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FOURIER SERIES ON COMPACT SYMMETRIC SPACES

GESTUR ÓLAFSSON AND HENRIK SCHLICHTKRULL

ABSTRACT. The Fourier coefficients of a function f on a compact symmetric space U/K are given by integration of f against matrix coefficients of irreducible representations of U . The coefficients depend on a spectral parameter μ , which determines the representation, and they can be represented by elements $\hat{f}(\mu)$ in a common Hilbert space \mathcal{H} .

We obtain a theorem of Paley-Wiener type which describes the size of the support of f by means of the exponential type of a holomorphic \mathcal{H} -valued extension of \hat{f} , provided f is K -finite and of sufficiently small support. The result was obtained previously for K -invariant functions, to which case we reduce.

1. INTRODUCTION.

The present paper is a continuation of our article [19]. We consider a compact symmetric space X , realized as the homogeneous space U/K of a compact Lie group U . Up to covering, U is the connected component of the group of isometries of X . As an example, we mention the sphere S^n , for which $U = \mathrm{SO}(n+1)$ and $K = \mathrm{SO}(n)$. In the cited paper, we considered K -invariant functions on U/K . The Fourier series of a K -invariant function f is

$$(1.1) \quad \sum_{\mu} a_{\mu} \psi_{\mu}(x),$$

where ψ_{μ} is the zonal spherical function associated with the representation of U with highest weight μ , and where the Fourier coefficients a_{μ} are given by

$$(1.2) \quad a_{\mu} = d(\mu) \tilde{f}(\mu) = d(\mu) \int_{U/K} f(x) \overline{\psi_{\mu}(x)} dx,$$

with $d(\mu)$ being the representation dimension, and dx being the normalized invariant measure on U/K . The main result of [19] is a local Paley-Wiener theorem, which gives a necessary and sufficient condition on the coefficients in the series (1.1) that it is the Fourier series of a smooth K -invariant function f supported in a geodesic ball of a given sufficiently small radius r around the origin in U/K . The condition is, that $\mu \mapsto a_{\mu}$ extends to a holomorphic function of exponential type r satisfying certain invariance under the action of the Weyl group. We refer to [3, 4, 5, 10] for previous results on special cases. The case of distributions will be treated in [21].

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In the present paper we consider the general case where the K -invariance is replaced by K -finiteness. Instead of being scalars, the Fourier coefficients take values in the Hilbert space $\mathcal{H} = L^2(K/M)$, where M is a certain subgroup of K . In case of $U/K = S^n$, we have $K/M = S^{n-1}$. Our main result is Theorem 7.2 below, which describes the set of Fourier coefficients of K -finite smooth functions on U/K , supported in a ball of a given sufficiently small radius. The corresponding result for Riemannian symmetric spaces of the non-compact type is due to Helgason, see [11].

Our method is by reduction to the previous case. For the reduction we use Kostant's description of the spherical principal series of a semisimple Lie group [16]. A similar reduction was found by Torasso [29] for Riemannian symmetric spaces of the non-compact type, thus providing an alternative proof of the mentioned theorem of Helgason.

2. BASIC NOTATION

We recall some basic notation from [19]. We are considering a Riemannian symmetric space U/K , where U is a connected compact semisimple Lie group and K a closed symmetric subgroup. By definition this means that there exists a nontrivial involution θ of U such that $K_0 \subset K \subset U^\theta$. Here U^θ denotes the subgroup of θ -fixed points, and $K_0 := U_0^\theta$ its identity component. The base point in U/K is denoted by $x_0 = eK$.

The Lie algebra of U is denoted \mathfrak{u} , and by $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ we denote the Cartan decomposition associated with the involution θ . We endow U/K with the Riemannian structure induced by the negative of the Killing form on \mathfrak{q} .

Let $\mathfrak{a} \subset \mathfrak{q}$ be a maximal abelian subspace, \mathfrak{a}^* its dual space, and $\mathfrak{a}_\mathbb{C}^*$ the complexified dual space. The set of non-zero roots of \mathfrak{a} is denoted by Σ . The roots $\alpha \in \Sigma \subset \mathfrak{a}_\mathbb{C}^*$ are purely imaginary valued on \mathfrak{a} . The corresponding Weyl group, generated by the reflections in the roots, is denoted W . We make a fixed choice of a positive system Σ^+ for Σ , and define $\rho \in i\mathfrak{a}^*$ to be half the sum of the roots in Σ^+ , counted with multiplicities. The centralizer of \mathfrak{a} in K is denoted $M = Z_K(\mathfrak{a})$.

Some care has to be taken because we are not assuming K is connected. We recall that if U is simply connected, then U^θ is connected and $K = K_0$, see [14], p. 320. We recall also that in general $K = MK_0$, see [19], Lemma 5.2.

In the following we shall need to complexify U and U/K . Since U is compact there exists a unique (up to isomorphism) connected complex Lie group $U_\mathbb{C}$ with Lie algebra $\mathfrak{u}_\mathbb{C}$, which contains U as a real Lie subgroup. Let \mathfrak{g} denote the real form $\mathfrak{k} + i\mathfrak{q}$ of $\mathfrak{u}_\mathbb{C}$, and let G denote the connected real Lie subgroup of $U_\mathbb{C}$ with this Lie algebra. It will be convenient to denote $\mathfrak{u}_\mathbb{C}$ and $U_\mathbb{C}$ by $\mathfrak{g}_\mathbb{C}$ and $G_\mathbb{C}$. The Cartan involutions of \mathfrak{u} and U extend to involutions of $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$, which we shall denote again by θ , and which leave \mathfrak{g} and G invariant. The corresponding Cartan decomposition of \mathfrak{g} is $\mathfrak{g} = \mathfrak{k} + i\mathfrak{q}$. It follows that $K_0 = G^\theta$ is maximal compact in G , and G/K_0 is a Riemannian symmetric space of the non-compact type.

We denote by $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{a} \oplus \mathfrak{n}$ and $G = K_0AN$ the Iwasawa decompositions of \mathfrak{g} and G associated with Σ^+ . Here $A = \exp(i\mathfrak{a})$ and $N = \exp \mathfrak{n}$. Furthermore,

we let $H: G \rightarrow \mathfrak{ia}$ denote the *Iwasawa projection*

$$K_0AN \ni k \exp Yn = g \mapsto H(g) = Y.$$

Let $K_{\mathbb{C}}$, $A_{\mathbb{C}}$, and $N_{\mathbb{C}}$ denote the connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}$, and put $K_{\mathbb{C}} = K_0K$. Then $G_{\mathbb{C}}/K_{\mathbb{C}}$ is a symmetric space, and it carries a natural complex structure with respect to which U/K and G/K_0 are totally real submanifolds of maximal dimension.

Lemma 2.1. *There exists an open $K_{\mathbb{C}} \times K$ -invariant neighborhood \mathcal{V}^a of the neutral element e in $G_{\mathbb{C}}$, and a holomorphic map*

$$(2.1) \quad H: \mathcal{V}^a \rightarrow \mathfrak{a}_{\mathbb{C}},$$

which agrees with the Iwasawa projection on $\mathcal{V}^a \cap G$, such that

$$(2.2) \quad u \in K_{\mathbb{C}} \exp(H(u))N_{\mathbb{C}}$$

for all $u \in \mathcal{V}^a$.

