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The head and tail conjecture for alternating knots

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THE HEAD AND TAIL CONJECTURE FOR ALTERNATING KNOTS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
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in

The Department of Mathematics

by

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Abstract

The colored Jones polynomial is an invariant of knots and links, which produces a sequence of Laurent polynomials. In this work, we study new power series link invariants, derived from the colored Jones polynomial, called its head and tail. We begin with a brief survey of knot theory and the colored Jones polynomial in particular. In Chapter 3, we use skein theory to prove that for adequate links, the n -th leading coefficient of the N -th colored Jones polynomial stabilizes when viewed as a sequence in N . This property allows us to define the head and tail for adequate links. In Chapter 4 we show a class of knots with trivial tail, and in Chapter 5 we develop techniques to calculate the head and tail for various knots and links using a graph derived from the link diagram.

Chapter 1

Introduction

The colored Jones polynomial is a link invariant which generalizes the Jones polynomial. It gives, for any knot or link, a sequence of Laurent polynomials, the first in the sequence being the original Jones polynomial. There are many equivalent formulations of the colored Jones polynomial, and in this paper we will discuss several, but the original definition comes from the use of Yang-Baxter operators. Given a Lie algebra and a representation of that algebra, one can use quantum groups to construct a solution to the Yang-Baxter equation. Turaev showed in [18] how such a solution produces a knot invariant. Invariants of this type are called quantum invariants. The N -th colored Jones polynomial is the quantum invariant coming from sl_2 and its N -th dimensional irreducible representation.

As the colored Jones polynomial is a sequence, naturally much work has gone into studying limits of this sequence. For instance, two well-known points of study include the Melvin-Morton conjecture and the Kashaev-Murakami-Murakami Volume conjecture. The Melvin-Morton conjecture, first rigorously proved by Bar-Natan and Garoufalidis in [3], states that a certain limit of the colored Jones polynomial results in the inverse of the Alexander polynomial. The Volume conjecture, which remains open, states that the growth rate of the colored Jones polynomial evaluated at a root of unity is equivalent to the volume of the knot complement.

The focus of study in this paper is on the first and last N coefficients of the N -th colored Jones polynomial, which is called the head and tail (irrespectively) of the colored Jones polynomial. The first result along this line of study is due to Kauffman in [11], who, while finding a bound on the degree of the Jones polynomial,

showed that the first and last coefficients of the Jones polynomial are ± 1 for all adequate links (a class of links generalizing alternating).

The first direct study of the head and tail of the colored Jones polynomial came from Dasbach and Lin, who in [5] found formulas for the first two and last two coefficients of the N -th colored Jones polynomial for adequate knots and showed that, up to sign, they do not depend on N . They also showed that the third coefficient does not depend on N up to sign, so long as it is greater than or equal to 3. They conjectured that this property holds for higher coefficients, and this property is the first main result of this paper, which will be proven in Chapter 3.

Main Theorem 1. *For an A -adequate link L , let $a_{j,N}$ denote the j -th coefficient of the N -th colored Jones polynomial of L . Then*

$$a_{j,N} = \pm a_{j,j}$$

when $j \leq N$.

In other words this theorem states that, for a fixed j , the sequence of coefficients $\{a_{j,N}\}$ viewed as a sequence with respect to N , stabilizes at $N = j$.

Another result concerning the head and tail is given in [4] by Champanerkar and Koffmann, who studied the colored Jones polynomials for the closure of positive braids with a full twist. In their work they studied more than the first N coefficients, but one of their results is that for such knots, the tail is trivial. Their theorem is generalized in Chapter 4 of this paper to show that the tail is trivial for all knots that can be written as the closure of a positive braid (no full twist necessary).

Main Theorem 2. *For knots which can be written as the closure of a positive braid,*

$$a_{1,N} = \pm 1, \quad a_{j,N} = 0$$

when $1 < j \leq N$.

To prove this theorem, a new description of the colored Jones polynomial will be developed, based on a quantum determinant description introduced in [9] by Huynh and Lê. This quantum determinant description involves a deformation of the Burau representation of braids. In [10], Vaughan Jones briefly mentioned a probabilistic interpretation of the Burau representation as walks along the braid, and in [13], Xiao-Song Lin, Feng Tian, and Zhenghan Wang used this interpretation to generalize the Burau representation to tangles by using walks along the tangles. Then in [14], Lin and Wang used this to calculate the colored Jones polynomial and derive a new proof of the Melvin-Morton-Rozansky conjecture. In Chapter 4, a different, but similar, generalization of this description coming from Huynh and Lê's result is used to give a geometric interpretation of the colored Jones polynomial in terms of walks along the braid.

The previously mentioned formulas for the the first three terms of the head and tail discovered by Dasbach and Lin in [5] depend only on two graphs derived from the knot, called the reduced A and B graphs. In Chapter 5, it is shown that, for adequate links, the head and tail in their entirety depend only on the reduced A and B graphs. Also techniques are developed to calculate heads and tails of a large class of knots and links based on their reduced A and B graphs. Chapter 5 consists of joint work with Oliver Dasbach.

At the end of chapter 5, the techniques developed are used to find the values of the heads and tails of all two bridge knots and links.

Chapter 2

Background and Preliminaries

2.1 Knot Theory

An l component link is an equivalence class of embeddings of l disjoint circles into S^3 (thought of as the one point compactification of \mathbb{R}^3) under ambient isotopy. A link with only a single component is called a *knot*. A *link diagram* is a projection of the knot into S^2 (thought of as the one point compactification of \mathbb{R}^2) with only transverse double points (called *crossings*) which come with information indicating which strand passes over the other strand. Two link diagrams represent the same link if and only if one can be transformed into the other by a sequence of *Reidemeister moves*.

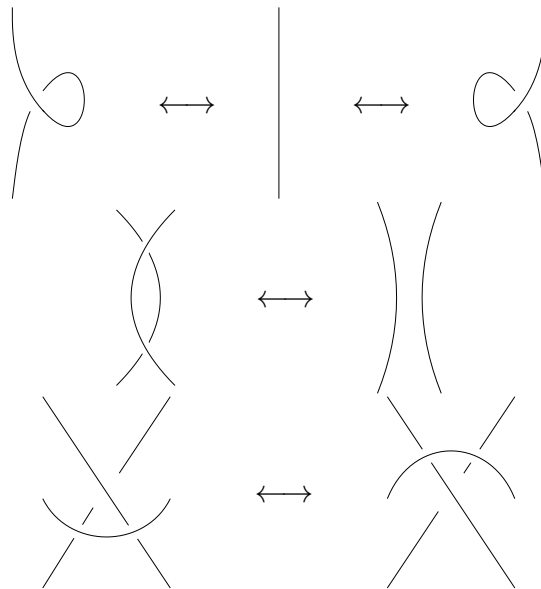


FIGURE 2.1: The Reidemeister moves Ω I, Ω II, and Ω III

An *oriented link* is a link together with an orientation. For a diagram D of an oriented link, one can define a number $\omega(D)$ called the *writhe* of the diagram

by summing the sign of each crossing, where the sign of a crossing is defined by Figure 2.2. For a knot, the writhe of a diagram is independent of the orientation.



FIGURE 2.2: The sign of a crossing

A *framed link* is a link together with a section of the normal bundle over the link called a *framing*. A link given by a diagram has a natural framing called the *blackboard framing* in which the vectors are perpendicular to the plane of projection. Two link diagrams each with the blackboard framing represent the same framed link if and only if one can be transformed into the other by a sequence of Reidemeister moves which only include moves of type II or III. Such a sequence is called a *regular isotopy*. We will always consider diagrams of framed links to have the blackboard framing.

A link diagram is called *alternating* if as one travels along the knot, the strand that is traveled along alternates between being the over and under strand. A knot is called *alternating* if it has an alternating diagram.

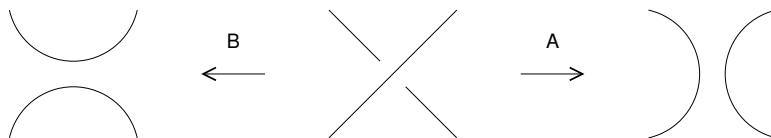


FIGURE 2.3: A and B smoothings

Crossings in a link diagram can be smoothed in two different ways as in Figure 2.3 to produce new diagrams. These are called the *A-smoothing* and *B-smoothing*. A *Kauffman state* (or simply *state*) S is a choice of A-smoothing or B-smoothing for each crossing in a diagram, resulting in a diagram with no crossings. In drawings of states, dotted lines are often used to indicate where crossings were in the original

diagram. A *state graph* G_S for a state S is the graph with vertex set, the collection of circles after applying the smoothings, and edges set, the set of crossings of the original diagram, where each edge connects the two vertices corresponding to the circles that the crossing meets. An example is given in Figure 2.4. The *reduced state graph* G'_S is obtained from the state graph G_S by replacing all multiple edge with single edges.

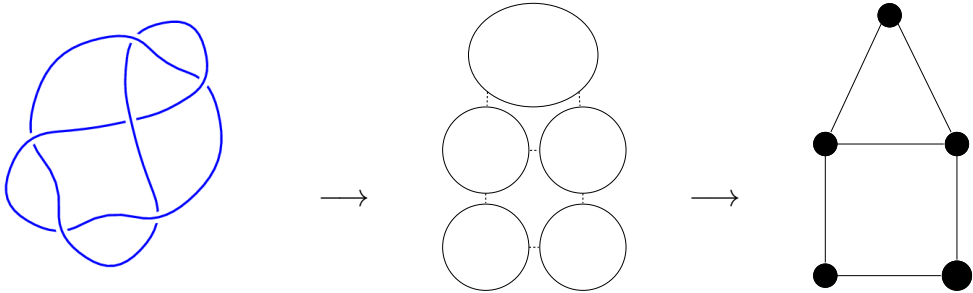


FIGURE 2.4: The All-B state graph G_B of 6_2

Two important states are the *all-A state* S_A and *all-B state* S_B , which are the states where the A-smoothing, or respectively the B-smoothing, is chosen for every crossing. The corresponding state graphs will be denoted G_A and G_B .

A link diagram is called *A-adequate* (respectively *B-adequate*) if G_A (respectively G_B) has no loops. A link diagram is called *adequate* if it is both A-adequate and B-adequate. A link is called *adequate* if it has an adequate diagram.

The most important property of A-adequate diagrams is that the number of circles in S_A is a local maximum. In other words, any state that has only a single B smoothing will have one fewer circle than the all-A smoothing. Similarly for B-adequate diagrams, the number of circles in S_B is a local maximum.

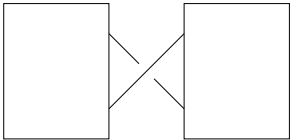


FIGURE 2.5: A nugatory crossing in a link diagram

A diagram is called *reduced* if it does not have any *nugatory* crossings, that is any crossings as in Figure 2.5. The following proposition is a well-known result.

Proposition 2.1. *A reduced alternating diagram is adequate.*

All alternating links have reduced alternating diagrams. Thus all alternating links are adequate.

Another important fact about adequate diagrams is that parallels of A-adequate diagrams are also A-adequate. Given a diagram D , the r -th parallel of D denoted D^r is the diagram formed by replacing D with r parallel copies of D .

An object related to knots and links is the *braid group*. The *braid group on m -strands* B_m is the group generated by σ_1 through σ_{m-1} satisfying the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$. A braid word can be expressed diagrammatically where σ_i is the diagram in Figure 2.6 and multiplication is represented by stacking.

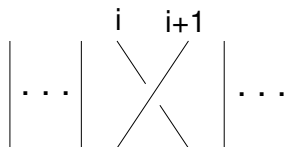


FIGURE 2.6: The generator σ_i of B_m

The *closure* of a braid β , denoted $\hat{\beta}$ is a link formed by attaching the top strands of β to the bottom strands of β without introducing any new crossings. It is a well-known theorem of Alexander's that any link can be expressed as the closure of a braid.

