorem.
The remainder of Chapter IV is devoted to proving an analog for algebraically irreducible semigroups of a theorem due to Phillips [44]. Phillips proved that the translate of a standard thread in a semigroup is an arc. We prove that the right translate of an algebraically irreducible semigroup is a compact bi-connected chain with respect to the left ideal quasi-order on $S$. 


PRELIMINARIES

Topological Preliminaries. Throughout this work, unless otherwise stated, a **space** will be a Hausdorff topological space. If A and B are subsets of a space X, then the complement of B in A will be denoted by A \ B. If A is a subset of the space X, then the closure of A in X will be denoted by A^*, the interior by A^o, and the boundary (A^* \ A^o) by F(A). A **continuum** is a compact connected Hausdorff space. An **arc** is a continuum with exactly two non-cutpoints. A function from one space to another is **light** if the inverse image of each point is totally disconnected and **open** if the image of each open set is open. The terms **map** and continuous function are synonymous. If a function is continuous, one-to-one, and open, it is a **homeomorphism**. The cartesian product of a space X with itself will be denoted by X^2, unless the set considered is a semigroup. In this case we use S x S. The **diagonal** of X^2 will be denoted by Δ_X and consists of \{(x,x) : x ∈ X\}. When no confusion can arise, we use Δ in place of Δ_X. Throughout this work, Σ and π will denote open sets containing the diagonal of some space. The term **iff** will mean 'if and only if'.

For further information, the reader is referred to [5], [18], [33], [34], [39], and [68].

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Relation Theoretic Preliminaries. A (binary) relation on $X$ is a subset of $X^2$. Without exception, $\Gamma$ and $\theta$ will be used to denote relations on spaces. If $\Gamma$ is a closed subset of $X^2$ we call $\Gamma$ a closed relation on $X$. If $\Gamma$ is a relation on $X$, then $\Gamma^{-1} = \{(y,x) : (x,y) \in \Gamma\}$. The relation $\Gamma$ on $X$ is reflexive if $\Delta \subseteq \Gamma$, symmetric if $\Gamma = \Gamma^{-1}$, anti-symmetric if $\Gamma \cap \Gamma^{-1} \subseteq \Delta$, and transitive if $(x,y) \in \Gamma$ and $(y,z) \in \Gamma$ imply $(x,z) \in \Gamma$. A quasi-order is a reflexive transitive relation. A partial order is an anti-symmetric quasi-order. An equivalence relation is a symmetric quasi-order. If $\Gamma$ is a relation on $X$ and $A \subseteq X$, then

$$L(A) = \{x : (x,a) \in \Gamma \text{ for some } a \in A\} \text{ and}$$

$$M(A) = \{y : (a,y) \in \Gamma \text{ for some } a \in A\}.$$  

For a fixed $x \in X$, $L(x)$ is called the set of predecessors of $x$ or the lower set of $x$ and $M(x)$ is called the set of successors of $x$ or the upper set of $x$. The level set $L_x$ of an element $x$ is $\{y : x \cup L(x) = y \cup L(y)\}$. If $\Gamma$ is a quasi-order then $L_x = L(x) \cap M(x)$. We denote by $I(x)$ the set $L(x) \setminus L_x$. If $A \subseteq X$ and $x, y \in A$, the interval between $x$ and $y$ in $A$ is $M(x) \cap L(y) \cap A$ and is denoted by $[x,y]_A$. If $A = X$, we simply use $[x,y]$. A chain in a quasi-ordered space is a subset $A$ such that $A^2 \subseteq \Gamma \cup \Gamma^{-1}$.

Our primary interest will be in quasi-orders and it will be convenient to write $x \leq y$ if $(x,y) \in \Gamma$ and $x < y$ if $(x,y) \in \Gamma \setminus \Gamma^{-1}$. We use these notations interchangeably when no ambiguity can arise.
A well known theorem due to Wallace [52] states that if $\Gamma$ is a closed quasi-order on a compact space $X$, then there exist $(\Gamma)$ minimal elements in $X$ ($x$ is minimal if $(y,x) \in \Gamma$ implies $(x,y) \in \Gamma$). It follows that if $\Gamma$ is a closed quasi-order on a space $X$, then each compact chain $A \subseteq X$ has a non-void sup and a non-void inf.

Let $\Gamma$ and $\Theta$ be quasi-orders on the spaces $X$ and $Y$, respectively. The function $f$ from $X$ into $Y$ is order preserving if $(x,y) \in \Gamma$ implies $(f(x), f(y)) \in \Theta$. The function $f$ is strictly order preserving if $f$ is order preserving and $(x,y) \in \Gamma \setminus \Gamma^{-1}$ implies $(f(x), f(y)) \in \Theta \setminus \Theta^{-1}$.

Let $\Theta$ be a quasi-order on $D$. Then, $\Theta$ directs $D$ if $\alpha, \beta \in D$ imply the existence of $\delta \in D$ such that $(\alpha, \delta) \in \Theta$ and $(\beta, \delta) \in \Theta$. If $\Theta$ directs $D$, the pair $(D, \Theta)$ is a directed set. A net in a space $X$ is a function from a directed set into $X$. If $(f, D, \Theta)$ is a net in $X$ and no confusion can arise, we say that $\{x_\alpha : \alpha \in D\}$ is the given net, where $x_\alpha = f(\alpha)$. For convenience, we sometimes simply write $\{x_\alpha\}$ is a net in $X$, making no reference to the associated directed set. A net $\{y_\delta : \delta \in E\}$ is a subnet of the net $\{x_\alpha : \alpha \in D\}$ if there exists a function $N$ on $E$ with values in $D$ such that

(a) $y_\delta = x_{N(\delta)}$ for each $\delta \in E$

(b) For each $\alpha \in D$ there is a $\delta \in E$ such that $\beta \geq \delta$ implies $N(\beta) \geq \alpha$ (recall that $\beta \geq \delta$ means $(\delta, \beta) \in E$).
If $D$ is a directed set, the subset $A$ of $D$ is cofinal (residual) in $D$ if $\beta \in D$ implies the existence of $\alpha \in A$ such that $\alpha \geq \beta$ ($\delta \geq \alpha$ implies $\delta \in A$). We say that the net

$[x_\alpha : \alpha \in D]$ is cofinally (residually) in a set $A$ if

$[\alpha : x_\alpha \in A]$ is cofinal (residual) in $D$. The net $[x_\alpha]$ clusters at (converges to) $x$ if it is cofinally (residually) in each open set $U$ containing $x$. A net $[x_\alpha]$ in $X$ is called a universal net if for each $A \subseteq X$ either $[x_\alpha]$ is eventually in $A$ or $[x_\alpha]$ is eventually in $X \setminus A$. Every net has a universal subnet [18;81]. Every net in a compact space $X$ clusters at some point $x \in X$.

Let $2^X$ denote the collection of subsets of $X$ and let $[A_\alpha : \alpha \in D]$ be a net in $2^X$. Then, we say that $[A_\alpha : \alpha \in D]$ cofinally (residually) meets the set $B$ if $[\alpha : A_\alpha \cap B \neq \emptyset]$ is cofinal (residual) in $D$ (here and throughout this work, $\emptyset$ denotes the empty set). The superior and inferior limits of $[A_\alpha]$ are defined by

$$\limsup A_\alpha = \{x : A_\alpha \text{ cofinally meets each open set } U \text{ containing } x\}$$

$$\liminf A_\alpha = \{x : A_\alpha \text{ residually meets each open set } U \text{ containing } x\}.$$ 

For any net $[A_\alpha]$ in $2^X$, $\liminf A_\alpha \subseteq \limsup A_\alpha$ and each is closed; $\limsup A_\alpha = \limsup A_\alpha^*$ and $\liminf A_\alpha = \liminf A_\alpha^*$ [33;242]. We say that $[A_\alpha]$ converges to $A$ if

$$\limsup A_\alpha = \liminf A_\alpha = A$$

and we write $\lim A_\alpha = A$. If $X$ is compact and $A_\alpha \neq \emptyset$ for
each $a$, then \( \limsup A_a \neq \emptyset \).

For further information, the reader is referred to [4], [6], [18], [33], [42], [51], and [68].

**Topological Algebraic Preliminaries.** Throughout this work a **topological semigroup** will be a Hausdorff space endowed with a continuous, associative multiplication. All semigroups we deal with will be compact topological semigroups, but the following concepts are defined for discrete semigroups also. An **idempotent** is an element \( e \) such that \( e^2 = e \). The set of idempotents will be denoted by \( E \). Each idempotent \( e \) belongs to a unique **maximal subgroup** of \( S \) and distinct idempotents have disjoint maximal groups. The maximal group containing \( e \) is denoted by \( H(e) \). The semigroup of positive powers of a fixed element \( x \), \( \{ x^n : n = 1, 2, \ldots \} \), is denoted by \( \theta(x) \); its closure by \( \Gamma(x) \). The set \( A \) is a **left** (right, two-sided) **ideal** of \( S \) provided \( SA \subseteq A \) (\( AS \subseteq A \), \( AS \cup SA \subseteq A \)).

If \( S \) has an identity, the relation \( \Gamma_L(S) (\Gamma_R(S), \Gamma_H(S), \Gamma_J(S)) \) defined by \( (x, y) \in \Gamma_L(S) (\Gamma_R(S), \Gamma_H(S), \Gamma_J(S)) \) iff \( x \in yS \) \((x \in yS, x \in yS \cap Sy, x \in SyS) \) is a quasi-order. Moreover, if \( S \) is compact, each of these quasi-orders is closed. The **centralizer** of a subset \( A \) of \( S \) is \( \{ x : xa = ax \text{ for each } a \in A \} \).

It is well known that a compact semigroup has minimal ideals of all types, each such is closed, and \( S \) has a unique minimal (two-sided) ideal which is a union of maximal groups [54]. The **minimal** (two-sided) **ideal** of a compact subsemigroup \( T \) of
S is denoted by Ker T; that of S by K. If T is compact and 
e \in E \cap \text{Ker } T, then H(e) = eTe. A clan is a compact connected 
topological semigroup with identity. A standard thread is an 
arc which is a semigroup such that one endpoint (non-cutpoint) 
acts as a zero and the other acts as an identity. A local 
one-parameter semigroup based at e is a continuous homomor-
phic (in-so-far as the word is meaningful) image of [0,a) 
under addition, where a > 0 and the image of 0 is e.

If x is a fixed element of a clan S, the H-class of x, 
denoted by H_x, is the equivalence class of x with respect to 
the equivalence relation \Gamma_H(S) \cap \Gamma_H(S)^{-1}. Let 
\[
T(H_x) = \{s : sH_x \subset H_x\}
\]
\[
\theta(H_x) = \{(s,t) : s, t \in T(H_x) \text{ and } sx = tx\}
\]
\[
\mathcal{Q}(H_x) = T(H_x) / \theta(H_x).
\]
Then, \(\mathcal{Q}(H_x)\) is a compact topological transformation group 
which acts simply transitively on H_x and is called the 
Schützenberger group of H_x (see e.g. [1]). The set H_x 
is homeomorphic to its Schützenberger group.

It should be pointed out that many of the concepts mentioned 
here are discussed in much greater generality in other works. 
However, for our purposes, the discussion given here suffices. 
Further references: [1], [3], [11], [38], and [54].
CHAPTER I
IMBEDDABILITY AND IRREDUCIBILITY

In the first part of this chapter we use results due to Nachbin to prove that every compact metric partially ordered space is imbeddable in the Hilbert cube by an order preserving homeomorphism. The second part deals with quasi-ordered spaces which are irreducible between two subsets.

Definition 1.1. If $\Gamma$ is a quasi-order on $X$, an order preserving real valued function is called increasing.

The following theorem follows immediately from Theorem 4 and Theorem 6 on pages 48 and 49 of [42].

Theorem 1.2. (Nachbin) Let $\Gamma$ be a closed partial order on the compact space $X$ and let $F$ be a closed subset of $X$. Then, every continuous, increasing, real valued function on $F$ can be extended to the entire space in such a way as to remain continuous and increasing.

Proofs of the following theorem can be found in both [42] and [58].

Theorem 1.3. Let $\Gamma$ be a closed partial order on the compact space $X$. Then, $X$ is imbeddable in a product $Y$ of closed
intervals by an order preserving homeomorphism (where $Y$ has the coordinatewise partial order).

We now show that, in the compact metric case, $Y$ may be taken as a countable product of intervals, hereafter called the **Hilbert cube**.

**Theorem 1.4.** If $\Gamma$ is a closed partial order on the compact metric space $X$, then $X$ is imbeddable in the Hilbert cube by an order preserving homeomorphism.

