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A Decomposition Theory for Rings of Operators.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENT</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>I PRELIMINARIES</td>
<td>6</td>
</tr>
<tr>
<td>II THE STRUCTURE OF DIAGONAL RINGS</td>
<td>31</td>
</tr>
<tr>
<td>III THE DIRECT SUM AND DIRECT INTEGRAL DECOMPOSITIONS</td>
<td>72</td>
</tr>
<tr>
<td>IV DIMENSION PROPERTIES OF THE DECOMPOSITIONS</td>
<td>117</td>
</tr>
<tr>
<td>APPENDIX, AN EXAMPLE</td>
<td>148</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>168</td>
</tr>
<tr>
<td>BIOGRAPHY</td>
<td>169</td>
</tr>
</tbody>
</table>
ABSTRACT

The object of this dissertation is to develop a decomposition theory for a particular kind of ring of operators on a Hilbert space, and to investigate the relationships between the ring, the Hilbert space, and the decomposition. This study has grown out of several previous theories of the same kind which were developed to treat the same problem. However, the development which is given here overcomes some of the main disadvantages of these previous theories, while preserving most of their important features.

The first chapter introduces the terminology and states the results from the theory of Banach symmetric rings and their representations, in the exact form which is to be used in this work. Most of this material is known and can be found in the references given in the bibliography, so that the proofs, with a few exceptions, are omitted.

In Chapter II, a study is made of a special kind of commutative ring $E$ of operators on a Hilbert space $H$, which is called a diagonal ring in this work. The topological structure of the maximal ideal space $M$ of $E$ is investigated, and it is shown how the invariant subspaces for the commutant $E'$ of $E$ are related to $M$. For the case where $H$ is cyclic for $E'$, the cyclic subspaces for $E$
are also studied. It is shown that a special class of linear functionals on $E$ comes about from a particular type of integral representation on its maximal ideal space $M$. This integral representation is used to relate the vectors of $\mathcal{H}$ to the structure of the cyclic subspaces for $E$ which they generate.

Chapter III, which contains the main part of this work, sets up a direct integral decomposition of a Hilbert space over the maximal ideal space $M$ of a diagonal ring $E$ whose commutant has a cyclic vector. This direct integral comes about from a particular kind of structural system related to the space and ring, which is called a canonical decomposition system. It is then shown that the operators of $E'$ and $E$ decompose in a special way in this direct integral, and the relationships between the operators and their decompositions are investigated. Finally, a relation is shown between the theory given here and that developed previously by Tomita and Segal.

In Chapter IV, a dimension function is defined on the maximal ideal space $M$ of $E$. Using this function, it is shown that certain dimension properties of the canonical decomposition system and direct integral of Chapter III are determined entirely by the space and diagonal ring on which they are defined.
For completeness, an appendix appears at the end of this work which contains an example to show the constructiveness of the theory, and also to give counterexamples to certain conjectures which were stated to be false in Chapter III.
INTRODUCTION

The problem to be studied in this work has been investigated in several previous theories. It is the problem of decomposing a given ring of operators on a Hilbert space into a direct integral of irreducible rings, or, at least, a direct integral of systems having properties which are simpler than the original. The basic difficulty involved is the actual formulation of what is to be meant by "direct integral of systems."

What is required first of all is a concept of direct integral of Hilbert spaces which generalizes, in a measure theoretic sense, the usual concept of direct sum. For a given measure space \((M, \mu)\), suppose a Hilbert space \(L_m\) is associated with each \(m \in M\). The direct integral of the \(L_m\) with respect to \(\mu\) is to consist of a system of vector valued functions \(\{\xi(m)\}\) defined almost everywhere on \(M\), with \(\xi(m) \in L_m\) where defined, which form a vector space with respect to pointwise operations in the "almost everywhere" sense. The integral \(\int (\xi(m), \eta(m))d\mu(m)\) is to exist for arbitrary \(\{\xi(m)\}, \{\eta(m)\}\) in the system, and to define an inner product which makes the system a Hilbert space. Then, starting with a symmetric ring \(R\) on a Hilbert space \(\mathcal{H}\), the object is to show that \(\mathcal{H}\) can be represented as such a direct integral, such that every operator \(A \in R\)
can be associated with a system \( \{A(m)\}_{m \in M} \) of operators with \( A(m) \in \mathcal{B}(\mathcal{L}_m) \). We require that if \( \{\xi(m)\} \) is any function in the direct integral, which is now identified with \( \mathcal{H} \), and \( A \in R \), then \( A(\{\xi(m)\}) = \{A(m) \xi(m)\} \).

The construction of the measure space \((M, \mu)\) for a given symmetric ring \( R \) on a space \( \mathcal{H} \) is not difficult, the usual procedure being as follows. First, it may be shown that the problem can be reduced to the case where \( \mathcal{H} \) is cyclic with respect to \( R \). Then a maximal commutative symmetric subring \( E \) of the commutant \( R' \) of \( R \) is chosen, usually called a "diagonal ring" for \( R \). The maximal ideal space of \( E \) is then used for the space \( M \) above. If \( \xi_0 \) is a cyclic vector for \( \mathcal{H} \) with respect to \( R \), then \( A \rightarrow (A\xi_0, \xi_0) \) defines a positive functional on \( E \), and \( \mu \) is chosen as the measure on \( M \) which corresponds in the natural way to this positive functional. It is sometimes possible and advantageous to take \( M \) as the maximal ideal space of a weakly dense subring of \( E \), rather than of \( E \) itself, in order to simplify its topological properties. With such a ring \( E \), it is not only possible to decompose the operators of \( R \), but also those of the Banach symmetric ring generated by \( R \cup E \), or even the weak closure of this ring, which is \( E' \). In this work, the procedure will be to start with the diagonal ring \( E \), and decompose the operators of \( E' \) directly.
A definition of direct integral was given originally by von Neumann in which the spaces \( \mathcal{L}_m \) are all subspaces of a given separable Hilbert space \( \mathcal{K} \), and are chosen in a special way with respect to a given orthonormal basis. (cf. [5], pp. 350 ff and chapter VIII). This definition is very workable for purposes of decomposing rings in the manner which we have described, however the separability of the \( \mathcal{L}_m \) limit the rings \( \mathcal{R} \) to which the theory is applicable. The separability requirement is, however, very important to the measure theoretic arguments which are involved.

To handle the general case, a different approach was developed by Segal [7] and Tomita [10]. An account of Tomita's theory is also given in chapter VIII of [4]. Although the procedures followed differ somewhat in these two works, the decompositions come about in essentially the same way. A concept of direct integral of Hilbert spaces is defined in an axiomatic manner. The method of arriving at such a direct integral from a ring \( \mathcal{R} \) involves the decomposition of a particular positive functional as an integral of a system of positive functionals \( \{f_m\}_{m \in M} \) over the maximal ideal space of the diagonal ring \( \mathcal{E} \). (See theorem 1.5(c) of this dissertation for a restatement of this decomposition.) The spaces \( \mathcal{L}_m \) of the direct integral are taken to be the spaces which occur in the representations of \( \mathcal{R} \) corresponding to the \( f_m \). The main disadvantage of this
procedure is that the properties of the individual spaces $L^m$ cannot be identified with each other, or with the original space $ℋ$ and the rings $R$ and $E$.

A certain portion of the Tomita theory was also shown later to be incorrect. What Tomita had hoped to show was that the Banach symmetric ring generated by $R$ and $E$ was actually decomposed into rings, almost all of which were irreducible, in the sense of the measure $μ$. However an error appeared in his work, and a counterexample to the result was given by Taylor [9]. Taylor also showed that the result was correct for the ring $E'$, which is the weak closure of the ring generated by $R \cup E$.

In this dissertation, we present an alternative theory to handle the general case. The direct integral decomposition of the Hilbert space $ℋ$ preserves most of the important features of the direct integral of von Neumann mentioned earlier. The measure theoretic difficulties for which the separability requirements were needed in that theory are overcome by using the very strong relationships which exist between the topological structure of $M$ and the measure $μ$. The theory is also related to that of Segal and Tomita as is pointed out in chapter III. This relationship shows some hope of giving insight into exactly what happens for a general ring concerning the error of Tomita's theory which was mentioned above, however no investigation of this is made
here. An important feature of the theory appears in chapter IV, where it is shown that certain properties of the decompositions are determined by the space and ring from which they came.

As a convenience to the reader, we adopt the practice of using the symbol □ to indicate the end of a proof.
The principal purpose of this chapter is to set forth the known results from the theory of symmetric rings in the form which they are to be used elsewhere in this work, and also to introduce the terminology which will be used in later chapters. Most of these results are published in at least one of the references given in the bibliography, the most important of these being [1], [3], and [5]. The proofs will, in most cases, be omitted, and a reference given whenever necessary.

By symmetric ring, we shall mean a system of elements $R$ furnished with several operations so that (1) $R$ is an algebraic ring, (2) a scalar multiplication is defined on $\mathbb{C} \times R$ into $R$, where $\mathbb{C}$ denotes the complex numbers, so that $R$ is a vector space over $\mathbb{C}$ relative to its ring addition operation, (3) the equation $a(xy) = (ax)y = x(ay)$ is satisfied whenever $x, y \in R$, $a \in \mathbb{C}$, and (4) a function $x \to x^*$ is defined on $R$ satisfying the properties $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$, whenever $x, y \in R$ and $\alpha, \beta \in \mathbb{C}$.

Any one of the three products shown in requirement (3) will be abbreviated, as usual, to $\alpha xy$. Of special importance are those symmetric rings containing an identity, which is an element $e$ of the ring which is an algebraic
identity. An element \( x \in R \) is said to be hermitian if \( x = x^* \). An identity element, whenever it exists, is always hermitian. Every element \( x \) of \( R \) can be written uniquely in the form \( x = x_1 + ix_2 \), where \( x_1, x_2 \) are hermitian; \( x_1, x_2 \) are determined from \( x \) by the equations \( x_1 = \frac{x + x^*}{2} \), \( x_2 = \frac{x - x^*}{2i} \).

A subring of a symmetric ring \( R \) is a subset of \( R \) which is, by itself, an algebraic ring under the induced operations, and is closed under scalar multiplication by the complex numbers. A symmetric subring is a subring which is also closed under the mapping \( x \to x^* \). Thus a symmetric subring of \( R \) is itself a symmetric ring. The subring may or may not have an identity independently of whether the original ring \( R \) has one, and an identity for a subring need not be an identity for the original ring. However if \( R \) has an identity element and the subring contains this element, then it is an identity for the subring. When this occurs, we shall state that the subring contains the identity of \( R \), or more simply that it contains the identity.

If a symmetric ring \( R \) is furnished with a non negative real valued function \( |\cdot| \) so that (1) \( R \), as a vector space, is a normed linear space with respect to \( |\cdot| \), and (2) \( |x^*| = |x| \) and \( |xy| \leq |x||y| \) for arbitrary \( x, y \in R \), then \( R \) will be called a normed symmetric ring. If, in addition, \( R \) is a Banach space with respect to \( |\cdot| \), i.e.,
it is complete in the induced metric, then $R$ will be called a **Banach symmetric ring**. A normed symmetric ring will be called **completely regular** if we have $|x^*x| = |x|^2$ for every $x \in R$. A symmetric subring of a normed symmetric ring is itself a normed symmetric ring with the induced norm. A symmetric subring of a Banach symmetric ring which is closed in the norm topology is a Banach symmetric ring. We shall call these, respectively, normed symmetric subrings and Banach symmetric subrings of the original ring $R$.

An important example of a completely regular Banach symmetric ring with identity is the system $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a given Hilbert space $\mathcal{H}$, with the algebraic operations and norm being defined in the usual way. The identity is the identity operator on $\mathcal{H}$, which we shall denote throughout by $I$. In accordance with what has been stated above, closed symmetric subrings of $\mathcal{B}(\mathcal{H})$ containing $I$ are Banach symmetric rings with identity.

Of special importance in a Banach symmetric subring $R$ of $\mathcal{B}(\mathcal{H})$ containing the identity are those operators $A$ for which $(A\xi, \xi) \geq 0$ for every $\xi \in \mathcal{H}$. Such operators are called **positive definite**. It is clear that $B^*B$ is a positive definite operator for every $B \in R$. Conversely, every positive definite operator of $R$ can be shown to have this form. We shall write $A \geq 0$, or $0 \leq A$, to denote the fact that $A$ is positive definite. We may extend
this definition to write \( A \succcurlyeq B \), or \( B \preccurlyeq A \), when \( A - B \) is positive definite. Then \( \preccurlyeq \) is a partial ordering of \( \mathbb{R} \).

A second important example of a completely regular Banach symmetric ring with identity is the system \( C(M) \) of continuous, necessarily bounded, complex valued functions on a compact Hausdorff space \( M \), with the algebraic operations and norm defined in the usual way. If a non-negative regular Borel measure \( \mu \) is given on \( M \), whose support is all of \( M \), then the system \( L_\infty(M,\mu) \) of essentially bounded \( \mu \)-measurable functions is also a completely regular Banach symmetric ring with identity, and \( C(M) \) can be considered a Banach symmetric subring containing the identity. The \( L_\infty(M,\mu) \) norm, as usual, is denoted by \( ||\cdot||_\infty \).

With regard to the ring \( \mathcal{B}(\mathcal{H}) \) of the first example, certain properties of closed linear subspaces of \( \mathcal{H} \) can be stated in terms of projection operators. A projection is an operator \( P \in \mathcal{B}(\mathcal{H}) \) for which \( P^2 = P = P^* \). For such an operator we necessarily have \( |P| = 1 \) unless \( P \) is the identically zero operator. Every closed linear subspace of \( \mathcal{H} \) is the range of a unique projection operator, and the range of a projection operator is always a closed linear subspace of \( \mathcal{H} \). If \( P \) is the unique projection having a given closed linear subspace \( \mathcal{M} \) for its range, we shall say that \( P \) is the projection onto \( \mathcal{M} \). For a single element \( \xi \)
to be in the range of a projection $P$, it is necessary and sufficient that $P\xi = \xi$, and also necessary and sufficient that $|P\xi| = |\xi|$. If $P$ is the projection onto a given closed linear subspace $M$, then $I - P$ is the projection onto its orthogonal complement, which we denote by $M^\perp$. If $P_1, P_2$ are projections onto closed linear subspaces $M_1, M_2$ respectively such that $P_1 P_2 = P_2 P_1$, then $P_1 (M_2) = P_2 (M_1) = M_1 \cap M_2$ which is also a closed linear subspace, and $P_1 P_2$ is the projection onto $M_1 \cap M_2$.

A subspace $M$ of $H$ is said to be invariant for an operator $A$ if it is a closed linear subspace such that $A(M) \subseteq M$. In this case, the restriction of $A$ to $M$, which we denote by $A|_M$, is a bounded linear operator on $M$. If, in addition, $M^\perp$ is also invariant for $A$, we say that $M$ reduces $A$. For $M$ to reduce $A$, it is necessary and sufficient that $M$ be invariant for both $A$ and $A^*$. In this case it follows that $(A|_M)^* = A^*|_M$. If $P$ is the projection onto $M$, then $M$ reduces $A$ if, and only if $AP = PA$. If $\{M_\alpha\}_{\alpha \in \Lambda}$ is any system of pairwise orthogonal invariant subspaces for $A$ and $\xi_\alpha \in M_\alpha$ for each $\alpha$ such that $\sum_{\alpha \in \Lambda} |\xi_\alpha|^2 < \infty$, then $A\left(\sum_{\alpha \in \Lambda} \xi_\alpha\right) = \sum_{\alpha \in \Lambda} A\xi_\alpha$, the vectors $A\xi_\alpha$ being in the respective spaces $M_\alpha$, hence pairwise orthogonal.

If $\alpha$ is a subset of $\Theta(H)$, a subspace $M$ of $H$ is said to be invariant for $\alpha$, or reducing for $\alpha$, if it is, respectively, invariant or reducing for every operator $A \in \alpha$. 
If $A^* \in \mathcal{A}$ whenever $A \in \mathcal{A}$, it follows from the above paragraph that every invariant subspace is necessarily reducing. This is the case, in particular, when $\mathcal{A}$ is a symmetric subring of $\mathcal{B}(\mathcal{H})$.

If $R_1, R_2$ are symmetric rings, a function $\phi: R_1 \to R_2$ is called a **symmetric homomorphism** if (1) $\phi$ is an algebraic ring homomorphism, and (2) $\phi(\alpha x) = \alpha \phi(x)$ and $\phi(x^*) = (\phi(x))^*$, whenever $\alpha$ is complex and $x \in R_1$. The range of $\phi$ is always a symmetric subring of $R_2$. If $R_1$ is a Banach symmetric ring and $R_2$ is a completely regular Banach symmetric ring, then $\phi$ is necessarily continuous and norm decreasing, i.e., $|\phi(x)| \leq |x|$ for every $x \in R_1$. If $R_1, R_2$ are both completely regular Banach symmetric rings, then the range of $\phi$ is necessarily a closed symmetric subring of $R_2$, hence a Banach symmetric ring.

A symmetric homomorphism $\phi: R_1 \to R_2$ is called a **symmetric isomorphism** if it is a one-to-one function. The **kernel** of a symmetric homomorphism $\phi: R_1 \to R_2$ is the subset of $R_1$ consisting of those elements $x$ for which $\phi(x) = 0$. In order for a symmetric homomorphism to be a symmetric isomorphism, it is necessary and sufficient that its kernel consist of the element 0 alone. If $R_1, R_2$ are both completely regular Banach symmetric rings and $\phi: R_1 \to R_2$ is a symmetric isomorphism, then $\phi$ is necessarily isometric, i.e., $|\phi(x)| = |x|$ for every $x \in R_1$. 
A symmetric homomorphism from a symmetric ring $R$ into a ring $\mathcal{B}(\mathcal{H})$ is called a representation of $R$. In the present discussion we shall consider only the case in which $R$ has an identity, and the image of the identity under the representation is the identity of $\mathcal{B}(\mathcal{H})$. In particular, if $R$ is a symmetric subring of $\mathcal{B}(\mathcal{H})$ containing the identity, then the identity function from $R$ to itself is a representation, called the identity representation of $R$. In dealing with representations of a symmetric ring $R$, a special symbol is usually used to denote the image of an element $x$ of $R$ rather than giving the representation a name such as $\Phi$. Symbols commonly used are $A_x$ for the image of $x$, or if $R$ is a ring of operators, the image of an operator $A$ of $R$ may be denoted by $A_1$, $A(m)$, or another notation which will be clear from the context. For the present discussion, we adopt the $x \rightarrow A_x$ notation to denote a given representation of a ring $R$ into a ring $\mathcal{B}(\mathcal{H})$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be cyclic for the representation $x \rightarrow A_x$ if $\mathcal{M}$ is invariant for all the $A_x$ and if there exists a single vector $\xi_0 \in \mathcal{M}$ such that $\mathcal{M}$ is the smallest invariant subspace for the $A_x$ which contains the vector $\xi_0$. Such a vector $\xi_0$ is called a cyclic vector for $\mathcal{M}$. We call $\mathcal{M}$ nontrivial if it is not the subspace consisting of $0$ alone. Every vector $\xi \in \mathcal{H}$ is a
cyclic vector for some cyclic subspace of the representation, this subspace being given by \( \mathcal{G}(A_x \xi \mid x \in \mathbb{R}) \), where \( \mathcal{G}(\mathcal{S}) \) denotes the smallest closed linear subspace containing the set \( \mathcal{S} \) for any subset \( \mathcal{S} \) of \( \mathcal{H} \). This space is called the \textbf{cyclic subspace generated by} \( \xi \) with respect to the representation. The set of vectors \( \{A_x \xi \mid x \in \mathbb{R}\} \) is a linear subspace of \( \mathcal{H} \), although usually not closed, which is a dense subspace of \( \mathcal{G}(A_x \xi \mid x \in \mathbb{R}) \). If the entire space \( \mathcal{H} \) is cyclic, the representation is called a \textbf{cyclic representation}. In any case, \( \mathcal{H} \) can always be written as a direct sum of cyclic subspaces for the given representation.

The cases of interest here will be those in which \( \mathcal{R} \) is a symmetric subring of \( \mathcal{B}(\mathcal{H}) \) with the identity, and the representation is the identity representation. A cyclic subspace in this case is said to be cyclic \textbf{for the ring} \( \mathcal{R} \). The general case can always be reduced to this by considering the image of the ring in \( \mathcal{B}(\mathcal{H}) \) under the representation.

We now state formally and prove a result on the number of cyclic subspaces into which \( \mathcal{H} \) can be decomposed as a direct sum.

\textbf{Theorem 1.1.} Suppose \( \mathcal{R} \) is a symmetric subring with the identity of the ring \( \mathcal{B}(\mathcal{H}) \) of bounded linear operators on a Hilbert space \( \mathcal{H} \). Suppose also that \( \mathcal{H} = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha \) and \( \mathcal{H}' = \sum_{\alpha' \in \Lambda'} \mathcal{H}'_{\alpha'} \) are two direct sum decompositions of \( \mathcal{H} \) into...
subspaces, each of which is a nontrivial cyclic subspace for \( R \). Then \( \text{card } \Lambda' \leq \aleph_0 \text{ card } \Lambda \) and \( \text{card } \Lambda \leq \aleph_0 \text{ card } \Lambda' \).

**Proof.** For each \( \alpha \in \Lambda \), let \( \xi_\alpha \) be a cyclic vector for \( \mathcal{H}_\alpha \), which is necessarily nonzero. Then \( \xi_\alpha = \sum_{\alpha' \in \Lambda'} \xi_{\alpha \alpha'} \), where \( \xi_{\alpha \alpha'} \) is the projection of \( \xi_\alpha \) onto \( \mathcal{H}_\alpha \), for each \( \alpha' \in \Lambda' \). Since \( \sum_{\alpha' \in \Lambda'} |\xi_{\alpha \alpha'}|^2 = |\xi_\alpha|^2 \), there exist, for fixed \( \alpha \), at most countably many indices \( \alpha' \in \Lambda' \) for which \( \xi_{\alpha \alpha'} \neq 0 \). Let \( \Lambda_\alpha \) denote this countable set of \( \alpha' \).

It is claimed that \( \bigcup_{\alpha \in \Lambda} \Lambda_\alpha = \Lambda' \). If this is not the case, there exists an \( \alpha' \in \Lambda' \) such that \( \xi_{\alpha \alpha'} = 0 \) for every \( \alpha \in \Lambda \). Let \( \eta \) be any vector of \( \mathcal{H}_\alpha \). Then \( (A\xi_\alpha, \eta) = (\xi_\alpha, A^*\eta) = 0 \) for every \( A \in R \) and \( \alpha \in \Lambda \), since \( \xi_\alpha \perp \mathcal{H}_\alpha \). This implies \( \eta \perp \mathcal{H}_\alpha \), since the vectors \( \{A\xi_\alpha \mid A \in R\} \) are dense in \( \mathcal{H}_\alpha \), hence \( \eta \perp \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha = \mathcal{H} \). Consequently \( \eta = 0 \).

We have thus shown that \( \mathcal{H}_\alpha \) has no nonzero vectors, contrary to the hypothesis that it be a nontrivial cyclic subspace. Thus \( \bigcup_{\alpha \in \Lambda} \Lambda_\alpha = \Lambda' \) as asserted, and since each \( \Lambda_\alpha \) is countable, \( \text{card } \Lambda' \leq \aleph_0 \text{ card } \Lambda \).

Since the hypothesis above is the same on the two direct sums, we have also that \( \text{card } \Lambda \leq \aleph_0 \text{ card } \Lambda' \).

A **positive functional** on a symmetric ring \( R \) is a linear functional \( f \) for which \( f(x^*x) \geq 0 \) for all \( x \in R \). If \( R \) has an identity \( e \), a positive functional \( f \) is said to be
normalized if \( f(e) = 1 \). A positive functional on a Banach symmetric ring with identity is necessarily continuous.

If \( R \) is a Banach symmetric ring with identity, \( x \to A_x \) is a representation of \( R \) in a ring \( \mathcal{B}(\mathcal{H}) \), and \( \xi \) is any vector of \( \mathcal{H} \), then \( f(x) = (A_x \xi, \xi) \) defines a positive functional, which is normalized if, and only if \( |\xi| = 1 \). (Recall \( A_e = I \) by our requirements.) Conversely, for every positive functional \( f \) on \( R \) there exists a representation of \( R \) such that \( f \) has the above form. The representation may be chosen so that \( \xi \) is a cyclic vector for \( \mathcal{H} \), and such a representation is unique, within equivalence, in the following sense: If \( x \to A_x^{(1)} \) and \( x \to A_x^{(2)} \) are representations of \( R \) in \( \mathcal{B}(\mathcal{H}_1) \) and \( \mathcal{B}(\mathcal{H}_2) \) respectively such that \( \xi_1 \in \mathcal{H}_1 \) is cyclic for \( \mathcal{H}_1 \), \( \xi_2 \in \mathcal{H}_2 \) is cyclic for \( \mathcal{H}_2 \), and \( (A_x^{(1)} \xi_1, \xi_1) = (A_x^{(2)} \xi_2, \xi_2) \) for every \( x \in R \), then there exists a unique unitary isomorphism \( U \) from \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that \( U \xi_1 = \xi_2 \) and \( UA_x^{(1)} = A_x^{(2)} U \) for every \( x \in R \).

Again, if \( x \to A_x \) is a representation of a symmetric ring \( R \) with identity in a ring \( \mathcal{B}(\mathcal{H}) \), a nontrivial subspace \( \mathcal{M} \) invariant for all of the \( A_x \) is said to be irreducible if there exist no subspaces of \( \mathcal{M} \) invariant for all the \( A_x \) except \( \mathcal{M} \) and \((0)\). There are several alternative ways of describing irreducibility of \( \mathcal{M} \), one being that every nonzero vector of \( \mathcal{M} \) is cyclic for \( \mathcal{M} \), another
being that there exist no operators in $\mathcal{B}(\mathcal{M})$, other than multiples of the identity, which commute with all operators $A_x \mid_m$. If the entire space $\mathcal{H}$ is irreducible, we say that $x \mapsto A_x$ is an irreducible representation.

We shall now consider commutative Banach symmetric rings $R$ with identity. A complex homomorphism of $R$ is a function $h : R \to C$ which is algebraically a ring homomorphism from $R$ to $C$, and also satisfies $h(ax) = ah(x)$ for $a \in C, x \in R$. Such a homomorphism is necessarily continuous, and $|h| \leq 1$. If $h(x) \neq 0$, then $h^{-1}(0)$ is a maximal ideal in $R$. All maximal ideals of $R$, by which we mean maximal proper ideals, are obtained in this fashion, and the correspondence between maximal ideals and complex homomorphisms which are not identically zero is one-to-one. We denote by $M$ the set of all maximal ideals of $R$, and for $x \in R, m \in M$, we denote by $\hat{x}(m)$ the complex number which is the image of $x$ under the corresponding homomorphism. For each $x$, we thus have a bounded, complex valued function $\hat{x}$ defined on $M$, called the Gelfand transform of $x$. The weakest topology on $M$ in which all of the functions $\hat{x}$ are continuous is called the Gelfand topology of $M$, and $M$ with this topology is called the maximal ideal space of $R$. $M$ is necessarily a compact Hausdorff space. Frequently, the space $M$ is easily recognized as being homeomorphic to another topological space associated in some way with
the ring. In such a case it is convenient to replace $M$ by its homeomorphic image, and identify the maximal ideals of $R$ and Gelfand transforms of elements of $R$ on this homeomorphic image accordingly.

If $R$ is a completely regular commutative Banach symmetric ring with identity, and in particular if $R$ is a commutative Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ containing the identity, then the mapping $x \rightarrow \hat{x}$ is a symmetric isomorphism from $R$ onto the ring $C(M)$. In this case, the dual space of $R$ as a Banach space can be identified through this isomorphism with the dual space of $C(M)$, so that every bounded linear functional has the form $f(x) = \int_M \hat{x}(m) d\mu(m)$, where $\mu$ is a complex valued regular Borel measure on $M$ which is uniquely determined by $f$. Since $x \rightarrow \hat{x}$ is a symmetric isomorphism from $R$ onto $C(M)$, an element $y$ has the form $x^*x$ exactly when $\hat{y}(m) \geq 0$ for all $m \in M$. Thus the positive functionals on $R$ correspond to the non negative measures $\mu$ on $M$ through the above identification.

If $R$ is a commutative Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ with the identity, then $P \in R$ is a projection operator by definition if $P^2 = P = P^*$. Under the Gelfand transform, this is equivalent to $\hat{P}^2(m) = \hat{P}(m) = \hat{P}(m)$ for all $m \in M$, which means that $\hat{P}(m) = 0$ or $1$ for all $m$, or that $\hat{P}$ is the characteristic function of some set $U$. 
Since \( \hat{P} = \chi_U \) is continuous, \( U \) is both closed and open in \( M \), which we abbreviate by calling \( U \) clopen. Conversely, if \( U \) is clopen in \( M \), then \( \chi_U \) is continuous, so there exists a \( P \in \mathbb{R} \) for which \( \hat{P} = \chi_U \), and \( P \) is necessarily a projection. Since the mapping \( A \rightarrow \hat{A} \) from \( \mathbb{R} \) to \( C(M) \) is one-to-one, the above correspondence between the projections of \( \mathbb{R} \) and the characteristic functions of clopen sets is one-to-one. For every clopen \( U \subset M \), we denote by \( P_U \) the unique projection of \( \mathbb{R} \) for which \( \hat{P}_U = \chi_U \). We note also that from previous observations, an operator \( A \in \mathbb{R} \) is positive definite if, and only if it can be written in the form \( B^*B \) for some \( B \in \mathbb{R} \). Thus we have in terms of the Gelfand transform that \( A \) is positive definite exactly when \( \hat{A}(m) \geq 0 \) for each \( m \in M \).

We now consider four topologies which are defined on the ring \( \mathcal{B}(\mathcal{H}) \) of bounded linear operators on a given Hilbert space \( \mathcal{H} \), which we shall call the four operator topologies of \( \mathcal{B}(\mathcal{H}) \). In each of these topologies, \( \mathcal{B}(\mathcal{H}) \) is a locally convex linear topological space.

The topologies and their descriptions are as follows.

A basis neighborhood of an operator \( A \) in the weak operator topology, sometimes called the weak topology, is determined by a number \( \epsilon > 0 \) and two sets of elements \( \xi_1, \cdots, \xi_n \) and \( \eta_1, \cdots, \eta_n \) from \( \mathcal{H} \), where \( n \) is some positive integer. It consists of all operators \( B \) for which \( |((A - B)\xi_j, \eta_j)| < \epsilon \) for \( j = 1, 2, \cdots, n \). The weak
topology is thus the weakest of all topologies on \( \mathcal{B}(\mathcal{H}) \) in which every linear functional of the form \( A \rightarrow (A\xi, \eta) \) is continuous.

A basis neighborhood of an operator \( A \) in the **strong operator topology**, or **strong topology**, is determined by a number \( \epsilon > 0 \) and a finite set \( \xi_1, \ldots, \xi_n \) from \( \mathcal{H} \). It consists of those operators \( B \) for which \( |(A - B)\xi_j| < \epsilon \) for \( j = 1, 2, \ldots, n \).

A basis neighborhood of an operator \( A \) in the **ultraweak topology** is determined by a number \( \epsilon > 0 \) and two sequences \( \{\xi_j\}_{j=1}^{\infty} \) and \( \{\eta_j\}_{j=1}^{\infty} \) in \( \mathcal{H} \) such that
\[
\sum_{j=1}^{\infty} |\xi_j|^2 < \infty, \quad \sum_{j=1}^{\infty} |\eta_j|^2 < \infty.
\]
It consists of those operators \( B \) for which
\[
\sum_{j=1}^{\infty} |(A - B)\xi_j, \eta_j| < \epsilon.
\]

A basis neighborhood of an operator \( A \) in the **ultrastrong topology** is determined by a number \( \epsilon > 0 \) and a sequence \( \{\xi_j\}_{j=1}^{\infty} \) in \( \mathcal{H} \) such that
\[
\sum_{j=1}^{\infty} |\xi_j|^2 < \infty.
\]
It consists of those operators \( B \) for which
\[
\sum_{j=1}^{\infty} |(A - B)\xi_j|^2 < \epsilon.
\]

The weak topology is the weakest of the four, and the ultrastrong topology is the strongest of the four. Also, each of the four operator topologies is weaker than the norm topology on \( \mathcal{B}(\mathcal{H}) \), hence if a linear subspace is closed in one of the operator topologies, it is norm closed also.
Operator multiplication in $\mathcal{B}(\mathcal{H})$ is continuous in each variable separately in any one of the four operator topologies, i.e., if $B$ is fixed in $\mathcal{B}(\mathcal{H})$, then the functions defined by $A \mapsto AB$ and $A \mapsto BA$ are continuous in each of the topologies. However, multiplication is not in general continuous in both variables simultaneously.

If $\mathcal{S}$ is an arbitrary subset of $\mathcal{B}(\mathcal{H})$, the **commutant** of $\mathcal{S}$, which is denoted by $\mathcal{S}'$, is defined to be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ for which $AB = BA$ and $AB^* = B^*A$ for every $B \in \mathcal{S}$. The commutant $\mathcal{S}'$ of an arbitrary set $\mathcal{S}$ is always a symmetric subring of $\mathcal{B}(\mathcal{H})$ containing the identity, and is closed in each of the four operator topologies, hence also in the norm topology. The commutant $\mathcal{S}''$ of the commutant $\mathcal{S}'$ is called the **double commutant** of $\mathcal{S}$, and clearly $\mathcal{S} \subseteq \mathcal{S}''$. A result of fundamental importance, known as the **double commutant theorem**, will now be stated as a formal result. The theorem and its proof, as well as related results, can be found in Chapter I, §3 of [1], and also in Chapter VII of [5].

**Theorem 1.2.** (a) If $R$ is a symmetric subring of $\mathcal{B}(\mathcal{H})$ containing the identity, then $R$ is closed in one of the four operator topologies if, and only if it is closed in each of the four simultaneously. For this to be the case, it is necessary and sufficient that $R = R''$. 
(b) The smallest weakly closed symmetric subring of $\mathcal{B}(\mathcal{H})$ with the identity which contains a given set $\mathcal{J}$ is $\mathcal{J}''$. If $\mathcal{J}$ is a symmetric subring with the identity, $\mathcal{J}''$ coincides with its weak closure.

We now suppose $R$ is a Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ with the identity having the property that $R$ contains every projection operator of $R''$. If $A$ is an arbitrary hermitian operator of $R''$, we have its spectral decomposition $A = \int dP(\lambda)$. Since $A$ commutes with every operator of $R'$, every $P(\lambda)$ commutes with all operators of $R'$, hence every $P(\lambda) \in R$ by hypothesis. The Riemann sums of the above integral approximate $A$ in the norm topology, hence $A \in R$, since $R$ is closed in norm. Since every $A \in R''$ is of the form $A_1 + iA_2$, where $A_1, A_2$ are hermitian operators of $R''$, we have $R'' \subseteq R$, in other words, $R$ is weakly closed. We have thus shown the following result.

**Theorem 1.3.** If $R$ is a Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ with the identity such that $R$ contains every projection of its weak closure $R''$, then $R$ is weakly closed.

If $E$ is a linear subspace of $\mathcal{B}(\mathcal{H})$, we denote by $E^*$ the dual space of $E$ as a normed linear space, i.e., the set of all linear functionals on $E$ which are continuous in the norm topology, so that $E^*$ is a Banach space in the usual norm. We denote by $E_*$ the set of all linear
functionals on $E$ continuous in the ultrastrong topology, and by $E_\sim$ the set of linear functionals on $E$ continuous in the strong topology. It is then clear that $E_\sim \subseteq E_* \subseteq E^*$. If $E$ is closed in the ultraweak topology, then several important relationships hold among these three spaces.

Theorem 1.4. Suppose $E$ is a linear subspace of $\mathcal{B}(\mathcal{H})$ which is closed in the ultraweak topology.

(a) The set of linear functionals on $E$ continuous in the weak topology coincides with the set $E_\sim$ of linear functionals continuous in the strong topology.

(b) The set of linear functionals on $E$ continuous in the ultraweak topology coincides with the set $E_*$ of linear functionals continuous in the ultrastrong topology.

(c) $E_*$ is a norm-closed linear subspace of $E^*$, and $E_\sim$ is dense in $E_*$ in the norm topology.

(d) The dual space of $E_*$ as a Banach space with the norm of $E^*$ is isometrically isomorphic to $E$, the isometric isomorphism from $E$ to $E^*$ being defined as follows: If $A \in E$, the linear functional on $E_*$ corresponding to $A$ is defined by the mapping $T \mapsto T(A)$.

A proof of this theorem and other results is given in Chapter I, §3 of [1].

The theorem will be of importance in this work for $E$ a weakly closed symmetric subring of $\mathcal{B}(\mathcal{H})$ with identity. According to Theorem 1.2, this is equivalent
to assuming $E$ ultraweakly closed.

We shall now consider a Banach symmetric subring $R$ of $\mathcal{B}(\mathcal{H})$ satisfying the property that $R' \subset R$. It follows that $R$ contains the identity, and if we denote the ring $R'$ by $E$, then $E$ is both the center and commutant of $R$. $E$ is a commutative, weakly closed symmetric ring containing the identity, and from Theorem 1.2(b), $E'$ is the weak closure of $R$. If $F$ is a positive functional on $R$, then $F$ is said to be centrally reducible if for every positive functional $G$ on $E$ for which $G(A*A) \leq F(A*A)$ for all $A \in E$, there exists a positive definite operator $B \in E$ such that $G(A) = F(BA)$ for all $A \in E$. It may be noted that the central reducibility of $F$ depends only on its behavior on $E$, and not on all of $R$.

We now state a formal result for a particular functional of this kind.

Theorem 1.5. Suppose $R$ is a Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ such that $R' \subset R$, and such that $\mathcal{H}$ is cyclic with respect to $R$. Let $\xi_0$ be a unit cyclic vector for $\mathcal{H}$ with respect to $R$, $E$ denote the commutative Banach symmetric ring $R'$, which contains the identity, and $M$ be the maximal ideal space of $E$.

(a) There exists a unique regular Borel measure $\mu$ on $M$ such that $(A\xi_0, \xi_0) = \int_M \hat{A}(m) d\mu(m)$ for every $A \in E$. 
\( \mu \) is necessarily non negative, \( \mu(M) = 1 \), and the support of \( \mu \) is all of \( M \).

(b) The functional \( F \) defined by \( F(A) = \langle A\xi_0, \xi_0 \rangle \) for all \( A \in R \) is a centrally reducible normalized positive functional.

(c) If \( \mu \) is the measure of part (a), then there exists a unique system \( \{f_m\}_{m \in M} \) of normalized positive functionals on \( R \) such that the following hold:

(i) \( f_m(A) \) is a continuous function on \( M \) for every \( A \in R \).

(ii) \( f_m(AB) = f_m(A)B(m) \) if \( A \in R, B \in E, m \in M \).

(iii) \( \langle A\xi_0, \xi_0 \rangle = \int_M f_m(A)d\mu(m) \) if \( A \in R \).

A proof of this result is given by Tomita [10]. An alternative presentation in which the notations and terminology are nearly the same as we are using here is given in Chapter VIII of [4]. The decomposition of part (c) was also obtained earlier by Segal [7].

The part of Theorem 1.5 which is of importance for our work is (a), and the ring \( R \) will be all of \( E' \). If Tomita's work is examined, it is seen that part (c) is an easy consequence of the central reducibility of part (b), however the establishment of the central reducibility is quite complicated. We shall therefore adopt the following plan. Part (a) will be shown immediately. In our development of the theory of Chapter II, which depends upon part (a), we
shall obtain the central reducibility of part (b) as a simple corollary. Thus with Tomita's derivation of part (c) from part (b), we will have a new and shorter proof of the theorem. Although part (c) is not of direct interest here, we shall use it later to show how our theory compares with that of Tomita and Segal.

Proof of (a). The existence and uniqueness of a measure \( \mu \) such that \((A\xi_o, \xi_o) = \int_M \hat{A}(m) d\mu(m)\) for all \( A \in E \) was seen earlier. That \( \mu \) is non negative follows from the fact that \( A \rightarrow (A\xi_o, \xi_o) \) is a positive functional on \( E \). The normalization on \( \mu \) follows from the equation

\[
\mu(M) = \int_M 1 d\mu(m) = \int_M \hat{1}(m) d\mu(m) = (I\xi_o, \xi_o) = |\xi_o|^2 = 1.
\]

We must show that the support of \( \mu \) is all of \( M \). This is equivalent to showing that \( \int_M \varphi(m) d\mu(m) > 0 \) whenever \( \varphi \) is a non negative continuous function which is not identically zero. But every such function \( \varphi \) is of the form \( \hat{A} \) for a unique positive definite operator \( A \in E \). Since \( \int_M \hat{A}(m) d\mu(m) = (A\xi_o, \xi_o) \), it suffices to show that the only positive definite operator \( A \in E \) for which \((A\xi_o, \xi_o) = 0\) is the zero operator.

Suppose, therefore, that \( A \in E \), \( A \) is positive definite, and \((A\xi_o, \xi_o) = 0\). If \( B, C \) are arbitrary in \( R \), it follows from the positive definiteness of \( A \) that \( |(AB\xi_o, \xi_o)|^2 = |(A\xi_o, B^*C\xi_o)|^2 \leq (A\xi_o, \xi_o)(AB^*C\xi_o, B^*C\xi_o) = 0 \), hence \((AB\xi_o, \xi_o) = 0\). Since \( \xi_o \) is a cyclic vector for \( \mathcal{H} \) with
respect to \( R \), we have \((A\xi, \eta) = 0\) for arbitrary \( \xi, \eta \in \mathcal{H} \), hence \( A = 0 \). 

The following convergence theorem on nets of positive definite operators will later be necessary. A proof of the theorem for a monotonic sequence of hermitian operators is given on p. 263 of [6], however, for completeness, we shall give a proof for the theorem exactly as it is stated here.

**Theorem 1.6.** Suppose \( \mathcal{S} \) is a directed system, and 
\((A_{\alpha})_{\alpha \in \mathcal{S}}\) is a net of positive definite operators in \( \mathcal{B}(\mathcal{H}) \) such that \( A_\alpha \geq A_\beta \) whenever \( \alpha \geq \beta \), and such that \(|A_\alpha|\) is uniformly bounded. Then the net \((A_{\alpha})_{\alpha \in \mathcal{S}}\) converges to a positive definite operator \( A \in \mathcal{B}(\mathcal{H}) \) in the strong, hence also in the weak topology of \( \mathcal{B}(\mathcal{H}) \). The operator \( A \) is a least upper bound for \((A_{\alpha})_{\alpha \in \mathcal{S}}\) in \( \mathcal{B}(\mathcal{H}) \), i.e., \( A \geq A_\alpha \) for every \( \alpha \in \mathcal{S} \), and if \( B \in \mathcal{B}(\mathcal{H}) \) such that \( B \geq A_\alpha \) for every \( \alpha \), then \( B \geq A \).

