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On density-critical matroids

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Abstract

For a matroid M having m rank-one flats, the density $d(M)$ is $\frac{m}{r(M)}$ unless $m = 0$, in which case $d(M) = 0$. A matroid is density-critical if all of its proper minors of non-zero rank have lower density. By a 1965 theorem of Edmonds, a matroid that is minor-minimal among simple matroids that cannot be covered by k independent sets is density-critical. It is straightforward to show that $U_{1,k+1}$ is the only minor-minimal loopless matroid with no covering by k independent sets. We prove that there are exactly ten minor-minimal simple obstructions to a matroid being able to

be covered by two independent sets. These ten matroids are precisely the density-critical matroids M such that $d(M) > 2$ but $d(N) \leq 2$ for all proper minors N of M . All density-critical matroids of density less than 2 are series-parallel networks. For $k \geq 2$, although finding all density-critical matroids of density at most k does not seem straightforward, we do solve this problem for $k = \frac{9}{4}$.

Mathematics Subject Classifications: 05B35

1 Introduction

Our notation and terminology follow Oxley [7]. For a positive integer k , let \mathcal{M}_k be the class of matroids M for which $E(M)$ is the union of k independent sets. We say such a matroid can be *covered* by k independent sets. Edmonds [3] gave the following characterization of the members of \mathcal{M}_k .

Theorem 1. *A matroid M has k independent sets whose union is $E(M)$ if and only if, for every subset A of $E(M)$,*

$$k r(A) \geq |A|.$$

Clearly, \mathcal{M}_k is closed under deletion. However, \mathcal{M}_k is not closed under contraction. For example, the 6-element rank-3 uniform matroid $U_{3,6}$ can be covered by two independent sets, yet contracting a point of this matroid gives $U_{2,5}$, which cannot. For all k , the loop is the unique minor-minimal matroid not in \mathcal{M}_k . On that account, we limit the types of obstructions we consider. We first examine the minor-minimal loopless matroids that are not in \mathcal{M}_k . We find the following result.

Proposition 2. *The unique minor-minimal loopless matroid that cannot be covered by k independent sets is $U_{1,k+1}$.*

Restricting attention to minor-minimal simple matroids not in \mathcal{M}_k , we find much more structure. We have the following collection of ten matroids for the case when k is two. In this result, $P(M_1, M_2)$ denotes the parallel connection of matroids M_1 and M_2 , this matroid being unique when both M_1 and M_2 have transitive automorphism groups. Geometric representations of the nine of these ten matroids of rank at most four are shown in Figure 1. A diagram representing the tenth matroid, $P(M(K_4), M(K_4))$ is also given where we note that this matroid has rank five.

Theorem 3. *The minor-minimal simple matroids that cannot be covered by two independent sets are $U_{2,5}$, $P(U_{2,4}, U_{2,4})$, O_7 , P_7 , F_7^- , F_7 , $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, $M^*(K_{3,3})$, and $P(M(K_4), M(K_4))$.*

The following consequence of Theorem 1 will be helpful.

Lemma 4. *Let M be a minor-minimal matroid that cannot be covered by k independent sets. Then*

$$k r(M) = |E(M)| - 1.$$

Moreover, M has no coloops.

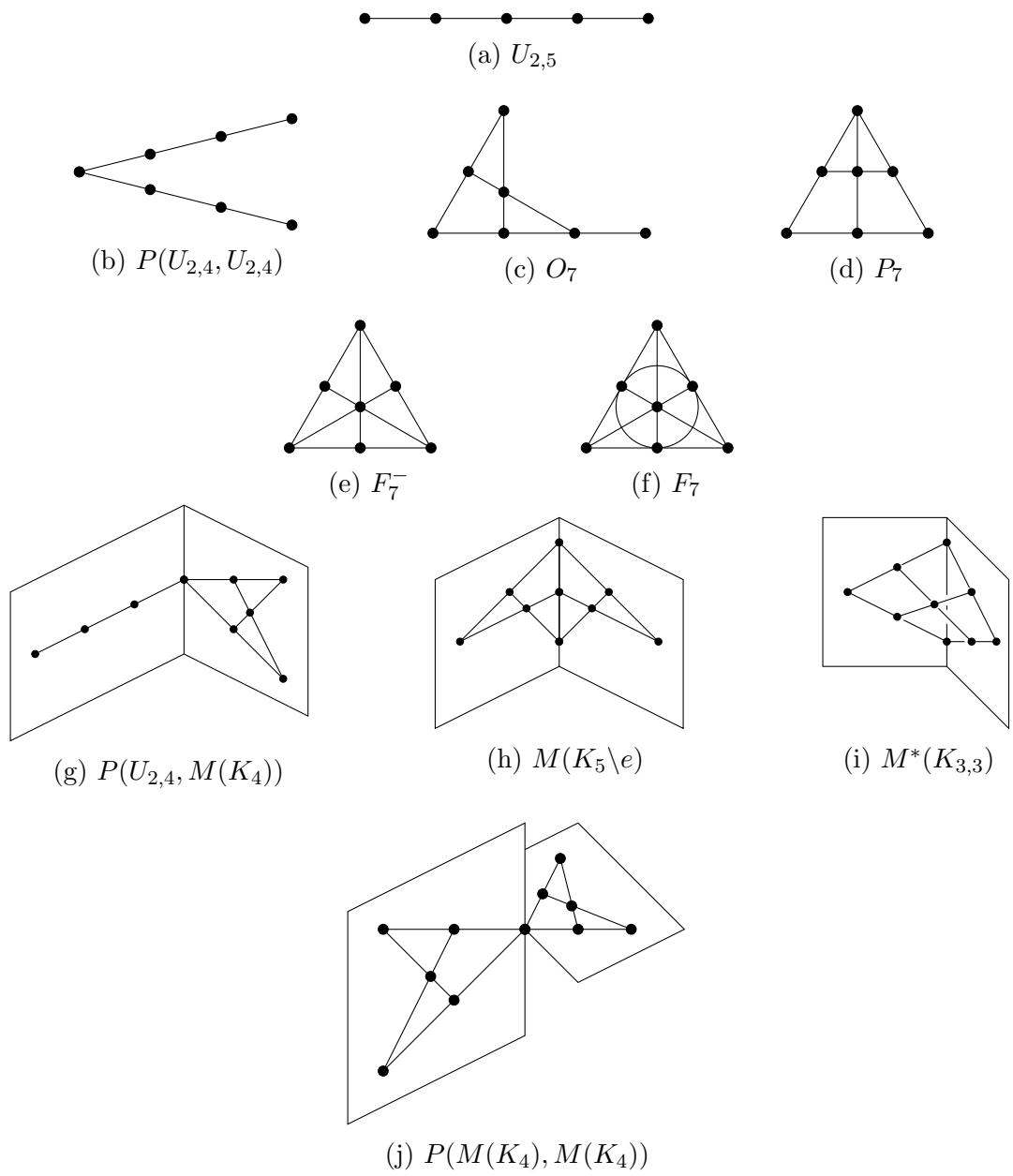


Figure 1: The minor-minimal simple matroids not in \mathcal{M}_2 .

