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Two dimensional sediment transport model using parallel computers

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TWO DIMENSIONAL SEDIMENT TRANSPORT MODEL USING PARALLEL COMPUTERS

A Thesis

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Master of Science in Civil Engineering
in

The Department of Civil and Environmental Engineering

by

Vikas Singh
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Abstract

Management and development of water bodies is vital for meeting domestic, agricultural, energy and industrial needs. To that end, dams, artificial channels, lakes and other water structures have been constructed. Management and development of these structures encounter problems of land erosion, reservoir silting, and degradation and aggradation of channel beds, which need to be addressed. Fundamental to these problems are sediment transport, erosion and deposition.

Numerical modeling of sediment transport is the best tool to simulate sediment transport in a water body. This study develops a vertically integrated two-dimensional numerical sediment transport model. Sediment transport is simulated in two parts in this model: suspended load and bed load. A fractional step approach is used to solve the two-dimensional advection diffusion equation, which splits the advection-diffusion equation into two separate parts: advection and diffusion. High resolution conservative algorithm is used to solve the advection part and a semi implicit finite difference scheme is used to solve the diffusion part. Different parallel numerical solvers are developed to solve linear system of equations resulting from diffusion part. Non-uniformity in sediment mixture which is quite common in real world problems is considered. The model is tested for different analytical and laboratory test cases. The model is coded for parallel computers so that enormous power of parallel computers can be exploited.
Chapter 1

Introduction

River management is as old as human civilization, as human settlements developed at the banks of rivers. Since ancient times rivers or water bodies have been used for irrigation, navigation, power generation, waste disposal, recreation and water supply. It is impossible to imagine life without rivers or other water bodies. Due to industrial growth, consumption of natural water resources has increased rapidly. Therefore, management and development of water resources is very important.

A river is a dynamic system. Rivers flow over loose erodible sediments and they carry, scour and deposit these sediments. As a result, rivers meander, braid and branch out. Contributing to these river characteristics are hydraulic and environmental factors. The time period could ranges from million of years to centuries to decades. Rivers alter the landscape of the earth’s surface, meaning that changes in river morphology have a direct impact on earth’s landscape.

Initially river flow was affected by natural and other climatic factors. As humans started using water for water supply, irrigation and navigation, the need for maintenance and development of water bodies became necessary. As a result they started constructing structures, such as reservoirs, dams, irrigation canals, and levees, etc. With increase in industrial and economic growth, especially over the past half a century, more and more man-made multipurpose structures were built. Nature and man-made structures affect the forces acting on the rivers and the way they respond. Perhaps the most common response is the change in sediment transport capacity which, in turn, changes river position and shape, leading to a multiple of engineering and environmental problems. Some of the common problems are:

1. land erosion and conservation,
2. silting of reservoirs,
3. degradation and aggradation of channel beds,
4. silt excluders,
5. navigation,
6. coastal erosion,
7. flooding, and
8. ecosystem changes

To address these problems it is important to investigate sediment transport for given conditions and how it changes with changes in conditions, so that the response of rivers or other water bodies can be predicted due to changes in natural or man-induced forces, and measures can be
taken to mitigate damages.

Sediment transport deals with both flow of water and sediment particles. Therefore, properties and theories of both water flow and sediment transport should be studied. In a water body sediments are transported as suspended and bed load, depending upon the sediment particle size, as shown in figure 1.1.

![Figure 1.1: Schematic diagram for different modes of sediment transport](image)

### 1.1 Developments in Modelling

Study of sediment transport can be traced back to 4000 years back in China. Very significant advances have been done in the last half a century. Initially experimentation was the main tool to analyze and investigate the behavior of river responses due to hydraulic or climate changes. Experimental studies are limited to laboratory experiments of real sites. These studies had limited value, as laboratory studies were done under controlled conditions which is highly uncommon in real life situations. On the other hand, it is difficult and expensive to do real site studies and conditions of one site may differ a lot from those of the other real site. Still these studies helped to understand the basic concepts of water flow and sediment transport. Many investigators have developed a wide range of theoretical and analytical models based on experimental studies. These analytical methods also can not applied effectively on real world problems. The reasons for this are that these studies make many simplifications and the dimensional scales of these studies are very small. Nevertheless importance of these physical and analytical studies cannot be neglected as these tools can be used to validate popular complex models of today.

With the advent of computers, a new era of sediment transport studies started in the form of computer models. Initially simpler analytical sediment transport were modeled on computers. These computer models proved useful and effective, as the use of computer for modeling or sim-
ulation cut down the time and man power required to do the computation. Also these computer model predictions and simulations were quite accurate.

Then came the era of numerical models, which completely changed the world of modeling and simulations. In these numerical methods a water system is represented by partial differential equations, which represent the conservation of mass and conservation of momentum. Numerical methods are used to solve these partial differential equations. The use of numerical methods for solving partial differential equations made it possible to solve complex partial differential equations, which were not able to be solved by analytical methods. Numerical modeling made real life system modeling possible due to its capability of solving complex partial differential equations, which can be used as a representative of real world problems. Another advantage of these numerical models is that it is easy to implement them on computers. These numerical models can simulate physical processes more accurately than experimental and analytical methods. For these reasons numerical modeling is widely used nowadays for all types of real life problems. With advances in technologies, computational power has been increasing continuously, so more sophisticated models are developing and getting implemented.

Still numerical modeling of physical processes on computer is a new and challenging field. Although a number of computer models have been developed for numerical modeling of sediment transport, there is still a lot of need in developing sediment transport model to analyze new problems. Another aspect in the field of modeling is to develop models which can be run on supercomputers, in order to utilize enormous computational power, hence cut down the simulation time tremendously or do a very large simulation.

1.2 Objectives

The objective of this study is to develop a vertically integrated two dimensional numerical sediment transport model. This model is divided in two parts: hydrodynamic modeling and sediment transport modeling. Hydrodynamic modeling simulates flow velocities which are then used in the sediment transport model to simulate sediment concentrations. To represent the sediment transport system in a flow, the conservative form of two dimensional advection diffusion equation is used. To solve this equation a fractional step method, also known as standard split approach (Sobey 1983, Dragsolav 2001), is used. This approach splits the advection diffusion equation in two parts: advection and diffusion, which are solved separately. To solve the advection part, a high resolution conservative algorithm for advection in incompressible flow developed by Leveque (1996) is used. To solve the diffusion part, a semi-implicit finite difference scheme is used. In this model different parallel numerical solvers are developed for solving the resulting linear system of equations. Another objective of this study is to parallelize the computer model to run on a supercomputer. The model also includes non-uniformity of sediment particle sizes.

To run the model a parallel cluster at LWRRI of Louisiana State University is used. The cluster is made of 32 node, each Pentium 2 350MZ with 2568 MB of RAM.
Chapter 2

Literature Review

As stated earlier, the oldest known sediment transport study was done around 4000 years ago in China. A significant work has been done in the last century in the field of sediment transport. All the studies can be classified in two broad categories: physical and mathematical. Figure 2.1 shows a schematic diagram for different kinds of sediment transport, erosion, deposition and bed change studies and their interrelationship.

2.1 Physical Studies

Physical studies are done by doing experiments in laboratory flumes or by taking field observations. Laboratory studies are not well representative of the river system as it is difficult to represent a river by a laboratory flume. So a lot of assumptions are usually incorporated in laboratory studies. Still these laboratory studies are important for verification of other studies and also to understand basic concepts of river flow and sediment transport. Many investigators have developed empirical methods to represent sediment transport phenomena using data obtained by laboratory studies.

Field studies by taking real time observations can be better tools to understand the complex real life river systems, as it is very difficult to take real time observations of river in the field and some time it is even impossible. Some of the widely used laboratory studies done till now are quoted below.

One of the oldest and still widely used studies was done by Newton (1951). The main objective of the study was to study the nature of degradation of the bed in an open channel. This study was done by uniform sediment size.

Bhamidipaty (1971) did extensive laboratory flume studies for three different sediment particle sizes using uniform sediment grain size for each experimental run. One of the objectives of the study was to investigate the degradation of the bed below a dam due to the release of comparatively less sediment loaded water from the reservoir. Another objective was to study the process of aggradation in a canal due to the difference in the sediment transport capacities between canal and the river from which canal was withdrawing the water.

One of the useful studies was done by Soni (1975). He did experiments to study the phenomena of aggradation in streams due to overloading of sediments. Soni used a mobile bed condi-
Figure 2.1: Schematic diagram for different sediment studies and their interrelationship

As stated earlier, these laboratory studies are vital to verify and validate any mathematical or analytical method to represent sediment transport phenomena. Most of the investigators who conducted laboratory studies and other investigators used these studies to develop empirical methods to represent sediment transport phenomena. Some of the empirical methods developed are...
Soni (1981) developed a similarity curve method using dimensional analysis and laboratory flume data to predict aggradation in channel due to excess sediment input and then equilibrium sediment transport capacity. Input to this method was equilibrium flow condition and excess sediment supply for estimation of aggradation.

Bhamidipaty (1971) developed an empirical relationship for estimating the bed profile of a degrading channel based upon channel length, sediment size particles and some parameters. These parameters were functions of initial shear stress due to grain roughness.

2.2 Mathematical Studies

Physical studies have the limitations due to the complexity of representing a real life river conditions through an experimental flume. Due to this restriction investigators made many assumptions during the experimental runs according to the requirement of the study. These assumptions limited the scope of these studies to apply them to real life problems.

To overcome this problem many investigators developed mathematical equations and their solutions to represent the sediment transport concepts in real life situations. All the mathematical models developed so far are based on the following five basic equations. These equations are written only in one dimension and can be extended for all three dimensions.

(1) Continuity equation for water flow

\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0
\]  

(2.1)

where

- \(Q\) = discharge
- \(A\) = cross-section area

(2) Momentum equation for water flow

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial z}{\partial x} = 0
\]

(2.2)

where

- \(g\) = gravitational acceleration
- \(z\) = flow depth

(3) Flow resistance equation

\[
U = aS^b
\]

(2.3)

where

- \(a, b\) = parameters
- \(S\) = bed slope

(4) Continuity equation for sediment

\[
\frac{\partial A}{\partial t} + \frac{1}{1 - \lambda} \frac{\partial G}{\partial x} = 0
\]

(2.4)
where
\( \lambda \) = porosity of sediment mixture
\( G \) = Sediment transport rate

(5) Sediment transport capacity equation

\[
G = cU^d
\]  \hspace{1cm} (2.5)

where
\( c, d \) = parameters
\( U \) = Mean flow velocity

Both analytical and numerical solutions have been developed to solve these equations by many investigators. Analytical solutions of these equations are useful due to their simplicity and effectiveness but analytical solutions can be developed and applied only in very simplified and simple cases. Numerical solutions are very effective in solving the complex differential equations in complicated conditions.

### 2.3 Analytical Sediment Transport Model

Normally, analytical solutions are developed in those cases where flow conditions are very simplified and can be lumped in one or two directions. It is difficult to develop analytical solutions for generalized two or three dimensional cases with complex conditions. Still analytical solutions are important to verify the numerical model as it is very difficult to obtain data for many conditions. Some of the well known analytical sediment models are summarized below.

One of the oldest models was developed by Tinney (1955). He solved a one dimensional differential equation analytically to simulate the degradation of bed composed of uniform sediment in an open channel. He compared his result with Newton (1951) and his result well fitted the data. Al-Khalif (1965) used the Einstein (1950) approach to develop a bed load function for a degrading channel and used that function to describe degradation. Jaramillo (1983) solved the linear parabolic sediment transport model to estimate bed load discharge for a semi-infinite and finite domain. He estimated the bed elevation using the expression of bed load discharge rate and sediment continuity equation. Gill (1983) developed a model to simulate the bed change in both aggradation and degradation using a linear parabolic bed elevation analytical model for a finite length channel. Jaramillo and Jain (1984) developed a nonlinear parabolic sediment model without considering flow non-uniformities. Zhang and Kahawita (1987) solved a nonlinear parabolic aggradation model and showed that bed elevation is a function of square root of time. Jain (1985) used the method of weighted residuals to solve a nonlinear parabolic aggradation model. De Vries (1973) used convection-acceleration and depth gradient terms and developed a linear hyperbolic bed elevation change model. Mosconi (1988) developed two different models separately for aggradation and degradation processes. He developed a linear hyperbolic analytical model for aggradation in the case of increase of sediment discharge and nonlinear parabolic analytical model for degradation in the case of reduction of sediment discharge.
2.4 Numerical Sediment Transport Models

All analytical mathematical models of sediment transport phenomena developed are based on the assumption of steady state or quasi steady state water flow, as unsteady state of water flow makes the system complex and it is difficult to develop an analytical solution for that complex system. This assumption is normally not valid in real life problems. To overcome this limitation investigators developed numerical methods to solve sediment transport equations in complex situations. This approach is further encouraged by advancement in the field of computers as these methods need enormous computation. Till now many numerical sediment transport models have been developed. All numerical models developed so far can be divided in three categories according to dimensions in one dimensional, two dimensional and three dimensional models. Some of the widely known and used numerical models are listed below.

2.4.1 One Dimensional Sediment Transport Model

In one dimensional sediment transport modeling concentration is averaged in lateral and vertical directions. This is the simplest mode of sediment transport modeling as it involves equations only in one direction. It is easy to implement this approach as analytical solutions can be developed easily for one dimensional differential equations, but this approach cannot be implemented in the case where longitudinal or vertical flow is also important. Many one dimensional analytical and numerical sediment transport models have been developed so far. Cunge (1980), Jansen (1979) and De Vries (1989) have reviewed one dimensional models. Garde (1965) developed a one dimensional numerical model for simulation for aggradation under the quasi-steady state flow conditions. Gesseler (1971) used a finite difference method to develop a numerical one dimensional model to predict aggradation and degradation. His scheme predictions are good in the case of rotational aggradation and degradation. De Vries (1967) used an explicit finite difference scheme to develop a numerical model to compute bed elevation and water surface profile during aggradation in the channel. This method was able to produce accurate results by imposing a restriction on the time step. Cunge (1973) used the same model and solved it with an implicit finite difference scheme to overcome the problem of time step restriction. Swamee (1974) developed a numerical model for aggradation at the upstream of a dam for constant discharge. He used the method of iteration and solved the equation for small time step value and then smoothened the final bed profile using Fourier sine series. Mahmood (1975) developed a numerical model by taking account of variation of suspended load with time and distance. He used an implicit finite difference scheme to solve the model numerically. Muskatirovic (1978) developed a model for the channel with depth varying along the cross section. He used the Preissmann four point scheme to numerically solve the partial differential equations.

Park and Jain (1986) developed a one dimensional model using the Preissmann scheme to simulate the change in the bed elevation due to overloading of the sediment. Bhallamudi and Chaudhry (1991) used the second order accurate explicit scheme to develop a one dimensional model. Some of the widely used one dimensional models are MIKE11 (DHI, 2003) and HEC-6 (USACE, 1993) for sediment transport, erosion and deposition in straight channels and rivers.
2.4.2 Two Dimensional Sediment Transport Model

In two dimensional sediment transport model sediment concentration is averaged in one direction, normally in vertical direction depending upon the flow characteristics and field requirements. Based on this integration two dimensional models can be classified as depth integrated and laterally integrated two dimensional models. In depth integrated models all the model parameters and variables are assumed to be the same throughout a water column. Application of two dimensional models is more complicated as compared to one dimensional models as this approach needs more resources in all aspects. Two dimensional models are most popular models than others as they provide enough information of the desired quantity for the project requirement in optimum resources. Some of the two dimensional models developed so far are described in the next section.

Struikisma (1985) developed a two dimensional sediment transport model to simulate the large scale bed change at Delft Hydraulics. Shimizu and Itakura (1989) developed a two dimensional bed load transport model for alluvial channels. Chaudhary (1996) developed a two dimensional bed load sediment transport model for straight and meandering channels. Some of the widely used two dimensional sediment transport models are MIKE21 (DHI 2003), TABS-MD (Thomas and McAnally, 1990), CCHE2D (Wu W., 2001) and HSCTM2D (Hayter, 1995).

One of the most popular sediment transport models is CCHE2D sediment transport model (Wu, 2001) developed at the National Center for Computational Hydroscience and Engineering, University of Mississippi. The CCHE2D model has a non equilibrium sediment transport model for suspended load and an equilibrium sediment transport model for bed load. The CCHE2D model is capable of taking account of non-uniform sediment mixtures with many size classes. In the CCHE2D model an exponential difference scheme is used to solve the suspended sediment transport equation and first order upwind scheme is used to solve the bed load transport equation.

HSCTM2D (Hydrodynamic, Sediment and Contaminant Transport Model) model was developed for U.S. Environmental Protection Agency. It is a finite element two dimensional, vertically integrated model for cohesive sediments. HSCTM2D is composed of two parts. The first is hydrodynamic modelling part named as HYDRO2D and second is contaminant and sediment transport modelling part known as CS2D. HSCTM2D can be used for both short term and long term simulations.

2.4.3 Three Dimensional Sediment Transport Model

Three dimensional sediment transport models are most informative as they include all the space dimensions. They are most complicated and resource consuming in implementation. Three dimensional models are avoided until very detailed distribution of desired quantity needs to be simulated and flow characteristics are important in all directions. Three dimensional models are mostly applied in the condition when flow is stratified like flow of fresh water over salt water or flow of warm water over cold water.

Many researchers have developed three dimensional models till now. Wang and Adeff (1986) developed a three dimensional finite element model for unsteady flow. Lin and Falconer (1996) developed a three dimensional model for estuaries and coasts. Van Rijn (1987) combined three dimensional sediment transport model and two dimensional depth integrated flow model. Demuren and Rodi (1986) developed a three dimensional flow and neutral tracer transport model
using $k-\varepsilon$ model. Demuren (1991) extended the model by including bed load transport and suspended load transport. Wu (2000) developed a three dimensional flow and sediment transport model for straight and meandering channels. Some of the most widely used three dimensional models are ECOMSED (HydroQual, Inc, 2003), CCHE3D (find out), Delft-3D (Delft Hydraulics, 2003).

ECOMSED sediment transport model was developed by HydroQual Inc. and named as SED module of ECOMSED(2002). This model was specially developed for estuaries and oceans. That is why it is applicable only up to a diameter size of 500\(\mu\)m and cannot be applied for bed load transport. The SED model is a three dimensional suspended sediment transport model for non-cohesive sediments. It takes account of cohesive sediment properties and treats cohesive and non cohesive sediments separately.
Chapter 3
Methodology

The sediment transport phenomena requires an understanding of physical properties of sediments, which are discussed here.

3.1 Physical Properties of Sediment

3.1.1 Size

Size is the most important parameter to describe any physical property of sediment particles. Many sediment properties depend primarily upon the sediment size. The sediment size can be measured by methods like sieve analysis, calipers, optical method, and photographic method. Lane (1947) divided sediments according to their sizes and that classification was adopted by American Geophysical Union and is still used by hydraulic engineers. Sediment particle size governs the mode of transport in water body. Bigger size particles like silt, sand etc. are transported as bed load and fine sediment particles like clay are transported as suspended load.