For a sequence $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_k)$ of pairs $\gamma_j = (i_j, \epsilon_j)$, $1 \leq i_j \leq m - 1$ and $\epsilon_j = \pm 1$ (The notation \pm and ± 1 will be used interchangeably as it should be clear from the context), let $\beta = \beta(\gamma)$ be the braid

$$\beta := \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \dots \sigma_{i_k}^{\epsilon_k}.$$

Denote $\omega(\gamma) = \sum_{j=1}^k \epsilon_j$. This is the writhe of $\beta(\gamma)$.

A particular class of links that are will be of interest are the (m, p) *torus links*, denoted $T(m, p)$. The link $T(m, p)$ is the closure of the braid $(\sigma_1 \sigma_2 \dots \sigma_{m-1})^p$. The number of components of $T(m, p)$ is the gcd of m and p . Thus, in particular, $T(2, p)$ is a knot if p is odd, and a 2 component link if p is even.

2.2 The Colored Jones Polynomial via Skein Theory

The material in this section can also be found in [12, 15]

The *Kauffman bracket skein module*, $S(M; R, A)$, of a 3-manifold M and ring R with invertible element A , is the free R -module generated by isotopy classes of framed links in M , modulo the submodule generated by the Kauffman relations:

$$\begin{array}{c} \diagdown \\ \diagup \end{array} = A \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) + A^{-1} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right), \quad \bigcirc = -A^2 - A^{-2}$$

If M has designated points on the boundary, then the framed links must include arcs which meet all of the designated points.

In this paper we will take $R = \mathbb{Q}(A)$, the field of rational functions in variable A with coefficients in \mathbb{Q} . We will be concerned with two particular skein modules: $S(S^3; R, A)$, which is isomorphic to R under the isomorphism sending the empty link to 1, and $S(D^3; R, A)$, where D^3 has $2n$ designated points on the boundary. With these designated points, $S(D^3; R, A)$ is also called the *Temperley-Lieb algebra* TL_n .

We will give an alternate explanation for the Temperley-Lieb algebra. First, consider the disk D^2 as a rectangle with n designated points on the top and n designated points on the bottom. Let TLM_n be the set of all crossing-less matchings on these points, and define the product of two crossing-less matchings by placing one rectangle on top of the other and deleting any components which do not meet

the boundary of the disk. With this product, TLM_n is a monoid, which we shall call the *Temperley-Lieb monoid*. It has generators h_i as in Figure 2.7, and following relations:

- $h_i h_i = h_i$
- $h_i h_{i\pm 1} h_i = h_i$
- $h_i h_j = h_j h_i$ if $|i - j| \geq 2$

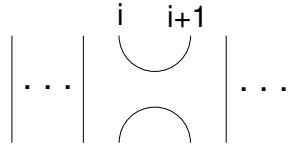


FIGURE 2.7: The generator h_i of TLM_n

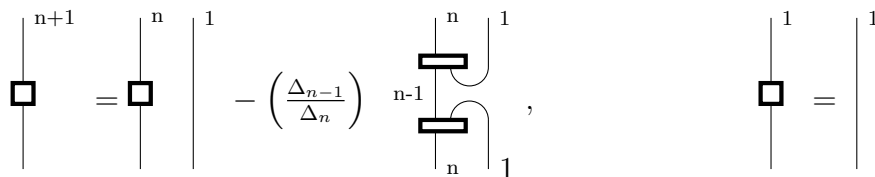
Any element in TL_n has the form $\sum_{M \in TLM_n} c_M M$, where $c_M \in \mathbb{Q}(A)$. Multiplication in TL_n is slightly different from multiplication in TLM_n , because $h_i h_i = (-A^2 - A^{-2})h_i$ in TL_n .

There is a special element in TL_n of fundamental importance to the colored Jones polynomial, called the *Jones Wenzl idempotent*, denoted $f^{(n)}$. Diagrammatically this element is represented by an empty box with n strands coming out of it on two opposite sides. By convention an n next to a strand in a diagram indicates that the strand is replaced by n parallel ones.

With

$$\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$$

and $\Delta_n! := \Delta_n \Delta_{n-1} \dots \Delta_1$ the Jones-Wenzl idempotent is defined recursively by



and satisfies the properties

The Jones-Wenzl idempotent also satisfies a change of framing relation:

If $M \in TLM_n$, define $f_M \in R$ as the coefficient of M in the expansion of the Jones-Wenzl idempotent. Thus $f^{(n)} = \sum_{M \in TLM_n} f_M M$. If e is the identity element of TLM_n , then $f_e = 1$.

We can use weighted trivalent graphs (often called *quantum spin networks*) to express certain elements in a skein module using the following correspondence:

This gives us powerful computational tools, such as the fusion relation, which allows two parallel strands to be fused together as in the following equation:

where the sum is over all c such that:

1. $a + b + c$ is even
2. $|a - b| \leq c \leq a + b$.

To define $\theta(a, b, c)$ let a , b and c be related as above and x , y and z be defined by $a = y + z$, $b = z + x$ and $c = x + y$ then

$$\theta(a, b, c) := \text{a} \begin{array}{c} \bullet \\ \bigcirc \\ \bullet \end{array} \text{c}$$

and one can show that

$$\theta(a, b, c) = \frac{\Delta_{x+y+z}! \Delta_{x-1}! \Delta_{y-1}! \Delta_{z-1}!}{\Delta_{y+z-1}! \Delta_{z+x-1}! \Delta_{x+y-1}!}.$$

Furthermore one has:

$$\begin{array}{c} \text{a} \quad \text{b} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \bullet \\ | \\ \text{c} \end{array} = (-1)^{\frac{a+b-c}{2}} A^{a+b-c+\frac{a^2+b^2-c^2}{2}} \begin{array}{c} \text{a} \quad \text{b} \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \text{c} \end{array}$$

We can now define the *un-normalized colored Jones polynomial* for framed links. The un-normalized colored Jones polynomial $\tilde{J}_{n,L}(A)$ of a framed link L can be defined as the value of the link with each component decorated by the n -th Jones-Wenzl idempotent $f^{(n)}$ viewed as an element in $S(\mathbb{R}^3; \mathbb{Q}(A), A) \cong \mathbb{Q}(A)$ under the isomorphism sending the empty link to 1. Although we have used the ring $\mathbb{Q}(A)$ it is a fact that $\tilde{J}_{n,L}(A)$ lies in $\mathbb{Z}[A, A^{-1}]$.

Another version of the colored Jones polynomial, that will be studied in this paper is the *normalized colored Jones polynomial*. In this version, we would like to consider links that do not come with a framing; this can be done by always assigning a framing coming from a diagram with writhe 0 to the link. We would also like to

change the variable involved; the difference in degree of any two terms in $\tilde{J}_{n,L}(A)$ is a multiple of 4, so it is reasonable to make the substitution $A^{-4} = q$. Finally, to obtain the normalized colored Jones polynomial we have to divide $\tilde{J}_{n,L}(A)$ by its value on the unknot. Thus

$$J_{n+1,L}(q) := \frac{\tilde{J}_{n,L}(A)}{\Delta_n} \Big|_{A=q^{-1/4}}.$$

Now the normalized colored Jones polynomial $J_{N,L}(q)$ for a link L is a sequence of Laurent polynomials in the variable $q^{1/2}$, i.e. $J_{N,L} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$. This sequence is defined for $N \geq 2$ so that $J_{2,L}(q)$ is the ordinary Jones polynomial, and $J_{N,U}(q) = 1$ where U is the unknot. For a link L with an odd number of components (including all knots), $J_{N,L}$ is actually in $\mathbb{Z}[q, q^{-1}]$. For links with an even number of components, $q^{1/2}J_{N,L} \in \mathbb{Z}[q, q^{-1}]$.

2.3 The Colored Jones Polynomial via the Huynh-Lê Quantum Determinant

An alternate formulation of the colored Jones polynomial which will be used in this paper is given by Huynh and Lê in [9]. Construct an ‘almost quantum’ matrix, and express the colored Jones polynomial as the inverse of the quantum determinant of this matrix evaluated in a particular way. This construction will only apply to knots, for which we have $J_{N,K}(q) \in \mathbb{Z}[q, q^{-1}] =: \mathcal{R}$.

A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is right quantum if

$$ac = qca$$

$$bd = qdb$$

$$ad = da + qcb - q^{-1}bc$$

An $m \times m$ matrix is right-quantum if all 2×2 submatrices of it are right-quantum.

If $A = (a_{ij})$ is right-quantum, then the quantum determinant is

$$\det_q(A) := \sum_{\pi \in \text{Sym}(m)} (-q)^{\text{inv}(\pi)} a_{\pi 1,1} a_{\pi 2,2} \cdots a_{\pi m,m}$$

where $\text{inv}(\pi)$ denotes the number of *inversions*, that is the number of pairs $i < j$ with $\pi(j) < \pi(i)$.

In general, when A is a right quantum matrix, $I - A$, where I is the identity matrix, is no longer right-quantum. So define

$$\widetilde{\det}_q(I - A) := 1 - C$$

where

$$C := \sum_{\emptyset \neq J \subset \{1,2,\dots,m\}} (-1)^{|J|-1} \det_q(A_J),$$

where A_J is the J by J submatrix of A , which is always right-quantum.

Before we define the right quantum matrix we will use, we must first define particular operators which will be the entries of this matrix. First define operators \hat{x} and τ_x and their inverses acting on the ring $\mathcal{R}[x^{\pm 1}, y^{\pm 1}, u^{\pm 1}]$:

$$\hat{x}f(x, y, \dots) = xf(x, y, \dots), \quad \tau_x f(x, y, \dots) = f(qx, y, \dots)$$

Also define \hat{y} , τ_y , \hat{u} , τ_u , and their inverses similarly.

Now let us define

$$a_+ = (\hat{u} - \hat{y}\tau_x^{-1})\tau_y^{-1}, \quad b_+ = \hat{u}^2, \quad c_+ = \hat{x}\tau_y^{-2}\tau_u^{-1},$$

$$a_- = (\tau_y - \hat{x}^{-1})\tau_x^{-1}\tau_u, \quad b_- = \hat{u}^2, \quad c_- = \hat{y}^{-1}\tau_x^{-1}\tau_u$$

If P is a polynomial operator in the operators a_{\pm} , b_{\pm} , and c_{\pm} then we get a polynomial $\mathcal{E}(P) \in \mathcal{R}[z^{\pm 1}]$ by having P act on the constant polynomial 1 and replacing x and y with z and replacing u with 1. We can get a polynomial $\mathcal{E}_N(P) \in \mathcal{R}$ by making the further substitution in $\mathcal{E}(P)$ replacing z with $q^{(N-1)}$.

For calculations using these operators, it is useful to observe the following relations:

$$a_+b_+ = b_+a_+, \quad a_+c_+ = qc_+a_+, \quad b_+c_+ = q^2c_+b_+$$

$$a_-b_- = q^2b_-a_-, \quad c_-a_- = qa_-c_-, \quad c_-b_- = q^2b_-c_-$$

Also we can write a formula for their evaluation:

Lemma 2.2 (Huynh, Lê).

$$\mathcal{E}_N(b_+^s c_+^r a_+^d) = q^{r(N-1-d)} \prod_{i=0}^{d-1} (1 - q^{N-1-r-i})$$

$$\mathcal{E}_N(b_-^s c_-^r a_-^d) = q^{-r(N-1)} \prod_{i=0}^{d-1} (1 - q^{r+i+1-N})$$

We are now prepared to define a right quantum matrix from a braid whose closure is a knot. First define matrices which are right quantum:

$$S_+ := \begin{pmatrix} a_+ & b_+ \\ c_+ & 0 \end{pmatrix} \quad S_- := \begin{pmatrix} 0 & c_- \\ b_- & a_- \end{pmatrix}$$

Given a braid $\beta(\gamma)$ defined as in Section 2.1, associate to each $\sigma_{i_j}^{\epsilon_j}$ the matrix which is the identity except for the 2×2 minor of rows $i_j, i_j + 1$ and columns $i_j, i_j + 1$ which is replaced by the matrix $S_{\epsilon_j, j}$.

