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Study of Active Vibration Control using the Method of Receptances

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STUDY OF ACTIVE VIBRATION CONTROL USING THE METHOD OF RECEPTANCES

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I would like to dedicate this dissertation to my family

Smt. Jyotsna Maha, Shri. Ajit Maha

and

Sau. Amruta Maha

for their unconditional support throughout my academic pursuits.

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Abstract

Using the Method of Receptance, it is possible in general to assign $2n$ poles (i.e. n conjugate pairs) of an axially vibrating rod of variable cross section by sensing the state at n points along the rod. Using the sensor data, closed loop feedback control gains can be calculated by solving a simple set of linear equations. The feedback gains can be applied to generate the required control force using an actuator. This partial pole assignment can be done without continuous or discrete analytical modeling. Therefore, the physical parameters such as the rigidity, the density, and the variable cross-sectional area of the rod may be considered unknown for calculating the gains for the control force. These control gains are purely determined by the measurement of receptances between the point of actuation and the points of sensing, which may be conducted experimentally. In the analytical arena, the estimates for the receptances are exact and therefore the assignment of the desired poles is also exact and suffers from no discretization or model reduction errors. However, in practical implementation, the control force affects the poles of the closed loop system. In practical implementation, the control force may also shift some of the poles which are not intended to be desirably placed or to remain unchanged. Although in this dissertation, the analysis is carried on an axially vibrating rod for simplicity, the Method of Receptances is generally applicable to other linear elastic structures of higher dimensions.

Chapter 1: Introduction

1.1. Overview of the Dissertation

This dissertation involves the application of the Method of Receptances to continuous structures. In particular, the Method of Receptance has been applied to axially vibrating rod. This method utilizes data from sensor measurements to provide a suitable set of closed loop feedback gains to the actuators, which in turn apply a control force to promote partial pole placement. It is shown that generally we may assign a set of $2n$ poles (i.e. n conjugate pairs) by using one actuator and n sensors measuring the displacement and velocities at n points along the rod. Using this method there is no need for any discretization or knowledge the model. The Method of Receptances was initially developed for discrete systems. This study has been implemented for the first time on a continuous system. This is an interesting technique to provide state feedback control to a continuous system without knowing physical parameters (density, rigidity, cross section area) or performing any model reduction analysis.

The closed loop feedback control of a an axially vibrating rod unit length rod with variable cross sectional area as shown in Figure 1.1 has been considered for applying the Method of Receptance. Appendix A shows the derivations for non-dimensional analysis of governing equations of motion.

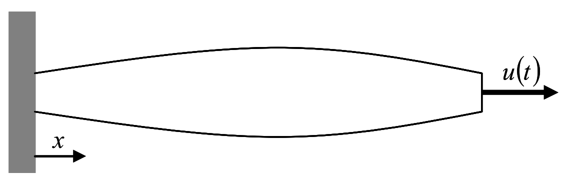

Figure 1.1: Closed loop system

The eigenvalues and the corresponding eigenfunctions of the free vibration motion is given by $\{\lambda_{\pm k}\}_{k=1,2,...}$ and $\{v_{\pm k}\}_{k=1,2,...}$, respectively. Suppose that a concentrated state feedback control force $u(t)$ is applied to the end of the rod such that the some of the undesired eigenvalues and the corresponding eigenfunctions are replaced by $\{\mu_{\pm k}\}_{k=1}^{\infty}$ and $\{w_{\pm k}\}_{k=1}^{\infty}$, respectively. In analytical examples, where the receptances do not suffer from experimental errors, the assignment of the desired poles is exact. However, the control affects the locations of the other poles of the rod.

This document is organized as follows. Section 1.2 covers literature survey providing some background on merits of using the method of receptances for discrete systems. In Chapter 2, the discussions for the background and theory are provided. They contain a brief introduction to the concept of Receptance, a simple illustration for discrete system and an overview of pole placement for continuous vibrating rod developed by Ram [28] and all the useful equations from this important paper. In chapter 3, the problem objective has been laid out and the analysis of receptance method for axially vibrating rod can be seen. In addition, the approach for two actuators, placed at two distinct locations, along the length of the rod has been shown. Chapter 4 provides analysis on natural frequency modification along with some numerical examples of how to apply the Method of Receptance. In addition, the effect of nominal random error has been considered in the numerical examples to show provide confidence for Receptance Method. Concluding remarks as well as recommendations are discussed in Chapter 5.

1.2. Literature Review

Vibration controls can be classified into passive and active controls. The uses of passive controls are very limited especially when the dynamic systems show undesirable vibrations. Therefore, extensive studies within the engineering communities were started from 1970s by using active control method to the problem of eigenvalue assignment. If a given vibrating system is controllable, then Wonham [42] in 1967 had shown that the system's poles or eigenvalues could be assigned by using an appropriate choice of state feedback system. It was also determined by Davison [5] that the eigenvalue assignment could be achieved under the conditions of which output feedback could be applied. A numerical method of determining pole-assignment to the state-feedback system was demonstrated by Kautsky et al. [9] using linear algebra. In addition, an effort to minimize the sensitivity of the pole assignment was also carried out.

Later, the engineering community started to focus on inverse eigenvalue problems. An effort is being carried out to assign natural frequencies as well as anti-resonances with the aid of structural modifications [3,16]. Experimental measurements were carried out by Mottershead et al. [17,11] to measure the rotational receptances of a structure and applied structural modifications to assign a set of the natural frequencies and antiresonances. Additional researches in structural modifications were carried out by Mottershead and Tehrani [22] on a helicopter tailcone to measure the rotational receptances. An inverse eignevalue problem for vibration absorption due to structural modification and using active control was studied by Mottershead and Ram [23]. The main advantage performing structural modification is to guaranteed the stability of the system. However, the disadvantages of the matrices of being symmetry, positive-definiteness, and the pattern of non-zero matrix terms are restrictive. In addition rotational receptances are very difficult to measure accurately as well as the eigenvalues needed to be assigned must match the rank modification.

Again recently, there has been a surge in the study of eigenvalue assignment for second order systems. A second-order matrix pencil relevant to the natural frequencies and damping of engineering structures were studied by Tisseur and Meerbergen [40] describing various methods. Additionally, numerically robust solutions to the quadratic eigenvalue problems were researched by Chu and Datta [4]. However, a method for partial pole placement was achieved by Datta et al. [5].

New methods are being developed and established for eigenvalue assignment or pole placement in vibrating structures. The poles and zeros of a vibrating system can be assigned by a state-feedback system with only a single input-state was shown in [32] and output feedback [21]. These methods use equations for controller gains for a single-input state feedback. The output feedback method generally offers eigenvalue placement at reduced control cost. These methods have been developed to take into account of controller time-delays [30]. It was also shown that the closed-loop system poles may be assigned robustly as well as being insensitive to uncertainties in the control gain terms [37].

The method of receptances for vibration control was introduced in [32] by implementing the Sherman-Morrison formula. This method was based entirely on measured vibration data, having significant modelling advantages over conventional matrix methods, including no requirement to know or to evaluate the $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ ($n \times n$ symmetric matrices), $\mathbf{M}$ is positive definite, $\mathbf{C}$ and $\mathbf{K}$ are positive semidefinite matrices. Thus there is no need for the estimation of the unmeasured state and no need for model reduction. However, the method did not lend itself naturally to *multi-input multi-output* control, as highlighted in [12]. The reformulation in [33], which was carried out by direct approach, i.e., without using the Sherman-Morrison formula, removed this limitation and allowed extension to the case of multi-input multi-output control. Implementation of the method in laboratory experiments was reported in [18], [19] and [36].

The theoretical developments described above were carried out on finite dimensional systems. In this dissertation, the use of the method of receptances was applied to the continuous model of an axially vibration rod. The extension of the analysis for other linear elastic systems in two or three spatial dimensions is straight forward. The details contained in this dissertation have also been published in [43].

Chapter 2: Background and Theory

2.1. Introduction to Receptance

The basic theoretical definition of receptance is that it is a matrix derived by the inverse of the *dynamic stiffness matrix*. In general, this is a measure of transfer function between input/output data including the dynamics of the sensors and actuators.

$$[\text{Receptance matrix}] = \mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}$$

In practice, receptances are the measurable quantities available from modal tests. The receptance equation is a displacement-type equation made complete by the non-zero force terms. The advantage is the ability to have displacement measurements at discrete locations and correlating this response to the input force. The receptance matrix $\mathbf{H}(s)$ may be expressed in terms of eigenvalues and eigenvectors (natural frequencies and mode shapes). The following analysis shows the derivation of the receptance matrix for a discrete system.