**Proof.** Let $\mathcal{V}$ be a countable base for the topology of $X$ and let $\mathcal{U}$ be the collection of finite unions of members of $\mathcal{V}$. Then, $\mathcal{U}$ is countable and so is $\mathcal{B} = \{(U,V) : U, V \in \mathcal{U}\}$. Hence, $\mathcal{B} = \{(U,V) : (U,V) \in \mathcal{B}, \text{ and } M(U^*) \cap L(V^*) = \emptyset\}$ is countable. Fix $(U,V) \in \mathcal{B}$. Since $M(U^*) \cap L(V^*) = \emptyset$, the real valued function $f(U,V)$ defined on $U^* \cup V^*$ by $f(U,V)(z)$ equals 1 if $z \in U^*$ and 0 if $z \in V^*$, is continuous and increasing. Hence, by Theorem 1.2, $f(U,V)$ can be extended to a continuous, increasing, real valued function $f(U,V)$ defined on all of $X$. Let $I(U,V)$ be a closed interval containing $f(U,V)(X)$ and let $Y$ be the Hilbert cube $\mathcal{P}(I(U,V) : (U,V) \in \mathcal{B})$. Let $\psi$ be the product of the functions $f(U,V)$ where $(U,V) \in \mathcal{B}$. Then, since each function $f(U,V)$ is continuous, $\psi$ is continuous. That $\psi$ is order preserving follows from the fact that each coordinate function is increasing and the fact that $Y$ has the coordinatewise partial order. We proceed to show that $\psi$ is one-to-one.
Suppose $x \neq y$. Then, we may assume that $(x,y) \notin \Gamma$. Thus, the closed sets $M(x)$ and $L(y)$ are disjoint. By normality, there exist disjoint open sets $A$ and $B$ containing $M(x)$ and $L(y)$, respectively. Now, by Proposition 5, page 45 of [42], there exist open sets $C$ and $D$ such that

$$M(x) \subseteq C = M(C) \subseteq A \quad \text{and} \quad L(y) \subseteq D = L(D) \subseteq B.$$ 

Again, by normality, there exist open sets $E$ and $F$ such that

$$M(x) \subseteq E \subseteq E^* \subseteq C \quad \text{and} \quad L(y) \subseteq F \subseteq F^* \subseteq D.$$ 

For each point $z \in M(x)$, let $U_z$ be a member of $U$ such that $z \subseteq U_z \subseteq E$. Then, the collection $\{U_z : z \in M(x)\}$ is an open cover of the compact set $M(x)$. Let $\{U_i : i = 1, 2, \ldots, n\}$ be a finite subcover and let $U = U[U_i : i = 1, 2, \ldots, n]$. Then, $U \in \mathcal{U}$ and $M(x) \subseteq U \subseteq E$. Similarly, there exists $V \in \mathcal{U}$ such that $L(y) \subseteq V \subseteq F$. Note that $U^* \subseteq E^* \subseteq C$ and $V^* \subseteq F^* \subseteq D$ so that $M(U^*) \cap L(V^*) = \emptyset$. Therefore, $(U,V) \in \mathcal{B}$. Hence,

$$f(U,V)(M(x)) \subseteq f(U,V)(U) \subseteq f(U,V)(U^*) = [1] \quad \text{and} \quad f(U,V)(L(y)) \subseteq f(U,V)(V) \subseteq f(U,V)(V^*) = [0].$$

This yields that $f(U,V)(x) \neq f(U,V)(y)$ and thus, $\psi(x) \neq \psi(y)$.

Hence, $\psi$ is a continuous, one-to-one function from the compact space $X$ into the Hausdorff space $Y$. It follows that $\psi$ is a homeomorphism and the proof is complete.

In [28], Koch and Mc Auley call a metric $d$ on a partially ordered space (with zero) radially convex if $(x,y) \in \Gamma$, $(y,z) \in \Gamma$, and $y \neq z$ imply $d(x,y) < d(x,z)$. The question has been raised by Koch (although it does not appear in print), 'Does a generalized tree admit a radially convex
metric?". The following corollary gives an affirmative answer to this question.

**Corollary 1.5.** If Γ is a closed partial order on the compact metric space X, then X admits a radially convex metric with respect to Γ.

**Proof.** Let Ψ be an order preserving homeomorphism from X into the Hilbert cube Y. We may assume that Y is a countable product of unit intervals and that the metric on Y is defined by \( d((x_n), (y_n)) = \sum |x_n - y_n| / 2^n \), where the summation is taken from \( n = 1 \) to \( \infty \). If \( x_n \leq y_n \) and \( y_n < z_n \) in the coordinatewise partial ordering on Y, then it follows immediately that \( d((x_n), (y_n)) < d((x_n), (z_n)) \). Hence, \( d \) is a radially convex metric on Y. Obviously, this property is hereditary and the proof is complete.

**Corollary 1.6.** Let Γ be a closed quasi-order on the compact metric space X. Then, there exists a continuous, order preserving function \( \chi \) taking X into the Hilbert cube, such that \( \chi^{-1} \chi(x) = L_x \) for each \( x \in X \). In particular, if \( x < y \) then \( \chi(x) < \chi(y) \) (\( \chi \) is strictly order preserving).

**Proof.** Let \( \eta \) be the natural map from X onto the quotient space \( X / (\Gamma \cap \Gamma^{-1}) = Y \). That Y is Hausdorff follows from the compactness of X and the fact that Γ is closed. It is well known that the Hausdorff continuous image of a compact metric space is compact and metric (see e.g. [18;148]). It is easy to see that \( \theta = (\eta \times \eta)(\Gamma) \) is a closed partial
order on $Y$. Hence, by Theorem 1.4, there exists an order preserving homeomorphism $\psi$ from $Y$ into the Hilbert cube. The composition of $\psi$ with $\eta$ has the desired properties.

The investigations made in the remainder of this chapter are along the same lines as those made by Krule and Wallace in [31], [32], [54], [55], [56], and [57]. Theorem 1.15 gives a number of order theoretic analogs of results due to Koch and Wallace [30].

**Definition 1.7.** The relation $\Gamma$ on $X$ is a *struct* provided it is non-void, closed, and transitive.

**Definition 1.8.** Following Wallace [55], we say that the relation $\Gamma$ on $X$ is *continuous* provided $L(A^*) \subseteq L(A)^*$ for each subset $A \subseteq X$.

**Definition 1.9.** The relation $\Gamma$ on $X$ is *monotone* provided $L(x)$ is connected for each $x \in X$.

**Definition 1.10.** If $\Gamma$ is a struct on $X$, the subset $A$ of $X$ is an $\Gamma$ *ideal* if $L(A) \subseteq A$.

In the remainder of this chapter, $K$ will denote the set of minimal elements of $X$ with respect to $\Gamma$. The following theorem may be found in [55].

**Theorem 1.11.** (Wallace) If $X$ is a continuum and if $\Gamma$ is a continuous monotone struct on $X$, then $K$ is a (non-void) continuum and is also an $\Gamma$ ideal.
Theorem 1.12. (Strother [51]) Let \( \Gamma \) be a continuous struct on \( X \). Then, if \( \{x_\alpha\} \) converges to \( x \), \( \{L(x_\alpha)\} \) converges to \( L(x) \).

Definition 1.13. The continuum \( X \) is irreducible between \( A \) and \( B \) if for any subcontinuum \( C \) with \( C \cap A \neq \emptyset \neq C \cap B \) we have \( C = X \).

Definition 1.14. The subset \( C \) of a continuum \( X \) is a \( C \)-set if \( C \neq X \) and for any subcontinuum \( A \) of \( X \) with \( A \cap C \neq \emptyset \) we have either \( A \subseteq C \) or \( C \subseteq A \).

Theorem 1.15. Let \( X \) be a continuum and let \( \Gamma \) be a continuous monotone reflexive struct on \( X \) such that \( K < X \). Suppose \( X \) is irreducible between \( K \) and \( A \). Then

1. If \( B \) is a closed \( \Gamma \) ideal containing \( K \), then \( x \in X \setminus B \) implies \( B \subseteq L(x) \).
2. Each closed \( \Gamma \) ideal is principal (generated by one element).
3. If \( x, y \in X \) then either \( L(x) \subseteq L(y) \) or \( L(y) \subseteq L(x) \).
4. The set \( I(x) = L(x) \setminus L_x \) is open for each \( x \in X \).
5. The continua which meet both \( K \) and \( X \setminus K \) are exactly the principal \( \Gamma \) ideals different from \( K \).
6. Each set \( L_x \), where \( x \in X \setminus K \), is a \( C \)-set in \( L(x) \).
7. For each \( x \in X \), \( L_x \) is connected.

Proof of (1). Let \( U \) be an open set containing \( x \) such that \( U^* \cap B = \emptyset \). It follows from (viii) of [55] that \( P = U \{ L(y) : y \in X \setminus U^* \} \) is a non-void, open \( \Gamma \) ideal. Note
that \( K \subseteq B \subseteq P \) and \( x \notin P^* \). By Theorem 1.11, \( K \) is connected and by monotonicity, \( L(y) \) is connected for each \( y \). It follows that \( P \) is connected. We observe that \( P^* \cap A = \emptyset \) since \( P^* \) is a continuum, \( P^* \cap K \neq \emptyset \), and \( P^* \) \( \subset \) \( X \). We now show that \( X \setminus P^* \) is connected. Assume \( X \setminus P^* = C \uplus D \) is a separation of \( X \setminus P^* \). Then, \( C \) and \( D \) are open in \( X \) and \( P^* \cup C = X \setminus D \); \( P^* \cup D = X \setminus C \) are closed and connected (since \( P^* \) is connected). Now, \( P^* \cap A = \emptyset \) implies either \( C \cap A \neq \emptyset \) or \( D \cap A \neq \emptyset \). If \( C \cap A \neq \emptyset \), then \( P^* \cup C \) is a continuum meeting both \( K \) and \( A \) so that \( P^* \cup C = X \) and \( D = \emptyset \). An analogous argument handles the case \( D \cap A \neq \emptyset \). Hence, \( X \setminus P^* \) is connected and we have \( (X \setminus P^*)^* \) is connected. Now, \( B \subseteq P \) and \( P \) is open so that \( \emptyset = B \cap F(P) = B \cap ((X \setminus P^*)^* \cap P^*) = B \cap (X \setminus P^*)^* \). Therefore, \( (X \setminus P^*)^* \) is a continuum missing \( B \). Also, \( x \in L(x) \cap (X \setminus P^*)^* \) which implies \( L(x) \cup (X \setminus P^*)^* \) is a continuum meeting \( K \) and \( A \). Hence, \( L(x) \cup (X \setminus P^*)^* = X \) and it follows that \( B \subseteq L(x) \).

**Proof of (2).** First we show that \( K = L(x) \) for each \( x \in K \).

We are given that \( K \) is closed and is not all of \( X \) so that \( K \) is not open. Thus, there exists a net \( \{x_\alpha\} \) in \( X \setminus K \) such that \( \{x_\alpha\} \) converges to \( x \in K \). Letting \( B = K \) in part (1) we have \( L(x_\alpha) \supset K \) for each \( \alpha \). By Theorem 1.12, \( \{L(x_\alpha)\} \) converges to \( L(x) \). Therefore, \( L(x) \supset K \). But, \( x \in K \) and \( K \) is a \( \Gamma \) ideal so that \( L(x) \subseteq K \). Hence, \( L(x) = K \). It follows easily that \( K = L(y) \) for each \( y \in K \). Now, let \( B \) be an arbitrary closed \( \Gamma \) ideal. It follows from [54] that \( B \cap K \neq \emptyset \). Fix \( y \in B \cap K \).
Then, $K = L(y) \subseteq L(B) = B$. If $B = X$, we let $a \in A$ and since $X$ is irreducible between $K$ and $A$, we have that $B = L(a)$. If $B < X$, then $B$ is not open and there exists a net $\{x_\alpha\}$ in $X \setminus B$ such that $\{x_\alpha\}$ converges to $x \in B$. Again, by Theorem 1.12 and part (1), $B \subseteq L(x)$. Consequently, $B = L(x)$.

**Proof of (3).** Fix $x$ and $y$ in $X$. If $z \in L(x) \setminus L(y)$, then, since $L(y)$ is a closed ideal, $L(y) \subseteq L(z)$ (see (1)).

Thus, by transitivity, $L(z) \subseteq L(x)$ and $L(y) \subseteq L(x)$. The conclusion follows immediately.

**Proof of (4).** Fix $x \in X$. If $y \in I(x)$, then $L_x \cap L(y) = \emptyset$.

Thus, there exists an open set $V$ containing $L(y)$ such that $L_x \subseteq X \setminus V^*$. As in the proof of (1), $P = \{L(z) : z \in V\}$ is an open ideal and $L_x \subseteq X \setminus P^*$. The continuity of $\Gamma$ insures that $P^*$ is a $\Gamma$ ideal. Hence, by part (1), $y \in P \subseteq P^* \subseteq L(x)$.

Finally, $P^* \subseteq V^*$ which misses $L_x$ so that $y \in P \subseteq I(x)$, yielding that $I(x)$ is open.

**Proof of (5).** Let $P$ be a continuum meeting both $K$ and $X \setminus K$. Then, $P \subseteq L(P)$ since $\Gamma$ is reflexive. Now, $L(P)$ is a closed ideal, which implies that $L(P) = L(x)$ for some $x \in X$. But, $x \in L(x) = L(P)$ implies that $x \in L(p)$ for some $p \in P$. Hence, $P \subseteq L(P) \subseteq L(x) \subseteq L(p)$. Note that $p \in X \setminus K$. We now show that $I(p) \subseteq P$. Fix $y \in I(p)$ and let $W$ be an open set containing $y$ such that $W \subseteq I(p)$ and $W^* \cap L_p = \emptyset$ (see (4)).

