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Conical representations for direct limits of symmetric spaces

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CONICAL REPRESENTATIONS FOR DIRECT LIMITS OF SYMMETRIC SPACES

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Abstract. We extend the definition of conical representations for Riemannian symmetric spaces to a certain class of infinite-dimensional Riemannian symmetric spaces. Using an infinite-dimensional version of Weyl's Unitary Trick, there is a correspondence between smooth representations of infinite-dimensional noncompact-type Riemannian symmetric spaces and smooth representations of infinite-dimensional compact-type symmetric spaces. We classify all smooth conical representations which are unitary on the compact-type side. Finally, a new class of non-smooth unitary conical representations appears on the compact-type side which has no analogue in the finite-dimensional case. We classify these representations and show how to decompose them into direct integrals of irreducible conical representations.

1. Introduction

Harmonic analysis and representation theory of topological groups have been very well-studied over the past century and have produced fruitful applications to areas such as PDEs and mathematical physics. Two broad developments in the theory are brought together in this paper: first, Helgason's theory of conical representations for noncompact-type Riemannian symmetric spaces and second, the more recent study of representation theory and harmonic analysis on infinite-dimensional Lie groups.

In the theory of Riemannian symmetric spaces, there are two crucially important dualities. One is the duality between compact-type and noncompact-type Riemannian symmetric spaces. The other is the duality between a noncompact-type Riemannian symmetric space and its horocycle space. These dualities are intimately connected to the representation theory of their corresponding isometry groups (see [15, 17, 18]). For instance, Weyl's unitary trick sets up a correspondence between finite-dimensional spherical representations for a compact-type symmetric space and finite-dimensional spherical representations for its corresponding noncompact-type symmetric space. In turn, the finite-dimensional spherical representations for a noncompact-type symmetric space are identical to the conical representations for its corresponding horocycle space. Furthermore, analysis on a noncompact-type Riemannian symmetric space and its corresponding horocycle space are connected by the famous Radon transform.

More recently, much progress has been made in the theory of infinite-dimensional Lie groups, that is, groups which are modeled on locally convex topological vector spaces in the same way that finite-dimensional Lie groups are modeled on finite-dimensional vector spaces. The simplest and "smallest" infinite-dimensional groups are the *direct-limit groups*, which are constructed by taking unions of increasing chains of finite-dimensional Lie groups. In a similar way, one can form an infinite-dimensional symmetric space by forming a direct limit of finite-dimensional symmetric spaces. Representation theory and even harmonic analysis questions for direct-limit groups and direct-limit symmetric spaces have been studied in some depth (e.g., see [2, 4, 35, 36, 32, 34, 45, 46, 47] for just a few examples). A good overview of the field may be found in [37].

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In particular, spherical representations for infinite-dimensional symmetric spaces are well-studied in the literature (e.g., see [7, 11, 36]). On the other hand, the theory of conical representations for infinite-dimensional Riemannian symmetric spaces appears to have been largely neglected up to this point. However, understanding conical representations is a necessary first step to constructing and studying a Radon transform for infinite-dimensional symmetric spaces. We point the reader to [19], for instance, in which the authors define and study a Radon transform between regular functions on certain direct limits of Riemannian symmetric spaces and regular functions on the corresponding horocycle spaces.

In this paper, we classify all of the smooth conical representations for direct limits of noncompact-type Riemannian symmetric spaces that satisfy certain technical conditions. Combined with the results of [7], we see that for infinite-dimensional symmetric spaces of infinite rank, none of the smooth conical representations are spherical, a situation which is in stark contrast with the classical result of Helgason that all finite-dimensional representations are spherical if and only if they are conical. We further demonstrate the existence, in certain cases, of nonsmooth unitary conical representations for direct limits of compact-type Riemannian symmetric spaces. This is a phenomenon which has no analogue for finite-dimensional symmetric spaces. We also show how these conical representations decompose into direct integrals of irreducible representations.

The outline is as follows. In Section 2, we quickly review the notation and relevant results on finite-dimensional noncompact-type Riemannian symmetric spaces. In Section 3, we review the notions of *propagated* direct limits of symmetric spaces (introduced in [34, 45, 47]) and *admissible* direct limits of symmetric spaces (introcuced in [19]), which provide the technical context for our results. It remains an open question whether every propagated direct limit is admissible, but in the appendix we have included a case-by-case proof that each of the classical examples of propagated direct limits of symmetric spaces are admissible. In Section 4, we continue by reviewing relevant results on representations of direct-limit groups and prove some new results which we will need later in the paper. Section 5 contains the main classification theorem for conical representations. Finally, in Section 6, we prove some decomposition theorems for general (not necessarily irreducible) conical representations. We end the paper with a summary in Section 7 and the aforementioned appendix containing the proof of admissiblility for the classical direct limits of symmetric spaces.

2. Finite-Dimensional Riemannian Symmetric Spaces

Riemannian symmetric spaces form a class of particularly well-behaved homogeneous spaces with a rich structure theory and relatively well-understood harmonic analysis. In particular, there is a beautiful duality between *compact-type* and *noncompact-type* Riemannian symmetric spaces.

In addition, the noncompact-type Riemannian symmetric spaces possess an associated homogeneous space called a *horocycle space*. The relationship between a Riemannian symmetric space and its horocycle space is analogous to, for instance, the relationship between points and hyperplanes in $\mathbb{R}^n$, or the relationship between points and horocycles of hyperbolic space (it is for this reason that the terminology *horocycle space* was originally chosen).

In the late 1950s, Gelfand and Graev developed a “horospherical method” which relates harmonic analysis on the noncompact-type Riemannian symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SU}(n)$ and harmonic analysis on its horocycle space (see [25, p. 283–287]). These ideas were generalized to all noncompact-type Riemannian symmetric spaces and developed quite completely in the pioneering work of Helgason (see [15], for instance). The relationship between symmetric spaces and horocycle spaces, together with its implications for representation theory, provides the primary context for this paper.

See [16] for a comprehensive overview of the structure theory for Riemannian symmetric spaces. See also [17] and [18] for applications of representation theory to analysis on Riemmanian symmetric spaces and horocycle spaces, respectively. An overview of this theory from the perspective of unitary group representations may be found in [31].

2.1. **Basic Definitions.** Suppose that G is a semisimple Lie group over $\mathbb{R}$ with finite center and that K is a closed subgroup. Furthermore, we suppose that there is an involutive automorphism $\theta : G \to G$ such that

$$(G^\theta)_0 \leq K \leq G^\theta, \tag{2.1}$$

where G^θ is the fixed-point subgroup for θ and $(G^\theta)_0$ is the connected component of the identity for G^θ. Then G/K is said to be a **symmetric space**.

The involution θ differentiates to an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G. By (2.1), the $+1$-eigenspace for θ is just $\mathfrak{k}$ (i.e., the Lie algebra for K). We denote the -1-eigenspace of θ by $\mathfrak{p}$. We may write down the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Due to the fact that θ is also a Lie algebra involution, one easily computes that $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Just as $\mathfrak{k}$ may be naturally identified with the tangent space T_eK, there is a natural identification of $\mathfrak{p}$ with the tangent space $T_{eK}G/K$ (see [16, p. 214]).

If U/K is a Riemannian symmetric space with U compact, then it is said to be a **compact-type Riemannian symmetric space**. On the other hand, if G/K is a Riemannian symmetric space such that G has no compact factors and has a finite center, then K is compact and G/K is said to be a **noncompact-type Riemmanian symmetric space**.

There is a beautiful duality between compact-type and noncompact-type Riemannian symmetric spaces. Suppose that U/K is a compact-type symmetric space with involution θ. We make the further simplifying assumption throughout this paper that U is simply-connected whenever we consider a compact-type symmetric space U/K. In particular, this assumption implies that $K = U^\theta$ is connected. As usual, θ admits an eigenspace decomposition:

$$\mathfrak{u} = \mathfrak{k} \oplus \widetilde{\mathfrak{p}},$$

where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\widetilde{\mathfrak{p}}$ are the $+1$ and -1 eigenspaces, respectively. We now consider the complexified Lie algebra $\mathfrak{u}_\mathbb{C} = \mathfrak{u} \otimes_\mathbb{R} \mathbb{C}$ and define a new real Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus i\widetilde{\mathfrak{p}}$$

Next we take the unique connected complex Lie group $U_\mathbb{C}$ with Lie algebra $\mathfrak{u}_\mathbb{C}$ such that U is the analytic subgroup of $U_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}_\mathbb{C}$. The Lie algebra involution θ on $\mathfrak{u}_\mathbb{C}$ integrates to an involution on $U_\mathbb{C}$ by Proposition 7.5 in [21]. We then consider the analytic subgroup $G \leq U_\mathbb{C}$ corresponding to the Lie algebra $\mathfrak{g} \subseteq \mathfrak{u}_\mathbb{C}$. By Proposition 7.9 in [21], we see that G is a closed subgroup of $U_\mathbb{C}$ and has a finite center. Putting everything together, we see that G/K is a *noncompact-type* Riemannian symmetric space, called the **c-dual** of U/K. To emphasize the symmetry between $\mathfrak{u}$ and $\mathfrak{g}$, we note that if we set $\mathfrak{p} = i\widetilde{\mathfrak{p}}$, then

$$\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$$
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

We need some well-known definitions from symmetric space theory. Fix a noncompact-type symmetric space G/K. As before, we can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. For each $\alpha \in \mathfrak{a}^*$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X\}.$$

We write $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \backslash \{0\} | \mathfrak{g}_\alpha \neq 0\}$ for the set of all restricted roots for $(\mathfrak{g}, \mathfrak{k})$. We can choose a positive subsystem Σ^+ of Σ. We denote the set of indivisible roots by Σ_0, and put $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$.

Next, we define $\mathfrak{m} = Z_\mathfrak{k}(\mathfrak{a})$ and note that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$ by the maximality of $\mathfrak{a}$ in $\mathfrak{p}$. Thus, if we put

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \text{ and } \overline{\mathfrak{n}} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}, \text{ then}$$
$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}.$$

Finally, we set $N = \exp(\mathfrak{n})$ and $M = Z_K(\mathfrak{a})$. Note that although $\mathfrak{m}$ is the Lie algebra of M, one generally does not have $M = \exp(\mathfrak{m})$ because M is typically disconnected.

Before proceeding, we lay down some standard notation which will be useful in the future. Suppose that $\dim \mathfrak{a} = n$. If $\Sigma(\mathfrak{g}, \mathfrak{a})$ is a root system of type A_n, then we identify $\mathfrak{a}$ with $\mathbb{R}^n$ by setting $\mathfrak{a} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$. Otherwise we make the identification $\mathfrak{a} = \mathbb{R}^n$. Set $e_1 = (1, \ldots, 0, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_r = (0, \ldots, 0, 1)$ where $r = n + 1$ for A_n and otherwise $n = r$. We view the vectors e_j also as elements in $\mathfrak{a}^*$ via the standard inner product in $\mathbb{R}^{n+1}$ in the case that $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type A_n and otherwise via the standard inner product in $\mathbb{R}^n$. Note that in the A_n-type case this defines a map $\mathbb{R}^{n+1} \to \mathfrak{a}^*$ which is not injective.

For the purpose of convenience, we choose an ordering on the vectors in $\mathfrak{a}^*$ (that is, a choice of Σ^+) such that the dominant weights have *increasing* coefficients with respect to the ordered basis $\{e_1, \ldots e_n\}$. Note that this is the opposite of the conventional choice. That is, we make the following choices for $\Sigma_0^+(\mathfrak{g}, \mathfrak{a})$ according to the Dynkin diagram Ψ of the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$:

Ψ	$\Sigma_0^+(\mathfrak{g}, \mathfrak{a})$
A_n	$\{e_j - e_i \mid i < j\}$
B_n	$\{e_j \pm e_i \mid i < j\} \cup \{e_i\}$
C_n	$\{e_j \pm e_i \mid i < j\} \cup \{2e_i\}$
D_n	$\{e_j \pm e_i \mid i < j\}$

2.2. **Finite-Dimensional Conical and Spherical Representations.** In this section, we review the basic results on finite-dimensional conical and spherical representations. We begin with their definitions:

Definition 2.1. *Suppose that G/K is a Riemannian symmetric space (of either compact or noncompact type). A representation $(\pi, \mathcal{H})$ of π on a Hilbert space $\mathcal{H}$ is **spherical** if there is $e \in \mathcal{H} \backslash \{0\}$ such that $\pi(K)e = e$, in which case e is said to be a **spherical vector** for π.*

Definition 2.2. *Suppose that G/K is a Riemannian symmetric space of noncompact type. Using the notation of Section 2.1, we say that a representation $(\pi, \mathcal{H})$ of π on a Hilbert space $\mathcal{H}$ is **conical** if there is $v \in \mathcal{H} \backslash \{0\}$ such that $\pi(MN)e = e$.*

In harmonic analysis on the homogeneous space G/MN (called the **horocycle space**), conical representations play a roll which is analogous to the roll played by spherical representations for harmonic analysis on the symmetric space G/K.

Note that if U/K is the compact c-dual of G/K, then every finite-dimensional representation of U extends by holomorphic continuation to a representation of G. By abuse of notation, we can thus consider a finite-dimensional representation (π, V) of U to be also a representation of G, and vice versa. Thus, a finite-dimensional representation of U is said to be conical if the corresponding representation of G is conical.

The problem of determining which finite-dimensional representations are spherical or conical is solved by the Cartan-Helgason Theorem on finite-dimensional spherical representations, which was first stated without proof in [44].

Theorem 2.3 (The Cartan-Helgason Theorem)**.** *(*[17, p. 535]*) Suppose that U/K is a compact-type symmetric space with c-dual G/K, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{u}$ containing $i\mathfrak{a}$, so that $\mathfrak{h}$ is θ-stable and $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{a}$ where $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Suppose further that U is simply-connected. If (π, V) is an irreducible representation of U with highest-weight $\lambda \in i\mathfrak{h}^*$, then the following are equivalent:*

1. *π is a spherical representation of U*
2. *$\pi(M)v = v$ for each highest-weight vector $v \in V_\lambda$.*
3. *$\lambda(\mathfrak{t}) = 0$ and also*

$$\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+$$

If π is spherical, we say by abuse of notation that $\lambda|_{i\mathfrak{a}}$ is the **highest weight** of π. Note that there is a natural identification of purely imaginary weights on $i\mathfrak{a}$ with purely real weights on $\mathfrak{a}$. Thus, the highest restricted roots may be identified with elements of $\mathfrak{a}^*$. We set

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{g},\mathfrak{a}) \equiv \left\{ \mu \in \mathfrak{a}^* \,\middle|\, \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}^+ \text{ for all } \alpha \in \Sigma^+ \right\}$$

and write π_λ for the unique spherical representation with highest weight $\lambda \in \Lambda^+$.

Moreover, Λ^+ is a semilattice. In fact, define linear functionals $\xi_j \in \mathfrak{a}^*$ by

$$\frac{\langle \xi_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j} \text{ for } 1 \le j \le r \quad , \tag{2.2}$$

where $\Psi = \{\alpha_1, \dots, \alpha_r\} \subseteq \Sigma_0^+$ is the set of positive simple roots. Then $\xi_1, \dots, \xi_r \in \Lambda^+$ and

$$\Lambda^+ = \mathbb{Z}^+\xi_1 + \cdots + \mathbb{Z}^+\xi_r = \left\{ \sum_{j=1}^r n_j \xi_j \,\middle|\, n_j \in \mathbb{Z}^+ \right\} .$$

The weights ξ_j are called the *fundamental weights* for $(\mathfrak{g},\mathfrak{a})$. Note that each element of Λ^+ corresponds to a unique irreducible spherical representation of U.

The Cartan-Helgason Theorem also gives a classification of conical representations of G by the following theorem of Helgason:

Theorem 2.4. *([15, p. 79]) Suppose that (π, V) is an irreducible finite-dimensional representation of G. Then π is spherical if and only if it is conical, in which case V^{MN} is one-dimensional and consists of the highest-weight vectors of π.*

Now that the irreducible finite-dimensional spherical and conical representations have been parameterized, one may ask more generally about finite-dimensional spherical and conical representations that may not be irreducible.