Here $S_{\pm, j}$ is the same as S_{\pm} with a_{\pm}, b_{\pm} , and c_{\pm} replaced by $a_{\pm, j}, b_{\pm, j}$, and $c_{\pm, j}$. These operators are defined by replacing the x, y, u and z in the above definition by x_j, y_j, u_j and z_j . These operators will act on $\bigotimes_{j=1}^k \mathcal{R}[x_j, y_j, u_j]$ where k is the number of crossings in $\beta(\gamma)$. It is immediate that any two of these operators with different indices will commute, and that for operators with the same indices, the previous relations still apply. Also the evaluation operator \mathcal{E}_N is multiplicative over different indices.

The matrix $\rho(\gamma)$ is the product of these matrices. The matrix $\rho'(\gamma)$ is $\rho(\gamma)$ with the first row and column removed.

Theorem 2.3 (Huynh, Lê). *If the closure of the braid $\beta(\gamma) \in B_m$ is a knot K , then*

$$J_{N,K}(q) = q^{(N-1)(\omega(\gamma)-m+1)/2} \mathcal{E}_N \left(\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))} \right)$$

Here

$$\frac{1}{\widetilde{\det}_q(I - q\rho'(\gamma))} = \sum_{n=0}^{\infty} C^n.$$

This sum is finite if the closure of $\beta(\gamma)$ is a knot.

Chapter 3

Existence of the Head and Tail

3.1 Introduction

In [5] and [6] Oliver Dasbach and Xiao-Song Lin showed that, up to sign, the first two coefficients and the last two coefficients of $J_{N,K}(q)$ do not depend on N for alternating knots. They also showed that the third (and third to last) coefficient does not depend on N so long as $N \geq 3$. This and computational data led them to believe that the k -th coefficient does not depend on N so long as $N \geq k$. In this chapter, we will prove this conjecture, which is Main Theorem 1 for all A -adequate links. It is also known that this property of $J_{N,L}(q)$ does not hold for all knots. In [2], with Oliver Dasbach, we examined the case of the $(4, 3)$ torus knot for which this property fails.

Definition 3.1. For two Laurent series $P_1(q)$ and $P_2(q)$ we define

$$P_1(q) \doteq P_2(q)$$

if $P_1(q) = \pm q^s P_2(q)$ for some power s .

Definition 3.2. For two Laurent series $P_1(q)$ and $P_2(q)$ we define

$$P_1(q) \doteq_n P_2(q)$$

if after multiplying $P_1(q)$ by $\pm q^{s_1}$ and $P_2(q)$ by $\pm q^{s_2}$, s_1 and s_2 some powers, to get power series $P'_1(q)$ and $P'_2(q)$ each with positive constant term, $P'_1(q)$ and $P'_2(q)$ agree $\pmod{q^n}$. For example $-q^{-4} + 2q^{-3} - 3 + 11q \doteq_5 1 - 2q + 3q^4$.

Another way of phrasing Definition 3.2 is that $P_1(q) \doteq_n P_2(q)$ if and only if their first n coefficients agree up to sign.

With this equivalence relation, we can now define the head and tail of the colored Jones polynomial $H_L(q)$ and $T_L(q)$.

Definition 3.3. *The tail of the colored Jones polynomial of a link L – if it exists – is a power series $T_L(q)$, with*

$$T_L(q) \doteq_N J_{L,N}(q), \text{ for all } N$$

Similarly, the head $H_L(q)$ of the colored Jones polynomial of L is the tail of $J_{L,N}(q^{-1})$, which is equal to the colored Jones polynomial of the mirror image of L .

Note that $T_L(q)$ exists if and only if $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$ for all N . For example, for the first few colors N the colored Jones polynomial of the knot 6_2 multiplied by q^{2N^2-N-1} is

$$\begin{aligned} N = 2 : & 1 - 2q + 2q^2 - 2q^3 + 2q^4 - q^5 + q^6 \\ N = 3 : & 1 - 2q + 4q^3 - 5q^4 + 6q^6 + \dots - q^{14} + 3q^{15} - q^{16} - q^{17} + q^{18} \\ N = 4 : & 1 - 2q + 2q^3 + q^4 - 4q^5 - 2q^6 + \dots - 2q^{29} - 3q^{30} + 3q^{32} - q^{34} - q^{35} + q^{36} \\ N = 5 : & 1 - 2q + 2q^3 - q^4 + 2q^5 - 6q^6 + \dots - 2q^{53} - q^{54} + 4q^{55} - q^{58} - q^{59} + q^{60} \\ N = 6 : & 1 - 2q + 2q^3 - q^4 - 2q^7 + q^8 + \dots - 3q^{82} + 3q^{84} + q^{85} - q^{88} - q^{89} + q^{90} \\ N = 7 : & 1 - 2q + 2q^3 - q^4 - 2q^6 + 4q^7 - 3q^8 + \dots + q^{119} + q^{121} - q^{124} - q^{125} + q^{126} \end{aligned}$$

This is exactly the property conjectured by Dasbach and Lin to hold for all alternating knots.

Theorem 3.4. *If L is an alternating link, then $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$.*

Because the mirror image of an alternating link is alternating, this theorem says that the head and the tail exists for all alternating links. Theorem 3.4 was simultaneously and independently proved by Stavros Garoufalidis and Thang Lê in [7] using alternate methods.

We are also able to prove a more general theorem about A -adequate links.

Theorem 3.5. *If L is a A -adequate link, then $J_{L,N}(q) \doteq_N J_{L,N+1}(q)$.*

Because all alternating links are A -adequate, Theorem 3.5 implies Theorem 3.4. Theorem 3.5 is a restatement of Main Theorem 1.

3.2 Additional Skein Theory Lemmas

In this section, we prove new skein theoretic lemmas which we will need in the proof of Theorem 3.5. As mentioned before, we will take $R = \mathbb{Q}(A)$, the field of rational functions in variable A with coefficients in \mathbb{Q} . As we are concerned with the lowest terms of a polynomial, we will need to express rational functions as Laurent series. This can always be done so that the Laurent series has a minimum degree.

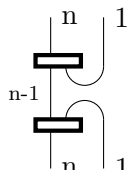
Definition 3.6. *Let $f \in \mathbb{Q}(A)$, define $d(f)$ to be the minimum degree of f expressed as a Laurent series in A .*

Note that $d(f)$ can be calculated without referring to the Laurent series. Any rational function f expressed as $\frac{P}{Q}$ where P and Q are both Laurent polynomials. Then $d(f) = d(P) - d(Q)$.

Recall from Section 2.2 that we express the Temperley-Lieb idempotent $f^{(n)}$ as $f^{(n)} = \sum_{M \in TLM_n} f_M M$.

Lemma 3.7. *If $M \in TLM_n$, then $d(f_M)$ is at least twice the minimum word length of M in terms of the h_i 's.*

Proof. This follows easily from the recursive definition of the idempotent by an

inductive argument. The only issue is that terms of the form  may have

a circle which needs to be removed. In this situation, the minimum degree of the coefficient is reduced by two, but the number of generators used is also reduced by one.

□

Using Lemma 3.7 we can find a lower bound for the minimum degree of any element of $S(S^3; R, A)$ which contains the Jones-Wenzl idempotent. Before we do this, consider a crossing-less diagram S in the plane consisting of arcs connecting Jones-Wenzl idempotents. We will define what it means for such a diagram to be adequate in much the same way that a knot diagram can be A or B- adequate.

Construct a crossing-less diagram \bar{S} from S by replacing each of the Jones-Wenzl idempotents in S by the identity of TL_n . Thus \bar{S} is a collection of circles with no crossings. Consider the regions in \bar{S} where the idempotents had previously been. S is adequate if no circle in \bar{S} passes through any one of these regions more than once. Figure 3.1a shows an example of a diagram that is adequate and Figure 3.1b shows an example of a diagram that is not adequate. In both figures every arc is labeled 1.

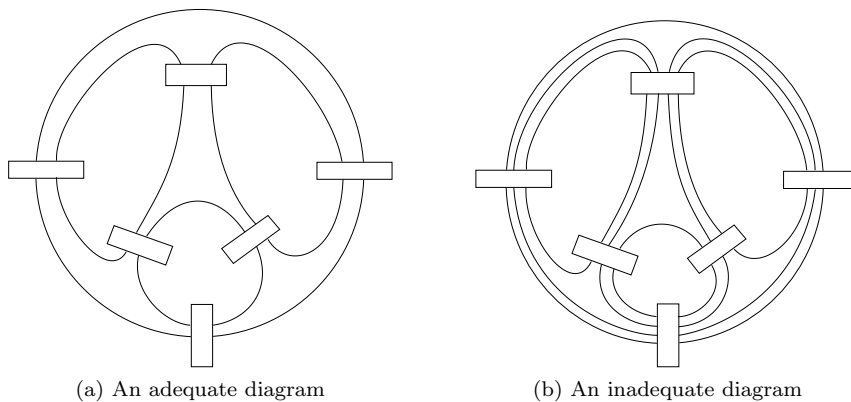


FIGURE 3.1: Example of adequate and inadequate diagrams

If S is adequate, then the number of circles in \bar{S} is a local maximum, in the sense that if the idempotents in S are replaced by other elements of TLM_n such that there is exactly one hook total in all of the replacements, then the number of circles in this diagram is one less than the number of circles in \bar{S} .

If the diagram S happens to have crossings in it, we can still construct the diagram \bar{S} , which is now a link diagram. Denote $D(S) := d(\bar{S})$.

Lemma 3.8. *If $S \in S(S^3; R, A)$ is expressed as a single diagram containing the Jones-Wenzl idempotent, then $d(S) \geq D(S)$.*

If the diagram for S is a crossing-less adequate diagram, then $d(S) = D(S)$.

Proof. First suppose that the diagram S has no crossings. We can get an expansion of S by expanding each of the idempotents that appear in the diagram. Consider a single term T_1 in this expansion. Unless all of the idempotents have been replaced by the identity in this term, then there will be a hook somewhere in the diagram. By removing a single hook, we get a different term T_2 in the expansion. The number of circles in T_1 differs from the number of circles in T_2 by exactly one. Also there are fewer hooks in T_2 , so by Lemma 3.7 and the fact that removing a circle results in multiplying by $-A^2 - A^{-2}$, the minimum degree of T_1 is at least as large as the minimum degree of T_2 . This tells us that the lowest degree amongst terms in the expansion of S is the degree of the term with the idempotents replaced by the identity, \bar{S} .

If S is adequate, then for any term T_1 with only a single hook, T_2 will be \bar{S} , and thus T_2 will have one more circle than T_1 . Therefore, $d(T_1) > d(\bar{S})$. This tells us that any term T in this expansion will have $d(T_1) > d(\bar{S})$, and thus $d(S) = d(\bar{S}) = D(S)$.

If there were crossings in S , then we can get an expansion of S by expanding the idempotents that appear in S and summing over all possible smoothings of the crossings. If we expand over the smoothings first, we get a collection of terms each of which is a coefficient times a crossingless diagram with idempotents. We can apply the previous argument to say that the minimum degree of each term is the minimum degree of that term with the idempotents replaced with the identity.

Now consider \bar{S} . By expanding \bar{S} by summing over all possible smoothings of \bar{S} , we get the same sum as before. Thus the minimum degree of \bar{S} agrees with the minimum degree of S . \square

Remark 3.9. Note that if L_1 and L_2 are two framed link whose underlying unframed links are the same, the $\tilde{J}_{n,L_1}(A) \doteq \tilde{J}_{n,L_2}(A)$. This means that we may use any framing of a link to calculate its colored Jones polynomial, if we are only concerned with its value up to the equivalence \doteq or \doteq_n .

In particular for B-adequate links, we will want to use its B-adequate diagram, even when this diagram does not have writhe zero.

3.3 The Main Lemma

In this section we will relate the tail of the colored Jones polynomial to a certain trivalent graph viewed as an element of the Kauffman bracket skein module of \mathbb{S}^3 . This construction was used in [2] to prove interesting properties of the head and the tail of the colored Jones polynomial.

Given a B-adequate diagram D of a link L , consider a negative twist region. Apply the identities of Section 2.2 to get the equation:

$$\begin{array}{c} \diagup \\ \vdots \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} = \sum_{j=0}^n (\gamma(n, n; 2j))^m \frac{\Delta_{2j}}{\theta(n, n, 2j)} \begin{array}{c} \diagup \\ | \\ \diagdown \end{array} \cdot$$

Here $\gamma(a, b; c) := (-1)^{\frac{a+b-c}{2}} A^{a+b-c+\frac{a^2+b^2-c^2}{2}}$.