Then, $(X \setminus L(W)^*)^* \cup P$ is a continuum which meets both $K$ and $A$. Hence, $(X \setminus L(W)^*)^* \cup P = X$. Now, $y \in W \subseteq L(W)$ implies
It follows that \( y \in P \) and \( I(p) \subseteq P \). Thus, \( I(p)^* \subseteq P \). Now, \( I(p) \) is open so that \( I(p)^* \setminus I(p) \neq \emptyset \) which implies \( I(p)^* \cap L_p \neq \emptyset \). Since \( I(p)^* \) is a \( \Gamma \) ideal, \( I(p)^* \) must contain \( L_p \). Therefore, \( I(p)^* = L(p) \subseteq P \) and by our previous remarks, \( L(p) = P \).

Proof of (6). Fix \( x \in X \setminus K \). Let \( Q \) be a subcontinuum of \( L(a) \) with \( Q \cap L_x \neq \emptyset \neq Q \cap I(x) \). Fix \( y \in Q \cap I(x) \). Then, \( L(y) \cup Q \) is a continuum which meets \( X \setminus K \) and \( K \). Hence, by (5), \( L(y) \cup Q \) is a principal \( \Gamma \) ideal. Also, \( L(y) \cup Q \) is contained in \( L(x) \) and meets \( L_x \). It follows that \( L(y) \cup Q = L(x) \).

Now, since \( L(y) \subseteq I(x) \subseteq X \setminus L_x \), \( L_x \subseteq Q \). Hence, \( L_x \) is a C-set in \( L(x) \).

Proof of (7). This follows directly from (6) and Lemma 1 of [56].

**Corollary 1.16.** If \( \Gamma \) is a continuous monotone reflexive structure on the continuum \( X \) and \( X \) is irreducible between \( K \) and \( A \), then \( X \) is a chain and has connected level sets.
In this chapter we establish certain inter-relationships between the concepts of order density, monotonicity, bi-connectivity, and order $\pi$-connectivity in quasi-ordered spaces. Some of the results relating order density to connectivity were given by Ward [59] in the partially ordered case. The main result of this chapter is Theorem 2.21, which gives a very useful equivalence to bi-connectivity in the case of compact chains.

Although some of the results in this chapter are valid under the weaker assumption that the quasi-orders are semi-continuous (see Ward [59]), we restrict our attention to quasi-orders which are closed. The following definitions may be found in [5] and [11]; see also [34].

**Definition 2.1.** If $\pi$ is an open subset of $X^2$ containing the diagonal $\Delta$, then $\{x_i : i = 1, \ldots, n\}$ is a $\pi$-chain provided $(x_i, x_{i+1}) \in \pi$ for each $i = 1, \ldots, n-1$. We say that $x$ and $y$ are joined by the $\pi$-chain $\{x_i : i = 1, \ldots, n\}$ if $x = x_1$ and $y = x_n$.

**Definition 2.2.** If $\pi$ is an open subset of $X^2$ containing $\Delta$, then the set $A \subset X$ is $\pi$-connected provided each pair of
points $x$ and $y$ in $A$ can be joined by a $\pi$ - chain completely contained in $A$.

One may think of a $\pi$ - connected set as one having 'gaps no larger than $\pi$'. In the metric case, if we let $\pi$ be $\{(x,y) : d(x,y) < \varepsilon\}$, then a $\pi$ - connected set is a set with gaps no larger than $\varepsilon$. Thus, in a sense, neighborhoods of the diagonal replace the notion of distance in non-metric spaces.

The next lemma is the non-metric analog of Theorem 9.1, page 14 in [68]. Regarding Theorem 2.4, see [5] and [11].

Lemma 2.3. Let $X$ be compact and let $\{A_\alpha : \alpha \in D\}$ be a net of subsets of $X$ such that for each open $\pi$ containing $\Delta$,

$\{\alpha : A_\alpha$ is $\pi$ - connected$\}$ is residual in $D$. If $\liminf A_\alpha \neq \emptyset$ then $\limsup A_\alpha$ is connected.

Proof. Suppose $\limsup A_\alpha = P \uplus Q$ where $P \neq \emptyset \neq Q$. Fix $\alpha \in \liminf A_\alpha$ and assume that $\alpha \in P$. Let $U$ and $V$ be open sets containing $P$ and $Q$ respectively such that $U^* \cap V^* = \emptyset$ and let $\pi = X^2 \setminus (U^* \times V^*)$. Obviously, $\pi$ is open and $\Delta \subset \pi$.

Thus, by hypothesis, $\{\alpha : A_\alpha$ is $\pi$ - connected$\}$ is residual in $D$. Since $X$ is compact $\{\alpha : A_\alpha \subset U \cup V\}$ is residual in $D$ and since $\alpha \in \liminf A_\alpha$, $\{\alpha : A_\alpha \cap U \neq \emptyset\}$ is residual in $D$. Now, $Q \neq \emptyset$ implies $\{\alpha : A_\alpha \cap V \neq \emptyset\}$ is cofinal in $D$. The combination of these facts gives the existence of $\beta \in D$ such that $A_\beta$ is $\pi$ - connected, $A_\beta \subset U \cup V$, and $A_\beta \cap U \neq \emptyset \neq A_\beta \cap V$.

Now, $A_\beta \subset U \cup V$ implies that any $\pi$ - chain contained in $A_\beta$
must be contained in either U or V (by definition of \( \pi \)).

However, \( A_\beta \cap U \neq \emptyset \neq A_\beta \cap V \) and \( A_\beta \) is \( \pi \)-connected imply the existence of a \( \pi \)-chain contained in \( A_\beta \) and stretching from U to V. This contradiction completes the proof.

**Theorem 2.4.** Let \( X \) be compact and let \( A \) be a subset of \( X \).

Then, T. A. E. (the following are equivalent)

1. \( A \) is compact and connected.
2. \( A \) is compact and \( \pi \)-connected for each open \( \pi \supset \Delta \).
3. \( A \) is the limit of a net \( \{ A_\alpha : \alpha \in D \} \) of sets such that if \( \pi \) is open containing \( \Delta \), then 
   \[ \{ \alpha : A_\alpha \text{ is } \pi \text{-connected} \} \text{ is residual in } D. \]

**Proof.** (1) \( \Rightarrow \) (2). Fix \( \pi \), an open set containing \( \Delta \). For fixed \( p \) in \( A \), let

\[ P = \{ x : x \in A \text{ and } p \text{ can be joined to } x \text{ by a } \pi \text{-chain in } A \}. \]

We first show that \( P \) is both open and closed in \( A \) so that \( P = A \). Fix \( y \in P \) and let \( \{ x_i : i = 1, \ldots, n \} \) be a \( \pi \)-chain from \( p \) to \( y \) which is contained in \( A \). Let \( U \) be an open set in \( A \) containing \( y \) such that \( U \times U \subset \pi \). Then, for each \( z \in U \) we have \( \{ x_i : i = 1, \ldots, n+1 \} \) is a \( \pi \)-chain from \( p \) to \( z \), where \( x_{n+1} = z \). Therefore, \( U \subset P \) and \( P \) is open in \( A \). Now, fix \( y \in P^* \) (note that \( P^* \subset A \)) and again let \( U \) be an open set in \( A \) containing \( y \) such that \( U \times U \subset \pi \). Since \( y \in P^* \), \( U \cap P \neq \emptyset \). Let \( z \in U \cap P \) and let \( \{ x_i : i = 1, \ldots, n \} \) be a \( \pi \)-chain from \( p \) to \( z \) which is contained in \( A \). Then, \( \{ x_i : i = 1, \ldots, n+1 \} \) is a \( \pi \)-chain from \( p \) to \( y \), where \( x_{n+1} = y \). Thus, \( y \in P \) and \( P \) is closed in \( A \). Finally, let
x and y be points of A and let \( \{x_i : i = 1, \ldots, n\} \) and 
\( \{y_i : i = 1, \ldots, m\} \) be \( \pi \) - chains from p to x and from p to y, respectively. Then \( \{z_i : i = 1, \ldots, n+m-1\} \) is a \( \pi \) - chain from x to y, where 
\[ z_i = x_{n+i-1} \] if \( 1 \leq i \leq n \) and 
\[ z_i = y_{i+n-m} \] if \( n \leq i \leq n+m-1 \). Thus, A is \( \pi \) - connected.

(2) \( \Rightarrow \) (3). This is obvious, letting D be the directed set with one element and letting \( A_1 = A \).

(3) \( \Rightarrow \) (1). If \( A = \emptyset \) we are finished. If \( A \neq \emptyset \), 
\[ \liminf A = \emptyset \] and by Lemma 2.3, \( \limsup A = A \) is connected.
After observing that the limsup of any net of sets is closed, 
and the fact that X is compact, the proof is complete.

Throughout the remainder of this chapter, X will be a quasi-ordered space with quasi-order \( \theta \).

The previous definitions and page 186 of [68] are the motivation for the following definitions.

Definition 2.5. If \( \pi \) is an open subset of \( X^2 \) containing \( \Delta \),
then \( \{x_i : i = 1, \ldots, n\} \) is an order \( \pi \) - chain provided 
\( (x_i, x_{i+1}) \in \theta \cap \pi \) for each \( i = 1, \ldots, n-1 \). We say that x and 
y are joined by the order \( \pi \) - chain \( \{x_i : i = 1, \ldots, n\} \) if 
x = x_1 and y = x_n.

Definition 2.6. If \( \pi \) is an open subset of \( X^2 \) containing \( \Delta \),
then the set \( A \subseteq X \) is order \( \pi \) - connected if each pair of points x and y in A with \( (x,y) \in \theta \) can be joined by an order
\(\pi\) - chain completely contained in \(A\).

Again, one may think of an order \(\pi\) - connected set as one in which each pair of comparable points are extremities of a \(\pi\) - chain which is contained in \(A\) and which proceeds directly from one point to the other relative to the quasi-order \(\theta\).

The next definition is that given by Koch [24], and is in no way related to that given by Kuratowski [33].

**Definition 2.7.** The set \(A \subset X\) is bi-connected if \(A\) is connected and \(L_x \cap A\) is connected for each \(x \in A\) (i.e. \(A\) is connected and has connected level sets).

**Note.** Any connected set which is partially ordered by \(\theta\) is bi-connected, since its level sets are points.

**Definition 2.8.** The set \(A \subset X\) is order dense if whenever \(x\) and \(y\) are in \(A\) with \(x < y\), there exists a \(z\) in \(A\) such that \(x < z < y\). (Birkhoff [4] calls such a set dense in itself; see Ward [59].)

**Theorem 2.9.** Each connected chain in \(X\) is order dense.

**Proof.** Let \(A\) be a connected chain in \(X\). Suppose there exist points \(x\) and \(y\) in \(A\) with \(x < y\) and such that 

\([x,y] \cap A \subset L_x \cup L_y\). Let \(P = \{p : p \in A\) and \(y \leq p]\) and let 

\(Q = \{q : q \in A\) and \(q \leq x\}\). Then, \(Q = L(x) \cap A\) by definition and \(A = P \cup Q\) since \(A\) is a chain. If \(p \in P\), then \(x < y \leq p\) so that \(p \not\in L(x)\). Now, \(L(x)\) is closed so there exists an
open set $U$ containing $p$ such that $U \cap L(x) = \emptyset$. Therefore, $U \cap Q = U \cap (L(x) \cap A) \subseteq U \cap L(x) = \emptyset$ so that $P$ is open in $A$. Similarly, $Q$ is open in $A$ contradicting the connectivity of $A$.

**Theorem 2.10.** Suppose $X$ is compact and $A$ is a compact, bi-connected, order dense subset of $X$. Then, for every $x$ and $y$ in $A$, each of the following sets is connected:

1. $L(y) \cap A$
2. $M(y) \cap A$
3. $[x,y] \cap A$

**Proof.** By considering the compact space $A$ and the quasi-order $\theta \cap (A \times A)$, we may assume that $A = X$, $\theta = \theta \cap (A \times A)$, and $L(y) \cap A = L(y)$ with no loss of generality.

We first show that for fixed $y \in A$, $L(y)$ is connected. Assume $L(y) = P \uplus Q$ where $P \neq \emptyset \neq Q$. Since $L_y$ is connected we may assume $L_y \subseteq Q$. Now, $P$ is a compact quasi-ordered space so that $(\sup P) \cap P \neq \emptyset$. Fix $p \in (\sup P) \cap P$; then, $(p,y) \in \theta$ since $P \subseteq L(y)$ and we consider $[p,y]$. Clearly, $[p,y] = (P \cap [p,y]) \uplus (Q \cap [p,y])$ and we observe that since $L_p$ is connected and $p \in P$, it must be true that $L_p \subseteq P$. As before, let $q \in \{\inf (Q \cap [p,y])\} \cap (Q \cap [p,y])$; then, $(p,q) \in \theta$ and $L_q \subseteq Q$. Moreover, $[p,q] = L_p \cup L_q$, for if there exists $z \in [p,q]$ such that $p < z < q$, then either $z \in P$ contrary to the choice of $p$ or $z \in Q \cap [p,y]$ contrary to the choice of $q$. However, this contradicts the order
density of $A$. This contradiction completes the proof.