Proof. (See [6] or [25].) We first assume that $K = U^{\theta}$. Then $K_{\mathbb{C}} = G_{\mathbb{C}}^{\theta}$. Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$, there exist an open neighborhood $T_{\mathfrak{n}_{\mathbb{C}}} \times T_{\mathfrak{a}_{\mathbb{C}}}$ of $(0, 0)$ in $\mathfrak{n}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ such that the map

$$\mathfrak{n}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}} \ni (X, Y) \mapsto \exp X \exp Y \cdot x_0 \in G_{\mathbb{C}}/K_{\mathbb{C}}$$

is a biholomorphic diffeomorphism of $T_{\mathfrak{n}_{\mathbb{C}}} \times T_{\mathfrak{a}_{\mathbb{C}}}$ onto an open neighborhood \mathcal{V} of $x_0 = eK_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}$. We assume, as we may, that $T_{\mathfrak{n}_{\mathbb{C}}}$ and $T_{\mathfrak{a}_{\mathbb{C}}}$ are invariant under the complex conjugation with respect to the real form \mathfrak{g} .

We denote by \mathcal{V}^a the open set $\{x \mid x^{-1}K_{\mathbb{C}} \in \mathcal{V}\} \subset G_{\mathbb{C}}$. The map

$$K_{\mathbb{C}} \times T_{\mathfrak{a}_{\mathbb{C}}} \times T_{\mathfrak{n}_{\mathbb{C}}} \ni (k, Y, X) \mapsto k \exp Y \exp X \in \mathcal{V}^a \subset G_{\mathbb{C}}$$

is then a biholomorphic diffeomorphism.

In particular, the map $H: \mathcal{V}^a \rightarrow \mathfrak{a}_{\mathbb{C}}$ defined by

$$k \exp Y \exp X \mapsto Y$$

for $k \in K_{\mathbb{C}}$, $Y \in T_{\mathfrak{a}_{\mathbb{C}}}$ and $X \in T_{\mathfrak{n}_{\mathbb{C}}}$, is holomorphic and satisfies (2.2).

The conjugation with respect to \mathfrak{g} lifts to an involution of $G_{\mathbb{C}}$ that leaves G pointwise fixed. Moreover, since this conjugation commutes with θ , it stabilizes $K_{\mathbb{C}}$. Hence it stabilizes \mathcal{V}^a . Let $u \in \mathcal{V}^a \cap G$ and write $u = k \exp Y \exp X$ with $k \in K_{\mathbb{C}}$, $Y \in T_{\mathfrak{a}_{\mathbb{C}}}$ and $X \in T_{\mathfrak{n}_{\mathbb{C}}}$. It follows that k , Y and X are fixed by the conjugation. In particular, $Y \in \mathfrak{ia}$ and $X \in \mathfrak{n}$, and hence $k = u \exp(-X) \exp(-Y) \in G \cap K_{\mathbb{C}} = K_0$. Therefore, $u = k \exp Y \exp X$ is the Iwasawa decomposition, and $H(u) = Y$ the Iwasawa projection, of u .

We postpone the condition of right- K -invariance and consider the general case where $K_0 \subset K \subset U^{\theta}$. We retain the sets $T_{\mathfrak{n}_{\mathbb{C}}}$ and $T_{\mathfrak{a}_{\mathbb{C}}}$ from above and recall that $K_{\mathbb{C}} = K_0K$ is an open subgroup of the previous $K_{\mathbb{C}}$. Again we define $\mathcal{V}^a = K_{\mathbb{C}} \exp(T_{\mathfrak{a}_{\mathbb{C}}}) \exp(T_{\mathfrak{n}_{\mathbb{C}}})$. This is an open subset of the previous \mathcal{V}^a . The restriction of the previous H to this set is obviously holomorphic, agrees with Iwasawa on $\mathcal{V}^a \cap G$, and it is easily seen to satisfy (2.2).

Finally, we note that \mathcal{V}^a contains an $\text{Ad}K$ invariant open neighborhood V of e in $G_{\mathbb{C}}$. Hence, for each $k \in K$, the set $\mathcal{V}^a k$ is left- $K_{\mathbb{C}}$ -invariant and contains V . The intersection $\bigcap_{k \in K} \mathcal{V}^a k$ is $K_{\mathbb{C}} \times K$ invariant and contains V . The interior of this set has all the properties requested of \mathcal{V}^a . \square

We call the map in (2.1) the *complexified Iwasawa projection*. A particular set \mathcal{V}^a as above can be constructed as follows. Let

$$\Omega = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma) |\alpha(X)| < \pi/2\}.$$

The set

$$\mathcal{V} = \mathrm{Cr}(G/K_0) = G \exp \Omega K_{\mathbb{C}} \subset G_{\mathbb{C}}/K_{\mathbb{C}},$$

called the *complex crown* of G/K_0 , was introduced in [1]. Its preimage in $G_{\mathbb{C}}$ is open and contained in $N_{\mathbb{C}}A_{\mathbb{C}}K_{\mathbb{C}} \subset G_{\mathbb{C}}$. This is shown for all classical groups in [17], Theorem 1.8, and in general in [15], Theorem 3.21. See also [9], [18]. Let $\mathcal{V}^a = \{x^{-1} \mid x \in \mathcal{V}\} \subset G_{\mathbb{C}}$. The existence of the holomorphic Iwasawa projection $\mathcal{V}^a \rightarrow \mathfrak{a}_{\mathbb{C}}$ is established in [17], Theorem 1.8, with a proof that can be repeated in the general case. It follows that \mathcal{V}^a has all the properties mentioned in Lemma 2.1.

One important property of the crown is that it is G -invariant (it is in fact maximal with this property, see [2]). However, this property plays no role in the present article, where we shall just assume that \mathcal{V}^a has the properties in Lemma 2.1, and $\mathcal{V} = (\mathcal{V}^a)^{-1}$.

3. FOURIER ANALYSIS

In this section we develop a local Fourier theory for U/K based on elementary representation theory. The theory essentially originates from Sherman [23].

An irreducible unitary representation π of U is said to be *spherical* if there exists a non-zero K -fixed vector e_{π} in the representation space V_{π} . The vector e_{π} (if it exists) is unique up to multiplication by scalars. By normalizing it with unit length we obtain the matrix coefficient

$$\psi_{\pi}(u) = \langle \pi(u)e_{\pi}, e_{\pi} \rangle$$

which is the corresponding *zonal spherical function*.

From the point of view of representation theory it is natural to define the Fourier transform of an integrable function f on U/K to be the map that associates the vector

$$\pi(f)e_{\pi} = \int_U f(u \cdot x_0)\pi(u)e_{\pi} du = \int_{U/K} f(x)\pi(x)e_{\pi} dx \in V_{\pi},$$

to each spherical representation, with a fixed choice of the unit vector e_{π} for each π (see [20] for discussion on the noncompact case). The corresponding Fourier series is

$$(3.1) \quad \sum_{\pi} d(\pi) \langle \pi(f)e_{\pi}, \pi(x)e_{\pi} \rangle$$

for $x \in U/K$. It converges to f in L^2 if f belongs to $L^2(U/K)$, and it converges uniformly if f has a sufficient number of continuous derivatives (see [28]).