Consider a two degree of freedom mass spring damper system as shown in Figure 2.1. Figure 2.1(a) shows the application of force on the first mass and Figure 2.1(b) shows the application of force on second mass. Then the receptance matrix for this system can be shown as follows:

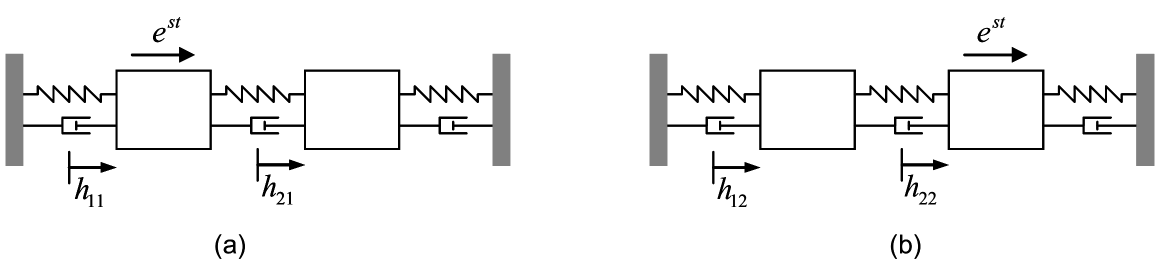

Figure 2.1: Two degree of freedom mass spring damper system

From Figure 2.1 (a)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{h}_{11} \\ \ddot{h}_{21} \end{pmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{pmatrix} \dot{h}_{11} \\ \dot{h}_{21} \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{st}$$

or $\quad \mathbf{M}\ddot{\mathbf{h}}_1 + \mathbf{C}\dot{\mathbf{h}}_1 + \mathbf{K}\mathbf{h}_1 = \mathbf{e}_1 e^{st}$

$$\mathbf{h}_1(t) = \mathbf{p}_1 e^{st} \quad \Rightarrow \quad \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{p}_1 e^{st} = \mathbf{e}_1 e^{st}$$

$$\Rightarrow \quad \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{p}_1 = \mathbf{e}_1$$

$$\Rightarrow \qquad \mathbf{p}_1 = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{e}_1 \tag{2.1}$$

From Figure 2.1 (b)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{h}_{12} \\ \ddot{h}_{22} \end{pmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{pmatrix} \dot{h}_{12} \\ \dot{h}_{22} \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{st}$$

$$\mathbf{M}\ddot{\mathbf{h}}_2 + \mathbf{C}\dot{\mathbf{h}}_2 + \mathbf{K}\mathbf{h}_2 = \mathbf{e}_2 e^{st}$$

$$\mathbf{h}_2(t) = \mathbf{p}_2 e^{st} \qquad \Rightarrow \qquad \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{p}_2 e^{st} = \mathbf{e}_2 e^{st}$$

$$\Rightarrow \qquad \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{p}_2 = \mathbf{e}_2$$

$$\Rightarrow \qquad \mathbf{p}_2 = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{e}_2 \tag{2.2}$$

Equations (2.1) and (2.2) can be written in the form

$$[\mathbf{p}_1 \quad \mathbf{p}_2] = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}[\mathbf{e}_1 \quad \mathbf{e}_2] \quad \text{or} \quad [\text{Receptance matrix}] = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}$$

2.2. Partial Pole Placement for Discrete Systems using Receptance Method

Consider the free vibration of the finite dimensional system

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0, \tag{2.3}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $n \times n$ symmetric matrices, $\mathbf{M}$ is positive definite, $\mathbf{C}$ and $\mathbf{K}$ are positive semidefinite matrices and $\mathbf{y}(t)$ is the free response. The dynamics of the open loop system (1) may be regulated by state feedback control leading to

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = \mathbf{b}u(t), \tag{2.4}$$

with

$$u(t) = \mathbf{f}^T\dot{\mathbf{z}} + \mathbf{g}^T\mathbf{z}, \tag{2.5}$$

where $\mathbf{z}(t)$ is the response of the closed loop system, $\mathbf{b}$ is a real constant input vector and $\mathbf{f}$ and $\mathbf{g}$ are real constant control gain vectors.

Substituting

$$\mathbf{y}(t) = \mathbf{v}e^{\lambda t} \tag{2.6}$$

in (2.3) gives the quadratic eigenvalue problem for the open loop system

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0}, \tag{2.7}$$

where $\mathbf{v}$ is a constant vector. Substituting

$$\mathbf{z}(t) = \mathbf{w}e^{\mu t} \tag{2.8}$$

in (2.4)-(2.5) gives the eigenvalue problem for the closed loop system

$$\left(\mu^2\mathbf{M} + \mu\mathbf{C} + \mathbf{K}\right)\mathbf{w} = \mathbf{b}\left(\mu\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}. \tag{2.9}$$

where $\mathbf{w}$ is a constant vector.

The poles of (2.7) are the roots $\{\lambda_k\}_{k=1}^{2n}$ of the open loop system, and the poles of (2.9) are the roots $\{\mu_k\}_{k=1}^{2n}$ of the closed loop systems. The control gain vectors $\mathbf{f}$ and $\mathbf{g}$ can be evaluated such that the poles of the closed loop system (2.9), $\{\mu_k\}_{k=1}^{2n}$ are assigned desirably. Since the numbering of poles is arbitrary we may, without loss of generality, require that m poles of the open loop system $\{\lambda_k\}_{k=1}^{m}$ are relocated by the using the desired control to the set $\{\mu_k\}_{k=1}^{m}$ while keeping the rest of the poles unaltered, i.e.,

$$\mu_k = \begin{cases} \mu_k & k = 1,2,\ldots,m \\ \lambda_k & k = m+1, m+2,\ldots,2n \end{cases}. \tag{2.10}$$

In order to have real solution for $\mathbf{f}$ and $\mathbf{g}$, that can be realized by the control, the sets $\{\lambda_k\}_{k=1}^{m}$ and $\{\mu_k\}_{k=1}^{m}$ have to be closed under conjugation.

The Receptance matrix for a discrete system is defined as

$$\mathbf{H}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1} \tag{2.11}$$

and let

$$\mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{b}. \tag{2.12}$$

Then, it has been shown in [7] that with $\mathbf{f}$ and $\mathbf{g}$ satisfying

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix}\begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{0} \end{pmatrix}, \tag{2.13}$$

will assign the new desired poles (2.10), where

$$\mathbf{P} = \begin{bmatrix} \mu_1 \mathbf{r}_1^T & \mathbf{r}_1^T \\ \mu_2 \mathbf{r}_2^T & \mathbf{r}_2^T \\ \vdots & \vdots \\ \mu_m \mathbf{r}_m^T & \mathbf{r}_m^T \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} \mu_{m+1} \mathbf{v}_1^T & \mathbf{v}_{m+1}^T \\ \mu_{m+2} \mathbf{v}_2^T & \mathbf{v}_{m+2}^T \\ \vdots & \vdots \\ \mu_{2n} \mathbf{v}_{2n}^T & \mathbf{v}_{2n}^T \end{bmatrix} \tag{2.14}$$

and where

$$\mathbf{e} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^T \in \Re^m . \tag{2.15}$$

2.3. Pole Placement for Continuous Vibrating Rod without spillover

The details provided in this section are from [28], which gives a theoretical method for obtaining partial pole placement without spill over for a continuous rod vibrating axially. The following are the important equations from that provide gain functions to achieve pole placement

If $\{\mu_{\pm k}\}_{k=1}^{m}$ be a self-conjugate set where

$$\mu_{-k} = \overline{\mu}_k \quad k = 1,2,...,m . \tag{2.16}$$

Then, it has been shown in [28] that with the control gain functions

$$f(x) = \sum_{\substack{k=-m \\ k \neq 0}}^{m} \beta_k a v_k \tag{2.17}$$

$$g(x) = \sum_{\substack{k=-m \\ k \neq 0}}^{m} \lambda_k \beta_k a v_k \tag{2.18}$$

the poles of the continuous rod are

$$\mu_{\pm k} = \begin{cases} \mu_{\pm k} & k = 1,...,m \\ \lambda_{\pm k} & m+1, m+2,... \end{cases} . \tag{2.19}$$

where

$$\beta_{\pm k} = \frac{\lambda_{\pm k} - \mu_{\pm k}}{\int_0^1 b v_{\pm k} dx} \prod_{\substack{r=-m \\ r \neq 0 \\ r \neq k}}^{m} \frac{(\lambda_{\pm k} - \mu_r)}{(\lambda_{\pm k} - \lambda_r)} \quad k = 1,2,...,m . \tag{2.20}$$

However, practical implementation of this method is cumbersome as measurements for every axial location along the length of the rod is not possible. Therefore it is difficult to evaluate the eigenfunction and use it to estimate the gains for the controller. In addition there are errors from instruments used for any practical application and this makes implementation process to be very challenging.

Therefore this problem has been revisited with the intension of providing state feedback control using the Method of Receptances.

Chapter 3: Receptance Method for Continuous System

3.1. Open Loop System

Consider the free axial vibration of a unit length rod with variable cross sectional area $a(x)$ and constant modulus of elasticity E and density ρ. The partial differential equation governing the axial motion is

$$\begin{array}{ll} a\ddot{y} = c^2(ay')' & 0 < x < 1 \quad t > 0 \\ y(0,t) = 0 & y'(1,t) = 0 \end{array}, \tag{3.1}$$

where $c = \sqrt{E/\rho}$ is the speed of sound in the rod, $y(x,t)$ is the displacement of $a(x)$ from its static equilibrium position, and dots and primes denote partial derivatives with respect t and x, respectively.

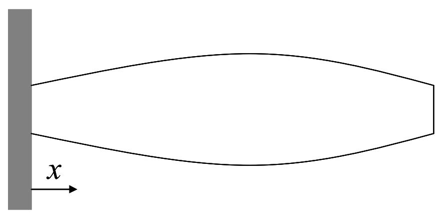

Figure 3.1: Open loop system

Separation of variables

$$y(x,t) = v(x)e^{\lambda ct} \tag{3.2}$$

Applied to (14) leads to the eigenvalue problem

$$\begin{cases} (av')' - \lambda^2 av = 0 & 0 < x < 1 \\ v(0) = 0 \quad v'(1) = 0 \end{cases}, \tag{3.3}$$

We denote the eigenvalues and the corresponding eigenfunctions of (3.3) by $\{\lambda_{\pm k}\}_{k=1,2,...}$ and $\{v_{\pm k}\}_{k=1,2,...}$, respectively.