To that end, suppose that $(\pi_\mu, \mathcal{H}_\mu)$ is an irreducible K-spherical representation of G with highest weight μ and that $(\sigma, \mathcal{H})$ is a unitary primary representation of G consisting of representations of type μ. By [14, Lemma 1.5], all cyclic primary representations of a compact group are finite-dimensional, and hence σ extends uniquely to a holomorphic spherical representation of $G^{\mathbb{C}}$. Because it is a finite-dimensional spherical representation, σ is automatically a conical representation of $G^{\mathbb{C}}$. In fact, as the following result shows, the MN-invariant vectors of σ are precisely the highest-weight vectors of irreducible subrepresentations of σ. The lemma is likely known by specialists, but the authors were not able to find an exact citation in the literature.

Lemma 2.5. *Suppose, as above, that $(\sigma, \mathcal{H})$ is a finite-dimensional, unitary primary representation of U consisting of representations with highest weight μ. If $v \in \mathcal{H}^{MN} \backslash \{0\}$, then v is a highest-weight vector that generates an irreducible spherical representation of U. Furthermore, if $v, w \in \mathcal{H}^{MN} \backslash \{0\}$ and $v \perp w$, then $\langle \pi(U)v \rangle \perp \langle \pi(U)w \rangle$.*

Proof. Let $v \in \mathcal{H}^{MN} \backslash \{0\}$, and consider $W = \langle \sigma(G)v \rangle$. We can write $W = W_1 \bigoplus \cdots \bigoplus W_n$ where each W_i gives an irreducible representation of G that is equivalent to $\mathcal{H}_\mu$. It must be a finite direct sum because all cyclic primary representations of compact groups are finite-dimensional (see [14]). For each i, let v_i be the orthogonal projection of v onto W_i. Then $v = v_1 + \cdots + v_n$. Since each W_i is a G-invariant subspace, it follows that each vector v_i is also invariant under MN. Because W_i is irreducible, we see that v_i must be a (nonzero) highest-weight vector of weight μ (see [13, Theorem 12.3.13]). Hence v is a weight vector of weight μ.

Suppose that W is not irreducible (that is, $n > 1$). Because W is cyclic, there must be $g_1, \dots, g_k \in G$ and $c_1, \dots, c_k \in \mathbb{C}$ such that $\sum_{i=0}^k c_i \pi(g_i)v = v_1$ (it is sufficient to consider finite linear combinations because W is finite-dimensional). It follows from the invariance of each space W_k that $\sum_{i=0}^k c_i \pi(g_i)v_1 = v_1$ and $\sum_{i=0}^k c_i \pi(g_i)v_2 = 0$. Because W_1 and W_2 give equivalent representations of G and all highest-weight vectors of an irreducible representation are constant multiples of each other, this is a contradiction. Thus W is irreducible and $v = v_1$ is a highest-weight vector for W.

Now suppose that v and w are nonzero MN-invariant vectors in $\mathcal{H}$ such that $v \perp w$. Write $V = \langle \pi(G)v \rangle$ and $W = \langle \pi(G)w \rangle$. By the above, we know that V and W are irreducible representations of G with highest-weight vectors v and w, respectively. Hence, either $V \cap W = \{0\}$ or $V = W$. Because the space of highest-weight vectors of an irreducible representation of G is one dimensional and $v \perp w$, we cannot have $V = W$. Thus $V \cap W = \{0\}$.

Now consider the invariant subspace $Z = V + W$ and the corresponding orthogonal projection $p : Z \to W$, which is an intertwining operator for π because W is an invariant subspace of Z. Hence, $p(v) \in \mathcal{H}^{MN}$ and so $p(v) = cw$ for some $c \in \mathbb{C}$. Since $v \perp w$, we see that $c = 0$ and thus $v \in \ker p$. Moreover, it is clear that $\ker p$ is a U-invariant subspace of Z, so it follows that $V = \langle \pi(U)v \rangle \subseteq \ker p$. Hence $V \perp W$ as we wished to show. □

We end this section with another useful lemma, which we will make use of several times. Once again, it is probably already known, but the authors were unable to find an exact citation in the literature.

Lemma 2.6. *Let G be a topological group and let $(\pi, \mathcal{H})$ be a unitary representation of G. Let $\mathcal{A}$ be a finite or countably infinite index set, and suppose that*

$$v = \sum_{i \in \mathcal{A}} v_i,$$

where $v_i \in \mathcal{H}$ for each $i \in \mathcal{A}$ and where $\langle \pi(G)v_i \rangle$ and $\langle \pi(G)v_j \rangle$ give mutually distinct irreducible representations of G for $i \neq j$. Then

$$\langle \pi(G)v \rangle = \bigoplus_{i \in \mathcal{A}} \langle \pi(G)v_i \rangle.$$

Proof. Write $V = \langle \pi(G)v \rangle$. The fact that $V_i = \langle \pi(G)v_i \rangle$ and $V_j = \langle \pi(G)v_j \rangle$ give disjoint representations of G for $i \neq j$ implies that $V_i \perp V_j$. It is obvious that

$$\langle \pi(G)v \rangle \subseteq \bigoplus_{i \in \mathcal{A}} \langle \pi(G)v_i \rangle,$$

so we prove the opposite containment. It suffices to show that $v_i \in V$ for all $i \in \mathcal{A}$.

Suppose that $v_i \notin V$ for some $i \in \mathcal{A}$. Define

$$w = \sum_{j \neq i} v_j \text{ and } W = \langle \pi(G)w \rangle \subseteq \bigoplus_{j \neq i} V_j.$$

Then $V_i \perp W$ and $v = v_i + w$. Furthermore, V_i and W give disjoint representations of G.

Now let $c_1, \ldots c_k \in \mathbb{C}$ and $g_1, \ldots g_k \in G$. Then

$$\sum_{j=1}^{k} c_j \pi(g_j) v = \left(\sum_{j=1}^{k} c_j \pi(g_j) v_i \right) + \left(\sum_{j=1}^{k} c_j \pi(g_j) w \right).$$

Because $v_i \notin V$ and V_i is irreducible, we see that $V \cap V_i = \emptyset$. It follows that

$$\sum_{j=1}^{k} c_j \pi(g_j) v_i = 0 \text{ if and only if } \sum_{j=1}^{k} c_j \pi(g_j) w = 0.$$

Hence there is a well-defined, nonzero intertwining operator $L : V_i \to W$ such that $L(v_i) = w$, which contradicts the fact that V_i and W give disjoint representations of G. □

3. Direct Limits of Groups and Symmetric Spaces

We refer the reader to [10] and [28] for a good overview of the basic properties of direct-limit groups. See [26] and [29] for some details about the construction of smooth manifold structures on direct-limit groups. See also [43] for an in-depth study of direct limits of abelian and nilpotent groups with applications to physics.

3.1. **Direct Systems of Riemannian Symmetric Spaces.** Suppose that $\{G_n/K_n\}_{n\in\mathbb{N}}$ is a sequence of semisimple Riemannian symmetric spaces such that G_n is a closed subgroup of G_m for $n \leq m$ and such that $K_m \cap G_n = K_n$ for $n \leq m$. We label the corresponding involutions by $\theta_n : G_n \to G_n$ and make the further assumption that $\theta_m|_{G_n} = \theta_n$ for all $n \leq m$. We thus obtain a direct system of homogeneous spaces $\{G_n/K_n\}_{n\in\mathbb{N}}$. Now construct the direct limits $G_\infty = \varinjlim G_n$, $K_\infty = \varinjlim K_n$, and $G_\infty/K_\infty = \varinjlim G_n/K_n$. Finally, one can show that $K_\infty = (G_\infty)^{\theta_\infty}$.

We say that G_∞/K_∞ is a **lim-Riemannian symmetric space**. If G_n/K_n is a noncompact-type Riemannian symmetric space for all $n \in \mathbb{N}$, then G_∞/K_∞ is said to be a **lim-noncompact Riemannian symmetric space**. Similarly, if G_n/K_n is a compact-type symmetric space for each $n \in \mathbb{N}$, then G_∞/K_∞ is said to be a **lim-compact Riemmanian symmetric space**. In this paper, all lim-compact Riemannian symmetric spaces will be limits of simply-connected spaces.

We now review how the notion of c-duals may be extended to lim-Riemannian symmetric spaces. Suppose that $\{U_n/K_n\}_{n\in\mathbb{N}}$ is a direct system of compact-type Riemannian symmetric spaces with involutions $\theta_n : U_n \to U_n$. One may construct a complexification $(U_\infty)_\mathbb{C} = \varinjlim (U_n)_\mathbb{C}$ for the lim-compact group $U_\infty = \varinjlim U_n$. To simplify notation we assume that $(U_n)_\mathbb{C} \subseteq (U_{n+1})_\mathbb{C}$ and therefore $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$. The involutions $\theta_n : U_n \to U_n$ extend to holomorphic involutions $\theta_n : (U_n)_\mathbb{C} \to (U_n)_\mathbb{C}$ such that $\theta_m|_{U_n} = \theta_n$ for $m \leq n$.

We write

$$\mathfrak{u}_n = \mathfrak{k}_n \oplus \widetilde{\mathfrak{p}}_n$$

for each $n \in \mathbb{N}$, where $\mathfrak{k}_n$ and $\widetilde{\mathfrak{p}}_n$ are the $+1$- and -1-eigenspaces of θ_n. It follows from the fact that $\theta_{n+1}|_{U_n} = \theta_n$ that

$$\mathfrak{k}_n = \mathfrak{k}_{n+1} \cap \mathfrak{u}_n \text{ and } \widetilde{\mathfrak{p}}_n = \widetilde{\mathfrak{p}}_{n+1} \cap \mathfrak{u}_n$$

and hence that $\mathfrak{k}_n \subseteq \mathfrak{k}_{n+1}$ and $\widetilde{\mathfrak{p}}_n \subseteq \widetilde{\mathfrak{p}}_{n+1}$. For each n, we construct the c-dual Lie algebra

$$\mathfrak{g}_n = \mathfrak{k}_n \oplus i\widetilde{\mathfrak{p}}_n \subseteq (\mathfrak{u}_n)_\mathbb{C}$$

and note that $\mathfrak{g}_n \subseteq \mathfrak{g}_{n+1}$. Finally, we construct the analytic subgroup G_n of $(U_n)_\mathbb{C}$ which corresponds to the Lie algebra $\mathfrak{g}_n$ and recall that G_n is closed in $(U_n)_\mathbb{C}$. Thus G_n is a closed subgroup of G_{n+1} for each n. It follows that the direct-limit group $G_\infty = \varinjlim G_n$ is a closed subgroup of $(U_n)_\mathbb{C}$ and possesses the direct-limit Lie algebra $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$.

Reviewing the construction of finite-dimensional c-dual spaces, we see that the complexified involution $\theta_n : (\mathfrak{u}_n)_\mathbb{C} \to (\mathfrak{u}_n)_\mathbb{C}$ restricts to an involution $\theta_n : \mathfrak{g}_n \to \mathfrak{g}_n$ and that $\mathfrak{k}_n$ and $i\widetilde{\mathfrak{p}}_n$ are the $+1$- and -1-eigenspaces of θ_n in $\mathfrak{g}_n$. Furthermore, because $\mathfrak{g}_n$ is θ_n-stable, the holomorphic involution $\theta_n : (U_n)_\mathbb{C} \to (U_n)_\mathbb{C}$ restricts to an involution $\theta_n : G_n \to G_n$ such that $(G_n)^{\theta_n} = K_n$. Thus $\{G_n/K_n\}_{n\in\mathbb{N}}$ is a direct system of noncompact-type Riemannian symmetric spaces. We say that $G_\infty/K_\infty = \varinjlim G_n/K_n$ is the **c-dual** of U_∞/K_∞.

We set $\mathfrak{p}_n = i\widetilde{\mathfrak{p}}_n$ for each n, so that

$$\begin{aligned}\mathfrak{g}_n &= \mathfrak{k}_n \oplus \mathfrak{p}_n\\ \mathfrak{u}_n &= \mathfrak{k}_n \oplus i\mathfrak{p}_n.\end{aligned}$$

Finally, we notice that

$$\begin{aligned}\mathfrak{g}_\infty &= \mathfrak{k}_\infty \oplus \mathfrak{p}_\infty\\ \mathfrak{u}_\infty &= \mathfrak{k}_\infty \oplus i\mathfrak{p}_\infty,\end{aligned}$$

where $\mathfrak{k}_\infty = \varinjlim \mathfrak{k}_n$ and $\mathfrak{p}_\infty = \varinjlim \mathfrak{p}_n$ are the $+1$-and -1-eigenspaces of θ_∞ in $\mathfrak{g}_\infty$.

3.2. **Propagated Direct Limits.** As before, we assume that G_∞/K_∞ is a lim-noncompact Riemannian symmetric space which is the c-dual of a direct limit U_∞/K_∞ of simply-connected compact Riemannian symmetric spaces. We need to put some further technical conditions on G_∞/K_∞ in order to prove our results about conical representations. The first condition is that of *propagation*, which was introduced in [34, 45, 47].

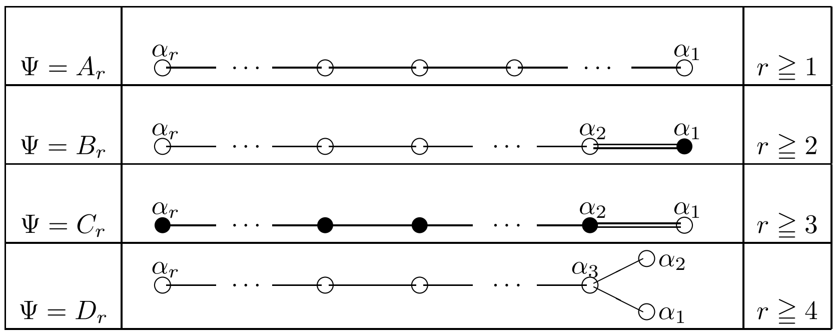
We begin this section by examining the restricted root data of G_∞/K_∞, using the notation of Section 3.1. We recursively choose maximal commutative subspaces $\mathfrak{a}_k \subseteq \mathfrak{p}_k$ such that $\mathfrak{a}_n \subseteq \mathfrak{a}_k$

for $n \leq k$ and define $\mathfrak{a}_\infty = \varinjlim \mathfrak{a}_n$. Note that $\mathfrak{a}_\infty^* \equiv \varprojlim \mathfrak{a}_n^*$, where the projective limit is given by projections $p_n : \mathfrak{a}_{n+1}^* \to \mathfrak{a}_n^*$ defined by $p_n : \alpha \mapsto \alpha|_{\mathfrak{a}_n}$. We then obtain the restricted root system $\Sigma_n = \Sigma(\mathfrak{g}_n, \mathfrak{a}_n)$ for each $n \in \mathbb{N}$. Note that $\Sigma_n \subseteq \Sigma_k|_{\mathfrak{a}_n} \backslash \{0\}$ whenever $n \leq k$.

Next, we choose positive subsystems $\Sigma_n^+ \subseteq \Sigma_n$ so that $\Sigma_n^+ \subseteq \Sigma_k^+|_{\mathfrak{a}_n} \backslash \{0\}$. The projective limit $\Sigma_\infty^+ = \varprojlim \Sigma_n^+$ plays the role of the positive root subsystem for $(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$.

For each $n \in \mathbb{N}$, we let $(\Sigma_n)_0$ denote the set of nonmultipliable roots in Σ_n and set $(\Sigma_n)_0^+ = (\Sigma_n)_0 \cap \Sigma_n^+$. Denote the set of simple roots in $(\Sigma_n)_0^+$ by $\Psi_n = \{\alpha_1, \ldots, \alpha_{r_n}\}$, where $r_n = \dim \mathfrak{a}_n$. Since we will be dealing with direct limits we may assume that Σ, and hence Σ_0, is one of the classical root systems. We number the simple roots in the following way:

(3.1)



We are now ready to introduce the definition of propagated direct-limits of symmetric spaces.