That $M(y)$ is connected for each $y$ follows directly from the consideration of the dual relation $\theta^{-1}$.

To see that each interval is connected, fix $x$ and $y$ in $A$. If $(x,y) \notin \theta$ then $[x,y] = \emptyset$ and we are finished. If $(x,y) \in \theta$, assume $[x,y] = P \uplus Q$ where $P \neq \emptyset \neq Q$. As before, we may assume $L_y = Q$. Let $p \in (\sup P) \cap P$ and let $q \in (\inf (Q \cap [p,y])) \cap (Q \cap [p,y])$. Now the argument proceeds in exactly the same way as that for $L(y)$.

Corollary 2.11. Suppose $X$ is compact and $A$ is a compact bi-connected chain. Then $\theta \cap (A \times A)$ is monotone and all intervals in $A$ are connected.

Proof. This follows directly from the theorem and Theorem 2.9 which gives the order density of $A$.

The following theorem is a partial converse of Corollary 2.11.

Theorem 2.12. Suppose $X$ is compact. If $A$ is a compact chain such that $\theta \cap (A \times A)$ is monotone, then $A$ is connected.

Proof. Let $m$ be a maximal element of $A$. Then, $L(m) = A$ is connected.

Next we give an example of a continuum which is order dense but not bi-connected and an example of a continuum which is bi-connected but not order dense.
Example 2.13. Let $A$ be the subset of the unit square consisting of the union of the diagonals. Define $\theta$ on $A$ by $\left(\begin{array}{l} (a,b), (c,d) \end{array}\right) \in \theta \text{ iff } b \leq d$. Then $A$ is a continuum which is order dense but not bi-connected. If $(a,b) \in A$ with $b < 1/2$, then $L[(a,b)] = [(0,0),(a,b)]$ is not connected.

Example 2.14. Let $I$ denote the unit interval with the usual ordering and let $X = I \times I$ with the product ordering. This is the usual lattice ordering on the 2-cell. Let $A$ be the union of the two straight line segments $B$ joining $(0,0)$ to $(1,0)$ and $C$ joining $(0,0)$ to $(1,1)$. Let $A$ have the partial order induced by the product order on $X$. Then, $A$ is a compact, bi-connected set which is not order dense. In this case we have $[(x,0),(x,x)] = \{(x,0)\} \cup \{(x,x)\}$ for each $x > 0$. Also, $M[(x,0)]$ and $[(x,0),(y,y)]$ are disconnected for each $x$ and $y$ such that $0 < x \leq y \leq 1$.

The next theorem points out the strong ties between order density, bi-connectivity, and order $\pi$-connectivity and is an important lemma for the main result of this chapter.

Theorem 2.15. Suppose $X$ is compact and $A$ is a compact and connected subset of $X$. Then, T.A.E.

1. $A$ is order dense and bi-connected.

2. $A$ is order $\pi$-connected for each open $\pi \supset A$.

Proof. $(1) \Rightarrow (2)$. By restricting our attention to $A$ and $\theta \cap (A \times A)$, we may assume that $X = A$ and that $\theta$ is a subset of $A \times A$. Fix $x$ and $y$ in $A$ with $(x,y) \in \theta$ and fix an open
set \( \pi \) containing \( \Delta \). If \((y,x) \in \theta\), then by Theorem 2.4 and the fact that \( L_x \) is connected, we may join \( x \) to \( y \) by a \( \pi \) - chain in \( L_x \) which is necessarily an order \( \pi \) - chain. If \((y,x) \notin \theta\), then \( L_y \) is a connected level set and by the above argument, \( L_y \) order \( \pi \) - connected. Hence,

\[ B = \{ B : B \text{ is an order } \pi \text{-connected chain in } [x,y] \} \]

contains the chain \( L_y \). Let \( M \) be a maximal tower of elements of \( B \) and let \( M = U\{B : B \in M\} \). It is obvious that \( M \) is an order \( \pi \) - connected chain contained in \([x,y]\). We now show that \( x \in M \) which will complete the proof of this part. If \( x \notin M \), then by the maximality of \( M \), \( M \cap L_x = \emptyset \) since lateral motion is possible in a connected level set by order \( \pi \) - chains. There are two cases to consider.

Case 1: \((\inf M) \cap M = \emptyset\).

Fix \( p \in (\inf M) \cap M \) (note that \( x \notin L_p \)). Then, as above, the maximality of \( M \) implies that \( L_p \subseteq M \). By Theorem 2.10, \([x,p]\) is connected. Obviously, \( L_p \) is a closed subset of \([x,p]\).

For each \( q \in L_p \) let \( V_q \) be an open set containing \( q \) such that \( V_q \cup V_q \subseteq \pi \). Let \( V = [x,p] \cap (\bigcup\{V_q : q \in L_p\}) \); then, \( V \) is open in \([x,p]\) and contains \( L_p \) so that \( V \neq L_p \). Therefore, there exists \( z \in V \) such that \( z < p \). Let \( M' = M \cup \{z\} \).

Since \( z \in V \), we have \( z \in V_q \) for some \( q \in L_p \). Thus, by choice of \( V_q \) and the fact that \( z < q \), we have \((z,q) \in \theta \cap \pi \). Now, if \( m \) is any element of \( M \), \( q \) can be joined to \( m \) by an order \( \pi \) - chain \([x_i : i = 1, \ldots, n]\) in \( M \). But then, \( z \) is joined to \( m \) by the order \( \pi \) - chain \([x_i : i = 0,1, \ldots, n]\), where \( x_0 = z \).
Therefore, \( M' \) is order \( \pi \) - connected, contradicting the maximality of \( M \).

Case 2: \((\inf M) \cap M = \emptyset\).

We know that \((\inf M) \cap M^* \neq \emptyset\), so fix \( p \in (\inf M) \cap M^* \) and let \( M' = M \cup \{p\} \). Obviously, \( M' \) is a chain in \([x, y]\) and we show that \( M' \) is order \( \pi \) - connected. Let \( U \) be an open set containing \( p \) such that \( U \times U \subset \pi \). Fix \( m \in M \); then, there exists \( n \in M \cap U \) such that \((n, m) \in \theta\). Otherwise, \((m, n) \in \theta\) for each \( n \in M \cap U \) so that \((m, p) \in \theta\) and \( m \in (\inf M) \cap M \), contrary to assumption. Therefore, we may find an order \( \pi \) - chain from \( p \) to \( m \) by taking \( x_0 = p \) and letting the remainder of the chain be an order \( \pi \) - chain joining \( n \) to \( m \). Again, this contradicts the maximality of \( M \).

\((2) = (1)\). To see that \( A \) is order dense, suppose there exist \( x \) and \( y \) in \( A \) with \( x < y \) and \([x, y] \cap A \subset L_x \cup L_y \). Let \( U \) and \( V \) be open sets containing \( L_x \) and \( L_y \), respectively, such that \( U^* \cap V^* = \emptyset \). Now, let \( \pi = X^2 \setminus (U^* \times V^*) \), so that \( \pi \) is an open set containing \( A \). Then, by the order \( \pi \) - connectivity of \( A \), we have the existence of an order \( \pi \) - chain \([x_i : i = 1, \ldots, n] \) in \( A \) joining \( x \) to \( y \). However, any order chain joining \( x \) to \( y \) in \( A \) must be contained in \( L_x \cup L_y \). Thus, by choice of \( \pi \), no order \( \pi \) - chain can join \( x \) to \( y \) in \( A \). This contradiction completes the proof that \( A \) is order dense.

That \( A \) is bi-connected follows from Theorem 2.4 as follows:
Each level set in an order \( \pi \) - connected set is \( \pi \) - connected. Therefore, each level set of \( A \) is \( \pi \) - connected for each open set \( \pi \) containing \( \Delta \), yielding the connectivity of each level set.

**Corollary 2.16.** If \( A \) is a continuum which is order \( \pi \) - connected for each open set \( \pi \) containing \( \Delta \), then \( \Theta \cap (A \times A) \) is monotone and all intervals in \( A \) are connected.

**Proof.** This follows directly from the Theorem in view of Theorem 2.10.

**Corollary 2.17.** Suppose \( X \) is compact and \( A \) is a compact chain. Then, T. A. E.

(1) \( A \) is bi-connected.

(2) \( A \) is order \( \pi \) - connected for each open set \( \pi \supset \Delta \).

**Proof.** (1) \( \Rightarrow \) (2). In view of Theorem 2.9, \( A \) is order dense and the theorem applies.

(2) \( \Rightarrow \) (1). This follows immediately from the theorem after the following observation has been made. If a chain is order \( \pi \) - connected, then, it is \( \pi \) - connected. Thus, \( A \) is \( \pi \) - connected for each \( \pi \) and by Theorem 2.4, \( A \) is connected. Again, the theorem applies.

In Theorem 2.21, we improve Corollary 2.17 and find a more usable equivalence.

One might ask, "What restrictions could be placed on \( \Theta \) to
insure that any bi-connected subset is order dense?". The next example shows that even with quite stringent restrictions on \( \theta \), one may still have bi-connected subsets which are not order dense. In fact, the example is a continuum \( A \) and a continuous, monotone, closed, partial order \( \theta \) on \( A \), with respect to which \( A \) is not order dense (and thus is not order \( \pi \)-connected for some open set \( \pi \) containing \( \Delta \)).

**Example 2.18.** As in Example 2.14, let \( X \) be the unit square with the usual lattice ordering. Let \( A \) be the subset of \( X \) consisting of the four straight line segments \( A_1 \) joining \( (0,0) \) to \( (1/4,3/4) \); \( A_2 \) joining \( (1/4,3/4) \) to \( (1,1) \); \( A_3 \) joining \( (0,0) \) to \( (3/4,1/4) \); and \( A_4 \) joining \( (3/4,1/4) \) to \( (1,1) \). Let \( \theta \) be the relation induced on \( A \) by the order on \( X \). Then, \( \theta \) is a closed partial order on the connected set \( A \) so that \( A \) is bi-connected. It can be easily shown that \( \theta \) is continuous and monotone. Clearly, \( A \) is not order dense and thus is not order \( \pi \)-connected for some \( \pi \).

**Lemma 2.19.** If \( \{A_\alpha : \alpha \in D\} \) is a net of chains in \( X \), then, liminf \( A_\alpha \) is also a chain.

**Proof.** Fix \( x \) and \( y \) in liminf \( A_\alpha \). Then, there exist nets \( \{x_\alpha : \alpha \in D\} \) and \( \{y_\alpha : \alpha \in D\} \) converging to \( x \) and \( y \) respectively, such that \( x_\alpha \) and \( y_\alpha \) are in \( A_\alpha \) for each \( \alpha \in D \). Since \( A_\alpha \) is a chain for each \( \alpha \), we have either \( \{\alpha : (x_\alpha, y_\alpha) \in \theta\} \) or \( \{\alpha : (y_\alpha, x_\alpha) \in \theta\} \) is cofinal in \( D \). If \( \{\alpha : (x_\alpha, y_\alpha) \in \theta\} \) is cofinal in \( D \) then, let \( U \) be an open set containing
(x,y) in X². Let U and V be open sets containing x and y, respectively, such that U x V ⊆ U. Then, we have that 
[a : (xₐ, yₐ) ∈ U x V] is residual in D and there exists an 
α ∈ D such that (xₐ, yₐ) ∈ θ ∩ (U x V). Therefore, due to 
the fact that θ is closed, (x,y) ∈ θ. A similar argument 
handles the case when [a : (yₐ, xₐ) ∈ θ] is cofinal in D.

Lemma 2.20. Let X be compact and let \{Aₐ : α ∈ D\} be a net 
of chains in X converging to A. If, for each open set τ ⊆ Δ, 
\{α : Aᵦ is order τ-connected\} is residual in D, then A has 
connected level sets.