3.1.2 Shape

The shape parameter of sediments is defined by the geometric formation regardless of sediment particle size and chemical composition. This shapes is important to describe many sediment properties, such as fall velocity, incipient motion, etc. Shape parameter affects the settling velocity and critical shear stress. The shape of a sediment does not match commonly used shapes such as cubic, circular, rectangular, etc. Hence many investigators have defined shape as a single parameter. Out of many derived shape parameters some mostly used parameters are as follows:

(1) Sphericity
(2) Roundness
(3) Schulz (1954) shape parameter

3.1.3 Density

The density of sediment particles primarily depends on their mineral composition. Usually, the specific gravity of sediment is used as an indicator of density. Specific gravity is defined as the
ratio of the specific weight or density of sediment and the specific weight or density of water. The value of specific gravity of sediment varies from 2.3 for coal to 7.6 for galena. For water borne sediments the value of specific gravity is normally taken as 2.65.

### 3.2 Sediment Transport

Sediment is transported in water bodies as suspended load and bed load. Bed load is defined as the sediment load which moves along the bed. Suspended load is defined as the sediment load which moves in suspension and occupies the entire flow depth above the bed load layer. According to the sediment particle contribution to bed evolution, the total sediment transport can be divided in bed material load or wash load. Wash load is that part of sediment load which washes through the channel. It consists of very fine silt and clay and they do not play a significant role in evolution of bed and because of that the percentage of these size particles in bed is relatively less. Bed load is that part of the sediment load that is mainly responsible for bed evolution. Bed material mainly consists of these sediment particles. Figure 3.1 shows a schematic diagram of total sediment load. In this figure it is shown that both wash load and bed material consist of suspended load and bed load.

![Figure 3.1: Different mode of sediment transport load](image)

As bed material load is most important in bed evolution, the bed material load transport
is simulated in the model. The bed load and suspended load of bed material load are simulated separately to take account of their individual properties. In model when mode of transport for a particular sediment size fraction is defined as a choice, it means that the user has to define the mode of transport.

### 3.3 Suspended Load Transport

A three dimensional conservation form of the advection-diffusion equation for sediment transport can be written as:

\[
\frac{\partial c_k}{\partial t} + \frac{\partial (uc_k)}{\partial x} + \frac{\partial (vc_k)}{\partial y} + \frac{\partial (wc_k)}{\partial z} - \frac{\partial (ws_c k)}{\partial z} = \frac{\partial}{\partial x} \left( k_x \frac{\partial c_k}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial c_k}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial c_k}{\partial z} \right)
\] (3.1)

Equation 3.1 can be converted in a pure advection system by neglecting the diffusion part and can be written as:

\[
\frac{\partial c_k}{\partial t} + \frac{\partial (uc_k)}{\partial x} + \frac{\partial (vc_k)}{\partial y} + \frac{\partial (wc_k)}{\partial z} = 0
\] (3.2)

Equation 3.1 can be converted in a pure diffusion system by neglecting the advection part and can be written as:

\[
\frac{\partial}{\partial x} \left( k_x \frac{\partial c_k}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial c_k}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial c_k}{\partial z} \right) = 0
\] (3.3)

To convert the three dimensional equation 3.1 into a two dimensional depth averaged equation, the depth averaged suspended load concentration is defined as

\[
C_k = \frac{1}{h-\delta} \int_{z_b+\delta}^{z_s} c_k dz
\] (3.4)

Integrating the three-dimensional advection-diffusion equation 3.1 over the suspended load zone,

\[
\int_{z_b+\delta}^{z_s} \frac{\partial c_k}{\partial t} + \int_{z_b+\delta}^{z_s} \frac{\partial (uc_k)}{\partial x} + \int_{z_b+\delta}^{z_s} \frac{\partial (vc_k)}{\partial y} + \int_{z_b+\delta}^{z_s} \frac{\partial (wc_k)}{\partial z} - \int_{z_b+\delta}^{z_s} \frac{\partial (ws_c k)}{\partial z}
\]

\[
= \int_{z_b+\delta}^{z_s} \frac{\partial}{\partial x} \left( k_x \frac{\partial c_k}{\partial x} \right) + \int_{z_b+\delta}^{z_s} \frac{\partial}{\partial y} \left( k_y \frac{\partial c_k}{\partial y} \right) + \int_{z_b+\delta}^{z_s} \frac{\partial}{\partial z} \left( k_z \frac{\partial c_k}{\partial z} \right)
\] (3.5)

Integration of equation 3.5 over the entire flow depth gives the following depth averaged two dimensional advection-diffusion equation:

\[
\frac{\partial}{\partial t} [(h - \delta)C_k] + \frac{\partial}{\partial x} [U(h - \delta)C_k] + \frac{\partial}{\partial y} [V(h - \delta)C_k]
\]

\[
= \frac{\partial}{\partial x} \left[ k_x (h - \delta) \frac{\partial C_k}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y (h - \delta) \frac{\partial C_k}{\partial y} \right] + E_k - D_k
\] (3.6)
where $U$ and $V$ are the depth averaged flow velocities in the X and Y, directions respectively; and $E_k$ and $D_k$ are erosion and deposition terms in upward and downward directions, respectively, and together known as source-sink term in the advection-diffusion equation. The source-sink term can be calculated as:

$$S_k = E_k - D_k = \alpha w_s (C_k^* - C_k)$$  \hspace{1cm} (3.7)

where $S_k = \text{the source sink term for specified sediment size}$, $\alpha$ is the non equilibrium adaptation coefficient, $w_s$ is the sediment particle settling velocity, and $C_k^*$ is the depth averaged sediment concentration under equilibrium condition or sediment transport capacity.

As the depth of the bed load zone is small compared to the flow depth $\delta << h$, equation 3.6 can be simplified as follows:

$$\frac{\partial}{\partial t} h C_k + \frac{\partial U h C_k}{\partial x} + \frac{\partial V h C_k}{\partial y} = \frac{\partial}{\partial x} \left[ k_x h \frac{\partial C_k}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y h \frac{\partial C_k}{\partial y} \right] + E_k - D_k$$  \hspace{1cm} (3.8)

To solve equation 3.8, a fractional step approach, also known as standard split approach (Sobey 1983), is used. In this approach both advection and diffusion parts of the advection diffusion equation are solved separately at each time step. Using a splitting approach very accurate numerical procedures can be used to solve advection and diffusion separately. A questionable part of this approach is that advection and diffusion parts are solved one after another, which makes them discrete, but in real life they occur simultaneously. This step introduces a splitting error in the solution irrespective of the accuracy of the schemes used to solve the advection and diffusion parts. However the magnitude of error is very less. This approach can be justified on the grounds that better and more accurate methods can be implemented for separate solutions of advection and diffusion parts. The fraction step method procedure is explained below. In general an advection diffusion transport equation can be written as:

$$\frac{\partial c}{\partial t} + L_a(C) - L_d(C) = 0$$  \hspace{1cm} (3.9)

where $L_a(C)$ is the advection part and $L_d(C)$ is the diffusion part including all source-sink terms. Equation 3.9 can be written using the Taylor series expansion for a nth time step as:

$$\frac{C^{n+1} - C^n}{\Delta t} + L_a(C^n) - L_d(C^n) = \frac{\partial^2 C^n}{\partial t^2} \frac{\Delta t}{2} + \ldots = O(\Delta t)$$  \hspace{1cm} (3.10)

Now introducing the fraction step approach and an intermediate variable $C'$, advection and diffusion parts can be written separately as:

$$\frac{C' - C^n}{\Delta t} + L_a(C^n) = \frac{\partial^2 C^n}{\partial t^2} \frac{\Delta t}{2} + \ldots = O(\Delta t)$$  \hspace{1cm} (3.11)

$$\frac{C^{n+1} - C'}{\Delta t} - L_d(C^n) = 0$$  \hspace{1cm} (3.12)

Equation 3.11 is a pure advection equation and so is equation 3.12. Both of these equations can be solved separately. The fraction-step procedure is independent of the scheme used for advection and diffusion parts. Numerical schemes used to solve for the advection and diffusion parts of the advection diffusion sediment transport equation are described below. Discretization of velocity, water depth and sediment concentration over the space is shown in the figure 3.2.
3.3.1 Advection Part

High resolution conservative algorithm for advection in incompressible flows developed by Leveque (1996) was used for solving the advection part. Leveque uses basic upwind method and proposed several correction terms to achieve better accuracy and stability. A conservative form of advection of a scalar concentration or density function $C(x,t)$ can be written in general as:

$$\frac{dC}{dt} + \nabla \cdot (\vec{u} C) = 0$$  \hspace{1cm} (3.13)

Assuming flow is incompressible

$$\nabla \cdot (\vec{u} x t) = 0$$  \hspace{1cm} (3.14)

From the generalized advection equation, two-dimensional advection equation can be written as:

$$c_t + (cu)_x + (cv)_y = 0$$  \hspace{1cm} (3.15)

and assuming flow is incompressible

$$u_x(x,y,t) + v_y(x,y,t) = 0 \quad \text{for all} \ x, y, t$$  \hspace{1cm} (3.16)

For incompressibility in discrete form for every cell in the discretized domain the following condition should satisfy:

$$(u^n_{i+1,j} - u^n_{i,j}) + (v^n_{i,j+1} - v^n_{i,j}) = 0$$  \hspace{1cm} (3.17)
To solve this conservative form of the advection equation Leveque (1996) used a basic upwind method in the flux differencing and later added correction terms to achieve better accuracy and stability. The upwind method is based on the flux calculation of the concentration at the cell interfaces and can be written as:

\[ \frac{C_{i,j}^{n+1} - C_{i,j}^n}{h} = \frac{k}{h}[F_{i+1,j} - F_{i,j} + G_{i,j+1} - G_{i,j}] \]  

(3.18)

where \( F_{i,j} \) represents the flux at the left interface of the cell \( C_{i,j} \) and \( F_{i+1,j} \) represents the flux at the right interface of the cell \( C_{i,j} \). Similarly \( G_{i,j} \) represents the flux at the bottom interface of the cell. \( C_{i,j} \) and \( G_{i,j+1} \) represent the flux at the top interface of the cell \( C_{i,j} \). Figure 3.3 shows the location of flux for a cell.

Figure 3.3: Representation of flux for a cell

These fluxes at the cell interfaces can be calculated as:

\[ F_{i,j} = u_{i,j}^n C_{i-1,j}^n \]
\[ G_{i,j} = v_{i,j}^n C_{i,j-1}^n \]  

(3.19)

In this whole section \( u \) and \( v \) are taken positive in the X and Y directions, respectively, and all the derivations are done by assuming that \( u \) and \( v \) are positive. In reality the directions of these fluxes at the interfaces depend upon the direction of the respective velocity vector. Thus equation 3.18 can be rewritten as:

\[ C_{i,j}^{n+1} = C_{i,j}^n - \frac{k}{h}[u_{i,j+1} C_{i,j}^n - u_{i,j} C_{i,j-1}^n + v_{i,j+1} C_{i,j}^n - v_{i,j} C_{i,j-1}^n] \]  

(3.20)

In this upwind method it is assumed that waves carrying differences \( (C_{i,j} - C_{i-1,j}) \) and \( (C_{i,j} - C_{i,j-1}) \) propagate perpendicular to the interfaces in the X and Y directions, respectively, at the speeds and directions given by velocities \( u \) and \( v \). This function can be achieved by using the wave propagation method assuming the above specified condition. In case of wave speed \((u,v)\) in the grid oblique to the interfaces a proper correction factor should be implemented. This correction can be incorporated by a two step procedure. In the first step the same upwind method is
used in which wave is propagated perpendicular to the interface and in the next step the remaining triangular part of the wave is used to update the flux between the cells due to its transverse motion. The area of the triangular part of the wave is $\frac{1}{2}k^2uv$ and due to this the cell average is modified by the value of $\frac{1}{2}h^2uv\Delta c$. In this quantity $\Delta c$ is the difference across the wave. This modification can be incorporated in the flux calculation of $F_{i,j}$ and $G_{i,j}$ as follows. For wave propagating,

$$F_{i,j} = F_{i,j} + u_{i,j}^{n+1}C_{i-1,j}$$

$$G_{i,j+1} = G_{i,j+1} - \frac{k}{2h}uv(C_i^n - C_{i-1,j}^n)$$

$$G_{i,j} = G_{i,j} + v_{i,j}^{n+1}C_{i,j-1}$$

$$F_{i+1,j} = F_{i+1,j} - \frac{k}{2h}uv(C_i^n - C_{i,j-1}^n)$$

(3.21)

The other $\frac{h}{k}$ term is incorporated in the flux differencing expression. This updated form of the upwind method which includes the transverse wave propagation is more stable and accurate than the original version of the upwind method specified in equation 3.19. This improved first order accurate method is known as the corner transport upwind method developed by Collela (1990).

To achieve second order accuracy in the algorithm, a second order Lax-Wendroff method is combined with the upgraded upwind method. The Lax-Wendroff method to calculate flux can be expressed as:

$$F_{i-1,j}^{LW} = \frac{1}{2}u_i(C_{i-1} + C_i) - \frac{k}{2h}u^2(C_i - C_{i-1})$$

(3.22)

The Lax-Wendroff scheme can also be rearranged as a combination of upwind method and a correction term as:

$$F_{i-1,j}^{LW} = u_iC_{i-1} + \frac{1}{2}|u| \left(1 - \frac{k}{h}|u|\right)(C_i - C_{i-1})$$

(3.23)

$$F_{i-1,j}^{LW} = F_{i-1,j}^{UP} + \frac{1}{2}|u| \left(1 - \frac{k}{h}|u|\right)(C_i - C_{i-1})$$

This approached is used to apply another correction term in the updated upwind method by adding the following term in the flux at the interface between $c_{i,j}$ and $c_{i-1,j}$. To avoid oscillation a flux limiting factor is also introduced in the term.

$$F_{i-1,j} = F_{i-1,j} + \frac{1}{2}|u| \left(1 - \frac{k}{h}|u|\right)(C_i - C_{i-1})\Phi_i$$

(3.24)

and similarly for the flux at the the interface between $C_{i,j}$ and $C_{i,j-1}$,

$$G_{i-1,j} = G_{i-1,j} + \frac{1}{2}|v| \left(1 - \frac{k}{h}|v|\right)(C_i - C_{i-1})\Phi_i$$

(3.25)

The flux limiting term $\Phi_i$ is defined as

$$\Phi_i = \phi(\theta_i), \quad \theta_i = \frac{q_i - q_{i-1}}{q_i - q_{i-1}}$$

(3.26)
where
\[
I = \begin{cases} 
  i - 1, & \text{if } u > 0 \\
  i + 1, & \text{if } u \leq 0
\end{cases}
\] (3.27)

Some standard limiters used in the algorithm are as follows:

- minmod: \( \phi(\theta) = \max(0, \min(1, \theta)) \),
- superbee: \( \phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)) \),
- van Leer: \( \phi(\theta) = \theta + |\theta| (1 + |\theta|) / 2 \),
- monotonized centered: \( \phi(\theta) = \max(o, \min(1 + \theta/2, 2, 2\theta)) \)

Leveque (1996) included the transverse propagation concept for the correction waves to increase the accuracy of the second order accurate method developed till now. This transverse motion of the correction wave at the interface between cells \( C_{i-1,j} \) and \( C_{i,j} \) modifies the flux \( F_{i,j} \) as described in the previous part and also modifies the fluxes \( G_{i-1,j+1} \) and \( G_{i,j+1} \) and can be calculated by the following expression. A flux limiter is also introduced in these expressions in the same way as in the previous expression to reduce oscillations. These flux limiters can be calculated in the same way as defined in equation 3.26:

\[
G_{i-1,j+1} = G_{i-1,j+1} - \frac{1}{2h} |u| v \left( 1 - \frac{k}{h} |u| \right) (c_{i,j} - c_{i-1,j}) \Phi_i
\] (3.28)

\[
G_{i,j+1} = G_{i,j+1} + \frac{1}{2h} |u| v \left( 1 - \frac{k}{h} |u| \right) (c_{i,j} - c_{i-1,j}) \Phi_i
\]

Similarly, transverse motion between cells \( C_{i,j-1} \) and \( C_{i,j} \) modifies the fluxes \( F_{i+1,j-1} \) and \( F_{i+1,j} \) and can be calculated using the following expression:

\[
F_{i+1,j-1} = F_{i+1,j-1} - \frac{1}{2h} v |u| \left( 1 - \frac{k}{h} v \right) (c_{i,j} - c_{i,j-1}) \Phi_i
\] (3.29)

\[
F_{i+1,j} = F_{i+1,j} + \frac{1}{2h} v |u| \left( 1 - \frac{k}{h} v \right) (c_{i,j} - c_{i,j-1}) \Phi_i
\]

These modifications in the flux calculations reduce the error. Leveque (1996) developed an algorithm by following all these steps one by one and that algorithm is shown in the appendix. The algorithm takes care of the directions of the velocity vectors.