Definition 3.1. *We say that a lim-noncompact symmetric space G_∞/K_∞ is* ***propagated*** *if*

(1) *For each simple root $\alpha \in \Psi_k$ there is a unique simple root $\widetilde{\alpha} \in \Psi_n$ such that $\widetilde{\alpha}|_{\mathfrak{a}_k} = \alpha$, whenever $k \leq n$.*
(2) *There is a choice of ordering on the roots in Ψ_k for each $k \in \mathbb{N}$ such that either $\mathfrak{a}_n = \mathfrak{a}_k$ or else Ψ_k extends Ψ_n for $n \leq k$ only by adding simple roots at the left end. (In particular, each Ψ_k has the same Dynkin diagram type.)*

Let $U_\infty = \varinjlim U_n$ be a direct limit of compact Lie groups. Each group U_n may be considered as a compact symmetric space $U_n \equiv U_n \times U_n/K_n$, where $K_n := \{(g,g) | g \in U_n\} \cong U_n$ is the fixed-point subgroup of the involution $\theta : (g,h) \mapsto (h,g)$ on $U_n \times U_n$. In this way, $(U_\infty \times U_\infty)/U_\infty \equiv U_\infty$ becomes a direct limit of symmetric spaces.

Definition 3.2. *We say that the lim-compact group U_∞ is* ***propagated*** *if the associated symmetric space $(U_\infty \times U_\infty)/U_\infty$ is propagated.*

Suppose that U_∞ is a propagated direct limit of compact, simply-connected semisimple Lie groups. Then each U_k may be decomposed into a product of compact simple Lie groups, say $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^{d_k}$. We can recursively choose Cartan subalgebras $\mathfrak{h}_k = \mathfrak{h}_k^1 \oplus \mathfrak{h}_k^2 \oplus \cdots \oplus \mathfrak{h}_k^{d_k}$ where each $\mathfrak{h}_k^i$ is a Cartan subalgebra of $\mathfrak{u}_k^i$. The definition of propagation then implies that $d_n = d_m \equiv d$ for each $n, m \in \mathbb{N}$ and that the indices may be ordered in such a way that $\{U_k^i\}_{n \in \mathbb{N}}$ is a propagated direct system of compact simple Lie groups for each $1 \leq i \leq d$.

3.3. **Admissible Direct Limits.** We continue to examine the root data for lim-noncompact symmetric spaces G_∞/K_∞. For each $k \in \mathbb{N}$ and each restricted root $\alpha \in \Sigma_k$, we define as before the root space

$$\mathfrak{g}_{k,\alpha} = \{Y \in \mathfrak{g}_k \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}_k\}.$$

Next we define the subalgebras

$$\mathfrak{n}_k = \bigoplus_{\alpha \in \Sigma_k^+} \mathfrak{g}_{k,\alpha}$$

and

$$\mathfrak{m}_k = Z_{\mathfrak{k}_k}(\mathfrak{a}_k)$$

of $\mathfrak{g}_k$. Similarly, we define the subgroups $N_k = \exp(\mathfrak{n}_k)$ and $M_k = Z_{K_k}(\mathfrak{a}_k)$ of G_k.

For each $k \in K$, the conical representations of G_k are the representations which possess a nonzero vector (or, more generally, distribution vector) which is invariant under the action of the group M_kN_k. Hence, in order to define conical representations of G_∞, one would like to define a subgroup $M_\infty N_\infty = \varinjlim M_nN_n$. In order for such a group to be well-defined, we need to introduce a technical condition, which was first considered in [19].

Definition 3.3. *A lim-noncompact symmetric space G_∞/K_∞ is said to be **admissible** if $M_kN_k \leq M_mN_m$ whenever $k \leq m$.*

As a consequence of the following lemmas, it is sufficient to assume that $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for $k \leq m$:

Lemma 3.4. *If G_∞/K_∞ is a lim-noncompact symmetric space, then $N_k \leq N_m$ for $k \leq m$.*

Proof. We will show that $\mathfrak{n}_k \subseteq \mathfrak{n}_m$. The result will then follow from the fact that $N_k = \exp \mathfrak{n}_k$ and $N_m = \exp \mathfrak{n}_m$.

In fact, it suffices to show that $\mathfrak{g}_{k,\alpha} \subseteq \mathfrak{n}_m$ for all $\alpha \in \Sigma_k^+$. Suppose that $X \in \mathfrak{g}_{k,\alpha}$. Consider the decomposition of X into $\mathfrak{a}_m$-root vectors:

$$X = \sum_{\beta \in \Sigma_m} X_\beta,$$

where $X_\beta \in \mathfrak{g}_{m,\beta}$ for each $(\mathfrak{g}_m, \mathfrak{a}_m)$-root β. Because this decomposition is unique and X is a root vector for $\mathfrak{a}_k \subseteq \mathfrak{a}_m$, it follows that $\beta|_{\mathfrak{a}_k} = \alpha$ for all $\beta \in \Sigma_m$ such that $X_\beta \neq 0$.

Now recall that we have made a consistent choice of positive root subsystems Σ_k^+ of Σ_k and Σ_m^+ of Σ_m. In other words, $\beta \in \Sigma_m$ is positive if $\beta|_{\mathfrak{a}_k}$ is positive. Since $\alpha \in \Sigma_k^+$, it follows that X is a sum of Σ_m^+-root vectors. Hence, $X \in \mathfrak{n}_m$. □

Due to the fact that M_k is typically a disconnected subgroup of G_n, it is not clear *a priori* that requiring $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for $k \leq m$ is sufficient to imply that $M_k \leq M_m$. However, the following lemma shows that this Lie algebra condition is, in fact, sufficient:

Lemma 3.5. *Suppose that G_∞/K_∞ is a propagated lim-noncompact symmetric space such that $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for all $k \leq m$. Then $M_k \leq M_m$ for $k \leq m$.*

Proof. By [21, Theorem 7.53] we see that for each $k \in \mathbb{N}$ there is a finite discrete subgroup $F_k \subseteq \exp(i\mathfrak{a}_k) \cap K_k$ such that $M_k = F_k(M_k)_0$, where $(M_k)_0 = \exp \mathfrak{m}_k$ is the connected component of the identity in M_k. Because $\mathfrak{m}_k \subseteq \mathfrak{m}_m$ for all $k \leq m$, we see that $(M_k)_0 \leq (M_m)_0$. It is thus sufficient to show that $F_k \leq M_m$ for $k \leq m$. In fact, since $F_k \subseteq \exp(i\mathfrak{a}_k) \subseteq \exp(i\mathfrak{a}_m)$, it is clear that F_k centralizes $\mathfrak{a}_k$. Since $F_k \subseteq K_k \subseteq K_m$, we see that $F_k \leq M_m$, and the result follows. □

It was not clear in [19] which lim-Riemannian symmetric spaces are admissible. While we still do not know know whether it is possible to show that all propagated direct systems of Riemannian symmetric spaces are admissible in the sense of 3.3, we have been able to show that each of the classical direct limits (see Table 8.1) are admissible on a case-by-case basis. The details of the proof are available in the appendix to this paper.

4. Representations of Direct-Limit Groups

In this section we review some important results about representations for direct-limit groups and lim-Riemannian symmetric spaces. See [37] for an overview of representation theory for classical direct limits of symmetric spaces. See also [10] and [28] for many basic results on representations of direct-limit groups.

4.1. **Direct Limits of Representations.** Suppose that $\{G_n\}_{n \in \mathbb{N}}$ is an increasing sequence of Lie groups (i.e., G_n is a closed subgroup of G_m for $n \leq m$) and that for each n we are provided with a continuous Hilbert representation $(\pi_n, \mathcal{H}_n)$ such that $(\pi_n, \mathcal{H}_n)$ is equivalent (by a unitary intertwining operator) to a subrepresentation of $(\pi_{n+1}|_{G_n}, \mathcal{H}_{n+1})$. Then one has a direct system of representations and may form a **direct-limit representation** $(\pi_\infty, \mathcal{H}_\infty)$ of G_∞ on the direct-limit vector space $\mathcal{H}_\infty \equiv \varinjlim \mathcal{H}_n$. Now $\mathcal{H}_\infty$ has a natural pre-Hilbert space structure, and if $\pi_\infty(g)$ is a bounded operator on $\mathcal{H}_\infty$ for all $g \in G_\infty$, then π_∞ extends to a continuous Hilbert representation of G_∞ on the Hilbert-space closure $\overline{\mathcal{H}_\infty} = \overline{\varinjlim \mathcal{H}_n}$.

Direct-limit representations are the easiest representations to construct for G_∞. The following theorem shows that they can in fact be used to construct a large class of irreducible unitary representations:

Theorem 4.1. *([22]) Suppose that $\{G_n\}_{n\in\mathbb{N}}$ is a direct system of locally compact groups and that $\{(\pi_n, \mathcal{H}_n)\}_{n\in\mathbb{N}}$ is a compatible direct system of irreducible unitary representations of G_n for each $n \in \mathbb{N}$. Then $(\pi, \mathcal{H}) \equiv (\varinjlim \pi_n, \overline{\varinjlim \mathcal{H}_n})$ is an irreducible unitary representation of G_∞.*

We caution the reader that there are many examples of irreducible representations of direct-limit groups which are not given by direct limits of irreducible representations (see [10, p. 971]).

4.2. **Smoothness and Local Finiteness.** Just as for finite-dimensional Lie groups, it is natural to try to gather information about a representation of a direct-limit group by differentiating it to obtain a representation of its Lie algebra. We begin by defining a notion of smoothness.

Definition 4.2. *Suppose that $(\pi, \mathcal{H})$ is a continuous representation of a direct-limit group $G_\infty = \varinjlim G_n$ on a Hilbert Space $\mathcal{H}$ and that $v \in \mathcal{H}$. We say that v is a **smooth vector** for π if it is a smooth vector for the restricted representation $(\pi|_{G_n}, \mathcal{H})$ of G_n for each $n \in \mathbb{N}$. We denote by $\mathcal{H}^\infty$ the space of all smooth vectors for π.*

*Similarly, we say that v is a **locally finite vector** for π if it is a G_n-finite vector for the restricted representation $(\pi|_{G_n}, \mathcal{H})$ of G_n for each $n \in \mathbb{N}$. We denote by $\mathcal{H}^{fin}$ the space of locally finite vectors for π. Note that $\mathcal{H}^{fin} \subseteq \mathcal{H}^\infty$.*

We remark that the question of how to put a smooth structure on direct limit groups such as G_∞ has been explored extensively in [12] and [26], where it is shown that under certain technical growth conditions on the G_n's, it is possible to put a smooth structure on G_∞ that is consistent with Definition 4.2.

It is not at all clear from the definition that a representation of G_∞ is guaranteed to possess any smooth vectors or locally-finite vectors. In fact, the existence of smooth vectors is far more subtle for representations of infinite-dimensional Lie groups than for finite-dimensional Lie groups, where every continuous representation on a Frechet space admits a dense subspace of smooth vectors. There are examples of unitary representations of Banach-Lie groups which do not possess any C^1 vectors, much less any smooth vectors (see [3]). For direct-limit groups, however, a beautiful theorem of Danilenko shows that unitary representations always admit smooth vectors.

Theorem 4.3. *([6]; see also [29, Theorem 11.3]) Suppose that $(\pi, \mathcal{H})$ is a unitary representation of a countable direct limit of Lie groups. Then $\mathcal{H}^\infty$ is a dense subspace of $\mathcal{H}$.*

Some representations may consist entirely of smooth vectors:

Definition 4.4. *Suppose that G_∞ is a direct-limit Lie group. We say that a continuous representation $(\pi, \mathcal{H})$ of G_∞ on a Hilbert space $\mathcal{H}$ is **smooth** if $\mathcal{H}^\infty = \mathcal{H}$. Similarly, we say that π is **locally finite** if $\mathcal{H}^{fin} = \mathcal{H}$.*

*If G_∞ is a direct limit of complex Lie groups, then a continuous Hilbert representation $(\pi, \mathcal{H})$ of G_∞ is **holmorphic** if $\pi|_{G_n}$ is holomorphic for each $n \in \mathbb{N}$.*

In fact, we will be primarily concerned with smooth representations in this paper. They play a role for direct-limit groups that is similar to the role played by finite-dimensional representations for finite-dimensional Lie groups. There are several conditions which are equivalent to smoothness, as we will soon see. First we need to reference a well-known lemma on smooth one-parameter semigroups.

Theorem 4.5. *([42, Theorem 13.36]; [41]) Let X be a Banach space and let $Q : [0, \infty) \to \mathcal{B}(X)$ be a strongly-continuous one-parameter semigroup with the (possibly unbounded) operator A as its infinitesmial generator. Then the following are equivalent*

(1) *The domain of A is all of X.*
(2) $\lim_{\epsilon\to 0} ||Q(\epsilon) - 1|| = 0$

(3) *$A \in \mathcal{B}(X)$ and $Q(t) = e^{tA}$ for all $t \geq 0$.*

This result about one-parameter semigroups straightforwardly generalizes to representations of Lie groups:

Theorem 4.6. *Let $(\pi, \mathcal{H})$ be a continuous Hilbert representation of a connected Lie group G. Then the following are equivalent:*

(1) *π is analytic.*
(2) *There is a Lie algebra representation $\mathrm{d}\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ (for which $\mathfrak{g}$ acts by bounded operators) such that*

$$\pi(\exp X) = \exp(\mathrm{d}\pi(X))$$

for each $X \in \mathfrak{g}$.
(3) *π is smooth.*
(4) *π is norm continuous.*

Proof. The result follows from judicious application of Theorem 4.5. The details are left to the reader. □

It is certainly possible to construct continuous unitary representations of direct-limit groups which possess no locally finite vectors. This behavior is already present for finite-dimensional Lie groups, however: an irreducible infinite-dimensional representation of a noncompact Lie group G does not possess any G-finite vectors. More surprisingly, it is possible to construct an irreducible unitary representation of a lim-compact group which has no locally finite vectors ([30]). However, Corollary 4.7 will show that smooth representations of connected lim-compact groups always consist entirely of locally finite vectors.

It is well known that every continuous, finite-dimensional representation of a Lie group is smooth. However, it is also possible to construct infinite-dimensional Hilbert representations which are smooth. Suppose that U is a compact Lie group and that (π, V) is a finite-dimensional representation of U. Without loss of generality, we may assume that π is unitary. Now consider the representation

$$(\infty \cdot \pi, \infty \cdot V) \equiv \left(\bigoplus_{n \in \mathbb{N}} \pi, \bigoplus_{n \in \mathbb{N}} V \right)$$

constructed by taking a Hilbert space direct sum of countably many copies of (π, V). For each $v \in \infty \cdot V$, we consider the closed invariant subspace

$$W = \overline{\langle (\infty \cdot \pi)(U) v \rangle}$$

generated by v. Then W gives a cyclic primary representation of U and decomposes into a direct sum of representations equivalent to (π, V). From [14] we see that every cyclic primary representation of the compact group U is finite-dimensional. Thus $\dim W < \infty$ and so v is a U-finite vector.

In fact, the next theorem shows that in a certain sense, primary representations (or more precisely, finite direct sums of them) provide the only way to obtain infinite-dimensional smooth representations of U:

Theorem 4.7. *Let $(\pi, \mathcal{H})$ be a unitary representation of a compact Lie group U. Then the following are equivalent.*

(1) *π is smooth.*
(2) *π decomposes into a finite direct sum of primary representations of U.*
(3) *π is locally finite.*

Proof. Let $(\pi, \mathcal{H})$ be a unitary representation of U. Then we can write

$$\mathcal{H} \cong_G \bigoplus_{\delta \in \widehat{G}} \mathcal{H}_\delta,$$

where $\mathcal{H}_\delta$ is the space of δ-isotypic vectors for each $\delta \in \widehat{G}$ (that is, vectors in $\mathcal{H}_\delta$ generate primary representations that are direct sums of copies of δ). Thus π is a finite direct sum of primary representations if and only if $\mathcal{H}_\delta = \{0\}$ for all but finitely many $\delta \in \widehat{G}$.

We begin by showing that (1) $\Longrightarrow$ (2). That is, suppose that π is smooth. Recall from the highest-weight theorem that irreducible representations of compact connected Lie groups are parametrized by a discrete semilattice $\Lambda^+(\mathfrak{g}, \mathfrak{h}) \subseteq i\mathfrak{h}^*$ of dominant integral weights, where $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$. Let $\mathcal{S}$ denote the set of all weights $\lambda \in \Lambda^+(\mathfrak{g}, \mathfrak{h})$ such that λ appears as the highest weight of a subrepresentation of π.