3.2. Objectives and Problem Definition

There are situations when some the natural frequencies of the rod $\{\lambda_{\pm k}\}_{k=1,2,..m.}$ are undesirable and we wish to eliminate these frequencies. In practice it is difficult to completely remove the undesirable frequencies. However, the approach is to shift the undesired poles to a different location such that the new locations damp out the system faster than the open loop system. This is achieved by performing a state-feedback approach to induce feedback force to set the system to desired frequencies.

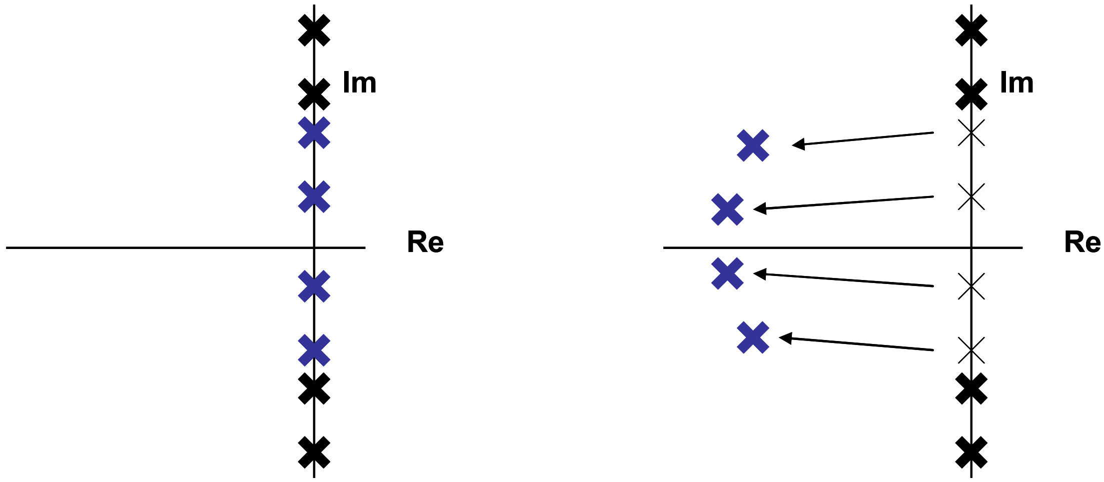

Figure 3.2: Partial Pole Placement

The problem is defined as how to achieve partial pole placement to move the undesired frequencies to a preferred location as shown in Figure 3.2.

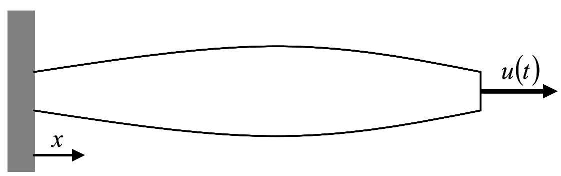

Figure 3.3: Closed loop system

Suppose that a concentrated state feedback control force

$$u(t) = Ea(1)\int_0^1 \left(c^{-1} f(x)\dot{z} + g(x)z\right)dx \tag{3.4}$$

is applied to the end of the rod as shown in Figure 3.3. Then the motion of closed loop rod is governed by

$$\begin{cases} a\ddot{z} = c^2 (az')' & 0 < x < 1 \quad t > 0 \\ z(0,t) = 0 & z'(1,t) = \int_0^1 \left(c^{-1} f(x)\dot{z} + g(x)z\right)dx \end{cases} . \tag{3.5}$$

Separation of variables

$$z(x,t) = w(x)e^{\mu ct} \tag{3.6}$$

applied to (3.5) leads to the eigenvalue problem of the closed loop system

$$\begin{cases} (aw')' - \mu^2 aw = 0 & 0 < x < 1 \\ w(0) = 0 \quad w'(1) = \int_0^1 (\mu fw + gw)dx \end{cases} \tag{3.7}$$

We denote the eigenvalues and the corresponding eigenfunctions of (3.7) by $\{\mu_{\pm k}\}_{k=1}^{\infty}$ and $\{w_{\pm k}\}_{k=1}^{\infty}$, respectively. In order to achieve partial pole placement, the poles of this closed loop system must be as follows

$$\mu_k \begin{cases} \mu_k & k = 1,2,...m \\ \lambda_k & k = m+1, m+2,...,2n \end{cases} \tag{3.8}$$

to move the first m poles to desired location and thus the conjugate pair of the these poles will also be relocated.

3.3. Receptances of the rod

Let α be a general point on the rod and suppose that a concentrated force $F(t) = Ea(\beta)e^{\xi ct}$ at $x = \beta$ excites the rod as shown in Figure 3.

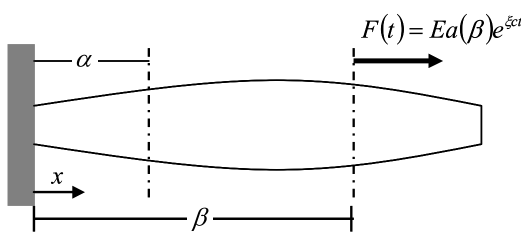

Figure 3.4: Exponential excitation

The governing differential equations for this case may be written in the form

$$\begin{cases} a\ddot{y}_1 = c^2(ay_1')' & 0 < x < \beta \\ a\ddot{y}_2 = c^2(ay_2')' & \beta < x < 1 \end{cases}. \tag{3.9}$$

with boundary conditions

$$y_1(0,t) = 0 \quad y_2'(1,t) = 0, \tag{3.10}$$

and matching conditions

$$y_1(\beta,t) = y_2(\beta,t) \quad (y_1'(\beta,t) - y_1'(\beta,t)) = e^{\xi ct} . \tag{3.11}$$

The solution for (3.9)-(3.11) takes the form

$$y_k(x,t) = h_k(x,\beta,\xi)e^{\xi ct} \quad k = 1,2 . \tag{3.12}$$

Substituting (3.12) in (3.9)-(3.11) gives

$$\begin{cases} (ah_1')' - \xi^2 ah_1 = 0 & 0 < x < \beta \\ (ah_2')' - \xi^2 ah_2 = 0 & \beta < x < 1 \\ h_1(0) = 0 & h_2'(1) = 0 \\ h_1(\beta) - h_2(\beta) = 0 & (h_1'(\beta) - h_2'(\beta)) = 1 \end{cases} \tag{3.13}$$

which determines the receptance functions $h_k(\alpha,\beta,\xi)$, $k = 1,2$.

For convenience we denote

$$h(\alpha,\beta,\xi) = \begin{cases} h_1(\alpha,\beta,\xi) & \alpha < \beta \\ h_2(\alpha,\beta,\xi) & \alpha > \beta \end{cases} . \tag{3.14}$$

In particular, when $\beta = 1$ the receptance $h(x,1,\xi)$ is determined by the solution of

$$\begin{cases} (ah'(x,1,\xi))' - a\xi^2 h(x,1,\xi) = 0 & 0 < x < 1 \\ h(0,1,\xi) = 0 \quad h'(1,1,\xi) = 1 \end{cases} \tag{3.15}$$

3.4. Solution for Continuous System

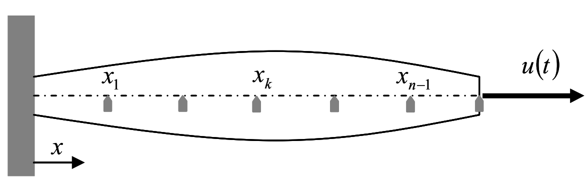

Figure 3.5: Controlled rod

Suppose that n sensors are located at the positions $\{x_k\}_{k=1}^n$ along the rod as shown in Figure 3.5. Let us define the vector of sensing positions as follows

$$\mathbf{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T \tag{3.16}$$

where $x_n = 1$, and denote

$$\mathbf{v}_k = \begin{pmatrix} v_k(x_1) & v_k(x_2) & \cdots & v_k(x_2) \end{pmatrix}^T \tag{3.17}$$

and

$$\mathbf{w}_k = \begin{pmatrix} w_k(x_1) & w_k(x_2) & \cdots & w_k(x_2) \end{pmatrix}^T \tag{3.18}$$

where $v_k(x)$ and $w_k(x)$ are the eigenfunctions of the open and closed loop systems corresponding to λ_k and μ_k, respectively.

Let

$$f(x) = \sum_{k=1}^{n} f_k \delta(x - x_k) \tag{3.19}$$

and

$$g(x) = \sum_{k=1}^{n} g_k \delta(x - x_k) \tag{3.20}$$

be the gain functions of the closed loop system where f_k and g_k, $k = 1,2,...,n$, are the k^{th} elements of the vectors $\mathbf{f}$ and $\mathbf{g}$, respectively.

Lemma 1

If

$$\left(\mu_k \mathbf{f}^T + \mathbf{g}^T\right)\mathbf{v}_k = 0 \tag{3.21}$$

then

$$\{\mu_k \quad w_k\} = \{\lambda_p \quad v_p\} \tag{3.22}$$

for some integer p. The physical meaning is that if (3.21) is satisfied then the eigenpair $\{\lambda_k \quad v_k\}$ is common to the open and the closed loop systems.