For a matroid M , we write $\varepsilon(M)$ for $|E(\text{si}(M))|$, the number of rank-one flats of M . The *density* $d(M)$ of M is $\frac{\varepsilon(M)}{r(M)}$ unless $r(M) = 0$. In the exceptional case, $\varepsilon(M) = 0$ and we define $d(M) = 0$. We say that M is *density-critical* when $d(N) < d(M)$ for all proper minors N of M . Note that all density-critical matroids are simple. By Lemma 4 and Theorem 1, M is a minor-minimal simple matroid that cannot be covered by k independent sets if and only if $d(M) > k$ but $d(N) \leq k$ for all proper minors N of M . Such matroids are strictly k -density-critical where, for $t \geq 0$, we say a matroid is *strictly t -density-critical* when its density is strictly greater than t while all its proper minors have density at most t . Thus Theorem 3 explicitly determines all ten strictly 2-density-critical matroids.

We propose the following.

Conjecture 5. For all positive integers k , there are finitely many minor-minimal simple matroids that cannot be covered by k independent sets.

More generally, we make the following conjectures. For $t > 0$, we say a matroid is *t -density-critical* when its density is at least t while all of its proper minors have density strictly less than t .

Conjecture 6. For all $t \geq 0$, there are finitely many strictly t -density-critical matroids.

Conjecture 7. For all $t > 0$, there are finitely many t -density-critical matroids.

We also propose the following weakening of the last conjecture.

Conjecture 8. For all $t \geq 0$, there are finitely many density-critical matroids with density exactly t .

We note that these conjectures hold over any class of matroids that is well-quasi-ordered with respect to minors. In particular, by a result announced by Geelen, Gerards, and Whittle (see, for example, [4]), these conjectures hold within the class of matroids representable over a fixed finite field.

Because the two excluded minors for series-parallel networks, $U_{2,4}$ and $M(K_4)$, have density exactly two, for $k < 2$, all density-critical matroids of density at most k are series-parallel networks. For $k > 2$, finding all density-critical matroids of density at most k does not seem straightforward. However, we were able to solve this problem when $k = \frac{9}{4}$. For all $n \geq 2$, we denote by P_n any matroid that can be constructed from n copies of $M(K_3)$ via a sequence of $n - 1$ parallel connections. In particular, $P_2 \cong M(K_4 \setminus e)$. There are two choices for P_3 depending on which element of $M(K_4 \setminus e)$ is used as the basepoint of the parallel connection with the third copy of $M(K_3)$. We denote by M_{18} the 18-element matroid that is obtained by attaching, via parallel connection, a copy of $M(K_4)$ at each element of an $M(K_3)$.

Theorem 9. *The following is a list of all pairs (M, d) where M is a density-critical matroid of density d and $d \leq \frac{9}{4}$: $(U_{1,1}, 1)$, $(U_{2,3}, \frac{3}{2})$, $(M(P_n), \frac{2n+1}{n+1})$ for all $n \geq 2$, $(U_{2,4}, 2)$, $(M(K_4), 2)$, $(P(M(K_4), M(K_4)), \frac{11}{5})$, $(P(U_{2,4}, M(K_4)), \frac{9}{4})$, $(M(K_5 \setminus e), \frac{9}{4})$, $(M^*(K_{3,3}), \frac{9}{4})$, $(M_{18}, \frac{9}{4})$.*

2 Preliminaries

This section proves some preliminary results beginning with two that were stated in the introduction.

Proof of Proposition 2. Clearly, $U_{1,k+1}$ is a minor-minimal loopless matroid that cannot be covered by k independent sets. Conversely, suppose that M is a minor-minimal loopless matroid that cannot be covered by k independent sets. Certainly, M contains some element e . Let $P \cup \{e\}$ be the parallel class of M that contains e where $P = \{e_1, e_2, \dots, e_\ell\}$ and $e \notin P$. Now $M/e \setminus P$ is loopless, so, by minimality, $M/e \setminus P$ can be covered by k independent sets $\{A_1, A_2, \dots, A_k\}$. Note that each $A_i \cup \{e\}$ is independent in M , so if $|P| = \ell \leq k - 1$, then $\{A_1 \cup \{e_1\}, A_2 \cup \{e_2\}, \dots, A_\ell \cup \{e_\ell\}, A_{\ell+1} \cup \{e\}, \dots, A_k \cup \{e\}\}$ is a set of k independent sets that covers M . Thus $|P| \geq k$, and so $M \cong U_{1,k+1}$. \square

Since $U_{1,k+1}$ is a $(k + 1)$ -element cocircuit, the matroids having no $U_{1,k+1}$ -minor are precisely the matroids for which every cocircuit has at most k elements.

Proof of Lemma 4. Take x in $E(M)$. Then $M \setminus x$ can be covered by k independent sets. Thus, by Theorem 1,

$$|E(M)| > kr(M) \geq kr(M \setminus x) \geq |E(M \setminus x)| = |E(M)| - 1.$$

We deduce that $kr(M) = |E(M)| - 1$ and $r(M) = r(M \setminus x)$ so M has no coloops. \square

Lemma 10. *Let M be a density-critical matroid of rank at least two. For each subset S of $E(M)$,*

$$|E(M)| - \varepsilon(M/S) > d(M)r(S).$$

In particular, every element of M is in a triangle and is in at least two triangles when $d(M) \geq 2$.

Proof. Since M is density-critical and therefore simple,

$$\frac{\varepsilon(M/S)}{r(M/S)} < \frac{\varepsilon(M)}{r(M)} = \frac{|E(M)|}{r(M)}.$$

Hence $r(M)\varepsilon(M/S) < |E(M)|(r(M) - r(S))$, so

$$r(M)d(M)r(S) = |E(M)|r(S) < r(M)(|E(M)| - \varepsilon(M/S)).$$

Thus $d(M)r(S) < |E(M)| - \varepsilon(M/S)$. In particular, $d(M) < |E(M)| - \varepsilon(M/e)$ for all e in $E(M)$. Hence every such element e is in at least one triangle, and e is in at least two triangles when $d(M) \geq 2$. \square

The next result will be useful in the proof of Theorem 3.

Lemma 11. *Let F be a $2k$ -element set $\{b_1, a_1, b_2, a_2, \dots, b_k, a_k\}$ in a 3-connected matroid M . Suppose $\{b_1, b_2, \dots, b_k\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a circuit for all i , where $b_{k+1} = b_1$. Then $M|F$ is a wheel of rank at least three or a whirl of rank at least two.*

Proof. Since M is 3-connected with at least four elements, it is simple. Now $M|F$ has $\{a_i, b_{i+1}, a_{i+1}\}$ as a triad, where $a_{k+1} = a_1$. By a result of Seymour [8] (see also [7, Lemma 8.8.5(ii)]), $M|F$ is a wheel or a whirl of rank k . \square

3 The matroids that cannot be covered by two independent sets

In this section, we prove Theorem 3, first restating it for convenience.