### 3.3.2 Diffusion Part

To solve for the diffusion part of the advection diffusion sediment transport equation, a semi-implicit finite difference scheme is used. The semi-implicit finite difference scheme is implemented in such a way that it can easily be converted to a completely explicit or completely implicit scheme. A finite difference representation of the diffusion part can be written as follows:

\[
\frac{\partial ch}{\partial t} = \frac{\partial}{\partial x} \left( k_x h \frac{\partial c}{\partial x} \right) + \left( k_y h \frac{\partial c}{\partial y} \right) + S
\] (3.30)

where \( S \) is the source-sink term, and \( k_x \) and \( k_y \) are the diffusivity coefficient in X and Y directions, respectively. Now we solve the above equation for time steps \( \Delta t \) using an explicit finite difference
scheme. In the following solution superscript \( n \) represents the \( n \)th time step. Introducing a new variable \( A \), equation 3.30 can be rewritten as follows:

\[
\frac{\partial Ch}{\partial t} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y \tag{3.31}
\]

where

\[
A_x = k_x h \frac{\partial c}{\partial x} \tag{3.32}
\]

\[
A_y = k_y h \frac{\partial c}{\partial y} \tag{3.33}
\]

Now, writing the finite difference of equation 3.31

\[
\frac{C^{n+1}(i,j) - C^n(i,j)}{\Delta t} = \frac{A_x(i+1,j) - A_x(i,j)}{\Delta x} + \frac{A_y(i,j+1) - A_y(i,j)}{\Delta y} \tag{3.34}
\]

Here, \( A_x(i,j) \) and \( A_y(i,j) \) can be calculated as:

\[
A_x(i,j) = K_x(i,j)h(i,j) \left[ \theta \left( \frac{C^{n+1}(i,j) - C^{n+1}(i-1,j)}{\Delta x} \right) \right. \\
+ (1 - \theta) \left( \frac{C^n(i,j) - C^n(i-1,j)}{\Delta x} \right) \right]
\]

\[
A_y(i,j) = K_y(i,j)h(i,j) \left[ \theta \left( \frac{C^{n+1}(i,j) - C^{n+1}(i,j-1)}{\Delta y} \right) \right. \\
+ (1 - \theta) \left( \frac{C^n(i,j) - C^n(i,j-1)}{\Delta y} \right) \right]
\]

Using these finite difference expressions equation 3.34 can be written as:

\[
\frac{c^{n+1} - c^n}{\Delta t} = \frac{1}{\Delta x} \left[ K_x(i+1,j)h(i+1,j) \left\{ \theta \left( \frac{C^{n+1}(i+1,j) - C^{n+1}(i,j)}{\Delta x} \right) \right\} \\
+ (1 - \theta) \left( \frac{C^n(i+1,j) - C^n(i,j)}{\Delta x} \right) \right] \\
- K_x(i,j)h(i,j) \left\{ \theta \left( \frac{C^{n+1}(i,j)C^{n+1}(i-1,j)}{\Delta x} \right) \right\} \\
+ (1 - \theta) \left( \frac{C^n(i,j) - C^n(i-1,j)}{\Delta x} \right) \right]
\]

\( \tag{3.37} \)
\[
\begin{align*}
+ \frac{1}{\Delta y} & \left[ K_y(i, j + 1) h(i, j + 1) \left\{ \theta \left( \frac{C^{n+1}(i, j + 1) - C^{n+1}(i, j)}{\Delta y} \right) \right\} \\
+ (1 - \theta) & \left( \frac{C^n(i, j + 1) - C^n(i, j)}{\Delta y} \right) \right] \\
- K_y(i, j) h(i, j) & \left\{ \theta \left( \frac{C^{n+1}(i, j) - C^{n+1}(i, j - 1)}{\Delta y} \right) \right\} \\
+ (1 - \theta) & \left( \frac{C^n(i, j) - C^n(i, j - 1)}{\Delta y} \right) \right] \\
\end{align*}
\]

(3.38)

In this expression all the nth time step terms are known and (n+1)th time step terms are not known. Thus, the above equation can be rearranged in the following way:

\[
\begin{align*}
\frac{C^{n+1} - C^n}{\Delta t} & = \frac{1}{\Delta x} \left[ K_x(i + 1, j) h(i + 1, j) \left\{ \theta \left( \frac{C^{n+1}(i + 1, j) - C^{n+1}(i, j)}{\Delta x} \right) \right\} \\
- K_x(i, j) h(i, j) & \left\{ \theta \left( \frac{C^{n+1}(i, j) - C^{n+1}(i - 1, j)}{\Delta x} \right) \right\} \right] \\
+ \frac{1}{\Delta y} & \left[ K_y(i, j + 1) h(i, j + 1) \left\{ \theta \left( \frac{C^{n+1}(i, j + 1) - C^{n+1}(i, j)}{\Delta y} \right) \right\} \\
- K_y(i, j) h(i, j) & \left\{ \theta \left( \frac{C^{n+1}(i, j) - C^{n+1}(i, j - 1)}{\Delta y} \right) \right\} \right] \\
+ \frac{1}{\Delta x} & \left[ K_x(i + 1, j) h(i + 1, j) \left\{ (1 - \theta) \left( \frac{C^n(i + 1, j) - C^n(i, j)}{\Delta x} \right) \right\} \\
- K_x(i, j) h(i, j) & \left\{ (1 - \theta) \left( \frac{C^n(i, j) - C^n(i - 1, j)}{\Delta x} \right) \right\} \right] \\
+ \frac{1}{\Delta y} & \left[ K_y(i, j + 1) h(i, j + 1) \left\{ (1 - \theta) \left( \frac{C^n(i, j + 1) - C^n(i, j)}{\Delta y} \right) \right\} \\
- K_y(i, j) h(i, j) & \left\{ (1 - \theta) \left( \frac{C^n(i, j) - C^n(i, j - 1)}{\Delta y} \right) \right\} \right] \\
\end{align*}
\]

(3.39)
Equation 3.40 can be written in the following form:

\[
C^{n+1}(i+1, j) \left[ \frac{\Delta t}{\Delta x^2} k_x(i+1, j) h(i+1, j) \right] + C^{n+1}(i-1, j) \left[ \frac{\Delta t}{\Delta x^2} k_x(i, j) h(i, j) \right] + C^{n+1}(i, j+1) \left[ \frac{\Delta t}{\Delta y^2} k_y(i, j) h(i, j) \right] + C^{n+1}(i, j-1) \left[ \frac{\Delta t}{\Delta y^2} k_y(i, j) h(i, j) \right]
\]

\[-C^{n+1}(i, j) \left[ \frac{\Delta t}{\Delta x^2} k_x(i+1, j) h(i+1, j) + \frac{\Delta t}{\Delta y^2} k_y(i, j) h(i, j) + \frac{\Delta t}{\Delta y^2} k_y(i, j+1) h(i, j+1) + \frac{\Delta t}{\Delta y^2} k_y(i, j-1) h(i, j-1) \right]
\]

\[
\Delta t \left[ K_x(i+1, j) h(i+1, j) \left\{ (1-\theta) \left( \frac{C^n(i+1, j) - C^n(i, j)}{\Delta x} \right) \right\} - K_x(i, j) h(i, j) \left\{ (1-\theta) \left( \frac{C^n(i, j) - C^n(i-1, j)}{\Delta x} \right) \right\} + \Delta y \left[ K_y(i, j+1) h(i, j+1) \left\{ (1-\theta) \left( \frac{C^n(i, j+1) - C^n(i, j)}{\Delta y} \right) \right\} - K_y(i, j) h(i, j) \left\{ (1-\theta) \left( \frac{C^n(i, j) - C^n(i, j-1)}{\Delta y} \right) \right\} \right] \right] = -C^n(i, j) h(i, j) + \]

\[
\frac{\Delta t}{\Delta x} \left[ K_x(i+1, j) h(i+1, j) \left\{ (1-\theta) \left( \frac{C^n(i+1, j) - C^n(i, j)}{\Delta x} \right) \right\} - K_x(i, j) h(i, j) \left\{ (1-\theta) \left( \frac{C^n(i, j) - C^n(i-1, j)}{\Delta x} \right) \right\} + \Delta y \left[ K_y(i, j+1) h(i, j+1) \left\{ (1-\theta) \left( \frac{C^n(i, j+1) - C^n(i, j)}{\Delta y} \right) \right\} - K_y(i, j) h(i, j) \left\{ (1-\theta) \left( \frac{C^n(i, j) - C^n(i, j-1)}{\Delta y} \right) \right\} \right] \right] \right] \right] \right] \right] \right] \right] \right] \right] \right]

(3.40)

Equation 3.40 can be written in the following form:

\[
czu2(i, j) C^{n+1}(i+1, j) + czu1(i, j) C^{n+1}(i-1, j) + czv2(i, j) C^{n+1}(i, j+1) + czv1(i, j) C^{n+1}(i, j-1) - C^{n+1}(i, j) = b(i, j)
\]

(3.41)

where \(czu2(i, j)\) = coefficient of \(C^{n+1}(i+1, j)\)

\[
= \left[ \frac{\Delta t}{\Delta x^2} k_x(i+1, j) h(i+1, j) \right]
\]

(3.42)

\(czu1(i, j)\) = coefficient of \(C^{n+1}(i, j-1)\)

\[
= \left[ \frac{\Delta t}{\Delta x^2} k_x(i, j) h(i, j) \right]
\]

(3.43)
\[ \text{czv2}(i,j) = \text{coefficient of } C^{n+1}(i,j+1) \]
\[
= \left[ \frac{\partial}{\partial x} k_x(i+1,j)h(i+1,j) + \frac{\partial}{\partial y} k_y(i,j)h(i,j) + \frac{\partial}{\partial x} k_x(i,j+1)h(i,j+1) + \frac{\partial}{\partial y} k_y(i,j)h(i,j) + h(i,j) \right]
\]

\[ (3.44) \]

\[ \text{czv1}(i,j) = \text{coefficient of } C^{n+1}(i,j-1) \]
\[
= \left[ \frac{\partial}{\partial y} k_y(i,j)h(i,j) \right]
\]
\[
= \left[ \frac{\partial}{\partial x} k_x(i+1,j)h(i+1,j) + \frac{\partial}{\partial y} k_y(i,j)h(i,j) + \frac{\partial}{\partial x} k_x(i,j+1)h(i,j+1) + \frac{\partial}{\partial y} k_y(i,j)h(i,j) + h(i,j) \right]
\]

\[ (3.45) \]

\[ \text{b}(i,j) = \text{known terms} \]
\[
= -C^n(i,j)h(i,j) + \frac{\Delta t}{\Delta x} \left[ K_x(i+1,j)h(i+1,j) \left\{ (1-\theta) \left( \frac{c^n(i+1,j) - c^n(i,j)}{\Delta x} \right) \right\} \right.
\]
\[
- K_x(i,j)h(i,j) \left\{ (1-\theta) \left( \frac{c^n(i,j) - c^n(i-1,j)}{\Delta x} \right) \right\} \]
\[
+ \frac{\Delta t}{\Delta y} \left[ K_y(i,j+1)h(i,j+1) \left\{ (1-\theta) \left( \frac{c^n(i,j+1) - c^n(i,j)}{\Delta y} \right) \right\} \right.
\]
\[
- K_y(i,j)h(i,j) \left\{ (1-\theta) \left( \frac{c^n(i,j) - c^n(i,j-1)}{\Delta y} \right) \right\} \] \]

\[ (3.46) \]

Equation 3.41 can be represented as \( Ax = b \), as it represents a linear system of equations. To solve for the diffusion term, this linear system of equations needs to be solved. To that end, the following numerical schemes was used.

### 3.4 Solver for Linear System of Equations

To solve the simultaneous linear system of equations, many algorithm have been developed. In this study following iterative solvers were used and implemented:

1. Jacobi
2. Red black gauss siedel
3. Succeseive over relaxation (SOR)
4. Bi-CGSTAB
5. Bi-CGSTAB-2

These methods are discussed in the next sections.
### 3.4.1 Jacobi Method

A linear system of equations generated from partial differential equations can be solved by using the Jacobi method, which can be written as:

\[
u_{i,j}^{n+1} = \frac{1}{4} \left[ u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n \right]
\]

(3.47)

where \(u_{i,j}^n\) denotes the \(n\)th iterative value of \(u_{i,j}\). Iteration error for the whole grid can be calculated as:

\[
\text{Error} = \sqrt{\sum (\text{abs}(u_{i,j}^{n+1} - u_{i,j}^n))^2}
\]

(3.48)

When error converges to a desired tolerance iteration can be ended. The Jacobi method is very slow in converging, so this can be used only for small grid size problems. The Jacobi method iteration can also be used as a preconditioner iteration method required in some other methods which are discussed further.

### 3.4.2 Successive Over-relaxation (SOR) Method

Successive over-relaxation method can be written as:

\[
u_{i,j}^{n+1} = (1 - \omega)u_{i,j}^n + \frac{1}{4} \omega \left[ u_{i-1,j}^{n+1} + u_{i+1,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n+1} \right]
\]

(3.49)

The rate of convergence of the SOR iteration method depends upon the choice of \(\omega\), which is called as accelerating factor and lies between 1 and 2. There is no way to estimate the value of \(\omega\) for an iteration process for a particular problem. The only way to estimate the value of \(\omega\) is by hit and trial method. Initially some value of \(\omega\) is assumed and then it is changed until the best converging rate is achieved. This method is also included in the model. This method is not very good as each time one has to estimate the value of \(\omega\) for best results. Iteration error for this method can be calculated in the same way as explained in the Jacobi method.

### 3.4.3 Red Black Gauss Seidel Method

The Red Black Gauss Seidel Method is derived from the Gauss Seidel Method. The Gauss Seidel method iterative formula can be written as:

\[
u_{i,j}^{n+1} = \frac{1}{4} \left[ u_{i-1,j}^{n+1} + u_{i+1,j}^{n+1} + u_{i,j-1}^{n+1} + u_{i,j+1}^{n+1} \right]
\]

(3.50)

The difference between Gauss Seidel and Jacobi method is that this method uses the latest iterative values available for the grid points, while the Jacobi method uses only old iterative values for all points. Due to this change, the Gauss Seidel method convergence increases many times more than the Jacobi method. The Red Black Gauss Seidel is a modification of the Gauss Seidel Method. In the Red Black Gauss Seidel method, iteration is done for alternate points in a row.
3.4.4 Bi-CGSTAB

Van der Vorst (1995) showed that Bi-CGSTAB method is a variant of Bi-CG and GMRES methods. In this method a preconditioner is used to speedup the convergence. Bi-CGSTAB smooth down the irregular convergence. The Bi-CGSTAB algorithm is shown in the appendix.

3.4.5 Bi-CGSTAB(2)

Bi-CGSTAB(2) is the most robust and fast numerical solver to solve the linear system of equations. Algorithm for this solver is shown in the appendix.

3.5 Bed Load Transport

The differential equation for the bed load transport can be derived by integrating the three dimensional sediment transport equation over the bed load zone. The integration of the equation leads to the continuity equation for the bed load transport and can be written as:

\[(1 - p') \frac{\partial z_{bk}}{\partial t} + \frac{\partial (\delta C_{bk})}{\partial t} + \frac{\partial q_{bkx}}{\partial x} + \frac{\partial q_{bky}}{\partial y} - E_{bk} + D_{bk} = 0\]  (3.51)

where

- \(p'\) = porosity of the bed material,
- \(\delta\) = depth of bed load zone,
- \(C_{bk}\) = average concentration of sediment over the bed load zone layer for the kth size fraction
- \(z_{bk}\) = depth of the bed layer
- \(q_{bkx}, q_{bky}\) = bed load transport rate components for the kth size fraction in the X and Y directions respectively, which can be calculated as:

\[q_{bkx} = \alpha_{bx} q_{bk}\]
\[q_{bky} = \alpha_{by} q_{bk}\]  (3.52)

where

- \(q_{bk}\) = bed load transport rate which is represented as

\[q_{bk} = \overline{U} \delta C_{bk}\]  (3.53)

\(\alpha_{bx}\) and \(\alpha_{by}\) = direction cosines in the X and Y directions, respectively, of bed load transport rate \(q_{bk}\).

It is assumed that the bed load transport rate is along the bed shear stress so that these direction cosines can be calculated using flow properties. In equation 3.51 the first term represents the change in the bed elevation in time which can be calculated as sum of bed change caused by suspended load and bed load.

\[(1 - p') \frac{\partial z_{bk}}{\partial t} = \alpha \omega_{bk} (C_k - C_k^*) + \frac{(q_{bk} - q_{bk}^*)}{L_t}\]  (3.54)
where
\( \alpha = \) non equilibrium adaptation coefficient.

Inserting equations 3.52 in equation 3.51, differential equation for bed load transport can be written as:

\[
\frac{\partial (\delta C_{bk})}{\partial t} + \frac{\partial (\alpha_q q_{bk})}{\partial x} + \frac{\partial (\alpha_q q_{bk})}{\partial y} + \frac{(q_{bk} - q_{\ast bk})}{L_t} = 0
\]

(3.55)

Using equation 3.53, equation 3.55 can be written as:

\[
\frac{\partial (\delta C_{bk})}{\partial t} + \frac{\partial (\alpha_q U \delta C_{bk})}{\partial x} + \frac{\partial (\alpha_q U \delta C_{bk})}{\partial y} + \frac{(U \delta C_{bk} - q_{\ast bk})}{L_t} = 0
\]

(3.56)

which is same as the suspended sediment transport equation with no diffusion part. Therefore the same fractional step method was used to solve this equation. In the first method only the advection part is solved using the algorithm described above and in the next part only source-sink term is added to obtain the new time step value.

### 3.6 Source-sink Term

In both suspended load and bed load transport equation the source-sink term needs to be calculated. The source-sink term for suspended load transport is defined as \( \alpha_w (C_k - C_k^\ast) \) and the source-sink term for bed load transport is defined as \( \frac{U \delta C_{bk} - q_{\ast bk}}{L_t} \). To calculate these terms the following parameters and sediment properties in the flow should be estimated: Expressions to calculate \( \alpha \) non-equilibrium adaptation coefficient, \( L_t \) non-equilibrium adaptation length, \( \alpha_{sk} \) settling velocity, and sediment load transport capacity (suspended load, bed load or bed material load)

### 3.7 Nonequilibrium Adaptation Length \( L_t \) and Coefficient \( \alpha \)

The nonequilibrium adaptation length \( L_t \) is defined as the length in which sediment concentration changes from nonequilibrium state to equilibrium state, In other words, it is the length in which the river bed adjust itself according to the nonequilibrium sediment concentration to achieve the equilibrium sediment concentration. This parameter is a very important parameter in the model. Till now many investigators Wang (1999), Philips and Sutherland (1989), Thuc (1991), Wu, Rodi and Wenka (2000), Rahuel (1989) and Fang (2000) among others have used this parameters for sediment transport modelling but have assigned significantly different values for the parameters and different bases for choosing those values. Bell and Sutherland (1983) used a time varying value of nonequilibrium adaptation length in his degradation experiment due to clear water over sediment bed. He expressed that sediment transport in the experiment was governed by the scour hole developed at the inlet of the water moving downstream with time, therefore, he used time dependent \( L_t \). Philips and Sutherland (1989), Thuc (1991), Wu, Rodi and Wenka (2000) used the length of sand ripples on the bed as the value of nonequilibrium adaptation length.
(1984) chose the length of sand dunes as the value for nonequilibrium adaptation length as in his case sand dunes were the most dominant bed form. Thus, as one can see that investigators have used a very wide range of values for the nonequilibrium adaptation length. So in this study the value of nonequilibrium adaptation length $L_t$ is taken as a user-defined parameter.

The nonequilibrium adaptation coefficient $\alpha$ is also assigned different values by different investigators in their studies. Han et al. (1980) and Wu and Li (1992) used $\alpha = 1$ for strong erosion, $\alpha = 0.25$ for strong deposition and $\alpha = 0.5$ for weak erosion and deposition. Yang (1998) used a very small value 0.001 for $\alpha$. Thus, the nonequilibrium adaptation coefficient $\alpha$ is also defined as a user-defined parameter in the model.

### 3.8 Sediment Settling Velocity

The settling velocity of sediment particle depends upon two forces which act on the sediment during fall in a quiescent column of water. These two forces are particle buoyant force and resisting force of water from fluid drag.

The fluid drag equation is defined as

$$F_D = C_D \rho A w_s^2$$  \hspace{1cm} (3.57)

The buoyant force of sediment particle is defined as

$$F_b = \frac{4}{3} r^3 \pi (\rho_s - \rho) g$$  \hspace{1cm} (3.58)

where

- $F_D =$ drag force
- $C_D =$ drag coefficient
- $\rho_s$, $\rho =$ density of sediment and water
- $A =$ projected area of particle in the direction of fall
- $w_s =$ settling velocity
- $r =$ particle radius

The sediment settling velocity can be determined once the value of drag coefficient is estimated. Since it is difficult to develop a relation for drag coefficient for all flow conditions and sediment, many investigators (Rubey (1933), Zhang (1989), Van Rijn (1989), Zhu and cheng (1993), Cheng (1997), Ahrens (2000), Chang and Liou (2001), Sha (1956), Ibade-Zade (1992), Burban (1990)) have developed empirical sediment settling velocity formulae. In the next section some of the empirical sediment settling velocity formulas are discussed.