Proof

With $g(x)$ and $f(x)$ as expressed by (3.19) and (3.20) we have

$$\int_0^1 (\mu_k f + g) w_k dx = \left(\mu_k \mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}_k = 0 \tag{3.23}$$

by virtue of (3.23). It thus follows that the closed loop eigenvalue problem (3.7) reduces to

$$\begin{cases}(aw_k')' - \mu_k^2 a w_k = 0 & 0 < x < 1 \\ w_k(0) = 0 \quad w_k'(1) = 0\end{cases} \tag{3.24}$$

which is the eigenvalue problem associated with the open loop system (16). □

Suppose that $\mu \notin \{\lambda_{\pm k}\}_{k=1}^{\infty}$. Then we define the receptance matrix

$$\mathbf{H}(\mu) = \begin{bmatrix} h(x_1,x_1,\mu) & h(x_1,x_2,\mu) & \cdots & h(x_1,x_n,\mu) \\ h(x_2,x_1,\mu) & h(x_2,x_2,\mu) & \cdots & h(x_2,x_n,\mu) \\ \vdots & \vdots & \vdots\vdots & \vdots \\ h(x_n,x_1,\mu) & h(x_n,x_2,\mu) & \cdots & h(x_n,x_n,\mu) \end{bmatrix}, \tag{3.25}$$

and denote

$$\mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{e}_n, \tag{3.26}$$

where $\mathbf{e}_r$ is the $r^{\mathbf{th}}$ unit vector.

Lemma 2

If

$$(\mu_k \mathbf{f}^T + \mathbf{g}^T)\mathbf{r}_k = 1 \tag{3.27}$$

then μ_k is a pole of the closed loop system (3.7).

Proof

Since $w(x)$ is an eigenfunction of (3.7) it may be scaled arbitrarily. We choose a scaling such that

$$\int_0^1 (\mu f + g) w dx = 1, \tag{3.28}$$

so that the closed loop eigenvalue problem (3.7) reduces to

$$\begin{cases}(aw')' - \mu^2 a w = 0 & 0 < x < 1 \\ w(0) = 0 \quad w'(1) = 1\end{cases} \tag{3.29}$$

Note that

$$h(x_i,1,\mu_k) = w(x_i) \quad i = 1,2,\ldots,n \quad k = 1,2,\ldots,m \tag{3.30}$$

by virtue of (3.15).

With $g(x)$ and $f(x)$ as expressed by (3.19) and (3.20) we have

$$\int_0^1 (\mu_k f + g) w_k dx = \left(\mu_k \mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w}_k = 1 . \tag{3.31}$$

The proof is completed by noting that

$$\mathbf{w}_k = \mathbf{r}_k \tag{3.32}$$

by virtue of the definitions (3.25), (3.26) and equation (3.30). □

Note that (3.23) and (3.27) may be written equivalently as

$$\begin{pmatrix} \lambda_p \mathbf{v}_p^T & \mathbf{v}_p^T \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = 0 \quad \mu_k = \lambda_p \tag{3.33}$$

and

$$\begin{pmatrix} \mu_k \mathbf{r}_k^T & \mathbf{r}_k^T \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = 1 \quad \mu_k \notin \{\lambda_{\pm i}\}_{i=1}^{\infty} . \tag{3.34}$$

It thus follows that the desired gain vectors $\mathbf{f}$ and $\mathbf{g}$ are determined by the set of linear equations obtained from equation (2.13).

3.5. Multiple Actuators

Consider the case were the control is done by more than one actuator. For the sake of simplicity we choose the two concentrated control forces

$$F_r(t) = b_r \delta(x - x_r) u(t) \qquad F_n(t) = b_n \delta(x - x_k) u(t) \ , \tag{3.35}$$

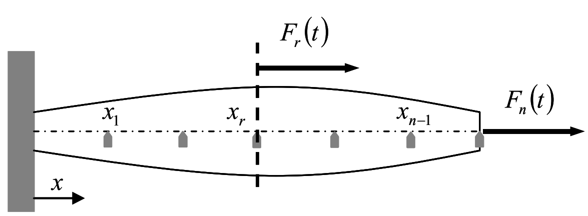

Figure 3.6: Rod with multiple actuators

shown in Figure 3.6, where b_r and b_n are some constants and $u(t)$ is given by (3.4). The extension of the analysis to control by more than two forces is self-explanatory. The eigenvalue problem of the closed loop system is

$$\begin{cases} (aw_A')' - \mu^2 a w_A = 0 & 0 < x < x_r \\ (aw_B')' - \mu^2 a w_B = 0 & x_r < x < 1 \\ w_A(0) = 0 & w_B'(1) = b_n\left(\mu \mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w} \\ w_A(x_r) - w_B(x_r) = 0 & \left(w_A'(x_r) - w_B'(x_r)\right) = b_r\left(\mu \mathbf{f}^T + \mathbf{g}^T\right)\mathbf{w} \end{cases} \tag{3.36}$$

where $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{w}$ are given by (3.19), (3.20) and (3.18), respectively, and

$$w(x) = \begin{cases} w_A(x) & 0 < x < x_r \\ w_B(x) & x_r < x < 1 \end{cases}. \tag{3.37}$$

Lemma 1 holds for this case since for each eigenpair $\{\mu_k \quad \mathbf{w}_k\}$ which satisfies (3.23) the eigenvalue problem of the closed loop system (3.36) reduces to that of its counterpart associated with the open loop system (3.3).

Suppose now that $\mu_k \notin \{\lambda_{\pm p}\}_{p=1}^{\infty}$. Then there exists receptance matrix $\mathbf{H}(\mu_k)$. Let

$$\mathbf{H}(\mu_k) = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \cdots \quad \mathbf{h}_n] \tag{3.38}$$

be the column partitioning of $\mathbf{H}$, and denote

$$\mathbf{b} = b_r \mathbf{e}_r + b_n \mathbf{e}_n\ . \tag{3.39}$$

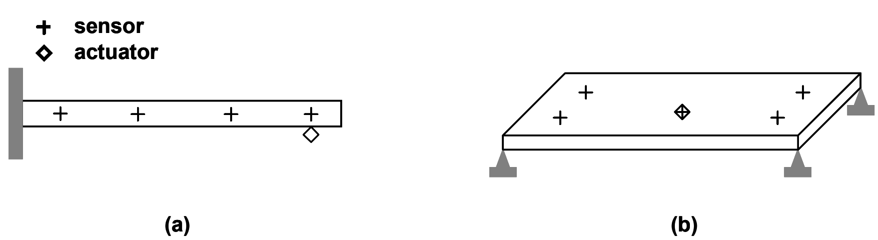

Figure 3.7: (a) Transversely vibrating beam, and (b) vibrating plate

With the normalization of $\mathbf{w}$ according to (3.31), the system of equations (3.36) yields

$$\mathbf{w}_k = b_r \mathbf{h}_r(\mu_k) + b_n \mathbf{h}_n(\mu_k) = \mathbf{H}(\mu_k)\mathbf{b}\ , \tag{3.40}$$

by virtue of (3.13), (3.25), (3.32) and the principle of superposition. It thus follows that Lemma 2, in conjunction with (2.12), holds for this case. We may also conclude that it is necessary to measure only the receptances between the points of actuation and the points of sensing.

It is clear from equation (3.40) that there is no unique solution for state feedback control using multiple actuations using the Method of Receptances. The coefficients b_r and b_n can be scaled arbitrarily depending on the weightage provided to the sensor data. Therefore there are many combinations that exist that can provide a required feedback to control the system.

Chapter 4: Natural Frequency Modification and Numerical Examples

4.1. Natural Frequency Modification

In this section, we will take a look at the modification of natural frequencies for axially vibrating rod.

Consider the case where

$$\mu_{-k} + \mu_k = 0 \quad k = 1,2,\ldots,m, \tag{4.1}$$

i.e., the $2m$ assigned eigenvalues are purely imaginary corresponding to undamped systems.

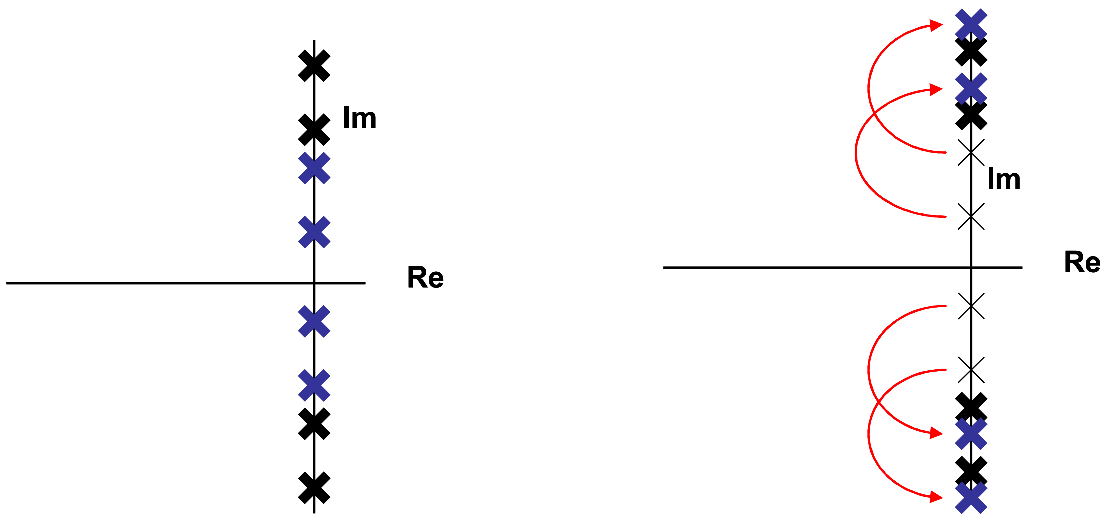

Figure 4.1: Natural Frequency Modification

Denote the natural frequencies of the open and closed loop systems by

$$\omega_k = \sqrt{-\lambda_{-k}\lambda_k} = |\lambda_k| \quad k = 1,2,\ldots, \tag{4.2}$$

and

$$\sigma_k = \sqrt{-\mu_{-k}\mu_k} = |\mu_k| \quad k = 1,2,\ldots, \tag{4.3}$$

respectively. Then eigenvalue problems for the open and closed loop systems are given by

$$\begin{cases} (a\hat{v}')' + \omega^2 a\hat{v} = 0 \quad 0 < x < 1 \\ \hat{v}(0) = 0 \quad \hat{v}'(1) = 0 \end{cases}, \tag{4.4}$$

and

$$\begin{cases} (a\hat{w}')' + \sigma^2 a\hat{w} = 0 & 0 < x < 1 \\ \hat{w}(0) = 0 & \hat{w}'(1) = \int_0^1 (\sigma f + g)\hat{w}dx \end{cases} . \tag{4.5}$$