In the case of the sphere S^2 , the expansion of f in spherical harmonics $Y_l^m(x)$ (with integral labels $|m| \leq l$) is obtained from this expression when we express $\pi(x)e_{\pi}$ by means of an orthonormal basis for the $2l + 1$ -dimensional representation space $V_{\pi} = V_l$.

For the purpose of Fourier analysis it is convenient to embed all the representation spaces V_{π} in a common Hilbert space \mathcal{H} , independent of π , such that

\widehat{f} can be viewed as an \mathcal{H} -valued function on the set of equivalence classes of irreducible spherical representations. This can be achieved as follows.

Recall that in the classification of Helgason, a spherical representation $\pi = \pi_\mu$ is labeled by an element $\mu \in \mathfrak{a}_\mathbb{C}^*$, which is the restriction from a maximal torus of the highest weight of π (see [13], p. 538). We denote by $\Lambda^+(U/K) \subset \mathfrak{a}_\mathbb{C}^*$ the set of these restricted highest weights, so that $\mu \mapsto \pi_\mu$ sets up a bijection from $\Lambda^+(U/K)$ onto the set of equivalence classes of irreducible K -spherical representations. According to the theorem of Helgason, every $\mu \in \Lambda^+(U/K)$ satisfies

$$(3.2) \quad \frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+,$$

for all $\alpha \in \Sigma^+$, where the brackets denote the inner product induced by the Killing form. Furthermore, if U is simply connected, then an element $\mu \in \mathfrak{a}_\mathbb{C}^*$ belongs to $\Lambda^+(U/K)$ if and only if it satisfies (3.2). For the description in the general case, one must add to (3.2) both the assumption that π_μ descends to U , and that the K_0 -fixed vector is also K -fixed.

For each $\mu \in \Lambda^+(U/K)$ we fix an irreducible unitary spherical representation (π_μ, V_μ) of U and a unit K -fixed vector $e_\mu \in V_\mu$. Furthermore, we fix a highest weight vector v_μ of weight μ , such that $\langle v_\mu, e_\mu \rangle = 1$. The following lemma is proved in [13], p. 535, in the case that U is simply connected.

Lemma 3.1. *Let $\mu \in \Lambda^+(U/K)$. Then $\pi_\mu(m)v_\mu = v_\mu$ for all $m \in M$, and the vectors $\pi_\mu(k)v_\mu$, where $k \in K_0$, span the space V_μ .*

Proof. Let $m \in M$ be given. Since m centralizes \mathfrak{a} and normalizes \mathfrak{n} , it follows that $\pi_\mu(m)v_\mu$ is again a highest weight vector of the same weight. Hence $\pi_\mu(m)v_\mu = cv_\mu$. By taking inner products with e_μ , which is M -fixed, it follows that $c = 1$. The statement about the span follows directly from the Iwasawa decomposition $G = K_0AN$. \square

It follows from Lemma 3.1 that the map $V_\mu \rightarrow L^2(K/M)$, $v \mapsto \langle v, \pi_\mu(\cdot)v_\mu \rangle$, is an injective K -map. We use the space $\mathcal{H} = L^2(K/M)$ as our common model for the spherical representations, and accordingly we define the *Fourier transform* of an integrable function f on U/K by

$$(3.3) \quad \begin{aligned} \tilde{f}(\mu, b) &= \int_{U/K} f(u) \overline{\langle \pi_\mu(u)e_\mu, \pi_\mu(k)v_\mu \rangle} du \\ &= \int_{U/K} f(u) \langle \pi_\mu(k)v_\mu, \pi_\mu(u)e_\mu \rangle du, \end{aligned}$$

for $\mu \in \Lambda^+(U/K)$ and $b = kM \in K/M$. We write $\tilde{f}(\mu)$ for the function $b \mapsto \tilde{f}(\mu, b)$ on K/M . If f is K -invariant, then $\tilde{f}(\mu)$ is independent of b . Integration over K then shows that this definition agrees with the spherical Fourier transform in (1.2).

It is easily seen that the Fourier transform $f \mapsto \tilde{f}(\mu)$ is intertwining for the left regular actions of K on U/K and K/M , respectively. In particular, it maps K -finite functions on U/K to K -finite functions on K/M .

We now invoke the complex group $G_{\mathbb{C}}$ and the complexified Iwasawa projection defined in the preceding section. Let $\mathcal{V}^a \subset G_{\mathbb{C}}$ and $H: \mathcal{V}^a \rightarrow \mathfrak{a}_{\mathbb{C}}$ be as in Lemma 2.1, and let $\mu \in \Lambda^+(U/K)$. Since π_{μ} extends to a holomorphic representation of $G_{\mathbb{C}}$, it follows from Lemma 2.1 that $\langle \pi_{\mu}(u)v_{\mu}, e_{\mu} \rangle = e^{\mu(H(u))}$ for all $u \in \mathcal{V}^a$. Let $\mathcal{V} = \{x^{-1} \mid x \in \mathcal{V}^a\} \subset G_{\mathbb{C}}$. Then

$$(3.4) \quad \langle \pi_{\mu}(k)v_{\mu}, \pi_{\mu}(u)e_{\mu} \rangle = e^{\mu(H(u^{-1}k))}$$

for $k \in K$, $u \in U \cap \mathcal{V}$ and $\mu \in \Lambda^+(U/K)$.

Lemma 3.2. *Let f be an integrable function on U/K with support in $U \cap \mathcal{V}$. Then*

$$(3.5) \quad \tilde{f}(\mu, k) = \int_{U/K} f(u) e^{\mu(H(u^{-1}k))} du,$$

for all $k \in K/M$, and the Fourier transform $\mu \mapsto \tilde{f}(\mu)$ extends to a holomorphic \mathcal{H} -valued function on $\mathfrak{a}_{\mathbb{C}}^*$, also denoted by \tilde{f} , satisfying the same equation (3.5). Moreover,

$$(3.6) \quad \pi_{\mu}(f)e_{\mu} = \int_{K/M} \tilde{f}(-\mu - 2\rho, b)\pi_{\mu}(k)v_{\mu} dk$$

for all $\mu \in \Lambda^+(U/K)$.

The measure on K/M used in (3.6) is the quotient of the normalized Haar measures on K and M .

Proof. The expression (3.5) follows immediately from (3.3) and (3.4). The integrand in (3.5) depends holomorphically on μ , locally uniformly with respect to u and k . Hence an analytic continuation is defined by this formula.