#### 3.8.1 Stokes Law

Stokes (1851) derived an expression for velocity of a sphere in a fluid. Stokes law is valid only up to Reynolds number equal to unity. Normally this expression is not used to determine the sediment fall velocity as it does not take into account many sediment properties, such as shape and
flow properties. Stokes law is represented as:

\[ w_s = \frac{(s-1)gd^2}{18\nu} \]  

where

\( w_s \) = settling velocity in m/sec  
\( d \) = sediment particle diameter in m  
\( s \) = specific gravity of sediment mixture  
\( \nu \) = Kinematic viscosity in \( m^2/sec \)  
\( g \) = acceleration due to gravity \( m/sec^2 \)

### 3.8.2 Rubey’s Formula

Rubey (1933) derived a fall velocity expression for non-cohesive sediments. The fall velocity using Rubey’s formula can be computed as

\[ w_s = F \left[ dg(s-1) \right]^{1/2} \]  

where \( F = 0.79 \) for particle size greater than 1mm and for particle less than 1mm \( F \) can be computed as

\[ F = \left[ \frac{2}{3} + \frac{36\nu^2}{gd^3(s-1)} \right]^{1/2} - \left[ \frac{36\nu^2}{gd^3(s-1)} \right]^{1/2} \]  

For particle size greater than 2mm this analytical expression can be simplified as:

\[ w_s = 3.32d^{1/2} \]  

where

\( w_s \) = settling velocity in m/sec  
\( d \) = sediment particle diameter in m  
\( s \) = specific gravity of sediment mixture  
\( \nu \) = Kinematic viscosity in \( m^2/sec \)  
\( g \) = acceleration due to gravity \( m/sec^2 \)

### 3.8.3 Sha Formula

Sha (1954) derived the following relationship between sediment settling velocity and sediment diameter:

\[ w_s = \frac{1}{24} \frac{\Delta gd^2}{\nu} \quad \text{if} \quad d < 10^{-4} \text{m} \]  

\[ w_s = 1.14 \sqrt{\Delta gd} \quad \text{if} \quad d > 2 \times 10^{-3} \text{cm} \]
\[
\left( \log \frac{w_s d}{v} + 3.79 \right)^2 + (\log d - 5.777)^2 = 39 \quad \text{if} \quad 10^{-4} m \geq d \leq 2 \times 10^{-3} m \quad (3.65)
\]

where

\( w_s \) = settling velocity in m/sec
\( d_s \) = dimensionless particle diameter is defined as

\[
d_s = \left( \frac{(s-1)g}{v^2} \right)^{1/3} d \quad (3.66)
\]

\( d \) = sediment particle diameter in m
\( s \) = specific gravity of sediment mixture
\( v \) = Kinematic viscosity in \( m^2/sec \)
\( g \) = acceleration due to gravity \( m/sec^2 \)

### 3.8.4 Ibade-zade Formula

Ibade-zade (1992) developed the following relationship for predicting the sediment settling velocity:

\[
w_s = \frac{1}{24} \frac{\Delta g d^2}{v} \quad \text{if} \quad d < 0.015 cm \quad (3.67)
\]

\[
w_s = 1.068 \sqrt{\Delta g d} \quad \text{if} \quad d > 0.15 cm \quad (3.68)
\]

\[
w_s = 67.6 \Delta d + 0.52 \Delta \left( \frac{T}{26} - 1 \right) \quad \text{if} \quad d = 0.015 - 0.15 cm \quad (3.69)
\]

where

\( w_s \) = settling velocity in m/sec
\( d \) = sediment particle diameter in m
\( \Delta = (s-1) \) \( s \) = specific gravity of sediment mixture
\( v \) = Kinematic viscosity in \( m^2/sec \)
\( g \) = acceleration due to gravity \( m/sec^2 \)

For Equation 3.69

\( T \) = temperature in \(^0\)C
\( d \) = sediment particle diameter in cm
\( w_s \) = settling velocity in cm/sec

### 3.8.5 Zhang Formula

Zhang (1989) proposed the following formula for calculating the sediment settling velocity:

\[
w_s = \sqrt{\left( 13.95 \frac{v}{d} \right)^2 + 1.09 \Delta g d - 13.95 \frac{v}{d}} \quad (3.70)
\]
where
\( w_s \) = settling velocity in m/sec
\( d \) = sediment particle diameter in m
\( \Delta = (s-1) \) \( s \) = specific gravity of sediment mixture
\( \nu \) = Kinematic viscosity in \( m^2/sec \)
\( g \) = acceleration due to gravity \( m/sec^2 \)

### 3.8.6 Van Rijn Formula

Van Rijn (1989) proposed the following formula to estimate the sediment settling velocity:

\[
w_s = \frac{1}{18} \frac{\Delta g d^2}{\nu} \quad \text{if} \quad d < 10^{-4} m \tag{3.71}
\]

\[
w_s = 1.1 \sqrt{\Delta g d} \quad \text{if} \quad d > 10^{-3} m \tag{3.72}
\]

\[
w_s = \frac{\nu d}{10} \left( \sqrt{1 + 0.01d^3_s} - 1 \right) \quad \text{if} \quad 10^{-4} m \geq d \leq 10^{-4} m \tag{3.73}
\]

where
\( w_s \) = settling velocity in m/sec
\( d \) = sediment particle diameter in m
\( \Delta = (s-1) \)
\( d_s \) = dimensionless particle diameter is defined as

\[
d_s = \left( \frac{(s-1)g}{\nu^2} \right)^{1/3} d \tag{3.74}
\]

\( s \) = specific gravity of sediment mixture
\( \nu \) = Kinematic viscosity in \( m^2/sec \)
\( g \) = acceleration due to gravity \( m/sec^2 \)

### 3.8.7 Zhu and Cheng Formula

Zhu and Cheng (1993) proposed the following formula to estimate the sediment settling velocity:

\[
w_s = \frac{\nu}{d} \left( \frac{-24 \cos^2 \alpha + \sqrt{576 \cos^6 \alpha + (18 \cos^3 \alpha + 3.6 \sin^2 \alpha)d^3_s}}{9 \cos^3 \alpha + 1.8 \sin^2 \alpha} \right) \tag{3.75}
\]

where
\[
\alpha = 0 \quad \text{for} \quad d_s \leq 1 \tag{3.76}
\]

\[
\alpha = \frac{\pi}{2 + 2.5(logd_s)^{-3}} \quad \text{for} \quad d_s > 1 \tag{3.77}
\]
\[ w_s = \text{settling velocity in m/sec} \]
\[ d = \text{sediment particle diameter in m} \]
\[ d_* = \text{dimensionless particle diameter is defined as} \]
\[ d_* = \left( \frac{(s - 1)g}{\nu^2} \right)^{1/3} d \]  
(3.78)

\[ s = \text{specific gravity of sediment mixture} \]
\[ \nu = \text{Kinematic viscosity in m}^2/\text{sec} \]
\[ g = \text{acceleration due to gravity m/sec}^2 \]

### 3.8.8 Cheng Formula

Cheng (1997) proposed following formula to calculate the sediment settling velocity:

\[ w_s = \frac{\nu}{d} \left( \sqrt{(25 + 1.2d_*^2)} - 5 \right)^{1.5} \]  
(3.79)

where
\[ w_s = \text{settling velocity in m/sec} \]
\[ d = \text{sediment particle diameter in m} \]
\[ d_* = \text{dimensionless particle diameter is defined as} \]
\[ d_* = \left( \frac{(s - 1)g}{\nu^2} \right)^{1/3} d \]  
(3.80)

\[ s = \text{specific gravity of sediment mixture} \]
\[ \nu = \text{Kinematic viscosity in m}^2/\text{sec} \]
\[ g = \text{acceleration due to gravity m/sec}^2 \]

### 3.8.9 Ahrens Formula

Ahrens (2000) used the Hallermeier (1981) approach to predict the sediment settling velocity:

\[ R = C_1A + C_t\sqrt{A} \]  
(3.81)

where
\[ R = \text{sediment Reynolds number} \]
\[ R = \frac{w_sd}{\nu} \]  
(3.82)

\[ A = \text{Archimedes buoyancy index} \]
\[ A = \frac{\Delta gd^3}{\nu^2} \]  
(3.83)
\( C_1, C_t = \) coefficients which are functions of \( A \)

Inserting the values of the above parameters in the equation 3.81, formula for the settling velocity can be written as

\[
  w_s = C_1 \Delta gd^2 / \nu + C_t + \sqrt{\Delta gd}
\]  

(3.84)

where the term \( C_1 \Delta gd^2 / \nu \) represents the laminar flow regime and the term \( C_t + \sqrt{\Delta gd} \) represents the turbulent flow regime. Ahrens estimated the values of coefficients \( C_1 \) and \( C_t \) as functions of \( A \) by trial and error and expressed them as follows:

\[
  C_1 = 0.055 \tanh \left[ 12A^{-0.59} \exp(-0.0004A) \right]
\]

\[
  C_t = 1.06 \tanh \left[ 0.016A^{0.5} \exp(-120/A) \right]
\]

(3.85)

where

- \( w_s = \) settling velocity in m/sec
- \( d = \) sediment particle diameter in m
- \( \Delta = (s-1) s = \) specific gravity of sediment mixture
- \( \nu = \) kinematic viscosity in \( m^2/sec \)
- \( g = \) acceleration due to gravity \( m/sec^2 \)

### 3.8.10 Chang and Liou Formula

Chang and Liou (2001) proposed the following formula to calculate the sediment settling velocity:

\[
  w_s = \frac{\nu}{d} \frac{\alpha A^\beta}{\chi (1 + \alpha A^{\beta-1})}
\]  

(3.86)

where \( \alpha, \beta, \chi \) are coefficients and suggested values of the coefficients are \( \alpha = 30.22, \beta = 0.463, \chi = 18.0 \). where

- \( A = \) Archimedes buoyancy index

\[
  A = \frac{\Delta gd^3}{\nu^2}
\]  

(3.87)

\( w_s = \) settling velocity in m/sec
\( d = \) sediment particle diameter in m
\( \Delta = (s-1) s = \) specific gravity of sediment mixture
\( \nu = \) Kinematic viscosity in \( m^2/sec \)
\( g = \) acceleration due to gravity \( m/sec^2 \)

All the above formulas are suitable for calculating the settling velocity of non-cohesive sediments but these formulas do not accurately predict the settling velocity for cohesive sediments. Cohesive sediments get flocculated during transport process and that is why the size of the falling sphere increases. Therefore, the cohesive sediments flocculation effect on settling speed should be incorporated. This property is effective only for sediment sizes less then 40 \( \mu m \).
3.8.11 Burban Formula

Burban (1990) did experiments on settling velocity of cohesive sediments and concluded that the settling speed of any falling sediment particle can be represented as:

\[ w_s = ad^m \] (3.88)

where \( a \) and \( m \) are coefficients. Burban compiled data for cohesive sediment settling velocity for different conditions in laboratory experiments and estimated the value of these coefficients. For fresh water \( a = 8.4 \times 10^{-3} \) and \( m = -0.024 \) For sea water \( a = 2.6 \times 10^{-3} \) and \( m = 0.28 \) For all water in general \( a = 4.5 \times 10^{-3} \) and \( m = 0.14 \). Burban also reported that in general the average settling velocity of all flocs can be taken as \( 8 \times 10^{-3} \text{cm/sec} \).

3.8.12 Migniot Formula

Migniot (1989) proposed a correction factor to convert the cohesive sediment settling velocity to floc settling velocity. The settling velocity of cohesive sediment flocs can be computed as

\[ w_{sf} = \frac{250}{d^2} w_s \] (3.89)

where \( w_s \) is the cohesive sediment settling velocity and \( w_{sf} \) is the floc settling velocity. The diameter of the sediment is taken as \( \mu m \).

3.9 Incipient Motion

To simulate sediment transport in any water body it is important to know the concept of incipient motion of sediment particles from the bed, because most of the sediment in any water body comes from its bed. Different investigators have used different criteria to define incipient motion. Some of those criteria are:

(1) single particle moving,
(2) few particles moving,
(3) initial motion,
(4) general motion on the bed, and
(5) limiting condition when the rate of sediment transport tends to zero.

In the following the sediment incipient motion criteria are discussed.

3.10 Forces on Sediment Particle

Figure 3.4 shows the forces which act on a sediment particle at bed. Component of gravitation force is not considered, as in most natural cases bed slope is small enough that gravitation force in the direction of flow is negligible. Major forces which act on the sediment particle are drag force \( F_D \), lift force \( F_L \), submerged weight \( W_s \) and resistance force \( F_R \). From the direction and
magnitude of these forces it can be stated that a sediment particle is in a state of incipient motion if one of the following conditions is satisfied:

\[ F_L = W_s \]
\[ F_D = F_R \]
\[ M_O = M_R \]

where
\( M_0 = \) overturning moment due to \( F_D \) and \( F_R \)
\( M_R = \) overturning moment due to \( F_L \) and \( W_s \)

Different approaches have been used by different investigators for defining the condition for incipient motion of sediment particles comprising the bed. These approaches are discussed below.

![Figure 3.4: Forces acting on a sediment particle](image)

3.10.1 Shear Stress Approach

In this approach, shear stress applied by the flowing water is mainly responsible for the movement of sediment particle in the bed. It is important to define the expression of shear stress (sometimes used as bed shear stress).
3.10.2 Shear Stress

For uniform flow shear stress can be calculated by considering the water flow section abcd shown in figure 3.5.

![Figure 3.5: Schematic diagram of a flow in a sloped channel](image)

For this section the force balance equation can be written as:

\[ \Sigma F = F_1 + W \sin \alpha - F_2 - \tau_0 \times \text{wetted area} = 0 \]  (3.90)

where
\[ \tau_0 = \text{average shear stress at the boundary} \]
\[ F_1 \text{ and } F_2 = \text{hydrostatic forces} \]
\[ W = \text{weight of the water in the considered section} = BDx\gamma \]
\[ B = \text{width of channel} \]
\[ x = \text{length of assumed section} \]
\[ \text{wetted area} = (B+2D)x \]

The depth of flow at both faces ab and cd is the same, so \( F_1 = F_2 \), and

\[ \tau_0 = \frac{W \sin \alpha}{(B+2D)x} \]
\[ \tau_0 = \frac{BDx\gamma \sin \alpha}{(B+2D)x} \]
\[ \tau_0 = \gamma R \sin \alpha \]  (3.91)

for small value of \( \alpha, \sin \alpha = \tan \alpha = S \), slope of the channel. Therefore,

\[ \tau_0 = \gamma RS \]  (3.92)

For wide channels hydraulic radius \( R \) can be taken as the depth of flow \( D \).

3.11 Empirical Formulas for Critical Shear Stress

Many investigators have developed empirical formulas for calculating critical shear stress using laboratory and field experiments. These empirical formulas define the relationship between
critical shear stress and sediment properties like diameter and flow properties. Some of these empirical formulas are discussed in the following sections.

3.11.1 USWES Formula

The United States Waterways Experiment Station (Garde and Ranga Raju (1978)) proposed the following formula for critical shear stress:

$$\tau_c = 0.00595 \left( (s - 1) \frac{d}{M} \right)^{1/2}$$  \hspace{1cm} (3.93)

where

$\tau_c =$ critical shear stress in lb/ft$^2$
$d =$ mean diameter of sediment in mm
$M =$ uniformity coefficient

This equation is valid for sediment particle size ranging from 0.205mm to 4.077mm and uniformity coefficient ranging from 0.280 to 0.643.

3.11.2 Chang’s Formula

Chang (Garde and Ranga Raju (1978)) proposed the following formula for estimating the critical shear stress value for a given diameter of sediment:

$$\tau_c = 0.0045 \left( (s - 1) \frac{d}{M} \right)^{1/2} \quad if \quad \left( (s - 1) \frac{d}{M} \right) > 2.0$$

$$\tau_c = 0.00635 \left( (s - 1) \frac{d}{M} \right) \quad if \quad \left( (s - 1) \frac{d}{M} \right) < 2.0$$

where

$\tau_c =$ critical shear stress in lb/ft$^2$
$d =$ mean diameter of sediment in mm
$M =$ uniformity coefficient

Chang’s formula is valid for sediment diameter ranging from 0.134mm to 8.09mm and therefore the uniformity coefficient varies from 0.23 to 1.0. This formula is valid for sediment specific gravity ranging from 2.05 to 3.89.

3.11.3 Krey’s Formula

Krey (Garde and Ranga Raju (1978)) proposed the following formula for critical shear stress for incipient motion for a given diameter of sediment particle:

$$\tau_c = (s - 1) \frac{d}{13}$$  \hspace{1cm} (3.94)

where

$\tau_c =$ critical shear stress in kg/m$^2$
$d =$ mean diameter of sediment in mm
3.11.4 Indri’s Formula

Indri (Garde and Ranga Raju (1978)) proposed the following formula for critical shear stress for incipient motion for sediment particle:

\[ \tau_c = 13.3d \left( \frac{s-1}{M} \right) + 12.16 \quad if \quad d < 1.0\text{mm} \]
\[ \tau_c = 54.85d \left( \frac{s-1}{M} \right) - 74.48 \quad if \quad d > 1.0\text{mm} \]  \hspace{1cm} (3.95)

where
\( \tau_c = \) Critical shear stress in gm/m\(^2\)
\( d = \) mean diameter of sediment in mm
\( M = \) uniformity coefficient

3.11.5 Aki and Sato’s Formula

Aki and Sato’s (Garde and Ranga Raju (1978)) proposed the following formula for shear stress calculation:

\[ \tau_c = 55.7(s-1)\lambda d \]  \hspace{1cm} (3.96)

where
\( \tau_c = \) Critical shear stress in gm/m\(^2\)
\( d = \) mean diameter of sediment in mm
\( \lambda = \) coefficient which depends upon size distribution of sediment

3.11.6 Sakai Formula

Sakai (Garde and Ranga Raju (1978)) proposed the following formula for critical shear stress:

\[ \tau_c = \frac{100(s-1)d^{6/5}}{3} \left( \frac{2+M}{1+2M} \right) \]  \hspace{1cm} (3.97)

where
\( \tau_c = \) Critical shear stress in gm/m\(^2\)
\( d = \) mean diameter of sediment in mm
\( M = \) uniformity coefficient

3.12 Shear Stress Formulas Based on Theoretical Analysis

3.12.1 Shields Diagram

Shields (1936) conducted laboratory studies and developed a relationship between sediment diameter and critical bed shear stress. To develop this relationship he used two non-dimensional
variables:
Particle Reynolds Number

\[ R_s = \frac{U_s D_s}{\nu} \]  (3.98)

Dimensional shear stress (Shield parameter)

\[ \theta = \frac{\tau}{(\gamma_s - \gamma)D_s} \]  (3.99)

The relationship between these two non-dimensionless variables was estimated by Shields (1936) experimentally. Further investigators fitted the curve to the data provided by Shields. Figure 3.6 shows the Shields diagram which was developed by Varoni (1975). The portion above the curve represents the values for which sediments will move and the portion below the curve represents the values for which sediments will not move. The curve values show the critical values at which sediments will start to move.