**Proof.** If \( \xi \) is any vector of \( \mathcal{H} \), we have 
\((A_{\alpha}\xi, \xi) \geq (A_{\beta}\xi, \xi)\) whenever \( \alpha \geq \beta \), and \((A_{\alpha}\xi, \xi) \leq K|\xi|^2\), where \( K = \sup_{\alpha \in A}\|A_\alpha\| \). Thus \( \lim_{\alpha \in \mathcal{S}} (A_{\alpha}\xi, \xi) \) exists for every vector \( \xi \in \mathcal{H} \). If \( \xi, \eta \) are arbitrary vectors, it follows from the equation
\[
(A_{\alpha}\xi, \eta) = \frac{1}{4} \left[ (A_{\alpha}(\xi + \eta), \xi + \eta) - (A_{\alpha}(\xi - \eta), \xi - \eta) 
+ i(A_{\alpha}(\xi + i\eta), \xi + i\eta) 
- i(A_{\alpha}(\xi - i\eta), \xi - i\eta) \right]
\]
that \( \lim_{\alpha \in \mathcal{A}} (A_{\alpha}, \xi, \eta) \) exists for arbitrary \( \xi, \eta \in \mathcal{H} \). Define
\[ \langle \xi, \eta \rangle = \lim_{\alpha \in \mathcal{A}} (A_{\alpha}, \xi, \eta). \]
Then \( \langle \cdot, \cdot \rangle \) is a bilinear form on \( \mathcal{H} \), and
\[ |\langle \xi, \eta \rangle| = \left| \lim_{\alpha \in \mathcal{A}} (A_{\alpha}, \xi, \eta) \right| \leq K |\xi| |\eta| \]
for all \( \xi, \eta \in \mathcal{H} \). Thus there exists a unique \( A \in \mathcal{B}(\mathcal{H}) \) for which
\[ \langle \xi, \eta \rangle = (A \xi, \eta) \]
for all \( \xi, \eta \in \mathcal{H} \). (cf. [3], pp. 38-39) From the equation \( (A \xi, \eta) = \lim_{\alpha \in \mathcal{A}} (A_{\alpha} \xi, \eta) \), which holds for arbitrary \( \xi, \eta \), it follows that \( \{A_{\alpha}\}_{\alpha \in \mathcal{A}} \) converges to \( A \) in the weak topology. If \( \xi \in \mathcal{H}, (A \xi, \xi) = \lim_{\alpha \in \mathcal{A}} (A_{\alpha} \xi, \xi) \geq 0 \), so
\( A \) is positive definite.

We show next that \( A \) is a least upper bound. If \( \alpha \)
is fixed in \( \mathcal{A} \), \( \xi \in \mathcal{H} \), and \( \beta \geq \alpha \), then \( (A_{\beta} \xi, \xi) \geq (A_{\alpha} \xi, \xi) \), hence \( (A \xi, \xi) = \lim_{\beta \in \mathcal{A}} (A_{\beta} \xi, \xi) \geq (A_{\alpha} \xi, \xi) \), so that \( A \geq A_{\alpha} \). If
\( B \in \mathcal{B}(\mathcal{H}) \) such that \( B \geq A_{\alpha} \) for every \( \alpha \in \mathcal{A} \), then for arbitrary \( \xi \in \mathcal{H}, (B \xi, \xi) \geq \lim_{\alpha \in \mathcal{A}} (A_{\alpha} \xi, \xi) = (A \xi, \xi) \). Thus
\( B \geq A \).

It remains to show that the net \( \{A_{\alpha}\}_{\alpha \in \mathcal{A}} \) converges strongly to \( A \). From the inequality \( |(A \xi, \eta)| = |
\langle \xi, \eta \rangle| \leq K |\xi| |\eta| \) derived above, we have \( |A| \leq K \). Using that \( A - A_{\alpha} \) is positive definite, we have \( |A - A_{\alpha}| \leq K \) for each \( \alpha \). Hence, for all \( \xi \in \mathcal{H}, \)
\[ |A \xi - A_{\alpha} \xi|^{\dagger} = |(A - A_{\alpha}) \xi|^{\dagger} = ((A - A_{\alpha}) \xi, (A - A_{\alpha}) \xi) \leq ((A - A_{\alpha}) \xi, \xi)((A - A_{\alpha})(A - A_{\alpha}) \xi, \xi) \]
\[ \leq ((A - A_{\alpha}) \xi, \xi) |A - A_{\alpha}|^{3} |\xi|^{2} \leq ((A - A_{\alpha}) \xi, \xi) K^{3} |\xi|^{2}. \]
But we already have \( \lim_{\alpha \in \mathcal{A}} ((A - A_{\alpha}) \xi, \xi) = 0 \), hence \( \lim_{\alpha \in \mathcal{A}} |A \xi - A_{\alpha} \xi| = 0 \). Since \( \xi \) was arbitrary,
\{A_\alpha\}_{\alpha \in \mathcal{J}} \text{ converges to } A \text{ strongly.}

The weak convergence of the net \{A_\alpha\}_{\alpha \in \mathcal{J}} will be sufficient in our applications of the theorem.

We shall now introduce a notion of direct sum of rings.

Definition 1.7. Suppose \{\mathcal{H}_\alpha\}_{\alpha \in \Lambda} is a system of Hilbert spaces, and \mathcal{R}_\alpha is a symmetric subring of \mathcal{B}(\mathcal{H}_\alpha) for every \alpha \in \Lambda. Let \mathcal{H} = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha. A symmetric subring \mathcal{R} of \mathcal{B}(\mathcal{H}) will be called a direct sum of the rings \mathcal{R}_\alpha if the following are true:

1.7(a) Every \mathcal{H}_\alpha is invariant for \mathcal{R}.

1.7(b) If A \in \mathcal{R}, then A\mathcal{H}_\alpha \in \mathcal{R}_\alpha for each \alpha \in \Lambda.

1.7(c) If \alpha \in \Lambda and A \in \mathcal{R}_\alpha, then there exists an operator \hat{A} \in \mathcal{R} for which \hat{A}\mathcal{H}_\alpha = A_\alpha.

The complete direct sum of the rings \mathcal{R}_\alpha is defined to be the set of all operators \hat{A} \in \mathcal{B}(\mathcal{H}) such that every \mathcal{H}_\alpha is invariant for \hat{A} and \hat{A}\mathcal{H}_\alpha \in \mathcal{R}_\alpha.

The complete direct sum is a direct sum. In fact, it is clear that the complete direct sum is a subring of \mathcal{B}(\mathcal{H}). Since each \mathcal{H}_\alpha is invariant for an operator \hat{A} of the complete direct sum, \mathcal{H}_\alpha \perp = \sum_{\beta \neq \alpha} \mathcal{H}_\beta is also invariant for \hat{A}, hence \mathcal{H}_\alpha reduces \hat{A}. Thus \mathcal{H}_\alpha reduces \hat{A}^*, and \hat{A}^*\mathcal{H}_\alpha = (A^*\mathcal{H}_\alpha)^* \in \mathcal{R}_\alpha. We have from this that the complete
direct sum is symmetric. It is clear that (1.7a) and (1.7b) are satisfied by the complete direct sum. To show (1.7c), we note that if $A_\alpha \in R_\alpha$ and $P_\alpha$ is the projection of $H$ onto $H_\alpha$, then $A_\alpha P_\alpha$ is in the complete direct sum and 

$$(A_\alpha P_\alpha)|_{H_\alpha} = A_\alpha.$$

If $A$ is any operator of the complete direct sum, and $A_\alpha = A|_{H_\alpha}$ for each $\alpha$, then $A_\alpha \in R_\alpha$ for each $\alpha$, and 

$$|A_\alpha| \leq |A|.$$  Conversely, if we are given a system of operators $\{A_\alpha\}_{\alpha \in \Lambda}$ such that $A_\alpha \in R_\alpha$ for each $\alpha$, and such that $\{|A_\alpha|\}_{\alpha \in \Lambda}$ is uniformly bounded, then 

$$A^\otimes = \sum_{\alpha \in \Lambda} A_\alpha P_\alpha$$ 

defines an operator in the complete direct sum, where $P_\alpha$ is the projection of $H$ onto $H_\alpha$. The correspondence $A \leftrightarrow \{A_\alpha\}_{\alpha \in \Lambda}$ is clearly unique, and 

$$|A| = \sup_{\alpha \in \Lambda} |A_\alpha|.$$  It follows easily that the complete direct sum is a Banach symmetric subring of $B(H)$ if, and only if each $R_\alpha$ is a Banach symmetric subring of $B(H_\alpha)$. The complete direct sum contains the identity of $B(H)$ if, and only if each $R_\alpha$ contains the identity of $B(H_\alpha)$. It is clear that every direct sum of the $R_\alpha$ is a symmetric subring of the complete direct sum.

For a particular type of direct sum, we shall introduce a special notation. Suppose $M$ is a compact Hausdorff space, and $\mu$ is a non negative regular Borel measure on $M$. 
Then $L_2(M,\mu)$ is a Hilbert space. If $\varphi \in L_\infty(M,\mu)$, we may define an operator on $L_2(M,\mu)$ by $Af(m) = \varphi(m)f(m)$. The operator $A$ will be denoted by $\varphi^*$ to signify its definition in terms of $\varphi$. Also, if $\{\mathcal{H}_\alpha\}_{\alpha \in \Lambda}$ is a system of subspaces of $L_2(M,\mu)$, each invariant for $\varphi^*$ for all $\varphi \in L_\infty(M,\mu)$, then $\varphi^*$ restricted to $\mathcal{H}_\alpha$ is an operator on $\mathcal{H}_\alpha$, and the set of such operators form a Banach symmetric subring $R_\alpha$ of $\mathcal{B}(\mathcal{H}_\alpha)$ with the identity, for each $\alpha$. We form the space $\mathcal{H} = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha$, and define for each $\varphi \in L_\infty(M,\mu)$ an operator on $\mathcal{H}$, to be denoted again by $\varphi^*$, by the equation $\varphi^*([f_\alpha]_{\alpha \in \Lambda}) = [\varphi^*f_\alpha]_{\alpha \in \Lambda}$, where $f_\alpha \in \mathcal{H}_\alpha$ for each $\alpha$, $\sum_{\alpha \in \Lambda} \|f_\alpha\|^2 < \infty$, and $\varphi^*f_\alpha$ on the right side of this equation is interpreted as above. The system of operators $\varphi^*$ will turn out to be a direct sum of the $R_\alpha$, but not in general the complete direct sum.
CHAPTER II

THE STRUCTURE OF DIAGONAL RINGS

This chapter is devoted to the study of a particular kind of commutative ring of operators. We shall show the relationships between such a ring, its spatial structure, and its maximal ideal space.

Definition 2.1. Suppose \( \mathcal{H} \) is a Hilbert space. A diagonal ring on \( \mathcal{H} \) is defined to be a weakly closed, symmetric commutative subring of \( \mathcal{B}(\mathcal{H}) \) which contains the identity.

Throughout this chapter, \( E \) will denote a fixed diagonal ring on a Hilbert space \( \mathcal{H} \), \( M \) its maximal ideal space. From the definition of commutant, we have \( E \subseteq E' \), and by Theorem 1.2(a), \( E'' = E \). Since \( E \) is completely regular, the Gelfand transform is a symmetric isomorphism of \( E \) onto \( C(M) \).

Theorem 2.2. \( M \) is extremely disconnected, i.e., the closure of an open set is open. Hence \( M \) is also totally disconnected.

Proof. Suppose \( U \) is an open subset of \( M \). Let \( \mathcal{F} \) denote the set of all functions \( f \) in \( C(M) \) for which \( 0 \leq f(m) \leq 1 \) for all \( m \), and \( f(m) = 0 \) for \( m \not\in U \). \( \mathcal{F} \) is then a directed system if we define \( f_1 \geq f_2 \) whenever \( f_1, f_2 \in \mathcal{F} \) and \( f_1(m) \geq f_2(m) \) for all \( m \in M \). For each \( f \in \mathcal{F} \),
let \( A_f \) denote the unique operator of \( E \) for which \( \hat{A}_f = f \).  

Then \( \{A_f\}_{f \in \mathcal{F}} \) is a net of operators for which \( 0 \leq A_f \leq I \) 
for all \( f \), and \( A_{f_1} \geq A_{f_2} \) whenever \( f_1 \geq f_2 \). By Theorem 1.6, 

\( \{A_f\}_{f \in \mathcal{F}} \) converges weakly to a positive definite operator 
\( P \) which is a least upper bound for the \( A_f \). Since \( E \) is 
weakly closed, \( P \in E \). It is claimed that \( \hat{P} = \chi_U \). In fact, 

since \( P \geq A_f \) for each \( f \in \mathcal{F} \), we have \( P(m) \geq A_f(m) = f(m) \)
for all \( m \in M \). If \( m_0 \in U \), there exists, by Urysohn's lemma, 
an \( f \in \mathcal{F} \) such that \( f(m_0) = 1 \). This shows \( \hat{P}(m_0) \geq 1 \). Since 
P is a least upper bound for the \( A_f \) and \( A_f \leq I \) for each 
f, \( \hat{P}(m_0) \leq \hat{I}(m_0) = 1 \). It follows that \( \hat{P}(m) = 1 \) for all 
m \in U. Since \( \hat{P} \) is continuous, \( \hat{P}(m) = 1 \) for all \( m \in U \).

Now suppose \( m_0 \not\in U \). Using Urysohn's lemma, there 
exists a continuous function \( g : M \to [0,1] \) such that 
\( g(m) = 1 \) for \( m \in U \), and \( g(m_0) = 0 \). Let \( \hat{A}_g \) be the operator 
in \( E \) such that \( \hat{A}_g = g \). As \( g(m) \geq f(m) \) for all \( m \in M \)
whenever \( f \in \mathcal{F} \), it follows that \( \hat{A}_g \geq A_f \) for \( f \in \mathcal{F} \). Since 
P is a least upper bound of \( \{A_f\}_{f \in \mathcal{F}} \), \( \hat{A}_g \geq P \), consequently 
g(m_0) = \hat{A}_g(m_0) \geq \hat{P}(m_0) \), hence \( \hat{P}(m_0) \leq 0 \). Since \( P \) is 
positive definite, \( \hat{P}(m_0) \geq 0 \), hence \( \hat{P}(m_0) = 0 \). Since \( m_0 \)
was arbitrary, \( \hat{P}(m) = 0 \) for \( m \not\in U \). We thus have \( \hat{P} = \chi_U \).

Since \( \hat{P} \) is continuous, \( U \) is necessarily an open set.

We now investigate the relationship between the subspaces 
of \( \mathcal{H} \) invariant for \( E' \) and the clopen subsets of \( M \).
Theorem 2.3. (a) The mapping \( U \rightarrow P_U(\mathcal{H}) \) defines a one-to-one correspondence from the clopen subsets of \( M \) onto the subspaces of \( \mathcal{H} \) invariant for \( E' \).

(b) If \( U_1, U_2 \) are clopen in \( M \), then \( P_{U_1} P_{U_2} = P_{U_1 \cap U_2} \), \( P_\emptyset = 0 \), and \( P_M = I \). Thus if \( P_{U_1}(\mathcal{H}) = \mathcal{M}_1 \), and \( P_{U_2}(\mathcal{H}) = \mathcal{M}_2 \), \( \mathcal{M}_1 \perp \mathcal{M}_2 \) if, and only if \( U_1 \cap U_2 = \emptyset \), and \( \mathcal{M}_1 \subset \mathcal{M}_2 \) if, and only if \( U_1 \subset U_2 \).

(c) \( \mathcal{M} = P_U(\mathcal{H}) \) is an irreducible subspace for \( E' \) if, and only if \( U \) consists of a single isolated point.

(d) If \( \{U_\alpha\}_{\alpha \in \Lambda} \) is a family of pairwise disjoint nonvoid clopen sets, and \( \mathcal{M}_\alpha = P_U(\mathcal{H}) \) for each \( \alpha \), so that \( \mathcal{M}_\alpha \perp \mathcal{M}_\beta \) if \( \alpha \neq \beta \), then \( \bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha = P(\bigcup_{\alpha \in \Lambda} U_\alpha)(\mathcal{H}) \).

(e) A subspace \( \mathcal{M} = P_U(\mathcal{H}) \) is generated by a cyclic vector with respect to \( E' \) if, and only if every family of pairwise disjoint, nonvoid clopen subsets of \( U \) is at most countable.

Proof of (a). Since \( E' \) is a symmetric ring, every invariant subspace for \( E' \) is necessarily reducing. But a closed linear subspace \( \mathcal{M} \) is reducing for \( E' \) if, and only if the projection \( P \) onto \( \mathcal{M} \) commutes with all operators of \( E' \), i.e., if and only if \( P \in E'' = E \). Every such \( P \) is of the form \( P_U \). Since distinct \( U \) determine distinct projections \( P_U \), the result follows.
Proof of (b). Taking Gelfand transforms, we have
\[
\mathcal{P}_{U_1} \mathcal{P}_{U_2} = \mathcal{P}_{U_1} \mathcal{P}_{U_2} = \chi_{U_1} \chi_{U_2} = \chi_{U_1} \cap \chi_{U_2} = \mathcal{P}_{U_1} \cap \mathcal{P}_{U_2},
\]
hence
\[
\mathcal{P}_{U_1} \mathcal{P}_{U_2} = \mathcal{P}_{U_1} \cap \mathcal{P}_{U_2}.
\]
That \( \mathcal{P}_\emptyset = 0 \) and \( \mathcal{P}_M = I \) are obvious, since \( \chi_\emptyset (m) = 0 \) and \( \chi_M (m) = 1 \).

The condition \( \mathcal{P}_{U_1} (\mathcal{H}) \perp \mathcal{P}_{U_2} (\mathcal{H}) \) is equivalent to
\[
\mathcal{P}_{U_1} \mathcal{P}_{U_2} = 0,
\]
which, by the above, is equivalent to
\[
U_1 \cap U_2 = \emptyset.
\]
The condition \( \mathcal{P}_{U_1} (\mathcal{H}) \subset \mathcal{P}_{U_2} (\mathcal{H}) \) is equivalent to
\[
\mathcal{P}_{U_1} \mathcal{P}_{U_2} = \mathcal{P}_{U_1},
\]
which, again using the above, is equivalent to
\[
U_1 \cap U_2 = U_1, \text{ or } U_1 \subset U_2.
\]

Proof of (c). By definition, \( \mathcal{M} = \mathcal{P}_U (\mathcal{H}) \) is irreducible for \( \mathcal{E}' \) if \( \mathcal{M} \neq (0) \), and no closed linear subspace of \( \mathcal{M} \), other than \( \mathcal{M} \) or \( (0) \), is invariant for \( \mathcal{E}' \). In view of part (b), this is equivalent to having \( U \neq \emptyset \), and \( U \) containing no clopen subsets other than \( U \) and \( \emptyset \). Since \( M \) is totally disconnected, this is equivalent to having \( U \) consist of a single isolated point.

Proof of (d). Since each \( \mathcal{M}_\alpha \) is invariant for \( \mathcal{E}' \), the direct sum \( \sum_{\alpha \in \Delta} \Theta \mathcal{M}_\alpha \) is invariant for \( \mathcal{E}' \), hence
\[
\sum_{\alpha \in \Delta} \Theta \mathcal{M}_\alpha = \mathcal{P}_U (\mathcal{H}) \text{ for some clopen set } U.
\]
Since
\[
\mathcal{P}_{U_\alpha} (\mathcal{H}) \subset \mathcal{P}_U (\mathcal{H}) \text{ we have } U_\alpha \subset U, \text{ for each } \alpha.
\]
Thus
\[
U \cup \bigcup_{\alpha \in \Delta} U_\alpha \subset U, \text{ and since } U \text{ is closed, } \bigcup_{\alpha \in \Delta} U_\alpha \subset U.
\]
By
Theorem 2.2, \( \bigcup_{\alpha \in \Lambda} U_\alpha \) is a clopen set, hence
\[ V = U - \bigcup_{\alpha \in \Lambda} (U_\alpha) \] is clopen. Since \( V \cap U_\alpha = \emptyset \) for each \( \alpha \), we have \( P_V(\mathcal{H}) \perp P_U(\mathcal{H}) \), hence \( P_V(\mathcal{H}) \perp \bigoplus_{\alpha \in \Lambda} P_U(\mathcal{H}) = P_U(\mathcal{H}) \), and \( V \cap U = \emptyset \). Since \( V \subseteq U \), necessarily \( V = \emptyset \).

Thus \( U = \bigcup_{\alpha \in \Lambda} U_\alpha \).

**Proof of (e).** We first assume that \( \mathcal{M} \) is generated by a cyclic vector \( \xi_0 \) with respect to \( E' \). Suppose \( \{U_\alpha\}_{\alpha \in \Lambda} \) is any family of pairwise disjoint nonvoid clopen subsets of \( U \). Then \( \{P_U(\mathcal{H})\}_{\alpha \in \Lambda} \) is a family of mutually orthogonal nonzero projection operators, so
\[ \sum_{\alpha \in \Lambda} |P_U(\xi_0)|^2 \leq |\xi_0|^2. \]
This implies that at most a countable number of \( P_U(\xi_0) \) are not zero. But if \( P_U(\xi_0) = 0 \) for some fixed \( \alpha \), then for every operator \( A \in E' \) we have \( P_U(A\xi_0) = AP_U(\xi_0) = 0 \), and since \( \mathcal{C} \{A\xi_0 | A \in E'\} = \mathcal{M}, P_U(\xi_0) = 0 \) for every \( \xi \in \mathcal{M} \). Since \( P_U \) is the projection onto \( \mathcal{M} \), this implies \( P_U P_U = 0 \). But \( U_\alpha \subset U \), hence \( U_\alpha = \emptyset \) contrary to hypothesis. The vectors \( P_U(\xi_0) \) are thus all nonzero, hence by the above, the family \( \{U_\alpha\}_{\alpha \in \Lambda} \) is at most countable.
Conversely, suppose every family of pairwise disjoint nonvoid clopen subsets of \( U \) is countable. Let 
\[ \mathcal{P}_U(\mathcal{H}) = \sum_{\alpha \in \Lambda} \mathcal{M}_\alpha, \]
where each \( \mathcal{M}_\alpha \) is a nontrivial subspace generated, with respect to \( E' \), by a cyclic vector \( \xi_\alpha \). We may assume the \( \xi_\alpha \) to be normalized so that \( |\xi_\alpha| = 1 \). Then \( \mathcal{M}_\alpha = \mathcal{P}_U(\mathcal{H}) \) for a unique nonvoid clopen subset \( U_\alpha \) of \( U \) for each \( \alpha \), and since the \( \mathcal{M}_\alpha \) are mutually orthogonal, the \( U_\alpha \) are disjoint. By hypothesis, the family \( \{U_\alpha\}_{\alpha \in \Lambda} \) is countable. We may thus assume that the index set \( \Lambda \) consists of either the positive integers, or the first \( n \) positive integers for some \( n \).

The corresponding cyclic vectors are thus likewise indexed, say \( \{\xi_j\}_{j \in \Lambda} \). Define \( \xi_0 = \sum_{j \in \Lambda} \frac{1}{j} \xi_j \). This sum converges, since the \( \xi_j \) are pairwise orthogonal, and
\[
\sum_{j \in \Lambda} \left| \frac{1}{j} \xi_j \right|^2 = \sum_{j \in \Lambda} \frac{1}{j^2} < \infty.
\]
Let \( \mathcal{K} = \mathcal{G}(\mathcal{A} \xi_0 | A \in E') \).

Since \( \xi_0 \in \sum_{j \in \Lambda} \mathcal{M}_j = \mathcal{M} \), necessarily \( \mathcal{K} \subset \mathcal{M} \). But if \( j \in \Lambda \), then
\[
\mathcal{J} \mathcal{P}_Uj \xi_0 = \mathcal{J} \mathcal{P}_Uj \sum_{k \in \Lambda} \frac{1}{k} \xi_k = \mathcal{J} \mathcal{P}_Uj \sum_{k \in \Lambda} \frac{1}{k} \mathcal{P}_Uk \xi_k = \mathcal{J} \sum_{k \in \Lambda} \frac{1}{k} \mathcal{P}_Uj \cap \mathcal{U}_k \xi_k = \xi_j, \quad \text{hence} \quad \xi_j \in \mathcal{K}, \quad \text{and}
\]
\[
\mathcal{M}_j = \mathcal{G}(\mathcal{A} \xi_j | A \in E') \subset \mathcal{K}. \quad \text{It follows that}
\]
\[
\mathcal{M} = \sum_{j \in \Lambda} \mathcal{M}_j \subset \mathcal{K}. \quad \text{Thus} \quad \mathcal{M} = \mathcal{K}, \quad \text{so we have shown that}
\]
\( \xi_0 \) is a cyclic vector for \( \mathcal{M} \).
Since, for every clopen subset \( U \) of \( M \) we have that \( P_U(\mathcal{H}) \) reduces \( A \) for all \( A \in E' \), \( A|_{P_U(\mathcal{H})} \) is a bounded linear operator on \( P_U(\mathcal{H}) \). We now make the following convention.

**Definition 2.4.** If \( U \) is a nonvoid clopen subset of \( M \) and \( \mathcal{A} \) is a subset of \( E' \), we denote by \( \mathcal{A}_U \) the family of all operators of \( \mathcal{B}(P_U(\mathcal{H})) \) which are of the form \( A|_{P_U(\mathcal{H})} \) for some \( A \in \mathcal{A} \).

**Theorem 2.5.** (a) The mapping \( A \to A|_{P_U(\mathcal{H})} \) is a symmetric homomorphism of the ring \( E' \) onto \((E')_U\). The image of the subring \( E \) is the subset \( E_U \) of \((E')_U\), and the identity of \( E' \) maps into the identity of \((P_U(\mathcal{H}))\). Hence \((E')_U\) and \( E_U \) are Banach symmetric subrings of \( \mathcal{B}(P_U(\mathcal{H})) \) containing the identity, and \( E_U \) is commutative.

(b) The mapping \( A \to AP_U \) is a symmetric isomorphism of \((E')_U\) onto a weakly closed (two sided) ideal of \( E' \) which consists of those operators \( B \) for which \( B = BP_U \). The image of the subring \( E_U \) under this mapping is a weakly closed ideal in \( E \), consisting of those operators \( B \) of \( E \) for which \( B = BP_U \).

(c) \((E_U)' = (E')_U\), and \((E')_U' = E_U\), hence \( E_U \), \((E')_U\) are weakly closed, and \( E_U \) is a diagonal ring on \( P_U(\mathcal{H}) \).
(d) The mapping \( A \mapsto (\hat{A}\mathcal{P}_U)(m) \) defines a nontrivial complex homomorphism on \( E_U \) for each \( m \in U \), and with this identification, \( U \) is the maximal ideal space of \( E_U \). We shall henceforth abbreviate \( \hat{A}\mathcal{P}_U(m) \) to \( \hat{A}(m) \) for \( A \in E_U \) and \( m \in U \).

(e) \( E_U \) consists of the ring of scalar multiplies of the identity if, and only if \( U \) consists of a single isolated point.

Proof of (a). If \( A, B \in E' \) and \( \alpha, \beta \) are complex, we have that \( (\alpha A + \beta B)|_{\mathcal{P}_U(\mathcal{H})} = \alpha A|_{\mathcal{P}_U(\mathcal{H})} + \beta B|_{\mathcal{P}_U(\mathcal{H})} \). Using the fact that \( \mathcal{P}_U(\mathcal{H}) \) reduces all operators of \( E' \), we have \( (AB)|_{\mathcal{P}_U(\mathcal{H})} = A|_{\mathcal{P}_U(\mathcal{H})} B|_{\mathcal{P}_U(\mathcal{H})} \) and \( A^*|_{\mathcal{P}_U(\mathcal{H})} = (A|_{\mathcal{P}_U(\mathcal{H})})^* \). Thus \( A \mapsto A|_{\mathcal{P}_U(\mathcal{H})} \) defines a symmetric homomorphism. That \( (E')_U \) and \( E_U \) are the respective images of \( E' \) and \( E \) is clear from their definitions. It is also clear that the restriction to \( \mathcal{P}_U(\mathcal{H}) \) of the identity operator of \( \mathcal{H} \) is the identity of \( \mathcal{P}_U(\mathcal{H}) \).

Proof of (b). Since \( \mathcal{P}_U \) commutes with all operators of \( E' \), it is clear that the set of operators \( B \in E' \) for which \( B = \mathcal{P}_U \) is a two-sided ideal. Since operator multiplication is continuous in each variable separately in the weak topology, it is clear that the above ideal is weakly closed. If \( B \in E' \) such that \( B = \mathcal{P}_U \), then
$B|_{PU}(\mathcal{H})$ is an operator $A$ of $(E')_{U}$ such that

$AP_{U} = B|_{PU}(\mathcal{H})P_{U} = BP_{U} = B$. Hence the above ideal is

contained in the image of $(E')_{U}$ under the mapping. Conversely, if $A \in (E')_{U}$, then there exists a $B \in E'$ for which

$B|_{PU}(\mathcal{H}) = A$, so that $AP_{U} = B|_{PU}(\mathcal{H})P_{U} = BP_{U}$, and since

$(BP_{U})P_{U} = BP_{U}^{2} = BP_{U}$, $AP_{U}$ is a member of the above ideal.

We have left to show that $A \rightarrow AP_{U}$ is a symmetric isomorphism.

If $A, B \in (E')_{U}$ and $\alpha, \beta$ are complex, then

$(\alpha A + \beta B)P_{U} = \alpha AP_{U} + \beta BP_{U}$, so the mapping is linear. If

$AP_{U} = A_{1}$, then $A_{1}|_{PU}(\mathcal{H}) = AP_{U}|_{PU}(\mathcal{H}) = A$, so that not

only is the mapping one-to-one, but its inverse is given by

$A_{1} \rightarrow A_{1}|_{PU}(\mathcal{H})$, defined, of course, on the operators $A_{1}$ of $E'$ for which $A_{1} = A_{1}P_{U}$. To prove that the mapping is a

symmetric homomorphism, suppose $A, B \in (E')_{U}$, $AP_{U} = A_{1}$,

$BP_{U} = B_{1}$. Then $(A_{1}B_{1})|_{PU}(\mathcal{H}) = A_{1}|_{PU}(\mathcal{H})B_{1}|_{PU}(\mathcal{H}) = AB$,

and since $A_{1}B_{1}P_{U} = A_{1}B_{1}$, we have $A_{1}B_{1}$ as the image of $AB$.

Also, $A_{1}^{*}|_{PU}(\mathcal{H}) = (A_{1}|_{PU}(\mathcal{H}))^{*} = A^{*}$, and since $A_{1}P_{U} = A_{1}$

and $P_{U}$ commutes with $A_{1}$, $A_{1}^{*}P_{U} = A_{1}^{*}$. Thus $A_{1}^{*}$ is the

image of $A^{*}$, and we have $A \rightarrow AP_{U}$ a symmetric isomorphism.

To show that the image of $E_{U}$ is as stated, suppose

$A \in E_{U}$. Then $A = B|_{PU}(\mathcal{H})$ for some $B \in E$, hence
\( A_{PU} = B_{P_U(H)} P_U = B P_U \in E, \) and \( (A_{PU}) P_U = A_{PU}. \) Conversely,

if \( B \in E \) such that \( B = B P_U \), set \( A = B_{P_U(H)} \in E_U. \) Then

\[ A_{PU} = B_{P_U(H)} P_U = B P_U = B, \]

so that \( B \) is in the image of \( E_U. \) The fact that the set of operators \( B \in E \) for which \( B = B P_U \) is a weakly closed ideal in \( E \) can be argued exactly as above for \( E' \), or we may simply observe that the image of \( E_U \) is the intersection with \( E \) of the image of \( (E')_U. \)

**Proof of (c).** Suppose \( A \in B(P_U(H)) \) such that \( A \in (E_U)' \). Then \( AB_{P_U(H)} = B_{P_U(H)} A \) for arbitrary \( B \in E. \) Set \( A_1 = A_{PU}. \) The above equation is then equivalent to \( A_1 B = BA_1 \) for every \( B \in E, \) since \( P_U(H) \) reduces \( E. \)

Hence \( A_1 \in E', \) and since \( A_1_{P_U(H)} = A, (E_U)' \subseteq (E')_U. \) The reverse inclusion \( (E'_U) \subseteq (E'_U)' \) is clear, since \( A \rightarrow A_{P_U(H)} \)

is a symmetric homomorphism, hence \( (E'_U)' = (E'_U). \)

The fact that \( (E'_U)' = E_U \) can be argued in exactly the same way, interchanging the roles of \( E' \) and \( E, \) and recalling that \( E'' = E. \)

**Proof of (d).** The mapping described is the composite of two homomorphisms, \( A \rightarrow A_{PU} \rightarrow \hat{A}_{PU}(m), \) hence it is a homomorphism. If \( I_U \) denotes the identity of \( P_U(H), \) then
\[ I_U P_U = P_U, \]

hence \( \hat{I}_U P_U(m) = \hat{P}_U(m) = \chi_U(m) = 1, \) since we have
assumed $m \in U$. Consequently the mapping is a nontrivial complex homomorphism, hence corresponds to a maximal ideal.

Conversely, suppose $h$ is a nontrivial complex homomorphism defined on $E_U$. Then, by composing homomorphisms, $A \rightarrow A|_{P_U(H)} \rightarrow h(A|_{P_U(H)})$ defines a complex homomorphism on $E$, which is nontrivial since the image of $E$ under the homomorphism $A \rightarrow A|_{P_U(H)}$ coincides with all of $E_U$. Consequently, there exists an $m \in M$ for which $\hat{A}(m) = h(A|_{P_U(H)})$ for all $A \in E$. But if $A \in E_U$, then $AP_U \in E$, and $AP_U|_{P_U(H)} = A$, so that $h(A) = h(AP_U|_{P_U(H)}) = (AP_U)(m)$. Since we have $\chi_U(m) = \hat{P}_U(m) = 1_{P_U}(m) = h(1_{P_U}) = 1$, $m \in U$. We have thus shown that $A \rightarrow (AP_U)(m)$ defines a nontrivial complex homomorphism on $E_U$ for each $m \in U$, and conversely every such homomorphism arises in this fashion.

If $m_1, m_2 \in U$ and $m_1 \neq m_2$, there exists an $A \in E$ for which $\hat{A}(m_1) \neq \hat{A}(m_2)$. Consequently $AP_U(m_1) = \widehat{AP_U}(m_1) \neq \widehat{AP_U}(m_2) = AP_U(m_2)$. Setting $A_1 = A|_{P_U(H)}$, $A_1P_U = AP_U$, hence $AP_U(m_1) \neq AP_U(m_2)$. The correspondence between the points of $U$ and the nontrivial complex homomorphisms of $E_U$ is thus one-to-one.

We shall now show that the topology on $U$ as a subset of $M$ agrees with its Gelfand topology as the maximal ideal space of $E_U$. In fact, the latter topology is the
weakest topology in which $\hat{A}P_U(m)$ defines a continuous function for every $A \in E_U$. Since the operators of $E_U$ are those of the form $A_1|_{P_U(H)}$ for $A_1 \in E$, the above means $$\hat{(A_1|_{P_U(H)}P_U)}(m) = \hat{A_1P_U}(m) = \hat{A_1}(m)\hat{P_U}(m) = \hat{A_1}(m)$$ must define a continuous function on $U$ for all $A_1 \in E$. But this is exactly the Gelfand topology of $M$ restricted to $U$.

Proof of (e). Since $E_U$ is a diagonal ring with maximal ideal space $U$, the Gelfand transform is a symmetric isomorphism from $E$ onto $C(U)$. For $E$ to consist of scalar multiples of the identity, it is necessary and sufficient that all continuous functions on $U$ be constant, which is the case exactly when $U$ consists of a single point. Since $U$ is clopen, such a point must be isolated.

We now have the necessary tools to show how our study can be reduced to the cyclic case.

Theorem 2.6. (a) There exists a family $\{U_\alpha\}_{\alpha \in \Lambda}$ of pairwise disjoint nonvoid clopen subsets of $M$, such that $P_{U_\alpha}(H)$ is cyclic with respect to $(E')_{U_\alpha}$ for every $\alpha \in \Lambda$.

Correspondingly, $H = \Sigma_{\alpha \in \Lambda} \Theta P_{U_\alpha}(H)$.

(b) For every system $\{U_\alpha\}_{\alpha \in \Lambda}$ as in (a), $E'$ is the complete direct sum of the rings $\{(E')_{U_\alpha}\}_{\alpha \in \Lambda}$, and $E$ is the complete direct sum of the rings $\{E_{U_\alpha}\}_{\alpha \in \Lambda}$. 
Proof of (a). \( \mathcal{H} \) can be written as a direct sum
\[
\sum_{\alpha \in \Lambda} \mathcal{H}_\alpha
\]
such that every \( \mathcal{H}_\alpha \) is a cyclic subspace for \( E' \).

From Theorem 2.3 (a), every \( \mathcal{H}_\alpha \) is of the form \( P_{U_\alpha} ( \mathcal{H} ) \) for
a unique clopen \( U_\alpha \subset M \). From Theorem 2.3 (b), \( U_\alpha \cap U_\beta = \emptyset \)
if \( \alpha \neq \beta \), and from Theorem 2.3 (d), \( \left( \bigcup_{\alpha \in \Lambda} U_\alpha \right) = M \) is equivalent to
\[
\sum_{\alpha \in \Lambda} P_{U_\alpha} ( \mathcal{H} ) = \mathcal{H}.
\]
\( P_{U_\alpha} ( \mathcal{H} ) \) is clearly cyclic for \( (E')_{U_\alpha} \)
for each \( \alpha \), since it is cyclic for \( E' \) as a subspace of \( \mathcal{H} \).

Proof of (b). From the definition, \( A \in E' \) implies
\[
A|_{P_{U_\alpha} ( \mathcal{H} )} \in (E')_{U_\alpha},
\]
and, correspondingly, \( A \in E \) implies
\[
A|_{P_{U_\alpha} ( \mathcal{H} )} \in E_{U_\alpha}
\]
for every \( \alpha \in \Lambda \).

Conversely, suppose \( A \in \mathcal{B}(\mathcal{H}) \) such that every \( P_{U_\alpha} ( \mathcal{H} ) \)
is invariant for \( A \) and \( A|_{P_{U_\alpha} ( \mathcal{H} )} \in (E')_{U_\alpha} \). If \( \alpha \in \Lambda \)
and \( B \in E \), it follows from Theorem 2.5 (c) that
\[
A|_{P_{U_\alpha} ( \mathcal{H} )} B|_{P_{U_\alpha} ( \mathcal{H} )} = B|_{P_{U_\alpha} ( \mathcal{H} )} A|_{P_{U_\alpha} ( \mathcal{H} )}.
\]
In other words \( AB \xi = BA \xi \) if \( \xi \in P_{U_\alpha} ( \mathcal{H} ) \). Since \( \alpha \) was arbitrary
and \( \mathcal{H} = \sum_{\alpha \in \Lambda} P_{U_\alpha} ( \mathcal{H} ) \), we have \( AB = BA \).
But \( B \) was
arbitrary in \( E \), hence \( A \in E' \). Thus \( E' \) is the complete
direct sum of the \( (E')_{U_\alpha} \).
If, in the above paragraph, we had in addition that 
\[ A|_{P_U \alpha} (Y) \in E_{U \alpha} \] for every \( \alpha \), the same reasoning would show
\[ A|_{P_U \alpha} (Y) \cup B|_{P_U \alpha} (Y) = B|_{P_U \alpha} (Y) \cup A|_{P_U \alpha} (Y) \] for arbitrary \( B \in E' \), hence we may conclude in the same manner that \( A \in E'' = E \). This shows that \( E \) is the complete direct sum of the \( E_{U \alpha} \).

If, in the above theorem, \( \mathcal{H} \) is cyclic for \( E' \), we could take the index set \( \Lambda \) to be one point, and the corresponding clopen set to be all of \( M \). The question arises as to whether it is advantageous to decompose \( M \) in some alternative way. We can give an affirmative answer in the event that \( M \) has isolated points, for if some \( U_\alpha \) consists of such an isolated point, then from Theorem 2.5 (e) \( E_{U_\alpha} \) consists of the scalar multiples of the identity, hence \( (E')_{U_\alpha} = (E_{U_\alpha})' = \mathcal{B}(P_U (\mathcal{H})) \). However, if \( M \) has no isolated points, then each system of \( (E')_U \) and \( E_U \) for \( U \) a nonvoid clopen subset of \( M \), resembles the original \( E' \) and \( E \) in most respects. The difficulty lies in the fact that a direct sum decomposition of the above kind is inadequate for purposes of completely decomposing \( E' \) and \( E \) at all points of \( M \) in a manner similar to what we have shown is possible at isolated points. This problem is the subject of the next chapter.
Theorem 2.6 does, however, show us how to reduce our study to the case where \( \mathcal{H} \) is cyclic for \( E' \). An alternative method of decomposing \( \mathcal{H} \) to obtain a diagonal ring from \( E \) on a subspace of \( \mathcal{H} \) in this case is given by the following.

**Theorem 2.7.** Suppose \( P_1 \) is a projection in \( E' \), and \( \eta \) is a cyclic vector for \( \mathcal{H} \) with respect to \( E' \) such that \( P_1 \eta = \eta \). Let \( \mathcal{H}_1 = P_1(\mathcal{H}) \), \( E_1 \) be the set of restrictions \( \{A|_{\mathcal{H}_1} : A \in E\} \). Then the mapping \( A \to A|_{\mathcal{H}_1} \) is a symmetric isomorphism of \( E \) onto \( E_1 \). \( E_1 \) is a diagonal ring on \( \mathcal{H}_1' \), \( E_1' = \{P_1A|_{\mathcal{H}_1} : A \in E'\} \), and the vector \( \eta \) is cyclic for \( \mathcal{H}_1 \) with respect to \( E_1' \).

**Proof.** Since \( P_1 \in E' \), \( \mathcal{H}_1 \) reduces all operators of \( E \). Thus, for \( \alpha, \beta \) complex and \( A, B \in E \), we have
\[
(\alpha A + \beta B)|_{\mathcal{H}_1} = \alpha A|_{\mathcal{H}_1} + \beta B|_{\mathcal{H}_1}, \quad (AB)|_{\mathcal{H}_1} = A|_{\mathcal{H}_1} B|_{\mathcal{H}_1}, \quad A^*|_{\mathcal{H}_1} = (A|_{\mathcal{H}_1})^*. 
\]
This shows \( A \to A|_{\mathcal{H}_1} \) to be a symmetric homomorphism. To show that the kernel consists of the zero operator alone, assume \( A \in E \) such that \( A|_{\mathcal{H}_1} = 0 \). This is equivalent to \( AP_1 = 0 \). If \( B \) is any operator of \( E' \), we thus have \( A(B\eta) = BA\eta = BAP_1\eta = 0 \). Since \( \eta \) was cyclic for \( \mathcal{H} \) with respect to \( E' \), it follows that \( A = 0 \). Consequently, \( A \to A|_{\mathcal{H}_1} \) is a symmetric isomorphism. \( I|_{\mathcal{H}_1} \) is clearly the identity on \( \mathcal{H}_1 \), and from the definition, \( E_1 \)
is the range of this isomorphism. It follows that $E_1$ is a Banach symmetric subring of $\mathcal{B}(\mathcal{H})$ containing the identity. We shall show subsequently that $E_1$ is weakly closed.