In order to establish the identity (3.6) it suffices to show that

$$\pi_{\mu}(u)e_{\mu} = \int_{K/M} e^{-(\mu+2\rho)H(u^{-1}k)} \pi_{\mu}(k)v_{\mu} dk$$

for $u \in U \cap \mathcal{V}$. The latter identity is easily shown to hold for $u \in G$ (use [13], p. 197, Lemma 5.19, and the fact that $K/M = K_0/(M \cap K_0)$). By analytic continuation it then holds for $u \in \mathcal{V}_0$, the identity component of \mathcal{V} . Since $\mathcal{V} = \mathcal{V}_0 K_{\mathbb{C}}$, it follows for all $u \in \mathcal{V}$. \square

Corollary 3.3. (Sherman) *Assume $f \in L^2(U/K)$ has support contained in $U \cap \mathcal{V}$. Then the sum*

$$\sum_{\mu \in \Lambda^+(U/K)} d(\mu) \int_{K/M} \tilde{f}(-\mu - 2\rho, k) \langle \pi_{\mu}(k)v_{\mu}, \pi_{\mu}(x)e_{\mu} \rangle dk, \quad x \in U/K,$$

converges to f in $L^2(U/K)$, and it converges uniformly if f has a sufficient number of continuous derivatives.

Proof. (See [23]). Follows immediately from (3.1) by insertion of (3.6). \square

In [24] the inversion formula of Corollary 3.3 is extended to global functions on U/K (for symmetric spaces of rank one). We shall not use this extension here. For the special case of the sphere $U/K = S^n$, see also [22], [26] and [31].

4. THE SPHERICAL PRINCIPAL SERIES

The space $\mathcal{H} = L^2(K/M) = L^2(K_0/(M \cap K_0))$ is the representation space for the spherical principal series for G . We denote by σ_λ this series of representations, given by

$$(4.1) \quad [\sigma_\lambda(g)\psi](k) = e^{-(\lambda+\rho)H(g^{-1}k)}\psi(\kappa(g^{-1}k))$$

for $\lambda \in \mathfrak{a}_\mathbb{C}^*$, $g \in G$, $\psi \in \mathcal{H}$ and $k \in K_0$. Here $\kappa: G \rightarrow K_0$ is the Iwasawa projection $kan \mapsto k$.

The embedding $v \mapsto \langle \pi_\mu(\cdot)v_\mu, v \rangle$ of V_μ into \mathcal{H} , used in the definition (3.3) of the Fourier transform, is related to the principal series representations. By extending each π_μ to a holomorphic representation of $G_\mathbb{C}$ and then restricting to G , we obtain a representation of G , which we again denote by π_μ . For $\mu \in \Lambda^+(U/K)$ define

$$(4.2) \quad h_v(k) = \langle \pi_\mu(k)v_\mu, v \rangle, \quad (k \in K).$$

We now have the following well-known result:

Lemma 4.1. *Let $\mu \in \Lambda^+(U/K)$. The antilinear map $v \mapsto h_v$ provides a G -equivariant embedding of the contragredient of the finite dimensional representation (π_μ, V_μ) into $\sigma_{-\mu-\rho}$.*

Proof. Since v_μ is a highest weight vector it follows easily from (4.1) that

$$\sigma_{-\mu-\rho}(g)h_v = h_{\pi_\mu(g^{-1})^*v}$$

for $g \in G$. □

The space $C^\infty(K/M) \subset \mathcal{H}$ carries the family of representations, also denoted by σ_λ , of $\mathfrak{g}_\mathbb{C}$ obtained by differentiation and complexification. Thus, although the representations σ_λ of G in general do not complexify to global representations of U , the infinitesimal representations σ_λ of \mathfrak{u} are defined for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$. We denote by $\mathcal{H}_\lambda^\infty$ the space $C^\infty(K/M)$ equipped with the representation σ_λ of $\mathfrak{u}_\mathbb{C} = \mathfrak{g}_\mathbb{C}$, and with the left regular representation of K .

Lemma 4.2. *The Fourier transform $f \mapsto \tilde{f}(\mu)$ defines a (\mathfrak{u}, K) -homomorphism from $C^\infty(U/K)$ to $\mathcal{H}_{-\mu-\rho}^\infty$, for all $\mu \in \Lambda^+(U/K)$. Moreover, the holomorphic extension, defined in Lemma 3.2, restricts to a (\mathfrak{u}, K) -homomorphism from*

$$\{f \in C^\infty(U/K) \mid \text{supp } f \subset U \cap \mathcal{V}\}$$

to $\mathcal{H}_{-\mu-\rho}^\infty$ for all $\mu \in \mathfrak{a}_\mathbb{C}^$.*

Proof. Since π_μ is a unitary representation of U it follows from Lemma 4.1 that $\sigma_{-\mu-\rho}(X)h_v = h_{\pi_\mu(X)v}$ for $X \in \mathfrak{u}$, $v \in V_\mu$. The first statement now follows, since

$$\tilde{f}(\mu) = \int_{U/K} f(u)h_{\pi_\mu(u)e_\mu} du.$$

It follows from Lemma 5.2 and Theorem 6.1 below, that the second statement can be derived from the first by analytic continuation with respect to μ , provided the support of f is sufficiently small. However, we prefer to give an independent proof, which only requires assumptions on the support of f as stated in the lemma.

Since the Fourier transform in (3.5) is clearly K -equivariant, it suffices to prove the intertwining property

$$(4.3) \quad [L(X)f]^\sim(\mu) = \sigma_{-\mu-\rho}(X)\tilde{f}(\mu)$$

for $X \in \mathfrak{q}$. By definition

$$[L(X)f](u) = \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tX)u)$$

and hence by invariance of the measure

$$[L(X)f]^\sim(\mu, k) = \int_{U/K} f(u) \left. \frac{d}{dt} \right|_{t=0} e^{\mu(H(u^{-1} \exp(-tX)k))} du.$$

Let $\mathfrak{p} = i\mathfrak{q}$ so that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of \mathfrak{g} , and write $X = iY$ for $Y \in \mathfrak{p}$. Since the complexified Iwasawa map H is holomorphic, it follows that

$$\left. \frac{d}{dt} \right|_{t=0} e^{\mu(H(u^{-1} \exp(-tX)k))} = i \left. \frac{d}{dt} \right|_{t=0} e^{\mu(H(u^{-1} \exp(-tY)k))}.$$

Furthermore

$$H(u^{-1} \exp(-tY)k) = H(u^{-1} \kappa(\exp(-tY)k)) + H(\exp(-tY)k)$$

and hence we derive

$$\begin{aligned} & [L(X)f]^\sim(\mu, k) \\ &= i \left. \frac{d}{dt} \right|_{t=0} \left[e^{\mu(H(\exp(-tY)k))} \int_{U/K} f(u) e^{\mu(H(u^{-1} \kappa(\exp(-tY)k))} du \right] \\ &= i \left. \frac{d}{dt} \right|_{t=0} \left[e^{\mu(H(\exp(-tY)k))} \tilde{f}(\mu, \kappa(\exp(-tY)k)) \right] \\ &= i \left. \frac{d}{dt} \right|_{t=0} \left[\sigma_{-\mu-\rho}(\exp(tY)) \tilde{f}(\mu) \right] (k). \end{aligned}$$

Since by definition $\sigma_{-\mu-\rho}(X) = i\sigma_{-\mu-\rho}(Y)$, the last expression is exactly $\sigma_{-\mu-\rho}(X)\tilde{f}(\mu)$ evaluated at k . \square

We recall that there exist normalized standard intertwining operators between the principal series:

$$\mathcal{A}(w, \lambda): \mathcal{H} \rightarrow \mathcal{H}, \quad w \in W,$$

such that

$$(4.4) \quad \sigma_{w\lambda}(g) \circ \mathcal{A}(w, \lambda) = \mathcal{A}(w, \lambda) \circ \sigma_\lambda(g)$$

for all $g \in G$. The normalization is such that

$$(4.5) \quad \mathcal{A}(w, \lambda)1 = 1$$

for the constant function 1 on K/M . The map $\lambda \mapsto \mathcal{A}(w, \lambda)$ is meromorphic with values in the space of bounded linear operators on \mathcal{H} .