If $\lambda \in \mathcal{S}$, then $\lambda(X)$ is an eigenvalue of $d\pi(X)$ for each $X \in \mathfrak{h}$. But since π is smooth, Theorem 4.6 implies that $||d\pi(X)|| < \infty$ for each $X \in \mathfrak{h}$. Thus, since $\Lambda^+(\mathfrak{g}, \mathfrak{h})$ is a semilattice, it follows that $\{\lambda(X) : \lambda \in \mathcal{S}\}$ is finite for each X. Hence $\mathcal{S}$ is finite because $\mathfrak{h}$ is finite-dimensional; that is, π decomposes as a direct sum of a finite number of primary representations.

Next we show that (2) $\Longrightarrow$ (3). Suppose that

$$\mathcal{H} \cong_U \bigoplus_{i=1}^n \mathcal{H}_{\delta_i},$$

where $\delta_i \in \widehat{U}$ for each i. We will show that π is smooth. For each $v \in \mathcal{H}$, we can write $v = v_1 + \cdots + v_n$, where $v_i \in \mathcal{H}_{\delta_i}$. Then

$$\langle \pi(U)v \rangle \subseteq \bigoplus_{i=1}^n \langle \pi(U)v_i \rangle.$$

However, because each space $\langle \pi(U)v_i \rangle$ gives a cyclic primary representation of U, we see that it is finite-dimensional (see [14]). Thus v is U-finite. Because $v \in \mathcal{H}$ was arbitrary, it follows that π is locally finite.

Finally, it is clear that (3) $\Longrightarrow$ (1); that is, if $\mathcal{H}^{\text{fin}} = \mathcal{H}$ then obviously $\mathcal{H}^\infty = \mathcal{H}$. □

Corollary 4.8. *Suppose that U is a compact Lie group and that $(\pi, \mathcal{H})$ is a holomorphic representation of $U_\mathbb{C}$. Then π is locally-finite.*

Proof. First we note that holomorphic representations are in particular smooth, and it follows that $\pi|_U$ is a smooth representation of U. Hence $\pi|_U$ is locally-finite by Theorem 4.7. Let $v \in \mathcal{H}$ and consider the finite-dimensional subrepresentation of $\pi|_U$ on $V = \langle \pi(U)v \rangle$. This representation has a unique holomorphic extension to a finite-dimensional representation of $U_\mathbb{C}$, which is thus a finite-dimensional subrepresentation of π which contains v. □

Corollary 4.9. *Suppose that $(\pi, \mathcal{H})$ is a continuous Hilbert representation of a connected lim-compact group U_∞. Then π is smooth if and only if it is locally finite. That is, $\mathcal{H}^\infty = \mathcal{H}$ if and only if $\mathcal{H}^{fin} = \mathcal{H}$.*

Proof. Fix $n \in \mathbb{N}$. Then representation $(\pi|_{U_n}, \mathcal{H})$ of U_n may be unitarized. Let $\mathcal{H}^{\infty,n}$ denote the space of U_n-smooth vectors and let $\mathcal{H}^{\text{fin},n}$ denote the space of U_n-finite vectors. By Theorem 4.7, it follows that $\mathcal{H} = \mathcal{H}^{\infty,n}$ if and only if $\mathcal{H} = \mathcal{H}^{\text{fin},n}$. The corollary then follows, since a vector in $\mathcal{H}$ is U_∞-smooth if and only if it is U_n-smooth for all $n \in \mathbb{N}$, and it is U_∞-finite if and only if it is U_n-finite for all $n \in \mathbb{N}$. □

The following corollaries restate the conclusions of the previous theorem in terms of weights. Different proofs of these corollaries may be found in Lemma 3.5 and Proposition 3.6 of [28].

Corollary 4.10. *Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{u}$, and suppose that $(\pi, \mathcal{H})$ is a unitary representation of U. Then π is smooth if and only if $\#\Delta(\pi) < \infty$ (that is, π has only finitely many weights).*

Proof. This follows immediately from the equivalence of conditions (1) and (2) in Theorem 4.7. □

Corollary 4.11. *Suppose that U_∞ is a lim-compact group. As before, we fix a subalgebra $\mathfrak{h}_\infty = \varinjlim \mathfrak{h}_n$ in $\mathfrak{u}_\infty$, where each $\mathfrak{h}_n$ is a Cartan subalgebra of $\mathfrak{u}_n$. Suppose that $(\pi, \mathcal{H})$ is a* *unitary representation of U. Then π is smooth if and only if $\#\Delta(\pi|_{U_n}) < \infty$ (that is, π has only finitely many weights) for each $n \in \mathbb{N}$.*

Proof. This result follows immediately from Corollary 4.10 and the fact that π is smooth if and only if $\pi|_{U_n}$ is smooth for each n. □

Suppose now that U_∞ is a propagated lim-compact group. We recursively choose a countable orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ for $\mathfrak{h}_\infty$ as in Section 3.2. Consider the supremum norm of a weight $\lambda \in i\mathfrak{h}_n^*$, given by

$$||\lambda||_\infty = \max_{1\le i\le r_n} |\lambda(e_i)|$$

We then obtain the following useful theorem, which is a slight generalization of [28, Proposition 3.14]:

Theorem 4.12. *A unitary representation $(\pi, \mathcal{H})$ of a propagated direct limit U_∞ of simply-connected compact semisimple Lie groups is smooth if and only if there is $M > 0$ such that for all n one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$.*

Proof. First we prove the theorem in the case that U_∞ is a direct limit of compact simple Lie groups.

Let $(\pi, \mathcal{H})$ be a unitary representation of U_∞. Suppose there is $M \in \mathbb{N}$ such that for all n one has $||\lambda||_\infty < M$ for each weight $\lambda \in i\mathfrak{h}_n^*$ that appears as the highest weight for an irreducible subrepresentation of $\pi|_{U_n}$. If $\lambda \in i\mathfrak{h}^*$ is a highest weight which appears in $\pi|_{U_n}$, then it has the form

$$\lambda = \sum_{i=1}^{r_n} a_i e_i, \text{ where } a_i \in \mathbb{Z} \text{ and } -M \le a_i \le M.$$

Thus, there are only $(2M)^{r_n}$ possible values for λ. In other words, $\pi|_{U_n}$ may be written as a direct sum of finitely many primary representations and is thus smooth by Theorem 4.7. Because $n \in \mathbb{N}$ was arbitrary, we have that π is smooth.

To prove the other direction, suppose that for each $M > 0$ there is $n \in \mathbb{N}$ and a highest weight $\lambda \in i\mathfrak{h}_n^*$ of an irreducible subrepresentation of $\pi|_{U_n}$ such that $||\lambda||_\infty > M$. Fix $M > 0$ and pick $n \in \mathbb{N}$ and $\lambda \in i\mathfrak{h}_n^*$ satisfying those conditions. Then $\lambda = \sum_{i=1}^{r_n} c_i e_i$, where $c_i \in \mathbb{Z}$ for each i. Because $||\lambda||_\infty > M$, we see that there is some index j such that $|c_j| > M$.

By considering the A_n, B_n, C_n, and D_n cases separately, we see that there is a Weyl group element $w \in W(\mathfrak{g}_n, \mathfrak{a}_n)$ such that $w(e_1) = e_i$ and $w(e_i) = e_1$. Then $|(w\lambda)(e_1)| = |c_j| > M$. By the Highest-Weight Theorem, we see that $w\lambda \in \Delta(\pi|_{U_n})$; that is, $w\lambda$ is an $\mathfrak{h}_n$-weight for $\pi|_{U_n}$. It is then clear that $(w\lambda)|_{\mathfrak{h}_k}$ is an $\mathfrak{h}_k$-weight for $\pi|_{U_k}$ whenever $k \le n$ (since every $w\lambda$-weight vector in $\mathcal{H}$ is automatically a $(w\lambda)_{\mathfrak{h}_k}$-weight vector). Furthermore, since $|(w\lambda|_{\mathfrak{k}_n})(e_1)| = |c_j| > M$, we see that $||w\lambda|_{\mathfrak{k}_n}||_\infty > M$.

Thus, if $k \in \mathbb{N}$ is fixed, then for each $M \in \mathbb{N}$ there is a weight $\lambda \in \Delta(\pi|_{U_k})$ such that $||\lambda|| > M$. Hence $\Delta(\pi|_{U_k})$ is not a finite set and thus by Corollary 4.11 it follows that π is not smooth.

Suppose more generally that U_∞ is a propagated direct limit of semisimple Lie groups. Then we can write $U_k = U_k^1 \times U_k^2 \times \cdots \times U_k^d$ for all $k \in \mathbb{N}$ in such a way that $\{U_n^i\}_{n\in\mathbb{N}}$ is a propagated direct system of compact simple Lie groups for each $1 \le i \le d$. We can then recursively choose Cartan subalgebras $\mathfrak{h}_n = \mathfrak{h}_n^i \oplus \mathfrak{h}_n^2 \oplus \cdots \oplus \mathfrak{h}_n^d$, where $\mathfrak{h}_n^i$ is a Cartan subalgebra of $\mathfrak{u}_n^i$ for each i and n. A weight in $\lambda \in i\mathfrak{h}_n^*$ is dominant integral if and only if $\lambda|_{\mathfrak{h}_n^i}$ is dominant integral for each $1 \le i \le d$. Since U_∞^i is a propagated direct limit of compact simple Lie groups, it follows that there is $M_i > 0$ such that for all $n \in \mathbb{N}$ one has that $||\lambda||_\infty < M_i$ for each highest weight $\lambda \in \mathfrak{h}_n^*$ appearing in $\pi|_{U_n^i}$. Since $\max\limits_{1\le i\le d} M_i < \infty$, we are done. □

We end the section with the following remarkable result, which implies that the smoothness of a representation of a direct limit of simple compact groups is controlled by the smoothness of the restriction to any nontrivial one-dimensional analytic subgroup.

Theorem 4.13. *Let U be a compact simple Lie group. Then a unitary representation $(\pi, \mathcal{H})$ of U is smooth if and only if there is $X \in \mathfrak{u}\backslash\{0\}$ such that $\mathrm{d}\pi(X)$ is a bounded operator on $\mathcal{H}$.*

Proof. One direction is obvious. To show the other direction, suppose that $(\pi, \mathcal{H})$ is a non-smooth unitary representation of U. We will show that $\mathrm{d}\pi(X)$ has an unbounded spectrum for any $X \in \mathfrak{u}\backslash\{0\}$. Let $\mathfrak{h}$ be any Cartan subalgebra for U.

Because π is not smooth, it follows that there is for each $M > 0$ weight $\lambda \in \Delta(\pi)$ with $||\lambda||_\infty > M$. As in the proof of Theorem 4.12, we see that for each Weyl-group element $w \in W(\mathfrak{u}, \mathfrak{h})$, the weight $w\lambda$ is in $\Delta(\pi)$. If we write $\lambda = \sum_{i=1}^r a_i e_i$, then there is some j such that $|a_j| > M$. We can use the Weyl group to permute the basis elements so that a_j appears as the i^{th} coefficient of a weight in $\Delta(\pi|_U)$. Thus we have that the set

$$\{\langle \lambda, e_i \rangle | \lambda \in \Delta(\pi)\}$$

of i^{th} coefficients of weights of π is unbounded for all $i \leq r$.

In other words, one has for each $n \in \mathbb{N}$ that the set of weights in $\Delta(\pi)$ is unbounded in every direction on $\mathfrak{h}$. It follows that $\mathrm{d}\pi(X)$ has an unbounded spectrum for all $X \in \mathfrak{h}$. Because every element of $\mathfrak{u}$ is contained in some Cartan subalgebra, the result follows. □

Corollary 4.14. *Let U_∞ be a direct limit of compact simple Lie groups. Then a unitary representation $(\pi, \mathcal{H})$ of U_∞ is smooth if and only if there is $X \in \mathfrak{u}\backslash\{0\}$ such that $\mathrm{d}\pi(X)$ is a bounded operator on $\mathcal{H}$.*

Proof. This corollary follows immediately by applying Lemma 4.13 to U_n for each n in $\mathbb{N}$. □

Note that this result is false for non-simple compact groups: suppose that J and T are compact Lie groups, that $(\pi, \mathcal{H})$ is a smooth unitary representation of J, and that $(\sigma, \mathcal{K})$ is a non-smooth unitary representation of T. Then the outer tensor product representation $(\pi \boxtimes \sigma, \mathcal{H} \otimes \mathcal{K})$ of $J \times T$ has the property that $\pi|_J$ is smooth but $\pi|_T$ is non-smooth.

4.3. **Generalizing Weyl's Unitary Trick.** Weyl's Unitary Trick plays a crucial role in understanding finite-dimensional representations of finite-dimensional Lie groups. There is a natural extension of Weyl's Unitary Trick to locally-finite representations of direct-limit groups. The first step is to extend Weyl's unitary trick to smooth representations of finite-dimensional groups. We begin with a well-known lemma on intertwining operators of smooth representations.

Lemma 4.15. *Suppose that $(\pi, \mathcal{H})$ is a smooth Hilbert representation of a Lie group G. Then the derived representation $\mathrm{d}\pi : \mathfrak{g} \to \mathcal{B}(\mathcal{H})$ possesses the same algebra of intertwining operators as π.*

Now we are ready to extend Weyl's Trick to smooth representations of finite-dimensional groups.

Theorem 4.16. *Suppose that U is a compact Lie group and that G is a (not necessarily compact) closed subgroup of $U_\mathbb{C}$ such that $U_\mathbb{C}$ is a complexification of G. There are one-to-one correspondences between the following categories of representations on $\mathcal{H}$ which preserve the algebras of intertwining operators:*

1. *Locally-finite representations of G on $\mathcal{H}$*
2. *Holomorphic representations of $U_\mathbb{C}$ on $\mathcal{H}$*
3. *Smooth representations of U on $\mathcal{H}$*

Proof. We begin by reminding the reader that a representation of U is smooth if and only if it is locally-finite (see Theorem 4.7) and that every holomorphic representation of $U_\mathbb{C}$ is locally-finite.

We will construct the correspondences (1) → (2) and (2) → (1). The proofs for (2) → (3) and (3) → (2) are identical.

One passes from (2) to (1) quite easily: if $(\pi, \mathcal{H})$ is a locally-finite representation of $U_\mathbb{C}$, then it is clear that $\pi|_G$ is a locally-finite representation of G.

To construct (1) $\to$ (2), we suppose that $(\pi, \mathcal{H})$ is a locally finite representation of G. We wish to construct a holomorphic representation $\pi_{\mathbb{C}}$ of $U_{\mathbb{C}}$ on $\mathcal{H}$ such that $\pi_{\mathbb{C}}|_G = \pi$. First we notice that each vector $v \in \mathcal{H}$ is contained in a finite-dimensional G-invariant subspace W. Write π^W for the subrepresentation of π corresponding to W. By the finite-dimensional Weyl Trick, we see that π^W uniquely extends to a holomorphic representation $\pi_{\mathbb{C}}^W$ of $U_{\mathbb{C}}$ on W. We define $\pi_{\mathbb{C}}(g)v = \pi_{\mathbb{C}}^W(g)v$ for each $v \in W$ and $g \in U_{\mathbb{C}}$. If V and W are finite-dimensional invariant subspaces of $\mathcal{H}$ and $v \in V \cap W$, then the uniqueness of the holomorphic extension shows that $\pi_{\mathbb{C}}^W(g)v = \pi_{\mathbb{C}}^V(g)v$ and thus $\pi_{\mathbb{C}}$ is well-defined and gives a locally-finite, holomorphic representation of $U_{\mathbb{C}}$.

That these one-to-one correspondences of representations preserve the algebra of intertwining operators follows from passing to the derived representation and using Lemma 4.15.

□

Our infinite-dimensional version of Weyl's Trick is then an immediate corollary (see [28, Proposition 3.6] for a partial version of this result and a different proof):

Corollary 4.17. *Suppose that G_∞/K_∞ is a lim-noncompact Riemannian symmetric space which is the c-dual of a lim-compact symmetric space U_∞/K_∞ where U_n/K_n and U_n are simply-connected for each n. Finally, let $\mathcal{H}$ be a Hilbert space. There are one-to-one correspondences between the following categories of representations on $\mathcal{H}$ which preserve the algebras of intertwining operators:*

(1) *Locally finite representations of G_∞ on $\mathcal{H}$*
(2) *Holomorphic representations of $(U_\infty)_{\mathbb{C}}$ on $\mathcal{H}$*
(3) *Smooth representations of U_∞ on $\mathcal{H}$*

Proof. This corollary follows immediately by applying Theorem 4.16 to representations of G_n, $(U_n)_{\mathbb{C}}$, and U_n on $\mathcal{H}$ for each $n \in \mathbb{N}$. □

4.4. **Highest-Weight Representations.** Now suppose that G_∞/K_∞ is an admissible lim-noncompact symmetric space which is the c-dual of a lim-compact symmetric space U_∞/K_∞. We wish to construct irreducible spherical and conical representations for G_∞/K_∞ and U_∞/K_∞. The most natural way to do this would be to construct a direct limit of spherical/conical representations. The following lemma provides the foundation for this construction and is a generalization of a result proved by Ólafsson and Wolf in [34, Lemma 5.8].