With

$$f(x) = 0 \quad g(x) = \sum_{k=1}^{m} \omega_k \beta_k a v_k \ , \tag{4.6}$$

where

$$\beta_k = \frac{\omega_k^2 - \sigma_k^2}{\int_0^1 b v_k dx} \prod_{\substack{r=1 \\ r \neq k}}^{m} \frac{\omega_k^2 - \sigma_r^2}{\omega_k^2 - \omega_r^2} \quad k = 1,2,\ldots,m \ , \tag{4.7}$$

the natural frequencies $\{\omega_k\}_{k=1}^m$ are assigned to $\{\sigma_k\}_{k=1}^m$ by virtue of (3.10)-(3.13).

4.2. Uniform Rod with Sinusoidal Excitation

Consider the axial vibration of a uniform rod with constant cross sectional area $a(x) = A$. For simplicity we address the case where the self conjugate set $\{\mu_{\pm k}\}_{k=1}^n$ consists of purely imaginary poles. In such a case it is suitable to express the open loop eigenvalue problem in terms of the natural frequency ω rather than the pole λ. With such a formulation the arithmetic involved in computing the eigenvalues and the eigenfunctions is real.

Substituting $\omega^2 = -\lambda^2$ in (3.3) gives the open loop eigenvalue problem

$$\begin{cases} \hat{v}'' + \omega^2 \hat{v} = 0 & 0 < x < 1 \\ \hat{v}(0) = 0 & \hat{v}'(1) = 0 \end{cases} . \tag{4.8}$$

The eigenpairs of (4.8) are real,

$$\left\{\omega_k^2 = (k - 1/2)^2 \pi^2 \quad \hat{v}_k = \sin \omega_k x\right\}_{k=1}^{\infty} . \tag{4.9}$$

The closed loop eigenvalue problem (3.7) is simplified to

$$\begin{cases} \hat{w}'' + \sigma^2 \hat{w} = 0 & \sigma^2 = -\mu^2 \\ \hat{w}(0) = 0 & \hat{w}'(1) = \int_0^1 g\hat{w}dx \end{cases} \tag{4.10}$$

since the closed loop is undamped. With the scaling

$$\int_0^1 g\hat{w}dx = 1, \tag{4.11}$$

the eigenvalue problem (4.10) reduces to the problem of solving a differential equation with non-homogeneous boundary conditions

$$\begin{cases} \hat{w}'' + \sigma^2 \hat{w} = 0 & 0 < x < 1 \\ \hat{w}(0) = 0 \quad \hat{w}'(1) = 1 \end{cases}. \tag{4.12}$$

We use the fact that in our problem the desired frequency σ is given and obtain from (4.12)

$$\hat{w} = \frac{\sin \sigma x}{\sigma \cos \sigma}. \tag{4.13}$$

To complete the analysis of the natural frequency assignment in terms of real arithmetic we consider the sinusoidal excitation

$$\hat{F} = \sin \sigma c t \tag{4.14}$$

at $x = \beta$ rather than the exponential excitation used in Section 3.3. The associated frequency response function is

$$\hat{h}(x,1,\sigma) = \frac{\sin \sigma x}{\sigma \cos \sigma} \tag{4.15}$$

since

$$\hat{h}(x,1,\sigma) = \hat{w}(x). \tag{4.16}$$

The frequency equation for the closed loop system is

$$\phi(\sigma) = \sigma \cos \sigma - \sum_{k=1}^{n} g_k \sin \sigma x_k = 0 \tag{4.17}$$

4.3. Exponential Rod

In this section we will consider the axial vibration of an exponential rod with cross sectional area $a(x) = e^x$. By using $\omega^2 = -\lambda^2$ the eigenvalue problem (3.3) gives

$$\begin{cases}(e^x\hat{v}')' + \omega^2 e^x \hat{v} = 0 & 0 < x < 1 \\ \hat{v}(0) = 0 \quad \hat{v}'(1) = 0\end{cases}. \tag{4.18}$$

Let γ_k be the k^{th} root of

$$\tan\gamma = 2\gamma\ . \tag{4.19}$$

Then

$$\omega_k = \sqrt{\gamma_{k+1}^2 + 1/4} \quad k = 1,2,... \tag{4.20}$$

is the k^{th} natural frequency of the rod. Note that $\omega^2 = 1/4$ corresponding to $\gamma_1 = 0$ in (4.20) leads to a trivial solution and it is therefore not an eigenvalue of (4.18). We therefore applied the shift of index from k in the left hand side of equation (4.20) to $k+1$ in its right hand side.

The eigenfunctions of (4.18) corresponding to ω_k are

$$\hat{v}_k(x) = e^{-x/2}\sin\sqrt{\omega_k^2 - 1/4}x\ . \tag{4.21}$$

The closed loop eigenvalue problem for the undamped exponential rod with sinusoidal excitation (4.15) is

$$\begin{cases}(e^x\hat{w}')' + \sigma^2 e^x \hat{w} = 0 & 0 < x < 1 \\ \hat{w}(0) = 0 \quad \hat{w}'(1) = \int_0^1 g\hat{w}dx\end{cases} \tag{4.22}$$

With the scaling (4.11) we obtain

$$\begin{cases}(e^x\hat{w}')' + \sigma^2 e^x \hat{w} = 0 & 0 < x < 1 \\ \hat{w}(0) = 0 \quad \hat{w}'(1) = 1\end{cases}, \tag{4.23}$$

which gives

$$\hat{w}(x) = \hat{h}(x,1,\sigma) = 2\frac{e^{s_1 x} - e^{s_2 x}}{\theta\left(e^{s_1} + e^{s_2}\right) - e^{s_1} + e^{s_2}} \tag{4.24}$$

where

$$\theta = \sqrt{1 - 4\sigma^2} \tag{4.25}$$

and

$$s_{1,2} = \frac{-1 \pm \sigma}{2}. \tag{4.26}$$

The frequency equation for the closed loop system is

$$\phi(\sigma) = \frac{1}{2} e^{-1/2} (\sin\psi + 2\psi\cos\psi) - \sum_{k=1}^{n} g_k e^{-x_k/2} \sin\psi x_k = 0 \tag{4.27}$$

where

$$\psi = \sqrt{\sigma^2 - 1/4} \tag{4.28}$$

4.4. Example 1: Natural Frequency Modification of Uniform Rod

Suppose that we wish to assign the natural frequency $\omega_1 = \pi/2$ of a uniform rod with $a = 1$ to the natural frequency $\sigma_1 = 3\pi/4$ by using the control (3.4), while leaving $\omega_2 = \sigma_2 = 3\pi/2$ and $\omega_3 = \sigma_3 = 5\pi/2$ unchanged. This objective could be done by using three sensing points. We choose

$$x_1 = 1/3 \quad x_2 = 2/3 \quad x_3 = 1$$

and obtain by (4.15)

$$\hat{h}_{13}(\sigma_1) = -\frac{4}{3\pi} \quad \hat{h}_{23}(\sigma_1) = -\frac{4\sqrt{2}}{3\pi} \quad \hat{h}_{33}(\sigma_1) = -\frac{4}{3\pi}$$

where $\hat{h}_{ij}(\sigma)$ is the $i-j$ entry of the sinusoidal transfer matrix $\hat{\mathbf{H}}(\sigma)$. It thus follows from (3.26) that

$$\hat{\mathbf{r}}_1 = -\frac{4}{3\pi}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}^T.$$

From (3.17) and (4.9) we have

$$\hat{\mathbf{v}}_k = \begin{pmatrix}\sin\omega_k x_1 & \sin\omega_k x_2 & \sin\omega_k x_3\end{pmatrix}^T \quad k = 1,2$$

which gives

$$\hat{\mathbf{v}}_2 = \begin{pmatrix}1 & 0 & -1\end{pmatrix}^T \quad \hat{\mathbf{v}}_3 = \frac{1}{2}\begin{pmatrix}1 & -\sqrt{3} & 2\end{pmatrix}^T .$$

Equation (2.13) reduces to

$$\mathbf{Ag} = \begin{bmatrix}\hat{\mathbf{r}}_1^T \\ \hat{\mathbf{v}}_2^T \\ \hat{\mathbf{v}}_3^T\end{bmatrix}\begin{pmatrix}g_1 \\ g_2 \\ g_3\end{pmatrix} = \begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix} \qquad \kappa(\mathbf{A}) = 1.88$$

which gives

$$\mathbf{g} = \frac{-3\pi}{4\left(3\sqrt{2}+2\sqrt{3}\right)}\begin{pmatrix}\sqrt{3} \\ 3 \\ \sqrt{3}\end{pmatrix} = \begin{pmatrix}-0.5295 \\ -0.9172 \\ -0.5295\end{pmatrix},$$

where $\kappa(\mathbf{A})$ is the condition number of $\mathbf{A}$. The system of linear equations for determining $\mathbf{g}$ in this case is well conditioned.