We need the following property of the *Poisson kernel*, which is defined for $x \in G$ and $k \in K_0$ by $e^{-(\lambda+\rho)H(x^{-1}k)}$. By Lemma 2.1 it is defined also for $x \in \mathcal{V}$ and $k \in K$.

Lemma 4.3. *The identity*

$$(4.6) \quad \mathcal{A}(w, \lambda)e^{-(\lambda+\rho)H(x^{-1}\cdot)} = e^{-(w\lambda+\rho)H(x^{-1}\cdot)},$$

of functions in \mathcal{H} , holds for all $x \in \mathcal{V}$.

Proof. The identity is well-known for $x \in G$. In fact in this case it follows easily from (4.1), (4.4) and (4.5). The map $x \mapsto e^{\mu(H(x^{-1}\cdot))}$ is holomorphic \mathcal{H} -valued on \mathcal{V} for each $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, because the complexified Iwasawa projection is holomorphic. Hence (4.6) holds for $x \in \mathcal{V}_0$ by analytic continuation, and then for $x \in \mathcal{V}$ by the obvious left- $K_{\mathbb{C}}$ -invariance of both sides with respect to x^{-1} . \square

5. THE K -FINITE PALEY-WIENER SPACE

For each irreducible representation δ of K_0 we denote by \mathcal{H}_{δ} the finite dimensional subspace of \mathcal{H} consisting of the functions that generate an isotypical representation of type δ . Likewise, for each finite set F of K_0 -types, we denote by \mathcal{H}_F the sum of the spaces \mathcal{H}_{δ} for $\delta \in F$. Obviously, the intertwining operators $\mathcal{A}(w, \lambda)$ preserve each subspace \mathcal{H}_F . Although we do not need it in the sequel, we remark that $\lambda \mapsto \mathcal{A}(w, \lambda)|_{\mathcal{H}_F}$ is a rational map from $\mathfrak{a}_{\mathbb{C}}^*$ into the space of linear operators on the finite dimensional space \mathcal{H}_F , for each F , see [30].

Note that since K/K_0 is finite, a function on $K/M = K_0/(K_0 \cap M)$ is K_0 -finite if and only if it is K -finite. We use the notations \mathcal{H}_{δ} and \mathcal{H}_F also for an irreducible representation δ of K , and for a set F of K -types.

Definition 5.1. For $r > 0$ the K -finite Paley-Wiener space $\text{PW}_{K,r}(\mathfrak{a})$ is the space of holomorphic functions φ on $\mathfrak{a}_{\mathbb{C}}^*$ with values in $\mathcal{H} = L^2(K/M)$ satisfying the following.

- (a) There exist a finite set F of K -types such that $\varphi(\lambda) \in \mathcal{H}_F$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.
- (b) For each $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$\|\varphi(\lambda)\| \leq C_k(1 + |\lambda|)^{-k} e^{r|\text{Re } \lambda|}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

- (c) The identity $\varphi(w(\mu + \rho) - \rho) = \mathcal{A}(w, -\mu - \rho)\varphi(\mu)$ holds for all $w \in W$, and for generic $\mu \in \mathfrak{a}_{\mathbb{C}}^*$.

We note that the norm on $\mathfrak{a}_{\mathbb{C}}$ used in (b) is induced by the negative of the Killing form on \mathfrak{a} . In particular we see that $\text{PW}_{K,r}(\mathfrak{a}) = \text{PW}_{K_0,r}(\mathfrak{a})$, that is, the K -finite Paley-Wiener space is the same for all the spaces U/K where $K_0 \subset K \subset U^{\theta}$.

Notice that the Paley-Wiener space $\text{PW}_r(\mathfrak{a})$ defined in [19] can be identified with the space of functions φ in $\text{PW}_{K,r}(\mathfrak{a})$, for which $\varphi(\lambda)$ is a constant function on K/M for each λ . This follows from the normalization (4.5).

The functions in the Paley-Wiener space are uniquely determined by their restriction to $\Lambda^+(U/K)$, at least when r is sufficiently small. This is seen in the following lemma.

Lemma 5.2. *There exists $R > 0$ such that if $\varphi \in \text{PW}_{K,r}(\mathfrak{a})$ for some $r < R$ and $\varphi(\mu) = 0$ for all $\mu \in \Lambda^+(U/K)$, then $\varphi = 0$.*

Proof. The relevant value of R is the same as in [19] Thm. 4.2 (iii) and Remark 4.3. The lemma follows easily from application of [19], Section 7, to the function $\lambda \mapsto \langle \varphi(\lambda, \cdot), \psi \rangle$ for each $\psi \in \mathcal{H}$. \square

Obviously $\text{PW}_{K,r}(\mathfrak{a})$ is K -invariant, where K acts by the left regular representation on functions on K/M . The following lemma shows that it is also a (\mathfrak{u}, K) -module.

Lemma 5.3. *Let $r > 0$, $\varphi \in \text{PW}_{K,r}(\mathfrak{a})$ and $X \in \mathfrak{u}_{\mathbb{C}}$. Then the function $\psi = \sigma(X)\varphi$ defined by*

$$\psi(\lambda) = \sigma_{-\lambda-\rho}(X)(\varphi(\lambda)) \in \mathcal{H}$$

for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, belongs to $\text{PW}_{K,r}(\mathfrak{a})$.

Proof. Recall that $\sigma_{-\lambda-\rho}(X)$ is defined by complexification of the infinitesimal action of \mathfrak{g} on the smooth functions in \mathcal{H} , and note that $\varphi(\lambda)$ is smooth on K/M , since it is K -finite. Hence we may assume $X \in \mathfrak{g}$. It is easily seen that $\psi(\lambda)$ is K_0 -finite of types, which occur in the tensor product of the adjoint representation Ad of K_0 on \mathfrak{g} with types from F . Hence condition (a) is valid for the function ψ . Condition (c) follows immediately from the intertwining property of $\mathcal{A}(w, \lambda)$. It remains to verify holomorphicity in λ , and the estimate in (b) for ψ .

By definition both the holomorphicity and norm in the estimate (b) refer to the Hilbert space $\mathcal{H} = L^2(K/M)$. However, because of condition (a) and since \mathcal{H}_F is finite dimensional, it is equivalent to require holomorphicity of $\psi(\lambda)(x)$ pointwise for each $x \in K/M$, and likewise to require the exponential estimate for $\psi(\lambda)(x)$ pointwise with respect to x . Thus let an element $x = kM \in K/M$ be fixed, where $k \in K_0$.