Theorem 4.18. *Let U_∞/K_∞ be a propagated lim-compact symmetric space such that U_n/K_n is simply connected for each $n \in \mathbb{N}$. Fix indices $n < m$ and dominant weights $\mu_n \in \Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ and $\mu_m \in \Lambda^+(\mathfrak{u}_m, \mathfrak{a}_m)$ such that $\mu_n|_{\mathfrak{a}_n} = \mu_m$. Consider the irreducible spherical representations $(\pi_{\mu_m}, \mathcal{H}_{\mu_m})$ and $(\pi_{\mu_n}, \mathcal{H}_{\mu_n})$ of U_m and U_n, respectively, with respective highest weights μ_m and μ_n. Let w be a highest-weight vector for π_{μ_m}.*

Then the representation of U_n on $W = \langle \pi_{\mu_m}(U_n)w\rangle$ is equivalent to π_{μ_n} (and, in particular, is irreducible).

Proof. We consider the action of U_n on W. For each dominant weight ν in $\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$, let w_ν be the orthogonal projection of w onto the space of ν-isotypic vectors. Then $w = \sum_\nu w_\nu$ (note that $w_\nu = 0$ for all but finitely many choices of ν). We also write $W_\nu = \langle \pi_{\mu_m}(U_n)w_\nu\rangle$ for each $\nu \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$.

Since w is a U_m-highest-weight vector for π_{μ_m}, we have that $\pi(M_mN_m)w = w$, and in particular, $\pi(M_nN_n)w = w$. Since the space of isotypic vectors in W of type μ_n is invariant under G_n, it follows that w_ν is fixed under M_nN_n for each $\nu \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$. Thus Lemma 2.5 shows that if $w_\nu \neq 0$, then W_ν is a U_n-irreducible subspace of W and that w_ν is a highest-weight vector for W_ν. In particular, w_ν is a weight vector of weight ν.

On the other hand, since w is a U_m-weight vector of weight μ_m, it follows that it is a U_n-weight vector of weight $\mu_n = \mu_m|_{\mathfrak{a}_n}$. But we also have that $w = \sum_\nu w_\nu$, where each w_ν is a weight vector of weight ν. Hence $w = w_{\mu_n}$ and $W = W_{\mu_n}$, and so we are done. □

We follow [45, p. 464–466] for the construction of highest-weight representations. For each n, we denote the set of fundamental weights by $\xi_{n,1}, \ldots \xi_{n,r_n}$, where $r_n = \dim \mathfrak{a}_n$ and where we have numbered the fundamental weights according to the roots as in Section 3.2. Suppose $k \leq n$. One can show that

$$\xi_{n,i}|_{\mathfrak{a}_k} = \xi_{k,i} \tag{4.1}$$

for all $n \in \mathbb{N}$ and $i \leq r_k$. Furthermore, one can check that $\xi_{n,i}|_{\mathfrak{a}_k} = 0$ for $r_k < i \leq r_n$. Thus

$$\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n) = \mathbb{Z}^+\xi_{n,1} + \cdots + \mathbb{Z}^+\xi_{n,r_n} = \left\{ \sum_{j=1}^{r_n} c_j \xi_{n,j} \;\middle|\; c_j \in \mathbb{Z}^+ \right\}, \tag{4.2}$$

and

$$\left(\sum_{j=1}^{r_n} c_j \xi_{n,j} \right)\Bigg|_{\mathfrak{a}_k} = \left(\sum_{j=1}^{r_k} c_j \xi_{k,j} \right) \in \Lambda^+(\mathfrak{g}_k, \mathfrak{a}_k) \tag{4.3}$$

whenever $k \leq n$.

We can thus form a projective limit

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty) = \varprojlim \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n).$$

We say that $\Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is the set of **dominant integral weights** for the restricted root system $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$. That is, Λ^+ consists of the elements λ of $\mathfrak{a}_\infty^* = \varprojlim \mathfrak{a}_n^*$ such that $\lambda|_{\mathfrak{a}_n}$ is dominant and integral for every n. Notice that (4.1) implies that for each $i \in \mathbb{N}$ there is a weight $\xi_i \in \mathfrak{a}_\infty^*$ such that $\xi_i|_{\mathfrak{a}_n} = \xi_{n,i}$ for each $n \in \mathbb{N}$.

Just as in the finite-dimensional case, weights in Λ^+ can be used to create highest-weight representations of U_∞. To see this, fix $\mu \in \Lambda^+$. For n in $\mathbb{N}$, let $(\pi_{\mu_n}, \mathcal{H}_{\mu_n})$ be the irreducible representation of U_n with highest weight $\mu_n \equiv \mu|_{\mathfrak{a}_n}$, and let $v_n \in \mathcal{H}_{\mu_n}$ be a nonzero highest-weight vector. By Theorem 4.18, we see that π_{μ_n} may be embedded unitarily into $\pi_{\mu_{n+1}}$ by identifying the respective highest-weight vectors v_n with v_{n+1}. The corresponding unitary representation of U_∞ constructed by the direct limit of π_{μ_n}, $n \in \mathbb{N}$ is denoted by

$$(\pi_\mu, \mathcal{H}_\mu) = \left(\varinjlim \pi_{\mu_n}, \overline{\varinjlim \mathcal{H}_{\mu_n}} \right),$$

where $\mathcal{H}_\mu = \overline{\varinjlim \mathcal{H}_{\mu_n}}$ is the Hilbert completion of the algebraic direct limit $\varinjlim \mathcal{H}_{\mu_n}$ of Hilbert spaces. We refer to π_μ as the **highest-weight representation with highest weight** μ. Note that a direct limit of irreducible representations of U_n is an irreducible representation of U_∞ by 4.1.

If $\dim \mathfrak{a}_\infty = \infty$, then we can write elements of $\mathfrak{a}^*$ as sequences $(a_i) \in \mathbb{Z}$ of integers, so that a sequence $(a_i) \in \mathbb{Z}$ corresponds to the formal sum $\sum_{i \in \mathbb{N}} a_i e_i \in \mathfrak{a}_\infty^*$. We now use this notation to write down the fundamental weights for $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ for some infinite Dynkin-diagram types.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type A_∞, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots)$$

where the first i entries in ξ_i are zeros.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type B_∞, then

$$\xi_1 = (1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, 2, \ldots) \text{ for } i > 1,$$

where the first $i-1$ entries in ξ_i are zero for $i > 1$.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type C_∞, then

$$\xi_i = (0, \ldots, 0, 2, 2, 2, \ldots),$$

where the first $i-1$ entries in ξ_i are zero.

If $\Sigma(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ has type D_∞, then

$$\xi_1 = (1, 1, 1, \ldots), \xi_2 = (-1, 1, 1, \ldots) \text{ and } \xi_i = (0, \ldots, 0, 2, 2, 2, \ldots) \text{ for } i \geq 3,$$

where the first $i-1$ entries in ξ_i are zero for $i \geq 3$.

By examining the fundamental weights in each case and extending them to weights on $\mathfrak{h}_\infty$, it follows from the boundedness condition in Theorem 4.12 that a highest-weight representation

$(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ will be smooth if and only if we can write λ as a finite linear combination

$$\lambda = \sum_{i=1}^{n} c_i \xi_i,$$

where $c_i \in \mathbb{N}$ for each n. In particular, if $\dim \mathfrak{a}_\infty < \infty$, then every highest-weight representation $(\pi_\mu, \mathcal{H}_\mu)$ for $\lambda \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$ is smooth.

5. Conical Representations for Admissible Direct Limits

In this section, we give a natural definition for conical representations of admissible lim-noncompact symmetric spaces G_∞/K_∞. As before, we assume that G_∞/K_∞ is the c-dual of a propagated lim-compact symmetric space U_∞/K_∞. By using the generalization of Weyl's Unitary Trick from the previous chapter, each smooth cyclic representation of U_∞ gives rise to a smooth cyclic representation of G_∞, and it is natural to say that a smooth cyclic representation of U_∞ is conical if the corresponding representation of G_∞ is conical.

In fact, we will see that in some cases it is possible to define nonsmooth unitary representations of U_∞ which are conical but do not correspond to continuous Hilbert representations of G_∞. This is a strange situation which does not occur in the finite-dimensional case.

With these definitions, we classify all of the irreducible cyclic unitary representations of U_∞ which are conical. Next we see that smooth conical unitary representations of U_∞ decompose into a discrete direct sum of highest-weight representations.

It will follow from this classification, together with [7, Theorem 4.5], that if Rank $U_\infty/K_\infty = \infty$, then there are no smooth unitary representations of U_∞ which are both spherical and conical. On the other hand, if Rank $U_\infty/K_\infty < \infty$, then we will see that a smooth irreducible unitary representation of U_∞ is spherical if and only if it is conical. This situation is also in stark contrast to the situation for finite-dimensional symmetric spaces, for which finite-dimensional representations are spherical if and only if they are conical.

5.1. Definition of Conical Representations.

We begin by presenting our definition of conical representations for lim-Riemannian symmetric spaces. Let G_∞/K_∞ be the c-dual of a propagated lim-compact symmetric space U_∞/K_∞ such that U_n/K_n and U_n are simply-connected for each n and assume that G_∞/K_∞ is admissible.

For finite-dimensional symmetric spaces, it is possible to consider a finite-dimensional conical representation to be a representation of either G or U (where G/K is the c-dual of the compact symmetric space U/K). On the one hand, many harmonic analysis applications of conical representations appear on the horocycle space G/MN, so in a certain sense it is most natural to speak of conical representations of G. On the other hand, these representations are only unitary if we move to the compact group U.

Similarly, because unitarity is crucially important in the arguments which follow, we will mainly consider unitary conical representations of U_∞. However, it is important to remember that smooth representations of U_∞ correspond to locally-finite representations of G_∞ under Theorem 4.17, and vice versa.

We are now ready to present the definition:

Definition 5.1. *A unitary representation $(\pi, \mathcal{H})$ of U_∞ is* **conical** *if there is a nonzero cyclic vector $v \in \mathcal{H}^{\text{fin}}$ such that $\pi(M_n N_n)v = v$ for all $n \in \mathbb{N}$. In that case, we say that v is a* **conical vector** *for π.*

Remark: Even though π is a representation of U_∞ in the above definition and $M_n N_n$ is a subgroup of the c-dual group G_∞, the fact that v is locally-finite implies that each $\pi|_{U_n}$ extends analytically to a representation of G_n on the finite-dimensional space $\langle \pi(U_n)v \rangle$, and so the condition that $\pi(M_n N_n)v = v$ makes sense.

Notice also that we do not require that conical representations of U_∞ be smooth. This opens the door to the aforementioned possibility of constructing conical representations of U_∞ which do not correspond to representations of G_∞ under the generalized unitary trick, and indeed we will construct many examples of such representations in Section 6.

5.2. **Classification of Conical Representations.** We now begin to classify the unitary conical representations of U_∞. We determine which representations are irreducible and show how conical representations decompose into subrepresentations.

Theorem 5.2. *Suppose that U_∞/K_∞ is a propagated lim-compact symmetric space with U_n and U_n/K_n simply-connected for each n and such that the c-dual G_∞/K_∞ is admissible. Suppose further that $(\pi, \mathcal{H})$ is a conical representation with a conical vector v. For each n, write $\Gamma_n(\pi, v)$ for the set of highest weights μ in $\Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ such that the projection $v_\mu = \mathrm{pr}_{\mathcal{H}_\mu} v$ of v onto the space of U_n-isotypic vectors of type μ is nonzero. Then*

1. *For each $n \in \mathbb{N}$ and $\mu \in \Gamma_n(\pi, v)$, the action of U_∞ on $\overline{\langle \pi(U_\infty)v_\mu\rangle}$ gives a conical representation of U_∞ with conical vector v_μ.*
2. *π decomposes into an orthogonal direct sum of disjoint conical representations as follows:*
$$\mathcal{H} = \overline{\langle \pi(U_\infty)v\rangle} = \bigoplus_{\mu\in\Gamma_n(\pi,v)} \overline{\langle \pi(U_\infty)v_\mu\rangle}$$
3. *If π is irreducible, then π is equivalent to a highest-weight representation π_μ for some $\mu \in \Lambda^+(\mathfrak{g}_\infty, \mathfrak{a}_\infty)$.*
4. *If π is irreducible, then $\dim \mathcal{H}^{M_\infty N_\infty} = 1$.*

Proof. For each $n \in \mathbb{N}$, the set $\Gamma_n(\pi, v)$ is finite because v is U_n-finite for all n. Then the decomposition of v into U_n-isotypic vectors may be written
$$v = \sum_{\mu\in\Gamma_n(\pi,v)} v_\mu,$$
where $v_\mu = \mathrm{pr}_{\mathcal{H}_\mu} v$. Since each isotypic subspace is U_n-invariant, it follows that $v_\mu \in \mathcal{H}^{M_nN_n}$ for each $\mu \in \Gamma_n(\pi, v)$. Note that $\langle \pi(U_n)v_\mu\rangle$ gives a primary representation of U_n of type μ. Hence, by Lemma 2.5, it is an irreducible representation with highest-weight vector v_μ.

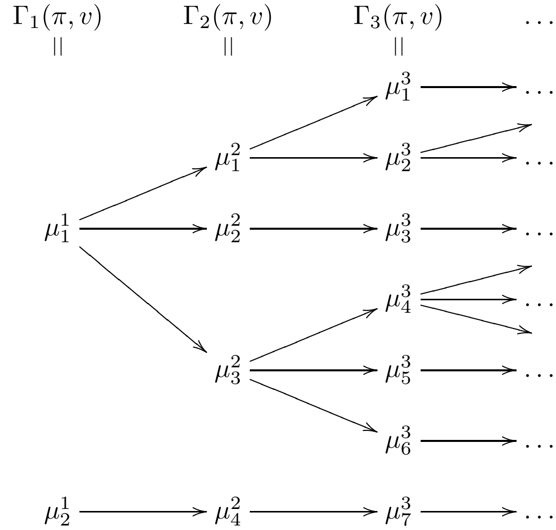
We repeat the same process for U_{n+1}, writing the decomposition of v into U_{n+1}-isotypic vectors as
$$v = \sum_{\lambda\in\Gamma_{n+1}(\pi,v)} v_\lambda \tag{5.1}$$
By Theorem 4.18 it follows for each $\lambda \in \Gamma_{n+1}(\pi, v)$ that $\langle \pi(U_n)v_\lambda\rangle$ is a U_n-irreducible subspace for which v_λ is a highest-weight vector of weight $\lambda|_{\mathfrak{h}_n}$. In other words, v_λ is also a U_n-isotypic vector, so $\lambda|_{\mathfrak{h}_n} \in \Gamma_n(\pi, v)$. Furthermore, since (5.1) is a decomposition of v into U_n- and U_{n+1}-isotypic vectors, we see that for each $\mu \in \Gamma_n(\pi, v)$ there is $\lambda \in \Gamma_{n+1}(\pi, v)$ such that $\lambda|_{\mathfrak{h}_n} = \mu$.

In other words, if we consider all the highest weights of irreducible subrepresentations $\pi(U_n)$ and allow $n \in \mathbb{N}$ to vary, then the highest weights may be naturally arranged into a tree, as in Figure 1.