Proof. Fix x ∈ A and assume \(L \cap A = P \cup Q\) where \(P \neq \emptyset \neq Q\). 
Assume x ∈ P and fix y ∈ Q. Let U and V be open sets con-
taining P and Q, respectively, such that \(U^* \cap V^* = \emptyset\) and let 
\(τ = X^2 \setminus (U^* \times V^*)\). Obviously,

\[R = \{α : Aₐ \cap U \neq \emptyset \neq Aₐ \cap V\}\] and 
\[S = \{α : Aₐ is order τ-connected\}\]

are residual in D. For each \(α \in R \cap S\), let \(xₐ \in Aₐ \cap U\) and 
\(yₐ \in Aₐ \cap V\). Since \(Aₐ\) is a chain for each \(α\), we may assume 
that \(C = \{α : (xₐ, yₐ) \in θ\}\) is cofinal in D. Let T be the 
cofinal subset \(R \cap S \cap C\) of D. Now, for fixed \(α \in T\), there 
exists an order τ-chain \(\{xₐ, i : i = 1, \ldots, n(α)\} \subseteq Aₐ\) from 
xₐ to yₐ. Thus, there must be an element \(zₐ \in Aₐ\) such that 
xₐ ≤ zₐ ≤ yₐ and \(zₐ \in X \setminus (U \cup V)\) (no τ-chain can stretch 
from U to V without meeting \(X \setminus (U \cup V)\)). Now, \(\{zₐ : α \in T\}\) 
clusters at \(z \in X \setminus (U \cup V)\) and since \(θ\) is closed we have 
x ≤ z ≤ y. Therefore, \(z \in Lₓ\). However, \(\{Aₐ : α \in D\}\) con-
verges to $A$ so that $z \in A$, contradicting the fact that $L_x \cap A \subseteq U \cup V$.

**Theorem 2.21.** Suppose $X$ is compact and $A \subseteq X$. Then, T. A. E.

1. $A$ is a compact bi-connected chain.
2. $A$ is a compact chain which is order $\pi$-connected for each open set $\pi$ containing $\Delta$.
3. $A$ is the limit of a net of chains $\{A_{\alpha} : \alpha \in D\}$ such that for each open set $\pi$ containing $\Delta$, $\{\alpha : A_{\alpha}$ is order $\pi$-connected$\}$ is residual in $D$.

**Proof.** (1) $\Rightarrow$ (2). This follows from Corollary 2.17. (2) $\Rightarrow$ (3). This is obvious. Let $\{A\}$ be a one element net. (3) $\Rightarrow$ (1). That $A$ is a chain and has connected level sets follows from Lemma 2.19 and Lemma 2.20. Now, $A_{\alpha}$ is a chain which is order $\pi$-connected and thus is $\pi$-connected for each open set $\pi$ containing $\Delta$. Thus, by Theorem 2.4, $A$ is connected. The limit of any net of sets is compact and the proof is complete.

**Corollary 2.22.** Suppose $X$ is compact and $A$ is a compact, bi-connected, order dense subset of $X$. If $x$ and $y$ are elements of $A$ such that $(x,y) \in \theta$, then, there exists a compact bi-connected chain $B \subseteq A$, such that $x \in \text{inf} B$ and $y \in \text{sup} B$.

**Proof.** Theorem 2.15 insures the existence of an order
\( \tau \)-connected chain \( C_{\tau} \) from \( x \) to \( y \) for each open set \( \tau \supset \Delta \). Letting \( (B_{\alpha}) \) be a universal subnet of \( (C_{\tau}) \), we show that \( (B_{\alpha}) \) converges. It suffices to show \( \limsup B_{\alpha} \subseteq \liminf B_{\alpha} \).

Fix \( x \in \limsup B_{\alpha} \) and fix an open set \( U \) containing \( x \). Let \( \mathcal{U} = \{ A : A \subseteq X \text{ and } A \cap U \neq \emptyset \} \). The net \( (B_{\alpha}) \) is cofinally in \( \mathcal{U} \) and since it is a universal net we have \( (B_{\alpha}) \) is residually in \( X \). That is, \( B_{\alpha} \cap U \neq \emptyset \) residually, yielding \( x \in \liminf B_{\alpha} \). Therefore, by the theorem, \( \lim B_{\alpha} = B \) is a compact bi-connected chain. Clearly, \( x \in \inf B \) and \( y \in \sup B \).

The following example shows that the chain restriction in Theorem 2.21 is necessary. That is, it is an example of a continuum which is the limit of a net of order \( \tau \)-connected sets which is not bi-connected.

**Example 2.23.** Let \( A \) be the interval \([-1,1]\) and define \( \theta \) by \((x,y) \in \theta \) iff one of these hold; (i) \( y = 0 \); (ii) \( x \) and \( y \) are negative and \( x < y \); (iii) \( x \) and \( y \) are positive and \( y < x \); or (iv) \( x = -1 \) and \( y = 1 \) or \( y = -1 \) and \( x = 1 \). One may think of this example as an inverted V with unique maximal element at the top, downward order on each arc, and only the endpoints comparing (each to the other). It is easily seen that \( \theta \) is a closed quasi-order on the continuum \( A \) and that \( A \) is order dense. There is exactly one non-trivial level set, namely, \([-1,1]\). Let \( A_{\perp} = (-1,1) \) be a single element net. Then, \( A_{\perp} \) is order \( \tau \)-connected for each open set \( \tau \) containing \( \Delta \), but, \( \lim A_{\perp} = A \) is not bi-connected.
Definition 2.24. A compact bi-connected set $X$ will be called a minimal compact bi-connected set if no proper compact bi-connected subset meets each level set of $X$.

Lemma 2.25. If $A \subseteq X$ and $\theta' = \theta \cap (A \times A)$, then $L_{a,\theta'} = L_{a,\theta} \cap A$ for each $a \in A$ (where $L_{x,\Gamma}$ is the level set of $x$ relative to $\Gamma$). Hence, $A$ is bi-connected relative to $\theta$ iff $A$ is bi-connected relative to $\theta \cap (A \times A)$.

Proof. The proof is straightforward and will be omitted.

Theorem 2.26. Each compact bi-connected subset $A$ of a quasi-ordered space $X$ contains a minimal compact bi-connected subset which meets each level set of $A$.

Proof. Let $\mathcal{M}$ be a maximal tower of compact bi-connected subsets of $A$, each of which meets every level set in $A$. Then, $B = \bigcap \{M : M \in \mathcal{M}\}$ is a continuum which meets each level set in $A$. Moreover, for each $b \in B$, $L_b \cap B = \bigcap \{L_b \cap M : M \in \mathcal{M}\}$ is also a continuum. Therefore, $B$ is a bi-connected subset of $A$. Now, using Lemma 2.25, we conclude that $B$ is a minimal compact bi-connected subset of $A$.

Corollary 1.16 shows that if $\theta$ is a continuous, monotone, reflexive struct on the continuum $X$ and $X$ is irreducible between $K$ and $A$, then $X$ is a bi-connected chain. It follows from the irreducibility that $X$ is a minimal compact bi-connected chain. It is conjectured here that the converse is
true. More precisely, we conjecture that each minimal compact bi-connected chain is irreducible between its inf and its sup.
CHAPTER III

APPLICATIONS TO TOPOLOGY

In this chapter our attention is focused on the existence of bi-connected chains in quasi-ordered spaces and some applications to topology. The first block of material, leading to Theorem 3.4, and its corollaries enable us to find bi-connected chains in certain subsets of quasi-ordered spaces. As a corollary, we obtain an alternate solution to a conjecture of Koch [24] recently settled by Ward [64]. Next, using Koch's order theoretic extension of an arc lifting theorem due to Whyburn, we prove that every tree is lift-able under a light open map (Theorem 3.28). Finally, we discuss possible extensions of our results on liftability.

From this point to Corollary 3.5, $X$ will be a space endowed with a relation $\theta$ which is a struct (non-void, closed, transitive relation) on $X$. Also, $C_x$ will denote the component of $L_x$ containing $x$.

Definition 3.1. (Koch [21]) The set $W \subseteq X$ has no (proper) local minima provided each open set containing a (non-minimal) member $x$ of $W$ also contains $y \in W$ such that $y < x$.

Definition 3.2. The set $W \subseteq X$ has no (proper) gaps if for
each (non-minimal) \( x \in W \), one of the following holds:

(i) \( C_x \cap X \setminus W \neq \emptyset \).

(ii) Each open set \( U \supset C_x \) contains also an \( y \in W \) such that \( y < x \).

**Note.** If \( W \) has no (proper) local minima, then \( W \) has no (proper) gaps. The converse is false.

**Theorem 3.3.** If \( W \) has no gaps with respect to \( \theta \), then \( (x,x) \in \theta \) for each \( x \in W^* \).

**Proof.** Fix \( x \in W \). If \( C_x \) is non-degenerate, let \( y \in C_x \) with \( y \neq x \). Then, \( x \in L(y) \) and \( y \in L(x) \) so that \( (x,y) \in \theta \) and \( (y,x) \in \theta \). By the transitivity of \( \theta \), \( (x,x) \in \theta \). If \( C_x \) is degenerate, each open set containing \( x \) contains an element \( y \) such that \( (y,x) \in \theta \). Thus, there exists a net \( \{y_\alpha\} \) converging to \( x \) such that \( (y_\alpha,x) \in \theta \) for each \( \alpha \). Now \( \{(y_\alpha,x)\} \) is a net in \( \theta \) converging to \( (x,x) \) and since \( \theta \) is closed, \( (x,x) \in \theta \). Finally, if \( x \in W^* \) and \( \{x_\alpha\} \) is a net in \( W \) converging to \( x \), then \( \{(x_\alpha,x_\alpha)\} \) is a net in \( \theta \) converging to \( (x,x) \). Again, \( (x,x) \) must belong to \( \theta \).

**Theorem 3.4.** Suppose \( W \) is an open subset of \( X \) with compact closure. If \( W \) has no gaps, then each \( a \in W \) lies in the sup of a compact, bi-connected chain \( A \subset W^* \) such that \( A \cap F(W) \neq \emptyset \).

**Proof.** Let \( \mathcal{D} \) be the collection of open sets in \( X^2 \) containing \( \Delta \). Fix \( \pi \in \mathcal{D} \) and let \( \mathcal{B}_\pi \) be the collection of all
subsets $B$ of $X$ satisfying the following conditions:

(1) $B \subseteq W^*$.

(2) $B$ is an order $\pi$-connected chain.

(3) $a \in \text{sup } B$.

Let $\mathcal{M}$ be a maximal tower of elements of $\mathcal{B}_\pi$ and let $B = \bigcup \{ M : M \in \mathcal{M} \}$. Clearly, $B \in \mathcal{B}_\pi$. We now claim that $B \cap F(W) \neq \emptyset$. Under the assumption that $B \cap F(W) = \emptyset$, we consider two cases.

Case 1: $(\inf B) \cap B \neq \emptyset$.

Clearly, if $x \in (\inf B) \cap B$, then $C_x \subseteq W$ and by the maximality of $B$, $C_x \subseteq B$. For each $y \in C_x$, let $U_y$ be an open set containing $y$ such that $U_y \times U_y \subseteq \pi$. Let $U = \{ U_y : y \in C_x \}$. Then, since $W$ has no gaps, there exists $z \in U \cap W$ such that $z < x$. Now, let $B' = B \cup \{ z \}$.

Obviously, $B'$ satisfies (1) and (3). To see that $B'$ satisfies (2) it suffices to show that $z$ can be joined to any $b \in B$ by an order $\pi$-chain. Fix $b \in B$. Since $z \in U$, there exists $y \in C_x$ such that $z \in U_y$. Therefore, $(z, y) \in \pi \cap \emptyset$.

Now, $y$ being in $(\inf B) \cap B$, there exists an order $\pi$-chain $\{ x_i : i = 1, \ldots, n \}$ joining $y$ to $b$. Hence, $\{ x_i : i = 0, \ldots, n \}$ is an order $\pi$-chain joining $z$ to $b$, where $x_0 = z$. However, this implies $B' \in \mathcal{B}_\pi^*$, contrary to the maximality of $B$.

Case 2: $(\inf B) \cap B = \emptyset$.

We know that $(\inf B) \cap B^* = \emptyset$, so let $x \in (\inf B) \cap B^*$. Let $U$ be an open set containing $x$ such that $U \times U \subseteq \pi$. We show
that for each $b \in B$, $U$ contains $y \in B$ such that $y \leq b$.

Fix $b \in B$ and assume that no such $y$ exists. Then, $b \leq y$ for each $y \in U \cap B$ ($B$ is a chain). However, there is a net $\{y_\alpha\}$ in $U \cap B$ which converges to $x$. Thus, $\{(b,y_\alpha)\}$ converges to $(b,x)$ yielding $(b,x) \in \theta$. This implies that $b \in (\inf B) \cap B$, contrary to assumption. Now, fixing $y \in U \cap B$ such that $(y,b) \in \theta$, we let $\{x_i : i = 1, \ldots, n\}$ be an order $\pi$-chain joining $y$ to $b$. Again, $x$ is joined to $b$ by the order $\pi$-chain $\{x_i : i = 0, \ldots, n\}$, where $x_0 = x$. Therefore, $B' = B \cup \{x\} \in B_\pi$, contrary to the maximality of $B$.

The claim is now established and we have the existence of a net $\{B_\pi : \pi \in \mathcal{J}\}$ such that for each $\pi \in \mathcal{J}$, $B_\pi$ satisfies (1) - (3) and $B_\pi \cap F(W) \neq \varnothing$. Let $\{B_\alpha\}$ be a universal subnet of $\{B_\pi : \pi \in \mathcal{J}\}$. As in the proof of Corollary 2.22, $\{B_\alpha\}$ converges. Let $A = \lim B_\alpha$. Then, by Theorem 2.21, $A$ is a compact, bi-connected chain. Clearly, $a \in \sup A$ and $A \cap F(W) \neq \varnothing$. The proof is complete.