To show that $E_1'$ is the set indicated, we note that for $B \in E'$, $A \in E$, we have $A|\mathcal{H}_1(P_1B) = (AP_1B)|\mathcal{H}_1 = (P_1BA)|\mathcal{H}_1 = (P_1B)|\mathcal{H}_1 A|\mathcal{H}_1$, showing that

$E_1' \supset \{(P_1B)|\mathcal{H}_1 | B \in E'\}$. Conversely, suppose $B_1 \in E_1'$.

Define $B = B_1P_1$, so that $B \in \mathcal{B}(\mathcal{H})$. Then $B|\mathcal{H}_1 = B_1$, and $B|\mathcal{H}_1' = 0$. If $A \in E$ and $\xi \in \mathcal{H}$, we have $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \mathcal{H}_1$, $\xi_2 \in \mathcal{H}_1'$. Then $AB\xi_1 = A|\mathcal{H}_1 B_1\xi_1 = B_1A|\mathcal{H}_1 \xi_1 = BA\xi_1$, and $AB\xi_2 = 0 = BA\xi_2$. Hence $AB\xi = BA\xi$.

and since $\xi$ was arbitrary, $BA = AB$. Since $A$ was arbitrary in $E$, $B \in E'$. Hence $E_1' \subset \{(P_1B)|\mathcal{H}_1| B \in E'\}$, so equality holds. To see that $\eta$ is cyclic for $\mathcal{H}_1$ with respect to $E_1'$, let $\xi \in \mathcal{H}_1$ and $\varepsilon > 0$. Then there exists a $B \in E'$ for which $|B\eta - \xi| < \varepsilon$. It follows that $|(P_1B)|\mathcal{H}_1 \eta - \xi| = |P_1B\eta - P_1\xi| = |P_1(B\eta - \xi)| \leq |B\eta - \xi| < \varepsilon$.

We shall now show $E_1$ to be weakly closed. Suppose $Q_1 \in E_1''$ and $Q_1$ is a projection operator. From the above paragraph, this means that $Q_1(P_1B)|\mathcal{H}_1 = (P_1B)|\mathcal{H}_1 Q_1$.
for each $B \in E'$. But then we have for arbitrary $B \in E'$ that

$$|BQ_1\eta|^2 = (BQ_1\eta, BQ_1\eta) = (Q_1\eta, B^*BQ_1\eta) = (Q_1\eta, (P_1B^*B)|_{\mathcal{H}_1} Q_1\eta) = (Q_1\eta, Q_1(P_1B^*B)|_{\mathcal{H}_1}\eta) = (Q_1\eta, (P_1B^*B)|_{\mathcal{H}_1}\eta) = (Q_1\eta, P_1B^*B\eta) = (Q_1\eta, B^*B\eta) = (BQ_1\eta, B\eta) \leq |BQ_1\eta| |B\eta|.$$  

From the first and last terms of this inequality we may cancel a factor of $|BQ_1\eta|$ to obtain $|BQ_1\eta| \leq |B\eta|$ for $B \in E'$. Thus the formula $Q(B\eta) = BQ_1\eta$ defines, unambiguously, a bounded linear operator $Q$ on the dense linear subspace of $\mathcal{H}$ consisting of the vectors $\{B\eta | B \in E'\}$, and $|Q| \leq 1$. $Q$ may thus be extended uniquely to a bounded linear operator on $\mathcal{H}$. For arbitrary $B, C \in E'$, we have $(QB)(C\eta) = QBC\eta = BCQ_1\eta = BQ(C\eta)$. Since the vectors $\{C\eta | C \in E'\}$ are dense in $\mathcal{H}$, we have $QB = BQ$. But since $B$ was arbitrary in $E'$, $Q \in E'' = E$.

Again, if $B$ is arbitrary in $E'$, $Q|_{\mathcal{H}_1}(P_1B)|_{\mathcal{H}_1}\eta = \eta \Rightarrow\eta$, $P_1B\eta = P_1BQ_1\eta = (P_1B)|_{\mathcal{H}_1} Q_1\eta = Q_1(P_1B)|_{\mathcal{H}_1} \eta$. From the above, the vectors $\{(P_1B)|_{\mathcal{H}_1}\eta | B \in E'\}$ are dense in $\mathcal{H}_1$. Thus $Q_1 = Q|_{\mathcal{H}_1}$, so $Q_1 \in E_1$.

Since $Q_1$ was an arbitrary projection operator of $E_1''$, $E_1$ is weakly closed by Theorem 1.3.

For the remainder of this chapter, we shall assume that $\mathcal{H}$ is generated with respect to $E'$ by a single unit cyclic vector, which we denote by $e_0$. Theorem 1.5 (a) may
then be applied to $E'$ and $E$. We denote by $\mu$ the unique non negative regular Borel measure on $M$ for which $(A\mathbb{E}_0, \mathbb{E}_0) = \int_M A(m)d\mu(m)$ for all $A \in E$. Then $\mu(M) = 1$, and $\mu$ has $M$ as its support. The terminology 'measurable' and 'almost everywhere' will mean, unless otherwise stated, with respect to the measure $\mu$. Most results will depend upon the particular $\mu$ or $\mathbb{E}_0$ involved, however those which are independent of $\mu$ or $\mathbb{E}_0$ will be indicated.

**Theorem 2.8.** (a) If $U$ is an open set in $M$, then $\mu(U) = \mu(\overline{U})$.

(b) If $K$ is a subset of $M$ such that $\overline{K}$ has void interior, then $\mu(K) = 0$.

(c) If $m \in M$, then $\mu([m]) > 0$ if, and only if $m$ is an isolated point.

(d) For every measurable set $S$, there exists a unique clopen set $U$ for which $\mu(S \Delta U) = 0$, where $S \Delta U = (S - U) \cup (U - S)$, the symmetric difference of $S$ and $U$.

(e) For every function $\varphi \in L_\infty(M, \mu)$, there exists a unique $B \in E$ and a unique $\psi \in C(M)$ such that $\hat{B} = \hat{\psi}$ and $\hat{\psi}(m) = \varphi(m)$ a.e. Furthermore, $|B| = \sup_{m \in M} |\psi(m)| = ||\varphi||_\infty$.

**Proof of (a).** If $U$ is open, denote by $\mathcal{V}$ the family of all clopen subsets $V$ of $U$. Then $\mathcal{V}$ is directed by inclusion, i.e., $V_1 \supseteq V_2$ if $V_1 \supseteq V_2$. If $V_1, V_2 \in \mathcal{V}$ and $V_1 \supseteq V_2$, then $P_{V_1} = \chi_{V_1} \supseteq \chi_{V_2} = P_2$, hence $P_{V_1} \supseteq P_{V_2}$. As
0 ≤ P_V ≤ I for V ∈ V, the net \{P_V\}_{V ∈ V} converges weakly, by Theorem 1.6, to a positive definite operator P which is a least upper bound for the P_V, and P ∈ E since E is weakly closed. Since P_V P_{M-U} = 0 for each V ∈ V and operator multiplication is continuous in each variable separately in the weak topology, P_{M-U} = 0. Consequently, if m ∉ U, then \hat{P}(m) = \hat{P}(m)P_{M-U}(m) = \hat{P}_{M-U}(m) = 0. If V is any fixed set in V, we have P_V P_{V_1} = P_{V_1} whenever V ∈ V and V ⊆ V_1. By the same reasoning as above, we have P_{V_1} = P_{V_1} for V_1 ∈ V. If m is an arbitrary point of U, there exists, by the total disconnectedness of M, a set V_1 ∈ V for which m ∈ V_1. It follows that \hat{P}(m) = \hat{P}(m)\hat{P}_{V_1}(m) = \hat{P}_{V_1}(m) = 1. Thus \hat{P}(m) = 1 for m ∈ U, and since \hat{P} is continuous, \hat{P} = \chi_U = \hat{P}_U, hence P = P_U.

From the definition of weak operator convergence, we have \lim_{V ∈ V} (P_V \xi_O, \xi_O) = (P_U \xi_O, \xi_O). It follows that
\[
\mu(U) = \int_M \chi_U(m) = \int_M P_U(m)d\mu(m) = (P_U \xi_O, \xi_O) = \lim_{V ∈ V} (P_V \xi_O, \xi_O) = \lim_{V ∈ V} \int_M P_V(m)d\mu(m) = \lim_{V ∈ V} \int_M \chi_V(m)d\mu(m) = \lim_{V ∈ V} \mu(V). \quad \text{Since } V ⊆ U \text{ for each } V ∈ V, \mu(V) ≤ \mu(U). \text{ Thus } \mu(U) ≤ \mu(U). \text{ But } \mu(U) ≤ \mu(U) \text{ is clear, so } \mu(U) = \mu(U).
\]

Proof of (b). If K has void interior, then (M - K) = M, and since M - K is open, we have from part (a) that \mu(M - K) = \mu(M), which implies \mu(K) = 0. Hence \mu(K) = 0 also.
Proof of (c). If \( m \) is an isolated point, then \( \{m\} \) is an open set. Since the support of \( \mu \) coincides with \( M \), \( \mu(\{m\}) > 0 \).

If \( m \) is not an isolated point, then \( \{m\} = \{\overline{m}\} \) has void interior, so from part (b), \( \mu(\{m\}) = 0 \).

Proof of (d). From the regularity of the measure \( \mu \), it is possible to find a sequence of compact sets \( Q_1 \subset Q_2 \subset \ldots \), such that \( Q_j \subset S \) for each \( j \), and

\[
\lim_{j \to \infty} \mu(Q_j) = \mu(S).
\]

Let \( K_j \) denote the closure of the interior of \( Q_j \) for each \( j \). Then \( K_j \subset Q_j \) for each \( j \), and \( K_1 \subset K_2 \subset \ldots \). Since \( M \) is extremely disconnected, each \( K_j \) is a clopen set, and \( Q_j - K_j \) is a closed set of void interior. Using part (b), \( \mu(Q_j - K_j) = 0 \), hence \( \mu(Q_j) = \mu(K_j) \) for each \( j \). Let \( V = \bigcup_{j=1}^{\infty} K_j \). Then \( V \) is open, \( V \subset S \), and

\[
\mu(V) = \lim_{j \to \infty} \mu(K_j) = \lim_{j \to \infty} \mu(Q_j) = \mu(S).
\]

From part (a), \( \mu(V) = \mu(\overline{V}) \). It follows that

\[
\mu(S \Delta \overline{V}) = \mu((S - \overline{V}) \cup (\overline{V} - S)) = \mu(S - \overline{V}) + \mu(\overline{V} - S) \leq \mu(S - V) + \mu(\overline{V} - V) = 0,
\]

so \( U = \overline{V} \) is a clopen set satisfying the given requirements.

If \( U_1 \) is a second clopen set satisfying the same condition, then

\[
\mu(U \Delta U_1) = \mu((U \Delta S) \Delta (S \Delta U_1)) \leq \mu(U \Delta S) + \mu(S \Delta U_1) = 0.
\]

Since \( U \Delta U_1 \) is clopen and the support of \( \mu \) is \( M \), \( U \Delta U_1 = \emptyset \), hence \( U = U_1 \).
Proof of (e). We first establish the uniqueness of $B$, $\psi$ whenever they exist, and the value of their norms. If $\hat{B}_1 = \psi_1$ and $\psi_1(m) = \varphi(m)$ a.e., where $B_1, \psi_1$ is a second pair satisfying the same conditions as a pair $B, \psi$ for which the conclusion holds, then $\psi_1(m) = \psi(m)$ a.e. Since $\psi_1, \psi$ are continuous and the support of $\mu$ is $M$, $\psi_1(m) = \psi(m)$ for all $m \in M$. Thus $\hat{B} = \hat{B}_1$, so $B = B_1$. Since $\psi(m) = \varphi(m)$ a.e., we have $||\psi||_\infty = ||\varphi||_\infty$, and since the support of $\mu$ is $M$, $|B| = \sup_{m \in M} |\psi(m)| = ||\psi||_\infty = ||\varphi||_\infty$.

We shall first establish the existence of $B$ and $\psi$ when $\varphi$ has the form $\chi_S$ for a measurable subset $S$ of $M$. In this case, let $U$ be the clopen set of part (d) for which $\mu(S \Delta U) = 0$. It follows that $\chi_U$ is continuous, $\hat{P}_U = \chi_U$, and $\chi_U(m) = \chi_S(m)$ whenever $m \notin S \Delta U$. Thus $B = P_U$ and $\psi = \chi_U$ satisfy the requirements in this case.

Next, suppose $\varphi$ is a finite linear combination of such characteristic functions, say $\varphi = \sum_{j=1}^{n} \alpha_j \chi_{S_j}$. By the above, there exists a pair $B_j, \psi_j$ satisfying the conclusion for each $\chi_{S_j}$. Set $B = \sum_{j=1}^{n} \alpha_j B_j$, $\psi = \sum_{j=1}^{n} \alpha_j \psi_j$. It is then clear that $B$ and $\psi$ satisfy the conclusion for $\varphi$.

Finally, if $\varphi$ is an arbitrary function of $L^\infty(M, \mu)$, there exists a sequence $\{\varphi_j\}_{j=1}^{\infty}$ of functions, such that
each $\varphi_j$ is a finite linear combination of characteristic functions, and $\lim_{j \to \infty} ||\varphi_j - \varphi||_\infty = 0$. Let $B_j$ and $\psi_j$ be chosen to satisfy the conclusion of the theorem for each $\varphi_j$, which is possible by what we have shown above. For every two integers $n_1, n_2$, it then follows that $B_{n_1} - B_{n_2}$, $\psi_{n_1} - \psi_{n_2}$ satisfy the conclusion for $\varphi_{n_1} - \varphi_{n_2}$, hence $|B_{n_1} - B_{n_2}| = \sup_{m \in M} |\psi_{n_1}(m) - \psi_{n_2}(m)| = ||\varphi_{n_1} - \varphi_{n_2}||_\infty$. But $\lim_{n_1, n_2 \to \infty} ||\varphi_{n_1} - \varphi_{n_2}||_\infty = 0$, so that $(B_j)_{j=1}^\infty$ is a norm convergent sequence of operators, and $(\psi_j)_{j=1}^\infty$ is a uniformly convergent sequence of continuous functions. If the respective limits of these two sequences are $B$ and $\psi$, it follows that $B \in \mathcal{E}$, and since $B_j = \psi_j$ for each $j$, $B = \psi$. For every $j$, $\psi_j(m) = \varphi_j(m)$ a.e., and also $\lim_{j \to \infty} \varphi_j(m) = \varphi(m)$ a.e. It follows that $\psi(m) = \lim_{j \to \infty} \psi_j(m) = \varphi(m)$ a.e. Thus $B$ and $\psi$ fulfill the requirements of the theorem for $\varphi$, and since $\varphi$ was arbitrary, the theorem is proved.

We can now give the simple proof of Theorem 1.5 (b) as promised in the last chapter.

**Corollary 2.8.1.** If $G$ is any positive functional on $\mathcal{E}$ such that $G(A^*A) \leq (A^*A_0, \xi_0)$ for every $A \in \mathcal{E}$, then there exists a positive definite operator $B \in \mathcal{E}$ such that $G(A) = (AB_0, \xi_0)$ for all $A \in \mathcal{E}$. 
Proof. Since \( G \) is a positive functional on \( E \), there exists a unique non-negative measure \( \nu \) on \( M \) for which
\[
G(A) = \int_M \hat{A}(m)d\nu(m), \text{ if } A \in E.
\]
From the hypothesis,
\[
\int_M |\hat{A}(m)|^2d\nu(m) \leq \int_M |\hat{A}(m)|^2d\mu(m) \text{ for every } A \in E,
\]
which is equivalent to \( \int_M f(m)d\nu(m) \leq \int_M f(m)d\mu(m) \text{ for every non-negative continuous function } f \text{ on } M. \)

Thus \( 0 \leq \nu(S) \leq \mu(S) \) for every Borel subset \( S \) of \( M \). From the Radon-Nikodym Theorem (cf. [2], p. 128) there exists a function \( \varphi \in L_1(M, \mu) \) for which
\[
\nu(S) = \int_S \varphi(m)d\mu(m) \text{ for every Borel subset } S \text{ of } M.
\]
The inequality \( 0 \leq \nu(S) \leq \mu(S) \) also shows that \( 0 \leq \varphi(m) \leq 1 \text{ a.e.} \)

By Theorem 2.8 (e) there exists a \( B \in E \) such that \( \hat{B}(m) = \varphi(m) \text{ a.e.} \), and since \( \hat{B} \) is continuous and the support of \( \mu \) is \( M, 0 \leq \hat{B}(m) \leq 1 \) for all \( m \in M \). Thus \( B \) is positive definite, and for arbitrary \( A \in E, G(A) = \int_M \hat{A}(m)d\nu(m) = \int_M \hat{A}(m)\hat{B}(m)d\mu(m) = (AB_0, \xi_0). \)

We shall need a generalization to arbitrary measurable functions of the above result that each essentially bounded measurable function is equal a.e. to a unique continuous function. For this purpose, we consider the one point compactification \( C \cup \{\infty\} \) of the complex numbers. The operations of \( C \) are extended to \( C \cup \{\infty\} \) in the usual way, i.e., \( a + \infty = \infty \text{ for } a \in C \cup \{\infty\}, a \cdot \infty = \infty \text{ for } a \in C \cup \{\infty\} \text{ and } a \neq 0, \frac{1}{\infty} = 0, \frac{1}{0} = \infty, |\infty| = \sqrt{\infty} = \infty. \)

We also define \( a < \infty \text{ if } a \) is any non-negative real number.
Definition 2.9. A function defined on M with values in $C \cup \{\infty\}$ will be called a continuous representative if it is continuous from M to $C \cup \{\infty\}$. If $\varphi$ is a function defined a.e. on M with values in $C \cup \{\infty\}$, and is measurable, a second function $\psi$ will be called a continuous representative of $\varphi$ if $\psi$ is a continuous representative, and $\varphi(m) = \psi(m)$ a.e.

"Continuous function" will continue to refer to a complex valued function, necessarily bounded, which is continuous in the usual sense, whereas "continuous representative" means that the function is continuous from M to $C \cup \{\infty\}$. This terminology was selected because of the following theorem, which states that an equivalence class of measurable functions can be represented by such a continuous representative.

Theorem 2.10. Every function $\varphi$ defined almost everywhere and measurable on M with values in $C \cup \{\infty\}$ is equal almost everywhere to a unique continuous representative $\psi$.

We shall henceforth refer to this unique $\psi$ as the continuous representative of $\varphi$.

Proof. Let $\varphi_1, \varphi_2$ be functions defined as follows.

$$\varphi_1(m) = \begin{cases} \varphi(m), & \text{if } |\varphi(m)| \leq 1 \\ 1, & \text{if } |\varphi(m)| > 1 \end{cases}$$
$$\varphi_2(m) = \begin{cases} \frac{1}{\varphi(m)}, & \text{if } |\varphi(m)| > 1 \\ 1, & \text{if } |\varphi(m)| \leq 1 \end{cases}$$

Since \( \varphi \) was measurable and defined almost everywhere, \( \varphi_1 \) and \( \varphi_2 \) are measurable, defined almost everywhere, and bounded. By Theorem 2.8 (e), there exist continuous functions \( \psi_1 \) and \( \psi_2 \) defined on \( M \), such that \( \varphi_1(m) = \psi_1(m) \) a.e. and \( \varphi_2(m) = \psi_2(m) \) a.e. Necessarily \( |\psi_1(m)| \leq 1 \) and \( |\psi_2(m)| \leq 1 \) for all \( m \in M \). Since the mapping \( x \rightarrow \frac{1}{x} \) is continuous from \( \mathbb{C} \cup \{\infty\} \) to \( \mathbb{C} \cup \{\infty\} \), \( \frac{1}{\psi_2} \) is a continuous representative. Then, since the mapping \( (x,y) \rightarrow x + y \) is continuous in both variables at all points of \( \mathbb{C} \times \mathbb{C} \cup \{\infty\} \), \( \psi_1 - 1 + \frac{1}{\psi_2} \) is a continuous representative. Let \( N \) be a fixed null set such that \( \varphi_1(m) = \psi_1(m) \) and \( \varphi_2(m) = \psi_2(m) \) for \( m \not\in N \). Then if \( m \not\in N \) and \( |\varphi(m)| \leq 1 \), we have

$$\psi_1(m) - 1 + \frac{1}{\psi_2(m)} = \varphi_1(m) - 1 + \frac{1}{\varphi_2(m)} = \varphi(m) - 1 + 1 = \varphi(m),$$

and if \( m \not\in N \) and \( |\varphi(m)| > 1 \), it follows that

$$\psi_1(m) - 1 + \frac{1}{\psi_2(m)} = \varphi_1(m) - 1 + \frac{1}{\varphi_2(m)} = 1 - 1 + \varphi(m) = \varphi(m).$$

Thus, \( \psi = \psi_1 - 1 + \frac{1}{\psi_2} \) is a continuous representative of \( \varphi \).

If \( \psi' \) is another continuous representative of \( \varphi \), then \( \psi(m) = \psi'(m) \) a.e. Since the set of \( m \in M \) for which \( \psi(m) \neq \psi'(m) \) is open and the support of \( \mu \) is all of \( M \), we have \( \psi(m) = \psi'(m) \) for all \( m \in M \).
Unfortunately, we cannot always perform the usual algebraic operations with continuous representatives and come back to continuous representatives. However, the following properties will usually suffice.

Theorem 2.11. Suppose $\varphi$ and $\psi$ are continuous representatives.

(a) If one of $\varphi, \psi$ is a continuous function, or if both $\varphi$ and $\psi$ take on values only in the extended non negative real numbers, (i.e., the non negative reals $\mathbb{R}^+\cup\{\infty\}$) then $\varphi + \psi$ is a continuous representative.

(b) If $\alpha$ is complex, then $\alpha \varphi$ is a continuous representative.

(c) $|\varphi|, \sqrt{|\varphi|}, |\varphi|^2$, and $\frac{1}{\varphi}$ are continuous representatives.

Proof. Part (a) follows from the fact that the mapping $(x,y) \mapsto x + y$ is continuous from $\mathbb{C} \times (\mathbb{C} \cup \{\infty\})$ to $\mathbb{C} \cup \{\infty\}$, and also continuous from $\mathbb{R}^+ \times \mathbb{R}^+ \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$, where $\mathbb{R}^+ \cup \{\infty\}$ denotes the extended non negative reals.

Part (b) follows from the fact that the mapping $(x,y) \mapsto xy$ is continuous in each variable separately on $(\mathbb{C} \cup \{\infty\}) \times (\mathbb{C} \cup \{\infty\})$. Part (c) is a consequence of the continuity of the four mappings defined by $x \mapsto |x|$, $x \mapsto \sqrt{|x|}$, $x \mapsto |x|^2$, and $x \mapsto \frac{1}{x}$, on $\mathbb{C} \cup \{\infty\}$.

From Theorem 2.8 (e), we have $L_\infty(M,\mu)$ identified, as a Banach space, with $C(M)$, which, in turn, is isomorphic
to $E$. But $L^\infty(M,\mu)$ is identified as the dual space of $L_1(M,\mu)$, and $E$ is identified as the dual space of $E_*$, according to Theorem 1.4 (d). We thus have $E$ identified as a dual space in two alternative ways. The following theorem shows these two identifications to be essentially the same.

**Theorem 2.12.** If $\varphi \in L_1(M,\mu)$, let $T_\varphi(A) = \int_M A(m)\varphi(m)d\mu(m)$ for all $A \in E$. Then $T_\varphi$ is a bounded linear functional on $E$ for each $\varphi \in L_1(M,\mu)$. Furthermore, the mapping $\varphi \to T_\varphi$ is an isometric isomorphism of $L_1(M,\mu)$ onto the subspace $E_*$ of $E^*$ consisting of those functionals which are continuous on $E$ relative to the ultrastrong topology.

**Proof.** If $\varphi \in L_1(M,\mu)$, then for every $A \in E$, $|T_\varphi(A)| = |\int_M A(m)\varphi(m)d\mu(m)| \leq \sup_{m \in M} |A(m)| \int_M |\varphi(m)|d\mu(m) = |A|||\varphi||_1$. $T_\varphi$ is clearly linear, so $T_\varphi \in E^*$, and $|T_\varphi| \leq ||\varphi||_1$. However, $|T_\varphi| = \sup_{|A| \leq 1} |\int_M A(m)\varphi(m)d\mu(m)|$

$$= \sup_{|\psi|_\infty \leq 1} |\int_M \psi(m)\varphi(m)d\mu(m)| = \sup_{|\psi|_\infty \leq 1} |\int_M \psi(m)\varphi(m)d\mu(m)|$$

$$= ||\varphi||_1.$$ The mapping $\varphi \to T_\varphi$ is clearly linear, hence it is an isometric isomorphism.

To show that $T_\varphi$ is ultrastrongly continuous for each $\varphi$, we consider first the case where $\varphi$ is essentially
bounded. Then, by Theorem 2.8 (e), there exists a \( B \in E \) such that \( \widehat{B}(m) = \varphi(m) \) a.e. We thus have \( T_\varphi(A) = \int_M \widehat{A}(m) \widehat{B}(m) d\mu(m) = \int_M \widehat{A}B(m) d\mu(m) = (AB, \xi) \), so that \( T_\varphi \) is continuous in the weak-topology, hence also in the ultrastong topology. Returning to the general case, if \( \varphi \) is an arbitrary function of \( L_1(M, \mu) \), there exists a sequence \( \{\varphi_n\} \) of essentially bounded functions such that
\[
\lim_{n \to \infty} ||\varphi_n - \varphi||_1 = 0.
\]
But then also \( \lim_{n \to \infty} |T - T_\varphi| = 0. \)

By Theorem 1.4 (c), \( E_* \) is a norm closed linear subspace of \( E^* \), hence \( T_\varphi \in E_* \).

Since the mapping \( \varphi \to T_\varphi \) is isometric, its range is a norm closed linear subspace of \( E_* \). If this subspace is proper, then by the Hahn-Banach theorem there exists a bounded linear functional \( \mathcal{L} \) on \( E_* \) such that \( \mathcal{L}(T_\varphi) = 0 \) for \( \varphi \in L_1(M, \mu) \), but \( \mathcal{L} \neq 0 \). By Theorem 1.4 (d), there exists an \( A \in E \) such that \( \mathcal{L}(T) = T(A) \) for all \( T \in E_* \). But then
\[
\mathcal{L}(T_\varphi) = T_\varphi(A) = \int_M \widehat{A}(m)\varphi(m) d\mu(m) = 0 \quad \text{for every } \varphi \in L_1(M, \mu).
\]
Since \( \widehat{A} \) is continuous, \( \widehat{A}(m) = 0 \), hence \( A = 0 \) and
\[
\mathcal{L}(T) = 0, \quad \text{contrary to assumption. Thus the range of}
\]
\( \varphi \to T_\varphi \) coincides with all of \( E_* \).

**Corollary 2.12.1.** If a linear functional \( T \in E^* \) is continuous in one of the four operator topologies, it is continuous in each of the four simultaneously. For this to be the case, it is necessary and sufficient that there
exist a $\varphi \in L_1(M,\mu)$ such that $T(A) = \int_M \hat{A}(m)\varphi(m)d\mu(m)$ for every $A \in E$.

**Proof.** Since the weak topology is weaker than the other three operator topologies, every linear functional which is continuous in the weak topology is continuous in all four topologies. Since the ultrastrong topology is the strongest of the four, a linear functional which is continuous in one of the four topologies is also continuous in the ultrastrong topology. By Theorem 2.12, the linear functionals continuous in the ultrastrong topology are exactly those having the above integral representation. It thus suffices to show that every functional having the integral representation is weakly continuous.

Suppose, therefore, that $T(A) = \int_M \hat{A}(m)\varphi(m)d\mu(m)$ for every $A \in E$, where $\varphi \in L_1(M,\mu)$. We may write $\varphi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$, where each $\varphi_j$ is a non-negative $L_1(M,\mu)$ function, and correspondingly write $T$ as a linear combination of the functionals $T_j$ defined by $T_j(A) = \int_M \hat{A}(m)\varphi_j(m)d\mu(m)$. It thus suffices to consider the case where the original function $\varphi$ is non-negative.

Define $\psi = \sqrt{\varphi}$ so that $\psi \in L_2(M,\mu)$. Let $[\varphi_n]$ be a sequence of bounded, non-negative, measurable functions such that $\varphi_n(m) \to \varphi(m)$ a.e. Then set $\psi_n = \sqrt{\varphi_n}$ for each $n$, so that $[\psi_n]$ is a sequence of bounded, non-negative, measurable
functions, and \( \psi_n(m) \sim \psi(m) \) a.e. Then we have

\[
\lim_{n \to \infty} \int_M |\phi(m) - \phi_n(m)| \, d\mu(m) = \lim_{n \to \infty} \int_M |\psi(m) - \psi_n(m)|^2 \, d\mu(m) = 0.
\]

For each \( n \), let \( B_n \) be chosen according to Theorem 2.8 (e) so that \( \hat{B}_n(m) = \psi_n(m) \) a.e., and set \( \zeta_n = B_n \xi_0 \). Then for every two integers \( n_1, n_2 \), we have

\[
|\zeta_{n_1} - \zeta_{n_2}|^2 = \int_M |\hat{B}_{n_1}(m) - \hat{B}_{n_2}(m)|^2 \, d\mu(m) = \int_M |\psi_{n_1}(m) - \psi_{n_2}(m)|^2 \, d\mu(m). \]

As \( n_1, n_2 \to \infty \), this last quantity approaches zero. The sequence \( \{\zeta_n\} \) is thus Cauchy, hence has a limit \( \zeta \) in \( \mathcal{H} \).

For each integer \( n \) and \( A \in E \), we have

\[
\int_M \hat{A}(m)\phi_n(m) \, d\mu(m) = \int_M \hat{A}(m)|\hat{B}_n(m)|^2 \, d\mu(m) = (AB_n^*B_n\xi_0, \xi_0) = (AB_n\xi_0, B_n\xi_0) = (A\zeta_n, \zeta_n), \]

hence

\[
|\int_M \hat{A}(m)\phi(m) \, d\mu(m) - (A\zeta, \zeta)| \leq |\int_M \hat{A}(m)(\phi(m) - \phi_n(m)) \, d\mu(m)| + |(A\zeta_n, \zeta_n) - (A\zeta, \zeta)|.
\]

As \( n \to \infty \), these two quantities approach zero, hence

\[
T(A) = \int_M \hat{A}(m)\phi(m) \, d\mu(m) = (A\zeta, \zeta). \]

Since this equation holds for arbitrary \( A \in E \), \( T \) is weakly continuous.

For every \( \xi \in \mathcal{H} \), \( A \to (A\xi, \xi) \) defines a weakly continuous linear functional on \( E \). Thus there exists a function \( \phi \in L_1(M, \mu) \), unique modulo the null functions, for which

\[
(A\xi, \xi) = \int_M \hat{A}(m)\phi(m) \, d\mu(m) \]

for every \( A \in E \). From Theorem 2.10, we may replace \( \phi \) by its continuous representative, which is
uniquely determined. Since \((A\xi,\xi) \geq 0\) for positive
definite \(A\), we have \(\int_M \psi(m) \varphi(m) d\mu(m) \geq 0\) for every continuous,
non negative function \(\psi\). This implies \(\varphi(m) \geq 0\) a.e., and
since \(\varphi\) is a continuous representative, \(\varphi(m) \geq 0\) every-
where. Also, since \(\varphi\) is a continuous representative,
\([m \in M|\varphi(m) \neq 0]\) is open, hence its closure is clopen.

**Definition 2.13.** If \(\xi\) is an arbitrary vector of \(H\),
we define \(\varphi_\xi\) to be the unique, necessarily extended non
negative valued, continuous representative for which
\((A\xi,\xi) = \int_M A(m) \varphi_\xi(m) d\mu(m)\) for all \(A \in E\), and \(S_\xi\) to be the
clopen set \([m \in M|\varphi_\xi(m) \neq 0]\).

These definitions depend upon the measure \(\mu\), which,
in turn, came from the cyclic vector \(\xi_0\) for \(H\) with
respect to \(E'\). If a second cyclic vector gives rise to the
same measure \(\mu\), the corresponding \(\varphi_\xi, S_\xi\) still remain the
same. However, if we choose a cyclic vector which gives
rise to a new measure, \(\varphi_\xi\) will, in general, change. \(S_\xi\)
will not, as will be seen in an independent characterization
below. Whenever the symbols \(\varphi_\xi, S_\xi\) are used, however, a
measure \(\mu\) which arises from a cyclic vector will be
specified in advance.

**Theorem 2.14.** (a) If \(U\) is a clopen subset of \(M\),
then \(P_U \xi = \xi\) if, and only if \(U \supset S_\xi\). \(P_U \xi \neq 0\) if, and only
if \(U \cap S_\xi \neq \emptyset\).
(b) \( S_\xi \) is the unique clopen subset of \( M \) for which 
\[ P_{S_\xi} (\mathcal{H}) = \mathbb{G} \{ \phi \xi \mid A \in E' \}. \]

(c) If \( A \in E' \), then \( \phi_{A\xi} (m) \leq |A|^2 \phi_\xi (m) \) for all \( m \in M \).

(d) If \( A \in E \), then \( \phi_{A\xi} (m) = |\hat{A}(m)|^2 \phi_\xi (m) \) a.e., and 
equality holds everywhere if \( \phi_\xi \) is a continuous function.

(e) If \( S_\xi \cap S_\eta = \emptyset \), then \( (A\xi, B\eta) = 0 \) for every \( A, B \in E' \).

(f) If \( (A\xi, \eta) = 0 \) for every \( A \in E \), then \( \phi_{\xi+\eta} (m) = \phi_\xi (m) + \phi_\eta (m) \) for all \( m \in M \).

(g) If \( \xi_1, \xi_2, \cdots \) is any (finite or infinite) sequence of mutually orthogonal vectors with \( \sum \xi_j^2 < \infty \), and
\[ \xi = \sum_j \xi_j, \] then \( S_\xi \subseteq \left( \bigcup_j S_{\xi_j} \right) \).

Proof of (a). From the definition of \( \phi_\xi \), \( (P_U\xi, \xi) = \int_M \chi_U(\xi) \phi_\xi (m) d\mu(m) \). Since \( P_U\xi = \xi \) is equivalent to
\( (P_U\xi, \xi) = (\xi, \xi) \), we have that \( P_U\xi = \xi \) exactly when
\[ \int_U \phi_\xi (m) d\mu(m) = \int_M \phi_\xi (m) d\mu(m), \] or equivalently,
\[ \int_{M-U} \phi_\xi (m) d\mu(m) = 0. \] Since \( \phi_\xi \) is an extended non negative valued continuous representative, this equation holds exactly when \( \phi_\xi (m) = 0 \) for \( m \notin U \), or equivalently,
\( \{ m \in M \mid \phi_\xi (m) \neq 0 \} \subseteq U \). Since \( U \) is closed, this
is equivalent to \( S_\xi = \{ m \in M \mid \phi_\xi (m) \neq 0 \} \subseteq U \).
To prove the second statement, we recall that for an arbitrary clopen set \( U \), \( P_{M-U} + P_U = I \), hence \( P_{M-U} \xi + P_U \xi = \xi \). Thus \( P_U \xi \neq 0 \) exactly when \( P_{M-U} \xi \neq \xi \). From what we have shown above, this is equivalent to \( \xi \not\subset M-U \), or \( U \cap \xi \neq \emptyset \).

**Proof of (b).** Let \( U \) be the unique clopen set of Theorem 2.3 (a) such that \( P_U(\mathcal{H}) = \mathcal{G}\{\xi A | A \in E'\} \). Then \( \xi \in P_U(\mathcal{H}) \), hence \( P_U \xi = \xi \) and \( U \supset \xi \) by part (a). On the other hand, \( P_\xi \xi = \xi \) by part (a), and \( P_\xi (\mathcal{H}) \) is an invariant subspace for \( E' \). Thus \( P_\xi (\mathcal{H}) = \mathcal{G}\{\xi A | A \in E'\} = P_U(\mathcal{H}) \), so \( \xi \supset U \) by Theorem 2.3 (b).

**Proof of (c).** For \( A \in E' \), we have for all \( B \in E \) that
\[
\int_M |\hat{B}(m)|^2 \varphi_{\xi A}(m) d\mu(m) = (B*BA \xi, A \xi) = (A*AB, B \xi) \leq |A*A|(B \xi, B \xi)
\]
\[
= |A|^2(B*B \xi, \xi) = |A|^2 \int_M |\hat{B}(m)|^2 \varphi_{\xi}(m) d\mu(m).
\]
As \( B \) ranges over \( E \), \( |\hat{B}|^2 \) ranges over all non negative continuous functions. Thus \( \varphi_{\xi A}(m) \leq |A|^2 \varphi_{\xi}(m) \) a.e. Since \( \varphi_{\xi A} \) and \( \varphi_{\xi} \) are both continuous representatives, we must have inequality everywhere.

**Proof of (d).** If \( A \in E \), then for arbitrary \( B \in E \) we have
\[
\int_M |\hat{B}(m)| \varphi_{A \xi}(m) d\mu(m) = (BA \xi, A \xi) = (BA*BA \xi, \xi) = \int_M |\hat{B}(m)| |\hat{A}(m)|^2 \varphi_{\xi}(m) d\mu(m).
\]
It follows that \( \varphi_{A \xi}(m) = |\hat{A}(m)|^2 \varphi_{\xi}(m) \) a.e. If, in addition, \( \varphi_{\xi} \) is a continuous function, then also \( |\hat{A}|^2 \varphi_{\xi} \) is a continuous function, hence a continuous representative. Equality must thus hold everywhere in this case.
Proof of (e). If \( A, B \in E' \), then \((A\xi, B\eta) = (AP_{\xi} P_{\eta}, BP_{\eta}) = (AP_{\xi} S_{\eta}, BP_{\eta}) \) using Theorem 2.3 (b). The hypothesis \( S_{\xi} \cap S_{\eta} = \emptyset \) makes \( P_{\xi} \cap S_{\eta} = 0 \), by the same theorem, hence \((A\xi, B\eta) = 0 \).

Proof of (f). The hypothesis \((A\xi, \eta) = 0 \) for every \( A \in E \) tells us that for \( A \in E \), \((A(\xi + \eta), \xi + \eta) = (A\xi, \xi) + (A\xi, \eta) + (A\eta, \eta) = (A\xi, \xi) + (A\eta, \eta) = \int_{M} \hat{A}(m)\varphi_{\xi}(m)dm(m) + \int_{M} \hat{A}(m)\varphi_{\eta}(m)dm(m) = \int_{M} \hat{A}(m)(\varphi_{\xi}(m) + \varphi_{\eta}(m))dm(m) \). By Theorem 2.11 (a), \( \varphi_{\xi} + \varphi_{\eta} \) is a continuous representative, hence \( \varphi_{\xi} + \eta = \varphi_{\xi} + \varphi_{\eta} \).

Proof of (g). \((U S_{\xi})_{j} \) is a clopen set which contains each \( S_{\xi} \). By part (a), \( P_{(U S_{\xi})_{j}} \xi_{k} = \xi_{k} \) for each \( k \). It follows that \( P_{(U S_{\xi})_{j}} \xi = P_{(U S_{\xi})_{j}} \xi = \sum_{k} \xi_{k} = \xi \). Hence, again by part (a), \( \xi \leq (U S_{\xi})_{j} \).

With the above notions, it is now possible to describe, within equivalence, the most general form for a subspace of \( \mathcal{H} \) which is cyclic with respect to \( E \). The first of the following theorems states that such a subspace is always generated by a special kind of cyclic vector.
Theorem 2.15. Suppose $\xi \in \mathcal{H}$. Then there exists $\zeta \in \mathcal{H}$ for which $\mathcal{G}(A\xi | A \in E) = \mathcal{G}(A\zeta | A \in E)$ with $S_\zeta = S_\xi$ and $\varphi_\zeta = \chi_{S_\zeta}$.

Proof. For every positive integer $n$, we define $T_n = \{m \in M | \frac{1}{n} < \varphi_\xi(m) < n\}$. Since $\varphi$ is an extended nonnegative valued continuous representative, we have that $\{m \in M | \frac{1}{n} < \varphi_\xi(m) < n\}$ is open for every $n$, hence $T_n$ is clopen by Theorem 2.2, and $\{m \in M | \frac{1}{n} \leq \varphi_\xi(m) \leq n\} \subset T_n \subset \{m \in M | \frac{1}{n} < \varphi_\xi(m) < n\} \subset S_\xi$ for each $n$. Hence $T_1 \subset T_2 \subset \ldots$, and $\bigcup_{n=1}^{\infty} T_n = \{m \in M | 0 < \varphi_\xi(m) < \infty\} \subset S_\xi$.

Since $\varphi_\xi \in L_1(M, \mu)$, $\mu(\{m \in M | \varphi_\xi(m) = \infty\}) = 0$, hence $\mu(\bigcup_{n=1}^{\infty} T_n) = \mu(\{m \in M | \varphi_\xi(m) \neq 0\})$. From Theorem 2.8 (a) and Definition 2.13, $\mu(\bigcup_{n=1}^{\infty} T_n) = \mu(S_\xi)$, or $\mu(S_\xi - (\bigcup_{n=1}^{\infty} T_n)) = 0$.

For each positive integer $n$, we define two functions $\psi_n$, $\varphi_n$ by

$$
\psi_n(m) = \begin{cases} 
\frac{1}{\sqrt[+]{\varphi_\xi(m)}} & , \text{if } m \in T_n \\
0, & \text{if } m \not\in T_n,
\end{cases}
$$

$$
\varphi_n(m) = \begin{cases} 
\varphi_\xi(m) & , \text{if } m \in T_n \\
0, & \text{if } m \not\in T_n.
\end{cases}
$$
It follows at once that $\psi_n$ and $\varphi_n$ are continuous functions bounded by $\sqrt{n}$ for every $n$, hence we may choose unique $B_n, C_n \in E$ for which $\hat{B}_n = \psi_n', \hat{C}_n = \varphi_n$. Furthermore, we have by direct computation from the above definitions that

$$\varphi_n|B_n| = \chi_{T_n}$$

if $n_1 > n_2$, which implies that

$$C_{n_2}B_{n_1} = P_{T_{n_2}}, \text{ if } n_1 > n_2.$$  

Also, $|\hat{B}_{n_1} - \hat{B}_{n_2}|^2 \varphi_5 = \chi_{T_{n_1}} - T_{n_2}$ if $n_1 > n_2$

and $|\hat{B}_n|^2 \varphi_5 = \chi_{T_n}$ for every integer $n$.