Theorem 12. *The minor-minimal simple matroids that cannot be covered by two independent sets are $U_{2,5}$, $P(U_{2,4}, U_{2,4})$, O_7 , P_7 , F_7^- , F_7 , $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, $M^*(K_{3,3})$, and $P(M(K_4), M(K_4))$.*

Proof. It is straightforward to check that each of the matroids listed is a minor-minimal simple matroid that cannot be covered by two independent sets. Now let M be such a matroid. The next two assertions are immediate consequences of Lemmas 4, 10, and Theorem 1. However, we include proofs independent of Edmonds's result for completeness.

12.1. *Every element of M is contained in at least two triangles.*

Let e be an element of M and let $M' = \text{si}(M/e)$. By minimality, M' has a partition into two independent sets A and B . Suppose e is not in a triangle. Then $E(M') = E(M) - \{e\}$ and we have $r_M(A \cup \{e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = |B| + 1$, so M is covered by the independent sets $A \cup \{e\}$ and $B \cup \{e\}$, which is a contradiction.

Now suppose e is in exactly one triangle $\{e, c, d\}$ of M . We may assume that $M' = M/e \setminus c$ and that $d \in A$. Then $r_M(A \cup \{c\}) = r_M(A \cup \{c, e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = r_{M'}(B) + 1 = |B| + 1$, so M is covered by the independent sets $A \cup \{c\}$ and $B \cup \{e\}$. This contradiction implies that 12.1 holds.

12.2. *$|E(M)| \leq 2r(M) + 1$ and $|A| \leq 2r(A)$ for every proper subset A of $E(M)$.*

Suppose A is a proper subset of $E(M)$. By the minimality of M , we can cover $M|A$ by two independent sets, and so $|A| \leq 2r(A)$. It follows easily that $|E(M)| \leq 2r(M) + 1$. Thus 12.2 holds.

We construct a simple auxiliary graph G from M , the vertices of which are the elements of M ; two such vertices are adjacent exactly when they share a triangle in M . Next, we show the following.

12.3. *Let Z be the vertex set of a component of G . Then $M|Z$ has a wheel or a whirl as a restriction.*

We may assume that $M|Z$ has no line with four or more points otherwise M has a rank-2 whirl as a restriction. For b_1 in Z , by 12.1, we can construct a maximal sequence $b_1, a_1, b_2, a_2, \dots, b_n$ of distinct elements such that $\{b_1, b_2, \dots, b_n\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a triangle for all i in $\{1, 2, \dots, n-1\}$. Then $n \geq 3$.

Now M has triangles $\{b_n, a_n, b_{n+1}\}$ and $\{b_0, a_0, b_1\}$ that differ from $\{b_{n-1}, a_{n-1}, b_n\}$ and $\{b_1, a_1, b_2\}$, respectively. Let $A' = \{b_1, a_1, b_2, a_2, \dots, b_{n-1}, a_{n-1}, b_n\}$. Assume that both $\{a_n, b_{n+1}\}$ and $\{a_0, b_0\}$ avoid A' . Then $|A' \cup \{a_n, b_{n+1}\}| = 2n + 1 = 2r(A' \cup \{a_n, b_{n+1}\}) + 1$. Thus, by 12.2, $A' \cup \{a_n, b_{n+1}\} = E(M)$. By symmetry, $A' \cup \{a_0, b_0\} = E(M)$. Hence $\{a_n, b_{n+1}\} = \{a_0, b_0\}$, so $\{b_n, a_n, b_{n+1}, b_1\}$ is a 4-point line, a contradiction.

We may now assume that b_{n+1} is a member c_i of $\{b_i, a_i\}$ for some i with $1 \leq i \leq n-1$. Then $\{c_i, b_{i+1}, b_{i+2}, \dots, b_n\}$ is an independent set in $M|Z$ such that every two consecutive elements in the given cyclic order are in a triangle. Thus, by Lemma 11, $M|Z$ has a wheel or whirl of rank $n - i + 1$ as a restriction. Hence 12.3 holds.

12.4. For some component of G having vertex set Z , the matroid $M|Z$ is not a wheel or a whirl.

Assume that this fails. Then, by 12.1, the only components of G are rank-2 whirls or rank-3 wheels. Assume there are s of the former and t of the latter. Then $|E(M)| = 4s + 6t = 2(2s + 3t)$. Clearly $r(M) \leq 2s + 3t$. By 12.2, equality must hold here. Hence each component of G corresponds to a wheel or whirl component of M . As each wheel and each whirl can be covered by two independent sets, so too can M , a contradiction. Thus 12.4 holds.

Now take a component of G having vertex set Z such that $M|Z$ is not a wheel or a whirl. By 12.3, consider a wheel or whirl restriction of $M|Z$ with basis $B = \{b_1, b_2, \dots, b_n\}$ and ground set $W = \{b_1, a_1, b_2, a_2, \dots, b_n, a_n\}$. Let $\{b_i, a_i, b_{i+1}\}$ be a triangle for all i where $b_{n+1} = b_1$. As $W \neq Z$, there is a point β_1 in W that is contained in a triangle $\{\beta_1, \alpha_1, \beta_2\}$ that is not a triangle of $M|W$. If $M|W$ is a rank-2 whirl or a rank-3 wheel, then, by symmetry, we may assume that $\beta_1 = a_1$. If, instead, $M|W$ is neither a rank-2 whirl nor a rank-3 wheel, then 12.1 guarantees that such a triangle $\{\beta_1, \alpha_1, \beta_2\}$ exists with $\beta_1 = a_1$. By repeatedly using 12.1, we can construct a sequence $\beta_1, \alpha_1, \dots, \beta_{m+1}$ where $\{\beta_i, \alpha_i, \beta_{i+1}\}$ is a triangle for all i in $\{1, 2, \dots, m\}$ and $B \cup \{\beta_2, \dots, \beta_{m+1}\}$ is dependent but $B \cup \{\beta_2, \dots, \beta_m\}$ is independent. By potentially interchanging α_m and β_{m+1} , we may assume that $\alpha_m \notin W$. Let $Q = \{\beta_1, \alpha_1, \dots, \beta_{m+1}\}$. Then

$$r(W \cup Q) = r(W \cup (Q - \{\beta_{m+1}\})) = n + m - 1. \quad (1)$$

As $|W \cup (Q - \{\beta_{m+1}\})| = 2(n + m - 1) + 1 = 2r(W \cup (Q - \{\beta_{m+1}\})) + 1$, we deduce, by 12.2, that

$$W \cup (Q - \{\beta_{m+1}\}) = E(M). \quad (2)$$

Hence

$$\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\}). \quad (3)$$

Assume that the theorem fails. We now show that

12.5. $M|Z$ has no wheel-restriction of rank exceeding three and no whirl-restriction of rank exceeding two.