Next we prove (1). First note that $V_\lambda = \langle \pi(U_\infty)v_\lambda\rangle$ is a U_∞-invariant subspace of $\mathcal{H}$ for each $\lambda \in \Gamma_n(\pi, v)$. Suppose $m > n$, and write
$$u_\lambda = \sum_{\nu\in\Gamma_m(\pi,v) \text{ s.t. } \nu|_{\mathfrak{a}_n}=\lambda} v_\nu$$
for each $\lambda \in \Gamma_n(\pi, v)$. Then u_λ is a U_n-isotypic vector of type λ. Because $v = \sum_{\nu\in\Gamma_m(\pi,v)} v_\nu$, we see that $v = \sum_{\lambda\in\Gamma_n(\pi,v)} u_\lambda$ since every U_m-highest-weight vector v_ν appears as a summand in exactly one u_λ. Since $v = \sum_{\lambda\in\Gamma_n(\pi,v)} v_\lambda$ is also a decomposition of v into U_n-isotypic vectors, it follows that $v_\lambda = u_\lambda$ for each $\lambda \in \Gamma_n(\pi, v)$. In particular, v_λ is M_mN_m-invariant for all $m \geq n$. It follows that $V_\lambda = \overline{\langle \pi(U_\infty)v_\lambda\rangle}$ gives a conical representation of U_∞, proving (1).

To prove (2), we need to show that $V_{\mu_1} \perp V_{\mu_2}$ for all $\mu_1 \neq \mu_2$ in $\Gamma_n(\pi, v)$. It is sufficient to show that $V^m_{\mu_1} = \langle \pi(U_m)v_{\mu_1}\rangle$ and $V^m_{\mu_2} = \langle \pi(U_m)v_{\mu_2}\rangle$ are orthogonal for all m. We apply Lemma 2.6 to see that
$$\langle \pi(U_m)v_\lambda\rangle = \bigoplus_{\nu\in\Gamma_m(\pi,v) \text{ s.t } \nu|_{\mathfrak{a}_n}=\lambda} \langle \pi(U_m)v_\nu\rangle.$$

FIGURE 1. Example of a highest-weight tree



It follows that $\langle\pi(U_m)v_{\mu_1}\rangle$ and $\langle\pi(U_m)v_{\mu_2}\rangle$ are orthogonal for all m and hence that $V = \bigcup_m \langle\pi(U_m)v_{\mu_1}\rangle$ and $W = \bigcup_m \langle\pi(U_m)v_{\mu_2}\rangle$ are orthogonal G-invariant subspaces of $\mathcal{H}$, proving (2). Figure 5.2 demonstrates how the decomposition of U_m-representations matches the tree structure of the highest weights that was exhibited in Figure 1.

To prove (3), we assume that π is irreducible. Suppose that there is n such that $\#\Gamma_n(\pi, v) > 1$ (that is, there is more than one U_m-highest weight in $\pi|_{U_m}$). Then (2) produces orthogonal, nonzero invariant subspaces of $\mathcal{H}$, which contradicts the assumption that π is irreducible. Hence $\#\Gamma_n(\pi, v) = 1$ for all m.

For each n, let μ_n refer to the single element of $\Gamma_n(\pi, v)$. From this it follows that v is a U_m-highest-weight vector of weight μ_m for each m with the property that $\mu_m|_{\mathfrak{a}_n} = \mu_n$ for $m \geq n$. Furthermore, $V_n = \langle\pi(U_n)v\rangle$ is a U_n-irreducible subspace of $\mathcal{H}$ for each n, and we can write $\pi = \varinjlim \pi_n$, where π_n is the representation of U_n on V_n induced by π. Thus π is a highest-weight representation and (3) is proved.

To prove that $\dim \mathcal{H}^{M_\infty N_\infty} = 1$, suppose that v and w are nonzero conical vectors for π such that $v \perp w$. Write $V_n = \langle\pi(U_n)v\rangle$ and $W_n = \langle\pi(U_n)w\rangle$ for each n. We see that V_n and W_n are both equivalent to π_{μ_n} and have v and w as respective highest-weight vectors. By Lemma 2.5, it follows that $V_n \perp W_n$ for each n, contradicting the irreducibility of π.

□

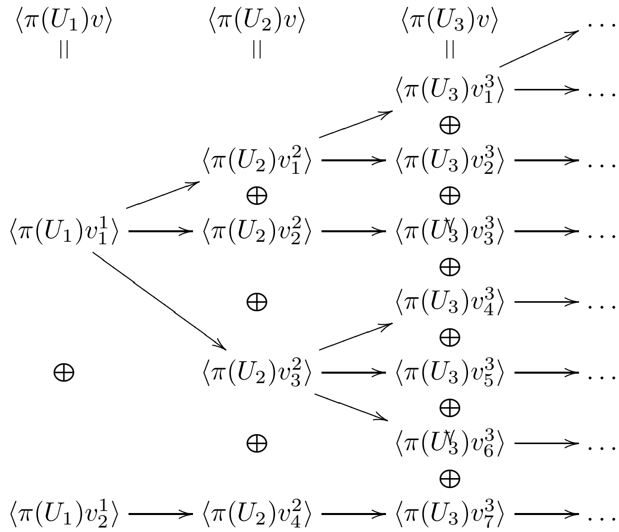
Notice that the maps $p_n^{n+1} : \Gamma_{n+1}(\pi, v) \to \Gamma_n(\pi, v)$ defined by $p_n(\lambda) = \lambda|_{\mathfrak{a}_n}$ define a projective system. We refer to the set $\Gamma(\pi, v) = \varprojlim \Gamma_n(\pi, v) \subseteq \Lambda^+(\mathfrak{u}_\infty, \mathfrak{a}_\infty)$ as the **highest-weight support** of π. If we arrange the highest weights in a tree as in Figure 1, then we see that elements of $\Gamma(\pi, v)$ correspond to infinite paths.

We now examine the connection between conical and spherical representations of G. Recall that for a finite-dimensional Riemannian symmetric space the irreducible finite-dimensional conical and spherical representations are identical. The situation is much different for infinite-dimensional symmetric spaces, as the following corollary shows.

Corollary 5.3. *If* $\mathrm{Rank}(U_\infty/K_\infty) < \infty$*, then a unitary irreducible representation is spherical if and only if it is conical. If* $\mathrm{Rank}(U_\infty/K_\infty) = \infty$*, then no unitary irreducible representation is both spherical and conical.*

Proof. By part (3) of Theorem 5.2, we see that the irreducible conical representations are precisely the highest-weight representations of U_∞ with highest weight $\mu \in \Lambda^+(U_\infty, K_\infty)$. By Theorem 4.5 in [7], it follows that these highest-weight representations of U_∞ are spherical if

FIGURE 2. Example of a decomposition of $\langle \pi(U_n)v\rangle$ into U_n-isotypic subspaces (direct sums are taken vertically)



and only if $\mathrm{Rank}(U_\infty/K_\infty) < \infty$. Furthermore, if $\mathrm{Rank}(U_\infty/K_\infty) < \infty$, then the spherical representations of U_∞ are exhausted by the irreducible highest-weight representations. □

5.3. **Highest-Weight Supports of Conical Representations.** In this section we explore some of the properties of the highest-weight trees associated with conical representations. These trees form an invariant for conical representations, but as we shall see it is possible for two distinct conical representations to possess the same highest-weight tree.

First we show that the tree set of a conical representation is independent of the choice of conical vector:

Theorem 5.4. *Let $(\pi, \mathcal{H})$ be a unitary conical representation of U_∞. Then $\Gamma_n(\pi, v) = \Gamma_n(\pi, w)$ for any conical vectors v, w in $\mathcal{H}$.*

Proof. Suppose that both v and w are conical vectors in $\mathcal{H}$ and that $\mu \in \Gamma_n(\pi, w)$ but $\mu \notin \Gamma_n(\pi, v)$. Write w_μ for the projection of w onto the μ-isotypic vectors in $\mathcal{H}$. Since $\mu \in \Gamma_n(\pi, w)$, it follows that $w_\mu \neq 0$. Define $W = \langle \pi(U_\infty)w_\mu\rangle$ and $V = \langle \pi(U_\infty)v\rangle$. We claim that $W \perp V$, which will be a contradiction since V is dense in $\mathcal{H}$.

Note that $W = \bigcup_{m\geq n}\langle \pi(U_m)w_\mu\rangle$ and $V = \bigcup_{m\geq n}\langle \pi(U_m)v\rangle$. It is sufficient to show that $\langle \pi(U_m)w_\mu\rangle \perp \langle \pi(U_m)v\rangle$ for $m \geq n$. As before, we see from Lemma 2.5 and Theorem 4.18 that

$$\langle \pi(U_m)v\rangle = \bigoplus_{\lambda\in\Gamma_m(\pi,v)} \langle \pi(U_m)v_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda\in\Gamma_m(\pi,v)} \mathcal{H}_\lambda$$

and

$$\langle \pi(U_m)w_\mu\rangle = \bigoplus_{\nu\in\Gamma^\mu_m(\pi,w)} \langle \pi(U_m)w_\nu\rangle \cong_{U_m} \bigoplus_{\nu\in\Gamma_m(\pi,w)} \mathcal{H}_\nu,$$

where $\Gamma^\mu_m(\pi, w) = \{\nu \in \Gamma_m(\pi, w) \text{ s.t. } \nu|_{\mathfrak{a}_n} = \mu\}$.

Fix $m \geq n$. Since $\mu \notin \Gamma_n(\pi, v)$, it follows that $\lambda|_{\mathfrak{a}_n} \neq \mu$ for all $\lambda \in \Gamma(\pi, v)$. Thus $\Gamma_m(\pi, v)$ and $\Gamma^\mu_m(\pi, w)$ are disjoint. This means that $\langle \pi(U_m)v_\lambda\rangle \perp \langle \pi(U_m)w_\nu\rangle$ for each $\lambda \in \Gamma_m(\pi, v)$ and $\nu \in \Gamma^\mu_m(\pi, w)$. Hence $\langle \pi(U_m)v\rangle \perp \langle \pi(U_m)w_\mu\rangle$ for all m, as we wanted to show. □

From now on, we write $\Gamma_n(\pi) \equiv \Gamma_n(\pi, v)$ and $\Gamma(\pi) = \varprojlim \Gamma_n(\pi, v)$, where v is any conical vector of a conical representation π of U_∞.

Corollary 5.5. *Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary conical representations of (U_∞, K_∞). If there is $n \in \mathbb{N}$ such that $\Gamma_n(\pi) \neq \Gamma_n(\rho)$, then $\pi \ncong \rho$.*

In particular, we have shown that having the same highest-weight tree is a necessary condition for two conical representations to be equivalent. Later we will provide examples of inequivalent conical representations with the same highest-weight trees. However, two conical representations with the same highest-weight trees are nonetheless *almost* equivalent in a certain sense, as the following theorem shows.

Theorem 5.6. *Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be conical representations of (U_∞, K_∞) with respective conical vectors v and w such that $\Gamma_n(\pi) = \Gamma_n(\rho)$ for each n. Consider the algebraic subrepresentations $V = \langle \pi(U_\infty)v\rangle$ and $W = \langle \rho(U_\infty)w\rangle$ generated by the action of U_∞ on v and w. Write π_V and ρ_W for the representations of U_∞ given by restricting π and ρ to the dense invariant subspaces V and W of $\mathcal{H}$ and $\mathcal{K}$, respectively. Then*

(1) $\pi_V \cong \rho_W$
(2) $\pi|_{U_n} \cong \rho|_{U_n}$ *for each n.*

Proof. We begin by proving (1). We claim that the map $L : V \to W$ induced by $\pi(g)v \mapsto \rho(g)w$ is a well-defined invertible U_∞-intertwining operator.

As before, write $V_m = \langle \pi(U_m)v\rangle$ and $W_m = \langle \pi(U_m)w\rangle$, so that $V = \bigcup_{m \geq n} V_m$ and $W = \bigcup_{m \geq n} W_m$. Then

$$V_m = \bigoplus_{\lambda \in \Gamma_m} \langle \pi(U_m)v_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda \in \Gamma_m} \mathcal{H}_\lambda$$

and

$$W_m = \bigoplus_{\lambda \in \Gamma_m} \langle \rho(U_m)w_\lambda\rangle \cong_{U_m} \bigoplus_{\lambda \in \Gamma_m} \mathcal{H}_\lambda,$$

where $\Gamma_m = \Gamma_m(\pi) = \Gamma_m(\rho)$. Thus V_m and W_m are U_m-isomorphic. We must show that there is an invertible U_m-intertwining operator $L^m : V_m \to W_m$ that maps v to w.

In fact, we note that for each $\lambda \in \Gamma_m$ there is a (not necessarily unitary) U_m-intertwining operator $L_\lambda : \langle \pi(U_m)v_\lambda\rangle \to \langle \rho(U_m)w_\lambda\rangle$ given by $\pi(g)v_\lambda \mapsto \rho(g)w_\lambda$. We can then define

$$L^m = \bigoplus_{\lambda \in \Gamma_m} L_\lambda : V_m = \bigoplus_{\lambda \in \Gamma_m} \langle \pi(U_m)v_\lambda\rangle \to \bigoplus_{\lambda \in \Gamma_m} \langle \rho(U_m)w_\lambda\rangle = W_m.$$

Hence $L^m v = L^m(\sum_{\lambda \in \Gamma_m} v_\lambda) = \sum_{\lambda \in \Gamma_m} w_\lambda = w$.

Since v and w are cyclic vectors in V_m and W_m, respectively, L^m is in fact uniquely determined as an intertwining operator by the fact that it maps v to w. In particular, $L^m|V_n = L^n$ for all $n \leq m$. Thus the family $\{L^m\}_{m \in \mathbb{N}}$ is a direct system of intertwining operators that induces a continuous U_∞-intertwining operator $L : V = \varinjlim V_m \to \varinjlim W_m = W$ such that $Lv = w$.

Next we prove (2). Fix $n \in \mathbb{N}$. Recursively define $\widetilde{V}_n = V_n$ and $\widetilde{V}_m = V_m \ominus V_{m-1}$ for $m > n$, where the orthogonal complement is taken with respect to the Hilbert space structure inherited by V_n as a closed subspace of $\mathcal{H}$. Notice that $\widetilde{V}_m$ is a finite-dimensional U_n-invariant subspace of $\mathcal{H}$ for each $m \geq n$. We define U_n-invariant spaces $\widetilde{W}_m \subseteq \mathcal{K}$ for each $m \geq n$ in exactly the same way.

Recall that V_m and W_m give equivalent representations of U_n for each $m \geq n$ under the intertwining operator L^m. It follows that $\widetilde{V}_m = V_m \ominus V_{m-1}$ and $\widetilde{W}_m = W_m \ominus W_{m-1}$ are U_n-isomorphic for all $m > n$. Note that

$$\mathcal{H} = \bigoplus_{m \geq n} \widetilde{V}_m \text{ and } \mathcal{K} = \bigoplus_{m \geq n} \widetilde{W}_m,$$

where the direct sums are orthogonal. Since there is a unitary U_n intertwining operator between $\widetilde{V}_m$ and $\widetilde{W}_m$ for all $m \geq n$, it follows that there is a unitary U_n-intertwining operator between $\mathcal{H}$ and $\mathcal{K}$. □

5.4. **Smooth Conical Representations.** Next we classify the smooth conical representations of U_∞. As mentioned previously, these are of interest because, by the generalized Weyl unitary trick, they are precisely the conical representations which extend to smooth conical representations of the c-dual G_∞. Our next theorem classifies the smooth representations.

Theorem 5.7. *Suppose that $(\pi, \mathcal{H})$ is a smooth conical representation of U_∞. Then π decomposes into a direct sum of irreducible smooth highest-weight representations.*

Proof. Let v be a conical vector for π. For each U_n, write

$$v = \sum_{\lambda \in \Gamma_n(\pi)} v_\lambda$$

as before. As in Section 3.2, we recursively construct a countable basis $\{e_i\}_{n\in\mathbb{N}}$ for $\mathfrak{a}_\infty$ such that $\{e_1, \ldots, e_{r_n}\}$ is a basis for $\mathfrak{a}_n$ for each n. For each $\lambda \in \mathfrak{a}_n^*$, write

$$||\lambda||_\infty = \max_{1\le i\le n} |\lambda(e_i)|.$$

In fact, if $\lambda \in \Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ and $\lambda = \sum_{i=1}^{r_n} a_i e_i$, then we see from the data at the end of Section 4.4 that $a_i \le a_j$ when $i \le j$; thus $||\lambda||_\infty = a_{r_n}$.

For each $\mu \in \Gamma_n(\pi)$, let $\Gamma^\mu_{n+1}(\pi) = \{\lambda \in \Gamma_{n+1}(\pi) : \lambda|_{\mathfrak{a}_n} = \mu\}$. Hence we have $||\lambda||_\infty \ge ||\mu||_\infty$ for each $\lambda \in \Gamma^\mu_{n+1}$,.