It is interesting to note that the approximating sets $B_\pi$ in the proof of Theorem 3.4 could have been chosen as finite sets. Let $B_\pi$ satisfy (1) - (3) and $B_\pi \cap F(W) \neq \varnothing$ and fix $x \in B_\pi \cap F(W)$. If we let $B'_\pi$ be an order $\pi$-chain joining $x$ to $a$ in $B_\pi$, then $B'_\pi$ is finite, meets $F(W)$, and satisfies (1) - (3). Theorem 3.4 has the following corollary, proved by Ward [64].
**Corollary 3.5.** Let $X$ be compact and suppose that $L_x$ is totally disconnected for each $x \in X$. If $W$ is an open set with no local minima, then each element of $W$ lies in the sup or an order arc $A \subset W^*$ such that $A \cap F(W) \neq \emptyset$.

**Proof.** In the case that $X$ has totally disconnected level sets, no gaps and no local minima are equivalent notions. Hence, by the theorem, each $a \in W$ lies in the sup of a compact, bi-connected chain $A$ meeting $F(W)$. Since level sets are totally disconnected, each level set in $A$ must be a point. It is well known that a compact, connected, partially ordered chain is an arc (see e. g. [62]) and the proof is complete.

In [65], Whyburn proved that any metric arc is liftable under a light open map. More precisely, if $f$ maps the the compact metric space $X$ onto the compact metric space $Y$, and if $f$ is light and open, then for any arc $B \subset Y$ and for any point $a \in f^{-1}(B)$, there exists an arc $A \subset X$ such that $a \in A$ and $f$ maps $A$ topologically onto $B$. Zippin proved the same theorem independently around 1936 (see [37]). Earlier papers by Stoilow [47], [48], and [49] contained this result in the case $f$ was a planer function. In [24], Koch obtained an order theoretic extension of this theorem to non-metric spaces. Our next block of material will be concerned with the ability to cover bi-connected chains by bi-connected chains under suitable maps. These investigations have Koch's
extension of Whyburn's theorem as a corollary. We first prove a useful result concerning continuous images of bi-connected chains.

**Theorem 3.6.** Suppose $\Gamma$ and $\Theta$ are closed quasi-orders on the compact spaces $X$ and $Y$, respectively. Let $f$ be a continuous, order preserving function from $X$ to $Y$. Then, the image of each compact bi-connected $\Gamma$ chain in $X$ is a compact bi-connected $\Theta$ chain in $Y$.

**Proof.** Let $A$ be a compact bi-connected $\Gamma$ chain in $X$. Fix an open set $\pi$ containing $\Delta_Y$. We show that $f(A)$ is a $\Theta$ order-connected chain. From the fact that $f$ is order preserving, it follows easily that $f(A)$ is a $\Theta$ chain. A simple continuity argument gives the existence of an open set $\Sigma$ containing $\Delta_X$ such that $(f \times f)(\Sigma) \subseteq \pi$. Now, fix $a, b \in f(A)$ such that $(a,b) \in \Theta$. There exist $x$ and $y$ in $A$ such that $f(x) = a$ and $f(y) = b$. We may assume $(x,y) \in \Gamma$. Now, by Theorem 2.21, there exists $\{x_i : i = 1, \ldots, n\}$ contained in $A$ such that $x_1 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in \Sigma \cap \Gamma$ for each $i = 1, \ldots, n-1$. Thus, $(f(x_i), f(x_{i+1})) \in \pi \cap \Theta \cap f(A)$ for each $i$. Therefore, $\{f(x_i) : i = 1, \ldots, n\}$ is a $\Theta$ order-$\pi$ chain joining $a$ to $b$ in $f(A)$. By Theorem 2.21, $f(A)$ is a compact bi-connected $\Theta$ chain.

Recall that $C_x$ is the component of $L_x$ containing $x$ and $I(x) = L(x) \setminus L_x$. 
Definition 3.7. Let \( \Gamma \) and \( \Theta \) be quasi-orders on \( X \) and \( Y \), respectively, and let \( f \) be an order preserving function from \( X \) to \( Y \). Then, \( f \) will be called dense from below if \( C_y \cap I(y)^* \neq \emptyset \) implies \( C_x \cap I(x)^* \neq \emptyset \) for each \( x \in f^{-1}(y) \).

The definition given above is a departure from that given by Koch in [24]. It follows from results due to Wallace [53], that each open, order preserving function from one quasi-ordered space to another is dense from below on the inverse of each subset of \( Y \). The following lemma is well known and its proof is omitted.

Lemma 3.8. Let \( X \) and \( Y \) be compact and suppose \( f \) is an open map from \( X \) onto \( Y \). If \( C \) is any continuum in \( Y \), then each component of \( f^{-1}(C) \) maps onto \( C \) under \( f \).

The proof of our next lemma is straightforward and will be omitted.

Lemma 3.9. If \( \{A_\alpha : \alpha \in D\} \) is a net of sets such that \( \alpha \leq \beta \) implies \( A_\alpha \subseteq A_\beta \), then \( \lim A_\alpha = [U\{A_\alpha : \alpha \in D\}]^* \).

Our next theorem is an extension of Corollary 3 of [24]. We point out that the function in question is not required to be strictly order preserving.

Theorem 3.10. Let \( \Gamma \) and \( \Theta \) be closed quasi-orders on the compact spaces \( X \) and \( Y \), respectively, and suppose \( Y \) is a bi-connected chain. Let \( f \) be a dense from below map from
X onto Y. Then each \( a \in f^{-1}(\sup Y) \) lies in the sup of a compact bi-connected chain \( A \) such that \( f(A) \cap I_y \neq \emptyset \) for each \( y \in Y \). Moreover, if \( Y \) is a minimal compact bi-connected chain (see Definition 2.24), then each such \( A \) maps onto \( Y \).

**Proof.** Let \( W = X \setminus f^{-1}(\inf Y) \). We claim that \( W \) has no gaps. For a fixed \( x \in W \) we have \( f(x) \neq \inf Y \). Also, \( L_f(x) = C_f(x) \) since \( Y \) is bi-connected. Now, \( C_f(x) \cap I(f(x))^* \neq \emptyset \), for otherwise, \( Y = M(f(x)) \cup I(f(x))^* \) is a separation of \( Y \).

Therefore, since \( f \) is dense from below, \( C_x \cap I(x)^* \neq \emptyset \). Thus, if \( U \) is open containing \( C_x \), there exists \( z \in U \cap W \) such that \( z < x \). This establishes the claim. It should be observed that, by definition of \( W \), \( L_x \subseteq W \) for each \( x \in W \).

Now, by Theorem 3.4, there exists a compact bi-connected chain \( A \subseteq W \) such that \( a \in \sup A \) and \( A \cap F(W) \neq \emptyset \). Clearly, \( F(W) \subseteq f^{-1}(\inf Y) \). Thus, \( f(A) \cap \inf Y \neq \emptyset \). Now, \( f(A) \) is bi-connected in view of Theorem 3.6, and meets both \( \inf Y \) and \( \sup Y \). It follows immediately that \( f(A) \cap L_y \neq \emptyset \) for each \( y \in Y \). This completes the proof of the first assertion. The second part is obvious.

**Corollary 3.11.** Let everything be as in Theorem 3.10. If \( f \) is strictly order preserving and \( f \) restricted to \( f^{-1}(L_y) \) is an open map from \( f^{-1}(L_y) \) onto \( L_y \) for each \( y \in Y \), then \( A \) may be chosen so that \( f(A) = Y \).

**Proof.** By Theorem 3.10, the collection \( \mathcal{A} \) of all sets \( A \),
such that A is a compact bi-connected chain, \( a \in \text{sup } A \), and \( f(A) \cap L_y \neq \emptyset \) for each \( y \in Y \) is non-void. Let \( \mathcal{M} \) be a maximal tower in \( \mathcal{B} \) and let \( A = \{ U(M : M \in \mathcal{M}) \}^* \). Then, by Lemma 3.9, \( A = \lim (M : M \in \mathcal{M}) \). Theorem 2.21 insures that each \( M \in \mathcal{M} \) is an order \( \pi \)-connected chain for each open set \( \pi \) containing \( A \). Again, by Theorem 2.21, \( A \) is a compact bi-connected chain. Clearly, \( a \in \text{sup } A \) and \( f(A) \cap L_y \neq \emptyset \) for each \( y \in Y \). The maximality of \( \mathcal{M} \) insures that for each \( x \in A \) we have \( C_x \subseteq A \). Now, since \( f \) is strictly order preserving and \( A \) is a chain \( f^{-1}(L_y) \cap A \subseteq L_x \) for some \( x \in A \). It is easy to see that \( C_x = f^{-1}(L_y) \cap A \) is a component of \( f^{-1}(L_y) \). Therefore, by Lemma 3.8, \( L_y = f(C_x) \subseteq f(A) \) and consequently, \( f(A) = Y \).

**Corollary 3.12.** Let everything be as in Theorem 3.10 and suppose that \( Y \) is an arc with the usual ordering. Then, there exists a compact bi-connected chain \( A \) in \( X \) such that \( a \in \text{sup } A \) and \( f(A) = Y \).

**Corollary 3.13.** Let \( X \) and \( Y \) be compact and let \( f \) be an open map from \( X \) onto \( Y \). If \( I \) is an arc in \( Y \) and \( a \in f^{-1}(I) \), then there exists a continuum \( A \) in \( X \) such that \( a \in A \), \( f(A) = I \), and \( f|_A \) is monotone (\( f^{-1}(y) \cap A \) is connected for each \( y \in I \)).

**Proof.** We may assume, without loss of generality, that \( Y = I \), \( X = f^{-1}(I) \), and \( a \in f^{-1}(\text{sup } Y) \). Define \( \theta \) on \( X \) by \( (x,y) \in \theta \) iff \( f(x) \leq f(y) \) in \( I \). That \( \theta \) is closed follows
from the continuity of $f$. It has already been mentioned that each order preserving open map is dense from below. Now, the theorem applies yielding the desired result.

**Corollary 3.14.** Let $X$ and $Y$ be compact and let $f$ be a light open map from $X$ onto $Y$. If $I$ is an arc in $Y$ and $a \in f^{-1}(I)$, then there exists an arc $A$ in $X$ such that $a \in A$ and $f$ takes $A$ topologically onto $I$.

**Proof.** Let $A$ be the continuum whose existence is insured by Corollary 3.13. Since $f$ is light and $f|_A$ is monotone, we have that $f^{-1}(y) \cap A$ is a point for each $y \in I$. Thus, $f|_A$ is a one-to-one, continuous function taking the compact set $A$ onto $I$, yielding the result.

Very simple examples exist to show that the conclusion of Theorem 3.10 is not valid if $Y$ is not a chain. One such is the following.

**Example 3.15.** Let $X$ be the union of the two straight line segments $A_1$ joining $(0,0)$ to $(0,1)$ and $A_2$ joining $(1,0)$ to $(1,1)$ in the plane. Define $\Gamma$ on $X$ by $
abla[(a,b),(c,d)] \in \Gamma$ iff $b < d$. Let $Y$ be the union of $A_2$ with the straight line segment $A_3$ joining $(0,1)$ to $(1,1)$. Define $\theta$ on $Y$ by $[(a,b),(c,d)] \in \theta$ iff $a < c$ and $b < d$. Define $f$ taking $X$ onto $Y$ by $f(a,b) = (b,1)$ if $(a,b) \in A_1$ and $f(a,b) = (a,b)$ if $(a,b) \in A_2$. Then $f$ is dense from below and $A_3$ is a compact bi-connected chain in $Y$. However, $(1,1) \in f^{-1}(\text{sup } A)$ and lies in no connected set mapping onto $A_3$. 
**Definition 3.16.** If \( X \) is a space and \( x, y \in X \), then
\[
E(x,y) = \{ z : z \text{ separates } x \text{ from } y \text{ in } X \}.
\]
That is, \( z \in E(x,y) \) iff there is a separation \( X \setminus \{ z \} = P \uplus Q \) where \( x \in P \) and \( y \in Q \).

**Definition 3.17.** A **tree** is a continuum \( X \) with the property that \( x \neq y \) in \( X \) implies \( E(x,y) \neq \emptyset \). A **dendrite** is a metric tree.

In [67], Whyburn proved that each dendrite is liftable under a light open map. Our next sequence of lemmas lead to the non-metric analog of this theorem (Theorem 3.24). A proof of the following lemma can be found in [59].

**Lemma 3.18.** A tree is locally connected.

**Lemma 3.19.** If \( X \) is a space and \( C \) is a connected subset of \( X \), then for each \( x, y \in C \) we have \( E(x,y) \subseteq C \).

**Proof.** If \( z \in X \setminus C \) and \( X \setminus \{ z \} = P \uplus Q \), then, since \( C \) is connected, either \( C \subseteq P \) or \( C \subseteq Q \). Thus, \( [x,y] \subseteq P \) or \( [x,y] \subseteq Q \), implying \( z \notin E(x,y) \).

The following lemma is a consequence of work due to Ward [61].