Assume that this fails. Then we may assume that $M|W$ is a wheel of rank at least four or a whirl of rank at least three. Now $r(W) = n$ and $r(Q) \leq m + 1$. By (1) and submodularity, $r(\text{cl}(W) \cap \text{cl}(Q)) \leq 2$. Assume W does not span M . Then, by (1) and (2), we see that $m > 1$ and the only possible elements of W that can lie in triangles with elements of $Q - W$ are β_1 and β_{m+1} . But a wheel of rank at least four and a whirl of rank at least three have at least three elements that are in unique triangles. Hence one of these elements will violate 12.1.

We now know that W spans M , so the unique element of $Q - W$ is α_1 . Each of a_1, a_2, \dots, a_n must be in a triangle with α_1 , the other element of which is in W . Assume both $\{a_1, \alpha_1, a_3\}$ and $\{a_1, \alpha_1, a_{n-1}\}$ are triangles. Then $n = 4$. Suppose $\{a_2, \alpha_1, a_4\}$ is also a triangle. Then, by Lemma 11, for each i in $\{2, 4\}$, deleting a_i from $M|(W \cup Q)$ gives a wheel or whirl of rank four. As $\{b_1, b_4, \alpha_1, a_2\}$ and $\{b_2, b_3, \alpha_1, a_4\}$ are circuits, both of

these deletions are wheels. It follows that $M|(W \cup Q) \cong M^*(K_{3,3})$, so $M \cong M^*(K_{3,3})$, a contradiction. Thus, we may assume that $\{a_2, \alpha_1, a_4\}$ is not a triangle. Since $\alpha_1 \notin \text{cl}(\{b_1, b_2, b_3\}) \cup \text{cl}(\{b_2, b_3, b_4\})$, there is no triangle containing $\{a_2, \alpha_1\}$, a contradiction.

We may now assume that $\{a_1, \alpha_1, a_3\}$ is not a triangle. Then, by 12.1, W has distinct elements x and y such that $\{a_1, \alpha_1, x\}$ and $\{a_3, \alpha_1, y\}$ are triangles. Thus $\{a_1, a_3, x, y\}$ contains a circuit. Now $\{a_1, a_3\}$ is not in a triangle of $M|W$. Moreover, if $\{a_1, x, y\}$ is a triangle, then $\{x, y\} = \{b_1, b_2\}$. Using the triangles, $\{a_1, \alpha_1, x\}$ and $\{a_3, \alpha_1, y\}$, we deduce that $a_3 \in \text{cl}(\{b_1, b_2\})$, a contradiction. It follows that $\{a_1, a_3, x, y\}$ is a circuit of M . Thus $M|W$ is either a rank-3 whirl or a rank-4 wheel.

Suppose $M|W$ is a rank-3 whirl. Then M is an extension of this matroid by α_1 in which every element is in at least two triangles. If $\{a_1, a_2, \alpha_1\}$ or $\{a_2, a_3, \alpha_1\}$ is a triangle, then one easily checks that $M \cong O_7$ or $M \cong P_7$, a contradiction. Hence we may assume that none of $\{a_1, a_2, \alpha_1\}$, $\{a_2, a_3, \alpha_1\}$, or $\{a_3, a_1, \alpha_1\}$ is a triangle. Then, to avoid having $U_{2,5}$ as a minor of M , we must have $\{a_1, b_3, \alpha_1\}$, $\{a_2, b_1, \alpha_1\}$, and $\{a_3, b_2, \alpha_1\}$ as triangles, that is, $M \cong F_7^-$, a contradiction.

We are left with the possibility that $M|W$ is a rank-4 wheel. Since it has $\{a_1, a_3, x, y\}$ as a circuit, it follows that $\{x, y\} = \{a_2, a_4\}$. Then M has either $\{a_1, a_2, \alpha_1\}$ and $\{a_3, a_4, \alpha_1\}$ as triangles or $\{a_1, a_4, \alpha_1\}$ and $\{a_2, a_3, \alpha_1\}$ as triangles. By symmetry, we may assume that we are in the second case. Then, by submodularity using the sets $\{b_1, b_2, a_1, a_4, b_4, \alpha_1\}$ and $\{b_2, b_3, a_2, a_3, b_4, \alpha_1\}$, we deduce that $r(\{b_2, b_4, \alpha_1\}) = 2$. It follows that $M \cong M(K_5 \setminus e)$, a contradiction. We conclude that 12.5 holds.

Now suppose that W spans Z . If $M|W$ is a rank-2 whirl, then $M|Z \cong U_{2,5}$, a contradiction. If $M|W$ is a rank-3 wheel, then one easily checks that $M|Z$ is isomorphic to one of O_7 , F_7^- , or F_7 , a contradiction.

We may now assume that W does not span Z . Then $m > 1$. By (3), $\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\})$. We will first suppose that $\beta_{m+1} = \beta_i$ for some i in $\{1, 2, \dots, m\}$. Then $\{\beta_i, \beta_{i+1}, \dots, \beta_m\}$ is an independent set and $\{\beta_j, \alpha_j, \beta_{j+1}\}$ is a triangle for all j in $\{i, i+1, \dots, m\}$. By 12.5 and Lemma 11, for $R = \{\beta_i, \alpha_i, \beta_{i+1}, \alpha_{i+1}, \dots, \beta_m, \alpha_m\}$, the matroid $M|R$ is a rank-3 wheel or a rank-2 whirl. Then the matroid obtained from $M|Z$ by contracting $\{\alpha_2, \alpha_3, \dots, \alpha_{i-1}\}$ and simplifying is the parallel connection of $M|W$ and $M|R$, that is, $M|Z$ has as a minor one of $P(U_{2,4}, U_{2,4})$, $P(U_{2,4}, M(K_4))$, and $P(M(K_4), M(K_4))$, a contradiction.

Finally, suppose that $\beta_{m+1} \notin \{\beta_1, \beta_2, \dots, \beta_m\}$. Then β_{m+1} is α_i for some $i \geq 1$, or $\beta_{m+1} \in W$. Consider the first case and take $\alpha_{m+1} = \beta_i$. Then, by 12.5 and Lemma 11, with $R = \{\beta_{i+1}, \alpha_{i+1}, \dots, \beta_{m+1}, \alpha_{m+1}\}$, we have that $M|R$ is a rank-3 wheel or a rank-2 whirl. Contracting $\{\alpha_2, \alpha_3, \dots, \alpha_{i-1}\}$ from $M|Z$ and simplifying, we obtain one of $P(U_{2,4}, U_{2,4})$, $P(U_{2,4}, M(K_4))$, and $P(M(K_4), M(K_4))$, a contradiction. In the second case, when $\beta_{m+1} \in W$, we recall that $\beta_1 = a_1$. Suppose that $\{\beta_1, \beta_{m+1}\}$ is not in a triangle of $M|W$. Then $M|W \cong M(K_4)$ and $\beta_{m+1} = b_3$. By assumption, $\{b_1, b_2, b_3\} \cup \{\beta_2, \dots, \beta_m\}$ is independent. By Lemma 11, the triangles $\{b_1, b_2, a_1\}$, $\{a_1, \alpha_1, \beta_2\}$, \dots , $\{\beta_m, \alpha_m, b_3\}$, $\{b_3, a_3, b_1\}$ imply that $M|Z$ has a wheel or whirl of rank at least four as a restriction, a contradiction. We deduce that $\{\beta_1, \beta_{m+1}\}$ is in a triangle of $M|W$. Then, by symmetry, we may assume that $\beta_{m+1} = b_1$. We let $\alpha_{m+1} = b_2$. Then, for $R = \{\beta_1, \alpha_1, \dots, \beta_{m+1}, \alpha_{m+1}\}$,

we have that $M|R$ is a rank-3 wheel or a rank-2 whirl. But $\alpha_1 \notin \text{cl}(W)$, so $M|R$ is a rank-3 wheel. If $M|W$ is a rank-2 whirl, then O_7 is a restriction of $M|Z$, a contradiction. If $M|W$ is a rank-3 wheel, then $M|(W \cup R)$ has rank four and consists of two copies of $M(K_4)$ sharing a triangle. This matroid is $M(K_5 \setminus e)$, a contradiction. \square