If $n_1 > n_2$, we thus have $|B_{n_1} \xi - B_{n_2} \xi|^2 = $

$$(B_{n_1} - B_{n_2})^*(B_{n_1} - B_{n_2}) \xi, \xi = \int_{M} |\hat{B}_{n_1}(m) - \hat{B}_{n_2}(m)|^2 \varphi_5(m) d\mu(m)$$

$= \int_{M} \chi_{T_{n_1}} - T_{n_2}(m) d\mu(m) = \mu(T_{n_1} - T_{n_2})$, hence

$$\lim_{n_1 \to n_2 \to \infty} |B_{n_1} \xi - B_{n_2} \xi|^2 = 0,$$

so $\{B_n \xi\}^\infty_{n=1}$ is a Cauchy sequence. Set $\zeta = \lim_{n \to \infty} B_n \xi$. We shall show that $\zeta$ is the desired vector. We first note that for arbitrary $A \in E$ and every integer $n$, $(A B_n \xi, B_n \xi) = (A B_n^* B_n \xi, \xi) = $$\int_{M} A(m) |\hat{B}_n(m)|^2 \varphi_5(m) d\mu(m) = \int_{M} A(m) \chi_{T_n}(m) d\mu(m)$, hence
\[ |(A\zeta, \zeta) - \int_M \hat{A}(m)x_{S_\zeta}(m)d\mu(m)| \leq \]

\[ |(A\zeta, \zeta) - (AB_n \xi, B_n \xi)| + \int_M \hat{A}(m)x_{T_n}(m)d\mu(m) - \int_M \hat{A}(m)x_{S_\xi}(m)d\mu(m)| \]

\[ = |(A\zeta, \zeta) - (AB_n \xi, B_n \xi)| + \int_M \hat{A}(m)x_{(S_\xi - T_n)}(m)d\mu(m)|. \]

If we take limits as \( n \to \infty \), the first term of this last expression approaches zero since \( B_n \xi \to \zeta \), and the second term approaches zero since \( \mu(S_\xi - T_n) \to \mu(S_\xi - \bigcup T_n) = 0. \)

It follows that \( (A\zeta, \zeta) = \int_M \hat{A}(m)x_{S_\xi}(m)d\mu(m) \). Since \( x_{S_\xi} \) is a continuous function, we have that \( \varphi_\zeta = x_{S_\xi} \), hence

\[ S_\zeta = S_\xi \text{ and } \varphi_\zeta = x_{S_\xi}. \]

From the above construction,

\[ \zeta = \lim_{n \to \infty} B_n \xi \in \mathcal{G}[A\xi | A \in E], \text{ hence } \mathcal{G}[A\zeta | A \in E] \subset \mathcal{G}[A\xi | A \in E]. \]

To prove the reverse inclusion, we note from the above that \( C_{n_2}B_{n_1} \xi = P_{T_{n_2}} \xi \) whenever \( n_1 > n_2 \). Taking limits as \( n_2 \to \infty \), this gives \( C_{n_2} \zeta = P_{T_{n_2}} \xi \) for every integer \( n_2 \). It follows that

\[ |\xi - C_{n_2} \zeta|^2 = |P_{S_\xi} \xi - P_{T_{n_2}} \xi|^2 = |P_{(S_\xi - T_{n_2})} \xi|^2. \]

\[ (P_{(S_\xi - T_{n_2})} \xi, \xi) = \int_M x_{(S_\xi - T_{n_2})}(m)\varphi_\xi(m)d\mu(m), \text{ for each integer } n_2. \]

Since \( \lim_{n_2 \to \infty} \mu(S_\xi - T_{n_2}) = \mu(S_\xi - \bigcup_{n=1}^\infty T_n) = 0 \), it follows
from the above relation that \( \lim_{n_2 \to \infty} |\xi - \zeta|_2^2 = 0. \) Hence

\[ \xi \in \mathcal{G}(A \xi | A \in \mathcal{E}), \]

which implies \( \mathcal{G}(A \xi | A \in \mathcal{E}) \subseteq \mathcal{G}(A \zeta | A \in \mathcal{E}). \) This gives \( \mathcal{G}(A \xi | A \in \mathcal{E}) = \mathcal{G}(A \zeta | A \in \mathcal{E}), \) and completes the proof.

**Definition 2.16.** If \( S \) is a clopen subset of \( M, \) then we denote by \( L_2(M, \mu, S) \) the set of all \( L_2(M, \mu) \) functions which vanish almost everywhere on \( M - S. \)

Since each \( L_2(M, \mu) \) equivalence class has a unique continuous representative, we shall frequently adopt the practice of using the continuous representative to represent the entire class. Of course, when we perform the usual pointwise operations with continuous representatives, the resulting function will usually require modification on a set of measure zero in order to obtain its continuous representative. A similar statement holds, regarding operators on \( L_2(M, \mu) \) of the form \( \varphi \cdot, \) where \( \varphi \in L_\infty(M, \mu). \) If \( f \in L_2(M, \mu) \) is a continuous representative, then the function defined by \( \varphi(m)f(m) \) may require modification on a set of measure zero to become a continuous representative, even if \( \varphi \) is continuous.

**Theorem 2.17.** Suppose \( \eta \in \mathcal{H} \) and \( \varphi_\eta = \chi_{S_\eta}. \) Then there exists a unitary isomorphism \( U \) from \( \mathcal{G}(A \eta | A \in \mathcal{E}) \) onto \( L_2(M, \mu, S_\eta) \) satisfying the following two properties:
(2.17.1). \( U\eta = \chi_{S^\eta} \)

(2.17.2). \( UA\xi = A \cdot U\xi \), if \( A \in E, \xi \in \mathcal{G}\{B\eta|B \in E\} \).

U, as a bounded linear operator, is uniquely determined by (2.17.1) and (2.17.2). Furthermore, if \( \xi \in \mathcal{G}\{A\eta|A \in E\} \), we have \( \varphi_\xi(m) = |U\xi(m)|^2 \) for all \( m \in M \) if \( U\xi \) denotes the continuous representative of its equivalence class.

**Proof.** We define \( U \) first on the dense subspace consisting of vectors of the form \( A\eta \), where \( A \in E \), by setting \( U(A\eta)(m) = \hat{A}(m)\chi_{S^\eta}(m) \). Then

\[
\int_M |U(A\eta)(m)|^2 d\mu(m) = \int_M |\hat{A}(m)|^2 \chi_{S^\eta}(m) d\mu(m) = \int_M |\hat{A}(m)|^2 \varphi_\eta(m) d\mu(m) = (A^*A\eta, \eta) = |A\eta|^2.
\]

It follows that the definition of \( U(A\eta) \) is unambiguous, and that \( U \) is linear and isometric on this dense subspace. The image of \( U \) as it has been defined at this stage is the dense linear subspace of \( L_2(M, \mu, S^\eta) \) consisting of the continuous functions which vanish on the complement of \( S^\eta \). Hence \( U \) possesses a unique continuous extension, which is isometric, from \( \mathcal{G}\{A\eta|A \in E\} \) onto \( L_2(M, \mu, S^\eta) \). \( U \) thus satisfies (2.17.1).

To show (2.17.2), suppose \( \xi \) is of the form \( B\eta \), where \( B \in E \), and \( A \) is an operator from \( E \). Then \( UA\xi(m) = UAB\eta(m) = \hat{A}(m)\hat{B}(m)\chi_{S^\eta}(m) = \hat{A}(m)U(B\eta)(m) = \hat{A} \cdot U\xi(m) \), or \( UA\xi = \hat{A} \cdot U\xi \) for \( \xi \in \mathcal{G}\{B\eta|B \in E\} \). This set is dense in \( \mathcal{G}\{B\eta, B \in E\} \), hence \( UA\xi = \hat{A} \cdot U\xi \) for all \( \xi \in \mathcal{G}\{B\eta|B \in E\} \).
To prove the uniqueness, suppose \( V \) is a second bounded linear operator which satisfies (2.17.1) and (2.17.2) in place of \( U \). Then for all \( B \in E \), \( V(B\eta)(m) = \hat{B} \cdot (V\eta)(m) = \hat{B}(m) \chi_{S_{\eta}}(m) = U(B\eta)(m) \). Again, since the vectors \( B\eta \) form a dense subset of \( \mathcal{S}(B\eta|B \in E) \), \( V = U \).

From Theorem 2.11 (c), we have that if \( U\xi(m) \) defines a continuous representative, so does \( |U\xi(m)|^2 \). For \( A \in E \), we have \( (A\xi, \xi) = (UA\xi, U\xi) = (A \cdot U\xi, U\xi) = \int_M \hat{A}(m)|U\xi(m)|^2 d\mu(m) \). Thus \( \varphi\xi(m) = |U\xi(m)|^2 \) for all \( m \in M \).

**Corollary 2.17.1.** If \( \xi \in \mathcal{S}(A\eta|A \in E) \), where \( \varphi_{\eta} = \chi_{S_{\eta}} \), then \( \xi \) can be represented in the form \( A\eta \) for some \( A \in E \) if, and only if \( \varphi_{\xi} \) is a continuous function. This representation is unique to the following extent: if \( A\eta = B\eta \), where \( A, B \in E \), then \( A S_{\eta} = B S_{\eta} \). In particular, if \( S_{\eta} = M \), then \( A\eta = B\eta \) if, and only if \( A = B \).

**Proof.** If \( \xi = A\eta \) for some \( A \in E \), then from Theorem 2.14 (d), we have \( \varphi_{\xi}(m) = |\hat{A}(m)|^2 \chi_{S_{\eta}}(m) \) for all \( m \), hence \( \varphi_{\xi} \) is a continuous function.

Conversely, suppose \( \varphi_{\xi} \) is a continuous function. If \( U \) is the operator of Theorem 2.17 and \( U\xi \) denotes the continuous representative of its equivalence class, then \( |U\xi(m)|^2 = \varphi_{\xi}(m) < \infty \) for all \( m \), so \( U\xi \) is a continuous function. Since \( U\xi \in L^2(M, \mu, S_{\eta}) \), we thus have \( U\xi(m) = 0 \).
for \( m \not\in S_\eta \), or \( U_\eta = U_\eta \chi_{S_\eta} \). Let \( A \) be the operator of \( E \) for which \( \hat{A} = U_\eta \). Then \( UA(m) = \hat{A}(m) \chi_{S_\eta}(m) = U_\eta(m) \chi_{S_\eta}(m) = U_\eta(m) \). Since \( U \) was one-to-one, \( A_\eta = \xi \).

To prove the assertion on the extent of the uniqueness, we note that if \( A_\eta = B_\eta \), then \( AP_{S_\eta} \eta = BP_{S_\eta} \eta \), hence
\[
(A - B)P_{S_\eta} \eta = 0.
\]
But this implies
\[
0 = ((A - B)P_{S_\eta} \eta, (A - B)P_{S_\eta} \eta) = \int \hat{A}(m) \hat{B}(m) \mu_\eta(m) dm, \text{ so that } \hat{A}(m) \chi_{S_\eta}(m) = \hat{B}(m) \chi_{S_\eta}(m) \text{ for all } m \in M. \text{ Hence } AP_{S_\eta} \eta = BP_{S_\eta} \eta. \]
CHAPTER III

THE DIRECT SUM AND DIRECT INTEGRAL DECOMPOSITIONS

In this chapter, we develop a method of decomposing a Hilbert space into a direct integral over the maximal ideal space of a diagonal ring whose commutant has a cyclic vector. We also identify the operators of the diagonal ring and of its commutant in terms of the direct integral. Throughout the chapter, we shall assume that $\mathcal{H}$ is a fixed Hilbert space, $\mathbb{E}$ is a diagonal ring on $\mathcal{H}$, and $\xi_0$ is a unit cyclic vector for $\mathcal{H}$ with respect to $\mathbb{E}'$. We shall denote the maximal ideal space of $\mathbb{E}$ by $\mathcal{M}$, and denote by $\mu$ the unique measure of Theorem 1.5 (a) associated with $\mathbb{E}'$, $\mathbb{E}$, and $\xi_0$. The expressions 'measurable' and 'almost everywhere' will always be used with respect to the measure $\mu$. For every vector $\xi \in \mathcal{H}$, $\phi_\xi$, and $s_\xi$ will be as in Definition 2.13, again with respect to the measure $\mu$.

The following concept will be studied in more detail in the next chapter, however we shall need it at the present in our main existence theorem.

Definition 3.1. The dimension of $\mathcal{H}$ relative to $\mathbb{E}$, which we denote by $\dim_\mathbb{E} \mathcal{H}$, is defined to be the smallest cardinal number $v$ such that $\mathcal{H}$ can be written as a direct sum $\sum_{\alpha \in \Lambda} \Phi \mathcal{H}_\alpha$, where each $\mathcal{H}_\alpha$ is a nontrivial cyclic
Corollary 3.1.1. \( \dim_E \mathcal{H} \) is less than or equal to the dimension of \( \mathcal{H} \) as a Hilbert space.

Proof. Let \( \mathcal{H} = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha \), such that \( \text{card} \Lambda = \dim_E \mathcal{H} \).
and each \( \mathcal{H}_\alpha \) is a nontrivial cyclic subspace for \( E \). For each \( \alpha \in \Lambda \), let \( e_{\alpha} \) be a unit vector in \( \mathcal{H}_\alpha \). Then the \( e_{\alpha} \) are pairwise orthogonal, hence \( \{e_{\alpha}\}_{\alpha \in \Lambda} \) is contained in an orthonormal basis. It follows that \( \dim_E \mathcal{H} = \text{card} \Lambda \leq \dim \mathcal{H} \).

We shall now adopt the following terminology on well orderings. Suppose \( \Lambda \) is a well ordered set. A segment in \( \Lambda \) is a subset \( I \) of \( \Lambda \) such that if \( \alpha \in I \) and \( \beta < \alpha \), then \( \beta \in I \). The segments \( I \) of \( \Lambda \) are easily seen to be of two kinds, (1) \( I = \Lambda \), or (2) \( I = \{\alpha \in \Lambda | \alpha < \alpha_0\} \), where \( \alpha_0 \) is a unique fixed element of \( \Lambda \). The segment \( \{\alpha \in \Lambda | \alpha < \alpha_0\} \) will be denoted by \( I(\alpha_0) \). In order to give the segment \( \Lambda \) this form also, we shall occasionally adjoin a single element \( \lambda_0 \) to \( \Lambda \), and define \( \alpha < \lambda_0 \) for each \( \alpha \in \Lambda \). Then \( \Lambda \cup \{\lambda_0\} \) is well ordered, and \( \Lambda = I(\lambda_0) \). The elements of \( \Lambda \) which have only finitely many predecessors will be denoted in the usual way by the integers, i.e., 1 is the smallest element of \( \Lambda \), 2 is the smallest element of \( \Lambda - \{1\} \), etc.

Our theory will be based on the following structural system, which is, by itself, of interest in the study of diagonal rings and their properties.
Definition 3.2. Suppose \( \{(m_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda} \) is a system of ordered pairs indexed by a well ordered set \( \Lambda \), such that \( m_\alpha \) is a subspace of \( \mathcal{H} \) and \( \eta_\alpha \) is a nonzero vector of \( \mathcal{H} \) for each \( \alpha \in \Lambda \). Such a system will be called a canonical decomposition system if the following five properties hold:

1. \( m_\alpha = \mathcal{C}(\mathcal{A}_\alpha | A \in E) \), for \( \alpha \in \Lambda \).
2. \( m_\alpha \perp m_\beta \), for \( \alpha \neq \beta \), if \( \alpha, \beta \in \Lambda \).
3. \( \mathcal{H} = \bigoplus_{\alpha \in \Lambda} m_\alpha \).
4. \( \phi_\eta_\alpha = \chi_\mathcal{S}_\eta_\alpha \), for \( \alpha \in \Lambda \).
5. \( \mathcal{S}_\eta_\alpha \supset \mathcal{S}_\eta_\beta \), if \( \alpha, \beta \in \Lambda \) and \( \alpha < \beta \).

According to the axiom of choice, every set can be well ordered, and hence well ordered sets exist which have any given cardinality. Using this fact, the following theorem guarantees the existence of a canonical decomposition system. Of equal importance for later purposes is the manner in which it allows us to choose such a system.

Theorem 3.3. Let \( \Lambda' \) be a well ordered set such that \( \text{card } \Lambda' = \dim_E \mathcal{H} \). Then there exists a canonical decomposition system \( \{(m_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda} \), where \( \Lambda \) is a segment in \( \Lambda' \). \( \Lambda \) must necessarily have the same cardinality as \( \Lambda' \).

Proof. By definition of \( \dim_E \mathcal{H} \), we may use \( \Lambda' \) as an index set to write \( \mathcal{H} = \bigoplus_{\alpha \in \Lambda'} \mathcal{H}_\alpha \), where each \( \mathcal{H}_\alpha \) is a cyclic
subspace for $E$ generated by a cyclic vector $\xi_\alpha$. We shall consider systems $\{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}$, such that $\Lambda$ is a segment in $\Lambda'$, $M_\alpha$ is a subspace of $H$ for each $\alpha \in \Lambda$, and $\eta_\alpha$ is a nonzero vector in $H$ for each $\alpha \in \Lambda$, which satisfy (3.2a), (3.2b), (3.2d), (3.2e), and the following two additional properties:

\[(3.3a) \text{ If } \beta \in \Lambda \text{ and } \xi_\beta \perp \sum_{\alpha \leq \beta} M_\alpha, \text{ then } S_\xi \subset S_\eta_\beta.\]

\[(3.3b) \sum_{\alpha \in \Lambda} M_\alpha \supset \sum_{\alpha \in \Lambda} H_\alpha.\]

Among these systems, we shall include the system in which $\Lambda$ is empty, with the interpretation that (3.2a), (3.2b), (3.2d), (3.2e), and (3.3a) are vacuous, and that both direct sums in (3.3b) consist of the trivial space $(0)$.

Let us define $\{(M_\alpha^{(1)}, \eta_\alpha^{(1)})\}_{\alpha \in \Lambda^{(1)}} \leq \{(M_\alpha^{(2)}, \eta_\alpha^{(2)})\}_{\alpha \in \Lambda^{(2)}}$ for two such systems whenever $\Lambda^{(1)} \subset \Lambda^{(2)}$, and for $\alpha \in \Lambda^{(1)}$ we have $M_\alpha^{(1)} = M_\alpha^{(2)}$ and $\eta_\alpha^{(1)} = \eta_\alpha^{(2)}$. Then $\leq$ is clearly a partial ordering. If $\{(M_\alpha^{(v)}, \eta_\alpha^{(v)})\}_{\alpha \in \Lambda^{(v)}}_{v \in V}$ is any family of systems forming a chain with respect to $\leq$, we define $\Lambda = \bigcup_{v \in V} \Lambda^{(v)}$. Then $\Lambda$ is a segment. For $\alpha \in \Lambda$, let $v \in V$ such that $\alpha \in \Lambda^{(v)}$, and set $M_\alpha = M_\alpha^{(v)}, \eta_\alpha = \eta_\alpha^{(v)}$. It is clear that this definition is independent of $v$ by the conditions imposed on the partial ordering $\leq$. The system
then satisfies (3.2a), (3.2b), (3.2d), (3.2e), (3.3a), and (3.3b), as is easily seen by using the fact that
\[\{(m_\alpha(v), \eta_\alpha(v))\}_{\alpha \in \Lambda(v)}\] is a chain of systems satisfying these properties. It is clear that
\[\{(m_\alpha(v), \eta_\alpha(v))\}_{\alpha \in \Lambda(v)} \leq \{(m_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}\] for every \(v \in \mathcal{V}\).

Consequently, every chain has an upper bound. By Zorn's Lemma, there exists a maximal system with respect to the partial ordering \(\leq\). Let \(\{(m_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}\) denote such a maximal system. We proceed to show that (3.2c) is satisfied.

Suppose, to the contrary, \(\Sigma \ominus m_\alpha \subset \mathcal{H}\). Since \(\Sigma \ominus \mathcal{H} = \mathcal{H}\), it follows from (3.3b) that \(\Lambda\) is a proper subset of \(\Lambda'\). If \(\alpha_0\) is the smallest element of \(\Lambda' - \Lambda\), we have that \(\Lambda = I(\alpha_0)\). Let \(\Lambda_0 = \Lambda \cup \{\alpha_0\}\), so that \(\Lambda_0\) is a segment which properly contains \(\Lambda\). For brevity, we set \(\eta = \Sigma \ominus m_\alpha\). Then \(\eta\) and \(\eta^\perp\) are invariant subspaces for \(E\).

Let \(\xi_{\alpha_0} = \xi_{\alpha_0}' + \xi_{\alpha_0}''\), where \(\xi_{\alpha_0}' \in \eta\) and \(\xi_{\alpha_0}'' \in \eta^\perp\). If \(\xi_{\alpha_0}'' \neq 0\), let \(\zeta\) be a vector for which
\[S_\zeta = S_{\xi_{\alpha_0}'}, \quad \psi_\zeta = \chi_{S_\zeta}, \quad \text{and} \quad \mathcal{G}(A\zeta|A \in E) = \mathcal{G}(A\xi_{\alpha_0}'|A \in E).\]
Such a vector exists by Theorem 2.15. If $\xi_o = 0$, let $p$ be an arbitrary nonzero vector of $\eta \perp$, and choose $\zeta$, by the same procedure, such that $S_\zeta = S_p$, $\varphi_\zeta = \chi_{S_\zeta}$, and

$$\mathcal{G}(A \mid A \in E) = \mathcal{G}(A_p \mid A \in E).$$

In either case, we have $\zeta \in \eta \perp$.

We now consider sets $W$ of nonzero vectors of $\mathcal{H}$ satisfying the following:

- (3.3c) $\zeta \in W$
- (3.3d) If $w \in W$, then $w \in \eta \perp$ and $\varphi_w = \chi_{S_w}$.
- (3.3e) $S_w \cap S_{w'} = \emptyset$ if $w, w' \in W$, and $w \neq w'$.

The set consisting of the single element $\zeta$ is one such set, and the system of all such sets is partially ordered by inclusion. If $\{W_v\}_{v \in V}$ is a chain with respect to the inclusion partial ordering, then $U_{v \in V} W_v$ satisfies

(3.3c), (3.3d), and (3.3e), and is an upper bound for the chain. Applying Zorn's Lemma, there exists a maximal set $W$ of nonzero vectors satisfying (3.3c), (3.3d), and (3.3e). We choose and fix such a maximal set $W$.

According to (3.3e), the mapping $w \rightarrow S_w$ is a one-to-one mapping from $W$ to a family of nonvoid, disjoint clopen subsets of $M$. Since $\mathcal{H}$ is cyclic with respect to $E'$, we have from Theorem 2.3 (e) that this family is countable, hence $W$ is also countable. Let $w_1, w_2, \ldots$ be an enumeration of $W$ into a finite or infinite sequence so
that \( w_1 = \zeta \). By (3.3e) and Theorem 2.14 (e), \( w_i \perp w_j \) for \( i \neq j \). Using (3.3d) and (3.3e), \( \Sigma |w_j|^2 = \Sigma (w_j, w_j) = \Sigma \int_M \chi_{S_{w_j}}(m) d\mu(m) = \Sigma \mu(S_{w_j}) = \mu(U S_{w_j}) < \infty \). Thus, \( \Sigma w_j \)

converges to a vector of \( \mathcal{H} \), which we define to be \( \eta_{\alpha_0} \). Set

\[ m_{\alpha_0} = \mathbb{C} \{ A \eta_{\alpha_0} | A \in \mathcal{E} \} \]. It is claimed that the system

\[ [(m_{\alpha}, \eta_{\alpha})]_{\alpha \in \Lambda_{\alpha}} \]

satisfies (3.2a), (3.2b), (3.2d), (3.2e), (3.3a), and (3.3b).

(3.2a) is clear from the definition of \( m_{\alpha_0} \).

(3.2b) is satisfied if \( \alpha < \alpha_0 \) and \( \beta < \alpha_0 \) trivially. If \( \alpha < \alpha_0 \) and \( \beta = \alpha_0 \), then \( m_\alpha \subset \eta \), and since \( w_j \in \eta^\perp \) for each \( j \), \( \eta_{\alpha_0} = \Sigma w_j \in \eta^\perp \). Since \( \eta^\perp \) is invariant for \( \mathcal{E} \),

\[ m_{\alpha_0} \subset \eta^\perp \], hence \( m_\alpha \perp m_{\alpha_0} \).

We need only show (3.2d) for \( \alpha_0 \). Here it is claimed

that \( S_{\eta_{\alpha_0}} = \overline{U S_{w_j}} \), and \( \varphi_{\eta_{\alpha_0}} = \chi_{\overline{U S_{w_j}}} \). \( \overline{U S_{w_j}} \) is, indeed, a clopen set, and these relationships follow from Theorem 2.8 (a), Definition 2.13, Theorem 2.14 (e), (3.3d), and (3.3e) in the following computation, which is valid for every \( A \in \mathcal{E} \):

\[ (A \eta_{\alpha_0}, \eta_{\alpha_0}) = (A(\Sigma w_j), \Sigma w_j) = \Sigma (A \eta_{w_j}, w_j) = \]
\[ \sum_j \int_M \hat{A}(m) \chi_{S^w_{wj}}(m) d\mu(m) = \int_M \hat{A}(m) \sum_j \chi_{S^w_{wj}}(m) d\mu(m) = \int_M \hat{A}(m) \chi_{U \bigcap \bigcup_j S^w_{wj}}(m) d\mu(m). \]

(3.2e) is clear for \( \alpha < \beta < \alpha_0 \). By the above, \( S_{\eta_\alpha} = \bigcup_j S^w_{wj} \). If \( \alpha \in A \), then for each \( j \) we have from

(3.3a) and (3.3d) that \( S^w_{wj} \subset S_{\eta_\alpha} \), hence \( \bigcup_j S^w_{wj} \subset S_{\eta_\alpha} \). Since \( S_{\eta_\alpha} \) is closed, \( S_{\eta_\alpha} \subset S_{\eta_\alpha} \).

(3.3a) holds, clearly, if \( \beta < \alpha_0 \). To show the result for \( \beta = \alpha_0 \), suppose, to the contrary, that there exists \( \xi \in \mathcal{H} \) with \( \xi \perp \sum_{\alpha \in \Lambda_0} \Theta M_\alpha \), and \( S_\xi \not\subset S_{\eta_\alpha} \). Let

\[ \xi' = P_{M - S_{\eta_\alpha}} \xi. \]

By Theorem 2.14 (a), \( \xi' \neq 0 \), and

\[ S_{\xi'} \subset M - S_{\eta_\alpha} . \]

Since \( \sum_{\alpha \in \Lambda_0} \Theta M_\alpha = \mathcal{N} \Theta M_\alpha \xi \in \mathcal{N} \perp \), hence \( \xi' \in \mathcal{N} \perp \). By Theorem 2.15, there exists a vector \( w_0 \in \mathcal{H} \) such that \( \mathcal{G}(Aw_0 | A \in E) = \mathcal{G}(A\xi' | A \in E) \), \( S_{w_0} = S_{\xi'} \), and \( \varphi_{w_0} = \chi_{S_{w_0}} \).

Then \( w_0 \in \mathcal{N} \perp \), \( w_0 \neq 0 \), and \( S_{w_0} \subset M - S_{\eta_\alpha} \). Since

\[ S^w_{wj} \subset S_{\eta_\alpha} \quad \text{for each } j \geq 1, \quad S_{w_0} \cap S^w_{wj} = \emptyset . \]

The system \( W \cup \{ w_0 \} \) thus satisfies (3.3c), (3.3d), and (3.3e), contrary to the maximality of \( W \). Hence (3.3a) also holds for \( \beta = \alpha_0 \).
To show (3.3b), we note that $\sum_{\alpha \in \Lambda_0} \Theta m_\alpha = \eta \otimes m_\alpha = \lambda \otimes m_\alpha$. Hence it suffices to show $\eta \otimes m_\alpha \supseteq \eta \otimes \Sigma_{\alpha \in \Lambda} \Theta h_\alpha$. Since $h_\alpha$ is generated by the vector $\xi_\alpha$ with respect to $E$, it suffices to show $\xi_\alpha \in \eta \otimes m_\alpha$.

Referring to our decomposition $\xi_\alpha = \xi_\alpha^1 + \xi_\alpha^2$, we have $\xi_\alpha \in \eta$, so it suffices to show $\xi_\alpha^2 \in m_\alpha$. If $\xi_\alpha^2 = 0$, this is trivial. Otherwise, we recall that $\zeta = w_1$ was chosen so that $\{A|A \in E \} = \{A\xi_\alpha^2|A \in E \}$. But

$$P_{S_{w_1}} \eta_\alpha = P_{S_{w_1}} (\Sigma P_{w_j} w_j) = \Sigma P_{S_{w_1}} P_{S_{w_j}} w_j = \Sigma P_{S_{w_1}} P_{S_{w_j}} w_j = P_{S_{w_1}} w_1 = w_1,$$

hence $w_1 = \zeta \in m_\alpha = \{A\eta_\alpha|A \in E \}$, so also $\xi_\alpha^2 \in m_\alpha$.

We have thus completed the proof of the claim that $((m_\alpha, \eta_\alpha))_{\alpha \in \Lambda_0}$ is a system satisfying (3.2a), (3.2b), (3.2d), (3.2e), (3.3a), and (3.3b). But it is clear that $((m_\alpha, \eta_\alpha))_{\alpha \in \Lambda} < ((m_\alpha, \eta_\alpha))_{\alpha \in \Lambda_0}$. This contradicts the maximality of the system $((m_\alpha, \eta_\alpha))_{\alpha \in \Lambda}$. Thus, (3.2c) necessarily had to be satisfied by this system beforehand. Hence, $((m_\alpha, \eta_\alpha))_{\alpha \in \Lambda}$ is the required canonical decomposition system.
That \( \text{card} \Lambda \leq \text{card} \Lambda' \) is immediate, and
\[ \text{card} \Lambda \geq \text{dim}_E \mathcal{H} = \text{card} \Lambda' \] by Definition 3.1. Consequently,
\[ \text{card} \Lambda = \text{card} \Lambda'. \]

We shall see in the next chapter that \( \text{card} \Lambda = \text{dim}_E \mathcal{H} \)
for an arbitrary canonical decomposition system, however we
do not assume this at the present. The following consequences
are, however, immediate from the definition, so we state
them at this time.

**Corollary 3.3.1.** Let \( \{(m_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda} \) be a canonical
decomposition system. Then

(a) \( S_{\eta_1} = M. \)

(b) At most countably many of the sets \( \{S_{\eta_\alpha}\}_{\alpha \in \Lambda} \) are
distinct.

(c) (3.3a) is satisfied by the system.

**Proof.** We shall establish (b) and (c) first, and then
return to (a). To prove (b), we note that the \( \{S_{\eta_\alpha}\}_{\alpha \in \Lambda} \)
are well ordered by \( \supset \), from (3.2e). Also, \( S_{\eta_\alpha} \rightarrow \mu(S_{\eta_\alpha}) \) is
a mapping from \( \{S_{\eta_\alpha}\}_{\alpha \in \Lambda} \) to the interval \([0,1]\), which maps
the ordering \( \supset \) into the usual ordering \( \geq \). If
\( S_{\eta_\alpha} \supset S_{\eta_\beta} \), then \( S_{\eta_\alpha} - S_{\eta_\beta} \) is a nonempty clopen set, hence
\( \mu(S_{\eta_\alpha} - S_{\eta_\beta}) > 0 \), since the support of \( \mu \) is \( M. \)
Consequently, \( \mu(S_{\eta_a}) > \mu(S_{\eta_B}) \), so the mapping is one-to-one.

But every subset of \([0,1]\) well ordered by \(>\) is at most countable, hence the family \(\{S_{\eta_a}\}_{\alpha \in \Delta}\) is at most countable.

To prove (c), suppose \( \xi \perp \sum_{\alpha < \beta} M_\alpha \) for some fixed \( \beta \).

Then \( \xi \in \sum_{\alpha > \beta} M_\alpha \), consequently \( \xi = \sum_{\alpha > \beta} \xi_\alpha \), where \( \xi_\alpha \in M_\alpha \) for each \( \alpha > \beta \), and \( \sum_{\alpha > \beta} |\xi_\alpha|^2 < \infty \). At most countably many of the \( \xi_\alpha \) are nonzero, so we enumerate them as \( \xi_{\alpha_1}, \xi_{\alpha_2}, \ldots \).

But \( M_\alpha = \mathcal{C}(A_{\eta_\alpha} | A \in E) \subseteq \mathcal{C}(A_{\eta_\alpha} | A \in E') = P_{S_{\eta_a}} (\mathcal{H}), \) according to Theorem 2.14 (b). Thus, \( S_{\xi_{\alpha_j}} \subseteq S_{\eta_{\alpha_j}} \subseteq S_{\eta_{\beta}} \) for each \( j \), so \( \bigcup_j S_{\xi_{\alpha_j}} \subseteq S_{\eta_{\beta}} \). Using Theorem 2.14 (g), and the fact that \( S_{\eta_{\beta}} \) is closed, we have \( S_{\xi} \subseteq \bigcup_j S_{\xi_{\alpha_j}} \subseteq S_{\eta_{\beta}} \).

Now (a) is easily established. \( M_1 = \mathcal{C}(A_{\eta_1} | A \in E) \subseteq \mathcal{C}(A_{\eta_1} | A \in E') = P_{S_{\eta_1}} (\mathcal{H}), \) by Theorem 2.14 (b). By Theorem 2.14 (a), we thus have \( P(M - S_{\eta_1}) \xi_0 \perp M_1 \), so that \( S_{P(M - S_{\eta_1}) \xi_0} \subseteq S_{\eta_1} \) by part (c), which we have just established. Applying Theorem 2.14 (a) again,
\[ P(M - S_{\eta_1})^5_0 = P_{S_{\eta_1}}^5 P(M - S_{\eta_1})^5_0 = P_{S_{\eta_1}} \cap (M - S_{\eta_1})^5_0 = 0. \]

Since \( \xi_0 \) is a cyclic vector for \( \mathcal{H} \) with respect to \( E' \), this implies \( M - S_{\eta_1} = \emptyset \), or \( S_{\eta_1} = M \).

For the remainder of the chapter, we shall assume that \[ \{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda} \] is a given canonical decomposition system. From (3.2a) and (3.2c), we have that the set of vectors which can be written in the form \( \sum B_1 \eta_{\alpha_1} + B_2 \eta_{\alpha_2} + \cdots + B_n \eta_{\alpha_n} \), where \( B_1, B_2, \ldots, B_n \in E \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda \), is a dense linear subspace of \( \mathcal{H} \). We shall denote this subspace by \( \mathcal{H}' \). Of course, \( \mathcal{H}' \) depends upon the particular canonical decomposition system which is chosen.

We now apply Theorem 2.17 to each \( (M_\alpha, \eta_\alpha) \) to choose the unitary isomorphism \( U_\alpha : M_\alpha \to L_2(M, \mu, \eta_\alpha) \) such that
\[ U_\alpha \eta_\alpha = \chi_{S_\alpha} \] and
\[ U_\alpha A\xi = \hat{A} \cdot U_\alpha \xi \] for \( \xi \in M_\alpha, A \in E \). We shall represent equivalence classes of \( L_2(M, \mu, \eta_\alpha) \) by their continuous representatives, and modify pointwise operations accordingly, as explained in Chapter II.

We now define \( U : \mathcal{H} \to \bigoplus_{\alpha \in \Lambda} L_2(M, \mu, \eta_\alpha) \) by the following. If \( \xi \in \mathcal{H} \), then \( \xi = \sum_{\alpha \in \Lambda} \xi_\alpha \), where \( \xi_\alpha \) is the projection of \( \xi \) onto \( M_\alpha \) for each \( \alpha \in \Lambda \). Let \( U\xi = \{U_\alpha \xi_\alpha\}_{\alpha \in \Lambda} \). Then \( U\xi \) is a system of functions \( \{f_\alpha\}_{\alpha \in \Lambda} \)
such that \( f_\alpha \) is a continuous representative in \( L^2(M, \mu, S_\alpha) \)
for each \( \alpha \in \Lambda \). Furthermore, it is clear that \( U \) is a
unitary isomorphism from \( \mathcal{H} \) onto \( \bigotimes_{\alpha \in \Lambda} L^2(M, \mu, S_\alpha) \). For
each fixed index \( \alpha_0 \), \( U\eta_{\alpha_0} \) is the system \( \{ f_\alpha \}_{\alpha \in \Lambda} \) such that
\[
f_{\alpha_0} = \chi_{S_\alpha}, \text{ and } f_\alpha = 0 \text{ if } \alpha \neq \alpha_0.
\]
Also, if \( \xi = \sum_{\alpha \in \Lambda} \xi_\alpha \),
where \( \xi_\alpha \in M_\alpha \) for each \( \alpha \), then \( U\xi = U \sum_{\alpha \in \Lambda} A\xi_\alpha =
\{ U\alpha A\xi \}_{\alpha \in \Lambda} = \hat{\Lambda} \cdot \{ U\alpha \xi \}_{\alpha \in \Lambda} = \hat{\Lambda} \cdot U\xi \).
These last two properties determine \( U \) uniquely as a bounded linear
operator.

In fact, suppose \( V : \mathcal{H} \to \bigotimes_{\alpha \in \Lambda} L^2(M, \mu, S_\alpha) \) is a
bounded linear operator such that \( V\eta_{\alpha_0} \) is the system
\[
\{ f_\alpha \}_{\alpha \in \Lambda}
\]
for which \( f_{\alpha_0} = \chi_{S_\alpha} \), \( f_\alpha = 0 \text{ if } \alpha \neq \alpha_0 \), and \( V\xi =
\hat{\Lambda} \cdot V\xi \) for \( \xi \in \mathcal{H} \). For each \( \alpha \), define \( V_\alpha : M_\alpha \to L^2(M, \mu, S_\alpha) \)
to be the operator for which \( V_\alpha \xi \) is the function in the \( \alpha \)-th
coordinate of \( V\xi \), for \( \xi \in M_\alpha \). Then \( V_\alpha \) satisfies
(2.17.1) and (2.17.2) for the vector \( \eta_\alpha \), hence \( V_\alpha = U_\alpha \).

If \( \alpha_0 \) is a fixed index and \( A \in E \), then from the above,
\( V\eta_{\alpha_0} = \hat{\Lambda} \cdot V\eta_{\alpha_0} \), hence \( V\eta_{\alpha_0} \) is zero in all coordinates \( \neq \alpha_0 \).
Since \( \{ A\eta_{\alpha_0} \mid A \in E \} \) is dense in \( M_{\alpha_0} \), it follows that \( V\xi \) is
zero in all coordinates $\neq \alpha_0$ if $\xi \in \mathcal{M}_{\alpha_0}$. Thus, for

arbitrary $\xi \in \mathcal{H}$, if $\xi = \sum_{\alpha \in \Lambda} \xi_{\alpha}$, where $\xi_{\alpha} \in \mathcal{M}_{\alpha}$ for each $\alpha$, we have

$$V\xi = V(\sum_{\alpha \in \Lambda} \xi_{\alpha}) = \{V_{\alpha} \xi_{\alpha} \}_{\alpha \in \Lambda} = \{U_{\alpha} \xi_{\alpha} \}_{\alpha \in \Lambda} = U\xi.$$ We thus have the following result.

**Theorem 3.4.** There exists a unitary isomorphism $U$ from $\mathcal{H}$ onto $\sum_{\alpha \in \Lambda} \otimes L_2(M, \mu, S_{\alpha})$ satisfying the following two properties:

1. $U\eta_{\alpha_0}$ is the system of functions $\{f_{\alpha} \}_{\alpha \in \Lambda}$ such that $f_{\alpha_0} = \chi_{\mathcal{S}_{\alpha_0}}$, $f_{\alpha} = 0$ for $\alpha \neq \alpha_0$, if $\alpha_0$ is any fixed index in $\Lambda$.

2. $U_{\alpha} \xi = \hat{\Lambda} \cdot U\xi$ for every $\alpha \in \Lambda$. As a bounded linear operator from $\mathcal{H}$ to $\sum_{\alpha \in \Lambda} \otimes L_2(M, \mu, S_{\alpha})$, $U$ is uniquely determined by these two properties.

**Definition 3.5.** The system $\sum_{\alpha \in \Lambda} \otimes L_2(M, \mu, S_{\alpha})$ will be called the canonical $L_2$ decomposition system corresponding to the canonical decomposition system $\{(\mathcal{M}_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda}$. The operator $U$ of Theorem 3.4 will be called the canonical $L_2$ isomorphism corresponding to the canonical decomposition system $\{(\mathcal{M}_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda}$.

We have thus shown that a diagonal ring $\mathcal{E}$ on a Hilbert space $\mathcal{H}$, such that $\mathcal{H}$ is cyclic for $\mathcal{E}'$, can be realized,
through spatial isomorphism, in a rather simple form. It is also relatively easy to construct examples of diagonal rings of this form directly. One such example is given in the appendix.

For the remainder of the chapter, \( \Sigma_{\alpha \in \Lambda} \Theta L_2(M, \mu, S_{\eta_\alpha}) \) and \( U \) will denote, respectively, the canonical \( L_2 \) decomposition system and the canonical \( L_2 \) isomorphism corresponding to the canonical decomposition system which we have fixed previously. If \( \xi \in \mathcal{H} \) and \( U\xi = \{ f_\alpha \}_{\alpha \in \Lambda} \), we note the following important properties of the system \( \{ f_\alpha \}_{\alpha \in \Lambda} \).

Since \( U \) is unitary, \( \int_M |f_\alpha(m)|^2 d\mu(m) = |\xi|^2 < \infty \).

Thus, there are at most a countable number of indices \( \alpha \) for which \( \int_M |f_\alpha(m)|^2 d\mu(m) \neq 0 \). But if \( \int_M |f_\alpha(m)|^2 d\mu(m) = 0 \), then \( f_\alpha(m) = 0 \) a.e., which implies \( f_\alpha(m) \equiv 0 \), since \( f_\alpha \) is a continuous representative. We have thus shown that there are at most a countable number of indices \( \alpha \) for which \( f_\alpha(m) \neq 0 \). From this, it follows that \( \Sigma_{\alpha \in \Lambda} |f_\alpha(m)|^2 \) defines a measurable function, and

\[
\int_M \Sigma_{\alpha \in \Lambda} |f_\alpha(m)|^2 d\mu(m) = \Sigma_{\alpha \in \Lambda} \int_M |f_\alpha(m)|^2 d\mu(m) = |\xi|^2 < \infty.
\]

Let \( \mathcal{L} \) be a fixed Hilbert space whose dimension is \( \text{card} \ \Lambda \), and let \( \{ e_\alpha \}_{\alpha \in \Lambda} \) be an orthonormal basis for \( \mathcal{L} \) indexed by \( \Lambda \). For each \( m \in M \), let \( \mathcal{L}_m \) be the subspace of
for each $m \in M$, we have from (3.2e) that the set \( \{ a \in \Lambda | m \in S_{\eta_a} \} \) is a segment in \( \Lambda \).