Now suppose that $\mu \in \Gamma_n(\pi)$ and that there are distinct weights $\lambda_1, \lambda_2 \in \Gamma^\mu_{n+1}(\pi)$. In this case we say that μ *splits* with respect to π. Because λ_1 and λ_2 in $\Lambda^+(\mathfrak{g}_n, \mathfrak{a}_n)$ are by assumption distinct and agree on the first r_n coordinates, we see that they must differ on a coordinate i with $r_n < i \le r_{n+1}$. Since the coefficients of dominant weights form an increasing sequence, we see that either $||\lambda_1||_\infty > ||\lambda_2||_\infty \ge ||\mu||_\infty$ or $||\lambda_2||_\infty > ||\lambda_1||_\infty \ge ||\mu||_\infty$

In other words, if a highest weight $\mu \in \Gamma_n(\pi)$ splits, then there is a U_{n+1}-highest weight in $\Gamma^\mu_{n+1}(\pi)$ with a coefficient which is strictly greater than all the coefficients in μ. It follows that unless there is a weight $\mu_n \in \Gamma_n(\pi)$ for some n which does not split and such that each $\lambda \in \Gamma^\mu_m(\pi)$ for any $m \ge n$ does not split, then we can repeat this process to obtain arbitrarily large coefficients of highest weights of representations appearing in π, contradicting Lemma 4.12. Hence, there is some highest weight $\mu \in \Gamma_n(\pi)$ such that, for each $m \ge n$, the vector v_μ is a U_m-highest-weight vector. Thus $\overline{\langle \pi(U_\infty)v_\mu \rangle}$ gives a highest-weight representation of U_∞.

We have shown that every smooth unitary conical representation possesses an irreducible subrepresentation and that the orthogonal complement is also a smooth unitary conical representation. A standard Zorn's Lemma argument then shows that $\mathcal{H}$ decomposes into an orthogonal direct sum of irreducible smooth conical representations. □

It follows from Theorems 4.12 and 5.7 that every smooth unitary conical representation $(\pi, \mathcal{H})$ of U_∞ is an orthogonal direct sum of smooth highest-weight representations:

$$\pi \cong \bigoplus_{i\in\mathcal{A}} \pi_{\mu_i},$$

where $\mu_i \in \Lambda^+ = \varprojlim \Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ for each $i \in \mathcal{A}$. We can write each highest weight μ_i in terms of fundamental weights as in Section 4.4:

$$\mu_i = \sum_{n=1}^{k_i} a^i_n \xi_i,$$

where $a^i_n \in \mathbb{N}$ for each i and n (each μ_i is a finite sum over the fundamental weights is finite because π_{μ_i} is a smooth highest-weight representation). By Theorem 4.12, the smoothness of π is equivalent to the existence of a bound $M > 0$ such that $\sum_{n=1}^{k_i} a^i_n < M$ for all $i \in \mathcal{A}$.

6. Disintegration of Conical Representations

If we remove the assumption in Theorem 5.7 that the conical representation $(\pi, \mathcal{H})$ is smooth, then we can no longer be assured that π has an irreducible subrepresentation. However, we would still like to describe general conical representations in terms of the irreducible ones. This sort of description is possible with a direct-integral decomposition.

Recall that

$$\Lambda^+ \equiv \Lambda^+(\mathfrak{u}_\infty, \mathfrak{a}_\infty) \equiv \varprojlim \Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n) \subseteq \mathfrak{a}^*_\infty$$

denotes the set of dominant integral weights for the root system $\Sigma(\mathfrak{u}_\infty, \mathfrak{a}_\infty)$. We start by putting a topology on Λ^+. Each lattice $\Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ carries the discrete topology. We then consider the projective limit topology on Λ^+. This topology is defined by a basis consisting of the cylinder sets $B_\lambda = \{\mu \in \Lambda^+ | \mu_{|\mathfrak{a}_n} = \lambda\}$, where λ is a dominant integral weight on $\mathfrak{a}_n$. It is clear that Λ^+ is second-countable under this topology, since there are only countably many dominant integral weights on $i\mathfrak{a}_n$, for each fixed $n \in \mathbb{N}$, so that our basis of cylinder sets is a countable union of countable sets. This projective-limit topology is also Hausdorff.

Next consider closed subsets Γ of Λ^+ with the property that, for each $n \in \mathbb{N}$, we have $\Gamma \bigcap B_\lambda = \emptyset$ for all but finitely many λ in Λ_n^+. We will refer to such sets as **tree sets** because, as we shall soon see, they are in one-to-one correspondence with trees of a certain type. We give each tree set Γ the subspace topology. Write $\Gamma^\lambda = B_\lambda \bigcap \Gamma = \{\mu \in \Gamma | \mu_{|\mathfrak{a}_n} = \lambda\}$ for each n and each $\lambda \in \Lambda_n^+$. We refer to these sets as **cylinder sets** for Γ. If $\lambda \in \Lambda_n^+$ and $\Gamma^\lambda \neq 0$ (that is, there is $\mu \in \Gamma$ such that $\mu|_{\mathfrak{a}_n} = \lambda$), then we say that λ is a **node** of the tree set Γ. We write $\Gamma_n = \{\mu_{|\mathfrak{a}_n} | \mu \in \Gamma\}$ for the set of all nodes of Γ that lie in Λ_n^+. One quickly sees that $\Gamma = \varprojlim \Gamma_n$ and that Γ has the corresponding projective limit topology. Thus Γ is a projective limit of compact topological spaces, from whih it follows that that Γ is compact.

It may be readily seen that if π is a conical representation of U_∞, then the highest-weight tree $\Gamma(\pi) \subseteq \Lambda^+$ is a tree set. This follows because each set $\Gamma_n(\pi) \subseteq \Lambda^+(\mathfrak{u}_n, \mathfrak{a}_n)$ is finite.

We spend a few moments explaining our tree-centric choice of terminology. For each tree set Γ, we can construct a tree as follows. Each element of Γ_n for each $n \in \mathbb{N}$ forms a node of the tree. Draw an edge from a node λ in Γ_n to a node μ in Γ_{n+1} if $\mu|_{\mathfrak{a}_n} = \lambda$. There is a correspondence between infinite paths in this tree and elements of Γ. Each infinite path $\{\lambda_n \in \Gamma_n\}_{n \in \mathbb{N}}$ of nodes of the tree defines a dominant weight $\lambda \in \Lambda^+$, since $\lambda_m|_{\mathfrak{a}_n} = \lambda_n$ for $m > n$. Because Γ is closed in the projective limit topology on Λ^+, it follows that $\lambda \in \Gamma$. Similarly, each dominant weight λ in Γ defines a path $\{\lambda|_{\mathfrak{a}_n} \in \Gamma_n\}_{n \in \mathbb{N}}$ in the tree. Hence, if λ is a node of Γ, then the cylinder set Γ^λ corresponds to the set of all infinite paths in the tree which pass through the node λ.

As usual, we give each tree set $\Gamma \subseteq \Lambda^+$ the Borel σ-algebra, which is generated by node sets. We can use Γ to define a measurable family of Hilbert spaces $\lambda \mapsto \mathcal{H}_\lambda$ over $\lambda \in \Gamma$. For each $\lambda \in \Gamma$, consider the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of U_∞ with highest-weight λ. For each such representation, pick out a unit highest-weight vector $v_\lambda \in \mathcal{H}_\lambda$.

To tie these Hilbert spaces together in a measurable way, we consider the family $\{s_g | g \in U_\infty\}$ of maps $s_g : \Gamma \to \dot{\bigcup}_{\lambda \in \Gamma} \mathcal{H}_\lambda$ given by $s_g(\lambda) = \pi_\lambda(g) v_\lambda$. Now choose a countable dense subset $E \subseteq U_\infty$ (recall that $U_\infty = \varinjlim U_n$ is separable) and consider the countable family

$$\{s_g | g \in E\}$$

of sections. We shall use this family as a measurable frame for our family of Hilbert spaces. Hence, we need to show that

$$\lambda \mapsto \langle s_g(\lambda), s_h(\lambda)\rangle = \langle \pi_\lambda(g) v_\lambda, \pi_\lambda(h) v_\lambda\rangle \tag{6.1}$$

is $\mathfrak{B}$-measurable for each $g, h \in E$. Suppose that $g, h \in U_n$ for some $n \in \mathbb{N}$. Then the representation of U_n on $\langle \pi_\lambda(U_n) v_\lambda\rangle$ is equivalent to $\pi_{\lambda|_{\mathfrak{a}_n}}$ for each λ. Thus the map in (6.1) is constant on each node set $\Gamma^{\lambda|_{\mathfrak{a}_n}}$ where $\lambda \in \Gamma$ and is hence $\mathfrak{B}$-measurable. Finally, note that $\langle \{s_g(\lambda) = \pi_\lambda(g) v_\lambda | g \in E\}\rangle$ is dense in $\mathcal{H}_\lambda$ since π_λ is irreducible and E is dense in U_∞. Thus, $\lambda \mapsto \mathcal{H}_\lambda$ is a measurable field of Hilbert spaces.

Next, we note that s_g is a measurable section for all $g \in U_\infty$. In fact, every $g \in U_\infty$ is a limit of a sequence $\{g_i\}_{i \in N} \subseteq E$. Hence, we have that

$$\lambda \mapsto \langle s_g(\lambda), s_h(\lambda)\rangle = \lim_{i \to \infty} \langle s_{g_i}(\lambda), s_h(\lambda)\rangle$$

is a measurable function for all $h \in E$, so that s_g is a measurable section.

In order to construct a direct integral of representations $(\pi_\lambda, \mathcal{H}_\lambda)$ over $\lambda \in \Gamma$, we still need a suitable choice of measure on $(\Gamma, \mathfrak{B})$. In particular, we need to choose a finite measure whose support is all of Γ (we will refer to such measures as having **full support**).

Given a finite Borel measure μ on Γ of full support, we may consider the direct integral $\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}_\mu d\mu(\lambda)$. Elements of this direct integral consist of measurable sections $x : \lambda \mapsto x(\lambda)$ of the field $\lambda \mapsto \mathcal{H}_\lambda$ such that the norm given by $||x||^2 = \int_\Gamma ||x(\lambda)||^2_{\mathcal{H}_\lambda} d\mu(\lambda)$ is finite.

Our next task is to show that $\lambda \to \pi_\lambda$ is a μ-measurable family of representations. Let $x \in \mathcal{H}$, and fix g in U_∞. We need to show that $\lambda \overset{\pi(x)}{\mapsto} \pi_\lambda(g)x(\lambda)$ is in $\mathcal{H}$. Now

$$\begin{aligned}\lambda \mapsto \langle \pi_\lambda(g)x(\lambda), s_h(\lambda)\rangle &= \langle \pi_\lambda(g)x(\lambda), \pi_\lambda(h)v_\lambda\rangle \\ &= \langle x(\lambda), \pi_\lambda(g^{-1}h)v_\lambda\rangle \\ &= \langle x(\lambda), s_{g^{-1}h}(\lambda)\rangle\end{aligned}$$

is measurable for all h in U_∞ since x is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Thus $\lambda \overset{\pi(g)x}{\mapsto} \pi_\lambda(g)x(\lambda)$ is a measurable section of $\lambda \mapsto \mathcal{H}_\lambda$. Furthermore, since each π_λ is unitary, it follows that $||\pi(g)x||_\mathcal{H} = ||x||_\mathcal{H} < \infty$. Hence $\pi = \int_\Gamma^\oplus \pi_\lambda d\mu(\lambda)$ is a unitary representation of U_∞. Our next task is to show that π is conical and classify all of its conical vectors.

Theorem 6.1. *Let Γ be a tree set and let μ be a finite Borel measure of full support on Γ. Consider the representation*

$$(\pi, \mathcal{H}) \equiv \left(\int_\Gamma^\oplus \pi_\lambda d\mu(\lambda), \int_\Gamma^\oplus \mathcal{H}_\lambda d\mu(\lambda) \right)$$

and suppose that w is any nonzero vector in $\mathcal{H}$. Then w generates a unitary conical representation of U_∞ if and only if there is $f \in L^2(\Gamma, \mu)$ such that $w = \int_\Gamma^\oplus f(\lambda)v_\lambda d\mu(\lambda)$.

In particular, π is a conical representation with conical vector $v = \int_\Gamma^\oplus v_\lambda d\mu(\lambda)$.

Proof. ($\Rightarrow$) Suppose that w is a conical vector for a subrepresentation of π and fix n in $\mathbb{N}$. Because conical representations are by definition locally finite, we see that $V_n \equiv \langle \pi(U_n)w\rangle$ is finite-dimensional, say with dimension d. We must show that $w(\lambda)$ is a conical vector in $\mathcal{H}_\lambda$ for almost all $\lambda \in \Gamma$. Our first task is to show that $V_n(\lambda) = \langle \pi(U_n)w(\lambda)\rangle$ is finite-dimensional for almost all $\lambda \in \Gamma$.

Write $d = \dim V_n$. Fix an orthonormal basis $w_1, \ldots w_d$ for V_n and write

$$W(\lambda) = \langle w_1(\lambda), \ldots w_d(\lambda)\rangle.$$

We will show that $W(\lambda) = V_n(\lambda)$ (and hence $\dim V_n(\lambda) \le d$) for almost all λ, and it will follow that $\dim V_n(\lambda) \le d$ for almost all λ. Apply a Gram-Schmidt orthonormalization process to the collection $w_1(\lambda), \ldots, w_d(\lambda)$ for each λ. We then obtain a collection $\widetilde{w}_1(\lambda), \ldots, \widetilde{w}_d(\lambda)$ with the property that $\langle \widetilde{w}_i(\lambda), \widetilde{w}_j(\lambda)\rangle = 0$ for $i \ne j$ and $\langle \widetilde{w}_i(\lambda), \widetilde{w}_i(\lambda)\rangle \in \{0, 1\}$. One can show that $\lambda \mapsto \widetilde{w}_i(\lambda)$ is measurable and thus that $\widetilde{w}_i \in \mathcal{H}$ for each i.

Now $W(\lambda) = V_n(\lambda)$ if and only if $\pi(g)w(\lambda) \in W(\lambda)$ for all g in U_∞. Choose a countable dense subset $\{g_n\}_{n\in\mathbb{N}}$ in U_∞ (one notes that U_∞ is separable because it is a countable direct union of separable spaces). By the strong continuity of π, we see that $W(\lambda) = V_n(\lambda)$ if and only if $\pi(g_m)w(\lambda) \in W(\lambda)$ for all m in $\mathbb{N}$ (recall that $W(\lambda)$ is closed because it is finite-dimensional). In turn, this happens exactly when $\pi(g_m)w(\lambda)$ is equal to its orthogonal projection onto $W(\lambda)$. In other words, $W(\lambda) = V_n(\lambda)$ if and only if $F_m(\lambda) = 0$ for all $m \in \mathbb{N}$, where F_m is the non-negative measurable function on Γ defined by

$$F_m : \lambda \mapsto ||\pi(g_m)w(\lambda)||^2 - \sum_{i=1}^d |\langle \pi(g_m)w(\lambda), \widetilde{w}_i(\lambda)\rangle|^2.$$

for all $m \in \mathbb{N}$.

Write $A = \{\lambda \in \Gamma | W(\lambda) \ne V_n(\lambda)\}$ and $A_m = \{\lambda \in \Gamma | \pi(g_m)w(\lambda) \notin W(\lambda)\}$. Then $A = \bigcup_{m\in\mathbb{N}} A_m$. Furthermore, A_m is measurable for each m since $A_m = F_m^{-1}(0)$ and F_m is a measurable function.

Suppose that it is not true that $W(\lambda) = V_n(\lambda)$ for almost all λ in Γ. Then $\mu(A) > 0$. Since $A = \bigcup_{m\in\mathbb{N}} A_m$, it follows that $\mu(A_m) > 0$ for some m. Since $\pi(g_m)w(\lambda) \notin W(\lambda)$ for all $\lambda \in A_m$, we see that $\pi(g_m)w \notin \langle w_1, \ldots, w_d\rangle$, which contradicts the assumption that $w_1, \ldots, w_d$ is a basis for $V_n = \langle \pi(g_m)w \rangle$. Therefore, $W(\lambda) = V_n(\lambda)$ (and, in particular, $\dim V_n(\lambda) \le d$) for almost all λ. In particular, $w(\lambda)$ is U_n-finite for almost all $\lambda \in \Gamma$.