The Shields diagram is still widely accepted and used to estimate the critical shear stress but it has a drawback which limits its use as it is not appropriate to use shear velocity \( u_* \) as an independent variable and shear stress \( \tau \) as a dependent variable as they are interrelated. The American
Society of Civil Engineers Task Committee recommended a third parameter to be incorporated in the Shields diagram as shown in the Figure 3.6. The parameter is

\[
\frac{D}{\nu} \left[ 0.1 \left( \frac{\sigma_s}{\gamma} - 1 \right) gd \right]^{1/2} \tag{3.100}
\]

This parameter is also included in the shields diagram. Now one can first estimate this variable and then accordingly the Shields parameter using the diagram.

It is difficult to use the Shields analysis directly in numerical modelling as Shields developed a diagram to estimate the critical shear stress and he did not specify any function to represent that graph. Many investigators have proposed different options which are more or less the same. Some of the established and popular options are discussed in the following section.

### 3.12.2 Chien and Wan Approach

Chien and Wan (1983) developed a relationship between two non dimensional parameters, Shields parameter which depends upon the critical shear stress and a non-dimensional representative diameter which depends upon sediment representative diameter \(d_{50}\) to represent the Shields curve. The critical shear stress can be estimated for a given sediment size class using this relation:

\[
\theta = \begin{cases} 
0.126D_s^{-0.44}, & D_s < 1.5 \\
0.131D_s^{-0.55}, & 1.5 \leq D_s < 10 \\
0.0685D_s^{-0.27}, & 10 \leq D_s < 20 \\
0.0173D_s^{0.19}, & 20 \leq D_s < 40 \\
0.0115D_s^{0.30}, & 40 \leq D_s < 150 \\
0.052, & D_s \geq 150
\end{cases} \tag{3.101}
\]

where

- \(D_s = \text{non dimensional representative diameter}\)
- \(\theta = \text{Shields parameter}\)

\[
D_s = d_s \left( \frac{g(s - 1)}{\nu^2} \right)^{1/3} \tag{3.102}
\]

\[
\theta = \frac{\tau_c}{(\gamma_s - \gamma)d_s} \tag{3.103}
\]

In this relation both parameters are dimensionless so consistent units can be used in the relationship. where

- \(\tau_c = \text{critical shear stress in kg/msec}^2\)
- \(\gamma_s, \gamma = \text{specific weight of sediment and water in KN/m}^3\)
- \(s = \text{specific gravity}\)
- \(g = \text{gravitational acceleration in m/sec}^2\)
- \(\nu = \text{kinematic viscosity in m}^2/\text{sec}\)
3.12.3 Yalin Approach

Yalin (1972) also showed that shields curve can be represented as a relationship between shields parameter $\theta$ and dimensionless particle diameter $D^*$. The relationship is shown as follows:

$$\theta = \begin{cases} 
0.24D_s^{-1}, & 1 < D_s \leq 4 \\
0.14D_s^{-0.64}, & 4 < D_s \leq 10 \\
0.045D_s^{-0.1}, & 10 < D_s \leq 20 \\
0.013D_s^{0.29}, & 20 < D_s \leq 150 \\
0.055, & D_s > 150 
\end{cases}$$

Both the non-dimensional parameters are defined in the same way as in Chien and Wan (1983) approach.

3.12.4 Madsen and Grant Approach

Madsen (1976) modified the Shields diagram for which he specified a well defined function to represent the relationship between critical Shields parameter $\theta_c$ and sediment fluid parameter $S^*$ which is defined as

$$S^* = \frac{d_s \sqrt{(s-1)gd}}{4\nu}$$

The relationship is defined as

$$\log_{10}\theta_c = 0.002235x^5 - 0.06043x^4 + 0.20307x^3 + 0.054252x^2 - 0.636397x - 1.03167$$

where $x = \log_{10}S^*$.

In this case also both Shields parameter and sediment fluid parameter are non-dimensional parameters so any consistent units can be used.

3.12.5 Shulits and Hill Approach

Shulits and Hill (1968) divided the Shields diagram in four parts and expressed each part by an expression. It is easy to estimate critical shear stress for incipient motion for given diameter of sediment.

$$\tau_c = 0.0215d_s^{0.25}; \quad \text{if} \quad 0.0003 < d_s < 0.0009 \text{ft}$$

$$\tau_c = 0.315d_s^{0.633}; \quad \text{if} \quad 0.0009 < d_s < 0.0018 \text{ft}$$

$$\tau_c = 16.8d_s^{1.262}; \quad \text{if} \quad 0.0018 < d_s < 0.022 \text{ft}$$

$$\tau_c = 16.8d_s^{1.262}; \quad \text{if} \quad 0.0018 < d_s < 0.022 \text{ft}$$

$$\tau_c = 6.18d_s; \quad \text{if} \quad d_s > 0.022 \text{ft}$$

where

$d_s$ = representative sediment particle diameter in ft

$\tau_c$ = critical shear stress in lb/ft$^2$
3.12.6 Guo Formula

Guo (1990) proposed the following empirical equation to represent the shields diagram for using directly in numerical modelling:

\[ \theta_c = \frac{0.11}{R_s} + 0.054 \left[ 1 - \exp \left( -\frac{4R_s^{0.52}}{25} \right) \right] \]  
(3.108)

where

\( \theta = \) Shields parameter

\( R_s = \) particle Reynolds number = \( \frac{U_s d_s}{\nu} \)

This equation needs to be solved iteratively, as there is no closed form solution for this equation. To get rid of this problem he used another non-dimensional parameter named as Rouse’s auxiliary parameter:

\[ R_s = \frac{d \sqrt{0.1(s-1)g d_s}}{\nu} \]  
(3.109)

Using this parameter equation 3.108 can be rewritten in the following form which has a closed form solution:

\[ \theta_c = \frac{0.1}{R_s^{2/3}} + 0.054 \left[ 1 - \exp \left( -\frac{R_s^{0.52}}{10} \right) \right] \]  
(3.110)

Guo (1990) also developed another relationship which is shown below to represent Shields diagram using a dimensionless diameter parameter:

\[ \theta_c = \frac{0.23}{d_s} + 0.054 \left[ 1 - \exp \left( -\frac{d_s^{0.85}}{23} \right) \right] \]  
(3.111)

where \( d_s \) is a dimensionless diameter parameter which is calculated as follows

\[ d_s = \left[ \left( \frac{(s-1)g}{\nu^2} \right)^{1/3} \right] \]  
(3.112)

In both relationships non-dimensional parameters are used, so any consistent units can be used.

3.13 Sediment Transport Capacity

To calculate the source-sink term in the suspended load or bed load transport equations sediment transport capacity should be estimated. It can be defined as the maximum amount of sediment, which water can carry at a given flow condition. Many investigators have developed formulas for estimating the sediment transport capacity for different loads, such as bed load, suspended load or bed-material load. Every formula has its own requirements and limitations. Out of the many formulas the following well established formulas were adopted in the model.
3.14  Sediment Transport Capacity Formulas

3.14.1 DuBoys Approach

DuBoys (1879) proposed that sediment particle move in layers along the bed due to the tractive force at the bed. Using this theory he defined the bed load transport capacity formula as:

\[ q^*_b = K\tau(\tau - \tau_c) \]  (3.113)

where \( K \) is a coefficient which depends upon the characteristics of the sediment particles. Straub (1935) defined \( K \) as a function of sediment particle as

\[ K = \frac{0.173}{\left(\frac{d}{s}\right)^{3/4}} \]  (3.114)

Hence the Duboys equation can be rewritten as

\[ q^*_b = \frac{0.173}{\left(\frac{d}{s}\right)^{3/4}} \tau_b (\tau_b - \tau_c) \]  (3.115)

where \( d_s \) = diameter of sediment particle in mm
\( \tau, \tau_c \) = bed and critical shear stress in lb/ft²
\( q^*_b \) = bed load transport capacity in (ft³/sec)/ft

3.14.2 Shields Approach

Shiels (1936) proposed the following formula to estimate the bed load transport capacity:

\[ \frac{q^*_b}{q_S} = 10 \frac{\tau_b - \tau_c}{(\gamma_s - \gamma)d} \]  (3.116)

Both sides of equation 3.116 are dimensionless so any consistent system of units can be used in the equation. In this equation \( q \) and \( q_b \) are, respectively, water discharge and sediment load per unit width.

3.14.3 Meyer-Peter Approach

Meyer-Peter (1934) derived the following formula for estimating the bed load transport capacity:

\[ \frac{0.4q^2/3_b}{d} = \frac{q^{2/3}S}{d} - 17 \]  (3.117)

where
\( q_b \) = bed load capacity in (kg/sec)/m
\( q \) = water discharge in (kg/sec)/m
\( S \) = slope
\( d \) = particle size in m

Meyer-Peter formula is valid only for sediment particle diameters greater then 3mm.
3.14.4 Meyer-Peter and Muller Approach

Meyer-Peter and Muller (1948) modified the Meyer-Peter (1934) formula and incorporated more parameters which affect the bed load transport capacity. The Meyer-Peter and Muller formula is defined as

$$\gamma \frac{K_s}{K_r}^{3/2} RS = 0.047(\gamma_s - \gamma)d_s + 0.25\rho^{1/3}q_b^{2/3}$$

(3.118)

$$\left(\frac{k_s}{k_r}\right)^{3/2} = \frac{S_r^2}{S}$$

(3.119)

$S_r$ from Strickler’s equation:

$$S_r = \frac{V^2}{k_r^2 R^{4/3}}$$

(3.120)

and value of $K_r$ is determined by Muller as:

$$k_r = \frac{26}{d_{90}^{1/6}}$$

(3.121)

Equation 3.118 in dimensionless form:

$$\left[\frac{q_b(\gamma_s - \gamma)}{\gamma_s}\right]^{2/3} \left(\frac{\gamma}{g}\right)^{1/3} \frac{0.25}{(\gamma_s - \gamma)d} = (k_s/k_r)^{3/2} \gamma RS \frac{(\gamma_s - \gamma)d}{(\gamma_s - \gamma)q_b^{2/3}} - 0.047$$

(3.122)

Equation 3.122 is in dimensionless form so any consistent units can be used. $q_b$ is bed load transport capacity per unit width.

3.14.5 Chien Approach

Chien (1956) used the Meyer-Peter and Muller (1948) approach and simplified the formula in the following form to estimate the bed load capacity:

$$q_b^* = \zeta(\tau_b - \tau_c)^{3/2}$$

(3.123)

where $\zeta$ is a coefficient, $\tau_b$ and $\tau_c$ are bed shear stress and critical shear stress, respectively, in $kg/m/sec^2$ and $q_b$ is bed load discharge in $m^3/sec$ per unit width.

3.14.6 Bagnold Approach

Bagnold (1966) used the energy concept to estimate the bed load transport capacity as a function of work done by the fluid. He proposed the following equation to estimate the bed load transport capacity:

$$q_b^* = \frac{e_b \tau_b U}{(\rho_s - \rho)g \cos \beta (\tan \phi - \tan \beta)}$$

(3.124)
where
\[\tan \phi = \text{dynamic friction factor equals to 0.6}\]
\[e_b = \text{efficiency factor ranging from 0.1 to 0.2}\]
\[\tan \beta = \text{bed slope}\]
\[U = \text{depth averaged mean velocity}\]
\[q_b^* = \text{bed load transport capacity per unit width in } m^3/sec\]
\[\tau_b = \text{bed shear stress in } kg/msec^2\]

3.14.7 Schoklitsch Approach

Schoklitsch (1934, 1949) proposed two bed load transport capacity formulae first in 1934 and second in 1949. The Schoklitsch (1934) formula is expressed as

\[q_b^* = 7000 \frac{S^{3/2}}{d_s^{1/2}} (q - q_c)\] (3.125)

where
\[d_s = \text{sediment particle in mm}\]
\[q_b^* = \text{bed load transport capacity in } (kg/sec)/m\]
\[q = \text{water discharge in } (m^3/s)/m\]
\[q_c = \text{critical water discharge at incipient motion in } (m^3/s)/m\]
and can be calculated as

\[q_c = \frac{0.00001944 d_s}{S}\] (3.126)

The Schoklitsch (1949) formula is defined as

\[q_b^* = 2500 S^{3/2} (q - q_c)\] (3.127)

where
\[d_s = \text{sediment diameter in m}\]
\[q_b^* = \text{bed load transport capacity in } (kg/sec)/m\]
\[q = \text{water discharge in } (m^3/s)/m\]
\[q_c = \text{critical water discharge at incipient motion in } (m^3/s)/m\]
and can be calculated as

3.14.8 Rottner Approach

Rottner (1959) proposed the following formula to estimate the bed load transport capacity:

\[q_b^* = \gamma_s [(s - 1)gD^3]^{1/2} \left\{ \frac{V}{[(s - 1)gD]^{1/2}} \left[ 0.667 \left( \frac{d_s}{D} \right)^{2/3} + 0.14 \right] - 0.778 \left( \frac{d_s}{D} \right)^{2/3} \right\}^3\] (3.128)

It is a dimensionally homogeneous equation. So any consistent units can be used in the equation.
3.14.9 Van Rijn Approach

Van Rijn (1984) proposed a bed load transport capacity formula for particle diameter ranging from 0.2mm to 2mm:

\[
q_b^* = 0.053(s - 1)^{0.5}g^{0.5}d_s^{1.5}D_*^{-0.3}T^{2.1} \text{ if } T < 3
\]

\[
q_b^* = 0.1(s - 1)^{0.5}g^{0.5}d_s^{1.5}D_*^{-0.3}T^{1.5} \text{ if } T \geq 3
\]

where

\[ T = \frac{\tau_b - \tau_c}{\tau_c} \]

\[ D_* = \text{dimensionless particle diameter defined as} \]

\[ D_* = d_s \left( \frac{(s - 1)g}{\nu^2} \right)^{1/3} \]

\( g \) = gravitational acceleration in m/sec\(^2\)

\( s \) = specific gravity

\( d_s \) = sediment particle diameter in m \( q_b \) = sediment load transport capacity per unit width in \( m^2/sec \)

3.14.10 Acker’s and White Modified Formula

Acker and White (1973) provided a transport capacity formula for uniform sediment. The sediment transport capacity for kth size fraction can be calculated as:

\[ d_{gr} = d_s \left[ \frac{g(s - 1)}{\nu^2} \right]^{1/3} \]  

If \( d_{gr} \geq 60 \)

\[ n = 0.0 \]

\[ A = 0.17 \]

\[ m = 1.5 \]

\[ C = 0.025 \]

If \( 1 < d_{gr} < 60 \)

\[ n = 1.00 - 0.56 \log d_{gr} \]

\[ A = 0.23d_{gr}^{-1/2} + 0.14 \]

\[ m = 9.66d_{gr}^{-1} + 1.34 \]

\[ \log C = -3.53 + 2.86 \log d_{gr} - (\log d_{gr})^2 \]
\[ F_{gr} = \frac{U^n}{[(\gamma_s/\gamma - 1)gd_j]^{1/2}} \left[ \frac{\bar{U}}{\sqrt{32\log(\alpha h/d_j)}} \right]^{(1-n)} \]  

(3.136)

\[ G_{gr} = C \left[ \frac{F_{gr}}{A} - 1 \right]^m \]  

(3.137)

\[ C_\ast = \frac{G_{gr}d_k\gamma_s}{\gamma h} \left( \frac{\bar{U}}{U_\ast} \right)^{1/n} \]  

(3.138)

where

d_s = sediment particle size in m
ν = kinematic viscosity in \( m^2/sec \)
\( \bar{U} \) = depth average velocity in m/sec
\( U_\ast \) = shear velocity in m/sec
\( \alpha \) = coefficient in rough turbulent equation = 10
h = flow depth in m
\( g \) = gravitational acceleration in \( m/sec^2 \)
s = specific gravity
\( \gamma_s, \gamma \) = specific weight of sediment and water
\( C_\ast \) = sediment transport capacity in part per million by weight

### 3.14.11 Yang Formula

Yang (1973) introduced a sediment transport capacity formula by assuming unit stream power as a dominant factor for calculation of sediment transport capacity. Yang described the unit stream power as the time rate of potential energy expenditure per unit weight of water in an alluvial channel. The sediment transport capacity using Yang’s formula can be written as:

\[ \log C_\ast = 5.435 - 0.286 \log \frac{\omega_s d}{\nu} - 0.457 \log \frac{U_\ast}{\omega_s} + \left( 1.799 - 0.4091 \log \frac{\omega_s d}{\nu} - 0.314 \log \frac{U_\ast}{\omega_s} \right) \log \left( \frac{\bar{U}S}{\omega_s} - \frac{V_{cr}S}{\omega_s} \right) \]

where \( V_{cr} \) is critical velocity and can be calculated as:

\[ V_{cr} = \begin{cases} \frac{2.5\omega}{\log \frac{U_\ast d}{\nu} - 0.06} + 0.66, & 0 < \frac{U_\ast d}{\nu} < 70 \\ 2.05\omega, & 70 < \frac{U_\ast d}{\nu} \end{cases} \]

where
\( U \) = depth averaged mean velocity in m/sec
S = bed slope
\( \nu \) = kinematic viscosity in \( m^2/sec \)
\( \omega_s \) = sediment settling velocity in m/sec
$U_s$ = shear velocity in m/sec  
$d_s$ = sediment diameter in m  
$C^*$ = sediment transport capacity in part per million by weight

### 3.14.12 Wu Wang and Jia Formula

Wu, Wang and Jia (2000) proposed separate formulae for calculating the bed load transport capacity and suspended load transport capacity.