4 The density-critical matroids of small density

In this section, we prove Theorem 9. The following result [6] (see also [7, Lemma 4.3.10]) will be used repeatedly in this proof.

Lemma 13. *In a connected matroid M with at least two elements, let $\{e_1, e_2, \dots, e_m\}$ be a cocircuit of M such that M/e_i is disconnected for all i in $\{1, 2, \dots, m-1\}$. Then $\{e_1, e_2, \dots, e_{m-1}\}$ contains a 2-circuit of M .*

We shall make repeated use of the following consequence of this lemma.

Corollary 14. *Let M be a simple connected matroid and Z be a non-empty subset of $E(M)$. Then M has a simple connected minor N such that $N|Z = M|Z$ and $r(N) = r_M(Z)$.*

Proof. We may assume that Z is non-spanning, otherwise we can take N to be M . Let C^* be a cocircuit of M that is disjoint from $\text{cl}(Z)$. As M is simple, it follows by Lemma 13 that there is an element e of C^* such that M/e is connected. Since $e \notin \text{cl}(Z)$, we see that $(M/e)|Z = M|Z$. Clearly we can label $\text{si}(M/e)$ so that its ground set contains Z . If $r(M) - r(Z) = 1$, then we take $N = \text{si}(M/e)$. Otherwise we repeat the above process using $\text{si}(M/e)$ in place of M . After $r(M) - r(Z)$ applications of this process, we obtain the desired minor N . \square

The next result, which was proved by Dirac [2], follows easily by induction after recalling that a connected matroid with no minor isomorphic to $U_{2,4}$ or $M(K_4)$ is isomorphic to the cycle matroid of a series-parallel network.

Lemma 15. *Let M be a simple matroid having no minor isomorphic to $U_{2,4}$ or $M(K_4)$. Then*

$$|E(M)| \leq 2r(M) - 1.$$

We omit the elementary proof of the next result a consequence of which is that every density-critical matroid is connected.

Lemma 16. *Let M_1 and M_2 be matroids of rank at least one. Then*

$$d(M_1 \oplus M_2) \leq \max\{d(M_1), d(M_2)\}.$$

Moreover, equality holds here if and only if $d(M_1) = d(M_2)$.

The next result will be useful in identifying the density-critical matroids of density at most two.

Lemma 17. *Let M be a density-critical matroid with $d(M) \leq 2$. If (X_1, X_2) is a 2-separation of M , then there is an element p in $\text{cl}(X_1) \cap \text{cl}(X_2)$, and $M = P(M|(X_1 \cup \{p\}), M|(X_2 \cup \{p\}))$.*

Proof. As (X_1, X_2) is a 2-separation of M , for some element q not in $E(M)$, we can write M as $M_1 \oplus_2 M_2$ where each M_i has ground set $X_i \cup \{q\}$. Let $|E(M_i)| = n_i$ and $r(M_i) = r_i$. Assume that both M_1 and M_2 are simple. Then $\frac{|E(M)|}{r(M)} > \frac{|E(M_1)|}{r(M_1)}$, so

$$\frac{n_1 + n_2 - 2}{r_1 + r_2 - 1} > \frac{n_1}{r_1}.$$

Hence

$$r_1 n_2 - 2r_1 > r_2 n_1 - n_1.$$

By symmetry,

$$r_2 n_1 - 2r_2 > r_1 n_2 - n_2.$$

Adding the last two inequalities gives $n_1 + n_2 > 2(r_1 + r_2)$, so $n_i > 2r_i$ for some i . Thus $d(M_i) > 2$. Since M is density-critical with density at most two, this is a contradiction. We conclude that M_1 or M_2 , say M_1 , is non-simple. Thus it has an element p in parallel with the basepoint q of the 2-sum. Hence $M = P(M|(X_1 \cup \{p\}), M|(X_2 \cup \{p\}))$. \square

Lemma 18. *Let N be a simple connected matroid in which all but at most one element is in at least two triangles. Then N has no 2-cocircuits. Moreover, if N has $\{a, b, c\}$ as a triad, then either*

- (i) $\{a, b, c\}$ is contained in a 4-point line and $N = P(U_{2,4}, N \setminus \{a, b, c\})$; or
- (ii) N has a triangle $\{x, y, z\}$ such that $N|_{\{a, b, c, x, y, z\}} \cong M(K_4)$ and N is the generalized parallel connection of $N|_{\{a, b, c, x, y, z\}}$ and $N \setminus \{a, b, c\}$ across the triangle $\{x, y, z\}$.

Proof. As N has at most one element that is not in at least two triangles, N has no 2-cocircuits. Suppose $\{a, b, c\}$ is a triad of N . If $\{a, b, c\}$ is also a triangle, then $\{a, b, c\}$ is 2-separating in N . Moreover, $\{a, b, c\}$ is contained in a 4-point line $\{a, b, c, d\}$ and (i) holds.

We may now assume that $\{a, b, c\}$ is not a triangle of N . Then, because at least two of a, b , and c are in at least two triangles, the hyperplane $E(N) - \{a, b, c\}$ of N contains distinct elements x, y , and z such that $\{a, b, z\}$, $\{a, y, c\}$, and $\{x, b, c\}$ are triangles. Now

$$\begin{aligned} r(\{x, y, z\}) &\leq r(E(N) - \{a, b, c\}) + r(\text{cl}(\{a, b, c\})) - r(N) \\ &= r(N) - 1 + 3 - r(N) = 2. \end{aligned}$$

Thus $\{x, y, z\}$ is a triangle of N and $N|_{\{a, b, c, x, y, z\}} \cong M(K_4)$. It follows by a result of Brylawski [1] (see also [7, Proposition 11.4.15]) that (ii) holds. \square

Corollary 19. *Let N be a simple connected matroid in which all but at most one element is in at least two triangles and $d(N) \leq \frac{9}{4}$. If $r(N) = 2$, then $N \cong U_{2,4}$. If $r(N) = 3$, then $N \cong M(K_4)$. If $r(N) = 4$, then $N \cong P(U_{2,4}, M(K_4)), M(K_5 \setminus e)$, or $M^*(K_{3,3})$.*

Proof. We omit the straightforward proof for the case when $r(N) \in \{2, 3\}$. Assume $r(N) = 4$. By Lemma 18, N has no 2-cocircuits. Now suppose N has $\{a, b, c\}$ as a triad. If (i) of Lemma 18 holds, then $N = P(U_{2,4}, N \setminus \{a, b, c\})$. By the result in the rank-3 case, $N \setminus \{a, b, c\} \cong M(K_4)$, so $N \cong P(U_{2,4}, M(K_4))$. If, instead, (ii) of Lemma 18 holds, then N is the generalized parallel connection across a triangle $\{x, y, z\}$ of $M(K_4)$ and $N \setminus \{a, b, c\}$. In the latter, $E(N \setminus \{a, b, c, x, y, z\})$ must be a triad of N , so $N \setminus \{a, b, c\} \cong M(K_4)$. Hence N is the generalized parallel connection across a triangle of two copies of $M(K_4)$, so $N \cong M(K_5 \setminus e)$.