We adjoin an element \( \lambda_0 \) to \( \Lambda \) in the manner described earlier. If $m \in M$ such that $m \notin S_{\eta_a}$ for at least one index \( \alpha \), let \( \gamma(m) \) denote the smallest index \( \alpha \) for which this is the case. If $m \in S_{\eta_a}$ for every $\alpha \in \Lambda$, let \( \gamma(m) = \lambda_0 \). It is then immediate that $m \in S_{\eta_a}$ if, and only if $\alpha \in I(\gamma(m))$. Thus, \( \{ e_\alpha | \alpha \in I(\gamma(m)) \} \) is an orthonormal basis for \( L_m \), so \( \dim L_m = \text{card}(I(\gamma(m))) \).

For $\xi \in \mathcal{H}$, we now define a vector valued function almost everywhere on $M$ as follows. Let \( U_\xi = \{ f_\alpha \}_{\alpha \in \Lambda} \).

We have shown above that

\[
\int_M \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 d\mu(m) < \infty, \text{ hence } \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 < \infty \text{ a.e.}
\]

Define \( \xi(m) = \sum_{\alpha \in \Lambda} f_\alpha(m)e_\alpha \) at all points $m \in M$ for which \( \sum_{\alpha} |f_\alpha(m)|^2 < \infty \). Since \( f_\alpha \in L_2(M,\mu,S_{\eta_a}) \) and is a continuous representative for each $\alpha \in \Lambda$, \( f_\alpha(m) = 0 \) if $m \notin S_{\eta_a}$. It follows that \( \xi(m) \in L_m \) where \( \xi(m) \) is defined.

**Definition 3.6.** If \( \{ \xi(m) \} \) is the almost everywhere defined vector valued function obtained from the vector \( \xi \) in the manner described above, we shall write
\[ \xi = \int_M \theta \xi(m) d\mu(m) \] to denote the relationship between \( \xi \) and \( \{\xi(m)\} \). The system of all such vector valued functions will be called a direct integral representation for \( \mathcal{H} \), and will be denoted by \( \int_M \theta \mathcal{L}_m d\mu(m) \). The spaces \( \mathcal{L}_m \) will be called the coordinate spaces of the direct integral.

\[ \int_M \theta \mathcal{L}_m d\mu(m) \] depends, of course, on the canonical decomposition system which we have fixed beforehand. Unless otherwise specified, for vectors \( \xi, \eta, \ldots, \in \mathcal{H} \), we denote \( \xi(m), \eta(m), \ldots \), the vectors in the coordinate spaces \( \mathcal{L}_m \), where they are defined, such that \( \xi = \int_M \theta \xi(m) d\mu(m), \eta = \int_M \theta \eta(m) d\mu(m), \ldots \).

**Theorem 3.7.** (a) If \( \xi, \eta \in \mathcal{H} \), \( \xi(m) + \eta(m) = (\xi + \eta)(m) \) a.e.

(b) If \( c \) is complex, \( (c\xi)(m) = c\xi(m) \) where \( \xi(m) \) is defined.

(c) If \( \xi = \sum_{j=1}^{n} B_j \eta_{\alpha_j} \in \mathcal{H}' \), where \( B_j \in \mathbb{E} \) and \( \alpha_j \in \Lambda \) for \( j = 1, 2, \ldots, n \), then \( \xi(m) \) is everywhere defined, and

\[ \xi(m) = \sum_{j=1}^{n} B_j(m) \chi_{\mathbb{H}_0}(m) \eta_{\alpha_j} \]

(d) If \( \xi \in \mathcal{H} \) and \( A \in \mathbb{E} \), then \( (A\xi)(m) = A(m)\xi(m) \) a.e.
(e) If $\xi, \eta \in \mathcal{H}$, then $(\xi(m), \eta(m))$ defines a summable function, and
\[
\int_{M} (\xi(m), \eta(m)) d\mu(m) = (\xi, \eta).
\]

(f) If $\{\xi_{j}\}_{j=1}^{\infty}$ is a sequence in $\mathcal{H}$ convergent to a vector $\xi$, then there exists a subsequence $\{\xi_{n_{j}}\}_{j=1}^{\infty}$ such that $\xi_{n_{j}}(m) \to \xi(m)$ a.e.

(g) If $\phi \in L_{\infty}(M, \mu)$ and $\xi \in \mathcal{H}$, there exists a vector $\eta \in \mathcal{H}$ such that $\phi(m)\xi(m) = \eta(m)$ a.e.

(h) For each fixed $m_{o} \in M$, the set of vectors $\xi(m_{o})$, where $\xi$ ranges over all vectors of $\mathcal{H}$ for which $\xi(m_{o})$ is defined, coincides with $\mathcal{L}_{m_{o}}$.

Proof of (a). If $U_{\xi} = \{f_{\alpha}\}_{\alpha \in \mathcal{A}}$, $U_{\eta} = \{g_{\alpha}\}_{\alpha \in \mathcal{A}}$, and $U(\xi + \eta) = \{h_{\alpha}\}_{\alpha \in \mathcal{A}}$, then for each $\alpha$, $h_{\alpha}$ is the continuous representative of $f_{\alpha} + g_{\alpha}$. From what we have observed previously, there exists a countable set $\alpha_{1}, \alpha_{2}, \ldots \in \mathcal{A}$ such that $f_{\alpha}(m) = g_{\alpha}(m) = 0$ if $\alpha \notin \{\alpha_{j}\}_{j}$.

Thus $h_{\alpha}(m) = 0$ for $\alpha \notin \{\alpha_{j}\}_{j}$. For every integer $j$ which is an index in the above, let $N_{j}$ be a null set such that $f_{\alpha_{j}}(m) + g_{\alpha_{j}}(m) = h_{\alpha_{j}}(m)$ for $m \notin N_{j}$. Then $\bigcup_{j} N_{j}$ is a null set, and it follows that $f_{\alpha}(m) + g_{\alpha}(m) = h_{\alpha}(m)$ for every $\alpha \in \mathcal{A}$ if $m \notin \bigcup_{j} N_{j}$. Let $N_{o}$ be a null set such that
\[
\sum_{\alpha \in \mathcal{A}} |f_{\alpha}(m)|^{2} < \infty \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}} |g_{\alpha}(m)|^{2} < \infty \quad \text{for} \quad m \notin N_{o},
\]
set $N = N_0 \cup \bigcup_{j} N_j$. Then $N$ is a null set, and for $m \not\in N$ we have
\[
\sum_{\alpha \in \Lambda} \left| h_\alpha(m) \right|^2 = \sum_{\alpha \in \Lambda} \left| f_\alpha(m) + g_\alpha(m) \right|^2 < \infty,
\]
and
\[
(\xi + \eta)(m) = \sum_{\alpha \in \Lambda} h_\alpha(m)e_\alpha = \sum_{\alpha \in \Lambda} (f_\alpha(m) + g_\alpha(m))e_\alpha =
\sum_{\alpha \in \Lambda} f_\alpha(m)e_\alpha + \sum_{\alpha \in \Lambda} g_\alpha(m)e_\alpha = \xi(m) + \eta(m).
\]

**Proof of (b).** If $U\xi = \{f_\alpha\}_{\alpha \in \Lambda}$, then from Theorem 2.11 (b) and the definition of $U$ we have $U(c\xi) = \{cf_\alpha\}_{\alpha \in \Lambda}$. Hence if $\sum_{\alpha \in \Lambda} \left| f_\alpha(m) \right|^2 < \infty$, then also $\sum_{\alpha \in \Lambda} \left| cf_\alpha(m) \right|^2 < \infty$, and
\[
(c\xi)(m) = \sum_{\alpha \in \Lambda} cf_\alpha(m)e_\alpha = c \sum_{\alpha \in \Lambda} f_\alpha(m)e_\alpha = c\xi(m).
\]

**Proof of (c).** We may assume, without loss of generality, that the $\alpha_j$ are distinct. Let $U\xi = \{f_\alpha\}_{\alpha \in \Lambda}$. Since $U\eta_{\alpha_j}$ is the system of functions with $\chi_{S \eta_{\alpha_j}}$ in the $\alpha_j$-th coordinate and zero in all other coordinates for $j = 1, 2, \ldots n$, $f_\alpha(m) = 0$ if $\alpha \not\in \{\alpha_j\}_{j=1}^n$, and $f_\alpha$ is the continuous representative of $\hat{B}_j \chi_{S \eta_{\alpha_j}}$. But this is a continuous function, hence $f_{\alpha_j}(m) = \hat{B}_j(m)\chi_{S \eta_{\alpha_j}}(m)$. Thus
\[
\sum_{\alpha \in \Lambda} \left| f_\alpha(m) \right|^2 = \sum_{j=1}^n \left| \hat{B}_j(m)\chi_{S \eta_{\alpha_j}}(m) \right|^2 < \infty \text{ for all } m \in M, \text{ and}
\]
\[ \xi(m) = \sum_{\alpha \in \Lambda} f_{\alpha}(m)e_{\alpha} = \sum_{j=1}^{n} B_j(m)\chi_{\eta_{\alpha_j}}(m)e_{\alpha_j}. \]

**Proof of (d).** Let \( U_{\xi} = \{f_{\alpha}\}_{\alpha \in \Lambda} \), and \( U_{\eta} = \{g_{\alpha}\}_{\alpha \in \Lambda} \).

For each \( \alpha \), \( g_{\alpha} \) is the continuous representative of the function defined by \( \hat{\Lambda}(\alpha)f_{\alpha}(m) \). Let \( \alpha_1, \alpha_2, \ldots \) be a countable set of indices such that \( f_{\alpha}(m) = 0 \) if \( \alpha \not\in \{\alpha_j\}_j \), and for each \( j \) let \( N_j \) be a null set such that \( g_{\alpha_j}(m) = \hat{\Lambda}(\alpha)f_{\alpha_j}(m) \) for \( m \not\in N_j \). Then \( g_{\alpha}(m) = \hat{\Lambda}(\alpha)f_{\alpha}(m) \) for all \( \alpha \) if \( m \not\in \bigcup_j N_j \), which is a null set. Let \( N_0 \) be a null set for which

\[ \sum_{\alpha \in \Lambda} |f_{\alpha}(m)|^2 < \infty \text{ for } m \not\in N_0. \]

It follows that \( N = N_0 \cup \bigcup_j N_j \) is a null set, and for \( m \not\in N \),

\[ \sum_{\alpha \in \Lambda} |f_{\alpha}(m)|^2 < \infty, \quad \sum_{\alpha \in \Lambda} |g_{\alpha}(m)|^2 = \sum_{\alpha \in \Lambda} |\hat{\Lambda}(\alpha)f_{\alpha}(m)|^2 = |\hat{\Lambda}(m)|^2 \sum_{\alpha \in \Lambda} |f_{\alpha}(m)|^2 < \infty, \]

and \( (\Lambda\eta)(m) = \sum_{\alpha \in \Lambda} \hat{\Lambda}(\alpha)f_{\alpha}(m)e_{\alpha} = \hat{\Lambda}(m) \sum_{\alpha \in \Lambda} f_{\alpha}(m)e_{\alpha} = \Lambda(m)\xi(m). \)

**Proof of (e).** We first consider the case where \( \xi = \eta \).

If \( U_{\xi} = \{f_{\alpha}\}_{\alpha \in \Lambda} \), then \( \xi(m) \) is defined where

\[ \sum_{\alpha \in \Lambda} |f_{\alpha}(m)|^2 < \infty, \text{ and } \xi(m) = \sum_{\alpha \in \Lambda} f_{\alpha}(m)e_{\alpha}. \]

Thus

\[ (\xi(m), \xi(m)) = \sum_{\alpha \in \Lambda} |f_{\alpha}(m)|^2 \]

where the series is convergent, hence by what we have noted above,

\[ \int_{\mathcal{M}} (\xi(m), \xi(m))d\mu(m) = \]
Proof of (f). Let \( n_1 < n_2 < \cdots \) be integers such that \( |\xi - \xi_{n_j}|^2 \leq 2^{-2j} \) for each \( j \). From part (e), this is equivalent to \( \int_M |\xi(m) - \xi_{n_j}(m)|^2 d\mu(m) \leq 2^{-2j} \) for each \( j \).

Since \( \xi(m), \xi_{n_1}(m), \xi_{n_2}(m), \cdots \) is a countable set of functions, each defined except on a null set, there exists a null set \( N_o \) such that each of \( \xi(m), \xi_{n_1}(m), \xi_{n_2}(m), \cdots \) is defined for \( m \notin N_o \). For every positive integer \( j \), define \( N_j = \{ m \in M | |\xi(m) - \xi_{n_j}(m)|^2 > 2^{-j} \} \). Then \( N_j \) is a measurable set, and \( 2^{-j} \mu(N_j) \leq \int_{N_j} |\xi(m) - \xi_{n_j}(m)|^2 d\mu(m) \leq \int_M |\xi(m) - \xi_{n_j}(m)|^2 d\mu(m) \leq 2^{-2j} \), so that \( \mu(N_j) \leq 2^{-j} \) for \( j = 1, 2, \cdots \).

Define \( N = N_o \cup (\cap_{k=1}^\infty \cup_{j=k}^\infty N_j) \). Then \( \xi(m), \xi_{n_1}(m), \xi_{n_2}(m), \cdots \) are all defined if \( m \notin N \). Since \( N_o \) is
a null set, we have, for every positive integer \( q \), \( \mu(N) = \mu(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} N_j) \leq \mu(\bigcup_{j=q}^{\infty} N_j) \leq \sum_{j=q}^{\infty} \mu(N_j) \leq \sum_{j=q}^{\infty} 2^{-j} = 2^{-q+1}.

Since \( q \) was arbitrary, \( \mu(N) = 0 \), hence \( N \) is a null set.

It is claimed that \( \xi_{n_j}(m) \to \xi(m) \) for \( m \notin N \). To show this, we note that if \( m \notin N \), then \( m \notin \bigcup_{j=k}^{\infty} N_j \) for some positive integer \( k \). Thus \( m \notin N_j \) if \( j \geq k \), and for such \( j \), \( \xi(m) \), \( \xi_{n_j}(m) \) are both defined, and

\[
|\xi(m) - \xi_{n_j}(m)|^2 \leq 2^{-j}
\]

from the definition of \( N_j \). This implies \( \lim_{j \to \infty} |\xi(m) - \xi_{n_j}(m)| = 0 \), the desired result.

Proof of (g). By Theorem 2.8 (e), there exists a \( B \in E \) such that \( \hat{B}(m) = \varphi(m) \) a.e. From part (d) above, \( (B\xi)(m) = \hat{B}(m)\xi(m) \) a.e., hence \( (B\xi)(m) = \varphi(m)\xi(m) \) a.e. Thus \( \eta = B\xi \) is the required vector.

Proof of (h). If \( \tau \) is any vector of \( \mathcal{X}_{m_0} \), then

\[
\tau = \sum_{\alpha \in \Gamma(y(m_0))} t_{\alpha} \quad \text{a.e., where } \{t_{\alpha}\}_{\alpha \in \Gamma(y(m_0))} \text{ is a system of complex numbers for which } \sum_{\alpha \in \Gamma(y(m_0))} |t_{\alpha}|^2 < \infty.
\]

Extend the definition of \( t_{\alpha} \) to all of \( \Lambda \) by setting \( t_{\alpha} = 0 \) for \( \alpha \notin y(m_0) \). Since \( |\eta_{\alpha}| \leq 1 \) for each \( \alpha \),

\[
\sum_{\alpha \in \Lambda} |t_{\alpha}|^2 < \infty, \quad \text{and since the } \eta_{\alpha} \text{ are mutually orthogonal,}
\]

\[
\sum_{\alpha \in \Lambda} |t_{\alpha} \eta_{\alpha}|^2 \leq \sum_{\alpha \in \Lambda} |t_{\alpha}|^2 < \infty.
\]
$\sum_{\alpha \in \Lambda} t_{\alpha} \eta_{\alpha}$ converges to a vector of $\mathcal{H}$, which we denote by $\xi$.

But from the definition of $U$, $U\xi = \{t_{\alpha} \chi_{S_{\eta_{\alpha}}}\}_{\alpha \in \Lambda}$, hence

$$\sum_{\alpha \in \Lambda} |t_{\alpha} \chi_{S_{\eta_{\alpha}}}(m)|^2 \leq \sum_{\alpha \in \Lambda} |t_{\alpha}|^2 < \infty \text{ for every } m \in M.$$ Thus $\xi(m)$ is everywhere defined, and $\xi(m) = \sum_{\alpha \in \Lambda} t_{\alpha} \chi_{S_{\eta_{\alpha}}}(m)e_{\alpha}$. For $m = m_0$, this equation becomes $\xi(m_0) = \tau$.

The above theorem shows that $\int_M \Theta \mathcal{L}_m d\mu(m)$ is a Hilbert space under the usual operations, which is isomorphic to $\mathcal{H}$, and that each coordinate space $\mathcal{L}_{m_0}$ is exploited to its full extent by the vector valued functions from the direct integral. A characterization of the vector valued functions in $\int_M \Theta \mathcal{L}_m d\mu(m)$, within almost everywhere equivalence, is given by the following.

**Theorem 3.8.** Suppose $F(m)$ is a vector valued function defined almost everywhere on $M$, with $F(m) \in \mathcal{L}_m$ where the function is defined. Suppose further that $(\eta_{\alpha}(m), F(m))$ defines a measurable function for each $\alpha \in \Lambda$, and that

$$\int_M |F(m)|^2 d\mu(m) < \infty,$$ where $\int_M \cdot$ denotes the upper Lebesgue integral. Then

(a) There exists a unique vector $\zeta \in \mathcal{H}$ such that $(\xi(m), \zeta(m)) = (\xi(m), F(m))$ a.e. for every $\xi \in \mathcal{H}$. Furthermore $|\zeta|^2 \leq \int_M |F(m)|^2 d\mu(m)$. 
(b) In order that $\zeta(m) = F(m)$ a.e., either one of the following two conditions is necessary and sufficient:

(i) There exists a null set $N$ and a countable set $\alpha_1, \alpha_2, \ldots$ of indices from $\Lambda$ such that $(\eta_{\alpha}(m), F(m)) = 0$ if $m \notin N$ and $\alpha$ is not one of the $\alpha_j$.

(ii) $|\xi|^2 = \int_M |F(m)|^2 d\mu(m)$.

In this case, $|F(m)|^2$ is a measurable function.

Proof. To prove (a), we note that if $B_1, B_2, \ldots$, $B_n \in E$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$, then $(\sum_{j=1}^n \hat{B}_j(m) \eta_{\alpha_j}(m), F(m)) = \sum_{j=1}^n \hat{B}_j(m)(\eta_{\alpha_j}(m), F(m))$ defines a measurable function on $E$.

Thus, from Theorem 3.7 (c), $(\xi(m), F(m))$ defines a measurable function if $\xi \in \mathcal{H}$. But if $\xi \in \mathcal{H}$, there exists a sequence $\{\xi_j\}$ in $\mathcal{H}$ converging to $\xi$, and according to Theorem 3.7 (f), we may assume, by replacing $\{\xi_j\}$ by a subsequence, that $\xi_j(m) \to \xi(m)$ a.e. If $N$ is a null set such that $\xi_j(m) \to \xi(m)$ and $F(m)$ is defined for $m \notin N$, it follows that $(\xi_j(m), F(m)) \to (\xi(m), F(m))$ for $m \notin N$. Thus, $(\xi(m), F(m))$ also defines a measurable function.

For arbitrary $\xi \in \mathcal{H}$, we have the inequality

$$
\int_M |(\xi(m), F(m))| d\mu(m) \leq \int_M |\xi(m)||F(m)| d\mu(m) \leq \int_M |\xi(m)|^2 d\mu(m) \frac{\frac{1}{2}}{\frac{1}{2}} \left( \int_M |F(m)|^2 d\mu(m) \right)^{\frac{1}{2}} = |\xi| \left( \int_M |F(m)|^2 d\mu(m) \right)^{\frac{1}{2}},
$$
so that \( L(\xi) = \int_M (\xi(m), F(m)) d\mu(m) \) defines a linear functional on \( \mathcal{H} \) with \(|L| \leq \left( \int_M |F(m)|^2 d\mu(m) \right)^{1/2} \). Thus, there exists \( \zeta \in \mathcal{H} \) such that \( L(\xi) = (\xi, \zeta) \) for every \( \xi \), and \(|\zeta|^2 = |L|^2 \leq \int_M |F(m)|^2 d\mu(m) \). Now let \( \xi \) be any fixed vector in \( \mathcal{H} \). If \( \phi \) is arbitrary in \( L_\infty(M, \mu) \), let \( \eta \) be a vector of \( \mathcal{H} \), which exists by Theorem 3.7 (g), such that \( \phi(m) \xi(m) = \eta(m) \) a.e. Applying the above formula to \( \eta \), we have \( \int_M \phi(m)(\xi(m), \zeta(m)) d\mu(m) = \int_M (\eta(m), \zeta(m)) d\mu(m) = (\eta, \zeta) = L(\eta) = \int_M (\eta(m), F(m)) d\mu(m) = \int_M \phi(m)(\xi(m), F(m)) d\mu(m) \).

Since \( \phi \) was arbitrary, \( (\xi(m), F(m)) = (\xi(m), \zeta(m)) \) a.e.

To show uniqueness, suppose \( \zeta' \) is a second vector such that \( (\xi(m), F(m)) = (\xi(m), \zeta'(m)) \) a.e. for every \( \xi \in \mathcal{H} \). Then \( (\xi(m), \zeta(m)) = (\xi(m), \zeta'(m)) \) a.e. for every \( \xi \), hence \( \zeta = \zeta' \) a.e. for every \( \xi \). The inequality stated in this part of the theorem was derived in the construction above.

We shall prove part (b) in the following cyclic manner:

Thus, suppose first that \( F(m) = \zeta(m) \) a.e. Let \( U_\zeta = \{ f_\alpha \}_{\alpha \in \Lambda} \), and \( \alpha_1, \alpha_2, \ldots \) be a countable set of distinct indices such that \( f_\alpha(m) = 0 \) if \( \alpha \not\in \{ \alpha_j \} \). Let \( N \) be a null
set such that \( \sum_{\alpha \in A} |f_\alpha(m)|^2 = \sum_{j} |f_{\alpha_j}(m)|^2 < \infty \) if \( m \not\in N \), and also such that \( F(m) \) is defined, and equal to \( \zeta(m) \) if \( m \not\in N \). Thus, if \( m \not\in N \) and \( \alpha \not\in \{\alpha_j\} \), \((\eta_\alpha(m), F(m)) = \langle \chi_{\eta_\alpha} \rangle(m)e_\alpha, \zeta(m)\rangle = \chi_{\eta_\alpha} \langle m \rangle(e_\alpha, \sum_{j} f_{\alpha_j}(m)e_{\alpha_j}) = 0 \). This proves that (i) is satisfied.

Now assume (i) is true. If \( m \in M \) such that \( F(m) \) is defined, we have, since \( \{e_\alpha\}_{\alpha \in I(\gamma(m))} \) is an orthonormal basis for \( \mathcal{X}_m \), \( F(m) = \sum_{\alpha \in I(\gamma(m))} (F(m), e_\alpha)e_\alpha \), and \( |F(m)|^2 = \sum_{\alpha \in I(\gamma(m))} |(F(m), e_\alpha)|^2 \). Since \( \eta_\alpha(m) = e_\alpha \) for \( \alpha \in I(\gamma(m)) \) and \( \eta_\alpha(m) = 0 \) for \( \alpha \not\in I(\gamma(m)) \), this may be rewritten

\[ |F(m)|^2 = \sum_{\alpha \in A} |(F(m), \eta_\alpha(m))|^2. \]

If the null set \( N \) and indices \( \alpha_1, \alpha_2, \ldots \) are as specified, it follows that

\[ |F(m)|^2 = \sum_{j} |(F(m), \eta_{\alpha_j}(m))|^2 \] for \( m \not\in N \). If \( \mathcal{U}_\zeta = \{f_\alpha\}_{\alpha \in A} \) then \( (F(m), \eta_{\alpha_j}(m)) = \langle \zeta(m), \eta_{\alpha_j}(m) \rangle = f_{\alpha_j}(m) \) a.e. for each \( j \). Since the set of indices \( j \) is countable, there exists a fixed null set \( N_1 \supset N \) such that

\( (F(m), \eta_{\alpha_j}(m)) = f_{\alpha_j}(m) \) for every \( j \) if \( m \not\in N_1 \), and hence

\[ |F(m)|^2 = \sum_{j} |f_{\alpha_j}(m)|^2 \] if \( m \not\in N_1 \). This proves \( |F(m)|^2 \) is
measurable. We thus have \( \int_M |F(m)|^2 d\mu(m) = \int_M |F(m)|^2 d\mu(m) = \sum_j \int_M |f_{\alpha_j}(m)|^2 d\mu(m) \leq \sum_{\alpha \in \Lambda} \int_M |f_{\alpha}(m)|^2 d\mu(m) = |\zeta|^2. \) Inequality in the other direction was shown in part (a), hence (ii) holds. We have also shown our final statement on the measurability of \( |F(m)|^2 \).

Finally, assume (ii). Let \( \mathcal{U} = \{ f_{\alpha} \}_{\alpha \in \Lambda} \), and \( \alpha_1, \alpha_2, \ldots \) be an enumeration of the indices \( \alpha \) for which \( f_{\alpha}(m) \neq 0 \), such that \( \alpha_i \neq \alpha_j \) if \( i \neq j \). Using the Fourier expansion for \( F(m) \) where it is defined, as above, we have
\[
|F(m)|^2 = \sum_j |(F(m), \eta_{\alpha_j}(m))|^2 + |F(m) - \sum_j (F(m), \eta_{\alpha_j}(m))\eta_{\alpha_j}(m)|^2.
\]
For each \( j \), we have \( (F(m), \eta_{\alpha_j}(m)) = (\zeta(m), \eta_{\alpha_j}(m)) = f_{\alpha_j}(m) \) a.e. Since there are at most a countable number of indices \( \alpha_j \), it follows, by the usual argument, that
\[
|F(m)|^2 = \sum_j |f_{\alpha_j}(m)|^2 + |F(m) - \sum_j f_{\alpha_j}(m)\eta_{\alpha_j}(m)|^2 \text{ a.e.}
\]
Since each term in the first sum on the right side of this equation is measurable, and there is at most a countable number of such terms
\[
\int_M |F(m)|^2 d\mu(m) = \sum_j \int_M |f_{\alpha_j}(m)|^2 d\mu(m) + \int_M |F(m)|^2 d\mu(m) - \sum_j f_{\alpha_j}(m)\eta_{\alpha_j}(m)|^2 d\mu(m).
\]
By assumption, $\int_M |F(m)|^2 d\mu(m) = |\zeta|^2 = \sum_{\alpha \in \Lambda} \int_M |f_\alpha(m)|^2 d\mu(m)$

$= \sum_j \int_M |f_{\alpha_j}(m)|^2 d\mu(m)$, hence we have

$\int_M |F(m) - \sum_j f_{\alpha_j}(m)\eta_{\alpha_j}(m)|^2 d\mu(m) = 0$. But, except on a set of measure zero, we have $\sum_j |f_{\alpha_j}(m)|^2 = \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 < \infty$,

and $\zeta(m) = \sum_{\alpha \in \Lambda} f_\alpha(m)e_\alpha = \sum_j f_{\alpha_j}(m)e_{\alpha_j} = \sum_j f_{\alpha_j}(m)\eta_{\alpha_j}(m)$. Thus,

$\int_M |F(m) - \zeta(m)|^2 d\mu(m) = 0$, which implies $F(m) = \zeta(m)$ a.e.

We note that in part (b) of the above theorem, condition (i) is always satisfied whenever $\Lambda$ is countable. In the general case, it is possible for the equivalent conditions in (b) to fail, and it is not possible to conclude that $|F(m)|^2$ is even measurable. An illustrative demonstration of this is given in the example in the appendix.

Theorems 3.7 and 3.8, together, show that

$\int_M \otimes \mathcal{L}_m d\mu(m)$ is a weak direct integral for $\mathcal{H}$ as defined by Segal in [7], p. 15. The additional condition it must satisfy to become a strong direct integral, namely condition 2(a) in the reference just stated, does not in general hold. However Theorem 3.8 (b) gives necessary and sufficient conditions that the strong direct integral condition hold for a fixed vector valued function.
We now show how the rings \( E \) and \( E' \) decompose in the above direct integral.

**Lemma 3.9.1.** If \( A \in E' \) and \( \xi \in \mathcal{H} \), then
\[
|A\xi(m)| \leq |A||\xi(m)| \quad \text{a.e.}
\]

**Proof.** For \( \epsilon > 0 \), define \( T_\epsilon = \{ m \in M | |A\xi(m)|^2 \geq |A|^2|\xi(m)|^2 + \epsilon \} \). Then by Theorem 2.8 (d), there exists a clopen set \( U_\epsilon \) such that \( \mu(U_\epsilon \Delta T_\epsilon) = 0 \). It follows that \( |A\xi(m)|^2 \geq |A|^2|\xi(m)|^2 + \epsilon \) for almost all \( m \in U_\epsilon \). Using Theorem 3.7 (d), \((P_{U_\epsilon} \xi)(m) = \chi_{U_\epsilon}(m)\xi(m) \quad \text{a.e.}, \)
and \((AP_{U_\epsilon} \xi)(m) = (P_{U_\epsilon} A\xi)(m) = \chi_{U_\epsilon}(m)(A\xi)(m) \quad \text{a.e.} \)
Then, from Theorem 3.7 (e), the inequality \( |AP_{U_\epsilon} \xi|^2 \leq |A|^2|P_{U_\epsilon} \xi|^2 \)
is equivalent to \( \int_M \chi_{U_\epsilon}(m)|A\xi(m)|^2d\mu(m) \leq |A|^2 \int_M \chi_{U_\epsilon}(m)|\xi(m)|^2d\mu(m). \)

But from the above, \( \int_M \chi_{U_\epsilon}(m)|A\xi(m)|^2d\mu(m) \geq |A|^2 \int_M \chi_{U_\epsilon}(m)|\xi(m)|^2d\mu(m) + \epsilon \mu(U_\epsilon), \)
hence \( \mu(T_\epsilon) = \mu(U_\epsilon) = 0. \)

Let \( \{ \epsilon_j \}_{j=1}^\infty \) be any sequence of positive numbers convergent to 0, and \( T = \{ m \in M | |A\xi(m)|^2 > |A||\xi(m)| \} \). Then \( T = \bigcup_{j=1}^\infty T_{\epsilon_j} \), hence \( \mu(T) = 0 \). If \( T_0 \) is a null set such that both \( \xi(m) \) and \( A\xi(m) \) are defined for \( m \not\in T_0 \), and \( N = T \cup T_0 \), then \( N \) is a null set, and \( |A\xi(m)| \leq |A||\xi(m)| \)
for \( m \not\in N \).
Theorem 3.9. Let $A$ be an operator of $E'$. Then there exists a unique operator valued function defined on all of $M$, whose value at each point $m$ we denote by $A(m)$, such that $A(m) \in B(L_m)$ for each $m \in M$, and the following hold:

(3.9a) $(A\xi)(m) = A(m)\xi(m)$ a.e. for every $\xi \in \mathcal{H}$.

(3.9b) $|A(m)|$ is bounded on $M$.

(3.9c) $(A(m)\eta_\alpha(m), \eta_\beta(m))$ defines a continuous function on $M$ for every $\alpha, \beta \in \Lambda$.

Furthermore, we have $(A\xi)(m) = A(m)\xi(m)$ everywhere if $\xi \in \mathcal{H}'$, and also that $\sup_{m \in M} |A(m)| = |A|$.

Proof. Let $\xi \in \mathcal{H}'$. Then $\xi$ can be written in the form $\sum_{j=1}^{n} B_j \eta_{\alpha_j}$, where $B_j \in E$ for $j = 1, 2, \ldots, n$, and the $\alpha_j$ are distinct indices in $\Lambda$. By Theorem 3.7 (c),

$$\xi(m) = \sum_{j=1}^{n} B_j(m)\chi_{\eta_{\alpha_j}}(m)$$

everywhere, hence $|\xi(m)|^2 = \sum_{j=1}^{n} |B_j(m)|^2 \chi_{\eta_{\alpha_j}}(m)$ everywhere. Let $U(A\xi) = \{f_\alpha\}_{\alpha \in \Lambda}$.

Then $|A\xi(m)|^2 = \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2$ where the sum is convergent, which is almost everywhere. By Lemma 3.9.1, it follows that

$$\sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 \leq |A|^2 \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{\eta_{\alpha_j}}(m)$$
a.e. This implies
that for each fixed index $\alpha$,
\[ |f_\alpha(m)|^2 \leq |A|^2 \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{\eta_{\alpha_j}}(m) \text{ a.e.} \]
But $f_\alpha$ is a continuous representative, hence $|f_\alpha|^2$ is an extended non negative valued continuous representative, and since the right side of this last equation defines a continuous function, we have inequality everywhere, hence $f_\alpha$ is finite valued. It follows that $f_\alpha$ is a continuous function for every index $\alpha$.

Now let $\beta_1, \ldots, \beta_q$ be any finite set of distinct indices in $\Lambda$. Then
\[ \sum_{j=1}^{q} |f_{\beta_j}(m)|^2 \leq \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 \leq \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{\eta_{\alpha_j}}(m) \text{ a.e.} \]
Since the first and last of these quantities define a continuous function on $M$, we have
\[ \sum_{j=1}^{q} |f_{\beta_j}(m)|^2 \leq \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{\eta_{\alpha_j}}(m) \text{ everywhere.} \]
Since the indices $\beta_1, \ldots, \beta_q$ were arbitrary, it follows that
\[ \sum_{\alpha \in \Lambda} |f_\alpha(m)|^2 \leq \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{\eta_{\alpha_j}}(m) \text{ for all } m \in M. \]

We now introduce a special notation for the vectors $\eta_\alpha$. For each $\alpha \in \Lambda$, let $\mathbb{U}_\alpha \eta_\alpha = \{a_{\alpha\beta}\}_{\beta \in \Lambda}$. Then $a_{\alpha\beta}(m)$ defines a continuous function on $M$ for every $\alpha, \beta \in \Lambda$, according to the above. Since $a_{\alpha\beta} \in L^2(M, \mu, S_{\eta_\beta})$,
\[ a_{\alpha \beta}(m) = 0 \text{ for } m \notin S_{\eta_\beta}. \text{ Also, } [a_{\alpha \beta}]_{\beta \in \Lambda} = U A \eta = U A P \eta \alpha = U A \eta \alpha = \hat{P} \eta \alpha. \text{ This implies that } a_{\alpha \beta} \text{ is the continuous representative of the function defined by } \chi_{S_{\eta_\alpha}}(m) a_{\alpha \beta}(m) \text{ for every } \alpha, \beta. \text{ As the latter defines a continuous function, } a_{\alpha \beta}(m) = \chi_{S_{\eta_\alpha}}(m) a_{\alpha \beta}(m), \text{ hence } a_{\alpha \beta}(m) = 0 \text{ for } m \notin S_{\eta_\alpha}, \text{ for arbitrary } \alpha, \beta \in \Lambda.

Returning to an arbitrary \( \xi \in \mathcal{H} \) of the form

\[ \sum_{j=1}^{n} B_j \eta_{\alpha_j}, \text{ where the } B_j \in \mathbb{E}, \text{ and the } \alpha_j \text{ are distinct,} \]

\[ U A \xi = U A \sum_{j=1}^{n} B_j \eta_{\alpha_j} = \sum_{j=1}^{n} U B_j \eta_{\alpha_j} = \sum_{j=1}^{n} \hat{B}_j \cdot U A \eta_{\alpha_j}. \text{ If } U A \xi = [f_{\alpha}]_{\alpha \in \Lambda}, \text{ we thus have, in terms of the function } a_{\alpha \beta} \text{ introduced above, } f_{\beta}(m) = \sum_{j=1}^{n} \hat{B}_j(m) a_{\alpha_j \beta}(m) \text{ a.e. for every } \beta \in \Lambda. \text{ Since both sides of this equation define continuous functions, equality holds everywhere. Our above inequality for the } f_{\alpha} \text{ thus becomes}

\[ \sum_{\beta \in \Lambda} \left| \sum_{j=1}^{n} \hat{B}_j(m) a_{\alpha_j \beta}(m) \right|^2 \leq |A|^2 \sum_{j=1}^{n} |\hat{B}_j(m)|^2 \chi_{S_{\eta_\alpha}}(m), \]

whenever \( B_1, \ldots, B_n \in \mathbb{E}, \alpha_1, \ldots, \alpha_n \) are distinct indices in \( \Lambda \), and \( m \in M \).
In particular, if \( B_j = t_j I \) for each \( j \), where the \( t_j \) are arbitrary complex numbers, we obtain

\[
\sum_{\beta \in \Lambda} \sum_{j=1}^{n} |t_j|^2 a_{\alpha_j \beta}(m)|^2 \leq |A|^2 \sum_{j=1}^{n} |t_j|^2 \chi_{S_{\eta_{\alpha_j}}}(m) \text{, for } m \in M,
\]

\( \eta_{\alpha_j} \) and \( \eta_{\beta} \) complex, and \( \alpha_1, \ldots, \alpha_n \) distinct indices in \( \Lambda \). In view of our observation above that \( a_{\alpha \beta}(m) = 0 \) for \( m \notin S_{\eta_{\alpha}} \cap S_{\eta_{\beta}} \), this inequality is equivalent to

\[
\sum_{\beta \in I(\gamma(m))} \sum_{j=1}^{n} |t_j|^2 a_{\alpha_j \beta}(m)|^2 \leq |A|^2 \sum_{j=1}^{n} |t_j|^2 \text{, whenever}
\]

\( t_1, \ldots, t_n \) are complex, and \( \alpha_1, \ldots, \alpha_n \) are distinct indices in \( I(\gamma(m)) \).

We now define a linear operator \( A(m) \) on a dense linear subspace of \( \mathcal{X}_m \), for each fixed \( m \), by

\[
A(m) = \sum_{j=1}^{n} t_j \alpha_j = \sum_{\beta \in I(\gamma(m))} (\sum_{j=1}^{n} t_j a_{\alpha_j \beta}(m)) \beta
\]

\( t_1, \ldots, t_n \) are complex, and \( \alpha_1, \ldots, \alpha_n \) are distinct in \( I(\gamma(m)) \).

Since \( \{ \alpha \} \in I(\gamma(m)) \) is an orthonormal basis for \( \mathcal{X}_m \), our above inequality shows that the right side of this expression converges, and \( |A(m)| \leq |A| \). Consequently, we may extend \( A(m) \) in a unique norm preserving manner to all of \( \mathcal{X}_m \). It is claimed that the \( A(m) \) defined in this manner for every \( m \) satisfy the conclusions of the
If $\xi$ is of the form $\sum_{j=1}^{n} B_j \eta_{\alpha_j}$ as above, we have from our above computations for $U_A \xi$ that $(A\xi)(m) =$

$$
\sum_{\beta \in \Lambda} (\sum_{j=1}^{n} B_j(m) a_{\alpha_j \beta}(m)) e_{\beta} \in \sum_{\beta \in I(\gamma(m))}^{n} (\sum_{j=1}^{n} B_j(m) a_{\alpha_j \beta}(m)) e_{\beta}
$$

$$
= A(m)(\sum_{j=1}^{n} B_j(m) x S J (m) e_{\alpha_j}) = A(m)\xi(m) \text{ for every } m \in M.
$$

We have thus obtained that part of our conclusion stating $(A\xi)(m) = A(m)\xi(m)$ for $\xi \in \mathcal{H}$. If $\xi$ is arbitrary in $\mathcal{H}$, there exists a sequence $\{\xi_j\}_{j=1}^{\infty}$ in $\mathcal{H}$ converging to $\xi$.

Since $A$ is bounded, $A\xi_j \to A\xi$, hence, by Theorem 3.7(f), we may replace $\{\xi_j\}_{j=1}^{\infty}$ by a subsequence and assume also that $\xi_j(m) \to \xi(m)$ and $(A\xi_j)(m) \to (A\xi)(m)$ for $m \not\in N$, where $N$ is a null set. By what we have just shown above, $(A\xi_j)(m) = A(m)\xi_j(m)$ for every $m \in M$ and every integer $j$. Hence, if $m \not\in N$, $(A\xi)(m) = \lim_{j \to \infty}(A\xi_j)(m) = \lim_{j \to \infty} A(m)\xi_j(m) = A(m)(\lim_{j \to \infty} \xi_j(m)) = A(m)\xi(m)$. This proves (3.9a).

From the above construction, we have seen that (3.9b) holds, with $\sup_{m \in M} |A(m)| \leq |A|$.

To prove (3.9c), we note that for arbitrary $m \in M$ and arbitrary $\alpha, \beta \in \Lambda$, we have $(A(m)\eta_{\alpha}(m), \eta_{\beta}(m)) =$
(A(m)x_{\alpha}, x_{\beta}) = (\sum_{\beta \in I(\gamma(m))} a_{\alpha\beta} (m) e_{\beta}) x_{\alpha} = a_{\alpha\beta}(m), which defines a continuous function.

Finally, we show that \( \sup_{m \in M} |A(m)| = |A| \). From the above, \( \sup_{m \in M} |A(m)| \leq |A| \). To prove the reverse inequality, we have from Theorem 3.7(e) and (3.9a),

\[
\int_M |(A\xi)(m)|^2 d\mu(m) = \int_M |A(m)\xi(m)|^2 d\mu(m) \leq \left( \sup_{m \in M} |A(m)| \right)^2 \int_M |\xi(m)|^2 d\mu(m) \]

for arbitrary \( \xi \in \mathcal{H} \). Hence \( |A| \leq \sup_{m \in M} |A(m)| \).

To prove uniqueness, suppose \( \{A_1(m)\}_{m \in M} \) is another system satisfying (3.9a), (3.9b), and (3.9c) for the operator \( A \). For each \( \alpha \), \( A_1(m)\eta_{\alpha}(m) = (A\eta_{\alpha})(m) = A(m)\eta_{\alpha}(m) \) a.e. by (3.9a), hence \( (A_1(m)\eta_{\alpha}(m), \eta_{\beta}(m)) = (A(m)\eta_{\alpha}(m), \eta_{\beta}(m)) \) a.e. for arbitrary \( \alpha, \beta \). Both sides are continuous, by (3.9c), hence equality holds everywhere.

For fixed \( m \), let \( \alpha, \beta \in I(\gamma(m)) \). Then the above becomes\( (A_1(m)e_\alpha, e_\beta) = (A(m)e_\alpha, e_\beta) \). Since \( \{e_\alpha | \alpha \in I(\gamma(m))\} \) is an orthonormal basis for \( \mathcal{L}_m \), and both \( A_1(m), A(m) \in \mathcal{B}(\mathcal{L}_m) \), necessarily \( A_1(m) = A(m) \). Since \( m \) was arbitrary, the uniqueness follows.