Note that by (4.1)

$$(\sigma(X)\varphi)(\lambda)(k) = \left. \frac{d}{dt} \right|_{t=0} e^{-(\lambda+\rho)(H(\exp(-tX)k))} \varphi(\lambda)(\kappa(\exp(-tX)k)).$$

Differentiating with the Leibniz rule, we obtain a sum of two terms.

The first term is

$$(5.1) \quad \left. \frac{d}{dt} \right|_{t=0} \left(e^{-(\lambda+\rho)(H(\exp(-tX)k))} \right) \varphi(\lambda)(k).$$

Let $\alpha(Z) = H(\exp(Z)k) \in i\mathfrak{a}$ for $Z \in \mathfrak{g}$, then $\alpha(0) = 0$ and it follows that (5.1) equals

$$(\lambda + \rho)(d\alpha_0(X)) \varphi(\lambda)(k)$$

where $d\alpha_0$ is the differential of α at 0. It is now obvious that (5.1) is holomorphic and satisfies the same the growth estimate as $\varphi(\lambda)(k)$. Hence (b) is valid for the first term.

The second term is

$$(5.2) \quad \left. \frac{d}{dt} \right|_{t=0} \varphi(\lambda)(\kappa(\exp(-tX)k)),$$

which we rewrite as follows. Let

$$\beta(Z) = \kappa(\exp(Z)k)k^{-1} \in K_0$$

for $Z \in \mathfrak{g}$, then $\beta(0) = e$ and

$$\varphi(\lambda)(\kappa(\exp(-tX)k)) = \varphi(\lambda)(\beta(-tX)k).$$

It follows that (5.2) equals

$$L(d\beta_0(X))(\varphi(\lambda))(k)$$

where $d\beta_0(X) \in T_e K_0 = \mathfrak{k}$. The linear operator $L(d\beta_0(X))$ preserves the finite dimensional space \mathcal{H}_F and hence restricts to a bounded linear operator on that space. It follows that (5.2) is holomorphic in λ and satisfies (b). \square

6. FOURIER TRANSFORM MAPS INTO PALEY-WIENER SPACE

In this section we prove the following result. Let $C_K^\infty(U/K)$ denote the space of K -finite smooth functions on U/K , and for each $r > 0$ let

$$C_{K,r}^\infty(U/K) = \{f \in C_K^\infty(U/K) \mid \text{supp } f \subset \text{Exp}(\bar{B}_r(0))\}$$

where $\bar{B}_r(0)$ denotes the closed ball in \mathfrak{q} of radius r and center 0, and Exp denotes the exponential map of U/K .

Theorem 6.1. *There exists a number $R > 0$ such that $\text{Exp}(\bar{B}_R(0)) \subset U \cap \mathcal{V}$ and such that the following holds for every $r < R$:*

If $f \in C_{K,r}^\infty(U/K)$, then the holomorphic extension of \tilde{f} from Lemma 3.2 belongs to $\text{PW}_{K,r}(\mathfrak{a})$.

In the proof we shall reduce to the case where $K = K_0$. The following lemma prepares the way for this reduction.

The projection $p : U/K_0 \rightarrow U/K$ is a covering map. Hence we can choose $R > 0$ such that p restricts to a diffeomorphism of the open ball $\text{Exp}(B_R(0))$ in U/K_0 onto the open ball $\text{Exp}(B_R(0))$ in U/K . It follows that for each $r < R$ a bijection $F \mapsto f$ of $C_{K_0,r}^\infty(U/K_0)$ onto $C_{K,r}^\infty(U/K)$ is defined by

$$f(u) = \sum_{v \in K/K_0} F(uv), \quad u \in U$$

for $F \in C_{K_0,r}^\infty(U/K_0)$, where for each u at most one term is non-zero. The inverse map is given by

$$F(u) = \begin{cases} f(p(u)), & u \in \text{Exp}(B_R(0)), \\ 0, & \text{otherwise,} \end{cases}$$

for $f \in C_{K,r}^\infty(U/K)$. Let $\mathcal{V}^a \subset G_{\mathbb{C}}$ be as in Lemma 2.1, and note that this set also satisfies the assumptions of that lemma for the symmetric space U/K_0 . As before, let $\mathcal{V} = \{x^{-1} \mid x \in \mathcal{V}^a\}$.

Lemma 6.2. *Let $f \in C_{K,r}^\infty(U/K)$ and $F \in C_{K_0,r}^\infty(U/K_0)$ be as above. Then f is supported in $U \cap \mathcal{V}$ if and only if F is supported in $U \cap \mathcal{V}$. In this case, the analytically continued Fourier transforms of these functions satisfy*

$$\tilde{f}(\mu) = c\tilde{F}(\mu)$$

for all $\mu \in \mathfrak{a}_{\mathbb{C}}^*$, where c is the index of K/K_0 .

Proof. It follows from the definition of the map $F \mapsto f$ that

$$\tilde{f}(\mu, k) = \int_U \sum_{v \in K/K_0} F(uv) e^{\mu(H(u^{-1}k))} du = c\tilde{F}(\mu, k)$$

by right- K -invariance of the Haar measure and left- K -invariance of H . \square

We can now give the proof of Theorem 6.1.

Proof. Property (a) in Definition 5.1 follows immediately from the fact that the Fourier transform is K -equivariant. Moreover, the transformation law for the Weyl group in Property (c) follows easily from Lemma 4.3 by integration over $U \cap \mathcal{V}$ against $f(u)$.

For the proof of Property (b), with r bounded by a suitable value R , we reduce to the case that K is connected. We assume that R is sufficiently small as described above Lemma 6.2. Then according to the lemma, given a function $f \in C_{K,r}^\infty(U/K)$, the function $F \in C_{K_0,r}^\infty(U/K_0)$ has the same Fourier transform up to a constant. The reduction now follows since $\text{PW}_{K,r}(\mathfrak{a}) = \text{PW}_{K_0,r}(\mathfrak{a})$, as mentioned below Definition 5.1. For the rest of this proof we assume $K = K_0$.

It is known from [19], Thm. 4.2(i), that the estimate in Property (b) holds for K -invariant functions on U/K . We prove the property in general by reduction to that case. In particular, we can use the same value of $R > 0$ (see [19], Remark 4.3).

Fix an irreducible K -type δ . It suffices to prove the result for functions f that transform isotypically under K according to this type.

We shall use Kostant's description in [16] of the K -types in the spherical principal series. We draw the results we need directly from the exposition in [12], Chapter 3. In particular, we denote by H_δ^* the finite dimensional subspace of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ which is the image under symmetrization of the space of harmonic polynomials on \mathfrak{p} of type δ , and we denote by E_δ the space

$$E_\delta = \text{Hom}_K(V_\delta, H_\delta^*),$$

of linear K -intertwining maps $V_\delta \rightarrow H_\delta^*$. It is known that E_δ has the same dimension as V_δ^M .