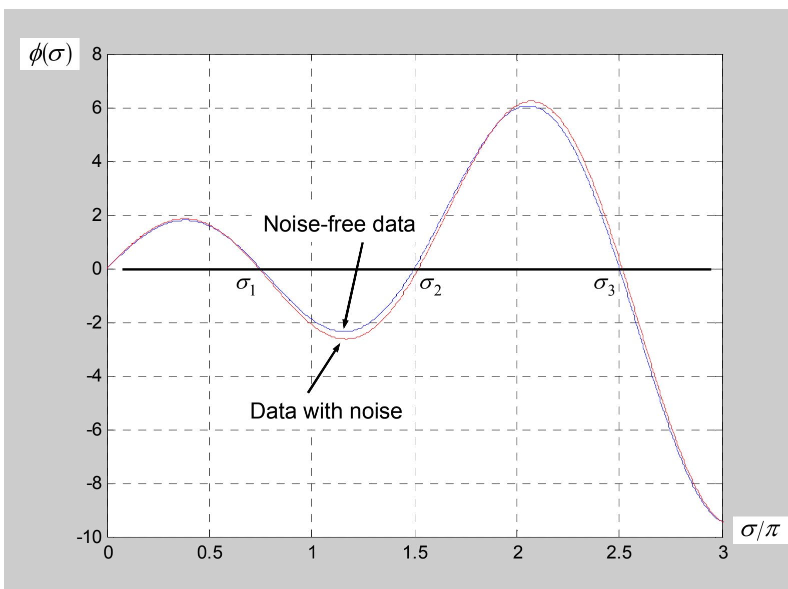

Figure 4.2: Frequency function of the closed loop uniform rod

To examine the sensitivity of the results to errors a random[1] noise was added to $\mathbf{g}$ of normal distribution, with zero mean and standard deviation 0.1, resulting in

[1] Using the first three numbers generated by MATLAB pseudo random number generator after invoking the software.

$$\tilde{\mathbf{g}} = \begin{pmatrix} -0.4758 \\ -0.7338 \\ -0.7554 \end{pmatrix} \qquad \frac{\|\tilde{\mathbf{g}} - \mathbf{g}\|}{\|\mathbf{g}\|} = 0.2499 .$$

The frequency functions (4.17) corresponding to the noise free and noise corrupted data are plotted in Figure 4.2. The frequencies σ_k, $k = 1,2,3$ have been assigned as desired by the noise free control. It is visually clear from this figure that the change in the assigned frequencies due to the noise is marginal in this case.

The next four natural frequencies, which were not intended to be controlled, are

$$\sigma_k = \begin{cases} \omega_k & k = 4,5 \\ 17.3678 & k = 6 \\ 20.4996 & k = 7 \end{cases} .$$

We note that $\omega_6 = 17.2788$ and $\omega_7 = 20.4204$. Hence, the control affected the locations of some of the unassigned natural frequencies.

4.5. Example 2: Natural Frequency Modification of Exponential Rod

We wish to assign the lowest three natural frequencies $\{\omega_k\}_{k=1}^{3}$ of the exponential rod with cross sectional area $a = e^x$ to $\sigma_1 = 3\pi/4$, $\sigma_2 = 3\pi/2$ and $\sigma_3 = 5\pi/2$, as in Example 1.

The sensing points are

$$x_1 = 1/3 \quad x_2 = 2/3 \quad x_3 = 1 .$$

Using (63), (37) and (38) we have

$$\begin{cases} \hat{\mathbf{r}}_1 = (-0.5072 \quad -0.6180 \quad -0.3894)^T \\ \hat{\mathbf{r}}_2 = (3.7196 \quad 0.0558 \quad -2.6644)^T \\ \hat{\mathbf{r}}_3 = (-1.8776 \quad 2.7443 \quad -2.6658)^T \end{cases}$$

Equation (2.13) for this case is

$$\mathbf{A}\mathbf{g} = \begin{bmatrix} \hat{\mathbf{r}}_1^T \\ \hat{\mathbf{r}}_2^T \\ \hat{\mathbf{r}}_3^T \end{bmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \kappa(\mathbf{A}) = 5.30$$

which gives

$$\mathbf{g} = (-0.3625 \quad -0.7552 \quad -0.8972)^T .$$

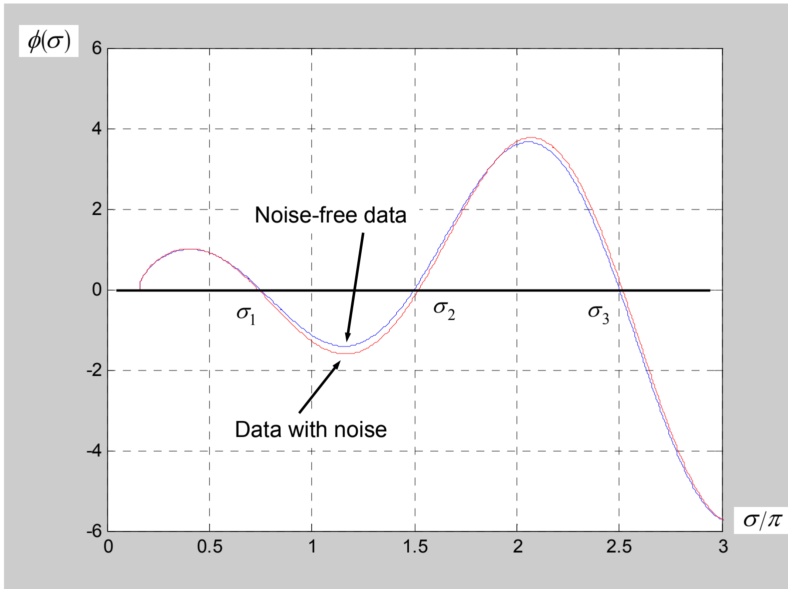

Figure 4.3: Frequency function of the closed loop exponential rod

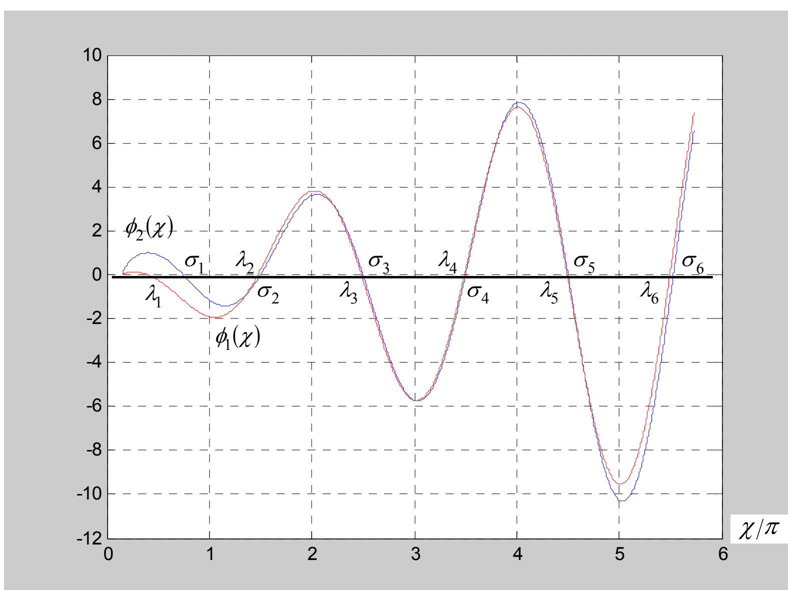

Figure 4.4: Frequency functions of the open and closed loops exponential rod

The same noise to $\mathbf{g}$ as in Example 1 has been introduced, resulting in

$$\tilde{\mathbf{g}} = \begin{pmatrix} -0.3088 \\ -0.5718 \\ -1.1231 \end{pmatrix} \qquad \frac{\|\tilde{\mathbf{g}} - \mathbf{g}\|}{\|\mathbf{g}\|} = 0.2410$$

The frequency equations of the closed loop system corresponding to $\mathbf{g}$ and the noise corrupted $\tilde{\mathbf{g}}$ are plotted in Figure 4.3. It shows that the frequencies σ_k, $k = 1,2,3$ have been assigned as required by the noise free control. It is visually clear from this figure that the change in the assigned frequencies due to the noise is marginal in this case.

The frequency equations ϕ_1 and ϕ_2 corresponding to the open loop and the closed loop systems, respectively, shown in Figure 4.4 indicate that the first six natural frequencies of the exponential rod were influenced by the control.

Chapter 5: Concluding Remarks and Future Work

5.1. Conclusion

The method of receptances is another technique to achieve partial pole placement using state feedback control of a linear elastic system by using measured experimental data, rather than analytical modeling. The original method was formulated in [32], and reformulated in [33], in the framework of a discrete n degrees of freedom system where all $2n$ poles (n conjugate pairs) are assigned desirably.

It would be therefore natural to assume that implementing the method of receptance to a continuous linear elastic systems, the assignment of poles would accrue some residual errors. This may be due to association with the incompleteness of the unmeasured points. It has been shown that in theory, where the receptances are exact, the assignment of the desired poles is also exact and suffers from no discretization or model reduction errors.

It has been shown that in general we may assign $2n$ poles (n conjugate pairs) of an axially vibrating non-uniform rod by sensing the state of n points without knowing the density, rigidity and the variable cross-sectional area of the rod. The required input is the measurements of receptances between the points of actuation and the sensing points. These functions may be measured experimentally.