We may now assume that N has no triads. Then every cocircuit of N has at least four elements. As N certainly has a plane that contains two intersecting triangles, $\{x, f_1, g_1\}$ and $\{x, f_2, g_2\}$, we deduce that $|E(N)| \geq 9$, so $|E(N)| = 9$. Let $\{a, b, c, d\}$ be the cocircuit $E(N) - \{x, f_1, f_2, g_1, g_2\}$. Because N has no plane with more than five points and has all but at most one element in two triangles, we may assume that $\{a, b, g_1\}$ and $\{a, c, g_2\}$ are triangles of N . Then $N \setminus d$ has $\{x, f_1, g_1\}, \{g_1, b, a\}, \{a, c, g_2\}, \{g_2, f_2, x\}$ as triangles. By Lemma 11, $N \setminus d$ is a rank-4 wheel or whirl. In this matroid, f_1, b, c , and f_2 are in unique triangles. It follows that N must have $\{d, f_1, c\}$ and $\{d, b, f_2\}$ as triangles. Thus $N \setminus d$ is a rank-4 wheel. Likewise, $N \setminus f_1$ and $N \setminus c$ are also rank-4 wheels, so $N \cong M^*(K_{3,3})$. \square

Lemma 20. *Let N be a simple matroid of rank at least three in which every element is in at least two triangles. Suppose $e \in E(N)$. Then*

- (i) e is in a plane of N having at least seven points; or
- (ii) every element of $\text{si}(N/e)$ is in at least two triangles; or
- (iii) N has a $U_{2,4}$ - or $M(K_4)$ -restriction using e .

Proof. Assume that neither (i) nor (iii) holds. We show that every element of $\text{si}(N/e)$ is in at least two triangles. First consider a triangle $\{e, c_1, c_2\}$ of N containing e . Let $\{c_1, d_1, f_1\}$ and $\{c_2, d_2, f_2\}$ be triangles of N where neither contains e . If $r(\{e, c_1, d_1, f_1, c_2, d_2, f_2\}) = 4$, then, in $\text{si}(N/e)$, the element c corresponding to c_1 and c_2 is in at least two triangles. Now suppose $r(\{e, c_1, d_1, f_1, c_2, d_2, f_2\}) = 3$. Since N has no plane with more than six points, we may assume that $f_1 = f_2$. Rename this element f . If $\{e, d_1, d_2\}$ is not a triangle, then $\text{si}(N/e)$ has a 4-point line containing c , so c is in at least two triangles of this matroid. If $\{e, d_1, d_2\}$ is a triangle of N , then $N \setminus \{e, c_1, c_2, d_1, d_2, f\} \cong M(K_4)$, a contradiction.

Now let f be an element of N that is not in a triangle with e . Let $\{f, g_1, h_1\}$ and $\{f, g_2, h_2\}$ be triangles of N . Then $\text{si}(N/e)$ has at least two triangles containing f otherwise $N \setminus \{e, f, g_1, g_2, h_1, h_2\} \cong M(K_4)$, a contradiction. \square

Recall that M_{18} is the 18-element matroid that is obtained by attaching, via parallel connection, a copy of $M(K_4)$ at each element of an $M(K_3)$.

Lemma 21. *Let N be a simple connected non-empty matroid in which every element is in a $U_{2,4}$ - or $M(K_4)$ -restriction. Assume that $d(N) \leq \frac{9}{4}$ but $d(N') < \frac{9}{4}$ for all proper minors N' of N . Then N is isomorphic to $U_{2,4}, M(K_4), P(U_{2,4}, M(K_4)), P(M(K_4), M(K_4)), M(K_5 \setminus e)$, or M_{18} .*

Proof. Since $d(N') \leq \frac{9}{4}$ for all minors N' of N , we see that, in any such N' , no line has more than four points and no plane has more than six points. Next we show the following.

21.1. *If N has a 4-point line, then N is isomorphic to $U_{2,4}$ or $P(U_{2,4}, M(K_4))$.*

This is immediate if $r(N) = 2$. Because N has no plane with more than six points, $r(N) \neq 3$. Let L be a 4-point line of N and let Z be a subset of $E(N)$ not containing L such that $N|Z$ is isomorphic to $U_{2,4}$ or $M(K_4)$. If $L \cap Z \neq \emptyset$, then again, since N has no plane with more than six points, we deduce that $N \cong P(U_{2,4}, M(K_4))$. We may now assume that $L \cap Z = \emptyset$. If $r(L \cup Z) \leq r(Z) + 1$, then N has a rank-3 or rank-4 restriction of density exceeding $\frac{9}{4}$, a contradiction. We deduce that $r(L \cup Z) = r(Z) + 2$.

By Corollary 14, N has a simple connected minor N' such that $N'|(L \cup Z) = N|(L \cup Z)$ and $r(N') = r(Z) + 2$. As N' is connected, it has an element x' that is not in the closure of L or of Z . Then N'/x' has $N|L$ and $N|Z$ as restrictions and has rank $r(Z) + 1$. Thus $\text{si}(N'/x')$ has either a plane with more than six points or has $P(U_{2,4}, M(K_4))$ as a restriction. Each possibility yields a contradiction, so 21.1 holds.

We may now assume that every element of N is in an $M(K_4)$ -restriction. We may also assume that N is not isomorphic to $M(K_4)$ or $P(M(K_4), M(K_4))$. Next we show the following.

21.2. *Let X and Y be distinct subsets of $E(N)$ such that both $N|X$ and $N|Y$ are isomorphic to $M(K_4)$. If $|X \cap Y| \geq 2$, then $N \cong M(K_5 \setminus e)$.*

Since N has no plane with more than six points, $r(X \cup Y) > 3$. As $|X \cap Y| \geq 2$, it follows by submodularity that $r(X \cup Y) = 4$ and $r(X \cap Y) = 2$. As $d(N|(X \cup Y)) \leq \frac{9}{4}$, we deduce that $|X \cup Y| = 9$, so $|X \cap Y| = 3$ and $N = N|(X \cup Y)$. Moreover, $N|X$ and $N|Y$ meet in a triangle Δ . By Lemma 18, N is the generalized parallel connection of $N|X$ and $N|Y$ across Δ . Thus $N \cong M(K_5 \setminus e)$ as each of $N|X$ and $N|Y$ is isomorphic to $M(K_4)$, so 21.2 holds.