Definition 3.10. The unique operator valued function
m → A(m) satisfying (3.9a) - (3.9c) for an operator A ∈ E' will be called its direct integral decomposition with respect to the direct integral \( \int_M \oplus \mathcal{L}_m d\mu(m) \). We shall write symbolically \( A = \int_M \oplus A(m)d\mu(m) \).

**Corollary 3.10.1.** If A ∈ E, then \( A = \int_M \oplus \hat{A}(m)I_m d\mu(m) \), where \( I_m \) denotes the identity operator on \( \mathcal{L}_m \) for every \( m ∈ M \).

**Proof.** The operator valued function \( m → \hat{A}(m)I_m \) satisfies (3.9a), according to Theorem 3.7(d). Clearly \( |\hat{A}(m)I_m| \leq |A| \), so (3.9b) holds. To prove (3.9c), we note that

\[
(\hat{A}(m)I_m \eta_\alpha(m), \eta_\beta(m)) = (\hat{A}(m) \chi_{S_\eta_\alpha}(m) \eta_\alpha, \chi_{S_\eta_\beta}(m) \eta_\beta) = \\
\hat{A}(m) \chi_{S_\eta_\alpha} \cap S_\eta_\beta(m)(\eta_\alpha, \eta_\beta)
\]

defines a continuous function for arbitrary \( \alpha, \beta ∈ \Lambda \).

Direct integral decompositions, and hence the operators from which they come, are relatively easy to construct, as shown by the following, which may also be considered a partial converse to Theorem 3.9.

**Theorem 3.11.** Suppose \( m → A(m) ∈ \Theta(\mathcal{L}_m) \) is an operator valued function defined everywhere on \( M \), such that (3.9b), (3.9c) hold. Then there exists an operator \( A ∈ E' \), necessarily unique, for which \( A = \int_M \oplus A(m)d\mu(m) \).

**Proof.** Let \( \xi, \eta ∈ \mathcal{H}' \), and suppose \( \xi = \sum_{j=1}^{n_1} B_j \eta_\alpha_j \).
\[ \eta = \sum_{j=1}^{n^2} c_j \eta_{j}^{\prime}, \] where the operators \( B_j, C_j \) are from \( E \), and the indices \( \alpha_j, \beta_j \) are from \( A \). By Theorem 3.7(c),

\[ \xi(m) = \sum_{j=1}^{n} B_j(m) \eta_{j}^{\prime} \] everywhere, and likewise, \( \eta(m) = \sum_{j=1}^{n^2} C_j(m) \eta_{j}^{\prime} \) everywhere. Hence

\[ (A(m)\xi(m), \eta(m)) = \sum_{j=1}^{n} \sum_{k=1}^{n^2} B_j(m) \bar{C}_k(m) (A(m)\eta_{j}^{\prime}(m), \eta_{\beta_k}^{\prime}(m)) \]
defines a continuous function, by our hypothesis that \( m \to A(m) \) satisfy (3.9c).

If \( \xi, \eta \in \mathcal{H} \), then since \( \mathcal{H} \) is dense in \( \mathcal{H} \), there exist sequences \( \{\xi_j\}_{j=1}^{\infty} \) and \( \{\eta_j\}_{j=1}^{\infty} \) which converge to \( \xi \) and \( \eta \) respectively. By Theorem 3.7(f), we may assume \( \xi_j(m) \to \xi(m) \) a.e. and \( \eta_j(m) \to \eta(m) \) a.e. Then it follows that \((A(m)\xi_j(m), \eta_j(m)) \to (A(m)\xi(m), \eta(m)) \) a.e., hence

\[ (A(m)\xi(m), \eta(m)) \] defines a measurable function for arbitrary \( \xi, \eta \in \mathcal{H} \). Also,

\[ \sup_{m \in M} |A(m)| \int_M |\xi(m)||\eta(m)| \, d\mu(m) \leq \]

\[ \left[ \sup_{m \in M} |A(m)| \right] \left( \int_M |\xi(m)|^2 \, d\mu(m) \right)^{\frac{1}{2}} \left( \int_M |\eta(m)|^2 \, d\mu(m) \right)^{\frac{1}{2}} = \]

\[ \left[ \sup_{m \in M} |A(m)| \right] ||\xi|| ||\eta||, \text{ for arbitrary } \xi, \eta \in \mathcal{H}. \]

The equation \( \langle \xi, \eta \rangle = \int_M (A(m)\xi(m), \eta(m)) \, d\mu(m) \) thus
defines a bounded bilinear form on $\mathcal{H}$, hence there exists a unique $A \in \mathcal{B}(\mathcal{H})$ such that $(A\xi, \eta) = \int_M (A(m)\xi(m), \eta(m))d\mu(m)$ for arbitrary $\xi, \eta \in \mathcal{H}$. If $B \in \mathcal{E}$, we have $(AB\xi, \eta) = \int_M (A(m)(B\xi)(m), \eta(m))d\mu(m) = \int_M (A(m)\hat{B}(m)\xi(m), \eta(m))d\mu(m) = \int_M (A(m)\xi(m), (B^*\eta)(m))d\mu(m)$

$= (A\xi, B^*\eta) = (BA\xi, \eta)$ for arbitrary $\xi, \eta$, hence $A \in \mathcal{E}'$. Let $A = \int_M \Phi A_1(m)d\mu(m)$ be the direct integral decomposition of $A$. We shall prove $A(m) = A_1(m)$ for all $m \in M$.

In fact, if $B \in \mathcal{E}$, and $\alpha, \beta \in \Lambda$, we have

$\int_M \hat{B}(m)(A(m)\eta_\alpha(m), \eta_\beta(m))d\mu(m) = \int_M (A(m)\hat{B}(m)\eta_\alpha(m), \eta_\beta(m))d\mu(m)$

$= \int_M (A(m)(B\eta_\alpha)(m), \eta_\beta(m))d\mu(m) = (AB\eta_\alpha, \eta_\beta)$

$= \int_M (A_1(m)(B\eta_\alpha)(m), \eta_\beta(m))d\mu(m) = \int_M (A_1(m)\hat{B}(m)\eta_\alpha(m), \eta_\beta(m))d\mu(m)$

$= \int_M \hat{B}(m)(A_1(m)\eta_\alpha(m), \eta_\beta(m))d\mu(m)$. Since $B$ was arbitrary from $\mathcal{E}$, it follows that $(A(m)\eta_\alpha(m), \eta_\beta(m)) = (A_1(m)\eta_\alpha(m), \eta_\beta(m))$ a.e. But both operator valued functions satisfy (3.9c), hence $(A(m)\eta_\alpha(m), \eta_\beta(m)) = (A_1(m)\eta_\alpha(m), \eta_\beta(m))$ for all $m \in M$ whenever $\alpha, \beta \in \Lambda$. If we fix $m$ and choose $\alpha, \beta \in I(\gamma(m))$, this becomes $(A(m)e_\alpha, e_\beta) = (A_1(m)e_\alpha, e_\beta)$. Since $

{e_\alpha | \alpha \in I(\gamma(m))}$ is an orthonormal basis for $\mathcal{L}_m$, we have $A(m) = A_1(m)$.

That an operator $A \in \mathcal{E}'$ is determined uniquely by its direct integral decomposition follows from Theorem 3.7(e)
and (3.9a), which show that $A$ is given from its decomposition by the formula 
\[
(A\xi, \eta) = \int_M (A(m)\xi(m), \eta(m)) \, d\mu(m).
\]

For the remainder of this chapter, we denote by $A(m), B(m), \ldots$ the respective values of the direct integral decompositions of the operators $A, B, \ldots \in E'$.

**Theorem 3.12.** Suppose $A, B \in E'$.

(a) If $a, b$ are complex, $(aA + bB)(m) = aA(m) + bB(m)$ for every $m \in M$.

(b) $A^*(m) = (A(m))^*$ for every $m \in M$.

(c) If $\xi \in \mathcal{H}$, then $(AB\xi)(m) = (AB)(m)\xi(m) = A(m)B(m)\xi(m)$ a.e.

(d) If $B \in E$, then $(AB)(m) = A(m)B(m) = B(m)A(m)$ for every $m \in M$.

**Proof of (a).** We shall prove that the function
\[
m \mapsto aA(m) + bB(m)
\]
satisfies (3.9a) - (3.9c) for the operator $aA + bB$. Thus, if $\xi \in \mathcal{H}$, 
\[
((aA + bB)\xi)(m) = a(A\xi)(m) + b(B\xi)(m) = aA(m)\xi(m) + bB(m)\xi(m)
\]
a.e., by Theorem 3.7 and the fact that $A(m), B(m)$ satisfy (3.9a).

Since $|A(m)| \leq |A|$ and $|B(m)| \leq |B|$, $|aA(m) + bB(m)| \leq |a||A(m)| + |b||B(m)| \leq |a||A| + |b||B|$ for all $m \in M$, hence (3.9b) holds. Since 
\[
(aA(m) + bB(m)) \eta_\alpha(m), \eta_\beta(m) = a(A(m)\eta_\alpha(m), \eta_\beta(m)) + b(B(m)\eta_\alpha(m), \eta_\beta(m))
\]
for $\alpha, \beta \in \Lambda$ and $m \in M$, and this last expression defines a continuous function on $M$ for fixed $\alpha, \beta$, (3.9c) is satisfied.
Proof of (b). Since $|A^*(m)| = |A(m)| \leq |A|$ for all $m$, the function $m \to A^*(m)$ satisfies (3.9b). Since

\[(A(m))^*\eta_\alpha(m), \eta_\beta(m)) = (\eta_\alpha(m), A(m)\eta_\beta(m))\]

for all $m$, it also satisfies (3.9c). By Theorem 3.11, there exists a unique operator $A_0 \in E'$ such that $A_0 = \int_M \phi(A(m))^* d\mu(m)$.

For arbitrary $\xi, \eta \in \mathcal{H}$, we have $(\eta, A_0 \xi) = (A_0 \eta, \xi) = \int_M ((A(m))^*\eta(m), \eta(m)) d\mu(m) = \int_M (\eta(m), A(m)\eta(m)) d\mu(m) = \int_M (A(m)\eta(m), \eta(m)) d\mu(m) = (A\eta, \xi)$. This shows $A_0 = A^*$, hence $A^* = \int_M \phi(A(m))^* d\mu(m)$.

Proof of (c). Since $A, B, \xi$ are fixed, we have from (3.9a), $(AB\xi)(m) = A(m)(B\xi)(m)$ a.e., $(B\xi)(m) = B(m)\xi(m)$ a.e., and $(AB\xi)(m) = (AB)(m)\xi(m)$ a.e. If we fix a null set $N$ such that all three of these relations hold whenever $m \notin N$, it follows that $(AB\xi)(m) = (AB)(m)\xi(m) = A(m)B(m)\xi(m)$ whenever $m \notin N$.

Proof of (d). From corollary 3.10.1, we have $B(m) = \hat{B}(m)I_m$, hence $A(m)B(m) = \hat{B}(m)A(m)$ for all $m$. By part (c) of this theorem, which we have just shown above, the function $m \to A(m)B(m)$ satisfies (3.9a) for the operator $AB$. Since $|A(m)B(m)| \leq |A(m)||B(m)| \leq |A||B|$ for all $m$, it satisfies (3.9b). To show (3.9c), we note that for $\alpha, \beta \in \Lambda$, $m \in M$, $(A(m)B(m)\eta_\alpha(m), \eta_\beta(m)) = (\hat{B}(m)A(m)\eta_\alpha(m), \eta_\beta(m)) = \hat{B}(m)(A(m)\eta_\alpha(m), \eta_\beta(m))$. For fixed
\(\alpha, \beta,\) this expression defines a continuous function on \(M.\)

All algebraic operations on \(E',\) except possibly multiplication, are thus carried over to the direct integral. One case in which we can be sure that multiplication holds is given by the following.

**Corollary 3.12.1.** If \(m_0\) is an isolated point of \(M,\) then \((AB)(m_0) = A(m_0)B(m_0)\) for every \(A, B \in E'.\)

**Proof.** By Theorem 3.12 (c), \((AB)(m)\xi(m) = A(m)B(m)\xi(m)\) a.e. for every \(A, B \in E',\xi \in \mathcal{H}.\) Since \(m_0\) is isolated, \(\mu([m_0]) > 0\) by Theorem 2.8 (c). Thus \((AB)(m_0)\xi_0 = A(m_0)B(m_0)\xi_0,\) both sides of the equation necessarily being defined. By Theorem 3.7 (h), the set of vectors \([\xi(m_0) | \xi \in \mathcal{H}]\) coincides with \(\mathcal{L}_{m_0}.\) Hence \((AB)(m_0) = A(m_0)B(m_0).\)

At non isolated points, multiplication may fail. In the example in the appendix, the space \(M\) has no isolated points, and two operators \(A, B\) are constructed such that \((AB)(m)\) fails to equal \(A(m)B(m)\) everywhere.

Because of this failure of pointwise multiplication, the mapping \(A \mapsto A(m_0),\) for fixed \(m_0,\) is not in general a homomorphism, hence we cannot conclude directly that its range is a subring of \(\mathcal{B}(\mathcal{L}_{m_0}).\) However, this turns out to be the case in a very strong way, as we now show.

**Theorem 3.13.** For fixed \(m_0 \in M,\) the range of the mapping \(A \mapsto A(m_0)\) coincides with \(\mathcal{B}(\mathcal{L}_{m_0}).\)
Proof. Suppose $A_1$ is any operator of $\mathcal{B}(L_{m_0})$. For every $m \in M$, define $P_{L_m}$ to be the projection operator from $L$ onto $L_m$. Define $A_o = A_1 P_{L_m}$, so that $A_o \in \mathcal{B}(L)$. Then, for every $m \in M$, define $A(m) = (P_{L_m} A_o)|_{L_m}$, so that $A(m) \in \mathcal{B}(L_m)$. We shall show that the function $m \mapsto A(m)$ satisfies (3.9b) and (3.9c).

(3.9b) is clear, since $|A(m)| \leq |P_{L_m} A_o| \leq |A_o| = |A_1|.

To prove (3.9c), we recall that $\{e_\alpha | \alpha \in \Lambda \}$ is an orthonormal basis for $L$, and that $L_m$ is the subspace generated by $\{e_\alpha | \alpha \in I(\gamma(m))\}$. Thus $P_{L_m}(e_\alpha) = e_\alpha$ if $\alpha \in I(\gamma(m))$, or equivalently, if $m \in S_{e_\alpha}$, and $P_{L_m}(e_\alpha) = 0$ if $m \notin S_{e_\alpha}$. We thus have, for $\alpha, \beta \in \Lambda$, $m \in M$,

$$(A(m)\eta_\alpha(m), \eta_\beta(m)) = (P_{L_m} A_o|_{L_m}) \chi_{S_{e_\alpha} \cap S_{e_\beta}}(m)(e_\alpha, e_\beta) = x_{S_{e_\alpha} \cap S_{e_\beta}}(m)(P_{L_m} A_o e_\alpha, e_\beta) = x_{S_{e_\alpha} \cap S_{e_\beta}}(m)(A_o e_\alpha, P_{L_m} e_\beta) = x_{S_{e_\alpha} \cap S_{e_\beta}}(m)(A_o e_\alpha, x_{S_{e_\beta}}(m)e_\beta) = x_{S_{e_\alpha} \cap S_{e_\beta}}(m)(A_o e_\alpha, e_\beta) \text{.}$$

For fixed $\alpha, \beta$, this defines a continuous function on $M$. 

By Theorem 3.11, there exists an operator $A \in E'$ such that $A = \int_M \phi A(m) d\mu(m)$. But $A(m_0) = P \mathcal{L}_{m_0} A_1 P \mathcal{L}_{m_0} \mid \mathcal{L}_{m_0} = A_1$.

Since $A_1$ was arbitrary, the result follows.

We conclude this chapter by showing a relationship between the decomposition given here and that developed by Segal and Tomita. By Corollary 3.3.1 (a) and Theorem 2.14 (b), $\eta_1$ is a cyclic vector for $\mathcal{H}$ with respect to $E'$. Since $\phi_{\eta_1} = \chi_{S_{\eta_1}} = \chi_{M'} |\eta_1|^2 = (\eta_1, \eta_1) = \int_M d\mu(m) = \mu(M) = 1$.

Theorem 1.5 is thus applicable to the rings $E'$ and $E$, and to the cyclic vector $\eta_1$. By definition of $\phi_{\eta_1}$, it follows that the unique measure of Theorem 1.5 (a) is the measure $\mu$ of the present chapter.

**Theorem 3.14.** Let $\{f_m\}_{m \in M}$ be the system of positive functionals of Theorem 1.5 (c) associated with the rings $E'$ and $E$, and with the vector $\eta_1$. Then $f_m(A) = (A(m)\eta_1, \eta_1)$ for all $m \in M$, $A \in E'$.

**Proof.** Let $A$ be an arbitrary operator of $E'$. By Theorems 3.7 (e), 3.9 (a), and 3.12 (d), and by conditions (ii) and (iii) of Theorem 1.5 (c), we have for arbitrary $B \in E$, $\int_M \hat{B}(m)(A(m)\eta_1(m), \eta_1(m))d\mu(m) = \int_M ((AB)(m)\eta_1(m), \eta_1(m))d\mu(m) = (AB\eta_1, \eta_1) = \int_M \hat{B}(m)f_m(A)d\mu(m)$. Since $B$ was arbitrary, we have $(A(m)\eta_1(m), \eta_1(m)) = f_m(A)$ a.e.
But both sides of this equation are continuous, the left by (3.9c), the right by condition (i) of Theorem 1.5 (c). Since \( \eta_1(m) = e_1 \) for all \( m \), this becomes \( f_m(A) = (A(m)e_1, e_1) \) for all \( m \in M \).

We have thus shown that for every \( m \in M \), \( A \to A(m) \) is a linear mapping from \( E' \) to \( \mathcal{B}(\mathcal{L}_m) \), which preserves adjoints, and satisfies the equation \( f_m(A) = (A(m)e_1, e_1) \) for every \( A \in E' \), where \( f_m \) is the positive functional associated with \( \eta_1 \) and \( m \) as above. The range of this mapping is \( \mathcal{B}(\mathcal{L}_m) \), hence \( e_1 \) is a cyclic vector for the space \( \mathcal{L}_m \) with respect to the range. Thus, \( A \to A(m) \) satisfies all properties of being the cyclic representation associated with the functional \( f_m \), except the multiplicative requirement \( (AB)(m) = A(m)B(m) \). We have remarked above that this requirement may fail to hold.

In some unpublished notes [8], J.L. Taylor has shown that it is possible to have points \( m \in M \) such that the Hilbert space in which the cyclic representation corresponding to \( f_m \) takes place has dimension greater than that of the original Hilbert space \( \mathcal{H} \). The dimensions of our spaces \( \mathcal{L}_m \) are always less than or equal to \( \text{card } \Lambda \), which will, in the next chapter, be shown equal to \( \dim_E \mathcal{H} \). Since \( \dim_E \mathcal{H} \leq \dim \mathcal{H} \) by corollary 3.1.1, the mapping \( A \to A(m) \) must necessarily fail to be multiplicative at points \( m \) of the kind just mentioned.
However, in sacrificing the multiplicativity, we have gained far greater control on the coordinate spaces of our direct integral.
CHAPTER IV

DIMENSION PROPERTIES OF THE DECOMPOSITIONS

The purpose of this chapter is to identify certain properties of the decompositions constructed in Chapter III in terms of the space and ring used in their construction. As usual, \( \mathcal{H} \) will denote a fixed Hilbert space, \( E \) a diagonal ring on \( \mathcal{H} \), \( \xi_0 \) a unit cyclic vector for \( \mathcal{H} \) with respect to \( E' \), \( M \) the maximal ideal space of \( E \), \( \mu \) the measure of Theorem 1.5(a) for \( E', E \), and \( \xi_0 \). For every vector \( \xi \in \mathcal{H} \); \( \varphi_\xi \) and \( S_\xi \) will be as in Definition 2.13, with respect to \( \mu \).

We shall have occasion to apply our theory to other diagonal rings constructed from \( E \) by the methods of Chapter II. Whenever this happens, we shall indicate the corresponding cyclic vectors, measures, etc., to be used in this new ring.

The concept on which our study depends is stated in Definition 3.1. A consequence of this Definition which handles the infinite case is as follows.

**Theorem 4.1.** If \( \dim_E \mathcal{H} \) is infinite, and \( \mathcal{H} = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha \) is an arbitrary decomposition of \( \mathcal{H} \) as a direct sum of nontrivial cyclic subspaces for \( E \), then \( \text{card} \, \Lambda = \dim_E \mathcal{H} \).

**Proof.** By Definition 3.1, there exists a second direct sum decomposition \( \mathcal{H} = \sum_{\alpha \in \Lambda'} \mathcal{H}'_\alpha \), where each
\( H'_\alpha \) is a nontrivial cyclic subspace for \( E \), and \( \text{card } \Lambda' = \dim_E H \). According to the definition, \( \text{card } \Lambda' = \dim_E H \leq \text{card } \Lambda \). But by Theorem 1.1, \( \text{card } \Lambda \leq \aleph_0 \text{card } \Lambda' \). By assumption, \( \text{card } \Lambda' \geq \aleph_0 \), hence \( \text{card } \Lambda \leq \text{card } \Lambda' \). Thus \( \text{card } \Lambda = \text{card } \Lambda' = \dim_E H \).

Definition 3.1 is applicable to the rings \( E_U \) on the spaces \( P_U(\mathcal{H}) \), for an arbitrary nonvoid clopen set \( U \) in \( M \). We shall now use this fact, together with the fact that the cardinal numbers are well ordered, in order to localize this concept of dimension.

Definition 4.2. Let \( m \) be a point of \( M \). We define \( d(m) \) to be the smallest of the cardinal numbers \( \dim_{E_U} P_U(\mathcal{H}) \), where \( U \) ranges over all clopen subsets of \( M \) which contain the point \( m \). \( d(m) \) will be called the \textit{local dimension} of \( \mathcal{H} \) relative to \( E \) at \( m \).

Lemma 4.3.1. If \( U_1, U_2 \) are nonvoid clopen subsets of \( M \) with \( U_1 \subset U_2 \), then \( \dim_{E_{U_1}} P_{U_1}(\mathcal{H}) \leq \dim_{E_{U_2}} P_{U_2}(\mathcal{H}) \).

Proof. Let \( P_{U_2}(\mathcal{H}) = \sum_{\alpha \in \Lambda} \mathcal{H}_\alpha \), where each \( \mathcal{H}_\alpha \) is a cyclic subspace for \( E_{U_2} \), and \( \text{card } \Lambda = \dim_{E_{U_2}} P_{U_2}(\mathcal{H}) \).

For each \( \alpha \in \Lambda \), let \( P_\alpha \) be the projection of \( P_{U_2}(\mathcal{H}) \) onto \( \mathcal{H}_\alpha \), and \( \xi_\alpha \) be a cyclic vector for \( \mathcal{H}_\alpha \) with respect
to $E_{U_2}$. Let $\Lambda_1$ denote the set of indices $\alpha \in \Lambda$ for which $P_{U_1}(H_0) \neq 0$. Set $\xi'_\alpha = P_{U_1} \xi_\alpha$, $H'_\alpha = P_{U_1}(H_0)$ for each $\alpha \in \Lambda_1$. It is claimed that $P_{U_1}(H_0) = 
abla_\alpha \in \Lambda_1$, and each $H'_\alpha$ is a cyclic subspace of $P_{U_1}(H_0)$ for $E_{U_1}$ with cyclic vector $\xi'_\alpha$.

First, $H'_\alpha = P_{U_1}(H_0) = P_{U_1} | P_{U_2}(H_0)(H_0) \subset H_0$, since $H_0$ is invariant for $E$. Thus, the $H'_\alpha$ are pairwise orthogonal. Since $H_0$ is invariant for $E_{U_2}$, $P_0 \in E_{U_2}$, hence $P_{U_1} | P_{U_2}(H_0) P_0 = P_0 P_{U_1} | P_{U_2}(H_0)$, which implies $H'_\alpha = P_{U_1}(H_0)$ is a closed linear subspace of $P_{U_1}(H_0)$, hence also a closed linear subspace of $P_{U_1}(H_0)$.

To prove $\{A_1 \xi'_\alpha \mid A_1 \in E_{U_1}\} = H'_\alpha$ for $\alpha \in \Lambda_1$, suppose first $A_1 \in E_{U_1}$. Then $A_1 = A_1|P_{U_1}(H_0)$ for some $A \in E$.

Consequently, if $A_2 = A_1|P_{U_2}(H_0)$, then $A_2 \xi'_\alpha \in H_0$, and $A_1 \xi'_\alpha = A_1 P_{U_1} \xi_\alpha = A_1 P_{U_1} \xi_\alpha = P_{U_1} A_2 \xi'_\alpha \in H'_\alpha$. Thus,

$\{A_1 \xi'_\alpha \mid A_1 \in E_{U_1}\} = \{A_1 \xi'_\alpha \mid A_1 \in E_{U_1}\} \subset H'_\alpha$. Conversely,
if \( \xi \in \mathcal{H}_\alpha' \subseteq \mathcal{H}_\alpha \), then for every \( \varepsilon > 0 \), there exists an operator \( A_2 \in E_{U_2} \) such that \( |A_2 \xi_\alpha - \xi| < \varepsilon \). If \( A_2 = A|_{P_{U_2}(\mathcal{H})} \) for \( A \in E \), let \( A_1 = A|_{P_{U_1}(\mathcal{H})} \). Then \( |A_1 \xi_\alpha' - \xi| = |A_{P_{U_1}} \xi_\alpha - \xi| = |P_{U_1} A \xi_\alpha - P_{U_1} \xi| = |P_{U_1} (A \xi_\alpha - \xi)| \leq |A \xi_\alpha - \xi| = |A_2 \xi_\alpha - \xi| < \varepsilon \). Hence \( \xi \in \mathcal{G}(A_1 \xi_\alpha'|A_1 \in E_{U_1}) \), and \( \mathcal{H}_\alpha' \) is the cyclic subspace for \( E_{U_1} \) generated by \( \xi_\alpha' \).

To show \( \sum_{\alpha \in \Lambda_1} \mathcal{H}_\alpha' = P_{U_1}(\mathcal{H}) \), let \( \xi \in P_{U_1}(\mathcal{H}) \subseteq P_{U_2}(\mathcal{H}) \), and \( \varepsilon > 0 \). Then there exist distinct indices \( \alpha_1, \ldots, \alpha_n \in \Lambda_1 \) and vectors \( \xi_j \in \mathcal{H}_{\alpha_j} \) for \( j = 1, 2, \ldots, n \), such that \( |\sum_{j=1}^{n} \xi_j - \xi| < \varepsilon \). Consequently, \( |\sum_{j=1}^{n} P_{U_1} \xi_j - \xi| = |P_{U_1}(\sum_{j=1}^{n} \xi_j - \xi)| \leq |\sum_{j=1}^{n} \xi_j - \xi| < \varepsilon \). If the indices have been ordered such that \( \alpha_1, \ldots, \alpha_r \in \Lambda_1 \), and \( \alpha_{r+1}, \ldots, \alpha_n \notin \Lambda_1 \), where \( 1 \leq r \leq n \), it follows that \( |\sum_{j=1}^{r} P_{U_1} \xi_j - \xi| < \varepsilon \). But \( P_{U_1} \xi_j \in \mathcal{H}_{\alpha_j'} \) for \( j = 1, \ldots, r \), and since \( \varepsilon \) was arbitrary, \( \sum_{\alpha \in \Lambda_1} \mathcal{H}_{\alpha'} = P_{U_1}(\mathcal{H}) \).

By definition, \( \dim_{E_{U_1}} P_{U_1}(\mathcal{H}) \leq \text{card} \Lambda_1 \leq \text{card} \Lambda \)
\[ = \dim_{E_{U_2}} P_{U_2}(\mathcal{H}). \]
Theorem 4.3. (a) For every \( m \in M \), there exists a clopen neighborhood \( U \) of \( m \) such that \( \dim_{E_{V}} P_{V}(\mathcal{H}) = d(m) \) whenever \( V \) is a clopen neighborhood of \( m \), and \( V \subseteq U \).

(b) For every cardinal number \( v \), \( \{ m \in M | d(m) \leq v \} \) is open.

(c) \( d(m) \leq \dim_{E} \mathcal{H} \) for every \( m \in M \), and there exists at least one point \( m_{0} \) for which \( d(m_{0}) = \dim_{E} \mathcal{H} \).

Proof of (a). Using the definition of \( d(m) \), choose \( U \) to be a clopen neighborhood of \( m \) such that \( d(m) = \dim_{E_{U}} P_{U}(\mathcal{H}) \).

If \( V \) is a clopen neighborhood of \( m \) and \( V \subseteq U \), then \( \dim_{E_{V}} P_{V}(\mathcal{H}) \geq d(m) \) by the definition of \( d(m) \), and \( \dim_{E_{V}} P_{V}(\mathcal{H}) \leq \dim_{E_{U}} P_{U}(\mathcal{H}) = d(m) \) by Lemma 4.3.1. Consequently \( \dim_{E_{V}} P_{V}(\mathcal{H}) = d(m) \).

Proof of (b). If \( v \) is a cardinal number, and \( m_{0} \in M \) such that \( d(m_{0}) \leq v \), there exists a clopen neighborhood \( U \) of \( m_{0} \) such that \( d(m_{0}) = \dim_{E_{U}} P_{U}(\mathcal{H}) \leq v \). But then, according to Definition 4.2, \( d(m) \leq \dim_{E_{U}} P_{U}(\mathcal{H}) \) for every \( m \in U \). It follows that every point of \( \{ m \in M | d(m) \leq v \} \) is an interior point, hence the set is open.

Proof of (c). Since \( M \) is a clopen neighborhood of every \( m \in M \), it follows from the definition that \( d(m) \leq \dim_{E_{M}} P_{M}(\mathcal{H}) = \dim_{E}(\mathcal{H}) \) for every \( m \in M \). To prove that
there exists an $m_0$ such that $d(m_0) = \dim_E \mathcal{H}$, we shall assume, to the contrary, that $d(m) < \dim_E \mathcal{H}$ for every $m \in M$.

For each cardinal number $v < \dim_E \mathcal{H}$, let $T_v = \{m \in M | d(m) \leq v\}$. By part (b), $T_v$ is open for every $v$, and it is clear that $T_{v_1} \subseteq T_{v_2}$ if $v_1 \leq v_2 < \dim_E \mathcal{H}$. The assumption that $d(m) < \dim_E \mathcal{H}$ for every $m \in M$ tells us that $\bigcup_{v < \dim_E \mathcal{H}} T_v = M$. Since $M$ is compact, and the sets $T_v$ form a tower according to the above, there exists a single fixed $v_0 < \dim_E \mathcal{H}$ for which $T_{v_0} = M$, or equivalently, $d(m) \leq v_0$ for every $m \in M$.

Using the definition of $d(m)$, we may choose, for each $m \in M$, a clopen neighborhood $U_m$ of $m$ such that $\dim_E \mathcal{H} \setminus \bigcup_{m} U_m = d(m) \leq v_0$. The family $\{U_m | m \in M\}$ is then an open cover of $M$, hence, by compactness, there exists a finite subcover $U_{m_1}, U_{m_2}, \ldots, U_{m_q}$. Among the sets $U_{m_1}, U_{m_2}, \ldots, U_{m_q}$, we denote by $V_1, V_2, \ldots, V_n$ those which are nonvoid. Then the sets $\{V_j\}_{j=1}^n$ are pairwise disjoint, nonvoid, clopen, and cover $M$. Each $V_j$ is contained in some $U_{m_k}$, hence $\dim_{E_{V_j}} \mathcal{H}$.
\[ \dim_{U_{\alpha}} P_{U_{\alpha}} (\mathcal{H}) \leq v_{o}, \text{ by Lemma 4.3.1.} \]

Let \( \Lambda' \) be a fixed set with \( \text{card} \Lambda' = v_{o}. \) For \( j = 1, 2, \ldots, n, \) let \( \Lambda_{j} \) be a subset of \( \Lambda' \) with \( \text{card} \Lambda_{j} = \dim_{E_{V_{j}}} P_{V_{j}} (\mathcal{H}). \) Then let \( \{\xi_{\alpha, j}\}_{\alpha \in \Lambda_{j}} \) and \( \{\mathcal{H}_{\alpha, j}\}_{\alpha \in \Lambda_{j}} \) be, respectively, systems of vectors and of subspaces of \( P_{V_{j}} (\mathcal{H}), \)

\[ \text{such that } \mathcal{H}_{\alpha, j} \text{ is the cyclic subspace for } E_{V_{j}} \text{ generated by } \xi_{\alpha, j}, \text{ and } P_{V_{j}} (\mathcal{H}) = \sum_{\alpha \in \Lambda_{j}} \mathcal{H}_{\alpha, j}. \]

These constructions are all possible by the definitions of \( \dim_{E_{V_{j}}} P_{V_{j}} (\mathcal{H}). \)

Let \( \Lambda = \bigcup_{j=1}^{n} \Lambda_{j}. \) If \( \alpha \in \Lambda - \Lambda_{j}, \) we define, for convenience, \( \xi_{\alpha, j} = 0, \) and \( \mathcal{H}_{\alpha, j} = (0). \) With this convention, \( \mathcal{H}_{\alpha, j} \)

is the cyclic subspace of \( P_{V_{j}} (\mathcal{H}) \) for \( E_{V_{j}} \) generated by \( \xi_{\alpha, j}, \) for \( j = 1, 2, \ldots, n, \) and \( \alpha \in \Lambda, \) and for every \( \alpha \in \Lambda, \) there is at least one \( j \) for which \( \xi_{\alpha, j} \neq 0. \) Furthermore, if \( j \neq k, \) then \( P_{V_{j}} (\mathcal{H}) \perp P_{V_{k}} (\mathcal{H}) \) by Theorem 2.3(b), and

\[ \mathcal{H}_{\alpha, j} \subset P_{V_{j}} (\mathcal{H}), \mathcal{H}_{\beta, k} \subset P_{V_{k}} (\mathcal{H}), \text{ we have } \mathcal{H}_{\alpha, j} \perp \mathcal{H}_{\beta, k} \text{ for every } \alpha, \beta \in \Lambda \text{ whenever } j \neq k. \]

From our choice of the \( \mathcal{H}_{\alpha, j} \), we also have \( \mathcal{H}_{\alpha, j} \perp \mathcal{H}_{\beta, k} \text{ if } j = k \text{ and } \alpha \neq \beta. \)
For each $\alpha \in \Lambda$, we define $\xi_\alpha = \sum_{j=1}^{n} \xi_{\alpha, j}$ and $\mathcal{H}_\alpha = \sum_{j=1}^{n} \mathcal{H}_{\alpha, j}$. Since, for fixed $\alpha$, the vectors $\{\xi_{\alpha, j}\}_{j=1}^{n}$ are pairwise orthogonal, and at least one is not zero, $\xi_{\alpha} \neq 0$.

Clearly $\xi_{\alpha} \in \mathcal{H}_\alpha$. It is claimed that $\mathcal{H}_\alpha$ is the cyclic subspace for $E$ generated by $\xi_{\alpha}$ for every $\alpha$, and that

$$\sum_{\alpha \in \Lambda} \mathcal{H}_\alpha = \mathcal{H}.$$ 

To prove $\mathcal{S}(A\xi_{\alpha} | A \in E) = \mathcal{H}_\alpha$, suppose first $A \in E$.

Then $A\xi_{\alpha} = A(\xi_{\alpha, 1} + \xi_{\alpha, 2} + \cdots + \xi_{\alpha, n}) = A|_{P_{V_1} H}\xi_{\alpha, 1} + A|_{P_{V_2} H}\xi_{\alpha, 2} + \cdots + A|_{P_{V_n} H}\xi_{\alpha, n} \in \mathcal{H}_\alpha, 1 \oplus \mathcal{H}_\alpha, 2 \oplus \cdots \oplus \mathcal{H}_\alpha, n = \mathcal{H}$. Thus $\mathcal{S}(A\xi_{\alpha} | A \in E) \subset \mathcal{H}_\alpha$. To show the reverse inclusion, we note that $\xi_{\alpha, j} = P_{V_j} \xi_{\alpha, j} = P_{V_j}(\xi_{\alpha, 1} + \cdots + \xi_{\alpha, n}) = P_{V_j} \xi_{\alpha} \in \mathcal{S}(A\xi_{\alpha} | A \in E)$, for $j = 1, 2, \cdots, n$, hence

$$\mathcal{H}_{\alpha, j} = \mathcal{S}(A\xi_{\alpha, j} | A \in E_{V_j}) = \mathcal{S}(A\xi_{\alpha, j} | A \in E) \subset \mathcal{S}(A\xi_{\alpha} | A \in E)$$

for $j = 1, 2, \cdots, n$. This implies $\mathcal{H}_\alpha = \mathcal{H}_\alpha, 1 \oplus \mathcal{H}_\alpha, 2 \oplus \cdots \oplus \mathcal{H}_\alpha, n \subset \mathcal{S}(A\xi_{\alpha} | A \in E)$, hence the equality $\mathcal{H}_\alpha = \mathcal{S}(A\xi_{\alpha} | A \in E)$ is established.

We have seen above that $\mathcal{H}_{\alpha, j} \perp \mathcal{H}_{\beta, k}$ if either $j \neq k$ or $\alpha \neq \beta$. Thus, for $\alpha \neq \beta$, $\mathcal{H}_\alpha = \mathcal{H}_\alpha, 1 \oplus \mathcal{H}_\alpha, 2 \oplus \cdots$
We may compute directly
\[ \sum_{\alpha \in \Delta} \mathcal{H}_\alpha \oplus \sum_{j=1}^{n} \sum_{\alpha \in \Delta} \mathcal{H}_\alpha \oplus \sum_{j=1}^{n} \mathcal{H}_\alpha \oplus \sum_{j=1}^{n} P_{V_j}(\mathcal{H}) = \mathcal{H}. \]

The above assertions have thus been verified. By Definition 3.1, \( \dim_E \mathcal{H} \leq \card \Delta \leq \card \Delta' = v_0 < \dim_E \mathcal{H} \), a contradiction. This shows our assumption \( d(m) < \dim_E \mathcal{H} \) for every \( m \in M \) to be false, hence \( d(m_0) = \dim E \mathcal{H} \) for at least one \( m_0 \).

**Corollary 4.3.2.** If \( m \) is an isolated point, then \( d(m) = \dim P_{\{m\}}(\mathcal{H}) \).

**Proof.** By Theorem 4.3(a), there exists a clopen neighborhood \( U \) of \( m \) such that \( d(m) = \dim_E P_V(\mathcal{H}) \) whenever \( V \) is clopen and \( m \in V \subset U \). In particular, we may take \( V = \{m\} \). Then \( \mathcal{E}_{\{m\}} \) consists of the multiples of the identity on \( P_{\{m\}} \mathcal{H} \), according to Theorem 2.5(e). Consequently, if \( P_{\{m\}} \mathcal{H} = \sum_{\alpha \in \Delta} \mathcal{H}_\alpha \), where \( \card \Delta = \dim_{E_{\{m\}}} P_{\{m\}}(\mathcal{H}) \), and each \( \mathcal{H}_\alpha \) is a nontrivial cyclic subspace for \( \mathcal{E}_{\{m\}} \), then each \( \mathcal{H}_\alpha \) is necessarily one dimensional. Thus \( d(m) = \card \Delta = \dim P_{\{m\}}(\mathcal{H}) \).

The following result is our main tool in connecting our theory of canonical decomposition systems with the concept of local dimension.
Theorem 4.4. Suppose U is a nonvoid clopen subset of M, and \{\(\mathcal{M}_\alpha, \eta_\alpha\)\}_{\alpha \in \Lambda} is a canonical decomposition system for \(\mathcal{H}\) with respect to \(E\). Let \(\xi_\perp = \frac{P_U \xi_0}{\|P_U \xi_0\|}\), \(\mu_\perp\) be the measure on U defined by \(\mu_\perp(S) = \frac{\mu(S)}{\mu(U)}\) for every \(\mu\)-measurable subset S of U. Then \(\xi_\perp\) is a unit cyclic vector for \(P_U(\mathcal{H})\) with respect to \((E_\perp)'\), and \(\mu_\perp\) is the measure for which \((A\xi_\perp, \xi_\perp) = \int_U \hat{A}(m) d\mu_\perp(m)\) for every \(A \in E_\perp\). Let \(\Lambda_\perp = \{\alpha \in \Lambda | S_{\eta_\alpha} \cap U \neq \emptyset\}, \eta'_\alpha = \frac{P_U \eta_\alpha}{\|P_U \xi_0\|}\) for \(\alpha \in \Lambda_\perp\), and \(\mathcal{M}_\alpha' = P_U(\mathcal{M}_\alpha)\) for \(\alpha \in \Lambda_\perp\). Then \(\Lambda_\perp\) is a segment in \(\Lambda\). If the quantities \(\varphi_\xi', S_\xi', \eta'_\alpha\) are defined for \(\xi \in P_U(\mathcal{H})\) as in Definition 2.13 for the ring \(E_\perp\), with the measure \(\mu_\perp\) which arises from the vector \(\xi_\perp\) above, then \{\(\mathcal{M}_\alpha', \eta'_\alpha\)\}_{\alpha \in \Lambda_\perp}\) is a canonical decomposition system for \(P_U(\mathcal{H})\) with respect to the diagonal ring \(E_\perp\). Furthermore, \(S_{\eta'_\alpha} = S_{\eta_\alpha} \cap U\) for every \(\alpha \in \Lambda_\perp\).

Proof. That \(|\xi_\perp| = 1\) is apparent. To prove that \(\xi_\perp\) generates \(P_U(\mathcal{H})\) with respect to \((E_\perp)' = (E')_U\), suppose \(\xi \in P_U(\mathcal{H})\) and \(\epsilon > 0\). Since \(\xi_0\) is cyclic for \(\mathcal{H}\) with respect to \(E\), there exists an \(A \in E'\) such that \(|A\xi_0 - \xi| < \epsilon\). Then \(|P_U \xi_0|A|P_U(\mathcal{H})\xi_\perp - \xi| = |A P_U \xi_0 - P_U \xi| = \)
For arbitrary $A \in E$, we have $(A|_{P_U(H)}|, \xi_1) = (A|_{P_U(H)}|, \xi_1)$

$$|P_\xi(A|_{P_U(H)}|, \xi_1)| \leq |A|_{P_U(H)}|, \xi_1| < \varepsilon.$$  

For arbitrary $A \in E$, we have $(A|_{P_U(H)}|, \xi_1) = (A|_{P_U(H)}|, \xi_1)$

$$= \frac{1}{|P_U\xi_0|} \int |A|_{P_U(H)}|, \xi_1| = \frac{1}{(P_U\xi_0, \xi_0)} =$$

$$= \frac{1}{\mu(U)} \int \mu(U) \chi_H(m) d\mu(m) = \frac{1}{\mu(U)} \int \mu(U) \chi_H(m) d\mu(m) =$$

$$= \int \mu(U) \chi_H(m) d\mu(m).$$  This shows $\mu_1$ to be the measure associated with $\xi_1$ and $E_U$.