5.2. Recommendations

The analysis revealed that this result is general as applicable to other linear elastic structures such as the transversely vibrating beam or the plate shown in Figure 3.7. The analysis presented in this dissertation does not address any optimization for multiple actuation. It is advised that a future study may provide insights and help to fully realize the implementation of this method for real systems.

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Appendix

A Non-Dimensional Equation of Motion of Axially Vibrating Rod

Consider a non-uniform cross-section rod vibrating axially. Let x_a be the axial location of any cross section a_a vibrating about its static equilibrium position with an amplitude of $z_a(x_a, t_a)$, where t_a represents the time evolution of the motion z_a. In addition, let E_a, ρ_a represent the elastic modulus and the density of the rod respectively. The subscript a represents variables corresponding to actual or physical rod under study. The variables without the subscript will represent normalized or non-dimensional values for scalability.

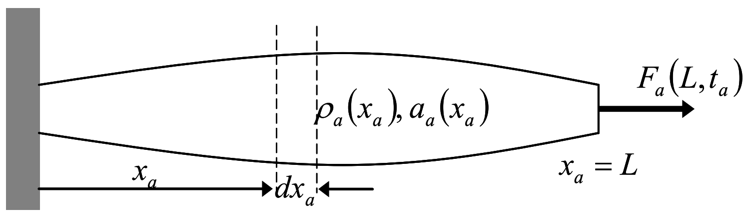

Figure A.1: Non-Uniform Rod

Consider an infinitesimal element dx_a, then we can represent the forces on a free body diagram and derive the equation of motion of the element.

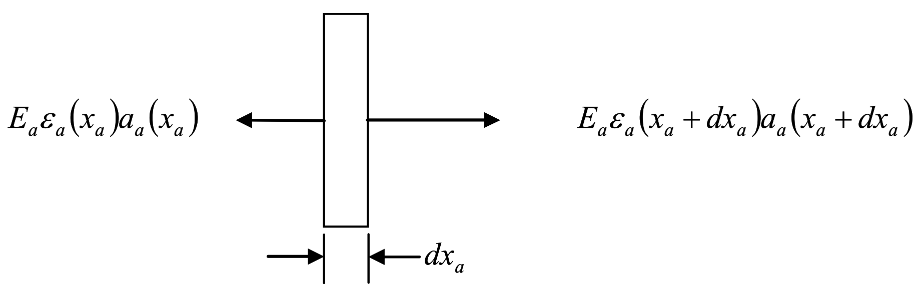

Figure A.2: Free Body Diagram of the element

Representing strain as $\varepsilon_a(x_a) = \dfrac{z_a(x_a)}{dx_a}$ we can then write the equation of motion using Newton's second law:

$$E_a a_a(x_a + dx_a)\varepsilon_a(x_a + dx_a) - E_a a_a(x_a)\varepsilon_a(x_a) = \rho_a(x_a)a_a(x_a)dx_a \frac{\partial^2 z_a}{\partial t_a^2}$$

Applying Taylor's Series Expansion and neglecting higher order terms, we get the final form of the solution as

$$\frac{\partial}{\partial x_a}\left(E_a a_a\left(x_a\right)\frac{\partial z_a}{\partial x_a}\right)=\rho_a a_a\left(x_a\right)\frac{\partial^2 z_a}{\partial t_a^2} \quad (1)$$

Then we can scale the above variables to represent them in non-dimensional (ND) terms by:

$$\text{ND axial location, } x = \frac{x_a}{L} \quad \frac{dx}{dx_a} = \frac{1}{L} \quad \frac{\partial}{\partial x_a} = \frac{\partial}{\partial x}.\frac{dx}{dx_a} = \frac{1}{L}\frac{\partial}{\partial x} \tag{2}$$

$$\text{ND axial vibration, } z = \frac{z_a}{L} \tag{3}$$

$$\text{ND cross-section area, } a(x) = \frac{a_a(x)}{L^2} \tag{4}$$

$$\text{Speed of sound, } c = \sqrt{E_a / \rho_a} \tag{5}$$

$$\text{ND time scale, } t = \frac{t_a}{L}\sqrt{\frac{E_a}{\rho_a}} \Rightarrow \frac{\partial t}{\partial t_a} = \frac{1}{L}\sqrt{\frac{E_a}{\rho_a}} \tag{6}$$

$$\text{ND Force, } F = \frac{F_a}{E_a L^2} \tag{7}$$

$$\text{ND frequency, } \omega = \omega_a L\sqrt{\frac{\rho_a}{E_a}} \tag{8}$$

$$\text{ND eigenvalue, } \lambda = \lambda_a L \tag{9}$$

Substituting for the variables in terms of non-dimensional variables, x, z, a, b, t in :

$$\frac{1}{L}\frac{\partial}{\partial x}\left(E_a L^2 a(x)\frac{L}{L}\frac{\partial z}{\partial x}\right) = \rho_a L^2 a(x)\frac{E_a}{L^2 \rho_a} L\frac{\partial^2 z}{\partial t^2} \tag{10}$$

Simplifying the equation leads to:

$$\frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) = a(x)\frac{\partial^2 z}{\partial t^2} \tag{11}$$

The boundary conditions will be transformed to:

$$\left.\frac{\partial z_a}{\partial x_a}\right|_{x_a=0} = 0 \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{x=0} = 0 \tag{12}$$

The second boundary condition is transformed to:

$$E_a a_a(x_a)\left.\frac{\partial z_a}{\partial x_a}\right|_{x_a=L} = F_a(x_a, t_a)\Big|_{x_a=L} \quad \Rightarrow \quad \left.\frac{\partial z(x,t)}{\partial x}\right|_{x=1} = F(x,t)\Big|_{x=1} \tag{13}$$

B Partial Pole Placement of Uniform Rod

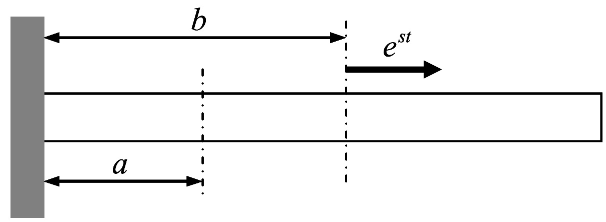

Figure B.1: Uniform Rod

Open loop system

The differential equation of motion

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1 \quad t > 0, \tag{1}$$

The two boundary conditions

$$u(0,t) = 0, \tag{2}$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=1,t} = 0 \tag{3}$$

Separation of variables applied to (1)

$$u(x,t) = v(x)\sin \omega t \tag{4}$$

gives

$$v'' + \omega^2 v = 0 \tag{5}$$

with the boundary conditions

$$v(0) = 0 \tag{6}$$

$$v'(1) = 0 \tag{7}$$

Sinusoidal form: The general solution to (5) takes the form

$$v(x) = A\sin \omega x + B\cos \omega x \tag{8}$$

The boundary condition (6) gives

$$v(0) = B = 0 \tag{9}$$

so that (8) reduces to

$$v(x) = A \sin \omega x \tag{10}$$

Substituting (10) in (7) gives

$$v'(1) = A\omega \cos \omega = 0 \tag{11}$$

so that the characteristic frequency equation is

$$\cos \omega = 0 \tag{12}$$

Its roots are

$$\omega_k = \frac{(2k-1)\pi}{2}, \qquad k = 1,2,3,.... \tag{13}$$

with corresponding eigenfunctions

$$v_k(x) = \sin \omega_k x \tag{14}$$

Exponential form: Separation of variables

$$u(x,t) = v(x)e^{st} \tag{15}$$

applied to (1) gives

$$v'' - s^2 v = 0 \tag{16}$$

with boundary conditions

$$v(0) = 0 \tag{17}$$

$$v'(1) = 0 \tag{18}$$

The general solution to (16) is

$$v(x) = Ae^{sx} + Be^{-sx} \tag{19}$$

The boundary condition (2) gives

$$v(0) = A + B = 0 \qquad \Rightarrow \qquad B = -A \tag{20}$$

so that (19) reduces to

$$v(x) = A\left(e^{sx} - e^{-sx}\right) \tag{21}$$

Substituting (21) in (3) gives

$$v'(1) = As\left(e^{s} + e^{-s}\right) = 0 \tag{22}$$

so that the characteristic frequency equation is

$$e^{s} + e^{-s} = 0 \tag{23}$$

With

$$s_k = \frac{(2k-1)i\pi}{2}, \qquad k = 1,2,3,.... \tag{24}$$

the roots of (23) are $\pm s_k$ with eigenfunctions

$$v_k(x) = \left(e^{s_k x} - e^{-s_k x}\right) \tag{25}$$

Closed loop system

The differential equation of motion is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1 \quad t > 0, \tag{26}$$

and the two boundary conditions are

$$u(0,t) = 0, \tag{27}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1,t} = \int_0^1 \left(f\ddot{u} + gu\right)dx. \tag{28}$$

Separation of variables

$$u(x,t) = v(x)e^{st} \tag{29}$$

applied to (26) gives

$$v'' - s^2 v = 0 \tag{30}$$

and the boundary conditions are

$$v(0) = 0 \tag{31}$$

$$v'(1) = \int_0^1 (sf + g) v dx \tag{32}$$

The general solution of (30) is

$$v(x) = Ae^{sx} + Be^{-sx} \tag{33}$$

From (27) we have

$$v(0) = A + B = 0 \tag{34}$$

so that (33) reduces to

$$v(x) = A\left(e^{sx} - e^{-sx}\right) \tag{35}$$

From (28) we determine the characteristic frequency equation

$$s\left(e^{s} + e^{-s}\right) = \int_0^1 (sf + g)\left(e^{sx} - e^{-sx}\right) dx \tag{36}$$

with eigenfunctions

$$v_k(x) = Ae^{s_k x} + Be^{-s_k x} \tag{37}$$

where s_k is the k-th root of (36)