We may now assume that $E(N)$ has at least three distinct subsets X with $N|X \cong M(K_4)$ and that no two such subsets meet in more than one element.

21.3. *N does not have $P(M(K_4), M(K_4))$ as a restriction.*

Assume that $N|X \cong P(M(K_4), M(K_4))$ and $N|Y \cong M(K_4)$ where $Y \not\subseteq X$. Suppose $|X \cap Y| = k$ where $k \in \{1, 2\}$. Then $r(X \cup Y) \leq 8 - k$ and $|X \cup Y| = 17 - k$, so

$$\frac{9}{4} \geq d(N|(X \cup Y)) \geq \frac{17 - k}{8 - k}.$$

Simplifying we obtain the contradiction that $4 \geq 5k \geq 5$. We deduce using 21.2 that $|X \cap Y| = 0$. Then $r(X \cup Y) = 8$ otherwise $d(N|(X \cup Y)) > \frac{9}{4}$.

By Corollary 14, N has a simple connected minor N' such that $N'|(X \cup Y) = N|(X \cup Y)$ and $r(N') = 8$. As $N|(X \cup Y)$ is disconnected, N' must contain an element that is not in $X \cup Y$. Hence $|E(N')| \geq 18$, so $d(N') \geq \frac{9}{4}$. Thus $N' = N$ and $|E(N)| = 18$, so N has a single element z that is not in $X \cup Y$. The $M(K_4)$ -restriction of N that contains z is forced to have more than one element in common with Y or one of the $M(K_4)$ -restrictions of $N|X$. This contradiction to 21.2 completes the proof of 21.3.

We now know that any two $M(K_4)$ -restrictions of N have disjoint ground sets. Let X , Y , and Z be distinct subsets of $E(N)$ such that each of $N|X$, $N|Y$, and $N|Z$ is isomorphic to $M(K_4)$. Next we show the following.

21.4. $r(X \cup Y) = 6$. Moreover, $r(X \cup Y \cup Z) = 9$ unless $N \cong M_{18}$.

As $|X \cup Y| = 12$ and $d(N|(X \cup Y)) < \frac{9}{4}$, we deduce that $r(X \cup Y) = 6$. The density constraint also means that $r(X \cup Y \cup Z) \geq 8$. Suppose $r(X \cup Y \cup Z) = 8$. Then $d(N|(X \cup Y \cup Z)) = \frac{9}{4}$, so $N = N|(X \cup Y \cup Z)$. Now $r(N/Z) = 5$. As $\frac{12}{5} > \frac{9}{4}$, we must have some parallel elements in N/Z . As Z is skew to each of X and Y , we know that $(N/Z)|X = N|X$ and $(N/Z)|Y = N|Y$. Thus there must be elements x of X and y of Y that are parallel in N/Z . If there is a second such parallel pair, then $r(N/Z) \leq 4$, a contradiction. In N , we see that $r(Z \cup \{x, y\}) = 4$. Hence, in N/x , we obtain a 7-point plane $Z \cup y$ unless $\{x, y, z\}$ is a triangle of N for some z in Z . Observe that each of N/x , N/y , and N/z is disconnected, so N is obtained from $M(K_3)$ by attaching a copy of $M(K_4)$ via parallel connection at each element. Thus $N \cong M_{18}$ and 21.4 holds.

By Corollary 14, N has a simple connected minor N' of rank 9 such that $N'|(X \cup Y \cup Z) = N|(X \cup Y \cup Z)$. As N' is connected, there is an element g of $E(N') - (X \cup Y \cup Z)$. Since N' has no plane with more than six points, g is not in the closure of any of X , Y , or Z in N' . As N'/g has rank 8 but has density less than $\frac{9}{4}$, the eighteen elements of $X \cup Y \cup Z$ cannot all be in distinct parallel classes of N'/g . Thus N' has a triangle $\{x, y, g\}$ where we may assume that $x \in X$ and $y \in Y$. Since $N'|(X \cup Y \cup Z \cup g)$ has Z as a component, there is an element h of $E(N')$ that is in neither $\text{cl}_{N'}(X \cup Y)$ nor $\text{cl}_{N'}(Z)$. As above, N' has a triangle $\{h, z, t\}$ where $t \in X \cup Y$ and $z \in Z$. Contracting g and h from $N'|(X \cup Y \cup Z \cup \{g, h\})$ and simplifying, we get a rank-7 matroid with 16 elements. As $\frac{16}{7} > \frac{9}{4}$, we have a contradiction that completes the proof of Lemma 21. \square

Lemma 22. *Let N be a simple connected matroid having an element z such that each of N and $\text{si}(N/z)$ has every element in at least two triangles. If $d(N) \leq \frac{9}{4}$ and $d(N') < \frac{9}{4}$ for all proper minors N' of N , then N is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$.*

Proof. We argue by induction on $r(N)$, which must be at least three. Suppose it is exactly three. Since $\text{si}(N/z)$ has density less than $\frac{9}{4}$, it is isomorphic to $U_{2,4}$. As $d(N) \leq \frac{9}{4}$, we see that $|E(N)| \leq 6$. By Lemma 18, N has no 2-cocircuits. Thus N has a triangle whose complement is a triad. By Lemma 18 again, $N \cong M(K_4)$ and we get a contradiction. Hence $r(N) \geq 4$. If $r(N) = 4$, then, by Corollary 19, N is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$.

Now assume the result holds for $r(N) < k$ and let $r(N) = k \geq 5$. Let $N_1 = \text{si}(N/z)$. Every element of N_1 is in at least two triangles. Let N_2 be a component of N_1 . By Lemma 20, either every element of N_2 is in a $U_{2,4}$ - or $M(K_4)$ -restriction, or N_2 has an element z_2 such that every element of $\text{si}(N_2/z_2)$ is in at least two triangles. If the latter occurs, then, by the induction assumption, N_2 is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$. Each of these matroids has density $\frac{9}{4}$, a contradiction. Thus every element of N_2 is in a $U_{2,4}$ - or $M(K_4)$ -restriction. As $d(N_2) < \frac{9}{4}$, Lemma 21 implies that N_2 , and hence each component of N_1 , is isomorphic to one of $U_{2,4}$, $M(K_4)$, or $P(M(K_4), M(K_4))$.

Suppose that $N_2 = N_1$. Then, as $r(N) \geq 5$, we deduce that $N_1 \cong P(M(K_4), M(K_4))$. As $N_1 = \text{si}(N/z)$, we see that $r(N) = 6$. Because $d(N) \leq \frac{9}{4}$, it follows that $|E(N)| \leq 13$. Since z is in at least two triangles of N , we deduce that $|E(N)| \geq |E(N_1)| + 3 = 14$, a contradiction.