If we apply this equation to every maximal negative twist region, then we get an embedded trivalent graph called Γ . We get a colored graph $\Gamma_{n,(j_1,\dots,j_k)}$ where k is the number of maximal negative twist regions and $0 \leq j_i \leq n$ by coloring the edges coming from the i -th twist region by $2j_i$ and coloring all of the other edges

by n . From the previous equation, it is clear that we get

$$\tilde{J}_{n,L} \doteq \sum_{j_1, \dots, j_k=0}^n \prod_{i=1}^k (\gamma(n, n; 2j_i))^m \prod_{i=1}^k \frac{\Delta_{2j_i}}{\theta(n, n, 2j_i)} \Gamma_{n, (j_1, \dots, j_k)}$$

The following Theorem is a useful tool to find properties of the head and tail of the colored Jones polynomial. In this chapter, it will be used to prove the existence of the tail for all B-adequate links:

Theorem 3.10. *If D is a B-adequate diagram of the link L , and $\Gamma_{n, (n, \dots, n)}$ is the corresponding graph, then*

$$\tilde{J}_{n,L} \doteq_{4(n+1)} \Gamma_{n, (n, \dots, n)}$$

This Theorem was proved for the case when D is a reduced alternating diagram in [2] as Theorem 4.3. The proof given there extends easily to B-adequate diagrams. We will present the proof again here with the modifications. In later sections, we shall denote $\Gamma_n := \Gamma_{n, (n, \dots, n)}$. The theorem will now follow from the following three lemmas.

Lemma 3.11.

$$\begin{aligned} d(\gamma(n, n; 2n)) &= d(\gamma(n, n; 2(n-1))) - 4n \\ d(\gamma(n, n; 2j)) &\leq d(\gamma(n, n; 2(j-1))) \end{aligned}$$

Lemma 3.12.

$$d\left(\frac{\Delta_{2j}}{\theta(n, n, 2j)}\right) = d\left(\frac{\Delta_{2(j-1)}}{\theta(n, n, 2(j-1))}\right) - 2$$

Lemma 3.13. *If Γ is the graph coming from a B-adequate diagram, then*

$$\begin{aligned} D(\Gamma_{n, (j_1, \dots, j_{(i-1)}, j_i, j_{i+1}, \dots, j_k)}) &= D(\Gamma_{n, (j_1, \dots, j_{(i-1)}, j_i-1, j_{i+1}, \dots, j_k)}) \pm 2 \\ d(\Gamma_{n, (n, \dots, n, \dots, n)}) &= D(\Gamma_{n, (n, \dots, n-1, \dots, n)}) - 2 \end{aligned}$$

Proof of Lemma 3.11.

$$\begin{aligned}\gamma(n, n; 2j) &= \pm A^{n+n-2j+\frac{n^2+n^2-(2j)^2}{2}} \\ &= \pm A^{2n-2j+n^2-2j^2}\end{aligned}$$

Clearly $d(\gamma(n, n; 2j))$ increases as j decreases. Furthermore:

$$\begin{aligned}d(\gamma(n, n; 2n)) &= -n^2 \\ d(\gamma(n, n; 2(n-1))) &= 2n - 2(n-1) + n^2 - 2(n-1)^2 \\ &= -n^2 + 4n\end{aligned}$$

□

Proof of Lemma 3.12. To calculate $\theta(n, n, 2j)$ note that in the previous formula for θ we get $x = j$, $y = j$, and $z = n - j$. Using this and the fact that $d(\Delta_n) = -2n$, we get:

$$\begin{aligned}d\left(\frac{\Delta_{2j}}{\theta(n, n, 2j)}\right) &= d\left(\frac{\Delta_{2j}\Delta_{n-1}!\Delta_{n-1}!\Delta_{2j-1}!}{\Delta_{n+j}!\Delta_{j-1}!\Delta_{j-1}!\Delta_{n-j-1}!}\right) \\ &= d\left(\frac{\Delta_{2j}\Delta_{2j-1}\Delta_{n-j}}{\Delta_{n+j}\Delta_{j-1}\Delta_{j-1}}\right) + d\left(\frac{\Delta_{2(j-1)}!\Delta_{n-1}!\Delta_{n-1}!}{\Delta_{n+j-1}!\Delta_{j-2}!\Delta_{j-2}!\Delta_{n-j}!}\right) \\ &= -4j - 2(2j-1) - 2(n-j) + 4(j-1) + 2(n+j) \\ &\quad + d\left(\frac{\Delta_{2(j-1)}}{\theta(n, n, 2(j-1))}\right) \\ &= -2 + d\left(\frac{\Delta_{2(j-1)}}{\theta(n, n, 2(j-1))}\right)\end{aligned}$$

□

Proof of Lemma 3.13. Consider $\Gamma_{n,(j_1,\dots,j_k)}$ as an element in $S(S^3; \mathbb{Q}(A), A)$. We must compare $D(\Gamma_{n,(j_1,\dots,j_{i-1},j_i,j_{i+1},\dots,j_k)})$ with $D(\Gamma_{n,(j_1,\dots,j_{i-1},j_{i-1},j_{i+1},\dots,j_k)})$. Recall that $D(S)$ is -2 times the number of circles in \bar{S} , where \bar{S} is obtained from S by replacing the idempotents in the diagram by the identity in TL_m . For

$\Gamma_{n,(j_1,\dots,j_{(i-1)},j_i,j_{i+1},\dots,j_k)}$ and $\Gamma_{n,(j_1,\dots,j_{(i-1)},j_{i-1},j_{i+1},\dots,j_k)}$, the number of circles in each diagram differ by 1. Thus

$$D(\Gamma_{n,(j_1,\dots,j_{(i-1)},j_i,j_{i+1},\dots,j_k)}) = D(\Gamma_{n,(j_1,\dots,j_{(i-1)},j_{i-1},j_{i+1},\dots,j_k)}) \pm 2.$$

For $\Gamma_{n,(n,\dots,n)}$, replacing the idempotents with the identity results in the all B-smoothing of the diagram D^n . Since D is a B-adequate diagram, so is D^n . For $\Gamma_{n,(n,\dots,n-1,\dots,n)}$, the replacement results in a smoothing of D^n with exactly one A smoothing. Thus the result of the replacement for $\Gamma_{n,(n,\dots,n)}$ will have one more circle than the result of the replacement for $\Gamma_{n,(n,\dots,n-1,\dots,n)}$, which give us $D(\Gamma_{n,(n,\dots,n,\dots,n)}) = D(\Gamma_{n,(n,\dots,n-1,\dots,n)}) - 2$.

Finally, since D^n is B-adequate, $\Gamma_{n,(n,\dots,n)}$ is adequate. Thus by Lemma 3.8,

$$d(\Gamma_{n,(n,\dots,n,\dots,n)}) = D(\Gamma_{n,(n,\dots,n,\dots,n)}).$$

□

3.4 Proof of Main Theorem 1

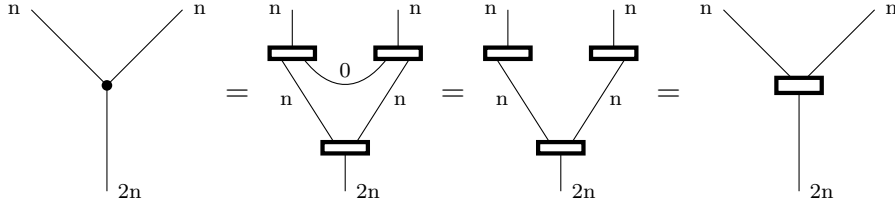
Using Theorem 3.10, Theorem 3.5 is equivalent to the following:

Theorem 3.14. *If D is a B-adequate diagram for a link L and Γ_n its corresponding graph, then*

$$\Gamma_n \stackrel{\cdot}{=}_{4(n+1)} \Gamma_{n+1}$$

Proof. We will first prove Theorem 3.14 in the case of D being a reduced alternating diagram, and then we will show how the proof can be modified to apply to any B-adequate diagram in general.

Interpreting Γ_n as a skein element, we may use the following simplification:



In general, Γ_n will reduce to a collection of circles coming from the all-B smoothing S_B , “fused” together with the Jones-Wenzl Idempotent colored $2n$ for each maximal negative twist region. We will call this reduced form $S_B^{(n)}$.

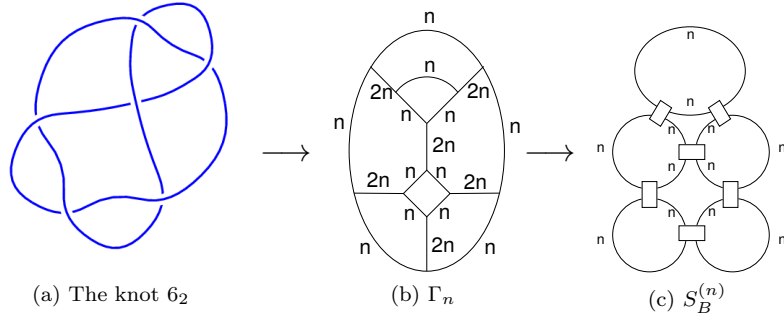


FIGURE 3.2: Example of the knot 6_2 along with Γ_n and $S_B^{(n)}$

We would now like to consider $S_B^{(n+1)}$ and show that we can reduce it to $S_B^{(n)}$ without affecting the lowest $4(n+1)$ terms. To do this we will first show a local relation which we will be able to use repeatedly.

Lemma 3.15.

$$\boxed{\text{Diagram with } n+1 \text{ and } k} = \boxed{\text{Diagram with } n \text{ and } k} + (-1)^{n+1} \left(\frac{\Delta_{k-1}}{\Delta_{n+k}} \right) \boxed{\text{Diagram with } n \text{ and } k}$$

Proof. Using the recursive formula for the Jones-Wenzl idempotent on the left of the left hand side of the identity we get:

$$\boxed{\text{Diagram with } n+1 \text{ and } k} = \boxed{\text{Diagram with } n \text{ and } k} - \left(\frac{\Delta_{n+k-1}}{\Delta_{n+k}} \right) \boxed{\text{Diagram with } n \text{ and } k}$$

Applying the recursive formula again on the middle idempotent of the right most diagram we get:

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} - \left(\frac{\Delta_{n+k-2}}{\Delta_{n+k-1}} \right) \text{Diagram 3} \\
 &= - \left(\frac{\Delta_{n+k-2}}{\Delta_{n+k-1}} \right) \text{Diagram 4} \\
 &= - \left(\frac{\Delta_{n+k-2}}{\Delta_{n+k-1}} \right) \text{Diagram 5}
 \end{aligned}$$

Now when we apply this recursive formula again, the first term will again be zero, and we can continue this process until we get:

$$\text{Diagram 1} = (-1)^n \left(\frac{\Delta_{k-1}}{\Delta_{n+k-1}} \right) \text{Diagram 6}$$

□

Consider a circle s in S_B . The circle s appears in $S_B^{(n+1)}$, although it runs through several idempotents. The goal of the argument is to remove one copy of the circle s from the idempotents. Once this is done for each circle in S_B , then $S_B^{(n+1)}$ will have been reduced to $S_B^{(n)}$.

Because the diagram D is alternating, the circle s bounds a disk which does not contain any of the other circles in S_B . This means that in $S_B^{(n+1)}$, the circle s looks like Figure 3.3. Here all of the arcs are labeled $n + 1$.

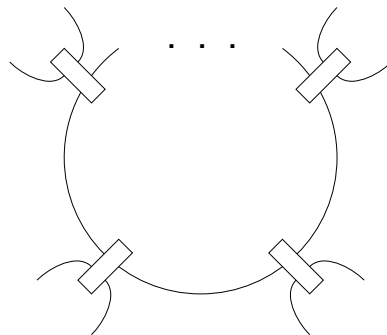


FIGURE 3.3: A circle from S_B seen in $S_B^{(n+1)}$

Apply Lemma 3.15 to get the following relation:

This argument will be applied to each circle in succession, so k is either n or $n + 1$ depending on whether the argument has been applied to that circle yet. All non-labeled arcs are either n or $n + 1$.