**Lemma 3.20.** If \( X \) is a tree and \( A \) is a subcontinuum of \( X \), then \( A \) is a tree.

The following lemma follows from work due to Ward [60] and Koch [21], as observed by Koch and Krule in [26].
Lemma 3.21. If X is a tree, then there is a unique arc joining each pair of points in X. That is, X is uniquely arcwise connected.

Lemma 3.22. If X is a tree and \([a,b] \) is the unique arc joining \(a\) to \(b\), then \(E(a,b) = (a,b) = [a,b] \setminus \{a\} \cup \{b\}\).

Proof. By Lemma 3.19, \(E(a,b) \subseteq (a,b)\). Suppose \(z \in (a,b)\) and \(z \notin E(a,b)\). Then, for any separation \(X \setminus \{z\} = P \uplus Q\), either \([a,b] \subseteq P\) or \([a,b] \subseteq Q\). Now, \(X \setminus \{z\}\) is an open subset of a locally connected space and thus is locally connected. Therefore, each component of \(X \setminus \{z\}\) is open in \(X \setminus \{z\}\). Since a component is always closed, we have that each component of \(X \setminus \{z\}\) is open and closed in \(X \setminus \{z\}\).

Let \(C\) be the component of \(X \setminus \{z\}\) containing \(a\). Then, \(X \setminus \{z\} = C \cup ([X \setminus \{z\}] \setminus C)\) is a separation of \(X \setminus \{z\}\). Hence, \(b \in C\). Now, for each \(x \in C\), let \(U_x\) be an open connected set containing \(x\) such that \(z \notin U_x^*\). There is a simple chain \(\{U_i : i = 1, \ldots, n\}\) of these open connected sets joining \(a\) to \(b\). Let \(T = [U_i : i = 1, \ldots, n]^*\). Then, \(T\) is a subcontinuum of \(X\) and by Lemma 3.20, \(T\) is a tree. Thus, there is an arc joining \(a\) to \(b\) in \(C \subseteq X \setminus \{z\}\). This contradicts the unique arcwise connectivity of \(X\) and the proof is complete.

Lemma 3.23. The intersection of any two connected subsets of a tree is a tree.
Proof. Let $C$ and $D$ be two connected subsets of a tree $T$. By Lemma 3.21, $T$ is uniquely arcwise connected. If $C \cap D = \emptyset$ we are finished. Otherwise, suppose $x$ and $y$ are members of $C \cap D$. Then, by Lemma 3.22, $E(x,y) = (x,y)$ and by Lemma 3.19, $[x,y] \subseteq C \cap D$. Therefore, $C \cap D$ is arcwise connected and the conclusion follows.

Theorem 3.24. Suppose $f$ is a light, open map from the compact space $X$ onto the compact space $Y$. If $T$ is a tree in $Y$ and if $a \in f^{-1}(T)$, then there exists a continuum $K \subseteq X$ such that $a \in K$ and $f$ maps $K$ topologically onto $T$.

Proof. Let $\mathcal{K}$ be the collection of all continua $K$ in $X$ such that $a \in K$ and $f$ restricted to $K$ is a homeomorphism into $T$. Then, $\{a\} \in \mathcal{K}$, so that $\mathcal{K} \neq \emptyset$. Let $\mathcal{M}$ be a maximal tower in $\mathcal{K}$, let $A = \bigcup \{M : M \in \mathcal{M}\}$, and let $K = A^*$. By Lemma 3.9, $K = \lim (M : M \in \mathcal{M})$.

Claim: $f$ is one-to-one on $K$.

Fix $p \in f(K)$. We show that $f^{-1}(p)$ meets $K$ in exactly one point. If $M \in \mathcal{M}$, the continuity of $f$ insures that $f(K)$ and $f(M)$ are continua. Let $\{U_\alpha : \alpha \in D\}$ be a basis for the topology of $T$ at $p$ consisting of open connected sets. In view of Lemma 3.23, $U_\alpha \cap f(M)$ is connected for each $M \in \mathcal{M}$. The remainder of the verification of the claim is presented in five parts:

(i) $f^{-1}(U_\alpha \cap f(A)) \cap A = \bigcup \{f^{-1}(U_\alpha \cap f(M)) \cap M : M \in \mathcal{M}\}$ for each $\alpha \in D$. 

(ii) \( U[[f^{-1}(U_\alpha \cap f(M)) \cap M] : M \in \mathcal{M}] \) is connected for each \( \alpha \in D \).

(iii) \( f^{-1}(p) \cap K \neq \emptyset \).

(iv) \( f^{-1}(p) \cap K \subseteq \liminf \{[f^{-1}(U_\alpha \cap f(A)) \cap A] : \alpha \in D\} \).

(v) \( \limsup f^{-1}(U_\alpha) \subseteq f^{-1}(p) \).

Once these are established, (i), (ii), (iii), and (iv) imply, by Lemma 2.3, that \( \limsup \{[f^{-1}(U_\alpha \cap f(A)) \cap A] : \alpha \in D\} \) is connected. This set is obviously contained in \( \limsup f^{-1}(U_\alpha) \).

Therefore, by (v), \( \liminf \{[f^{-1}(U_\alpha \cap f(A)) \cap A] : \alpha \in D\} \) is a connected subset of the totally disconnected set \( f^{-1}(p) \) and must be a point. Hence, by (iv), \( f^{-1}(p) \cap K \) is a point, establishing the claim.

Verification of (i):

\[ x \in f^{-1}(U_\alpha \cap f(A)) \cap A \]

\[ \Rightarrow x \in f^{-1}(U_\alpha \cap f(M)) \cap M' \text{ for some } M, M' \in \mathcal{M}. \]

If \( M \subseteq M' \), then \( f(M) \subseteq f(M') \)

\[ = U_\alpha \cap f(M) \subseteq U_\alpha \cap f(M') \]

\[ = f^{-1}(U_\alpha \cap f(M)) \subseteq f^{-1}(U_\alpha \cap f(M')) \]

\[ = f^{-1}(U_\alpha \cap f(M)) \cap M' \subseteq f^{-1}(U_\alpha \cap f(M')) \cap M' \]

\[ = x \in U[[f^{-1}(U_\alpha \cap f(M)) \cap M] : M \in \mathcal{M}] \].

If \( M' \subseteq M \), then \( x \in f^{-1}(U_\alpha \cap f(M)) \cap M' \)

\[ = x \in f^{-1}(U_\alpha \cap f(M)) \cap M \]

\[ = x \in U[[f^{-1}(U_\alpha \cap f(M)) \cap M] : M \in \mathcal{M}] \].

\[ \supseteq \] This containment follows immediately from the fact that \( M \subseteq A \) for each \( M \in \mathcal{M} \).
Verification of (ii): Since $f|_M$ is a homeomorphism and $U_\alpha \cap f(M)$ is connected, we have that $f^{-1}(U_\alpha \cap f(M)) \cap M$ is connected for each $M \in \mathcal{M}$. This collection of sets is towered and hence, their union is connected.

Verification of (iii): This is obvious, as $p$ was chosen in $f(K)$.

Verification of (iv): Fix $x \in f^{-1}(p) \cap K$ and let $W$ be an open set containing $x$. Then, $f^{-1}(U_\alpha) \cap W$ is an open set containing $x$ for each $\alpha \in D$. Therefore,

$$\bigcup_{\alpha \in D} \{ f^{-1}(U_\alpha) \cap W \} \cap A \neq \emptyset$$

Thus, $x \in \liminf \{ f^{-1}(U_\alpha) \cap f(A) \cap A : \alpha \in D \}$.

Verification of (v): We first show that if $\{ y_\alpha \}$ is any net in $Y$ which converges to $p$, then $\limsup f^{-1}(y_\alpha) \subset f^{-1}(y)$.

Fix $z \in \limsup f^{-1}(y_\alpha)$. We show that $f(z) = p$. Fix an open set $V$ containing $f(z)$. Then, $z \in f^{-1}(V)$ which is open by the continuity of $f$. Thus, $f^{-1}(V) \cap f^{-1}(y_\alpha) \neq \emptyset$ cofinally which implies $f(f^{-1}(V)) = V$ contains $y_\alpha$ cofinally. Therefore, $f(z) \in \limsup \{ y_\alpha \}$. Since $X$ is Hausdorff and $\{ y_\alpha \}$ converges to $p$, $\limsup \{ y_\alpha \} = \{ p \}$. Hence, $f(z) = p$. Now, suppose $x \in \limsup f^{-1}(U_\alpha)$. Then, there exists a net $\{ x_\alpha \}$, where $x_\alpha \in f^{-1}(U_\alpha)$, such that $\{ x_\alpha \}$ clusters at $x$. By the continuity of $f$, $\{ f(x_\alpha) \}$ clusters at $f(x)$ and $f(x_\alpha) \in U_\alpha$. Thus, $f(x) \in \limsup \{ U_\alpha \}$ which is easily seen to be $\{ p \}$, in view of the fact that $\{ U_\alpha \}$ is a neighborhood basis at $p$. 
Now that the claim has been established, we proceed to show that \( f(K) = T \). Assume that \( f(K) < T \) and fix \( y \in T \setminus f(K) \). If \( e \) is fixed in \( f(K) \), there is a unique arc \([y,e]\) joining \( y \) to \( e \) in \( T \) in view of Lemma 3.21. Let \( y \in \text{inf}([y,e] \cap f(K)) \) (relative to the cutpoint ordering on \([y,e]\) with minimal element \( y \)). Then, \( z \in f(K) \) and we let \( b = f^{-1}(z) \cap K \). By Corollary 3.14, there exists an arc \( B \) in \( X \) such that \( b \in B \) and \( f \) maps \( B \) homeomorphically onto \([y,z]\). Letting \( K' = K \cup B \), we have that \( K' \) is a continuum containing \( a \) and \( f|_{K'} \) is a homeomorphism into \( T \). This, however, contradicts the maximality of \( M \) and the assumption that \( f(K) < T \) is false.

The proof of the theorem is complete.

One might reasonably ask what spaces are always liftable under light open maps. In particular, is there a non-locally connected space which is always liftable? Probably the simplest example of a non-locally connected continuum is the so called 'whisk broom' \( W \) \([34;214]\). That is, \( W \) is the union of the straight line segments \( A_n \) joining \((0,0)\) to \((1,1/n)\) in the plane along with the unit segment \( A_0 \) joining \((0,0)\) to \((1,0)\). In \([8]\), Cornette describes a very elegant example of a continuum \( X \) and a light open map from \( X \) onto a continuum \( Y \), very closely related to \( W \), with the property that no subcontinuum of \( X \) maps homeomorphically onto \( Y \). In fact, \( Y \) is homeomorphic to \( W \cup W' \), where \( W' \) is the image of \( W \) under a \( 180^\circ \) rotation of the plane about the line \( y = x - 1 \).
Proposition 3.25. Let \( W \) be the whisk broom. Let \( U \) be an open set containing \( a \) where \( a \in A_0 \setminus (0,0) \). Then, there exists a continuum \( X \) and a light open map \( f \) from \( X \) onto \( W \) such that \( f \) is a homeomorphism on \( X \setminus f^{-1}(U) \) and no sub-continuum of \( X \) maps homeomorphically onto \( W \) under \( f \).

Verification. We assume that \( a \neq (1,0) \). An analogous example can be constructed in the case \( a = (1,0) \). There exists a neighborhood \( V \) such that \( a \in V \subset U \) and \( V \) is homeomorphic to \( Z = [-1/2,1/2] \times \{1/n : n = 1,2,...\} \cup \{0\} \). The space \( Z \) is a sequence of arcs converging to an arc. We now describe a space \( Y \) and a light open map taking \( Y \) onto \( Z \). For the sake of convenience, the space \( Y \) is illustrated by diagram. We begin with the two adjacent hollow cylinders in three space with radius \( 1/8 \), centers at \((-1/8,0,0) \) and \((1/8,0,0) \), and height \( 3/2 \).
We assume that \( a_n = (-1/2,0,1/n) \), \( x_n = (-1/4,0,1/n) \), 
\( y_n = (1/4,0,1/n) \), and \( b_n = (1/2,0,1/n) \) for each \( n = 1,2,... \) 
and also that \( p_n \neq q_n \) and both are in \([0] \times [0] \times [0,3/2]\).

Finally, we assume that \( a_0 = (-1/2,0,0) \), \( x_0 = (-1/4,0,0) \), 
\( p_0 = (0,0,0) \), \( y_0 = (1/4,0,0) \), and \( b_0 = (1/2,0,0) \). The set 
\( B_1 \) consists of the union of the arcs \([a_1, x_1]\) and \([y_1, b_1]\) 
with the curves \([x_1, p_1]\), \([x_1, q_1]\), \([p_1, y_1]\), and \([q_1, y_1]\) for 
each positive \( i \). The set \( B_0 \) consists of the union of the 
two arcs \([a_0, x_0]\) and \([y_0, b_0]\) with the two base circles.