That $\Lambda_1$ is a segment in $\Lambda$ follows from (3.2a), since $S^*_\alpha \cap U \neq \emptyset$ and $\beta < \alpha$ imply $S^*_\beta \supset S^*_\alpha$, hence $S^*_\beta \cap U \neq \emptyset$.

For every $\alpha \in \Lambda_1$, let $P_\alpha$ be the projection of $H$ onto $M_\alpha$. Since $M_\alpha$ is invariant for $E$, $P_\alpha \in E'$, hence $P_\alpha P_U = P_U P_\alpha$. It follows that $M_\alpha' = P_U(M_\alpha)$ is a closed linear subspace of $H$, hence also a closed linear subspace of $P_U(H)$. By the invariance of $M_\alpha$ for $E$, we also have $M_\alpha' = P_U(M_\alpha) \subset M_\alpha$, hence $M_\alpha' \perp M_\beta'$ for $\alpha \neq \beta$ and $\alpha, \beta \in \Lambda_1$. Thus the spaces $M_\alpha'$ satisfy (3.2b).

To show (3.2a) for $M_\alpha'$ and $\eta_\alpha'$, we first note that

for $A \in E$, $A|_{P_U(H)}|, \eta_\alpha' = A\eta_\alpha' = A(P_U|_{P_U\xi_0} - \xi_0) = P_U(A|_{P_U\xi_0} - \xi_0)$

$\in M_\alpha'$, hence $\mathcal{G}(A|_{P_U(H)}|, \xi_1, A \in E_U) \subset M_\alpha'$. To show the reverse
inclusion, suppose \( \xi \in \mathcal{M}_\alpha' \) and \( \epsilon > 0 \). Then \( \xi \in \mathcal{M}_\alpha \),

hence there exists \( A \in E \) such that \( |A \eta_\alpha - \xi| < \epsilon \). It follows that

\[
|P_U\xi - A|P_U(\mathcal{H})\eta_\alpha' - \xi| = |A \eta_\alpha - P_U\xi|
\]

\[
= |P_U(A \eta_\alpha - \xi)| \leq |A \eta_\alpha - \xi| < \epsilon. \text{ Hence } \mathcal{M}_\alpha' = \mathcal{S}\{A \eta_\alpha' | A \in E\}.
\]

To prove (3.2c), we first make the following observation.

If \( S \cap U = \emptyset \), then \( P_U \eta_\alpha = 0 \) by Theorem 2.14(a). Hence, for every \( A \in E \), \( P_U A \eta_\alpha = A P_U \eta_\alpha = 0 \), which implies \( P_U(\mathcal{M}_\alpha) = 0 \). Now if \( \xi \) is an arbitrary vector of \( P_U(\mathcal{H}) \), \( \xi = \sum \xi_\alpha \) where \( \xi_\alpha \) is the projection of \( \xi \) onto \( \mathcal{M}_\alpha \). Applying \( P_U \), and recalling that \( \mathcal{M}_\alpha \) is invariant for \( P_U \), \( \xi = P_U \xi = \sum \xi_\alpha \), consequently \( P_U \xi_\alpha = \xi_\alpha \) for every \( \alpha \). It follows that \( \xi_\alpha \in \mathcal{M}_\alpha' \) if \( \alpha \in \Lambda_1 \), and, by what we have shown above, \( \xi_\alpha = 0 \) for \( \alpha \not\in \Lambda_1 \). Consequently \( \xi = \sum \xi_\alpha \), where \( \xi_\alpha \in \mathcal{M}_\alpha' \), and since \( \xi \) was arbitrary, (3.2c) is satisfied.

(3.2d), (3.2e), and our statement that \( S \eta_\alpha' = S \eta_\alpha \) for \( \alpha \in \Delta_1 \), all follow from the following computation, which is valid for arbitrary \( A \in E, \alpha \in \Delta_1 \):

\[
|P_U\xi - A|P_U(\mathcal{H})\eta_\alpha' - \eta_\alpha = \frac{1}{|P_U\xi - A|P_U(\mathcal{H})\eta_\alpha' - \eta_\alpha} = \frac{1}{(P_U\xi, \xi)}(P_U(\mathcal{H})\eta_\alpha' - \eta_\alpha).
\]
Theorems 4.1 and 4.4 will allow us to analyze immediately those points of $M$ for which $d(m)$ is infinite. For the points at which $d(m)$ is finite, the following more delicate results will be needed.

**Proof.** From Definition 3.1, the hypothesis implies $q$ elements. We shall verify the following two statements:

(ii) $q = n$.

Let $k$ be an integer, and be an nonvoid clopen subset of $M$ chosen in the following manner. If (i) holds, let $k = q$, and $M = \emptyset$. If (i) does not hold, then by (3.2e) and Corollary 3.3.1(a) we have that at least one of the clopen sets $S_j = S_{\rho_j}$, for $j = 1, 2, \ldots, q - 1$, is not empty. Let $k$ be an integer among $1, 2, \ldots, q - 1$, such that $S_{\rho_k} \neq \emptyset$, and $M_1 = S_{\rho_k}$.
Regardless of which method was used above in the choice of \( k \) and \( M_1 \), we have \( S_{\rho_j} \cap M_1 = M_1 \) for \( 1 \leq j \leq k \), and 
\[ S_{\rho_j} \cap M_1 = \emptyset \text{ for } k < j \leq q. \]

We now apply Theorem 4.4 to each of the above canonical decompositions and the clopen set \( M_1 \). Let \( \xi_1 = \frac{P_{M_1} \xi_0}{|P_{M_1} \xi_0|} \), \( \mu_1 \) be the measure defined for \( \mu \)-measurable subsets \( S \) of \( M_1 \) by \( \mu_1(S) = \frac{\mu(S)}{\mu(M_1)} \). For brevity, we define \( E_1 = E_{M_1}, \mathcal{H}_1 = P_{M_1}(\mathcal{H}) \). Let \( \eta_j = \frac{P_{M_1} \eta_j}{|P_{M_1} \xi_0|} \) and \( M_j = P_{M_1}(M_j) \) for \( j = 1,2,\ldots,n \). Also, let \( \rho_j = \frac{P_{M_1} \rho_j}{|P_{M_1} \xi_0|} \) and \( \nu_j = P_{M_1}(\nu_j) \) for \( j = 1,2,\ldots,k \). Then \( \{(M_j, \eta_j)\}_{j=1}^n \) and 
\[ \{(\eta_j, \rho_j)\}_{j=1}^k \] 
are canonical decomposition systems in \( \mathcal{H}_1 \) with respect to \( E_1 \), where the corresponding \( \varphi_\xi \) and \( S_\xi \) of Definition 2.13 are defined with respect to the measure \( \mu_1 \). Furthermore, \( S_{\eta_j} = M_1 \) for \( j = 1,2,\ldots,n \), and \( S_{\rho_j} = M_1 \) for \( j = 1,2,\ldots,k \).

For each \( j = 1,2,\ldots,n \), we have \( \eta_j = \sum_{\nu=1}^k \eta_j^{\nu} \), where \( \eta_j^{\nu} \) is the projection of \( \eta_j \) onto \( \nu_j \). Since the spaces
\( \eta_j \) are orthogonal and invariant for \( E \), we have, by Theorem 2.14 (f), \( \varphi' \eta_{j, v} = \sum_{v=1}^{k} \varphi' \eta_{j, v} \). But \( \varphi' \eta_{j, v} = 1 \) for all \( m \), and since every \( \varphi' \eta_{j, v} \) is non-negative extended real valued, each is a continuous function. By corollary 2.17.1, there exists, for every \( j, v \), an operator \( B_{j, v} \in E_k \) such that \( B_{j, v} \varphi' v = \eta_{j, v} \). Thus \( \eta_{j} = \sum_{v=1}^{k} B_{j, v} \varphi' v \) for \( j = 1, 2, \ldots, n \).

We denote by \( \mathcal{B} \) the \( n \) by \( k \) matrix of operators \( \{B_{j, v}\} \).

The roles of the \( \varphi' v \) and \( \eta_{j, v} \) may now be reversed in the above construction. For every \( v = 1, 2, \ldots, k \), we have \( \varphi' v = \sum_{j=1}^{n} \rho_{v, j} \), where \( \rho_{v, j} \) is the projection of \( \varphi' v \) on \( \mathcal{M}'_{v} \).

Theorem 2.14 (f) may again be applied to obtain \( \varphi' \rho_{v, j} = \sum_{j=1}^{n} \varphi' \rho_{v, j} \), and hence that every \( \varphi' \rho_{v, j} \) is a continuous function. Applying corollary 2.17.1 again, there exists an operator \( C_{v, j} \in E_k \) for every \( v \) and \( j \) such that \( \rho_{v, j} = C_{v, j} \eta_{j} \). Hence \( \rho' v = \sum_{j=1}^{n} C_{v, j} \eta_{j} \) for \( v = 1, 2, \ldots, k \). Let \( \mathcal{C} \) denote the \( k \) by \( n \) matrix of operators \( \{C_{v, j}\} \).

Let \( \rho \) denote the column of vectors \( \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_k \end{bmatrix} \), and \( \eta \) denote \( \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix} \).
the column of vectors \[
\begin{bmatrix}
\eta_1^i \\
\vdots \\
\eta_n^i
\end{bmatrix}
\]. The above equations relating
the \( \eta_j^i \), \( \rho_j^i \), \( B_j^i \), \( \gamma_j^i \) and \( C_j^i \) then have the following form in
matrix notation:
\[
\begin{align*}
\eta &= \Theta \rho, \\
\rho &= C \eta.
\end{align*}
\]
If we multiply the first of these equations by \( C \), and the
second by \( \Theta \), we thus obtain
\[
\begin{align*}
\rho &= C \Theta \rho, \\
\eta &= \Theta C \eta.
\end{align*}
\]
Let \( \{V_i^j\}_{i,j=1}^n \) denote the entries of the \( n \) by \( n \) matrix
\( \Theta C \), and \( \{W_i^j\}_{i,j=1}^k \) denote the entries of the \( k \) by \( k \)
matrix \( C \Theta \). The above matrix relations then yield
\[
\begin{align*}
\rho_j^i &= \sum_{v=1}^k W_j^v \rho_v^i, \quad j = 1, 2, \ldots, k, \\
\eta_j^i &= \sum_{v=1}^n V_j^v \eta_v^i, \quad j = 1, 2, \ldots, n.
\end{align*}
\]
Each of the spaces \( \mathcal{N}_j^i \) and \( \mathcal{M}_j^i \) is invariant for \( E \). The
first of these relations therefore represents \( \rho_j^i \) as a
direct orthogonal sum of elements from the spaces \( \mathcal{N}_j^i \), and
the second represents \( \eta_j^i \) as a direct and orthogonal sum of
elements from the spaces \( \mathcal{M}_j^i \). Consequently
\( W_{j,\nu} \rho_{\nu}' = 0 \), for \( j,\nu = 1,2,\ldots,k \), if \( j \neq \nu \),
\( W_{j,j} \rho_{j}' = \rho_{j}' \), for \( j = 1,2,\ldots,k \),
\( V_{j,\nu} \eta_{\nu}' = 0 \), for \( j,\nu = 1,2,\ldots,n \), if \( j \neq \nu \),
\( V_{j,j} \eta_{j}' = \eta_{j}' \), for \( j = 1,2,\ldots,n \).

Since \( S_{j}' = S_{\rho_{\nu}'} = M_1 \) whenever \( j = 1,2,\ldots,n \), and \( \nu = 1,2,\ldots,k \), we have, from the uniqueness assertion in Corollary 2.17.1, \( W_{j,\nu} = \delta_{j,\nu} I \) for \( j,\nu = 1,2,\ldots,k \) and \( V_{j,\nu} = \delta_{j,\nu} I \) for \( j,\nu = 1,2,\ldots,n \). It follows that \( \Theta C \) and \( C \Theta \) are, respectively, the \( n \) by \( n \) and \( k \) by \( k \) identity matrices with entries from the commutative ring \( E_1 \). From the theory of matrices, this implies \( k = n \).

From our previous constructions, assertions (i) and (ii) are now seen to hold. In fact, if (i) had been false, the integer \( k \) would have been chosen so that \( k \leq q-1 < q \leq n \), contrary to \( k = n \) as we have just shown. Thus (i) holds, and from our choice of \( k \), we have \( n = k = q = \dim E \mathcal{H} \), which proves (ii), and also the conclusion of the lemma.

Lemma 4.5.2. Suppose \( \dim E \mathcal{H} = n \), where \( n \) is a positive integer. Then if \( \{(m_{\alpha},n_{\alpha})\}_{\alpha \in \Lambda} \) is an arbitrary canonical decomposition system, either \( \Lambda \) has at most \( n \) elements, or \( S_{\eta_{n+1}} \notin M \).
Proof. Assume, to the contrary, that \( n+1 \in \Lambda \) and \( S_{n+1} = M \). Let \( \mathcal{H}_l = \sum_{j=1}^{n+1} \mathcal{M}_j \), and \( P \) be the projection of \( \mathcal{H} \) onto \( \mathcal{H}_l \). Since every \( \mathcal{M}_j \) is invariant for \( \mathcal{E} \), \( \mathcal{H}_l \) is also, hence \( P \in \mathcal{E}' \). From Theorem 2.14 (b) and Corollary 3.3.1 (a), \( \eta_1 \) is a cyclic vector for \( \mathcal{H} \) with respect to \( \mathcal{E}' \), and from our construction, \( \eta_1 \in \mathcal{H}_l \). Let \( \mathcal{E}_1 \) be the set of restrictions \( \{ A|_{\mathcal{H}_l} | A \in \mathcal{E} \} \). Then, by Theorem 2.7, \( \mathcal{E}_1 \) is a diagonal ring on \( \mathcal{H}_l \), the mapping \( A \to A|_{\mathcal{H}_l} \) is a symmetric isomorphism from \( \mathcal{E} \) onto \( \mathcal{E}_1 \), and \( \eta_1 \) is a cyclic vector for \( \mathcal{H}_l \) with respect to \( \mathcal{E}_1 \). We shall identify the maximal ideal spaces of \( \mathcal{E} \) and \( \mathcal{E}_1 \) through the above isomorphism, i.e., for \( m \in M \), let \( (A|_{\mathcal{H}_l})(m) = \hat{A}(m) \) whenever \( A \in \mathcal{E} \).

For an arbitrary \( A \in \mathcal{E} \), it follows that \( (A|_{\mathcal{H}_l}, \eta_1, \eta_1) = (A\eta_1, \eta_1) = \int_M \hat{A}(m)\eta_1(m)d\mu(m) = \int_M \hat{A}(m)d\mu(m) = \int_M (A|_{\mathcal{H}_l})(m)d\mu(m) \).

It follows that \( \mu \) is the measure on \( M \) corresponding to \( \mathcal{E}_1 \), \( \mathcal{E}_1 \), and \( \eta_1 \) as in Theorem 1.5 (a). Also, for every other vector \( \xi \in \mathcal{H}_l \), and for every \( A \in \mathcal{E} \), \( (A|_{\mathcal{H}_l}, \xi, \xi) = (A\xi, \xi) = \int_M \hat{A}(m)\xi(m)d\mu(m) = \int_M (A|_{\mathcal{H}_l})(m)\xi(m)d\mu(m) \). It follows that if we define \( \varphi_\xi, S_\xi \) for vectors \( \xi \in \mathcal{H}_l \).
according to Definition 2.13 on the maximal ideal space $M$ of $E_1$, identified as above, where the measure under consideration comes from $\eta_1$, then these quantities are, respectively, the same as our original $\phi_\xi \cdot S_\xi$. We shall thus, in the remainder of this proof, use the same symbols $\phi_\xi \cdot S_\xi$ for vectors $\xi \in \mathcal{H}_1$ to denote the quantities of Definition 2.13 for $\xi$, whether considered on $M$ as the maximal ideal space for $E$, or on $M$ considered as the maximal ideal space of $E_1$.

It then follows that $\left( \left( M_j, \eta_j \right) \right)_{j=1}^{n+1}$ is a canonical decomposition system on $\mathcal{H}_1$ with respect to $E_1$. In fact, (3.2a), (3.2b), (3.2d), and (3.2e) follows from the observation that $\left( \left( M_\alpha, \eta_\alpha \right) \right)_{\alpha \in \Delta}$ is a canonical decomposition for $\mathcal{H}$ with respect to $E$, together with the fact that $\phi_\xi \cdot S_\xi$ are the same for vectors $\xi \in \mathcal{H}_1$ whether considered in $\mathcal{H}_1$ or $\mathcal{H}$. (3.2c) is a consequence of our definition of $\mathcal{H}_1$. By our assumption and (3.2e), we have $\eta_1 = \eta_2 = \cdots = \eta_{n+1} = M$.

Thus, by Lemma 4.5.1, $\dim_{E_1} \mathcal{H}_1 = n+1$.

By hypothesis, we also have $\dim_{E} \mathcal{H} = n$. Thus, according to Theorem 3.3, there exists a canonical decomposition system $\left( \left( \eta_j, p_j \right) \right)_{j=1}^{n}$ for $\mathcal{H}$, with respect to $E$, which has $n$ elements. Set $p_j = P_1 p_j$ for $j = 1, 2, \ldots, n$. It is claimed that $\mathcal{G} \left( A_1 p_1^1 + A_2 p_2^2 + \cdots + A_n p_n^1 \right) |_{A_1, \ldots, A_n \in E_1}$
To show this, suppose $\xi \in H_1$, $\epsilon > 0$. Then there exist operators $B_1, \ldots, B_n \in E$ such that $|B_1\rho_1 + \cdots + B_n\rho_n - \xi| < \epsilon$. If $A_j = B_j | H_1$ for $j = 1, 2, \ldots, n$, it follows that $|A_1\rho_1 + \cdots + A_n\rho_n - \xi| = |B_1\rho_1 + \cdots + B_n\rho_n - \xi| = |B_1\rho_1 + \cdots + B_n\rho_n - \xi| < \epsilon$, which proves the above claim.

We now define vectors $\xi_1, \xi_2, \ldots, \xi_n$ and subspaces $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$ of $H_1$ by induction, to have the following properties:

(i) $\mathcal{K}_j = \subseteq \{A| A \in \mathcal{F}_1\}$, for $j = 1, 2, \ldots, k$.

(ii) $\mathcal{K}_i \perp \mathcal{K}_j$ if $i \neq j$, and $i, j = 1, 2, \ldots, k$.

(iii) $\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_j \supseteq \subseteq \{A_1\rho_1 + \cdots + A_j\rho_j|A_1, \ldots, A_j \in E\}$ for $j = 1, 2, \ldots, k$.

For $k = 1$, set $\mathcal{K}_1 = \mathcal{F}_1$, $\mathcal{K}_1 = \subseteq \{A| A \in \mathcal{F}_1\}$. Assuming $\xi_1, \ldots, \xi_k$ and $\mathcal{K}_1, \ldots, \mathcal{K}_k$ to be defined so that (i), (ii), and (iii) hold for the integer $k$, where $1 \leq k < n$, define $\xi_{k+1}$ to be the projection of $\rho_{k+1}$ on $(\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k)^\perp$, and $\mathcal{K}_{k+1} = \subseteq \{A| A \in \mathcal{F}_1\}$. Then (i) is clearly satisfied for $j = k+1$. Since $(\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k)^\perp$ is invariant for $E_1$, $\mathcal{K}_{k+1} \subseteq (\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k)^\perp$, hence (ii) holds for integers $\leq k+1$. To show that (iii) is valid whenever $j \leq k+1$, it suffices, by the induction hypothesis, to show that $A_\rho_{k+1} \in \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{k+1}$ whenever $A \in E_1$. By definition,
\[ \rho_{k+l} = \xi_{k+l} + \eta, \text{ where } \eta \in \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k \text{ and} \]

\[ \xi_{k+l} \perp \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k. \text{ Thus } A\eta \in \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k, \text{ and} \]

\[ A\xi_{k+l} \in \mathcal{K}_{k+1} \text{ from the definition. It follows that } A\rho_{k+l} = A\xi_{k+l} + A\eta \in \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{k+1}, \text{ and the induction hypothesis is satisfied for } k+1. \]

Taking \( k = n \) in (iii), we have \( \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_n = \mathcal{H}_n \).

If \( q \leq n \) and \( k_1, k_2, \ldots, k_q \) are the integers for which \( \mathcal{K}_{k_j} \neq 0 \), then also \( \mathcal{K}_{k_1} \oplus \cdots \oplus \mathcal{K}_{k_q} = \mathcal{H}_l \). By (i), every \( \mathcal{K}_{k_j} \) is cyclic for \( \mathcal{E}_1 \), hence by Definition 3.1, \( \dim \mathcal{E}_1 \mathcal{H}_l \leq q \leq n < n+1 \), which contradicts our above statement that \( \dim \mathcal{E}_1 \mathcal{H}_l = n+1 \). This contradiction shows our assumption \( S_{\eta_{n+1}} = M \) to be false, and proves the lemma.

We are now ready to prove our main result.

**Theorem 4.5.** Suppose \( \{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Delta} \) is a canonical decomposition system for \( \mathcal{H} \) with respect to \( \mathcal{E} \),

\[ \int_M \otimes \mathcal{L}_m \, d\mu(m) \text{ is the corresponding direct integral. Then} \]

(a) \( \text{card } \Delta = \dim \mathcal{E} \mathcal{H} \).

(b) \( \dim \mathcal{L}_m = d(m) \) for every \( m \in M \).

(c) If \( n \) is an integer, \( d(m) = n \) if, and only if \( m \in S_{\eta_n} \setminus S_{\eta_{n+1}} \).

(d) If \( n \) is an integer, \( M - S_{\eta_{n+1}} = \{ m \in M | d(m) \leq n \} \).
(e) If $\alpha \in \Delta$ is arbitrary, then 
\[ \{ m \in M \mid d(m) < \text{card } I(\alpha) \} \subseteq M - S^\alpha \subseteq \{ m \in M \mid d(m) \leq \text{card } I(\alpha) \}. \]

(f) $d(m)$ takes on at most countably many values.

Proof of (a). Since $H = \sum_{\alpha \in \Delta} \mathcal{M}_\alpha$ and each $\mathcal{M}_\alpha$ is a nontrivial cyclic subspace for $E$, we have, by Definition 3.1, $\dim_E H \leq \text{card } \Delta$. If $\dim_E H$ is infinite, then $\dim_E H = \text{card } \Delta$ by Theorem 4.1. Suppose, therefore, that $\dim_E H = n$, where $n$ is a positive integer, and assume $n+1 \in \Delta$.

By Theorem 3.3, there exists a canonical decomposition system $\{ (\mathcal{M}_j, \rho_j) \}_{j=1}^n$ with $n$ elements. By (3.2e) and corollary 3.3.1 (a), the sets $S_{\rho_1} - S_{\rho_2}, S_{\rho_2} - S_{\rho_3}, \ldots, S_{\rho_{n-1}} - S_{\rho_n}, S_j$ are disjoint, clopen, and cover $M$. For brevity, we set $\rho_{n+1} = 0$, so that $S_{\rho_{n+1}} = \emptyset$. Thus, at least one of the sets $(S_{\rho_j} - S_{\rho_{j+1}}) \cap S$ is not empty. Let $k$ be a fixed integer among $1, 2, \ldots, n$ such that $(S_{\rho_k} - S_{\rho_{k+1}}) \cap S_{\eta_{n+1}} \neq \emptyset$, and define $M_1$ to be the nonvoid clopen set $(S_{\rho_k} - S_{\rho_{k+1}}) \cap S_{\eta_{n+1}}$.

It follows that $S_{\rho_j} \cap M_1 = M_1$ for $j = 1, 2, \ldots, k$, $S_{\rho_j} \cap M_1 = \emptyset$ for $j = k+1, \ldots, n$. Also, $S_{\eta_{\alpha}} \cap M_1 = M_1$ for
α = 1, 2, …, n+1, hence if \( A_1 = \{ α \in \Delta | S_{ν_α} \setminus M ≠ ∅ \} \), then

\[
1, 2, \cdots, n+1 \in A_1. \text{ Let } g_1 = \frac{P_{M_1} g_0}{|P_{M_1} g_0|}, \mu_1(S) = \frac{μ(S)}{μ(M_1)} \text{ for every } μ\text{-measurable subset } S \text{ of } M_1, \rho_j = \frac{P_{M_1} ρ_j}{|P_{M_1} g_0|} \text{ for } j = 1, 2, \cdots, k, \text{ and } M_α' = P_{M_1}(M_α) \text{ for } α \in A_1. \]

By Theorem 4.4, \( \{( \eta_j', ρ_j') \}_{j=1}^k \) and \( \{( M_α', ν_α') \}_{α \in A_1} \) are both canonical decomposition systems for \( P_{M_1}(\mathcal{H}) \) with respect to \( E_{M_1} \), where the quantities \( φ_s', S_{ς} \) of Definition 2.13 are with respect to \( μ_1 \). From that theorem, we also have \( S_ρ' = S_ν' = M_1 \) whenever \( j = 1, 2, \cdots, k, \) and

\( α = 1, 2, \cdots, n+1. \) By Lemma 4.5.1 applied to the canonical decomposition system \( \{( \eta_j', ρ_j') \}_{j=1}^k \), we have \( \dim_{E_{M_1}} P_{M_1}(\mathcal{H}) = k. \) Then, applying Lemma 4.5.2, \( S_ν' \_{k+1} \notin M_1, \) which is contrary to the above. Thus, \( A \) can have at most \( n \) elements.

Proof of (b). If \( m \in M \), let \( U \) be a clopen neighborhood of \( m \), chosen according to Theorem 4.3 (a), such that \( \dim_{E_{V}} P_{V}(\mathcal{H}) = d(m) \) whenever \( V \) is clopen and \( m \in V \subset U. \)

Then set \( V = (M - S_ν(m)) \cap U. \) Then \( V \) is a clopen subset
of $U$, and from the definition of $\gamma(m)$, $m \notin S_{\eta_{\gamma(m)}}$, hence $m \in V$, and $\dim_{E_V} P_V(\mathcal{H}) = d(m)$.

If $\alpha \in \Lambda$ and $\alpha < \gamma(m)$, then $m \in S_{\eta_{\alpha}}$, so that $S_{\eta_{\alpha}} \cap V \neq \emptyset$. If $\alpha \geq \gamma(m)$, then $S_{\eta_{\alpha}} \subseteq S_{\eta_{\gamma(m)}}$ by (3.2e), hence $S_{\eta_{\alpha}} \cap V = \emptyset$. It follows that $\{m \in M \mid S_{\eta_{\alpha}} \cap V \neq \emptyset\} = \mathcal{I}(\gamma(m))$.

Let the quantities $\mu_1$, $\xi_1$, $\mathcal{M}_1$, $\eta_1$ be defined as in Theorem 4.4 for the clopen set $V$ in place of $U$. Then, by the conclusion to that theorem, $\left\{ (\mathcal{M}_1, \eta_1) \right\}_{\alpha \in \mathcal{I}(\gamma(m))}$ is a canonical decomposition system for $P_V(\mathcal{H})$ with respect to $E_V$. Applying part (a) of the present theorem, which we have just shown, we have $d(m) = \dim_{E_V} P_V(\mathcal{H}) = \text{card } \left( \mathcal{I}(\gamma(m)) \right) = \dim \mathcal{L}_m$.

Proof of (c). By part (b) above, $d(m) = \dim \mathcal{L}_m = \text{card } \left( \mathcal{I}(\gamma(m)) \right)$. Thus, if $n$ is an integer, $d(m) = n$ if, and only if $\gamma(m) = n+1$. From the definition of $\gamma(m)$, this holds if, and only if $m \in S_{\eta_n}$ and $m \notin S_{\eta_{n+1}}$.

Proof of (d). If $n$ is an integer, we have from part (c), $\{m \in M \mid d(m) \leq n\} = \bigcup_{j=1}^{n} \{m \in M \mid d(m) = j\} = \bigcup_{j=1}^{n} (S_{\eta_j} - S_{\eta_{j+1}})$. By (3.2e) and corollary 3.3.1 (a), this set is exactly $M - S_{\eta_{n+1}}$. 
Proof of (e). Suppose \( m \in M \) and \( d(m) \leq \text{card} \ I(\alpha) \). From part (b) above, \( d(m) = \dim \mathcal{L}_m = \text{card} \ I(\gamma(m)) \), so \( \text{card} \ I(\gamma(m)) \leq \text{card} \ I(\alpha) \), which implies \( \gamma(m) \leq \alpha \). By (3.2e), \( S_{\eta_\alpha} \subseteq S_{\eta_\gamma(m)} \), and since \( m \notin S_{\eta_\alpha} \), we have \( m \in M - S_{\eta_\alpha} \).

This proves the first inclusion relation.

To prove the second, suppose \( m \in M - S_{\eta_\alpha} \). Then \( m \notin S_{\eta_\alpha} \), so by the definition of \( \gamma(m) \), \( \gamma(m) \leq \alpha \). Thus \( d(m) = \dim \mathcal{L}_m = \text{card} \ I(\gamma(m)) \leq \text{card} \ I(\alpha) \).

Proof of (f). Let \( \Delta = \{ \alpha \in \Lambda | S_{\eta_\beta} \not\supseteq S_{\eta_\alpha} \text{ if } \beta < \alpha \} \). Then the mapping \( \alpha \to S_{\eta_\alpha} \) defined from \( \Delta \) to the subsets of \( M \) is one-to-one, and the range of this mapping is \( \{ S_{\eta_\alpha} \}_{\alpha \in \Delta} \). By Corollary 3.3.1 (b), it follows that \( \Delta \) is countable.

From the definition of \( \gamma(m) \) as the least index \( \alpha \) for which \( m \notin S_{\eta_\alpha} \), or \( \gamma(m) = \lambda_0 \) if \( m \in S_{\eta_\alpha} \) for all \( \alpha \), it follows that \( \gamma(m) \in \Delta \cup \{ \lambda_0 \} \). Since \( d(m) = \dim \mathcal{L}_m = \text{card} \ I(\gamma(m)) \) from part (b), \( d(m) \) must always be one of the values in the countable set \( \{ \text{card} \ I(\alpha) | \alpha \in \Delta \cup \{ \lambda_0 \} \} \).

The above theorem shows that the cardinality of \( \Lambda \), the sets \( S_{\eta_n} \) for integers \( n \), and the dimensions of the coordinate spaces \( \mathcal{L}_m \) of the direct integral are all determined by the ring \( E \) and the space \( \mathcal{H} \), independently.
of the choice of the canonical decomposition system 

\[ \{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}. \]

Part (e) shows that the sets \( S_\eta \) cannot be completely arbitrary. However, when \( I(\alpha) \) is infinite, they cannot always be determined, independently of the canonical decomposition system, as in the finite case. The intuitive reason as to why this happens is that infinite sets always have many non-equivalent well orderings. However the uniqueness or non-uniqueness of the \( S_\eta \) is also related to the structure of the local dimension function \( d(m) \). We shall conclude this chapter by stating a partial result which relates the properties of \( E \) and \( \# \) to the uniqueness or non-uniqueness of the \( S_\eta \), and to the different well orderings which are possible on the sets \( \Lambda \) of canonical decomposition systems.

If \( \Lambda \) is any given infinite set, there exists a well ordering \( < \) on \( \Lambda \) satisfying the condition that \( \text{card } I(\alpha) < \text{card } \Lambda \) for every \( \alpha \in \Lambda \). \( \Lambda \), with a well ordering \( < \) as described, then has the following important property: if \( \Lambda' \) is any well ordered set with \( \text{card } \Lambda' = \text{card } \Lambda \), then \( \Lambda \) is isomorphic, in the order preserving sense, to a segment in \( \Lambda' \).
Theorem 4.6. Suppose $\dim_{E} \mathcal{H}$ is infinite.

(a) There exists a canonical decomposition system
\[ \{(m_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda} \] such that $\text{card} I(\alpha) < \text{card} \Lambda$ for every $\alpha \in \Lambda$.

(b) If the set $\{m \in M | d(m) = \dim_{E} \mathcal{H} \}$ has void interior, then for every canonical decomposition system
\[ \{(m_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda} \] it is true that $\text{card} I(\alpha) < \text{card} \Lambda$ for every $\alpha \in \Lambda$.

(c) If the interior of the set $\{m \in M | d(m) = \dim_{E} \mathcal{H} \}$ is nonvoid, suppose $\{(m_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda}$ is a canonical decomposition system such that $\text{card} I(\alpha) < \text{card} \Lambda$ for every $\alpha \in \Lambda$, $\Lambda'$ is any well ordered system with $\text{card} \Lambda' = \text{card} \Lambda$ such that $\Lambda$ is contained in $\Lambda'$ as a segment, and $U$ is an arbitrary nonvoid clopen subset of $\{m \in M | d(m) = \dim_{E} \mathcal{H} \}$. (Recall, such $U$ exist, since $M$ is totally disconnected.) Then there exists a second canonical decomposition system
\[ \{(m_{\alpha}', \eta_{\alpha}')\}_{\alpha \in \Lambda'} \] such that $\eta_{\alpha} = \eta_{\alpha}'$ whenever $\alpha \in \Lambda$, and $\eta_{\alpha}' = U$ whenever $\alpha \in \Lambda' - \Lambda$.

Proof of (a). As we have noted above, there exists a well ordered set $\Lambda'$ with $\text{card} \Lambda' = \dim_{E} \mathcal{H}$, such that $\text{card} I(\alpha) < \text{card} \Lambda'$ for every $\alpha \in \Lambda'$. By Theorem 3.3, there exists a canonical decomposition system
\[ \{(m_{\alpha}, \eta_{\alpha})\}_{\alpha \in \Lambda} \] where $\Lambda$ is a segment in $\Lambda'$, which has the same cardinality as $\Lambda'$. From the way the well ordering
was chosen, necessarily \( \Lambda = \Lambda' \), hence card \( I(\alpha) < \text{card } \Lambda \) for every \( \alpha \in \Lambda \).

**Proof of (b).** We shall show that the existence of a canonical decomposition system \( \{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda} \) such that card \( I(\alpha) = \dim_E \mathcal{H} \) for at least one \( \alpha \in \Lambda \) implies that \( \{m \in M | d(m) = \dim_E \mathcal{H} \} \) has nonvoid interior. In fact, if card \( I(\alpha) = \dim_E \mathcal{H} \), then for each \( m \in S_{\eta_\alpha} \) we have \( \gamma(m) > \alpha \). Considering the corresponding direct integral, we have, by Theorem 4.5(c), \( d(m) = \dim_L m = \text{card } I(\gamma(m)) \geq \text{card } I(\alpha) = \dim_E \mathcal{H} \) for every \( m \in S_{\eta_\alpha} \). Since \( d(m) < \dim_E \mathcal{H} \) for every \( m \in M \) by Theorem 4.3(c), it follows that \( S_{\eta_\alpha} \subset \{m \in M | d(m) = \dim_E \mathcal{H} \} \). Since \( S_{\eta_\alpha} \) is open, the result follows.

**Proof of (c).** By Theorem 4.5(e), \( M - S_{\eta_\alpha} \subset \{m \in M | d(m) \leq \text{card } I(\alpha)\} \), or equivalently, \( S_{\eta_\alpha} \supset \{m \in M | d(m) > \text{card } I(\alpha)\} \) for every \( \alpha \in \Lambda \). From our hypothesis, card \( I(\alpha) < \dim_E \mathcal{H} \) for every \( \alpha \in \Lambda \), hence \( U \subset \{m \in M | d(m) = \dim_E \mathcal{H} \} \subset S_{\eta_\alpha} \) for every \( \alpha \in \Lambda \). Let \( F \) be a one-to-one function from \( \Lambda' \) onto \( \Lambda \). For brevity in the following, we define \( \eta_\alpha = 0, M_\alpha = (0) \) for every \( \alpha \in \Lambda' - \Lambda \). For such \( \alpha \), \( S_{\eta_\alpha} = \emptyset \).

For every \( \alpha \in \Lambda' \), let \( P_\alpha \) be the projection of \( \mathcal{H} \)
onto $\mathcal{M}_\alpha$. Since $\mathcal{M}_\alpha$ is invariant for $E$, $P_\alpha \in E'$, hence $P_U P_\alpha = P_\alpha P_U$ and $P_{M-U} P_\alpha = P_\alpha P_{M-U}$. Thus $P_U(\mathcal{M}_\alpha)$, $P_{M-U}(\mathcal{M}_\alpha)$ are closed linear subspaces of $\mathcal{H}$, and by Theorem 2.3(b), $P_U(\mathcal{M}_\alpha) \perp P_{M-U}(\mathcal{M}_\beta)$ for arbitrary $\alpha, \beta \in \Delta'$. We define $\mathcal{M}_\alpha' = P_U(\mathcal{M}_F(\alpha)) \oplus P_{M-U}(\mathcal{M}_\alpha)$ and $\eta_\alpha' = P_U\eta_F(\alpha) + P_{M-U}\eta_\alpha \in \mathcal{M}_\alpha'$ for every $\alpha \in \Delta'$. It is claimed that $\{(\mathcal{M}_\alpha', \eta_\alpha')\}_{\alpha \in \Delta'}$ is the required canonical decomposition system. Since $F(\alpha) \in \Delta$ for every $\alpha \in \Delta'$, $S_{\eta_F(\alpha)} \supset U \neq \emptyset$ by the above, 

hence $P_U\eta_F(\alpha) \neq 0$ by Theorem 2.14(a). Then $P_U\eta_F(\alpha) \perp P_{M-U}\eta_\alpha$ implies $\eta_\alpha' \neq 0$. We proceed to show (3.2a) - (3.2e).

If $A \in E$ is arbitrary, then $A \eta_\alpha' = AP_U\eta_F(\alpha) + AP_{M-U}\eta_\alpha$

$= P_U A \eta_F(\alpha) + P_{M-U} A \eta_\alpha \in P_U \mathcal{M}_F(\alpha) \oplus P_{M-U} \mathcal{M}_\alpha = \mathcal{M}_\alpha'$, so $G\{A \eta_\alpha' | A \in E\} \subset \mathcal{M}_\alpha'$. Conversely, if $\xi \in \mathcal{M}_\alpha'$, $\xi$ has the form $P_U \xi_1 + P_{M-U} \xi_2$, where $\xi_1 \in \mathcal{M}_F(\alpha), \xi_2 \in \mathcal{M}_\alpha$.

If $\epsilon > 0$, let $A_1, A_2 \in E$ so that $|A_1 \eta_F(\alpha) - \xi_1| < \frac{\epsilon}{2}$ and $|A_2 \eta_\alpha - \xi_2| < \frac{\epsilon}{2}$. Set $A = A_1 P_U + A_2 P_{M-U} \in E$. Then $|A \eta_\alpha' - \xi|$

$= |(A_1 P_U + A_2 P_{M-U})(P_U \eta_F(\alpha) + P_{M-U} \eta_\alpha) - P_U \xi_1 - P_{M-U} \xi_2| =$

$|A_1 P_U \eta_F(\alpha) + A_2 P_{M-U} \eta_\alpha - P_U \xi_1 - P_{M-U} \xi_2| =$

$|P_U(A_1 \eta_F(\alpha) - \xi_1) + P_{M-U}(A_2 \eta_\alpha - \xi_2)| \leq |A_1 \eta_F(\alpha) - \xi_1| + |A_2 \eta_\alpha - \xi_2| < \epsilon$, where we have used Theorem 2.3(b) to
conclude $P_{U}P_{M-U} = 0$. Since $e$ was arbitrary, $\mathcal{C}(A\eta_{\alpha}' | A \in E) \Rightarrow \mathcal{M}_{\alpha}'$, so equality holds. We have thus established (3.2a).

If $\alpha, \beta \in \Lambda'$, $\alpha \neq \beta$, $\xi \in \mathcal{M}_{\alpha}'$, $\zeta \in \mathcal{M}_{\beta}'$, let $\xi = P_{U}\xi_{1} + P_{M-U}\xi_{2}$, where $\xi_{1} \in \mathcal{M}_{F}(\alpha)$, $\xi_{2} \in \mathcal{M}_{\alpha}$, and $\zeta = P_{U}\zeta_{1} + P_{M-U}\zeta_{2}$, where $\zeta_{1} \in \mathcal{M}_{F}(\beta)$, $\zeta_{2} \in \mathcal{M}_{\beta}$. Using again the relation $P_{U}P_{M-U} = 0$, we have $(\xi, \zeta) = (P_{U}\xi_{1}, P_{U}\xi_{1}) + (P_{M-U}\xi_{2}, P_{M-U}\xi_{2})$. The first of these terms is zero, since $P_{U}\xi_{1} \in \mathcal{M}_{F}(\alpha)$, $P_{U}\xi_{1} \in \mathcal{M}_{F}(\beta)$, and $\mathcal{M}_{F}(\alpha) \perp \mathcal{M}_{F}(\beta)$. The second term is zero since $P_{M-U}\xi_{2} \in \mathcal{M}_{\alpha}$, $P_{M-U}\xi_{2} \in \mathcal{M}_{\beta}$, and $\mathcal{M}_{\alpha} \perp \mathcal{M}_{\beta}$. Since $\xi, \zeta$ were arbitrary, we have $\mathcal{M}_{\alpha}' \perp \mathcal{M}_{\beta}'$, hence (3.2b) holds.

To show (3.2c), suppose $\alpha \in \Lambda$ is arbitrary, and $\xi \in \mathcal{M}_{\alpha}$. Then $\xi = P_{U}\xi + P_{M-U}\xi$. $P_{U}\xi \notin P_{U}(\mathcal{M}_{\alpha}) = P_{U}(\mathcal{M}_{F}(F^{-1}(\alpha))) \subset \mathcal{M}_{F}^{-1}(\alpha)$, and $P_{M-U}\xi \notin P_{M-U}(\mathcal{M}_{\alpha}) \subset \mathcal{M}_{\alpha}'$. Thus, $\mathcal{M}_{\alpha} \subset \mathcal{M}_{\alpha}' \otimes \mathcal{M}_{F}^{-1}(\alpha), \subset \Sigma_{\beta \in \Lambda'} \mathcal{M}_{\beta}'$. Since $\alpha$ was arbitrary,

$\mathcal{H} = \Sigma_{\alpha \in \Lambda} \otimes \mathcal{M}_{\alpha} \subset \Sigma_{\beta \in \Lambda'} \otimes \mathcal{M}_{\beta}'$, consequently $\Sigma_{\alpha \in \Lambda} \mathcal{M}_{\alpha}' = \mathcal{H}$.