Now $S_B^{(n+1)}$ is expressed as the sum of two terms, and the claim is that the minimum degree of the second term is at least the minimum degree of the first term plus $4(n + 1)$. Thus the equation simplifies to:

To see that the claim is true, first note that

$$d \left(\frac{\Delta_{k-1}}{\Delta_{n+k}} \right) = 2(n + 1).$$

Now we need to compare the degree of the two diagrams involved. By Lemma 3.8 we can get a lower bound for the minimum degrees of these diagrams, and since the knot diagram that these came from was B-adequate, the first term will be an adequate diagram, and thus, the lower bound will be equal to the actual minimum degree. Note that the element in TL_{n+2} shown in Figure 3.4 can be expressed as $h_{n+1}h_n \dots h_1$. In the previous equation, this element appears in the right most term. When comparing this terms to the first term, each h_i merges two circles into

one circle. Thus the number of circles in the diagrams differ by $n + 1$. And, finally, a circle can be removed and replaced with a factor of $-A^2 - A^{-2}$. This tells us that the difference in the minimum degrees of the diagrams is at least $2(n + 1)$. Putting this together with the difference in degrees of the coefficients, the difference in minimum degrees of the terms themselves is at least $4(n + 1)$.

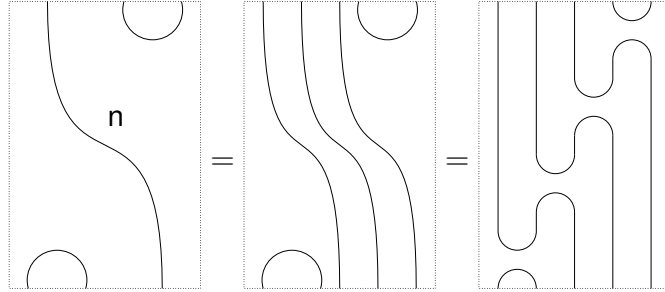


FIGURE 3.4: Multiple pictures expressing $h_{n+1}h_n \dots h_1$

Apply this argument around the circle up to the final idempotent connected to that circle. Now the diagram looks like Figure 3.5.

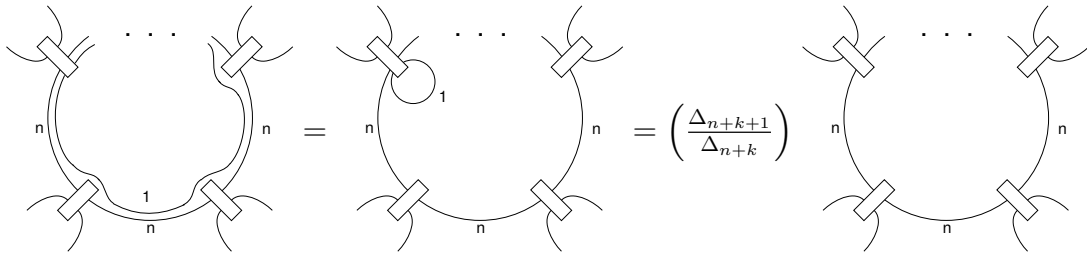


FIGURE 3.5: Reducing $S_B^{(n+1)}$ to $S_B^{(n)}$

We can rewrite the coefficient here:

$$\left(\frac{\Delta_{n+k+1}}{\Delta_{n+k}} \right) = (-1) \frac{A^{2(n+k+2)} - A^{-2(n+k+2)}}{A^{2(n+k+1)} - A^{-2(n+k+1)}} = (-A^{-2}) \frac{A^{4(n+k+2)} - 1}{A^{4(n+k+1)} - 1} \doteq_{4(n+1)} 1$$

Now applying this argument to every circle in $S_B^{(n+1)}$, we see that

$$S_B^{(n+1)} \doteq_{4(n+1)} S_B^{(n)}$$

and thus,

$$\Gamma_n \stackrel{\cdot}{=}_{4(n+1)} \Gamma_{n+1}.$$

This proves Theorem 3.14 in the case of reduced alternating diagrams. For the case when the diagram D is B-adequate, most of the proof still applies. The only thing that goes wrong is that Figure 3.3 is not accurate because a circle s in S_B might not bound a disk, and thus in $S^{(n+1)}$, may have idempotents which alternate which side of the circle it fuses to other circles. Figure 3.6 shows a non-alternating B-adequate diagram of the trefoil where the dotted circle is an example such a circle s . We would still like to pull out one copy of s , but in this case we must be more careful while doing so.

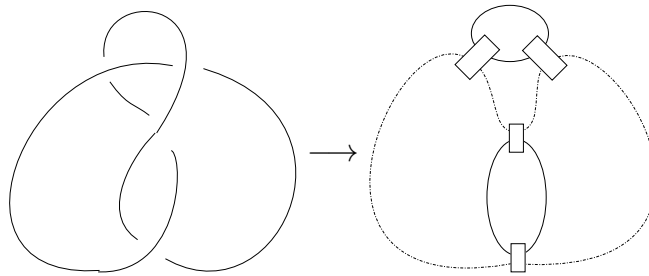


FIGURE 3.6: Example of a non-alternating B-adequate knot diagram

First we will modify the diagram as in Figure 3.7 by adding crossings along the circle s between any pair of idempotents which alternate which side of s is the outer side. The procedure is to modify the diagram so that the outer strand passes over all of the other copies of s , so that it is still the outer strand when it meets the next idempotent in line. Call this new diagram T . When expanding T by summing over all possible smoothings of the crossings, only one state is non-zero, and that state is $S^{(n+1)}$. Since this particular smoothing has an equal number of A and B smoothings, we get that $S^{(n+1)} = T$.

Now we will still apply the procedure to try to pull out one copy of s as in the alternating case. The argument still applies with no modification, as long as we

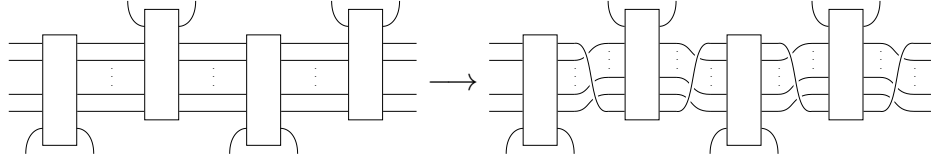


FIGURE 3.7: Modifying $S^{(n+1)}$ to get T

note that in \bar{T} , there are still $n + 1$ parallel copies of s , and they are still unknotted because one of the copies is completely over the other copies. Thus that copy can be straightened out to lie next to the other copies. When applying Lemma 3.15, the second term is still has $n + 1$ fewer circles than \bar{T} because of the same argument.

Finally we have the equation:

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} \\
 & = \text{Diagram 3} \\
 & = \text{Diagram 4} \\
 & = \binom{\Delta_{n+k+1}}{\Delta_{n+k}} \text{Diagram 5} \\
 & \stackrel{\cdot}{=}_{4(n+1)} \text{Diagram 6}
 \end{aligned}$$

which completes the argument.

□

Chapter 4

Walks Along Braids and the Closure of Positive Braids

4.1 Walks

In this Chapter, we will use the Huynh-Lê quantum determinant description of the colored Jones polynomial to derive an equivalent combinatorial model involving walks along braids. This new model will then be used to prove Main Theorem 2.

Definition 4.1. *A path along the braid $\beta(\gamma)$ from i to j is defined as follows:*

Beginning at the bottom of the i -th strand, follow the braid along a strand until you begin to cross over another strand. At this over crossing there is a choice to either continue along the strand or jump down to the strand below and continue following along the braid. Continue to the top of the braid ending at the j -th strand.

Each path is given a weight defined as follows:

At the j -th crossing, (i.e. the crossing corresponding to γ_j):

- If the path jumps down, assign a_{j,ϵ_j} .
- If the path follows the lower strand, assign b_{j,ϵ_j} .
- If the path follows the upper strand, assign c_{j,ϵ_j} .

The weight of the path is the product of the weights of the crossings. These weights are the same as the operators defined in Section 2.3. Recall that for different values of j , they all commute with each other, so the order of the product can be taken arbitrarily.

A walk W along β consists of a set $J \subset \{1, \dots, m\}$, permutation π of J , and a collection of paths where there is exactly one path from j to $\pi(j)$ in the collection, for each $j \in J$.

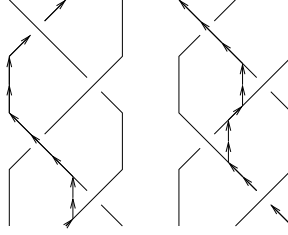


FIGURE 4.1: A path from 2 to 2 and from 3 to 1

The weight assigned to a walk is $(-1)(-q)^{|J|+\text{inv}(\pi)}$ times the product of the weights of the paths in the collection. Here the order is important since a single value of j can appear multiple times. The order is taken to be the same as the order of the starting positions at the bottom of the walk. An example of walks along the braid $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ will follow Theorem 4.2.

Note that we are reading the braid in two different directions. When writing the braid as a product of generators the braid is read from the top down. When walking along the braid it is read from the bottom up. This is important to get the order of the paths correct in the product of the weights.

Finally a stack of walks is an ordered collection of walks, and the weight of the stack is the product of the weights of walks in the appropriate order. We call this a stack because when considering a picture of these objects it can be thought of as simply stacking the walks on top of each other.

For a stack of walks W , we will also consider the local weight at a crossing j , denoted W_j , which is the product of the weights of the crossing for each walk in the stack. Note that with this notation we can express the weight of W as $(-1)^n(-q)^{\sum_k(|J_k|+\text{inv}(\pi_k))} \prod_j W_j$, where the sum ranges over the walks in the stack and n is the number of walks in the stack.

It will be useful to talk about an ordering on the paths that make up a stack. If two paths belong to two different walks at different level of the stack, then the path in the higher walk (that is the walk whose weight is multiplied to the left

of the other) is said to be above the other path. If the two paths are in the same walk, then the path which begins to the left of the other path is said to be above the other path.

We can now state the theorem relating the colored Jones polynomial to walks. This is simply a reinterpretation of Theorem 2.3. The proof will be presented in Section 4.2.

Theorem 4.2. *Given a braid $\beta(\gamma)$ whose closure is the knot K ,*

$$\begin{aligned} J_{N,K}(q) &= q^{(N-1)(\omega(\beta)-m+1)/2} \sum_{n=0}^{\infty} \mathcal{E}_N(C^n) \\ &= q^{(N-1)(\omega(\beta)-m+1)/2} \mathcal{E}_N(\mathcal{S}) \end{aligned}$$

where the polynomial C is the sum of the weights of walks on $\beta(\gamma)$ with $J \subset \{2, \dots, m\}$. Furthermore, $\mathcal{S} = \sum_{n=0}^{\infty} C^n$ is the sum of the weights of the stacks of walks on $\beta(\gamma)$ with $J \subset \{2, \dots, m\}$.

Before presenting an example, there is a simplification we can make to this theorem. It turns out that in general there will be several canceling terms in this sum. The following lemma states what some of these cancellations are.

Definition 4.3. *A simple walk is a walk in which no two paths in the collection traverse the same point on the braid.*

Lemma 4.4. *a) For any nonsimple walk $\beta(\gamma)$, there is another walk whose weight is the negative of the original. The nonsimple walks occur in canceling pairs.*

b) For any stack of walks which traverse the same point on N different levels and has weight W , the evaluation $\mathcal{E}_N(W)$ of that weight will be zero.

In other words, part a) of this lemma tells us that in Theorem 4.2 the occurrences of the word “walks” may be replaced with “simple walks”. In later sections, when Theorem 4.2 is applied, we will assume all walks are simple. Part b) assures us

that the sum will be finite and gives us a limit on the stacks of walks we need to consider. The proof of this lemma will follow an example.