We may now assume that N_1 has more than one component. Hence, for some $k \geq 2$, there is a collection N^1, N^2, \dots, N^k of connected matroids such that $E(N^i) \cap E(N^j) = \{z\}$ for all $i \neq j$, the matroid N^i/z is connected for all i , and N is the parallel connection of N^1, N^2, \dots, N^k across the common basepoint z . As noted above, each $\text{si}(N^i/z)$ is isomorphic to one of $U_{2,4}$, $M(K_4)$, or $P(M(K_4), M(K_4))$. As every element of N is in at least two triangles, every element of each N^i except possibly z is in at least two triangles of N^i . Thus, by Corollary 19, $N^i \cong M(K_4)$; or $r(N^i) = 4$ and $|E(N^i)| = 9$; or $r(N^i) > 4$. In the first case, $\text{si}(N^i/z) \not\cong U_{2,4}$; in the second case, $d(N^i) = \frac{9}{4}$. Both of these possibilities give contradictions, so $\text{si}(N^i/z) \cong P(M(K_4), M(K_4))$ for each i . As z is in at least two triangles of N , we may assume the elements of two such triangles lie in $E(N^1) \cup E(N^2)$. As $|E(\text{si}(N^i/z))| = 11$ and $r(N^i/z) = 5$, we see that $|E(N^1) \cup E(N^2)| \geq 25$ and $r(E(N^1) \cup E(N^2)) = 11$. But $\frac{25}{11} > \frac{9}{4}$, a contradiction. \square

We conclude the paper by proving Theorem 9. In this proof, we will make extensive use of the Cunningham-Edmonds canonical tree decomposition of a connected matroid. The definition and properties of this decomposition may be found in [7, Section 8.3]. In brief, associated with each connected matroid M , there is a tree T that is unique up to the labelling of its edges. Each vertex of T is labelled by a circuit, a cocircuit, or a 3-connected matroid with at least four elements. Moreover, no two adjacent vertices of T are labelled by circuits and no two adjacent vertices are labelled by cocircuits. For an edge e of T whose endpoints are labelled by matroids M_1 and M_2 , the ground sets of these two matroids meet in $\{e\}$. When we contract e from T , the composite vertex that results by identifying the endpoints of e is labelled by the 2-sum of M_1 and M_2 . By repeating this process, contracting all of the remaining edges of T one by one, we eventually obtain a single-vertex tree. Its vertex is labelled by M .

Each edge f of T induces a partition of $E(M)$. This partition is a 2-separation of M displayed by f . The remaining 2-separations of M coincide with those that are displayed by those vertices of T that are labelled by circuits or cocircuits. For such a vertex v having label N , there is a partition $\{X_1, X_2, \dots, X_k\}$ of $E(M) - E(N)$ induced by the components of $T - v$. A partition (X, Y) of $E(M)$ is displayed by the vertex v if each X_i is contained in X or Y . Every such partition of $E(M)$ with both X and Y having at least two elements is a 2-separation of M and these 2-separations along with those displayed by the edges of T are all of the 2-separations of M . Recall that, for all $n \geq 2$, we denote by P_n any matroid that can be constructed from n copies of $M(K_3)$ via a sequence of parallel connections.

Proof of Theorem 9. Let M be a density-critical matroid with $d(M) \leq \frac{9}{4}$. Suppose $d(M) \geq 2$. By Lemma 10, every element of M is in at least two triangles. By Corollary 19, if $r(M) \in \{2, 3\}$, then M is $U_{2,4}$ or $M(K_4)$. We may now assume that $r(M) \geq 4$. By Lemma 20, either every element of M is in a $U_{2,4}$ - or $M(K_4)$ -restriction, or, for some

element z of M , every element of $\text{si}(M/z)$ is in at least two triangles. In the first case, by Lemma 21, M is isomorphic to $P(U_{2,4}, M(K_4))$, $P(M(K_4), M(K_4))$, $M(K_5 \setminus e)$, or M_{18} . In the second case, by Lemma 22, M is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$. Thus the theorem identifies all possible density-critical matroids with density in $[2, \frac{9}{4}]$ and one easily checks that each of the matroids identified is indeed density-critical.

Now suppose that $d(M) < 2$. By Lemma 16, M is connected. Clearly, if $r(M)$ is 1 or 2, then M is isomorphic to $U_{1,1}$ or $U_{2,3}$. As $U_{2,4}$ and $M(K_4)$ both have density 2, M is a series-parallel network (see, for example, [7, Corollary 12.2.14]). Thus, in the Cunningham-Edmonds canonical tree decomposition T of M , every vertex is labelled by a circuit or a cocircuit. Since M is simple, for every vertex of T that is labelled by a cocircuit C^* , at most one element of C^* is in $E(M)$. Let e be an edge of T that meets the vertex labelled by C^* . Then, for the 2-separation (X, Y) of M that is displayed by e , Lemma 17 implies that M has an element p in $\text{cl}(X) \cap \text{cl}(Y)$. Thus $p \in C^*$, so C^* contains exactly one element of M .

Now take a vertex of T that is labelled by a circuit C where $C = \{e_1, e_2, \dots, e_k\}$ and suppose that $k \geq 4$. Suppose $e_1 \in E(M)$. Then M/e_1 is simple having rank $r(M) - 1$. As $\frac{|E(M)|-1}{r(M)-1} < \frac{|E(M)|}{r(M)}$, we obtain the contradiction that $|E(M)| < r(M)$. We deduce that $C \cap E(M) = \emptyset$. Now $T \setminus e_1, e_2$ has exactly three components. Let T' be the one containing e_3 and let X be the subset of $E(M)$ corresponding to T' . Then $(X, E(M) - X)$ is a 2-separation of M . By Lemma 17, there is an element p of M that is in $\text{cl}(X) \cap \text{cl}(E(M) - X)$. But the tree decomposition implies that there is no such element. We deduce that C has exactly three elements. Thus every vertex of T that is labelled by a circuit is labelled by a triangle. Since every vertex of T that is labelled by a cocircuit has exactly one element of $E(M)$ in that cocircuit, a straightforward induction argument establishes that, for some $n \geq 2$, the matroid M is obtained from n copies of $M(K_3)$ by a sequence of $n - 1$ parallel connections. Thus $M \cong P_n$.

Finally, we show by induction that P_n is density-critical. This is true for $n = 1$. Assume it true for $n < m$ and let $n = m \geq 2$. Take x in $E(P_n)$. Assume first that x is in exactly one triangle $\{x, y, z\}$. Then $\text{si}(P_n/x) \cong P_n/x \setminus z$. As the last matroid is easily seen to be isomorphic to the density-critical matroid P_{n-1} and $d(P_{n-1}) < d(P_n)$, every minor of P_n/x has density less than $d(P_n)$. Now assume x is in at least two triangles of P_n . Then $\text{si}(P_n/x)$ is easily seen to be the direct sum of a collection of matroids each of which is isomorphic to some P_k with $k < n$ or to $U_{1,1}$. By Lemma 16 and the induction assumption, every minor of P_n/x has density less than $d(P_n)$. We conclude that P_n is density-critical, so the theorem is proved. \square

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