We shall now compute $\phi_{\eta_{\alpha}}', S_{\eta_{\alpha}}'$ for $\alpha \in \Lambda'$. If $A \in E$ is arbitrary, we use again the relation $P_{U}P_{M-U} = 0$ to compute $(A\eta_{\alpha}', \eta_{\alpha}') = (A(P_{U}\eta_{F}(\alpha) + P_{M-U}\eta_{\alpha})P_{U}\eta_{F}(\alpha) + P_{M-U}\eta_{\alpha})$

$= (A_{P_{U}}\eta_{F}(\alpha), P_{U}\eta_{F}(\alpha)) + (A_{P_{M-U}}\eta_{\alpha}, P_{M-U}\eta_{\alpha}) = (A_{P_{U}}\eta_{F}(\alpha), \eta_{F}(\alpha))$
\[ + \langle \mathbf{M}_{\mathbf{F} \mathbf{H}} \triangleright \eta_{\alpha}, \eta_{\alpha} \rangle = \int_{\mathcal{M}} \hat{\mathbf{A}}(m) \chi_{U(m)} \chi_{S} \eta_{F}(\alpha) \eta_{\alpha}(m) d\mu(m) + \int_{\mathcal{M}} \hat{\mathbf{A}}(m) \chi_{M-U(m)} \chi_{S} \eta_{\alpha}(m) d\mu(m). \]

We have shown above that \( S_{\eta_{\alpha}} \triangleright U \) for every \( \alpha \in \Lambda \), and since the range of \( F \) is \( \Lambda \), \( S_{\eta_{\alpha}} \triangleright U \) for every \( \alpha \in \Lambda' \). The above thus becomes

\[ (\Lambda_{\eta_{\alpha}'}, \eta_{\alpha}') = \int_{\mathcal{M}} \hat{\mathbf{A}}(m) \chi_{U(m)} d\mu(m) + \int_{\mathcal{M}} \hat{\mathbf{A}}(m) \chi_{S} \eta_{\alpha} \cap (M-U)(m) d\mu(m) \]

\[ = \int_{\mathcal{M}} \hat{\mathbf{A}}(m) \chi_{S} U U(m) d\mu(m) \text{ for every } \alpha \in \Lambda'. \]

This shows \( S_{\eta_{\alpha}'} = S_{\eta_{\alpha}} \cup U \), \( \varphi_{\eta_{\alpha}'} = \chi_{S} \eta_{\alpha}' \), establishing (3.2d). Again, using what we have shown above, \( S_{\eta_{\alpha}} \triangleright U \) if \( \alpha \in \Lambda \), and \( S_{\eta_{\alpha}} = \emptyset \) if \( \alpha \in \Lambda' - \Lambda \), hence \( S_{\eta_{\alpha}'} = S_{\eta_{\alpha}} \) if \( \alpha \in \Lambda \), and \( S_{\eta_{\alpha}'} = U \) if \( \alpha \in \Lambda' - \Lambda \). (3.2e) thus holds trivially, and the sets \( S_{\eta_{\alpha}'} \) have the form required in the conclusion of the theorem.
APPENDIX
AN EXAMPLE

In chapter III, we have defined the concept of a canonical decomposition system for a diagonal ring $E$ on a Hilbert space $H$ such that $H$ is cyclic for $E'$, and developed our decomposition theory on the basis of it. The concept of such a system and the relationship of the diagonal ring to it is rather easy to grasp in view of the isomorphism set up in Theorem 3.4. Examples of the structure which comes about through this isomorphism are also relatively easy to set up directly, and in these examples the diagonal ring and canonical decomposition systems are easily recognizable. The hard part of the theory, which we have developed in this work, shows that every diagonal ring comes about, within isomorphism, from examples of this kind.

We shall now go about constructing one such example of a diagonal ring in this way, and to identify a canonical decomposition system for it. With this, we shall investigate some of the properties of the corresponding direct integrals. There are two main purposes in doing this. First, we wish to demonstrate the ease with which examples of the theory can be constructed. Secondly, we wish to display counterexamples to conjectures which were asserted to
be false in this work. The counterexamples are illustrative of what goes wrong in the general case.

The first step will be to obtain the maximal ideal space for the ring $E$, and the measure of Theorem 1.5(a).

**Theorem A.1.** There exists a compact Hausdorff space $M$, and a non negative regular Borel measure $\mu$ on $M$ whose support is all of $M$, and satisfies $\mu(M) = 1$, such that:

(a) Every $\mu$-essentially bounded measurable function on $M$ is equal almost everywhere $[\mu]$ to a continuous function.

(b) $M$ is a perfect set and $\mu$ is a continuous measure on $M$.

(c) $M$ contains a non $\mu$-measurable set.

**Proof.** Let $M$ be the maximal ideal space of the Banach ring $L_\infty[0,1]$, where the measure on $[0,1]$ is understood to be Lebesgue measure, which we denote by $\sigma$. Since $L_\infty[0,1]$ is completely regular, the Gelfand transform $f \mapsto \hat{f}$ is a (norm preserving) symmetric isomorphism of $L_\infty[0,1]$ onto $C(M)$. The equation $T(\hat{f}) = \int_0^1 f(x)dx$ defines a normalized positive functional on $C(M)$, hence there exists a unique non negative regular Borel measure $\mu$ on $M$ for which $T(\hat{f}) = \int_M \hat{f}(m)d\mu(m) = \int_0^1 f(x)dx$ for every $f \in L_\infty[0,1]$, and $\mu(M) = 1$. It will be shown that $M$ and $\mu$ satisfy the requirements of the theorem.
To prove that the support of $\mu$ is $M$, suppose $g \in C(M)$, $g \geq 0$, and $g(m) \neq 0$. Then there exists an $f \in L_\infty[0,1]$ for which $\hat{f} = g$. The above conditions on $g$ imply $f(x) \geq 0$ a.e., and $f$ is not a null function. Consequently, $\int_M g(m)d\mu(m) = \int_0^1 f(x)dx > 0$. Since $g$ was arbitrary, the support of $\mu$ is all of $M$.

To prove (a), suppose $\phi \in L_\infty(M, \mu)$. For each $f \in L_\infty[0,1]$, define $F(f) = \int_M \hat{f}(m)\phi(m)d\mu(m)$. Then $|F(f)| = \int_M |\hat{f}(m)\phi(m)|d\mu(m)$}
\begin{equation*}
\leq \int_M |\hat{f}(m)|d\mu(m) ||\phi||_\infty = \int_0^1 |f(x)|dx ||\phi||_\infty.
\end{equation*}
$F$ is therefore a linear functional on $L_\infty[0,1]$ which is continuous in the $L_1[0,1]$ norm, hence it can be extended uniquely to a bounded linear functional on $L_1[0,1]$. Thus, there exists a $g \in L_\infty[0,1]$ for which $F(f) = \int_0^1 f(x)g(x)dx$ for all $f \in L_1[0,1]$. In particular, if $f \in L_\infty[0,1]$, $\int_M \hat{f}(m)\phi(m)d\mu(m) = F(f) = \int_0^1 f(x)g(x)dx = \int_M \hat{f}(m)\hat{g}(m)d\mu(m)$. Since this holds for all $f \in L_\infty[0,1]$, $\phi(m) = g(m)$ a.e. $[\mu]$, and $g$ is continuous.

For part (b), suppose, to the contrary, that $M$ has an isolated point $m_0$. Then $\chi_{\{m_0\}}$ is continuous, hence there exists an $f \in L_\infty[0,1]$ such that $\hat{f} = \chi_{\{m_0\}}$. As $\chi_{\{m_0\}} = \overline{\chi_{\{m_0\}}} = (\chi_{\{m_0\}})^2$, we have $f(x) = \overline{f(x)} = f(x)^2$ a.e.,
so by changing \( f \) on a null set, we may assume \( f = \chi_S \),
where \( S \) is a measurable subset of \([0,1]\). Since \( \chi_S = \chi_{\{m_0\}} \),
\( \chi_S \) is not a null function, and \( \sigma(S) > 0 \). Consequently,
there exists a measurable set \( S_1 \subset S \) such that \( 0 < \sigma(S_1) < \sigma(S) \).
Then \( 0 < \chi_{S_1} < \chi_S \), hence if \( g = \chi_{S_1} \), \( 0 \leq g \leq \chi_{\{m_0\}} \).
Since \( \chi_{S_1}^2 = \chi_{S_1} \), we have \( g^2 = g \), hence \( g(m_0) = 0 \) or 1.
If \( g(m_0) = 0 \), then \( g = 0 \), and \( \chi_{S_1} \) is a null function,
contrary to \( \mu(S_1) > 0 \). If \( g(m_0) = 1 \), then \( g = \chi_{\{m_0\}} \),
and \( \chi_{S_1} (x) = \chi_S (x) \) a.e., contrary to \( \sigma(S_1) < \sigma(S) \). It
follows that \( m_0 \) cannot be isolated.

The fact that \( \mu \) is continuous now follows readily.
Suppose \( \mu(\{m_0\}) > 0 \). By part (a), there exists a continuous
function \( g \) such that \( g(m) = \chi_{\{m_0\}}(m) \) a.e. \([\mu] \).
Since \( \mu(\{m_0\}) > 0 \), \( g(m_0) = 1 \), hence there exists an open neighborhood
\( U \) of \( m_0 \) on which \( |g(m)| > \frac{1}{2} \). Since \( m_0 \) is not isolated,
\( U - \{m_0\} \) is nonvoid, and \( 0 = \int_M |g(m) - \chi_{\{m_0\}}(m)|d\mu(m) \)
\( \geq \int_{U - \{m_0\}} |g(m) - \chi_{\{m_0\}}(m)|d\mu(m) = \int_{U - \{m_0\}} |g(m)|d\mu(m) \)
\( \geq \frac{1}{2} \mu(U - \{m_0\}) \). This last quantity is positive since
the support of \( \mu \) is all of \( M \), which yields the desired contradiction.
The proof of (c) is more difficult, and will require some preliminary constructions. The ring \( C[0,1] \) is a Banach symmetric subring of \( L^\infty[0,1] \) which contains the identity. Thus, every maximal ideal \( m \) of \( L^\infty[0,1] \) determines a maximal ideal \( m \cap C[0,1] \) in \( C[0,1] \). The maximal ideal space of \( C[0,1] \) is however, identified with the interval \([0,1]\) in the usual way, hence \( m \cap C[0,1] \) consists of the functions \( f \in C[0,1] \) which vanish at some point \( x \) of \([0,1]\), and \( \hat{f}(m) = f(x) \) for every \( f \in C[0,1] \). Denote by \( t(m) \) the point \( x \) of \([0,1]\) which corresponds to \( m \) in this fashion. Thus, by definition, \( \hat{f}(m) = f(t(m)) \) for every \( f \in C[0,1] \). From the definition of the Gelfand Topologies, it is clear that the function \( t : M \to [0,1] \) is continuous.

Since the Shilov boundary of \( C[0,1] \) coincides with its entire maximal ideal space \([0,1]\), every maximal ideal of \( C[0,1] \) has the form \( m \cap C[0,1] \), where \( m \) is a maximal ideal of \( L^\infty[0,1] \). (cf. [5], p. 214 and p. 218) This shows the range of the mapping \( t \) to be all of \([0,1]\).

We shall now show that for every \( \mu \)-measurable subset \( S \) of \( M \), \( \mu(S) \leq \sigma^\ast(t(S)) \), where \( \sigma^\ast \) denotes inner Lebesgue measure on \([0,1]\). To see this, suppose first that \( S \) is compact. Then \( t(S) \) is compact, since \( t \) is continuous. Using the regularity of \( \sigma \), together with Urysohn's lemma,
we have that for every $\epsilon > 0$ there exists a continuous function $f$ on $[0,1]$ such that $0 \leq f(x) \leq 1$ for all $x$, $f(x) = 1$ for $x \in t(S)$, and $\sigma(t(S)) > \int_0^1 f(x)dx - \epsilon$. From the definition of $t$, $\hat{f}(m) = f(t(m))$ for all $m$, hence $0 \leq \hat{f}(m) \leq 1$, and $\hat{f}(m) = 1$ for $m \in S$. Thus $\mu(S) \leq \int_M \hat{f}(m)d\mu(m) = \int_0^1 f(x)dx < \sigma(t(S)) + \epsilon$. Since $\epsilon$ was arbitrary, $\mu(S) \leq \sigma(t(S))$.

If $S$ is now an arbitrary $\mu$-measurable set, then for every $\epsilon > 0$ there exists a compact $K \subset S$ for which $\mu(K) > \mu(S) - \epsilon$. From the above, $\mu(K) \leq \sigma(t(K))$, and since $K \subset S$, we have $t(K) \subset t(S)$. Thus $\mu(S) - \epsilon < \mu(K) \leq \sigma(t(K)) \leq \sigma_*(t(S))$, and since $\epsilon$ was arbitrary, $\mu(S) \leq \sigma_*(S)$ as asserted.

Non $\mu$-measurable sets in $M$ may now be found as follows. There exist sets $Q_1, Q_2 \subset [0,1]$ with $Q_1 \cup Q_2 = [0,1]$, $Q_1 \cap Q_2 = \emptyset$, such that $\sigma_*(Q_1) = \sigma_*(Q_2) = 0$. (cf [2], p. 70) Let $S_1 = t^{-1}(Q_1)$, $S_2 = t^{-1}(Q_2)$. Then $S_1 \cup S_2 = M$, $S_1 \cap S_2 = \emptyset$, $t(S_1) = Q_1$, $t(S_2) = Q_2$. If one of $S_1, S_2$ is measurable, then both are necessarily measurable, and $\mu(S_1) + \mu(S_2) = \mu(S_1 \cup S_2) = 1$. However, $\mu(S_1) \leq \sigma_*(t(S_1)) = 0$ and $\mu(S_2) \leq \sigma_*(t(S_2)) = 0$ from our above result. Hence, both $S_1, S_2$ are non $\mu$-measurable.
We shall henceforth denote by \( M \) and \( \mu \), respectively, a space and measure satisfying the conclusions of Theorem A.1, and by \( Z \) the positive integers. The terminology 'measurable' and 'almost everywhere' will always refer to the measure \( \mu \).

Let \( < \) be a well ordering of \( M \), and consider the positive integers \( Z \) with the usual well ordering. Let \( \Lambda = M \times Z \). Then the well orderings on \( M \) and \( Z \) induce the lexicographic well ordering on \( \Lambda \) defined as follows: if \( \alpha_1 = (m_1, n_1) \in \Lambda \) and \( \alpha_2 = (m_2, n_2) \in \Lambda \), then \( \alpha_1 < \alpha_2 \) if either \( m_1 < m_2 \) or \( m_1 = m_2 \) and \( n_1 < n_2 \). We shall adopt the notations and conventions for the well ordered set \( \Lambda \) as explained in Chapter III.

For every \( \alpha \in \Lambda \), let \( L_2(M, \mu)_\alpha \) be a copy of the space \( L_2(M, \mu) \), and define \( \mathcal{H} = \bigoplus_{\alpha \in \Lambda} L_2(M, \mu)_\alpha \). Then \( \mathcal{H} \) is a Hilbert space, and a vector of \( \mathcal{H} \) consists of a system of \( L_2 \) functions \( \{f_\alpha\}_{\alpha \in \Lambda} \) for which \( \sum_{\alpha \in \Lambda} \int_M |f_\alpha(m)|^2 d\mu(m) < \infty \). For each \( \alpha \in \Lambda \), let \( \mathcal{M}_\alpha \) denote \( L_2(M, \mu)_\alpha \) identified as a closed linear subspace of \( \mathcal{H} \), i.e., all systems \( \{f_\beta\}_{\beta \in \Lambda} \) in \( \mathcal{H} \) such that \( f_\beta = 0 \) if \( \beta \neq \alpha \). Also, for each \( \alpha \), let \( \eta_\alpha \) denote that vector \( \{f_\beta\}_{\beta \in \Lambda} \) for which \( f_\beta = 0 \) for \( \beta \neq \alpha \), and \( f_\alpha(m) = 1 \).
Every function $\varphi \in C(M)$ determines an operator $\varphi^*$ on $\mathcal{H}$ as described in Chapter I. We denote the set of all such operators by $\mathcal{E}$.

**Theorem A.2.** (a) $\mathcal{E}$ is a diagonal ring on $\mathcal{H}$, and the mapping $\varphi \mapsto \varphi^*$ is a symmetric isomorphism from $C(M)$ onto $\mathcal{E}$.

(b) $M$ is the maximal ideal space of $\mathcal{E}$ under the identification $(\varphi^*)(m) = \varphi(m)$. Furthermore $\eta_1$ is a cyclic vector for $\mathcal{H}$ with respect to $\mathcal{E}'$, and $\mu$ is the measure such that $(A\eta_1, \eta_1) = \int_M \hat{A}(m)d\mu(m)$ for $A \in \mathcal{E}$.

(c) The system $\{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}$ is a canonical decomposition system for $\mathcal{H}$ with respect to $\mathcal{E}$, where the measure used to define $\varphi_\xi$ and $S_\xi$ is $\mu$. $S_\eta = M$ for every $\alpha \in \Lambda$. Moreover, if we represent $L^2(M, \mu)$ equivalence classes by their continuous representatives, then the identity function on $\mathcal{H}$ is the canonical $L^2$ isomorphism corresponding to $\{(M_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}$.

**Proof.** The mapping $\varphi \mapsto \varphi^*$ is clearly linear, and preserves multiplication. Also, for arbitrary $[f_\alpha], [g_\alpha] \in \mathcal{H}$, $(\varphi \cdot [f_\alpha], [g_\alpha]) = \sum_{\alpha \in \Lambda} \int_M \varphi(m)f_\alpha(m)\overline{g_\alpha(m)}d\mu(m) = \sum_{\alpha \in \Lambda} \int_M f_\alpha(m)\overline{\varphi(m)}g_\alpha(m)d\mu(m) = ([f_\alpha], \overline{\varphi} \cdot [g_\alpha]).$ Consequently, $\overline{\varphi^*} = (\varphi^*)^*$, so the mapping is a symmetric homomorphism.

If $\varphi^* = 0$, then $0 = |\varphi \cdot \eta_1|^2 = (\varphi \cdot \eta_1, \varphi \cdot \eta_1) = \int_M \varphi(m)\overline{\varphi(m)}d\mu(m)$.
\[ \int_M |\varphi(m)|^2 \, d\mu(m). \]

Since \( \varphi \) is continuous, it follows that \( \varphi(m) = 0 \). This proves that the mapping is a symmetric isomorphism. Since the image of the identity in \( C(M) \) under this mapping is the identity in \( \mathcal{B}(\mathcal{H}) \), and since both \( C(M) \) and \( \mathcal{B}(\mathcal{H}) \) are completely regular, it follows that \( E \) is a commutative Banach symmetric subring of \( \mathcal{B}(\mathcal{H}) \) with the identity. All assertions of part (a) thus hold, except possibly the requirement that \( E \) be weakly closed in order to be a diagonal ring. This will be shown later in the proof.

To prove the first statement of part (b), we recall that \( M \) can be identified as the maximal ideal space of \( C(M) \), where the Gelfand transform is the identity function on \( C(M) \). Since \( \varphi \to \varphi^* \) is a symmetric isomorphism from \( C(M) \) onto \( E \), the statement follows. To show the asserted relationship between \( \eta_\perp \) and \( \mu \), suppose \( A \in E \) is arbitrary. Then \( A = \varphi^* \) for a unique \( \varphi \in C(M) \), consequently \( (A\eta_\perp, \eta_\perp) = (\varphi \cdot \eta_\perp, \eta_\perp) = \int_M \varphi(m) d\mu(m) = \int_M (\varphi^*)^*(m) d\mu(m) = \int_M \hat{A}(m) d\mu(m) \). The statement that \( \eta_\perp \) is cyclic for \( \mathcal{H} \) with respect to \( E \) will require some preliminary results.

First we show \( \mathcal{G}(\{A\eta_\alpha | A \in E\}) = \mathcal{M}_\alpha \) for every \( \alpha \in \Lambda \).

In fact, if \( \varphi \in C(M) \), then \( \varphi \cdot \eta_\alpha = [f_\beta]_{\beta \in \Lambda} \), where \( f_\beta = 0 \) for \( \beta \neq \alpha \) and \( f_\alpha = \varphi \). The set of vectors \( \{A\eta_\alpha | A \in E\} \)
thus consists of all \([f_{\beta}]_{\beta \in \Lambda}\) such that \(f_{\beta} = 0\) for \(\beta \neq \alpha\), and \(f_{\alpha}\) is an arbitrary continuous function. Since the continuous functions form a dense linear subspace of \(L_2(M, \mu)\), it follows that \(\mathcal{C}\{A \eta_{\alpha} \mid A \in \mathcal{B}\} = \mathcal{M}_{\alpha}\).

Next, we define a system of operators \(V_{\alpha \beta}\), for \(\alpha, \beta \in \Lambda\) as follows. If \([f_{\gamma}]_{\gamma \in \Lambda} \in \mathcal{H}\), then \(V_{\alpha \beta} [f_{\gamma}]_{\gamma \in \Lambda} = [g_{\gamma}]_{\gamma \in \Lambda}\), where \(g_{\gamma} = 0\) for \(\gamma \neq \alpha\), and \(g_{\alpha} = f_{\beta}\). It is clear that \(V_{\alpha \beta} \in \mathcal{B}(\mathcal{H})\) for every \(\alpha, \beta\), and \(|V_{\alpha \beta}| \leq 1\). The following are claimed for the \(V_{\alpha \beta}\):

(i) \(V_{\alpha \beta} \in \mathcal{B}^{1}\) for every \(\alpha, \beta \in \Lambda\).
(ii) \(V_{\alpha \beta} V_{\beta \gamma} = V_{\alpha \gamma}\) for every \(\alpha, \beta, \gamma \in \Lambda\).
(iii) \(V_{\alpha \beta} = V_{\beta \alpha}^{*}\) for every \(\alpha, \beta \in \Lambda\).
(iv) \(V_{\alpha \alpha}\) is the projection of \(\mathcal{H}\) onto \(\mathcal{M}_{\alpha}\) for every \(\alpha \in \Lambda\).
(v) \(V_{\alpha \beta}\) is a partial isometry with initial domain \(\mathcal{M}_{\beta}\) and terminal domain \(\mathcal{M}_{\alpha}\) for every \(\alpha, \beta \in \Lambda\). Furthermore \(V_{\alpha \beta} \eta_{\beta} = \eta_{\alpha}\).

Statement (i) follows readily. In fact, if \(\varphi \in C(M)\) and \([f_{\gamma}]_{\gamma \in \Lambda} \in \mathcal{H}\), then \(V_{\alpha \beta} \varphi [f_{\gamma}] = \varphi V_{\alpha \beta} [f_{\gamma}] = [g_{\gamma}]\), where \(g_{\gamma} = 0\) for \(\gamma \neq \alpha\), and \(g_{\alpha} = \varphi f_{\beta}\). (ii) is immediate from the definition of the \(V_{\alpha \beta}\). (iii) follows by using the definition of \(V_{\alpha \beta}\) to make the computation \((V_{\alpha \beta} [g_{\gamma}], [f_{\gamma}])\)
\[ = \int_M g_\alpha(m) f_\alpha(m) d\mu(m) = ([g_\gamma], V_\beta [f_\gamma]) \]

which is valid for arbitrary \([g_\gamma], [f_\gamma] \in \mathcal{H}\). (iv) follows from the definition of \(V_{\alpha\alpha}\). To prove (v), we have from (ii), (iii), and (iv) that \(V_{\alpha\beta} V_{\alpha\beta} = V_{\beta\beta} = P_\beta\), \(V_{\alpha\beta} V_{\alpha\beta} = V_{\alpha\alpha} = P_\alpha\), where \(P_\beta\) and \(P_\alpha\) are the respective projections onto \(\mathcal{M}_\beta\) and \(\mathcal{M}_\alpha\). The fact that \(V_{\alpha\beta} \eta_\beta = \eta_\alpha\) follows from the definition of \(V_{\alpha\beta}\).

We now prove \(\mathcal{G}(A_{\eta_1} | A \in E') = \mathcal{H}\). Since \(V_{\alpha_1} \in E'\) for every \(\alpha \in \Lambda\), it follows that \(\eta_\alpha = V_{\alpha_1} \eta_1 \in \mathcal{G}(A_{\eta_1} | A \in E')\). Since \(E\) is commutative, \(E \subseteq E'\), hence for every \(\alpha \in \Lambda\), \(\mathcal{M}_\alpha = \mathcal{G}(A_{\eta_\alpha} | A \in E) \subseteq \mathcal{G}(A_{\eta_\alpha} | A \in E') \subseteq \mathcal{G}(A_{\eta_1} | A \in E')\). Since \(\alpha\) was arbitrary, we have \(\mathcal{H} \subseteq \bigoplus_{\alpha \in \Lambda} \mathcal{M}_\alpha \subseteq \mathcal{G}(A_{\eta_1} | A \in E')\), so the result follows, and all assertions of (b) hold.

The above will now be used to prove that \(E\) is weakly closed. Suppose \(B\) is an arbitrary operator of \(E''\). Then \(B\) commutes with every operator of \(E'\), and in particular, with every operator of the form \(\varphi\cdot \). Let \(B\eta_1 = \{b_\alpha\}_{\alpha \in \Lambda}\). If \(\varphi_1, \varphi_2 \in C(M)\) are arbitrary, then \((B\varphi_1 \cdot \eta_1, \varphi_2 \cdot \eta_1) = (B\eta_1, (\bar{\varphi_1} \cdot \varphi_2) \cdot \eta_1) = \int_M b_1(m) \varphi_1(m) \varphi_2(m) d\mu(m)\). We shall show that this implies \(b_1 \in L_1(M, \mu)\). In fact, if \(\psi\) is an arbitrary continuous function on \(M\), define \(\varphi_2 = \sqrt{|\psi|}\),
\[ \varphi_1(m) = \frac{\psi(m)}{\sqrt{\psi(m)}} \] where \( \psi(m) \neq 0 \), \( \varphi_1(m) = 0 \) where \( \psi(m) = 0 \).

Then \( \varphi_1, \varphi_2 \) are continuous, and from the above computation,
\[
|\int_M b_1(m)\psi(m)d\mu(m)| = |\int_M b_1(m)\varphi_1(m)\overline{\varphi_2(m)}d\mu(m)| = |(B\varphi_1, \eta_1, \varphi_2, \eta_1)| \leq |B| |\varphi_1, \eta_1| |\varphi_2, \eta_1| =
\]
\[
|B| \left[ \int_M |\varphi_1(m)|^2d\mu(m) \right]^{\frac{1}{2}} \left[ \int_M |\varphi_2(m)|^2d\mu(m) \right]^{\frac{1}{2}} =
\]
\[
|B| \int_M |\psi(m)|d\mu(m). \text{ Since } \psi \text{ was arbitrary, this shows}
\]
\[
||b_1||_\infty \leq |B|.
\]

From Theorem A.1(a), \( b_1 \) is equal to a continuous function almost everywhere, hence we may assume that \( b_1 \) is, itself, continuous. We shall show that \( b_1 \cdot = B \).

Since we have from the above that \( \mathcal{G}(\mathcal{A} \eta, |A \in E) = \mathcal{M}_\alpha \),
and \( \sum_{\alpha \in \Lambda} \mathcal{M}_\alpha = \mathcal{H} \), it suffices to show that \( (b_1 \cdot \varphi_1, \eta_\alpha, \varphi_2, \eta_\beta) = (B\varphi_1, \eta_\alpha, \varphi_2, \eta_\beta) \) for arbitrary \( \varphi_1, \varphi_2 \in \mathcal{C}(M) \), and arbitrary \( \alpha, \beta \in \Lambda \). But this equation is equivalent to \( (b_1 \cdot \eta_\alpha, (\overline{\varphi_1}\varphi_2) \cdot \eta_\beta) = (B\eta_\alpha, (\overline{\varphi_1}\varphi_2) \cdot \eta_\beta) \), hence it suffices to show \( (b_1 \cdot \eta_\alpha, \varphi \cdot \eta_\beta) = (B\eta_\alpha, \varphi \cdot \eta_\beta) \) for arbitrary \( \varphi \in \mathcal{C}(M) \), \( \alpha, \beta \in \Lambda \).

For the case where \( \alpha = \beta \), we have from the definition of \( b_1 \) and the fact that \( B \) commutes with all operators
\[
V_\alpha, (B\eta_\alpha, \varphi \cdot \eta_\alpha) = (B\varphi \eta_\alpha, \varphi \cdot \eta_\alpha) = (B\varphi \eta_\alpha, \varphi \cdot \eta_\alpha) =
\]
\[
(V_\alpha B\eta_\alpha, \varphi \cdot \eta_\alpha) = (B\eta_\alpha, V_\alpha \varphi \cdot \eta_\alpha) = (B\eta_\alpha, \varphi \cdot V_\alpha \eta_\alpha) = (B\eta_\alpha, \varphi \cdot \eta_\alpha)
\]
If \( \alpha \neq \beta \), then
\[
(B\eta_\alpha, \varphi \cdot \eta_\beta) = (BV_{\alpha\alpha} \eta_\alpha, \varphi \cdot V_{\beta\beta} \eta_\beta) = (V_{\alpha\alpha} B\eta_\alpha, V_{\beta\beta} \varphi \cdot \eta_\beta) = \]
\[
(B\eta_\alpha, V_{\alpha\alpha} V_{\beta\beta} \varphi \cdot \eta_\beta) = 0, \text{ since } V_{\alpha\alpha}, V_{\beta\beta} \text{ are projections}
\]
onto mutually orthogonal subspaces. Likewise, \((b_1 \cdot \eta_\alpha, \varphi \cdot \eta_\beta) = 0\) by direct computation. We thus have \((b_1 \cdot \eta_\alpha, \varphi \cdot \eta_\beta) = (B\eta_\alpha, \varphi \cdot \eta_\beta)\) in all cases, showing that \(b_1 \cdot = B\), hence that \(B \in E\), and \(E\) is weakly closed.

To prove (c), we note that (3.2b) and (3.2c) are satisfied by the system \(\{(\mathbb{M}_\alpha, \eta_\alpha)\}_{\alpha \in \Lambda}\) by construction, while (3.2a) was shown above. For arbitrary \(\alpha \in \Lambda\) and \(\varphi \in \mathbb{C}(M)\), we have by direct computation that
\[
\varphi \cdot \eta_\alpha(m) = \int_M \varphi(m) d\mu(m) \]
and \(S_{\eta_\alpha} = M\). This verifies (3.2d), (3.2e), and our assertion that \(S_{\eta_\alpha} = M\) for every \(\alpha \in \Lambda\).

The fact that the identity operator on \(\mathcal{H}\) is the canonical \(L^2\) isomorphism follows from the fact that it satisfies (3.4.1) and (3.4.2).

Now let \(\int_M \Theta \mathcal{L}_m d\mu(m)\) be the direct integral corresponding to this canonical decomposition system. Let \(\mathcal{L}\) be the fixed Hilbert space with \(\dim \mathcal{L} = \text{card } \Lambda\), and \(\{e_\alpha\}_{\alpha \in \Lambda}\) be the orthonormal basis indexed by \(\Lambda\) in the construction.
of the direct integral. It follows that $\mathcal{L}_m = \mathcal{L}$ for every $m \in M$, since $m \in M = \mathcal{S}_{\eta}$ for every $\alpha \in \Lambda$, and $\eta_{\alpha}(m) = e_{\alpha}$ for every $\alpha$.

We can now construct vector valued functions which fail to satisfy the equivalent conditions of Theorem 3.8(b).

**Theorem A.3.** If $Q$ is an arbitrary subset of $M$, there exists an everywhere defined function $F(m)$ on $M$, with $F(m) \in \mathcal{L}_m = \mathcal{L}$ for each $m$, such that $(\xi(m), F(m)) = 0$ a.e. for every $\xi \in \mathcal{H}$ and $|F(m)|^2 = \chi_Q(m)$.

**Proof.** We recall that $\Lambda = M \times \mathbb{Z}$, so the indices $\alpha$ are ordered pairs $(m, n)$. Define $F(m) = \chi_Q(m)e_{(m, 1)}$ for all $m$. The conclusion that $|F(m)|^2 = \chi_Q(m)$ is then immediate. To show $(\xi(m), F(m)) = 0$ a.e. for every $\xi \in \mathcal{H}$, we first prove the assertion whenever $\xi = \eta_{\alpha}$ for some $\alpha$. Then $\alpha = (m_0, n_0)$ for some $m_0 \in M$, $n_0 \in \mathbb{Z}$, and $\eta_{\alpha}(m) = e_{(m_0, n_0)}$. Thus $(\eta_{\alpha}(m), F(m)) = \chi_Q(m)(e_{(m_0, n_0)}, e_{(m, 1)})$.

This expression can be nonzero for at most one $m$, namely $m = m_0$, and then only if $m \in Q$ and $n_0 = 1$. Hence $(\eta_{\alpha}(m), F(m)) = 0$ a.e. since $\mu$ is continuous. If, as in Chapter III, we denote by $\mathcal{H}'$ the set of vectors $\xi$ having the form

$$\sum_{j=1}^{n} B_j \eta_{\alpha_j},$$

where $B_1, \ldots, B_n \in E$ and $\alpha_1, \ldots, \alpha_n \in \Lambda$, then by Theorem 3.7(c), for $\xi$ of this form, $(\xi(m), F(m)) =$
\[ \sum_{j=1}^{n} \hat{B}_j(m)(\eta_j(m),F(m)). \] From the above, we thus have \((\xi(m),F(m)) = 0 \text{ a.e. if } \xi \in \mathcal{H}'.\) Now if \(\xi\) is arbitrary in \(\mathcal{H},\) there exists a sequence \(\{\xi_j\}_{j=1}^{\infty}\) in \(\mathcal{H}'\) which converges to \(\xi,\) and by Theorem 3.7(f), we may assume \(\xi_j(m) \to \xi(m) \text{ a.e.}\) Then \((\xi_j(m),F(m)) \to (\xi(m),F(m)) \text{ a.e., and since } (\xi_j(m),F(m)) = 0 \text{ a.e. for every } j, \text{ we have } (\xi(m),F(m)) = 0 \text{ a.e.}\)

Regardless of which set \(Q\) is chosen, it is clear that \(F(m)\) satisfies the hypothesis of Theorem 3.8. The unique vector \(\zeta\) of part (a) of that theorem is necessarily the zero vector. It is then clear that \(F(m) = \zeta(m)\) only for \(m \notin Q,\) so that if \(Q\) is not a null set, none of the equivalent conditions of Theorem 3.8(b) hold. Condition (ii) in this case is simply a restatement of the fact that \(Q\) must be a null set.

Since we can take \(Q\) to be a non-measurable set, by Theorem A.1(c), it follows that \(|F(m)|^2\) can be a non-measurable function.

We now turn our attention to showing that the pointwise product of direct integral decompositions of operators in \(E'\) need not yield the direct integral decomposition of the product.

**Theorem A.4.** There exist two operators \(A,B \in E'\) such that if \(A = \int_{M} \otimes A(m)d\mu(m), B = \int_{M} \otimes B(m)d\mu(m),\)
and \( AB = \int_M \mathcal{A}(m \otimes \mathcal{B})(m) d\mu(m) \), then \( (AB)(m) \) is nowhere equal to \( A(m)B(m) \).

Proof. We have \( \mu([m]) = 0 \) for every \( m \in M \), and also that \( M \) is totally disconnected. From the regularity of \( \mu \), it is thus possible to select a sequence of clopen sets \( \{U_{m,j}\}_{j=1}^\infty \) for each \( m \), such that \( m \in U_{m,j} \) for every \( j, M = U_{m,1} \supset U_{m,2} \supset \cdots \), and \( \mu(\bigcap_{j=1}^\infty U_{m,j}) = \lim_{j \to \infty} \mu(U_{m,j}) = 0 \). We select and fix such a sequence for every \( m \in M \), and set \( K_m = \bigcap_{j=1}^\infty U_{m,j} \). Then \( K_m \) is a closed set of measure zero containing \( m \).

Define \( f_{m,j} = \chi(U_{m,j} - U_{m,j+1}) \) for each integer \( j \) and each \( m \in M \). Since the \( U_{m,j} \) are clopen, the \( f_{m,j} \) are continuous functions. For every fixed \( m_0 \in M \) and \( m \in M \), we have that \( f_{m_0,j}(m) = 1 \) for exactly one integer \( j \) if \( m \in K_{m_0} \), the rest of the \( f_{m_0,j}(m) \) then being zero. If \( m \in K_{m_0} \), then \( f_{m_0,j}(m) = 0 \) for every integer \( j \).

For each \( m \in M \), the system \( \{e_{(m',n)}(m) \mid m' \in M, n \in \mathbb{Z}\} \) is an orthonormal basis for \( \mathcal{X}_m = \mathcal{L} \). The set of finite linear combinations of the form \( \sum_{j=1}^q \sum_{k=1}^r t_{j,k} e_{(m_j,k)} \), where the \( t_{j,k} \) are complex, and \( m_j \neq m_k \) for \( j \neq k \), thus
form a dense linear subspace of $\mathcal{L}_m$. We define

$$A(m)(\sum_{j=1}^q \sum_{k=1}^r t_{j,k} e^{(m_j,k)}) = \sum_{j=1}^q \sum_{k=1}^r f_{m_j,k}^{(m)} e^{(m_j,k)}.$$ 

Then $A(m)$ is a linear operator on the above mentioned dense linear subspace. From our observation above, we have, for each index $j$ appearing on the right side of this expression, that there is at most one $k$ for which $f_{m_j,k}^{(m)} \neq 0$, in which case $f_{m_j,k}^{(m)} = 1$. Thus we have

$$|A(m)(\sum_{j=1}^q \sum_{k=1}^r t_{j,k} e^{(m_j,k)})|^2 = \sum_{(j,k) | f_{m_j,k}^{(m)} \neq 0} \sum_{j=1}^q \sum_{k=1}^r |t_{j,k}|^2.$$ 

It follows that $|A(m)| \leq 1$, hence $A(m)$ has a unique norm-preserving extension to all of $\mathcal{L}_m$, which we denote again by $A(m)$.

We shall show that $m \mapsto A(m)$ satisfies (3.9b) and (3.9c). (3.9b) is clear, since $|A(m)| \leq 1$ for all $m$. Recalling that $\eta_\alpha(m) = e_\alpha$ for every $\alpha$, suppose $\alpha = (m_1, n_1)$, $\beta = (m_2, n_2) \in \Lambda$. Then $(A(m)\eta_\alpha(m), \eta_\beta(m)) = (A(m)e^{(m_1, n_1)}, e^{(m_2, n_2)}) = (f_{m_1, n_1}^{(m)} e^{(m_1, 1)}, e^{(m_2, n_2)}) = f_{m_1, n_1}^{(m)}(e^{(m_1, 1)}, e^{(m_2, n_2)})$. This expression defines a continuous function, either $f_{m_1, n_1}^{(m)}$ or the identically zero function. By Theorem 3.11, there exists an $A \in E'$
such that \( A = \int_M \Omega A(m) d\mu(m). \)

We construct \( B \) in a similar manner. For each \( m \in M \), let \( B(m) \) be defined on the above dense linear subspace of \( \mathcal{L}_m \) by

\[
B(m)(\sum_{j=1}^{r} \sum_{k=1}^{r} t_{j,k} e(m_{j,k})) = \sum_{j=1}^{r} \sum_{k=1}^{r} f_{m_{j,k}}(m) e(m_{j,k}).
\]

In the infinite sum in the index \( k \) on the right side of this expression, at most one \( f_{m_{j,k}}(m) \) is not zero for fixed \( j \), and if such a term occurs, \( f_{m_{j,k}}(m) = 1 \). \( B(m) \) is clearly linear, and

\[
|B(m)(\sum_{j=1}^{r} \sum_{k=1}^{r} t_{j,k} e(m_{j,k}))|^2 = \sum_{j=1}^{r} \sum_{k=1}^{r} |t_{j,k}|^2 \leq \sum_{j=1}^{r} \sum_{k=1}^{r} |t_{j,k}e(m_{j,k})|^2. \text{ Thus } |B(m)| \leq 1,
\]

and \( B(m) \) has a unique norm-preserving extension to all of \( \mathcal{L}_m \), which we again denote by \( B(m) \).

\( m \to B(m) \) satisfies (3.9b) with \( |B(m)| \leq 1 \) for every \( m \). To show (3.9c), suppose \( \alpha = (m_1, n_1), \beta = (m_2, n_2) \in \Lambda \).

If \( n_1 \neq 1 \), then \( B(m)\eta_\alpha(m) = B(m)e(m_1, n_1) = 0 \), hence

\[
(B(m)\eta_\alpha(m), \eta_\beta(m)) \text{ defines the identically zero function.}
\]

If \( n_1 = 1 \), then \( (B(m)\eta_\alpha(m), \eta_\beta(m)) = (B(m)e(m_1, 1), e(m_2, n_2)) = (\sum_{k=1}^{\infty} f_{m_1,k}(m)e(m_1,k), e(m_2, n_2)) = f_{m_1,n_2}(m)(e(m_1, n_2), e(m_2, n_2)), \)
which defines either $f_{m_1,n_2}$ or the identically zero function.

Applying Theorem 3.11 again, there exists a $B \in E'$ for which $B = \int_M \Theta B(m) d\mu(m)$.

Let $AB = \int_M \Theta (AB)(m) d\mu(m)$, and let $m_0$ be an arbitrary point of $M$. Applying Theorem 3.12(c) to $A$, $B$, and the vector $\eta(m_0,l)$, we have $(AB)(m)\eta(m_0,l)(m) = A(m)B(m)\eta(m_0,l)(m)$ a.e. Since $\eta(m_0,l)(m) = e(m_0,l)$, this implies

$$(AB)(m)e(m_0,l)e(m_0,l) = (A(m)B(m)e(m_0,l)e(m_0,l)) \text{ a.e.}$$

The left side of this equation is continuous by (3.9c) applied to the operator $AB$. If we can show that the right side is 1 almost everywhere, it will follow that the left side is identically 1. But if $m$ is not in the null set $K_{m_0}$, there is a unique integer $j$, depending on $m$, for which $f_{m_0,j}(m) = 1$. By the definitions of $A(m)$ and $B(m)$, it follows that $(A(m)B(m)e(m_0,l)e(m_0,l)) = (A(m)e(m_0,l)e(m_0,l)) = 1$. Since this computation is possible for all $m \notin K_{m_0}$, it follows that $((AB)(m)e(m_0,l)e(m_0,l)) = 1$. In particular, taking $m = m_0$, we have $((AB)(m_0)e(m_0,l)e(m_0,l)) = 1$, hence

$$(AB)(m_0)e(m_0,l) \neq 0.$$
However, \( m_0 \in K_{m_0} \), so that \( f_{m_0, j}(m_0) = 0 \) for every integer \( j \). Hence, from the definition, \( B(m_0)e_{(m_0, 1)} = 0 \). Thus \( A(m_0)B(m_0)e_{(m_0, 1)} = 0 \neq (AB)(m_0)e_{(m_0, 1)} \), so that \( A(m_0)B(m_0) \neq (AB)(m_0) \). Since \( m_0 \) was arbitrary, we have the result. \( \square \)
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