Example 4.5. Let $\gamma = ((1, +), (2, -), (1, +), (2, -))$. Thus $\beta = \beta(\gamma) = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ and K , the closure of β , is the figure-eight knot. We also have that $m = 3$ and $\omega(\beta) = 0$. The only two walks along β which do not start or end at the first strand are presented in the following figure:

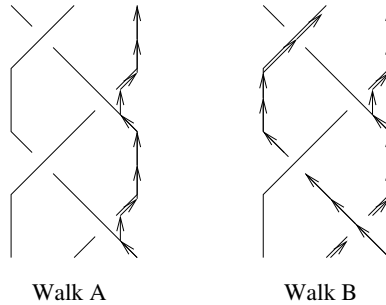


FIGURE 4.2: Two Example Walks

Notice that the walk A consists of a single path from 3 to 3 and walk B consists of two paths, one from 2 to 3 and one from 3 to 2. The only possible walks which do not start at the first strand are paths from 2 to 2, of which there are none, paths from 3 to 3, the only one being A , and walks consisting of two paths, one of which starts at 2 and the other starts at 3, and they end at 2 and 3 (not necessarily respectively); B is the only walk of this kind. It is an easy exercise to confirm that there are no other such walks.

We will often not distinguish between the weight of a walk and the walk itself as it should be clear from context. For instance, in this example $A = qa_{2,-}a_{4,-}$ and $B = q^3a_{2,-}b_{4,-}c_{1,+}b_{3,+}c_{4,-}$.

Thus by Theorem 4.2 and Lemma 4.4

$$J_{N,K}(q) = q^{(1-N)} \sum_{n=0}^{N-1} \mathcal{E}_N((qa_{2,-}a_{4,-} + q^3a_{2,-}b_{4,-}c_{1,+}b_{3,+}c_{4,-})^n)$$

The reason the sum stops at $N - 1$ is because both A and B traverse the bottom right corner of the braid, and thus any stack with more than $N - 1$ levels will evaluate to zero by part b) of Lemma 4.4.

We will use Lemma 2.2 to evaluate this sum for given values of N .

First for $N = 2$,

$$\mathcal{E}_2(qa_{2,-}a_{4,-}) = q(1 - q^{-1})^2$$

$$\begin{aligned} \mathcal{E}_2(q^3a_{2,-}b_{4,-}c_{1,+}b_{3,+}c_{4,-}) &= \mathcal{E}_2(q^3c_{1,+}a_{2,-}b_{3,+}b_{4,-}c_{4,-}) \\ &= q^3 * q * (1 - q^{-1}) * q^{-1} \\ &= q^3(1 - q^{-1}) \end{aligned}$$

Thus

$$\begin{aligned} J_{2,K}(q) &= q^{-1}(1 + q(1 - q^{-1})^2 + q^3(1 - q^{-1})) \\ &= q^2 - q + 1 - q^{-1} + q^{-2} \end{aligned}$$

which is the ordinary Jones polynomial for the figure-eight knot.

For $N \geq 2$, we need to expand the binomial. Since operators with different subscripts commute, this means that $AB = qBA$ because $a_{4,-}b_{4,-}c_{4,-} = qb_{4,-}c_{4,-}a_{4,-}$.

Thus

$$(A + B)^n = \sum_{k=0}^n \binom{n}{k}_q B^k A^{n-k}$$

$$J_{N,K}(q) = q^{(1-N)} \sum_{n=0}^{N-1} \sum_{k=0}^n \binom{n}{k}_q q^{n+k(k+1)} \left[\prod_{j=1}^n (1 - q^{j-N}) \right] \left[\prod_{i=1}^{n-k} (1 - q^{k+i-N}) \right]$$

where $\binom{n}{k}_q$ is the q-binomial coefficient

$$\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{1 - q^{n-i}}{1 - q^{i+1}}$$

proof of Lemma 4.4. Part a: Consider a walk W on the braid β where a point on the braid is traversed more than once, such as the walk in Figure 4.3. There may in general be many such points that are traversed more than once. Consider the highest crossing (equivalently the crossing with the lowest index number) in which two paths in W separate and call it's index I . For example, in Figure 4.3, $I = 2$ because the two paths separate at the second crossing from the top, which is crossing 2. There is another walk, call it W' which passes through the same points as W , but at crossing I , the two paths that separate take the opposite direction than was taken in W . Figure 4.3 shows a pair of walks which differ in this way. Either W or W' has the property that the two paths which pass through crossing I separate so that the one that began to the left of the other one is to the left of the other one immediately after crossing I . Denote the walk with this property as $W^{(1)}$ and the other as $W^{(2)}$. Also denote the previously mentioned path that begins to the left of the other one as $X^{(i)}$ and the path that begins to the right of the other one as $Y^{(i)}$, so that $X^{(i)}$ and $Y^{(i)}$ are paths in $W^{(i)}$. In Figure 4.3, the walk on the left is $W^{(1)}$ and the walk on the right is $W^{(2)}$. Also, the paths consisting of the straight arrows are the $X^{(i)}$'s and the paths consisting of the round arrows are the $Y^{(i)}$'s.

The claim is that the weight of $W^{(1)}$ is the negative of the weight of $W^{(2)}$, and thus when they are added together in the colored Jones polynomial, they will

cancel out. This is clearly a bijection from the set of nonsimple walks to itself, which means we need not consider any walks of this type.

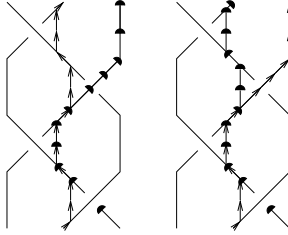


FIGURE 4.3: A Pair of Nonsimple Walk

Denote $W_j^{(i)}$ for the local weight of $W^{(i)}$ at crossing j . Observe that at crossing I , the walks $W^{(1)}$ and $W^{(2)}$ consists of an a_{I,ϵ_I} and c_{I,ϵ_I} in some order. If $\epsilon_I = +$, then $W_I^{(1)}$ is $a_{I,+}c_{I,+}$ and $W_I^{(2)}$ is $c_{I,+}a_{I,+} = q^{-1}a_{I,+}c_{I,+}$. If $\epsilon_I = -$, then $W_I^{(1)}$ is $c_{I,-}a_{I,-}$ and $W_I^{(2)}$ is $a_{I,-}c_{I,-} = q^{-1}c_{I,-}a_{I,-}$. In both cases, $W_I^{(2)} = q^{-1}W_I^{(1)}$.

The paths $X^{(i)}$ and $Y^{(i)}$ may continue to cross above crossing I , but since they cannot meet again it must be with b 's and c 's. If they cross at a positive crossing j with $X^{(i)}$ over $Y^{(i)}$, then $W_j^{(i)}$ will be $c_{j,+}b_{j,+}$. If they cross at a positive crossing j with $Y^{(i)}$ over $X^{(i)}$, then $W_j^{(i)}$ will be $b_{j,+}c_{j,+} = q^2c_{j,+}b_{j,+}$. If they cross at a negative crossing j with $Y^{(i)}$ over $X^{(i)}$, then $W_j^{(i)}$ will be $b_{j,-}c_{j,-}$. And if they cross at a negative crossing j with $X^{(i)}$ over $Y^{(i)}$, then $W_j^{(i)}$ will be $c_{j,-}b_{j,-} = q^2b_{j,-}c_{j,-}$.

If the paths $X^{(i)}$ and $Y^{(i)}$ cross an even number of times above crossing I , then the paths $X^{(i)}$ and $Y^{(i)}$ will contribute one inversion in the permutation associated to $W^{(2)}$ which is not in $W^{(1)}$. If the paths $X^{(i)}$ and $Y^{(i)}$ cross an odd number of times above crossing I , then the paths $X^{(i)}$ and $Y^{(i)}$ will contribute one inversion in $W^{(1)}$ which is not in $W^{(2)}$. Assuming no other paths cross $X^{(i)}$ or $Y^{(i)}$ then in both cases we see that $W^{(2)} = -W^{(1)}$, and the proof is complete.

There are, however, several possible cases where another path crosses one of X_i or Y_i . We will work through one case, and the other cases can be worked through similarly.

Suppose there is a path Z in $W^{(i)}$ that begins between $X^{(i)}$ and $Y^{(i)}$, and ends between $X^{(i)}$ and $Y^{(i)}$. The only difference in the weights of $W^{(1)}$ and $W^{(2)}$ comes from the crossings between Z and $X^{(i)}$ and $Y^{(i)}$, as well as two additional inversions in the permutation of either $W^{(1)}$ or $W^{(2)}$. The difference between the weights at crossings above I involving Z and one of the special paths will again be $b_\epsilon c_\epsilon$ versus $c_\epsilon b_\epsilon$ as described earlier. The path Z must cross one of $X^{(1)}$ and $Y^{(1)}$ an even number of times and the other an odd number of times. Thus change in the weights at crossings involving Z will be $q^{\pm 1}$, and the change coming from the inversions involving Z will be $q^{\mp 1}$. Thus in this case we again have $W^{(2)} = -W^{(1)}$.

Part b: Suppose a stack of walks W traverses a point on the braid on N or more different levels. Starting at that point follow along the braid until you reach an over-strand of a crossing. If the top of the braid is reached first, then the same position at the bottom of the braid will have the same starting positions there as there were ending positions, so we can consider the walks to continue from the bottom. Thus we can follow along the braid until we reach an over-strand. The weight at this over-strand will be the product of N or more c 's and a 's with possibly some additional b 's. If there is at least one a and less than N c 's, then by Lemma 2.2, the evaluation of the local weight $\mathcal{E}_N(W_j)$ at that crossing will have a factor of $(1 - q^0)$ and thus $\mathcal{E}_N(W) = 0$. If the number of c 's at this crossing is N or more however, continue along the over-strand until the next over-strand, possibly starting from the bottom if you reach the top of the braid again. If this process continues until you reach the original point where this process started without ever having come across an over-strand with less than N c 's taken, then the part

of the braid traversed through this process will be a component of the closure of the braid. However, the closure of this braid is a knot, and there is a point on the closure of the braid that could not have been traversed corresponding to the lower left starting position on the braid. Thus, the traversed area could not be a component and thus there must be a crossing with $\mathcal{E}_N(W_j) = 0$.

□

4.2 Proof of Theorem 4.2

If K is the closure of a braid $\beta(\gamma) \in B_m$, recall from Theorem 2.3:

$$J_{N,K}(q) = q^{(N-1)(\omega(\beta)-m+1)/2} \sum_{n=0}^{\infty} \mathcal{E}_N(C^n)$$

where

$$C = \sum_{\emptyset \neq J \subset \{1,2,\dots,m\}} (-1)^{|J|-1} \det_q(A_J)$$

and A_J is the J by J submatrix of A .

In order to prove Theorem 4.2, we need to show that the polynomial C in Theorem 2.3 defined by the quantum determinant, is the same as the polynomial C in Theorem 4.2 which was defined to be the sum of the weights of the walks along $\beta(\gamma)$.

Step 1: The main idea is that the matrix multiplication corresponds to the choices made during a walk. More explicitly, if $\rho(\gamma) = (M_{i,j})$, then $M_{i,j}$ is the sum of the weights of the paths from j to i . We will show this by induction on the length of the braid word.

Base case: $\rho(\emptyset) = I_m$, also $\beta(\emptyset) = e_m$, where I_m is the $m \times m$ identity matrix, and e_m is the identity braid on m strands. It is straight-forward to see that the sum of the weights of paths from j to i along the identity braid is 1 if $i = j$ and 0 otherwise.

Inductive step: Now suppose the claim is true for braid words of length k and consider a braid word $\beta(\gamma)$ of length $k + 1$. If $\gamma_1 = (l, +)$, (i.e. the first letter in $\beta(\gamma)$ is σ_l) then

$$\rho(\gamma) = (1 \oplus \dots \oplus 1 \oplus \begin{pmatrix} a_{1,+} & b_{1,+} \\ c_{1,+} & 0 \end{pmatrix} \oplus 1 \oplus \dots \oplus 1)(\rho(\gamma'))$$

where γ' has length k .

If $\rho(\gamma) = (M_{i,j})$ and $\rho(\gamma') = (M'_{i,j})$, then

$$M_{i,j} = M'_{i,j} \text{ for } i \neq l, l + 1$$

$$M_{l,j} = a_{1,+}M'_{l,j} + b_{1,+}M'_{(l+1),j}$$

$$M_{(l+1),j} = c_{1,+}M'_{l,j}$$

Let us now compare this with the paths along $\beta(\gamma)$.