**Suspend load transport capacity formula**

The following steps are used to calculate the suspended load sediment transport capacity:

**Sediment settling velocity using Zhang’s formula:**

$$\omega_{sj} = \sqrt{\left(13.95 \frac{\nu}{d_k}\right)^2 + 1.09 \left(\frac{\gamma_s}{\gamma} - 1\right) gd_k - 13.95 \frac{\nu}{d_k}}$$  

Compute hiding and exposure factor as:

$$P_{hk} = \sum_{j=1}^{n} P_j \frac{d_j}{d_j + d_k}$$  

$$P_{ek} = \sum_{j=1}^{n} P_j \frac{d_k}{d_j + d_k}$$  

(3.139)

where  
$p_j$ = probability or percent of particle $d_j$  
$d_k$ = diameter of desired sediment particle  
$d_j$ = diameter of other sediment particle

Compute critical shear stress:

$$\tau_c = 0.03(\gamma_s - \gamma)d_k \left(\frac{p_{hk}}{p_{ek}}\right)^{0.6}$$  

(3.140)

compute dimensional stress:

$$\phi_{sk} = 0.0000262 \left[\left(\frac{\tau_b}{\tau_{ck}} - 1\right) \frac{U}{\omega_s}\right]^{0.6}$$  

(3.141)

and then suspended load sediment transport capacity by the following equation:

$$C^*_k = \phi_{sk} p_{sk} \sqrt{\left(\frac{\gamma_s}{\gamma} - 1\right) gd_k^3}$$

where  
$C^*_k$ = suspended load sediment transport capacity per unit width for kth size fraction in $m^2/sec$
\( d_k \) = sediment diameter of kth size fraction  
\( g \) = gravitation acceleration  

**Bed load transport capacity formula**  
The Following steps must be followed for calculating bed load transport capacity:  
Calculate \( n' \) which is Maning’s coefficient corresponding to the grain roughness  
\[
n' = \frac{d_{50}^{1/6}}{20} \tag{3.142}
\]

Calculate the critical shear stress as defined in the suspended load transport capacity formula.  
Compute dimensional stress:  
\[
\phi_{bk} = 0.0000262 \left( \frac{\tau_b}{\tau_{ck}} \right) \left( \frac{U}{w_{sk}} \right)^{1.74} \tag{3.143}
\]

and then bed load sediment transport capacity by following equation:  
\[
q_k^* = \phi_{bk} \rho_{bk} \sqrt{\left( \frac{\gamma_s}{\gamma} - 1 \right) g d_k^3}
\]

where  
\( q_k^* \) = bed load sediment transport capacity per unit width for kth size fraction in \( m^2/sec \)  
\( d_k \) = sediment diameter of kth size fraction  
\( g \) = gravitation acceleration

### 3.14.13 Sediment Transport Capacity for Weak Cohesive Sediment

In this model, weak cohesive sediments are taken as those sediments in which cohesive sediment sediments concentration percentage is less than 30. In this case still the transport capacity will be governed by the same phenomenen as in case of non-cohesive sediment but to account for the cohesive sediment properties cohesive sediment settling velocity formulae are used.  

The Migniot (1989) formula is used to calculate the cohesive sediment settling velocity:  
\[
\omega_{sc} = \frac{250}{d^2} \omega_s \tag{3.144}
\]

where  
\( \omega_k \) = settling velocity of cohesive sediment flocs  
\( \omega_{sc} \) = settling velocity of single cohesive sediment Stokes Law is used to calculate the single cohesive sediment particle  
\[
\omega_s = \frac{gd^2}{18\mu}(\rho_s - \rho) \tag{3.145}
\]
3.15 Bed Elevation Change

As described in the equation 3.54 the bed elevation change due to sediment erosion and deposition can be represented by the following equation:

\[ (1 - p') \frac{\partial z_{bk}}{\partial t} = \alpha \omega_{sk} \left( C_k - C^*_k \right) + \frac{(q_{bk} - q^*_{bk})}{L_t} \]  

(3.146)

Writing this equation in discrete form for incorporating in the model for calculating the bed elevation change after each time step,

\[ (1 - p') \frac{\Delta z_{n+1}^k}{\Delta t} = \alpha \omega_{sk} \left[ C_k - C^*_k \right] + \frac{(q_{bk} - q^*_{bk})}{L_t} \]  

(3.147)

\[ \Delta z_{n+1}^k = \frac{\alpha \omega_{sk} \left( C_k - C^*_k \right) \Delta t + (q_{bk} - q^*_{bk}) \Delta t / L_t}{1 - p'} \]  

(3.148)

where \( \Delta z_{n+1}^k \) is the fractional change in bed elevation at the end of nth time step due to the kth fraction of sediment. The total bed elevation change after each time step can be calculated by summing all the fractional bed changes due to all sediment fractions:

\[ \Delta z^{n+1} = \sum \Delta z_{n+1}^k \]  

(3.149)

So at new time step the bed elevation can be computed as

\[ z^{n+1} = z^n + \Delta z^{n+1} \]  

(3.150)

3.16 Bed Material Sorting

If the bed is made of non-uniform sediments then during sediment transport, the fraction of each sediment class which makes the bed, changes due to erosion and deposition. In this model the bed layer is divided into vertical layers. The bottom most layer is nonerodable layer, above that subsurface layer and the uppermost is the mixing layer, which is in direct contact with the water and all erosion and deposition happens in this layer. To estimate the bed material fraction in the mixing layer, the Wu and Li (1992) concept is used which can be written as:

\[ \frac{\partial (\delta_m p_{bk})}{\partial t} = \frac{\partial z_{bk}}{\partial t} + p_{bk}^* \left( \frac{\partial \delta_m}{\partial t} - \frac{\partial z_b}{\partial t} \right) \]  

(3.151)

where

\[ p_{bk} = \text{bed material gradation in the mixing layer} \]
\[ p_{bk}^* = \text{bed material gradation in the subsurface layer} \]
\[ \delta_m = \text{thickness of mixing layer} \]

Discretizing equation 3.151 in space leads to following expression:

\[ p_{bk}^{n+1} = \frac{\Delta z_{bk}^{n+1} + \delta_m p_{bk}^n + p_{bk}^n (\delta_m - \delta_m^n - \Delta z_b^{n+1})}{\delta_m^{n+1}} \]  

(3.152)
where

\[ p_{bk}^{*n} = p_{bk}^n \quad \text{when} \quad \delta_m^{n+1} - \delta_m^n - \Delta z_{bk}^{n+1} \leq 0 \]
\[ p_{bk}^{*n} = p_{bk}^{*n} \quad \text{when} \quad \delta_m^{n+1} - \delta_m^n - \Delta z_{bk}^{n+1} \geq 0 \]  

(3.153)

The thickness of the mixing layer for the current time step can be calculated as

\[ \delta_m^{n+1} = \max(\Delta z_{bk}^{n+1}, \Delta_m, \delta_{mt}) \]  

(3.154)

where \( \delta_{mt} \) is the instantaneous mixing layer thickness, which is taken as \( 2d_s \) and \( \Delta_m \) is the maximum mixing thickness reached by all sediment fractions and can be calculated as follows:

\[ \Delta_m = \delta_m^n + \max_{k=1}^{n} \left( 0, -\frac{\Delta z_{bk}^{n+1}}{p_{bk}^n} \right) \]  

(3.155)

### 3.17 Model Approach

As described earlier, sediments are transported as suspended load and bed load. In the proposed model both approaches are used and selection of an approach depends upon the sediment particle size. Normally the sediment mixture comprises a wide range of sediments from very small sand particles to boulders. Initially a sediment mixture should be divided in some size fraction classes according to the degree of variation present in the mixture. Then, each size fraction representative mean diameter should be calculated precisely, as that diameter will be used in all the calculations. According to the representative diameter of the size fraction class a suitable approach is implemented in the model as bed material load. So the model is divided in two different approaches: bed load type model and suspended load type model. In the bed load type model approach the bed load transport model is applied as the bed material load for the desired size fraction. In another approach the bed material load for a particular size fraction can be simulated as suspended load.

The flow of model can be represented by the flow chart shown in Figure 3.7. First of all, initial conditions and other parameter values are read and then hydrodynamic model calculates velocity profile based upon initial conditions. After that sediment transport model uses the velocity profile as input and uses the suspended load type model or bed material type model according to the sediment fraction size. In sediment transport model, first advection part is calculated. Then source sink terms are calculated based upon the sediment concentration and then diffusion part is run. Finally the change in bed elevation and sediment concentration at each grid point is calculated.

### 3.18 Parallel Computing

Parallel computing uses resources of more than one processor for a single problem. Parallel computers, also known as super computers, are the fastest computers available. Supercomputers can be classified in two categories.

(1) Shared Memory

(2) Distributed Memory
3.18.1 Shared Memory Machines

These machines are based on Parallel Random Access Memory (PARM) model. In this approach all processors have access to a common shared memory. This approach can be further divided into two parts, Concurrent Read Exclusive Write (CREW) PRAM model and Exclusive Read Exclusive Write (EREW). CREW PRAM allows different processors to access the same location of memory but in EREW PRAM allows one processor at a time to access a memory location.

Shared memory machines are normally faster for a given number of processors as access to local memory is many times faster than the access to remote memory. Due to the complex architecture and cutting edge technology, shared memory machines are built by big companies. The drawback with shared memory machines is that they are not scalable for large number of processors.

3.18.2 Distributed Memory Machines

Distributed memory machines are different processors which have their local memory connected through network. In these machines communication has to be done explicitly. Parallel clusters are distributed memory machines. Don Becker at NASA developed first parallel cluster and named as Beowulf cluster, which is still used as a name for parallel clusters. Figure 3.8 shows the schematic diagram of a distributed memory parallel computer. MPI (Message Passing Interface) is used for communication between different nodes of a cluster.

3.18.3 Message Passing Interface (MPI)

MPI is a programming library for parallelizing the code on parallel computers. MPI is a standard parallel programming library used across the world on different kinds of parallel machines. Parallel codes written using MPI are portable and can be run on any kind of parallel machine. Normally MPI is used for distributed memory machines, but the code written on distributed memory machines can be executed on shared memory machines without any modifications. MPI can also be used for heterogeneous parallel cluster which contains machines of different architecture. The parallel code should be written in Fortran, C and C++ to use MPI library. MPI takes care of all kinds of communication between processors. MPI also takes care of domain decomposition and accordingly the processor topologies. Figure 3.9 shows a domain of 9x9 grid size for a problem for a sequential program to run on a single computer. To run the same problem with same domain on a parallel computer using 4 processors, the domain has to decomposed in 4 parts as shown in figure 3.10. According to domain decomposition processor topology should be determined. This approach is used to discretize sediment transport model’s two dimensional problem space over processors.
Figure 3.7: Flow chart of sediment transport model
Figure 3.8: Domain representation for a problem on one processor
Figure 3.9: Domain representation for a problem on one processor
Figure 3.10: Domain representation for a problem on one processor
Chapter 4

Results and Discussion

In this chapter sediment transport model is tested for different kinds of problems. Initially both diffusion and advection parts are checked for a test problem. Then combined advection-diffusion scheme was checked against analytical solutions for some test problems. Then sediment transport model is checked for different types of sediment transport conditions. The model is checked for aggradation and degradation of bed in case of uniform sediment, then the model is checked for suspended load transport. Finally the model is checked for aggradation problem due to non-uniform sediment mixture.

4.1 Testing of Numerical Solver

As mentioned before, different numerical solvers were developed to solve linear systems of equations. These solvers were developed to run on super computers. All of these methods are checked for accuracy and speed. For that Poisson’s equation is used as a problem and analytical solutions are developed for it. A finite difference method is used to discretize Poisson’s equation and to develop the linear system of equations, to which different numerical solvers can be applied. In the following paragraphs these tests are discussed.

Poisson’s equation is a second-order partial differential equation, which represents pure diffusion transport in a system. Two dimensional Poisson’s equation can be expressed as follows. It represents the profile of a density function $\phi(x,y)$ over a two dimensional plane according to the defined boundary conditions of the plane:

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = F(x,y) \quad (4.1)$$

Poisson’s equation is used for testing the numerical solvers for linear systems of equations, because analytical solutions are available for it.

4.1.1 Numerical Solution of Poisson’s Equation

A finite difference method is used to discretize Poisson’s equation over a plane at grid points. Discretization of a density function $\phi(x,y)$ in a plane is shown in Figure 4.1
Application of finite difference method for Poisson’s equation is shown below:

\[ \frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} = F(x, y) \quad (4.2) \]

Taylor’s series expansion is used for writing a finite difference approximation of the derivatives in the partial differential equation. Taylor’s series expansion for \( \phi(x + h) \) can be written as:

\[ \phi(x + h) = \phi(x) + h\phi'(x) + \frac{h^2}{2}\phi''(x) + \frac{h^3}{6}\phi'''(x) \quad (4.3) \]

Similarly Taylor’s series expansion for \( \phi(x - h) \) can be written as:

\[ \phi(x - h) = \phi(x) - h\phi'(x) + \frac{h^2}{2}\phi''(x) - \frac{h^3}{6}\phi'''(x) \quad (4.4) \]

By adding equations 4.3 and 4.4 and neglecting the second order and higher order terms by assuming that the value of \( h \) is small, a finite difference approximation of the second derivative of the density function is obtained as:

\[ \phi''(x) = \frac{\phi(x - h) - 2\phi(x) + \phi(x + h)}{h^2} \quad (4.5) \]

Equation 4.5 can be rewritten in terms of one dimensional discretized space in grids as:

\[ \phi''(i) = \frac{\phi(i - 1) - 2\phi(i) + \phi(i + 1)}{\delta x^2} \quad (4.6) \]
where i is the indexing of the grid in one dimensional space. Using equation 4.6, the second partial derivative in both directions of two dimensional density function $\phi(x, y)$ can be written for two dimensional discretized space as shown in figure 4.1 as follows:

\[
\begin{align*}
\frac{\partial^2 \phi}{\partial x^2} &= \frac{\phi(i-1, j) - 2\phi(i, j) + \phi(i+1, j)}{\Delta x^2} \\
\frac{\partial^2 \phi}{\partial y^2} &= \frac{\phi(i, j-1) - 2\phi(i, j) + \phi(i, j+1)}{\Delta y^2}
\end{align*}
\] (4.7)

Inserting the values of second partial derivatives from equation 4.7 into Poisson’s equation 4.1

\[
\frac{\phi(i-1, j) - 2\phi(i, j) + \phi(i+1, j)}{\Delta x^2} + \frac{\phi(i, j-1) - 2\phi(i, j) + \phi(i, j+1)}{\Delta y^2} = F(i, j)
\] (4.8)

Rearranging equation 4.8 as:

\[
\frac{1}{\Delta x^2} \phi(i-1, j) + \frac{1}{\Delta x^2} \phi(i+1, j) + \frac{1}{\Delta y^2} \phi(i, j-1) + \frac{1}{\Delta y^2} \phi(i, j+1) - \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \phi(i, j) = F(i, j)
\] (4.9)

which can be written as:

\[
C_1 \phi(i-1, j) + C_2 \phi(i+1, j) + C_3 \phi(i, j-1) + C_4 \phi(i, j+1) + C_5 \phi(i, j) = F(i, j)
\] (4.10)

where

\[
C_1 = C_2 = \frac{1}{\Delta x^2} \\
C_3 = C_4 = \frac{1}{\Delta y^2} \\
C_5 = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}
\] (4.11)

Equation 4.10 represents a linear system of equations to which different numerical solvers can be applied at grid points for estimating the profile of the density function over a two dimensional plane. This numerical solution is independent of the Dirichelet or Neumann boundary conditions.

4.1.2 Analytical Solution of Poisson’s Equation

For analytical solution in Dirichelet condition, Poisson’s equation is simplified by assuming that the right hand term $F(x, y)$ equals to zero. This simplified equation is known as Laplace equation, and written as follows.

\[
\frac{\partial^2 \phi(x, y)}{\partial x^2} + \frac{\partial^2 \phi(x, y)}{\partial y^2} = 0
\] (4.12)

This equation represents the profile of a density function over a two dimensional plane with specified boundary conditions without source-sink. A two dimensional plane with assumed
dimensions with Dirichelet boundary conditions is shown in the figure 4.2. Boundary conditions for the Laplace equation are taken as follows as a test problem.

\[
\begin{align*}
\phi(0,y) &= F_1 \quad 0 < y < b \\
\phi(a,y) &= F_2 \quad 0 < y < b \\
\phi(x,0) &= F_3 \quad 0 < y < b \\
\phi(x,b) &= F_4 \quad 0 < x < a
\end{align*}
\]

(4.13)

![Figure 4.2: Plane with Dirichelet boundary for Poission’s equation](image)

Analytical solution of Poisson’s equation for the above specified Dirichlet boundary can be written as:

\[
\phi(x,y) = \phi_1(x,y) + \phi_2(x,y) + \phi_3(x,y) + \phi_4(x,y)
\]

(4.14)
where $\phi_1(x,y), \phi_2(x,y), \phi_3(x,y)$ and $\phi_4(x,y)$ can be written as:

\[
\phi_1(x,y) = \frac{2F_1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n \sinh(n\pi/b)(a-x)\sin(n\pi/b)y}{n \sinh(n\pi/b)a}
\]
\[
\phi_2(x,y) = \frac{2F_2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n \sinh(n\pi/b)x\sin(n\pi/b)y}{n \sinh(n\pi/b)a}
\]
\[
\phi_3(x,y) = \frac{2F_3}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n \sin(n\pi/a)x\sinh(n\pi/a)y}{n \sinh(n\pi/a)b}
\]
\[
\phi_4(x,y) = \frac{2F_4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n \sin(n\pi/a)x\sinh(n\pi/a)(b-y)}{n \sinh(n\pi/a)b}
\] (4.15)

To calculate the accuracy of the numerical solvers, the cumulative error is calculated for the whole grid points by the following way for Jacobi and Red Black Gauss Seidel methods:

\[
\text{Error} = \sqrt{\sum (\text{Numerical solution}(i,j) - \text{analytical solution}(i,j))^2}
\] (4.16)

Now, a test problem was selected to compare numerical solvers against analytical solvers. In the test problem the dimension of the plane was taken 10m x 10m. The convergence of the Jacobi and Red Black Gauss Seidel methods are slow than Bi-CGSTAB and Bi-CGSTAB(2) methods. For Jacobi and Red Black Gauss Seidel methods the grid size was taken 100 x 100 and Bi-CGSTAB abd Bi-CGSTAB(2) methods grid size was taken 500 x 500, so that proper scalability of the numerical solver could be checked. Boundary conditions were taken the same for all methods. The Dirichlet boundary condition

\[
\begin{align*}
\phi(0,y) &= 2 \quad 0 < y < b \\
\phi(a,y) &= 3 \quad 0 < y < b \\
\phi(x,0) &= 5 \quad 0 < x < a \\
\phi(x,b) &= 8 \quad 0 < x < a
\end{align*}
\] (4.17)

Figures 4.3 and 4.4 show the scalability of numerical solvers and also comparison of speed for different numerical solvers.

From test results the following conclusions can be drawn. The Jacobi method is very slow in convergence and it is not advisable to use this method for large grid size problems. The Red Black Gauss Seidel method is faster but still not fast enough to use for large grid size problem. Bi-CGSTAB method is quite faster and can be used for larger problems and also the use of preconditioners improves the performance of the method and speeds up the convergence. Bi-CGSTAB(2) is the fastest method among all. This method is also most robust method.

### 4.2 Testing of Advection Algorithm

Advection algorithm described in previous section was also tested for a test problem shown in Leveque (1996), who developed the advection algorithm. In the test problem a plane of dimensions 1 x 1 with grid sizes of 100 x 100, which make grid dimensions of 0.01 x 0.01. In this plane a
value of density function was assigned in such a way that it formed a disk in the z direction, which is shown in the figure 4.6. This method is called solid body rotation test. Now a non-constant velocity profile is specified in the plane as:

$$u = -(y - 1/2), \quad v = (x - 1/2)$$  \hspace{1cm} (4.18)

Initial data of density function in the form of disk is centered at $x_0 = 0.5$ and $y_0 = 0.75$ with a radius of 0.15. A time step value for advection iteration was chosen 0.01. In this test problem pure advection is assumed and the velocity profile is taken in such a way that disk should come back at its original position with the same density function values as at points. This problem was tested by method four with all limiters. The result after one revolution using the fourth method is shown in figure 4.7.