It is easy to see that \( \{B_n : n = 1,2,...\} \) converges to \( B \) 
in \( Y \) if \( Y = \bigcup \{B_n : n = 0,1,2,...\} \). Define \( g \) from \( Y \) to \( Z \) by 
\( g(x,y,z) = (x,1/n) \) if \( (x,y,z) \in B_n \) for some positive \( n \) and 
\( g(x,y,z) = (x,0) \) if \( (x,y,z) \in B_0 \). It can be argued without 
great difficulty that \( g \) is light and open. Now, we define 
\( X \) by replacing \( V \) by \( Y \) in \( W \). Let \( f \), taking \( X \) onto \( W \), be de­
defined by \( f(x) = x \) if \( x \in X \setminus Y \) and \( f(x) = g(x) \) if \( x \in Y \).

Then \( f \) is a light open map from the continuum \( X \) onto \( W \). If 
we let \( A \) be a continuum in \( X \) such that \( f|_A \) is a homeomor­
phism on \( ([0,1/2] \times ([1/n : n = 1,2,...] \cup \{0\})) \cap A \subset Z \), it 
follows easily that \( f|_A \) is two-to-one on the set 
\( ((-1/4,0) \times ([1/n : n = 1,2,...] \cup \{0\})) \cap A \subset Z \). Thus, no 
continuum in \( X \) maps homeomorphically onto \( W \) under \( f \) and the 
proposition is verified.

In view of the theory of continua of convergence [68,20], 
it is conjectural that Proposition 3.25 is valid if \( W \) is 
replaced by an arbitrary continuum \( C \) and \( a \) by a point at
which $C$ is not connected im kleinen [39].
CHAPTER IV
APPLICATIONS TO SEMIGROUPS

In this chapter we turn our attention to applications of the results of Chapter II to topological semigroups. We first give an alternate proof of a theorem, due to Koch [23], on the existence of threads in compact semigroups. Next, we sketch a proof of the result of Hofmann and Mostert [11] that every algebraically irreducible clan is abelian. A result of Phillips [44] motivates our last theorem on translates of algebraically irreducible semigroups.

Lemma 4.1. If $S$ is a compact semigroup and $S = ES$ or $S = SE$, then each $H$-class in $S$ is the continuous image of some group in $S$.

Proof. We consider the case $S = ES$, the other case being the dual. Fix an $H$-class $H$ and let $h \in H$. Then, $eh = h$ for some $e \in E$. Let $f$ be a minimal element of $E$ relative to $fh = h$. Then, by Proposition 11 of [1], there is a subgroup of $H(f)$ which maps homomorphically and continuously onto the Schutzenberger group of $H$. Now, the $H$-class $H$ is homeomorphic to the Schutzenberger group of $H$ and the composition of the homeomorphism with the homomorphism
yields the desired function.

**Theorem 4.2.** (Koch) Let $S$ be a clan with minimal ideal $K \neq S$. If each subgroup of $S$ is totally disconnected, then there is an (abelian) standard thread $T$ in $S$ with $1 \in T$ and $T \cap K \neq \emptyset$.

**Proof.** Let $\theta$ be the $H$ quasi-order $\Gamma_H(S)$ on $S$. Since $S$ has an identity, $(x,y) \in \theta$ iff $x \in yS \cap Sy$. Fix an open set $\pi$ containing $\Delta$ and let $B_\pi$ be the collection of subsets $B$ of $S$ satisfying the following four conditions:

1. $1 \in B$.
2. $B$ is an abelian subsemigroup of $S$.
3. $B$ is an order $\pi$-connected chain.
4. $\text{Ker } B$ is a point (i.e., $B$ has a zero).

Note that $\{1\} \in B_\pi$ so that $B_\pi \neq \emptyset$. Let $\mathcal{M}$ be a maximal tower in $B_\pi$ and let $A = \bigcup \{M : M \in \mathcal{M}\}$. It is obvious that $A$ satisfies (1), (2), and (3). Now, $A^*$ is a compact abelian semigroup and thus, $\text{Ker } A^*$ is a group. Let $e$ be the identity of $\text{Ker } A^*$. Clearly, $e \leq f_M = \text{Ker } M$ for each $M \in \mathcal{M}$ so that $eM = ef_M = ef_M = e$ for each $M \in \mathcal{M}$. Therefore, $eA = e$ and $e$ is a zero for $A$. Hence, $A \in B_\pi$.

We now show that $e \in A$. Assume $e \not\in A$ and let $B = A \cup \{e\}$. Obviously, $B$ satisfies (1), (2), and (4). To see that $B$ satisfies (3) it clearly suffices to show that $e$ can be joined to each element of $A$ by an order $\pi$-chain. The argument is very similar to the one given in the proof of
Theorem 2.15 and will be omitted.

We now show that \( A \cap K \neq \varnothing \). Under the assumption that \( A \cap K = \varnothing \) we consider two cases.

Case 1: The element \( e \) is isolated in \( E \cap eSe \).

Then, by [41] or [11], there exists a local one-parameter semigroup in \( eSe \) based at \( e \) which meets the complement of \( H(e) \). This local semigroup may be extended by standard methods (see e. g. [41]; a very thorough treatment of this extension is given in [25]), yielding an abelian semigroup \( P \) with connected kernel. That \( P \) is order \( \pi \) - connected follows easily from the method in which the local one-parameter semigroup is extended (\( P \) is, in fact, compact and bi-connected). Thus if \( B = A \cup P \), then \( B \) satisfies (1)-(4), contrary to the maximality of \( \mathcal{M} \).

Case 2: The element \( e \) is not isolated in \( E \cap eSe \).

Let \( U \) be an open set containing \( e \) such that \( U \times U \subset \pi \). Let \( f \in U \cap E \cap eSe \) with \( f \neq e \). Then \( B = A \cup \{f\} \) is easily seen to satisfy (1) - (4), again contrary to the maximality of \( \mathcal{M} \).

Therefore, \( A \cap K \neq \varnothing \) and we have the existence of a net of sets \( \{A_\pi\} \) such that \( A_\pi \) satisfies (1) - (4) and \( A_\pi \cap K \neq \varnothing \).

Let \( \{B_\alpha\} \) be a universal subnet of \( \{A_\pi\} \). Then, as in Corollary 2.22, \( \{B_\alpha\} \) converges to \( T = \limsup B_\alpha \). Clearly, \( T \) is an abelian semigroup containing \( 1 \) and meeting \( K \). By
Theorem 2.21, T is a bi-connected chain relative to \( \theta \) and in particular, each group in T is connected. Hence, each group in T is trivial and by Lemma 4.1, each H-class in T is trivial. Thus, T is an arc and the proof is complete.

**Corollary 4.3.** (Faucett [9]) Let S be a semigroup whose underlying space is an arc. If one endpoint acts as a zero and the other acts as an identity, then S is commutative.

**Definition 4.4.** The clan S with identity 1 is said to be **algebraically irreducible** if no proper subclan of S contains 1 and meets K.

**Theorem 4.5.** (Hofmann and Mostert [11]) Let S be a clan and let G be a compact, connected, abelian group of automorphisms of S. Then, the set of fixed points of G on S is a compact connected semigroup which meets the minimal ideal.

**Theorem 4.6.** (Hofmann and Mostert [11]) If S is a clan with kernel \( K \neq S \), there is a sub-semigroup T of S satisfying the following:

1. T is compact and connected.
2. T is abelian.
3. \( 1 \in T \).
4. \( T \cap K \neq \emptyset \).
5. Each group in T is connected.

**Sketch of Proof.** Let \( \theta \) be the H quasi-order \( \Gamma_H(S) \) on S. Fix an open set \( \pi \) containing \( \Delta \) and let \( \mathcal{B}_\pi \) be the collection
of subsets $B$ of $S$ satisfying the following conditions:

(a) $1 \in B$.

(b) $B$ is an abelian sub-semigroup of $S$.

(c) $B$ is an order $\pi$-connected chain.

(d) $\text{Ker} B$ is connected.

As in the proof of Theorem 4.2, we let $\mathcal{M}$ be a maximal tower in $\mathcal{B}_\pi$ and let $A = \bigcup \{ M : M \in \mathcal{M} \}$. Then, it can be shown that $A \cap K \neq \emptyset$ in almost exactly the same way as in the proof of Theorem 4.2. The difference is in the fact that you must be able to extend abelian semigroups within the centralizer of their kernel. This is where Theorem 4.5 comes into play. In one case one extends a local one-parameter semigroup yielding a compact bi-connected semigroup $P \subseteq F \subseteq eSe$, where $F$ is the centralizer of $\text{Ker} A$ under the inner automorphism action on $eSe$. Then, $A$ can be extended to $A' = A \cup P(\text{Ker} A)$ which satisfies (a) - (d), contrary to the maximality of $\mathcal{M}$. In the other case we may choose $f < e = E \cap \text{Ker} A$, such that $f \in F$ and $(f,e) \in \pi$. Then, $A$ can be extended to $A' = A \cup f(\text{Ker} A)$, again contrary to the maximality of $\mathcal{M}$. The remainder of the proof is exactly like that of Theorem 4.2.

We observe that the above approach yields immediately the connectivity of all groups in algebraically irreducible semigroups. Hofmann and Mostert go to some trouble to verify this result. The result is actually due to Hunter [17].
In [44], Phillips proves that the translate of any standard thread in a semigroup is an arc. Our next theorem gives an analogous result for translates of algebraically irreducible clans.

**Theorem 4.7.** Let S be a clan with kernel $K \neq S$. If $T$ is algebraically irreducible between $K$ and 1, then $T$ is simultaneously a bi-connected $\Gamma_R(S)$, $\Gamma_L(S)$, $\Gamma_H(S)$, and $\Gamma_J(S)$ chain. Moreover, for each $x \in S$, the set $xS$ ($Sx$) is a bi-connected $\Gamma_R(S)$ ($\Gamma_L(S)$) chain.

**Proof.** By Theorem 4.6, $T$ is abelian so that $\Gamma_R(T) = \Gamma_L(T) = \Gamma_H(T) = \Gamma_J(T)$. Theorem 4.6 also gives that $T$ is a bi-connected $\Gamma_H(T)$ chain. Thus, $T$ is simultaneously a compact bi-connected $\Gamma_R(T)$, $\Gamma_L(T)$, $\Gamma_H(T)$, and $\Gamma_J(T)$ chain.

We consider the right ideal order only, the other cases being analogous.

First, we show that $T$ is a $\Gamma_R(S)$ chain. If $(a,b) \in \Gamma_R(T)$, then $a \in bT$ so that $a \in bS$ and $(a,b) \in \Gamma_R(S)$. Thus, $\Gamma_R(T) \subseteq \Gamma_R(S)$ and it follows that $T$ is a $\Gamma_R(S)$ chain.

Next we show that each $R$-class $A$ in $S$ meets $T$ in a connected set. Suppose $a$ and $b$ are in $A \cap T$ and $(a,b) \in \Gamma_R(T)$. Fix $c \in T$ such that $(a,c)$ and $(c,b)$ are in $\Gamma_R(T)$. Then, since $\Gamma_R(T) \subseteq \Gamma_R(S)$, we have $(a,c)$, $(c,b)$, and $(b,a)$ are in $\Gamma_R(S)$. Therefore, $(a,c) \in \Gamma_R(S) \cap [\Gamma_R(S)]^{-1}$. Hence, $c \in A \cap T$ and $A \cap T$ is a closed convex subset of $T$. This
clearly implies that $A \cap T$ is an interval in $T$ relative to $\Gamma_{R}(T)$. Now, by Theorem 2.9 and Theorem 2.10, each interval is connected. Therefore, $T$ is a bi-connected $\Gamma_{R}(S)$ chain.

Finally, we show that $xT$ is a bi-connected $\Gamma_{R}(S)$ chain. If $(a,b) \in \Gamma_{R}(S)$, then $a \in bS$ so that $xa \in xbS$ and $(xa,xb) \in \Gamma_{R}(S)$. Therefore, the function $f$ taking $S$ into $S$ defined by $f(a) = xa$ is order preserving relative to $\Gamma_{R}(S)$. Hence, by Theorem 3.6, $f(T) = xT$ is a compact bi-connected $\Gamma_{R}(S)$ chain.
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BIography

James Harvey Carruth was born on August 17, 1938, in Baton Rouge, Louisiana. He attended the public schools of Maplewood, Lake Charles, Lake Arthur, and Houma, Louisiana, where he graduated in June, 1957. He did his undergraduate work in the field of electrical engineering at Louisiana State University and received his Bachelor of Science degree in June, 1961. From 1961 until 1964 he held a National Defense Education Act Graduate Fellowship, during which time he earned his Master of Science degree in mathematics, also at Louisiana State University.

In September, 1965, he married Kayla Ann Bernard of Natchitoches, Louisiana.

During the academic year 1965-66, he held an instructorship in the Mathematics Department at Louisiana State University. He is presently a candidate for the degree of Doctor of Philosophy in Mathematics.
EXAMINATION AND THESIS REPORT

Candidate: James Harvey Carruth

Major Field: Mathematics - Topology

Title of Thesis: Topics in Quasi-ordered Spaces

Approved:

[Signatures]

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

[Signatures]

Date of Examination:

July 20, 1966