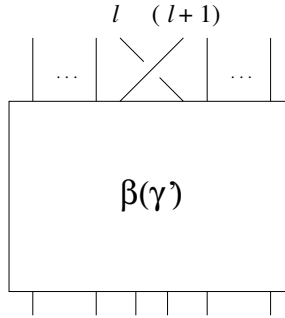


FIGURE 4.4

By induction, $M'_{i,j}$ is the sum of the weights of the paths along $\beta(\gamma')$ from j to i , so the sum of the weights along $\beta(\gamma)$ from j to i when $i \neq l, (l+1)$ is also $M'_{i,j}$. The paths from j to l come in two types: those that walk along $\beta(\gamma')$ from j to l and then jump down at crossing 1, and those that walk along $\beta(\gamma')$ from j to $(l+1)$ and follow along the lower strand of crossing 1. Thus the sum of the weights of these paths is $a_{1,+}M'_{l,j} + b_{1,+}M'_{(l+1),j}$. Finally, the paths from j to $(l+1)$ consists of

paths along $\beta(\gamma')$ from j to l and then following along the upper strand of crossing 1. Thus the sum of the weights of these paths is $c_{1,+}M'_{l,j}$. This completes step 1.

Step 2: The rest of the proof is simply following through the definitions of the weights of walks and the inverse of the quantum determinant.

The polynomial C from Theorem 2.3 is the sum

$$C = \sum_{\emptyset \neq J \subset \{1,2,\dots,m\}} (-1)^{|J|-1} \det_q(\rho'(\gamma)_J).$$

Since $\rho'(\gamma)$ is just $\rho(\gamma)$ with the first row and column removed, we can write C as

$$C = \sum_{\emptyset \neq J \subset \{2,\dots,m\}} (-1)^{|J|-1} \det_q(\rho(\gamma)_J).$$

For a particular subset J , the expression $(-1)^{|J|-1} \det_q(\rho(\gamma)_J)$ is precisely the sum of the weights of the walks for the given J . Thus C is the sum of the weights of all walks for all J .

4.3 Positive Braids: Proof of Main Theorem 2

Suppose that $\beta(\gamma)$ is a positive braid, meaning that $\epsilon_j = +$ for all j . To prove Main Theorem 2 we will show that, in the sum $\sum_n \mathcal{E}_N(C^n)$, every monomial with $n > 0$ has degree at least N .

Let us consider lowest terms in the evaluation of a stack of walks W . In order to apply Lemma 2.2 we need to rearrange the order of the product. Since the operators corresponding to different crossings commute, we can rearrange the product and evaluate the weight at each crossing.

Define W_j to be the product of the local weights at crossing j . Let A_j , B_j , and C_j be the number of a 's, b 's, and c 's respectively in the local weight W_j at crossing j . Also let A_{B_j} be the number of pairs of an a and a b in the local weight at crossing j such that the a is to the left of the b , in other words, the number of “commutations” that would need to be made to arrange the letters so that all b 's

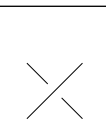
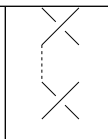
appear to the left of all a 's. Define, similarly, notation for all combinations of A , B , and C .

For a stack of walks W , by Lemma 2.2 the lowest degree will be:

$$\sum_k (|J_k| + \text{inv}(\pi_k)) + \sum_j w_j$$

where $w_j = A_{C_j} - 2C_{B_j} - C_j A_j + (N - 1)C_j$

The goal now is to find a useful lower bound for this sum. Firstly, now that we have an explicit sum for the minimum degree, we will modify the terms in the sum without changing the total value. At each crossing, we will add or subtract 1 to the term w_j every time two paths cross each other. If the path originally on the left is above the other path, then add 1; if the path originally on the right is above the other path subtract 1. The first column of the following table shows how paths might cross each other and the result of adding or subtracting 1 from each crossing of paths that occurs at that crossing in the braid. The second column shows the situation where two paths may come together (this can only happen if the two paths are on different level of the stack) and then separate, which may or may not count as a crossing of paths. If the paths separate without crossing then we may add 1 at one of the crossings and subtract 1 at the other, so that resultant sum is not changed. If the paths do cross from this situation, then we will add or subtract 1 at only one of the crossings.

| | | |
|-------------------|---|--|
| |  |  |
| L over R (top) | $+C_{B_j}$ | $+C_{A_j}$ |
| R over L (bottom) | $-B_{C_j}$ | $-B_{A_j}$ |

By adding or subtracting 1 in this way throughout, we can describe how the total sum will change. The original sum was $\sum_j w_j$. Call the new sum $\sum_j v_j$, where v_j

is the new term coming from crossing j after all of the additions and subtractions are applied. If you restrict to only paths on the same walk, say the k -th walk, then the result of all of the additions and subtractions will be a change by $\text{inv}(\pi_k)$. If you restrict to two different walks and consider only the ± 1 's coming which occurs between paths on these two different levels, then all of these additions and subtractions will result in no change to the total sum. This can be seen as follows:

Each of the walks themselves can be thought of as braids. The closure of these braids are links. If we stack the two links coming from these two walks the way the walks are stacked, then we can see that the additions and subtractions are just the calculation of the sum of the linking numbers of different components of these links, where the different components come from the different links. This is obviously 0 since one link is entirely above the other.

Thus

$$\sum_j v_j = \sum_k \text{inv}(\pi_k) + \sum_j w_j$$

To every crossing, we have added: $C_B - B_C - B_A + C_A$

Thus

$$v_j = (N - 1)C_j - C_j B_j - B_{Aj}$$

Now, because $\sum_j B_j = \sum_j C_j$, we can define

$$\begin{aligned} u_j &:= (N - 1)B_j - C_j B_j - B_{Aj} \\ &\geq (N - 1)B_j - C_j B_j - B_j A_j \\ &= (N - 1 - C_j - A_j)B_j \\ &\geq 0 \end{aligned}$$

And we get

$$\sum_j u_j = \sum_j v_j$$

There will necessarily be a crossing which has some number of b 's and no a 's or c 's. Call this crossing ι and we then have

$$u_\iota \geq (N - 1 - C_\iota - A_\iota)B_\iota \geq N - 1$$

Finally we get that the lowest degree in $\mathcal{E}_N(W)$ is

$$\begin{aligned} & \sum_k (|J_k| + \text{inv}(\pi_k)) + \sum_j w_j \\ &= \sum_k |J_k| + \sum_j v_j \\ &= \sum_k |J_k| + \sum_j u_j \\ &\geq 1 + u_\iota \\ &\geq N \end{aligned}$$

Chapter 5

The Head and Tail and the Reduced A and B-Graphs

5.1 The Reduced A-Graph and Torus Knots

This chapter consists of joint work with Oliver Dasbach.

The following theorem is an immediate corollary of Theorem 3.10:

Corollary 5.1. *If two alternating links L_1 and L_2 with reduced alternating diagrams D_1 and D_2 such that the reduced A-graphs of D_1 and D_2 coincide, then $T_{L_1}(q) = T_{L_2}(q)$.*

Let L_p be the $(2, p)$ torus link. The reduced A-graph of L_p is simply two vertices connected by an edge, and the reduced B-graph of L_p is C_p , the cycle of length p . Because L_p is the closure of a positive braid, it follows from Main Theorem 2 that $T_{L_p}(q) = 1$. We will now find a formula for $H_{L_p} = T_{\bar{L}_p}$.

Definition 5.2. 1. *The general (two variable) Ramanujan theta function (e.g. [1]):*

$$\begin{aligned} f(a, b) &:= \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} \\ &= \sum_{k=0}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} + \sum_{k=1}^{\infty} a^{k(k-1)/2} b^{k(k+1)/2} \end{aligned}$$

Note that $f(a, b) = f(b, a)$.

2. *The false theta function (e.g. [16]):*

$$\Psi(a, b) := \sum_{k=0}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} - \sum_{k=1}^{\infty} a^{k(k-1)/2} b^{k(k+1)/2}$$

Note that $\Psi(a, b) - 2 = -\Psi(b, a)$.

There are already known formulas for the colored Jones polynomial of Torus links. The first, due to Morton [17], applies to all torus knots, while the second, due to Hikami [8], applies to $(2, p)$ torus links for all p .

Theorem 5.3 (Morton). For $K = T(m, -p)$,

$$(q^{N/2} - q^{-N/2})J_{N,K}(q) = q^{-mp(c^2+c)-1/2} \sum_{r=-c}^c (q^{r^2 mp+rp+rm+1} - q^{r^2 mp-rp+rm})$$

where $c = (N - 1)/2$

Theorem 5.4 (Hikami). For $L = T(2, -2k)$,

$$(q^{N/2} - q^{-N/2})J_{N,L}(q) = q^{-\frac{k(N^2-1)+1}{2}} \left(\sum_{r=0}^{N-1} q^{kr^2+(k+1)r+1} - \sum_{r=0}^{N-1} q^{kr^2+(k-1)r} \right)$$

The following theorem is essentially a corollary to Theorems 5.3 and 5.4.

Theorem 5.5. 1. Let $K = T(2, -(2k + 1))$,

$$T_K(q) = f(-q^{2k}, -q)$$

2. Let $L = T(2, -2k)$,

$$T_L(q) = \Psi(q^{2k-1}, q)$$

Proof. Let $p := 2k + 1$. By Theorem 5.3 we have

$$\begin{aligned} (q^N - 1)J_{N,K}(q) &\doteq \sum_{r=-(N-1)/2}^{(N-1)/2} q^{p(2r^2+r)} q^{2r+1} - q^{p(2r^2-r)} q^{2r} \\ &\doteq \sum_{R=-N+1}^N (-1)^R q^{p(R^2-R)/2} q^R \\ &\doteq \sum_{R=-N+1}^N (-1)^R q^{k(R^2-R)} q^{(R^2+R)/2} \end{aligned}$$

Since $k(R^2 - R) + (R^2 + R)/2$ is increasing in $|R|$ the first result follows from the definition of $f(a, b)$.

Now let $p = 2k$. By Theorem 5.4:

$$(q^N - 1)J_{N,L}(q) \doteq \left(\sum_{r=0}^{N-1} q^{kr^2+(k+1)r+1} - \sum_{r=0}^{N-1} q^{kr^2+(k-1)r} \right)$$

Together with

$$\begin{aligned}
\Psi(q^{2k-1}, q) &= \sum_{r=0}^{\infty} q^{(2k-1)r(r+1)/2} q^{r(r-1)/2} - \sum_{r=1}^{\infty} q^{(2k-1)r(r-1)/2} q^{r(r+1)/2} \\
&= \sum_{r=0}^{\infty} q^{kr^2+(k-1)r} - \sum_{r=1}^{\infty} q^{k(r^2-r)+r} \\
&= \sum_{r=0}^{\infty} q^{kr^2+(k-1)r} - \sum_{r=0}^{\infty} q^{kr^2+(k+1)r+1}
\end{aligned}$$

the result follows. □

5.2 Products of Tails

In this section we present a method to reduce certain graphs into the product of two simpler graphs, which will then allow us to calculate tails of certain knots and links from simpler knots and links whose tails are already known.

Theorem 5.6. *Let L_1 and L_2 be alternating links. Any alternating link L_3 whose reduced A -graph can be formed by gluing the reduced A -graphs of L_1 and L_2 along a single edge (as in Figure 5.1) satisfies the statement:*

$$T_{L_1}(q)T_{L_2}(q) = T_{L_3}(q)$$

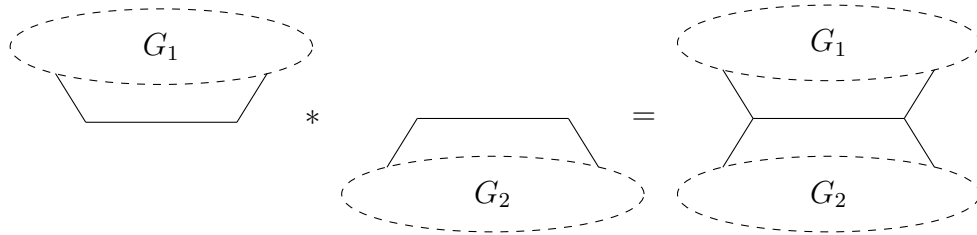


FIGURE 5.1: Product of two checkerboard graphs

Proof. The proof of this theorem uses Theorem 3.10, so because in the skein picture $q = A^{-4}$ we will consider the mirror images K_1^*, K_2^* and K_3^* of K_1, K_2 , and K_3 . Thus it is their reduced B -graphs which are related as in the statement of the theorem.