From these results the following conclusions can be drawn. The advection algorithm can accurately simulate the algorithm. The first limiter provides the worst results in all and the second limiters provide the best results but this conclusion cannot be taken as a rule to always use the second limiters. The choice of limiters will vary according to the problem conditions.
Figure 4.4: Scalability and comparison of speed for Bi-CGSTAB using different preconditioner choices and Bi-CGSTAB(2) methods

4.3 Advection-diffusion Combined Test

After testing advection and diffusion schemes separately, the combined advection-diffusion scheme was tested. The combined advection-diffusion scheme was tested for Wexler (1992) analytical solution of two dimensional advection-diffusion including source-sink term and Zoppou’s (1997) analytical solution without source-sink term. These tests solution are shown in the next paragraphs.

4.3.1 Zoppou Solution

Zoppou (1997) developed an analytical solution to two-dimensional advection diffusion equation for spatially variable diffusion coefficient with no source-sink term. He simplified the two dimensional advection-diffusion in the following way:

$$\frac{\partial hC}{\partial t} + \frac{\partial U hC}{\partial x} + \frac{\partial V hC}{\partial y} = \frac{\partial}{\partial x} \left[ k_x h \frac{\partial C}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y h \frac{\partial C}{\partial y} \right] + E - D$$

(4.19)
Figure 4.5: Scalability and comparison of speed for Bi-CGSTAB using different preconditioner choices and Bi-CGSTAB(2) methods

Assuming the unit flow depth and no source-sink term, equation 4.19 can be written as

$$\frac{\partial C}{\partial t} + \frac{\partial U C}{\partial x} + \frac{\partial V C}{\partial y} = \frac{\partial}{\partial x} \left[ k_x \frac{\partial C}{\partial x} \right] + \frac{\partial}{\partial y} \left[ k_y \frac{\partial C}{\partial y} \right]$$

(4.20)

He specified the following spatial variation of velocity and diffusion coefficients:

$$u = u_0 x$$

$$\nu = -v_0 y$$

$$D_x = D_0 u_0^2 x^2$$

$$D_y = D_0 u_0^2 y^2$$

(4.21)

where $u_0, D_0$ are parameters and x, y are distances along X and Y dimensions, respectively. Inserting the above specified spatial variation of velocity and diffusion coefficient in equation 4.20

$$\frac{\partial C}{\partial t} + \frac{\partial u_0 x C}{\partial x} + \frac{\partial - u_0 y C}{\partial y} = \frac{\partial}{\partial x} \left[ D_0 u_0^2 x^2 \frac{\partial C}{\partial x} \right] + \frac{\partial}{\partial y} \left[ D_0 u_0^2 y^2 \frac{\partial C}{\partial y} \right]$$

(4.22)
Zoppou (1997) proposed following analytical solution for estimating the concentration profile over a two-dimensional plane.

\[ C(x, y, t) = \frac{1}{4\pi D_0 u_0^2 t \sqrt{xyx_0y_0}} \left( \frac{x_0}{x} \right)^{1/(2u_0D_0)} \exp \left( \frac{-\rho^2 - 2(1 + D_0^2u_0^2)t^2}{4D_0t} \right) \]  

(4.23)

where

\[ \rho = \frac{\sqrt{\ln^2(x/x_0) + \ln^2(y/y_0)}}{u_0} \]  

(4.24)

Now a instantaneous release of unit mass was assumed at \( x_0 = 4.75 \) and \( y_0 = 4.75 \) with velocity coefficient \( u_0 = 1 \) and diffusion coefficient \( D_0 = 2 \). Figure 4.8 shows the 3D plot of concentration at \( t = 0.05 \) seconds using the analytical method and figure 4.9 shows the 3D plot of concentration at \( t = 0.05 \) second using the combined advection-diffusion numerical scheme used in the model. Figure 4.10 shows the 3D plot of concentration at \( t = 0.1 \) second using the analytical method and figure 4.11 shows the 3D plot of concentration at \( t = 0.1 \) second using the combined advection-diffusion numerical scheme. Figure 4.12 shows the contour plot of concentration profile at 0.05 seconds. In the figure filled contours represent the analytical solution and line contours represents the numerical solution. Similarly figure 4.13 shows the contour plot of concentration profile at 0.1 second. In all the graphs the combined numerical advection-diffusion scheme predictions are very well matched with analytical solutions.
Figure 4.7: Profile of density function after one revolution using 3rd method
Figure 4.8: 3D plot of concentration profile using analytical method at 0.05 second
Figure 4.9: 3D plot of concentration profile using advection-diffusion numerical scheme at 0.05 second
Figure 4.10: 3D plot of concentration profile using analytical method at 0.1 second
Figure 4.11: 3D plot of concentration profile using advection-diffusion numerical scheme at 0.1 second
Figure 4.12: Concentration profile contour using analytical method and numerical scheme at 0.05 second. Filled contour - Analytical method. Line contour - Numerical scheme.
Figure 4.13: Concentration profile contour using analytical method and numerical scheme at 0.1 second. Filled contour - Analytical method. Line contour - Numerical scheme.
4.3.2 Wexler Solution

Wexler (1992) developed an analytical solution for two-dimensional advection-diffusion equation including source-sink term. Diffusion coefficients and velocity profile were taken constant spatially. He developed an analytical solution for the following form of the advection-diffusion equation, which can be derived by assuming unit depth and spatially constant diffusion coefficients and velocity profile.

\[
\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} + V \frac{\partial C}{\partial y} = k_x \left[ \frac{\partial^2 C}{\partial x^2} \right] + \left[ \frac{\partial^2 C}{\partial y^2} \right] + E - D \tag{4.25}
\]

He assigned the following for source-sink term:

\[
E - D = -\lambda C \tag{4.26}
\]

where \( \lambda \) is a coefficient. Using this source-sink term equation 4.25 can be rewritten as

\[
\frac{\partial C}{\partial t} + U \frac{\partial C}{\partial x} + V \frac{\partial C}{\partial y} = k_x \left[ \frac{\partial^2 C}{\partial x^2} \right] + \left[ \frac{\partial^2 C}{\partial y^2} \right] - \lambda C \tag{4.27}
\]

He assigned the following boundary and initial conditions for the test.

Boundary conditions:

\[
C = C_0, \quad x = 0 \text{ and } y_1 < y < y_2 \tag{4.28}
\]
\[
C = 0, \quad x = 0 \text{ and } y < y_1 \text{ or } y > y_2 \tag{4.29}
\]
\[
\frac{\partial C}{\partial y} = 0, \quad y = 0 \tag{4.30}
\]
\[
\frac{\partial C}{\partial y} = 0, \quad y = W \tag{4.31}
\]
\[
\frac{\partial C}{\partial y} = 0, \quad X = L \tag{4.32}
\]

where

\( W = \) width of the two-dimensional problem area
\( L = \) length of the two-dimensional problem area

Initial conditions:

\[
C = 0, \quad 0 < x < L \text{ and } 0 < y < W \tag{4.33}
\]

He proposed the following analytical solution for equation 4.27 to calculate the concentration profile in the two-dimensional space.

\[
C(x, y, t) = C_0 \sum_{n=0}^{\infty} L_n P_n \cos(\eta y) \left\{ \exp \left[ \frac{x(U - \beta)}{2k_x} \right] \operatorname{erfc} \left[ \frac{x - \beta t}{2\sqrt{k_xt}} \right] + \exp \left[ \frac{x(U + \beta)}{2k_x} \right] \operatorname{erfc} \left[ \frac{x + \beta t}{2\sqrt{k_xt}} \right] \right\} \tag{4.34}
\]
where

\[ L_n = \begin{cases} \frac{1}{2}, & n = 0 \\ 1, & n > 0 \end{cases} \]  
(4.35)

\[ P_n = \begin{cases} \frac{y_2 - y_1}{\eta}, & n = 0 \\ \frac{\sin(\eta y_2) - \sin(\eta y_1)}{n\pi}, & n > 0 \end{cases} \]  
(4.36)

\[ \eta = \frac{n\pi}{W}, \quad n = 0, 1, 2, 3, \ldots \]

\[ \beta = \sqrt{U^2 + 4k_x(\eta^2 k_y + \lambda)} \]  
(4.37)

\[ \text{erfc}(y) = \text{complementary error function defined as} \]

\[ \text{erfc}(y) = 1 - \text{erf}(y) \]  
(4.38)

\[ \text{erf}(y) = \text{error function defined as} \]

\[ \text{erf}(y) = \frac{2}{\pi} \int_0^y \exp(-\epsilon^2) d\epsilon \]  
(4.39)

The following numerical values were assumed for a sample problem to check the combined numerical advection-diffusion scheme with analytical solution:

- Width = 3000ft
- \( y_1 = 400 \text{ft}, y_2 = 2000 \text{ft} \)
- X-axis velocity U = 1ft/day
- Diffusion coefficient in x direction = 200 ft\(^2\)/day
- Diffusion coefficient in x direction = 60 ft\(^2\)/day
- Boundary concentration \( C_0 = 1000 \text{mg/l} \)

Initially source-sink terms were assumed to be zero, \( \lambda = 0 \). The numerical advection-diffusion scheme used in the model was used to simulate the above specified sample problem and compared with analytical solution. Figure 4.14 shows the contour plot of concentration profile in two dimensional space after 1500 days using both analytical and numerical solutions. In the figure filled contours represent the analytical solution and lined contours represent the concentration profile using the model.

Now the source-sink term was included for the next sample problem and coefficient \( \lambda \) was assigned a value of 0.001. Again the model was used to simulate the concentration profile including the above specified source-sink term and compared with the analytical solution. Figure 4.15 shows the contour plot of the concentration profile in two dimensional space after 1500 days and figure 4.16 shows the contour plot of the concentration profile after 3000 days using both analytical and numerical solutions. In the figures filled contours represent the analytical solution and lined contours represent the concentration profile using the model. In all contour plots the concentration profile calculated by numerical advection-diffusion scheme is well matched with concentration calculated by the analytical solution.
Figure 4.14: Concentration profile contour using analytical method and numerical scheme after 1500 days with no source-sink term (Filled contour - Analytical method Line contour - Numerical scheme)
Figure 4.15: Concentration profile contour using analytical method and numerical scheme after 1500 days including source-sink term (Filled contour - Analytical method Line contour - Numerical scheme)
Figure 4.16: Concentration profile contour using analytical method and numerical scheme after 3000 days including source-sink term (Filled contour - Analytical method Line contour - Numerical scheme)
4.4 Sediment Transport Model Testing

In the following paragraphs testing of the sediment transport model is shown. For this many laboratory studies were used. These laboratory studies were normally done to estimate the degradation or aggradation in a river for given condition. To that end flumes were used as a representative of river and to maintain the river dynamics as much possible.

4.5 Newton Degradation Test

In nature streams achieve an equilibrium state which is represented as no major change in bed elevation for given flow and sediment and other conditions. Any construction of a control structure like dam changes the equilibrium conditions in the stream and forces stream to adjust itself for the new sediment or flow conditions. In that case degradation at the downstream of the control structure starts in a way as to achieve the equilibrium state for the new conditions.

Newton (1951) did laboratory experiments to develop and verify analytical methods for calculating the expected degradation in nature. The main objective of the study was to analyze the degradation processes of the bed in an open channel when a control structure at the upstream changes the sediment load condition in the stream. He used a recirculating, open channel laboratory flume for the experiments. The flume was 9.14m long, 0.3048m wide, and 0.6097m deep. Figure 4.17 shows a cross sectional view of the flume. Sediment was supplied through a sediment feed elevator at the upstream. Transported sand through flume was collected in a bucket at the downstream which was attached with an elevator to flush out the sediment at the downstream.

![Figure 4.17: Newton Sediment Degradation Experiment Flume Section](image)

Every experiment consisted of two parts. The first part was to develop an equilibrium condition for a specified discharge rate Q and sediment feed rate G. After establishment of the equilibrium condition in the flume, sediment feed was cut off in the second part, so that degradation of the bed would start to achieve a new equilibrium condition. Flume was assumed to be in
equilibrium state when sediment collection rate in the downstream trap bucket became equal to the sediment feed rate at the upstream and when bed surface and water surface reached constant slopes and elevations. Normal flow conditions had been established at the equilibrium state.

20-30 Ottawa well rounded uniform sand was used in this experiment. The mean size of the sand was 0.69mm. The specific gravity of sand was 2.65. The void ratio of sand before feeding in to flume was between 0.36 and the void ratio of bed developed in the flume was 0.39. The depth of flow at the downstream was also maintained constant by placing the weir at a constant height for an experiment. A constant sediment feed rate \( G \) was maintained during the first part of the experiment to establish an equilibrium state in the flume. A constant discharge was maintained both before and after the establishment of equilibrium state in the experiment.

Newton did four experiments for different combinations of discharge and initial sediment feed rate. Comparison of measured values and simulated values for three out of the four experiments are shown below.

### 4.5.1 Experiment No.1

Initial flow and sediment conditions are as follows.

- Discharge \( Q = 0.00566 \text{m}^3/\text{sec} \)
- Sediment feed rate \( G = 0.88652 \text{kg/m}^3 \)
- Constant downstream water depth \( d = 0.0411 \text{m} \)
- Slope of bed \( S_b = 0.00416 \)
- Mean Velocity of flow \( V = 0.45 \text{m/sec} \)

It took 25.5 hours for the experiment flume to reach the equilibrium state and the following flow condition was measured at the equilibrium state. The initial flow conditions are simulated through the numerical flow model. The flow model calculated flow depth 0.0412m and flow velocity 0.049m/sec, which match well with measured initial flow conditions.

For second stage of the experiment sediment feed at the upstream was stopped. Due to this, bed degradation started. To simulate bed degradation sediment transport model was started and upstream sediment concentration was defined zero.

Figure 4.18 shows a plot of comparison between measured and simulated bed elevations using the model after 1 hour of degradation. In the plot simulated bed elevation is calculated using different sediment transport capacity formulas. As is seen, the modified calibrated Chien formula predicted most accurately and according to the trend of the degrading bed.

Figure 4.19 compares measured and computed bed elevations at 1hr, 2hr and 3hr of degradation period using the modified calibrated Chien method. The computed bed elevation was not well matched at the upstream of the flume. Due to very high erosion at the upstream a big scour hole was generated at the upstream. It is because sediment transport capacity was assumed to be equilibrium transport capacity throughout the flume, which is not a good assumption. In case of water flows from non-erodible surface to erodible surface water does not attain the equilibrium sediment transport capacity. It takes some distance and time for water to reach its equilibrium sediment transport capacity.
Figure 4.18: Computed bed elevation at 1hr using different sediment transport capacity formulas
So non-equilibrium sediment transport capacity is applied to simulate the degrading bed elevation. Bell and Sutherland (1983) proposed that the water sediment transport capacity depends on both time and distance from the upstream and can be calculated by the following formula:

\[
\frac{q_n^s(x,t)}{q_s(x,t)} = 1 - e^{-C(t)(x-x_0)}
\]

(4.40)

where \(q_n^s\) and \(q_s\) are, respectively, non-equilibrium and equilibrium sediment transport capacities and \(C(t)\) is a decreasing function of time which is not well defined yet and should be estimated by trial and error. For this simulation the following form of decay function was implemented:

\[
C(t) = \frac{1}{1+at}
\]

(4.41)

where \(t\) is time in hours and \(a\) is a parameter which was optimized to get the best results. Figure 4.20 compares measured and computed bed elevations using the non equilibrium sediment transport capacity approach. As can be seen, the calculated values very well match with measured values.
**4.5.2 Experiment No.2**

Initial flow and sediment conditions are:
- Discharge \( Q = 0.0113 \text{ m}^3/\text{sec} \)
- Sediment feed rate \( G = 0.9492 \text{ kg/m}^3 \)
- Constant downstream water depth \( d = 0.0487\text{m} \)
- Slope of bed \( S_b = 0.00438 \)
- Mean Velocity of flow \( V = 0.761\text{m/sec} \)

It took 9 hours for the experiment flume to reach the equilibrium state and the flow condition was measured at the equilibrium state. Initial flow conditions were simulated through the numerical flow model. The flow model calculated the flow depth of 0.048m and flow velocity of 0.762m/sec, which well matched with measured initial flow conditions.

For the second stage of the experiment sediment feed at the upstream was stopped. Due to
this bed degradation started. To simulate bed degradation the sediment transport model was started and upstream sediment concentration was defined zero. The non-equilibrium sediment transport capacity approach was applied for this run to calculate the degrading bed elevation. Figure 4.21 compares measured and calculated bed elevation for run no. 2 at 0.5hr, 1hr, 2hr. As can be seen, calculated values are well matched by measured values.

![Figure 4.21: Comparison between measured and calculated values](image)

### 4.5.3 Experiment No.3

Initial flow and sediment conditions are:

- Discharge $Q = 0.00564 m^3/sec$
- Sediment feed rate $G = 1.83 kg/m^3$
- Constant downstream water depth $d = 0.039 m$
- Slope of bed $S_b = 0.00607$
- Mean Velocity of flow $V = 0.474 m/sec$
It took 16 hours for the experimental flume to reach the equilibrium state and the flow condition was measured at the equilibrium state. Initial flow conditions were simulated through the numerical flow model. The flow model calculated the flow depth 0.04m and flow velocity 0.48m/sec, which are well matched with measured initial flow conditions.

For second stage of the experiment sediment feed at the upstream was stopped. Due to this, bed degradation started. To simulate bed degradation the sediment transport model was started and upstream sediment concentration was defined zero. The non-equilibrium sediment transport capacity approach was applied for this run to calculate the degrading bed elevation. Figure 4.22 compares measured and calculated bed elevation for run no. 3 at 0.5hr, 1.5hr and 3hr. As can be seen calculated values are well matched by measured values.

Figure 4.22: Comparison between measured and calculated values
4.6 Soni Aggradation Test

Soni (1981) performed laboratory experiments to study aggradation in natural rivers due to the increase in the stream sediment load. Recirculatory flume used by Soni was 30.0 m long, 0.20 m wide and 0.50m deep as shown in figure 4.23 located in the Hydraulics Laboratory of the University of Roorkee, Roorkee, India. The water discharge was controlled by a valve and measured by a calibrated orifice meter. To measure the bed elevation and water surface elevation during the experiment a pointer gauge with a least count of 0.01cm was mounted on a movable carriage. An adjustable gate was provided at the downstream end of the flume to maintain a constant water depth at the downstream end for a particular water discharge. A sediment sampler was installed at the downstream end to measure the sediment transport rate at the downstream end. A floating wooden wave suppressor was used at the entrance of the flume to dampen the disturbance at the free surface.

The sand used for bed material and sediment feed in the experiments had a mean diameter of 0.32m, gradation coefficient of 1.30 and specific gravity of 2.65. Soni performed experiments in the mobile bed condition to better represent natural rivers. To attain mobile bed in the flume before starting the experiment the following steps were followed. Initially flume was given a desired slope using the specified sand. Then the recirculatory flume was filled slowly with water and control valve was used to attain the specified discharge. The tail gate height was adjusted in a way so that uniform flow was obtained in the flume by allowing the bed to adjust by erosion or deposition.