We denote by $\text{Hom}^*(V_\delta^M, E_\delta)$ the space of *anti-linear* maps $V_\delta^M \rightarrow E_\delta$. The principal result we need is Theorem 2.12 of [12], p. 250, according to which there exists a rational function $P = P^\delta$ on $\mathfrak{a}_\mathbb{C}^*$ with values in $\text{Hom}^*(V_\delta^M, E_\delta)$ such that

$$(6.1) \quad \int_{K/M} e^{-(\lambda+\rho)H(x^{-1}k)} \langle v, \delta(k)v' \rangle dk = [L(P(\lambda)(v')(v))\varphi_\lambda](x)$$

for all $v \in V_\delta$, $v' \in V_\delta^M$ and $x \in G/K$. Here L denotes the action of the enveloping algebra from the left on functions on G/K , and φ_λ denotes the spherical function

$$\varphi_\lambda(x) = \int_{K/M} e^{-(\lambda+\rho)H(x^{-1}k)} dk$$

on G/K .

The equality (6.1) is valid for $x \in U \cap \mathcal{V}$ by analytic continuation. Let $f \in C_{K,r}^\infty(U/K)_\delta$, where $r < R$, then

$$f(x) = d(\delta) \int_K \chi_\delta(l) f(l^{-1}x) dl$$

for all $x \in U$, where χ_δ is the character of δ . It follows that

$$\tilde{f}(\mu, k) = d(\delta) \int_{U/K} \int_K \chi_\delta(l) f(l^{-1}u) dl e^{\mu(H(u^{-1}k))} du$$

and hence by Fubini and invariance of measures

$$\tilde{f}(\mu, k) = d(\delta) \int_{U/K} \int_{K/M} \int_M \chi_\delta(kml^{-1}) dm e^{\mu(H(u^{-1}l))} dl f(u) du.$$

The inner expression $\int_M \chi_\delta(kml^{-1}) dm$ is a finite sum of matrix coefficients of the form $\langle \delta(k)v, \delta(l)v' \rangle$ with $v \in V_\delta$ and $v' \in V_\delta^M$, and hence it follows from (6.1) that $\tilde{f}(\mu, k)$ is a finite sum of expressions of the form

$$\int_{U/K} [L(P(-\mu - \rho)(v'))(\delta(k)v)] \varphi_{-\mu-\rho}(u) f(u) du$$

with v and v' independent of μ and k . In these expressions the right invariant differential operators $L(P(-\mu - \rho)(v'))(\delta(k)v)$ can be thrown over, by taking adjoints, so that we obtain

$$(6.2) \quad \int_{U/K} \varphi_{-\mu-\rho}(u) [L(P(-\mu - \rho)(v'))(\delta(k)v)]^* f(u) du.$$

Notice that (6.2) is exactly the spherical Fourier transform from [19], Section 6, of the function in square brackets averaged over K . It follows that $\tilde{f}(\mu, k)$ is a finite sum in which each term has the form of the spherical Fourier transform applied to an averaged derivative of f by a differential operator with coefficients that depends rationally on μ (more precisely, the element $P(-\mu - \rho)(v')$ in the finite dimensional space E_δ depends rationally on μ). The application of a differential operator to f does not increase the support, hence it follows from the estimates in [19] that each term is a rational multiple of a function of μ of exponential type. It then follows from [12] Lemma 5.13, p. 288, that the Fourier transform $\tilde{f}(\mu, k)$ itself is of the same exponential type. We have established Property (b) in Definition 5.1 for \tilde{f} . □

7. FOURIER TRANSFORM MAPS ONTO PALEY-WIENER SPACE

Let $\varphi \in \text{PW}_{K,r}(\mathfrak{a})$ for some $r > 0$ and consider the function f on U/K defined by the Fourier series

$$(7.1) \quad f(x) = \sum_{\mu \in \Lambda^+(U/K)} d(\mu) \int_{K/M} \varphi(-\mu - 2\rho, k) \langle \pi_\mu(k)v_\mu, \pi_\mu(x)e_\mu \rangle dk.$$

It follows from the estimate in Property (b) of Definition 5.1 that the sum converges and defines a smooth function on U/K (see [27]).

Theorem 7.1. *There exists a number $R > 0$ such that $\text{Exp}(\bar{B}_R(0)) \subset U \cap \mathcal{V}$ and such that the following holds for every $r < R$. For each $\varphi \in \text{PW}_{K,r}(\mathfrak{a})$ the function f on U/K defined by (7.1) belongs to $C_{K,r}^\infty(U/K)$ and has Fourier transform $\tilde{f} = \varphi$.*

Proof. Again we first reduce to the case that K is connected. Assuming that the theorem is valid in that case, we find a number $R > 0$ such that every function $\varphi \in \text{PW}_{K_0,r}(\mathfrak{a})$, where $r < R$, is of the form \tilde{F} for some $F \in C_{K_0,r}^\infty(U/K_0)$. We may assume that R is as small as explained above Lemma 6.2. Let $\varphi \in \text{PW}_{K,r}(\mathfrak{a})$ be given and recall that $\text{PW}_{K,r}(\mathfrak{a}) = \text{PW}_{K_0,r}(\mathfrak{a})$. Let $F \in C_{K_0,r}^\infty(U/K_0)$ with $\tilde{F} = c^{-1}\varphi$, and construct $f \in C_{K,r}^\infty(U/K)$ as in Lemma 6.2. It follows from the lemma that $\tilde{f} = c\tilde{F} = \varphi$, and then it follows from Corollary 3.3 that f is the function given by (7.1). This completes the reduction.

For the rest of this proof, we assume that $K = K_0$. The value of R that we shall use is the same as in [19], Thm. 4.2(ii) and Remark 4.3. We may assume that $\varphi(\lambda, \cdot)$ is isotypical of a given K -type δ for all $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

For $v \in V_\delta$ and $v' \in V_\delta^M$ we denote by $\psi_{v,v'}$ the matrix coefficient

$$\psi_{v,v'}(k) = \langle v, \delta(k)v' \rangle$$

on K/M . By the Frobenius reciprocity theorem it follows that these functions $\psi_{v,v'}$ span the space \mathcal{H}_δ . Moreover, it follows from the definition of the standard intertwining operators by means of integrals over quotients of $\theta(N)$, that these operators act on each function $\psi_{v,v'}$ only through the second variable. That is, there exists a linear map

$$B(w, \lambda): V_\delta^M \rightarrow V_\delta^M$$

such that

$$(7.2) \quad \mathcal{A}(w, \lambda)\psi_{v,v'} = \psi_{v, B(w, \lambda)v'}$$

for all v, v' . Notice that the dependence of $B(w, \lambda)$ on λ is anti-meromorphic.

It follows (by using a basis for V_δ) that we can write $\varphi(\mu, k)$ as a finite sum of functions of the form

$$\psi_{v, v'(\mu)}(k)$$

where $v \in V_\delta$ is fixed and where $v': \mathfrak{a}_\mathbb{C}^* \rightarrow V_\delta^M$ is anti-holomorphic of exponential type r and satisfies the transformation relation in Definition 5.1 (c), that is,

$$(7.3) \quad v'(w(\mu + \rho) - \rho) = B(w, -\mu - \rho)v'(\mu)$$

for $w \in W$.