In Figures 5.2 and 5.3, the interior of the dotted regions represent $S(D^2, n, n, 2n)$ the Kauffman Bracket Skein Module of the disk with three colored points. This is known to be one dimensional when the three colored points are admissibly colored, generated by a single trivalent vertex [12]. Thus any element of $S(D^2, n, n, 2n)$ is some rational function times the generator. Let $\bar{\Gamma}$ be the closure of Γ by filling in the outside of the dotted circle by a single trivalent vertex as in Figure 5.3. Also define a bilinear pairing $\langle \Gamma_1, \Gamma_2 \rangle$ which identifies the boundaries of Γ_1 and Γ_2 as in Figure 5.2.

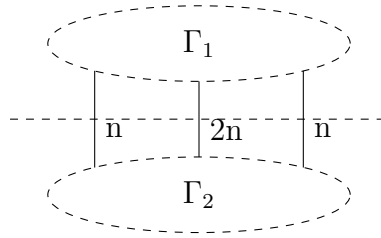


FIGURE 5.2: The bilinear pairing $\langle \Gamma_1, \Gamma_2 \rangle$

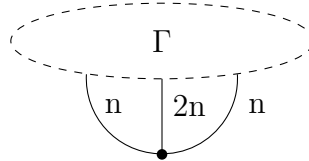


FIGURE 5.3: The closure of Γ

By Theorem 3.10, there is a Γ_1 and Γ_2 , such that

$$\begin{aligned} \Delta_n J_{n+1, K_1^*}(A) &\doteq_{4(n+1)} \bar{\Gamma}_1 \\ \Delta_n J_{n+1, K_2^*}(A) &\doteq_{4(n+1)} \bar{\Gamma}_2 \\ \Delta_n J_{n+1, K_3^*}(A) &\doteq_{4(n+1)} \langle \Gamma_1, \Gamma_2 \rangle \end{aligned}$$

By the fact that $S(D^2, n, n, 2n)$ is one-dimensional, there are rational functions R_i for $i = 1, 2$ such that $\Gamma_i = R_i*$ the trivalent vertex. Thus

$$\begin{aligned}\bar{\Gamma}_1 &= R_1\theta(n, n, 2n) \\ \bar{\Gamma}_2 &= R_2\theta(n, n, 2n) \\ \langle \Gamma_1, \Gamma_2 \rangle &= R_1R_2\theta(n, n, 2n)\end{aligned}$$

Therefore

$$\begin{aligned}J_{n+1, K_1^*}(A)J_{n+1, K_2^*}(A) &\stackrel{\cdot}{=}_{4(n+1)} R_1R_2 \left(\frac{\theta(n, n, 2n)}{\Delta_n} \right)^2 \\ &= R_1R_2 \left(\frac{\Delta_{2n}}{\Delta_n} \right)^2 \\ &= R_1R_2 \left(\frac{A^{2(2n+1)} - A^{-2(2n+1)}}{A^{2(n+1)} - A^{-2(n+1)}} \right)^2 \\ &\stackrel{\cdot}{=}_{4(n+1)} J_{n+1, K_3^*}\end{aligned}$$

This last line is true because $\frac{A^{2(2n+1)} - A^{-2(2n+1)}}{A^{2(n+1)} - A^{-2(n+1)}} \stackrel{\cdot}{=}_{4(n+1)} 1$.

□

Example 5.7. Figure 5.4 depicts the knot 9_{20} together with its reduced A and B-graphs. In the sense of Theorem 5.6 the reduced B-graph is the product of two squares and a triangle. The reduced A-graph is the product of two triangles.

Thus the head and tail functions of the colored Jones polynomial of 9_{20} are:

$$T_{9_{20}}(q) = f(-q^2, -q)^2$$

and

$$H_{9_{20}}(q) = \Psi(q^3, q)^2 f(-q^2, -q).$$

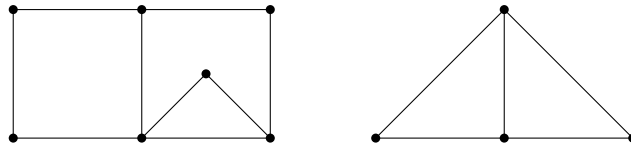
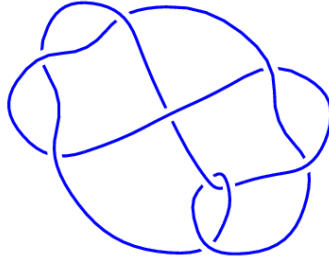


FIGURE 5.4: The knot 9_{20} and its reduced A and B-graphs. The reduced B-graph is on the left and the reduced A-graph on the right

5.3 Heads and Tails of 2-Bridge Links

The product structure described in Section 5.2 can be used to calculate the heads and tails of a large class of knots and links. One commonly studied family of knots and links that fall under this class is that of 2-bridge links. Every 2-bridge link has a diagram of the form of one of the two types illustrated in Figure 5.5 where each a_i is positive and the boxes represent twist regions as in Figure 5.6; thus the diagram is a reduced alternating diagram.

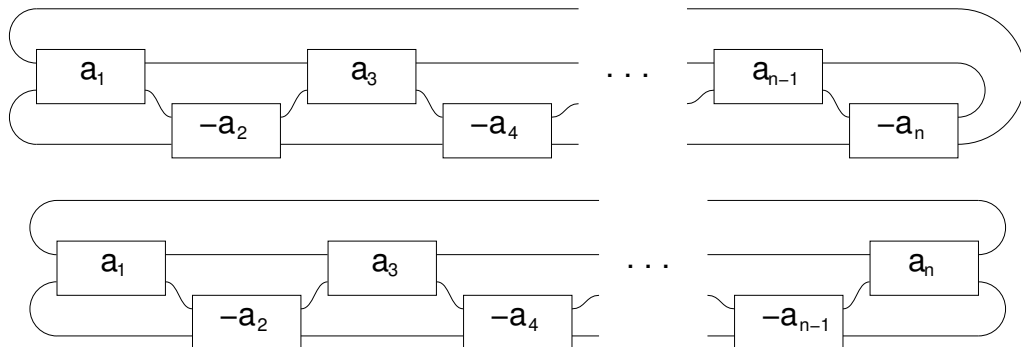


FIGURE 5.5: Diagrams of 2-bridge links with an even and odd number of twist regions

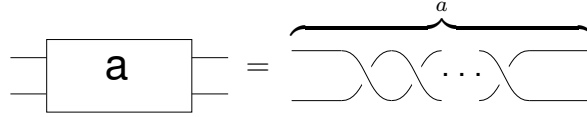


FIGURE 5.6: A twist region

The A and B-graphs of a diagram as in Figure 5.5 is simple to determine. Figure 5.7 gives an example of a 2-bridge knot and its reduced A and B-graphs.

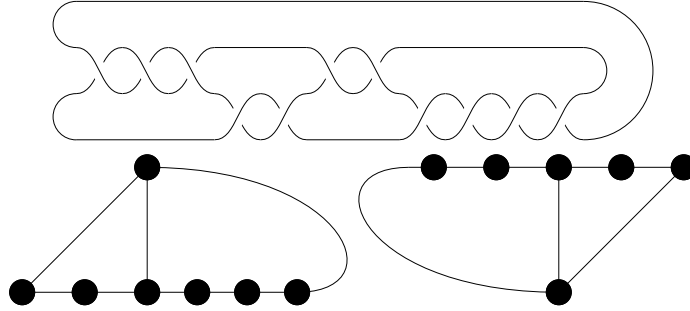


FIGURE 5.7: A 2-bridge knot and its reduced A and B-graphs. The reduced A-graph is on the left and the reduced B-graph on the right

Using Theorems 5.5 and 5.6 we can calculate the heads and tails of 2-bridge knots. To simplify the statement of the theorem, let us first define:

Definition 5.8.

$$h_k(q) := \begin{cases} f(-q^{k-1}, -q) & \text{if } 2 < k \text{ is odd} \\ \Psi(q^{k-1}, q) & \text{if } 2 < k \text{ is even} \\ 1 & \text{if } k = 2 \end{cases}$$

Theorem 5.9. *If L is the 2-bridge link corresponding to the sequence (a_1, \dots, a_n) , then:*

a) if n is even, then

$$T_L(q) = h_{a_2+2}(q)h_{a_4+2}(q) \dots h_{a_{n-2}+2}(q)h_{a_n+1}(q)$$

and

$$H_L(q) = h_{a_1+1}(q)h_{a_3+2}(q)h_{a_5+2}(q) \dots h_{a_{n-1}+2}(q)$$

b) if n is odd, then

$$T_L(q) = h_{a_2+2}(q)h_{a_4+2}(q) \cdots h_{a_{n-1}+2}(q)$$

and

$$H_L(q) = h_{a_1+1}(q)h_{a_3+2}(q)h_{a_5+2}(q) \cdots h_{a_{n-2}+2}(q)h_{a_n+1}(q)$$

References

- [1] G. E. Andrews, and B. C. Berndt, *Ramanujan's Lost Notebook: Part I., Volume 1*, Springer, 2005.
- [2] C. Armond, and O. T. Dasbach, *Rogers-Ramanujan Type Identities and the Head and Tail of the Colored Jones Polynomial*, arXiv:1106.3948v1
- [3] D. Bar-Natan, and S. Garoufalidis, *On the Melvin-Morton-Rozansky Conjecture*, Invent. Math. 125 (1996) 103-133.
- [4] A. Champanerkar, and I. Kofman, *On the Tail of Jones Polynomials of closed braids with a full twist*,
- [5] O. Dasbach, and X.-S. Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math., 142 (2006), No. 5, 1332-1342.
- [6] O. Dasbach, and X.-S. Lin, *A Volum-ish Theorem for the Jones polynomial of alternating knots*, Pacific J. Math., 231 (2007), No. 2, 279-291.
- [7] S. Garoufalidis, and T.T.Q. Le, *Nahm sums, stability and the colored Jones polynomial*, arxiv1112.3905v1.
- [8] K. Hikami, *Quantum Invariant for Torus Links and Modular Forms*, Communications in Mathematical Physics **246** (2004), no. 2, 403-426.
- [9] V. Huynh, and TTQ. Lê, *On the colored Jones polynomial and the Kashaev invariant*, Fundam. Prikl. Mat. **11** (2005), no. 5, 57-78.
- [10] V. Jones, *Hecke algebra representations of braid group and link polynomials*, Ann. Math. **126** (1987), 335-388.
- [11] L. H. Kauffman, *States models and the Jones polynomial*, Topology **26** (1987) 395-407.
- [12] W. B. R. Lickorish, *An Introduction to knot theory*, Springer, 1997.
- [13] X.-S. Lin, F. Tian, and Z. Wang, *Bureau representation and random walk on a string link*, Pacif. J. Math., **182** (1998), 289-302.
- [14] X.-S. Lin, Z. Wang, *Random walk on knot diagrams, colored Jones polynomial and Ihara-Selberg zeta function*, Proceedings of the Birmanfest, AMS/IP Studies in Advanced Mathematics, vol. 24, 2001, pp. 107-121.
- [15] G. Masbaum, and P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J. Math., 164 (1994), No. 2, 361-381.

- [16] J. Mc Laughlin, A. V. Sills, and P. Zimmer, *Rogers-Ramanujan-Slater Type Identities*, Electronic Journal of Combinatorics **15** (2008), no. DS15, 1-59.
- [17] H. R. Morton, *The coloured Jones function and the Alexander polynomial for torus links*, Mathematical Proceedings of the Cambridge Philosophical Society **117** (1995), no. 01, 129-135 (English).
- [18] V. G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988) no. 3, 527-553.

Vita

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