The Problem: find $f(x)$ and $g(x)$ such that the roots of (36) are:

$$s = \pm\frac{3\pi i}{4},\ s = \pm\frac{(2k-1)\pi i}{4},\ k = 2,3,4,... \tag{38}$$

Comment: If $f(x) = g(x) = 0$ then the roots of (36) are

$$s = \pm \frac{(2k-1)\pi i}{4}, \quad k = 1,2,3,... \tag{39}$$

Solution:

$f(x) = 0$

$$g := -\frac{5}{16}\pi^2 \sin\left(\frac{\pi x}{2}\right)$$

Characteristic frequency equation

$$eq1 := s\,(\mathbf{e}^{s} + \mathbf{e}^{(-s)}) + \frac{5}{4}\,\frac{\pi^2\, s\,(1 + \mathbf{e}^{(2\,s)})\,\mathbf{e}^{(-s)}}{4\,s^2 + \pi^2}$$

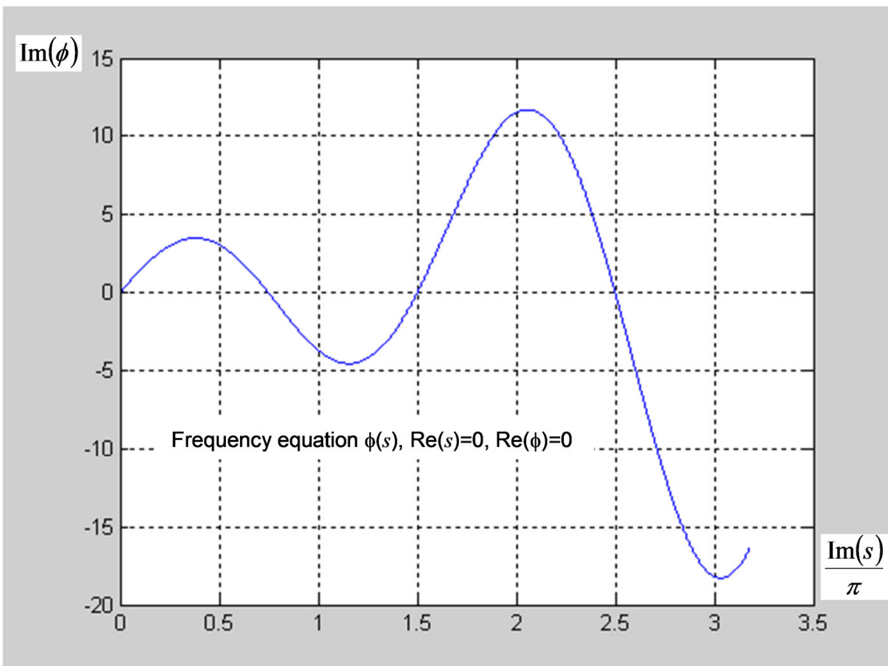

Figure B.2: Frequency function of Uniform Rod

Sinusoidal characteristic equation

$$\phi(\omega) = \frac{\omega(9\pi^2 - 16\omega^2)}{4(\pi^2 - 4\omega^2)}\cos\omega$$

$$\frac{1}{4}\frac{\cos(w)\,w\,(9\,\pi^2-16\,w^2)}{\pi^2-4\,w^2}$$

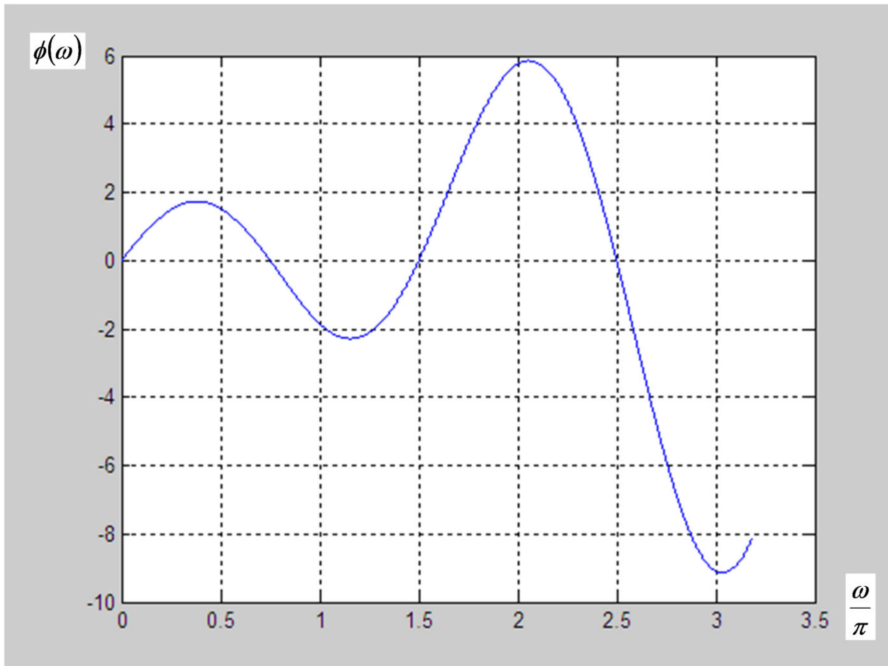

Figure B.3: Frequency function of Uniform Rod

C MATLAB Code for Example problems

C.1 MATLAB code: Example 1

Using randn function of MATLAB, the noise level was generated

NOISE: 0.1*randn(3,1)
0.053766713954610
0.183388501459509
-0.225884686100365

```
clear all
x1=1/3;x2=2/3;x3=1;
s=[0:0.01:20];
g1=-3/4*3^(1/2)/(3*2^(1/2)+2*3^(1/2))*pi;
g1a=g1+0.1*randn(size(g1));
g2=-9/4/(3*2^(1/2)+2*3^(1/2))*pi;
g2a=g2+0.1*randn(size(g2));
g3=-3/4*3^(1/2)/(3*2^(1/2)+2*3^(1/2))*pi;
g3a=g3+0.1*randn(size(g3));
g=[g1,g2,g3]';
ga=[g1a,g2a,g3a]';
w1=sin(s*x1);
w2=sin(s*x2);
w3=sin(s*x3);
f=s.*cos(s)-(g1*w1+g2*w2+g3*w3);
fa=s.*cos(s)-(g1a*w1+g2a*w2+g3a*w3);
plot(s/pi,f,'b',s/pi,fa,'r')
grid on
axis([0 3 -10 8])
P=[-4/3/pi -4*sqrt(2)/3/pi -4/3/pi;
    1 0 -1;
    1/2 -sqrt(3)/2 1];
b=[1;0;0];
```

C.2 MATLAB code: Example 2

Using randn function of MATLAB, the noise level was generated

NOISE: 0.1*randn(3,1)
0.053766713954610
0.183388501459509
-0.225884686100365

```
clear all
x1=1/3;x2=2/3;x3=1;
s=[0.5:0.01:18]';
g1=-0.362516921073658516;
g2=-0.755219694840847944;
g3=-0.897243428247868290;
g=[g1,g2,g3]';
w1=exp(-1/2*x1)*sin(1/2*(4*s.^2-1).^(1/2)*x1);
w2=exp(-1/2*x2)*sin(1/2*(4*s.^2-1).^(1/2)*x2);
w3=exp(-1/2*x3)*sin(1/2*(4*s.^2-1).^(1/2)*x3);
f=-1/2*exp(-1/2)*sin(1/2*(4*s.^2-1).^(1/2))...
    +1/2*exp(-1/2)*cos(1/2*(4*s.^2-1).^(1/2)).*(4*s.^2-1).^(1/2)...
    -(g(1)*w1+g(2)*w2+g(3)*w3);
fa=-1/2*exp(-1/2)*sin(1/2*(4*s.^2-1).^(1/2))...
    +1/2*exp(-1/2)*cos(1/2*(4*s.^2-1).^(1/2)).*(4*s.^2-1).^(1/2);
plot(s/pi,f,'b',s/pi,fa,'r')
grid on
P=[-0.5072243244 -0.6179648916 -0.3894409608;
    3.719586204 0.05583565952 -2.664361814;
    -1.877562190 2.744307642 -2.665839660];
```

Vita

Amit Maha was born in Hyderabad, India. He completed his schooling from Sri Vidya Senior Secondary School, Hyderabad and then finished studying at MES Indian School, Doha, Qatar. He then joined Louisiana State University, Baton Rouge, LA, in 1996 and earned a Bachelor of Science degree in Mechanical Engineering in December 2000 along with a minor in Computer Science. After his graduation, he joined the Graduate Program in the Mechanical Engineering Department at Louisiana State University and earned his degree as a Master of Science in Mechanical Engineering in December 2005 with concentration in the area of Thermo-Heat Transfer and Fluid Mechanics. He joined as a Research Associate at Southern University, Baton Rouge LA. In 2007, he joined Louisiana State University to work towards Doctor of Philosophy degree in Mechanical Engineering in the area of Mechanical Systems. He also worked for a year as an Engineer at Maxim Watermakers LLC, Shreveport, LA (from May 2012 – Aug 2013). He returned back to Louisiana State University in fall 2013 to continue with this Ph.D. degree program. Amit is expected to graduate with his Doctor of Philosophy degree in Mechanical Engineering in August 2015 and pursue a career in the multidisciplinary aspects of Mechanical Engineering involving Machine Design, Vibrations, Controls, Fluid Mechanics, Thermodynamics and Heat Transfer.