Since the Poisson transformation for G/K is equivariant for the left action and injective for generic λ , it follows from (6.1), by applying the inverse Poisson transform on both sides, that

$$(7.4) \quad \psi_{v,v'} = \sigma_\lambda(P(\lambda)(v')(v))1$$

for all $v \in V_\delta$, $v' \in V_\delta^M$ (see also [12], Thm. 3.1, p. 251), and for generic values of λ . Here 1 denotes the constant function with value 1 on K/M . We apply

(7.4) for $\lambda = -\mu - \rho$ to the function $\psi_{v, v'(\mu)}$ and thus obtain our Paley-Wiener function $\varphi(\mu, \cdot)$ as a finite sum of elements of the form

$$\sigma_{-\mu-\rho}(P(-\mu - \rho)(v'(\mu))(v))1.$$

The functions $P: \mathfrak{a}_{\mathbb{C}}^* \rightarrow \text{Hom}^*(V_{\lambda}^M, E_{\delta})$ satisfy the following transformation property

$$(7.5) \quad P(w\lambda) \circ B(w, \lambda) = P(\lambda).$$

Indeed, it follows from (7.4), (7.2) and (4.4) that

$$\sigma_{w\lambda}(P(w\lambda)(B(w, \lambda)v')(v))1 = \sigma_{w\lambda}(P(\lambda)(v')(v))1$$

for all v and v' , and generic λ . The identity (7.5) follows, since the map $u \mapsto \sigma_{\nu}(u)1$ is injective from H_{δ}^* to \mathcal{H} for generic ν according to [12], Thm. 3.1, p. 251 (alternatively, (7.5) follows from [12], Thm. 3.5, p. 254).

It follows from (7.5) combined with (7.3) that the function

$$\mu \mapsto u(\mu) := P(-\mu - \rho)(v'(\mu))(v) \in H_{\delta}^*$$

satisfies $u(w(\mu+\rho)-\rho) = u(\mu)$ for all μ , that is, the shifted function $\lambda \mapsto u(\lambda-\rho)$ is W -invariant. Notice that u is a rational multiple of a holomorphic function of μ , since $P(-\mu - \rho)$ is antilinear in v' , and v' is antiholomorphic in μ .

It follows from [12], Prop. 4.1, p. 264, that $\lambda \mapsto P(-\lambda)$ is non-singular on an open neighborhood of the set where

$$\text{Re} \langle \lambda, \alpha \rangle \geq 0$$

for all roots $\alpha \in \Sigma^+$. Hence $u(\lambda - \rho)$ is holomorphic on this set. By the above-mentioned W -invariance the function is then holomorphic everywhere. Since it is a rational multiple of a function of exponential type r , we conclude from [12], Lemma 5.13, p. 288, that it has exponential type r .

Since H_{δ}^* is finite dimensional we thus obtain an expression for $\varphi(\lambda, \cdot)$ as a finite sum of functions of the form

$$\varphi_i(\lambda)\sigma_{\lambda}(u_i)1,$$

with scalar valued functions φ_i on $\mathfrak{a}_{\mathbb{C}}^*$ which are W -invariant (for the action twisted by ρ) and of exponential type r , and with $u_i \in H_{\delta}^*$.

According to the theorem proved in [19], each function φ_i is the spherical Fourier transform of a K -invariant smooth function $f_i \in C_r^{\infty}(U/K)$. The function $L(u_i)f_i$ also belongs to $C_r^{\infty}(U/K)$, and by Lemma 4.2 it has Fourier transform $\varphi_i(\lambda)\sigma_{\lambda}(u_i)1$. We conclude that if f is the sum of the $L(u_i)f_i$, then $\tilde{f} = \varphi$, as desired.

Finally, it follows from Corollary 3.3 that f is identical to the function given by the Fourier series (7.1). \square

We combine Theorems 6.1 and 7.1 to obtain the following.

Theorem 7.2. *There exists a number $R > 0$ such that the Fourier transform is a bijection of $C_{K,r}^{\infty}(U/K)$ onto $\text{PW}_{K,r}(\mathfrak{a})$ for all $r < R$.*

We note the following corollary, which is analogous to a result of Torasso in the non-compact case (see [12], Cor. 5.19, p. 291).

Corollary 7.3. *There exists $r > 0$ such that each function in $C_{K,r}^\infty(U/K)$ is a finite linear combination of derivatives of K -invariant functions in $C_r^\infty(U/K)$ by members of $\mathcal{U}(\mathfrak{g})$, acting from the left.*

Proof. More precisely, the proof above shows that if $f \in C_{K,r}^\infty(U/K)$ is K -finite of isotype δ , then $f = \sum_i L(u_i)f_i$ with $u_i \in H_\delta^*$ and $f_i \in C_r^\infty(U/K)^K$.

8. FINAL REMARKS

Every function $f \in C^\infty(U/K)$ can be expanded in a sum of K -types,

$$(8.1) \quad f = \sum_{\delta \in \hat{K}} f_\delta$$

where $f_\delta \in C_\delta^\infty(U/K)$ is obtained from f by left convolution with the character of δ (suitably normalized). It is easily seen that f is supported in a given closed geodesic ball B around x_0 , if and only if each f_δ is supported in B . The following is then a consequence of Theorem 7.2.

Corollary 8.1. *There exists $R > 0$ with the following property. Let $f \in C_R^\infty(U/K)$ and $r < R$. Then $f \in C_r^\infty(U/K)$ if and only if the Fourier transform \tilde{f}_δ of each of the functions f_δ allows a holomorphic continuation satisfying the growth estimate (b) of Definition 5.1 (with constants depending on δ).*

For example, in the case of the sphere S^2 , the expansion (8.1) of f reads $f = \sum_{m \in \mathbb{Z}} f_m$, and the Fourier transform of f_m is the map

$$(8.2) \quad l \mapsto \begin{cases} c_{l,m} & \text{for } l \geq |m| \\ 0 & \text{for } 0 \leq l < |m| \end{cases}$$

where $c_{m,l}$ are the coefficients of the spherical harmonics expansion

$$f = \sum_{l=0}^{\infty} (2l+1) \sum_{|m| \leq l} c_{l,m} Y_l^m.$$

The condition in Corollary 8.1 is thus that the map (8.2) has a holomorphic extension to $l \in \mathbb{C}$ of the proper exponential type, for each $m \in \mathbb{Z}$.

It is an obvious question, whether the assumption of K -finiteness can be removed in Theorem 7.2. It is not difficult to remove it from Theorem 7.1. Assume that φ satisfies Properties (b) and (c) in Definition 5.1 for a suitably small value of r . Define a function $f : U/K \rightarrow \mathbb{C}$ by (7.1). Using the arguments from [27, 28] it follows that $f \in C^\infty(U/K)$. By expanding f as in (8.1) it follows from Corollary 8.1 that f has support inside the ball of radius r . It also follows that $\tilde{f} = \varphi$.

The nontrivial part would be to remove the assumption from Theorem 6.1. At this point we do not know if the Fourier transform actually maps all non- K -finite functions of small support into the space of functions satisfying the estimate in Property (b). The ingredients in our proof, in particular the matrices $P(\lambda)$, depend on the K -types. We would like to point out that for the noncompact dual G/K , this direction is proved in [12], p. 278, using the Radon transform. It has been suggested to us by Simon Gindikin that [7] might be used in such an argument for U/K .

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