For a desired discharge a uniform flow condition in the flume was achieved when the measured bed and water surface were parallel to each other. After reaching the uniform flow condition, sediment was dropped at the upstream of the flume at a constant rate. The sediment injection section was located far enough from the entrance of the flume to avoid entrance disturbances. The aggradation in the bed started due to the excess load of the sediment. Bed and water surface elevation were measured at regular time intervals. Downstream sediment sampler was used continuously.

Figure 4.23: Experiment flume used by Soni
to remove sediment from the recirculatory water.

### 4.6.1 Experiment Run

For experimental run a constant discharge of 0.0071 m$^3$/sec was maintained and by adjusting the tail water gate height a uniform flow was achieved at the mean water depth of 0.072m and slope of 0.00427 at the mean flow velocity of 0.493m/sec. This initial flow condition was estimated correctly using the flow model by specifying the upstream discharge and tailgate water depth. Computed initial mean water depth was 0.073m and mean velocity was 0.49m/sec.

After achieving the equilibrium condition in the flow, sediment injection was started at the upstream at a constant rate of 4.88kg/m$^3$. Due to this excess sediment load the aggradation stared in the bed. This aggradation process was simulated using the sediment transported model. Figure 4.24 shows the comparison between measured and computed bed elevation using the calibrated Chien sediment transport capacity formula. As can be seen, the computed bed elevation is well matched with measured bed elevation.

![Comparison between measured and computed values for Soni aggradation test](image-url)

Figure 4.24: Comparison between measured and calculated values for Soni aggradation test
4.7 Seal Aggradation Test for Non-uniform Sediments

Seal (1997) performed a aggradation test in a flume using non uniform sediment. The flume used for experiment was 45m long and 0.3 m wide and 1.2m deep. At the end a ramp at 45° was attached which was used to generate a free fall at the downstream end. A tail gate was placed at 3 m from the downstream end of the channel, which was used to maintain the constant water elevation at the downstream end. The schematic diagram of the flume is shown in figure 4.25 with aggraded bed sue to non uniform sediment.

Figure 4.25: Schematic diagram of flume used by Seal with aggraded bed due to non uniform sediment

In the flume the discharge was maintained using a 10cm pipe and measured by an orifice and manometer. To measure the bed elevation during the experiment a cart on rails was mounted on flume, on which standard point gauge was attached.

4.7.1 Experiment Run

For the experiment, initially a steady uniform discharge of 0.049 m³/sec was maintained. The discharge was kept constant throughout the experiment. The downstream water elevation was kept at a constant height of 0.4m throughout the experiment. The sediment mixture used for the experiment was comprised of a wide range of sediment sizes ranging from 0.125mm to 64mm. The size distribution of the mixture is shown in figure 4.26. The feed rate of sediment mixture was maintained at a constant rate of 3.87 kg/m³ throughout the experiment.

Now the sediment transport model was run with a hydrodynamic model. Figure 4.27 compares measured and computed bed elevations along the flume after 2hr and 5hr. As can be seen, computed values are well matched by measured values.
4.8 Wang and Ribberink Experiment

Wang and Ribberink (1986) performed experiments in a flume to study sediment transport phenomena. The laboratory flume used in the experiment was 30m long, 0.5m wide and 0.5 deep shown in figure 4.28. The flume was divided in three sections. The first section was the inflow section with a rigid bed of 10m length. The second section was test section with perforated bed of 16m length and outflow section of 4m length.

The test section was made up of perforated plates to avoid erosion. In order to make sure for no flow development in the chamber below perforated plates, the chamber was subdivided in compartments of 0.5m length and width equal to the flume. Rigid bed was given artificial bed roughness equal to the perforated bed roughness to minimize the change in flow conditions due to bed change from rigid to perforated bed. Sediment concentration was measured by a sediment sampler which was able to take 8 samples at a cross section in vertical direction simultaneously. The depth average sediment concentration was calculated using those 8 samples.

4.8.1 Experiment Run

Uniform and steady flow was maintained throughout the experiment. It had a constant discharge of 0.0601m³/sec was and mean water depth of 0.216m, and a bed slope of 0.00097 was kept constant in the flume. The sand used for the experiment had a mean diameter of 0.1mm and average fall velocity of 0.7cm/sec. Sediment injection rate at the upstream was kept at a constant rate of 70.8kg/hr.
As in this experiment a constant flow condition was maintained, so there is no need of flow model. Specified flow conditions were used for the sediment transport model. Figure 4.29 shows a comparison of measured and computed sediment concentrations after reaching the equilibrium state. As can be seen, computed sediment concentration profile is well matched by the measured profile.

### 4.9 Sensitivity and Error Analysis

#### 4.9.1 Sensitivity Analysis for Suspended Load Transport

Wang and Ribberink (1986) test was used to study the sensitivity and error analysis of different parts of suspended load transport. In this test sediment settling velocity and non-equilibrium adaptation coefficient $\alpha$ were the only parameter to achieve the results.

Settling velocity measured by Wang during experiment was 0.7 cm/sec. Now all the different formulas for calculating sediment settling velocity in model were tested for the Wang test particle size. Figure 4.30 compares the settling velocity calculated by different methods with measured values. Stoke’s, Rubey, Van Rijn, Chng and Liou and Aherens predicted settling velocity are very well matched with the measured one. Zhang, Sha, Ibade Zade, Zhu and Cheng and Cheng predicted little less values but in acceptance range. Figure 4.31 compares the final concentration using different settling velocity formulas. From graph it can be concluded that all the settling velocity formulas give acceptable results for this test.

Now sensitivity analysis of non-equilibrium adaptation coefficient $\alpha$ was done. In the model $\alpha = 2$ was taken to achieve the best results. Figure 4.32 compares the final concentration
in the flume using different values of $\alpha$. The figure shows the decrement in the value of $\alpha$ leads to increased sediment concentration in the flume and increment in $\alpha$ results in less concentration. This can be justified by the equation 4.42 used to calculate the source sink term:

$$Source - sink = \alpha w_s (C_k - C_k^*)$$  \hspace{1cm} (4.42)

In this case perforated bed was used so there was no erosion which simplified equation 4.42 in the following manner:

$$Source - sink = \alpha w_s C_k$$ \hspace{1cm} (4.43)

From figure 4.32 it can be concluded that non-equilibrium adaptation $\alpha$ is an important parameter and should be tuned to achieve the results.

Now for sensitivity analysis of diffusion coefficient, final sediment concentration was predicted using 50% increased and decreased values of the diffusion coefficient used for simulation. Figure 4.33 shows plots of sediments concentration using diffusion coefficient equal to 0.25, 0.5, 0.75. From the graph it can be concluded that change in diffusion coefficient does not affect much the final sediment concentration for this test.

**4.9.2 Sensitivity Analysis for Bed Load Transport**

Soni (1981) aggradation test was used for sensitivity analysis of bed load transport. In bed load transport critical shear stress is an important parameter. In this model many options have been incorporated to calculate the critical stress. Figure 4.34 compares the critical shear stress calculated by all options for 0.32mm sediment mixture used by Soni in his experiment. Out of all options the Krey method predicts a little more value of critical stress.

Another parameter used in the bed load transport part was non-equilibrium adaptation length coefficient $L_t$. Figure 4.35 compares the bed depth after 1hr using different values of the
Figure 4.29: Comparison between measured and computed sediment concentration along the flume coefficient. From the figure it can be concluded that change in bed elevation is not very sensitive to value of coefficient, even $L_t = 3$ gave the best fit.

Now different sediment transport capacity formulas were tested. Bed elevation after 1hr was simulated using different formulas and then compared with the measured bed elevation. Figure 4.36 compares the bed elevation after 1hr using different sediment transport capacity formulas.
Figure 4.30: Comparison between measured and computed sediment settling velocity for 0.1mm sediment particle

Figure 4.31: Computed sediment concentration using different settling velocity formulas
Figure 4.32: Computed sediment concentration using different values of non-equilibrium adaptation coefficient

Figure 4.33: Computed sediment concentration using different values of diffusion coefficient
Figure 4.34: Comparison between calculated critical shear stress using different methods for 0.32mm sediment particle

Figure 4.35: Bed elevation after 1hr using different values of non-equilibrium adaptation length $L_t$
Figure 4.36: Bed elevation after 1hr using different sediment transport capacity formulas
Chapter 5

Summary and Conclusion

5.1 Summary

A vertically integrated two dimensional sediment transport model has been developed. In model suspended load and bed load are simulated separately, so that the dynamics of both types of transport can be taken into account. Fraction step approach is used to solve the sediment transport equation, which leads to separate advection and diffusion part with source and sink. Leveque (1996) high resolution conservative algorithm is used to solve the advection part. To solve the diffusion part, a semi implicit finite difference scheme is employed in such a way that it can be converted in complete implicit or complete explicit scheme. Parallel numerical solvers are developed to solve the linear system of equations generated from application of finite difference scheme to the diffusion part. A modular approach is used in developing the model, so that model can be incorporated in any hydrodynamic model. The Model is able to take account of non-uniformity of sediment mixture. In the model many options have been provided to calculate sediment settling velocity, critical shear stress and sediment transport capacity.

The Model has been tested for different conditions. Initially both advection and diffusion parts are tested separately. Solid body rotation test was used to validate the advection part. The results from test are shown in section 4.2, in which the advection part simulates pure advection accurately.

To test the diffusion part and parallel numerical solvers, two dimensional Laplace equation was used as a problem which represents a pure diffusion problem. Analytical solutions for Laplace equation were developed for Dirichelet boundary conditions and then compared with numerical solvers. Comparison of analytical and numerical solutions is discussed in section 4.1. Scalability analysis of Jacobi, Red Black Gauss Seidel, Bi-CGSTAB, Bi-CGSTAB(2) numerical solvers is also discussed in the section. Scalability analysis shows that a particular problem scales up to some number of processors depending upon the problem.

The combined advection and diffusion part was tested using an analytical method for two dimensional advection diffusion equation without source sink, developed by Zoppou (1997). Analytical solution developed by Wexler (1992), including source source-sink term, was also used to test the combined advection-diffusion scheme. Section 4.3 discusses and compares the numerical solution with analytical solution. An instantaneous release of unit mass was simulated using nu-
The numerical advection-diffusion scheme and analytical method and compared. The numerical scheme simulated the concentration profile accurately.

Then the sediment transport model was tested for different flow and sediment transport conditions like suspended load and bed load transport. Newton’s (1951) degradation test was simulated for three different flow conditions for uniform sediment mixture. Soni’s (1981) bed aggradation test was simulated using the model for uniform sediment mixture. Seal’s (1997) bed aggradation test for non-uniform sediment mixture was simulated. Wang and Ribberink’s (1986) suspended load transport test was simulated using model.

Newton (1951) performed bed degradation test using a uniform sediment mixture. Three bed degradation tests were simulated using the model for prediction of change in bed elevation due to degradation. Simulated bed elevations were compared with measured bed elevation. This comparison is shown in section 4.5. Model predicted accurate bed elevation in all three cases.

Soni (1981) performed a bed aggradation test by increasing the sediment feed rate at the upstream for a uniform sediment mixture. Simulated aggraded bed elevation were compared with measured values. This comparison is shown in section 4.6. The model predicted aggradation in bed elevation accurately.

Seal (1997) did an aggradation test using a non-uniform sediment mixture comprised of a wide range of sediment sizes ranging from 0.125mm to 64mm. The bed elevation was simulated using the model and compared with the measured bed elevation. Section 4.7 discusses and compares the results. The model was able to predict the change in bed elevation for non-uniform sediment mixture accurately.

Wang and Ribberink (1986) did a suspended load transport test with uniform sediment mixture using perforated bed to eliminate erosion from bed. Steady state sediment concentration was simulated using the model and compared with measured concentration, which is shown in section 4.8. Simulated steady state concentration was simulated accurately by the model.

Sensitivity and error analysis of model was done and is discussed in section 4.9. Different options used for calculation of settling velocity and critical shear stress were compared. The effect of non-equilibrium adaptation length $L_t$ on predictions of bed elevation was analyzed. Effect of sediment transport capacity formulas on prediction of bed elevation was also analyzed. The effect of settling velocity, non-equilibrium adaptation coefficient $\alpha$ and diffusion coefficient on suspended load concentration was analyzed.

5.2 Conclusions

From this study following conclusions can be drawn.
1. The model can simulate suspended load and bed load transport separately.
2. The model is sensitive to the non-equilibrium adaptation length ($\alpha$), which is used as a parameter in suspended load transport simulation. So the value of $\alpha$ should be assigned with great care.
3. Use of different settling velocity formulas leads to different simulated suspended sediment concentration. So the settling velocity formula should be chosen by hit and trial method for a particular case.
4. The model is sensitive to the non-equilibrium adaptation length $L_t$, which is used as a parameter in bed load transport simulation.
5. Bed load transport simulation is sensitive to different bed load transport capacity formulas. So a great care should be taken in choosing the bed load transport capacity formula.
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Appendix: Algorithms

Advection algorithm

for each i, j do

\[ F_{i,j} = 0, \quad G_{i,j} = 0 \]

Calculate the flux based on the interfaces in x-direction and update increments.

for each i,j do

# Considering interface between cells \( C_{i-1} \) and \( C_{i,j} \)

\[ U = u^{n+1}_{i,j} \]
\[ V = v^{n+1}_{i,j} \]
\[ R = c_{i,j} - c_{i-1,j} \]

if \( U > 0 \) then \( I = i - 1 \) else \( I = i \)

\[ F_{i,j} = F_{i,j} + U q_{I,j} \]

# If method = 1 then end loop here

if \( U > 0 \) then \( I = i \) else \( I = i - 1 \)
if \( V > 0 \) then \( J = j + 1 \) else \( J = j \)

\[ G_{I,J} = G_{I,J} - \frac{1}{2} \frac{k}{h} U VR \]

# If method = 2 then end loop here

R = Limited version of R (Apply one of the four flux limiters)

\[ S = \frac{1}{2} |U| (1 - \frac{k}{h} |U|) R \]

\[ F_{i,j} = F_{i,j} + S \]

# If method = 3 then end loop here

\[ G_{i,J} = G_{i,J} + \frac{k}{h} VS \]

\[ G_{i-1,J} = G_{i-1,J} - \frac{k}{h} VS \]
# If method = 4 then end loop here
# To update increments and fluxes based on interfaces in Y direction, follow the same above steps
# with roles of i and j, u and v, F and G switched and replace $h_x$, length of cell in X-direction by $h_y$, 
# length of cell in Y-direction. # Update the value of $c$

for each i, j do

$$c_{i,j}^{n+1} = c_{i,j}^n - \frac{k}{h_x}[F_{i+1,j} - F_{i,j}] - \frac{k}{h_y}[G_{i,j+1} - G_{i,j}]$$

Bi-CGSTAB Algorithm

This algorithm is a iterative solver for solving the linear system of equation $Ax=b$, with 
preconditioner $K$.

Make an initial guess $x_0$

$$r_0 = b - Ax_0$$

$r_0$ is an arbitrary vector, such that

$$(r_0, r_0) \neq 0, \text{ e.g., } r_0 = r_0$$

$\rho_{-1} = \alpha_{-1} = \omega_{-1} = 1$;

$v_{i-1} = p_{-1} = 0$;

for $i = 0, 1, 2,...$

$$\rho_i = (r_0, r_i);$$

$$\beta_{i-1} = (\frac{\rho_i}{\rho_{i-1}})(\frac{\alpha_{i-1}}{\omega_{i-1}})$$

$$p_i = r_i + \beta_{i-1}(p_{i-1} - \omega_{i-1}v_{i-1})$$

solve $\hat{p}$ from $K\hat{p} = p_i$:

$$v_i = A\hat{p};$$

$$\alpha_i = \frac{\rho_i}{(r_0), v_i};$$

$$s = r_i - \alpha_i v_i;$$

if $\|s\|$ small enough then

$$x_{i+1} = x_i + \alpha_i \hat{p}; \text{ quit;}$$

solve $z$ from $Kz = s$;

$$t = Az;$$

$$\omega_i = \frac{(t,s)}{(t,t)}$$

$$x_{i+1} = x_i + \alpha_i \hat{p} + \omega_i z;$$
if $x_{i+1}$ is accurate enough then quit;

$$r_{i+1} = s - \omega_i t;$$

end

**Bi-CGSTAB**

This algorithm is an iterative solver for solving the linear system of equation $Ax=b$. Make an initial guess $x_0$

$$r_0 = b - Ax_0$$

$\overline{r}_0$ is an arbitrary vector., such that

$$(\overline{r}_0, r_0) \neq 0, \text{ e.g., } \overline{r}_0 = r_0$$

$\rho_0 = \omega_2 = 1; u = \alpha = 0;$

for $i = 0, 1, 2,...$

$$\rho_0 = -\omega_2 \rho_0;$$

even BiCG step:

$$\rho_1 = (\overline{r}_0, r_i);$$

$$\beta = \left( \frac{\alpha \rho_1}{\rho_0} \right);$$

$$\rho_0 = \rho_1;$$

$$u = r_i - \beta u;$$

$$v = Au;$$

$$\gamma = (v, \overline{r}_0);$$

$$\alpha = \frac{\rho_0}{\gamma};$$

$$r = r_i - \alpha v;$$

$$s = Ar;$$

$$x = x_i + \alpha u;$$
Odd BICG Step:

\[ \rho_1 = (\hat{r}_0, s); \]
\[ \beta = \alpha \frac{\rho_1}{\rho_0}; \]
\[ \rho_0 = \rho_1; \]
\[ v = s - \beta v; \]
\[ \omega = A v; \]
\[ \gamma = (\omega, \hat{r}_0); \]
\[ \alpha = \frac{\rho_0}{\gamma}; \]
\[ u = r - \beta u; \]
\[ r = r - \alpha v; \]
\[ s = s - \alpha \omega; \]
\[ t = A s; \]

GCER(2) - part:

\[ \omega_1 = (r, s); \]
\[ \mu = (s, s); \]
\[ v = (s, t); \]
\[ \tau = (t, t); \]
\[ \omega_2 = (r, t); \]
\[ \tau = \tau - \frac{v^2}{\mu}; \]
\[ \omega_2 = \frac{(\omega_2 - v \omega_1 / \mu)}{\tau}; \]
\[ \omega_1 = \frac{(\omega_1 - v \omega_2)}{\mu}; \]
\[ x_{i+2} = x + \omega_1 r + \omega_2 s + \alpha u; \]
\[ r_{i+2} = r - \omega_1 s - \omega_2 t; \]

if \( x_{i+2} \) accurate enough then quit

\[ u = u - \omega_1 v - \omega_2 w \]

end
Vita

Vikas Singh was born on December 4, 1980, in Utterpradesh, India to Mrs Santosh Singh and Mr Jaipal Singh. He graduated from Institute of Technology, Banaras Hindu University, with a Bachelor of Technology degree in Civil Engineering in the year 2002. He came to the United States to join the graduate school at Louisiana State University Agricultural and Mechanical College in the Department of Civil Engineering in spring 2003. He defended his master’s thesis on 11th of October 2004.