Fix $n \in \mathbb{N}$. Since $\pi(M_n)w = w$, it follows that $\pi(M_n)w(\lambda) = w(\lambda)$ for almost all λ. Next, $\pi(\mathfrak{n}_n)w = w$ because $\pi(N_n)w = w$. In fact, $\pi(X)w = \int_\Gamma^\oplus \pi(X)w(\lambda)d\mu(\lambda)$ for $X \in \mathfrak{u}_n^{\mathbb{C}}$ by [1]. Thus $\pi(\mathfrak{n}_n)w(\lambda) = w(\lambda)$ for almost all λ, from which it follows that $\pi(N_n)w(\lambda) = w(\lambda)$ for almost all λ.

Since $\pi(M_nN_n)w(\lambda) = w(\lambda)$ for all n and almost all $\lambda \in \Gamma$, it follows from part (4) of Theorem 5.2 that for almost all λ there is $f(\lambda) \in \mathbb{C}$ such that $w(\lambda) = f(\lambda)v_\lambda$. Since $\lambda \mapsto f(\lambda) = \langle w(\lambda), v_\lambda \rangle$ is measurable and

$$||f||^2 = \int_\Gamma |f(\lambda)|^2 d\mu(\lambda) = \int_\Gamma ||w(\lambda)||^2 d\mu(\lambda) = ||w||^2,$$

we see that $f \in L^2(\Gamma, \mu)$, as was to be shown.

($\Leftarrow$) Now suppose that $w = \int_\Gamma^\oplus f(\lambda)v_\lambda d\mu(\lambda)$, where $f \in L^2(\Gamma, \mu)$. We show that w generates a conical representation of U_∞ with highest-weight support ess supp f.

Consider $V_n = \langle \pi(U_n)w \rangle$. We will show that V_n is finite-dimensional. As before,

$$\pi \cong \bigoplus_{\mu \in \Gamma_n} \left(\int_{\Gamma^\mu}^\oplus \pi_\lambda d\mu(\lambda) \right).$$

Write $w = \sum_{\mu \in \Gamma_n} w_\mu$, where $w_\mu = 1_{N_\mu} w \in \int_{\Gamma^\mu}^\oplus \mathcal{H}_\lambda d\mu(\lambda) \subseteq \mathcal{H}_\Gamma$ for each μ.

Of course, if $f|_{\Gamma^\mu} = 0$, then $w_\mu = 0$. On the other hand, we claim that if $f|_{\Gamma^\mu} \neq 0$, then $\langle \pi(U_n)w_\mu \rangle \cong_{U_n} \pi_\mu$. In fact,

$$\sum_{i=1}^k c_i \pi(g_i) w_\mu = \int_{\Gamma^\mu} \sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda d\mu(\lambda).$$

where $c_i \in \mathbb{C}$ and $g_i \in U_n$. Fix $\lambda \in \Gamma^\mu$ such that $f(\lambda) \neq 0$. Since $\lambda|_{\mathfrak{a}_n} = \mu$, we see that $\langle \pi(U_n) f(\lambda) v_\lambda \rangle$ is U_n-isomorphic to π_μ.

Now $\sum_{i=1}^k c_i \pi(g_i) w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$ in $\mathcal{H}_\lambda$ for μ-almost all λ in Γ^μ. For any λ in Γ^μ such that $f(\lambda) = 0$, it follows automatically that $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$. But for any fixed λ in Γ^μ such that $f(\lambda) \neq 0$, we see that $\sum_{i=1}^k c_i \pi(g_i) f(\lambda) v_\lambda = 0$ in $\mathcal{H}_\lambda$ if and only if $\sum_{i=1}^k c_i \pi(g_i) v_\mu = 0$ in $\mathcal{H}_\mu$.

Since f is not almost-everywhere zero on Γ^μ, we see that $\sum_{i=1}^k c_i \pi(g_i) w_\mu = 0$ in $\mathcal{H}$ if and only if $\sum_{i=1}^k c_i \pi(g_i) v_\mu = 0$ in $\mathcal{H}_\mu$. Hence there is an injective U_n-intertwining operator $L : \langle \pi(U_n) w_\mu \rangle \to \mathcal{H}_\mu$ with the property that $L w_\mu = v_\mu$. Since π_μ is irreducible, it follows that $\langle \pi(U_n) w_\mu \rangle \cong_{U_n} \pi_\mu$, as we wanted to show.

It follows from Lemma 2.6 that

$$\langle \pi(U_n) w \rangle \cong_{U_n} \bigoplus_{\mu \in \Gamma_n \text{ s.t. } w_\mu \neq 0} \langle \pi(U_n) w_\mu \rangle.$$

Furthermore, since $w = \sum_{\mu \in \Gamma_n} w_\mu$ and each w_μ is M_nN_n-invariant, we see that w is M_nN_n-invariant. Since this holds for all n, it follows that w generates a conical subrepresentation of π. The fact that this subrepresentation has highest-weight support ess supp f follows from the fact that $w_\mu = 0$ if and only if $f|_{\Gamma^\mu} = 0$ (recall that w_μ is the projection of w onto the μ-isotypic vectors in $\mathcal{H}$). □

In fact, for each conical vector identified by the previous theorem, it is possible to describe the subrepresentation that it generates. We first remind the reader that the **essential support** of a function $f : \Gamma \to \mathbb{C}$ is defined to be the complement in Γ of the union of all open sets on which f vanishes μ-almost everywhere. That is, ess supp $f = \Gamma \setminus \bigcup \{A \subseteq \Gamma | A \text{ is open and } f|A = 0 \text{ a.e.}\}$.

Theorem 6.2. *As before, let Γ be a tree set, let μ be a finite Borel measure of full support on Γ, and consider the representation* $(\pi, \mathcal{H}) \equiv \left(\int_\Gamma^\oplus \pi_\lambda d\mu(\lambda), \int_\Gamma^\oplus \mathcal{H}_\lambda d\mu(\lambda)\right)$.

Suppose that $f \in L^2(\Gamma, \mu)$, and put $w = \int_\Gamma^\oplus f(\lambda) v_\lambda d\mu(\lambda)$. Then the conical representation generated by w has highest-weight support equal to $\operatorname{ess\,supp} f$ *and*

$$\overline{\langle \pi(U_\infty) w\rangle} = \int_{\Gamma \setminus f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$$

Proof. It suffices to show that

$$\overline{\langle \pi(U_\infty) w\rangle}^\perp = \int_{f^{-1}(0)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda).$$

One direction of containment is clear: for any $x \in \overline{\langle \pi(U_\infty) w\rangle}$, we see that $x(\lambda) = 0$ for almost all λ such that $f(\lambda) = 0$ (since $w(\lambda) \equiv f(\lambda) v_\lambda = 0$ if and only if $f(\lambda) = 0$). Hence, if $y \in \mathcal{H}$ such that $y|_{\Gamma \setminus f^{-1}(0)} = 0$, then $\langle x, y\rangle = \int_\Gamma \langle x(\lambda), y(\lambda)\rangle d\mu(\lambda) = 0$. In other words, $\int_{f^{-1}(0)}^\oplus \mathcal{H}_\lambda d\mu(\lambda) \subseteq \overline{\langle \pi(U_\infty) w\rangle}^\perp$.

Now suppose that $x \perp \overline{\langle \pi(U_\infty) w\rangle}$. In Lemma 6.3, we will show that $hw \in \overline{\langle \pi(U_\infty) w\rangle}$ for all $h \in L^\infty(\Gamma, \mu)$. We define $h \in L^\infty(\Gamma, \mu)$ by

$$h(\lambda) = \frac{\overline{\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle}}{|\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle|}.$$

Then

$$0 = \langle x, \pi(g) h w\rangle = \int_\Gamma |\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle| d\mu(\lambda).$$

for all g. Hence, for almost all λ, $\langle x(\lambda), \pi_\lambda(g) f(\lambda) v_\lambda\rangle = 0$ for all $g \in U_\infty$. It follows that, for almost all λ, either $x(\lambda) = 0$ or $f(\lambda) = 0$. Hence, $x(\lambda) = 0$ for almost all λ such that $f(\lambda) \neq 0$. In other words, $x \in \int_{f^{-1}(0)}^\oplus \mathcal{H}_\lambda d\mu(\lambda)$, and we are therefore done. ☐

We now prove the lemma that we used in the proof of Theorem 6.2:

Lemma 6.3. *Suppose that $f \in L^2(\Gamma, \mu)$ and put $w = \int_\Gamma^\oplus f(\lambda) v_\lambda d\mu(\lambda)$. Then $hw \in \overline{\langle \pi(U_\infty) w\rangle}$ for all $h \in L^\infty(\Gamma, \mu)$.*

Proof. We begin by showing that $1_{\Gamma^\mu} w \in \langle \pi(U_\infty) w\rangle$ for every node set Γ^μ. As before, we choose $c_1, \ldots, c_d \in \mathbb{C}$ and $g_1, \ldots, g_d \in U_\infty$ such that $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\mu = v_\mu$ and $\sum_{i=1}^k c_i \pi_\nu(g_i) v_\nu = 0$ for all $\nu \neq \mu$ in Γ_n. We claim that $1_{\Gamma^\mu} w = \sum_{i=1}^k c_i \pi_\mu(g_i) w$. If $f(\lambda) = 0$, then $w(\lambda) = 0$ and hence equality holds automatically. On the other hand, if $f(\lambda) \neq 0$, then recall that $\langle \pi(U_n) w\rangle$ is equivalent to $\pi_{\lambda|_{\mathfrak{a}_n}}$ by identifying $w(\lambda) = f(\lambda) v_\lambda$ with $v_{\lambda|_{\mathfrak{a}_n}}$. Hence $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\lambda = v_\mu$ if $\lambda|_{\mathfrak{a}_n} = \mu$ (i.e., if $\lambda \in \Gamma^\mu$) and $\sum_{i=1}^k c_i \pi_\mu(g_i) v_\lambda = 0$ otherwise. Thus $1_{\Gamma^\mu} w = \sum_{i=1}^k c_i \pi_\mu(g_i) w$ and so $1_{\Gamma^\mu} w \in \langle \pi(U_\infty) w\rangle$.

Next we see that $1_A w \in \overline{\langle \pi(U_\infty) w\rangle}$ for all open sets A in Γ. Every open set A can be written as a disjoint union $A = \bigcup_{i=1}^\infty N_i$ of node sets. Write $A_n = \bigcup_{i=1}^n N_i$ for each n and note that $1_{A_n} = \sum_{i=1}^k 1_{N_i}$ is in $\langle \pi(U_\infty) v\rangle$ by the previous paragraph. One then sees that

$$\int_\Gamma^\oplus 1_{A_n}(\lambda) f(\lambda) v_\lambda d\mu(\lambda) = 1_{A_n} w \to 1_A w = \int_\Gamma^\oplus 1_A(\lambda) f(\lambda) v_\lambda d\mu(\lambda)$$

in $\mathcal{H}$ since $1_{A_n} f \to 1_A f$ in $L^2(\Gamma, \mu)$. Thus $1_A v \in \overline{\langle \pi(U_\infty) v\rangle}$.

Next we show that $1_B v \in \overline{\langle \pi(U_\infty) v\rangle}$ for every Borel set B in Γ. This follows since

$$\begin{aligned}\mu(B) &= \inf\left\{ \mu\left(\bigcup_{i=1}^\infty F_i\right) \middle| B \subseteq \bigcup_{i=1}^\infty F_i \text{ and } F_i \in \mathfrak{F}\right\} \\ &= \inf\{\mu(A) | B \subseteq A \text{ and } A \text{ open}\}.\end{aligned}$$

Thus $1_B f$ can be approximated in $\overline{L^2(\Gamma,\mu)}$ by a sequence $1_{A_n} f$ given by open sets A_n, so that $1_{A_n} w \to 1_B w$ in $\mathcal{H}$. Hence $1_B w \in \overline{\langle \pi(U_\infty)w \rangle}$.

Finally, note that if $h_n \to h$ in $L^\infty(\Gamma,\mu)$, then $h_n f \to hf$ in $L^2(\Gamma,\mu)$ and hence $h_n w \to hw$ in $\mathcal{H}_\Gamma$. Because the measurable simple functions are dense in $L^\infty(\Gamma,\mu)$ (recall that μ is a finite measure), we see that $hw \in \overline{\langle \pi(U_\infty)w \rangle}$ for all $h \in L^\infty(\Gamma,\mu)$.

□

Finally, we show that every unitary conical representation of U_∞ disintegrates into a direct integral of highest-weight representations.

Theorem 6.4. *Suppose that $(\pi,\mathcal{H})$ is a unitary conical representation of U_∞ and $w \in \mathcal{H}\backslash\{0\}$ is a conical vector. Then there is a unique Borel measure μ on its highest-weight support $\Gamma(\pi)$ such that there is a unitary intertwining operator*

$$U : \mathcal{H} \to \int_{\Gamma(\pi)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$$

defined by $Uw = \int_{\Gamma(\pi)}^{\oplus} v_\lambda d\mu(\lambda)$.

Proof. Without loss of generality, suppose that $||w|| = 1$. We begin by constructing a suitable measure μ. For each λ in $\Gamma_n(\pi)$, define $\mu(\Gamma^\lambda) = ||w_\lambda||^2$. Observe that $w_\lambda = \sum_{\nu\in\Gamma_m^\lambda} w_\nu$ and hence

$$\mu(\Gamma^\lambda) = ||w_\lambda||^2 = \sum_{\nu\in\Gamma_m^\lambda(\pi)} ||w_\nu||^2 = \sum_{\nu\in\Gamma_m^\lambda(\pi)} \mu(\Gamma^\lambda).$$

Similarly,

$$\sum_{\nu\in\Gamma_n(\pi)} \mu(\Gamma^\nu) = \sum_{\nu\in\Gamma_n(\pi)} ||w_\nu||^2 = ||w||^2 = 1$$

Thus μ extends uniquely to a Borel measure on $\Gamma(\pi)$.

Consider the representation $(\widetilde{\pi},\widetilde{\mathcal{H}}) \equiv \left(\int_{\lambda\in\Gamma(\pi)}^{\oplus} \pi_\lambda d\mu(\lambda), \int_{\lambda\in\Gamma(\mathcal{H})}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)\right)$ and let $\widetilde{w} \equiv \int_{\Gamma(\pi)} v_\lambda d\mu(\lambda)$. Then $\widetilde{\pi}$ is conical with conical vector $\widetilde{w}$ and highest-weight support $\Gamma(\pi)$. We construct a unitary intertwining operator $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ such that $Uw = \widetilde{w}$.

By Theorem 5.6 (i), there is a U_∞-intertwining operator $L : \langle \pi(U_\infty)w \rangle \to \langle \widetilde{\pi}(U_\infty)\widetilde{w} \rangle$ given by $Lw = \widetilde{w}$. For each n and each $\nu \in \Gamma_n(\pi)$, L restricts to an intertwining operator between $\langle \pi(U_n)w_\nu \rangle$ and $\langle \widetilde{\pi}(U_n)\widetilde{w}_\nu \rangle$ such that $L(w_\nu) = \widetilde{w}_\nu$. Furthermore,

$$||\widetilde{w}_\nu||^2 = \int_{\Gamma^\nu} ||\widetilde{w}_\lambda||^2 d\mu(\lambda) = \int_{\Gamma^\nu} 1 d\mu(\lambda) = \mu(\Gamma^\nu) = ||w_\nu||^2.$$

Hence, L restricts to a unitary operator on $\langle \pi(U_n)w_\nu \rangle$ for every n and every $\nu \in \Gamma_n(\pi)$. Because $\langle \pi(U_\infty)w_\nu \rangle$ and $\langle \pi(U_\infty)\widetilde{w}_\nu \rangle$ are dense in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively, L extends to a unitary intertwining operator from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.

The uniqueness of the measure follows from standard results from direct-integral theory (see [9], for instance). □

A corollary of Theorems 6.1 and 6.4 is that unitary conical representations of U_∞ are multiplicity-free (and thus of type I).

Corollary 6.5. *Every unitary conical representation of U_∞ is multiplicity-free.*

Proof. Let $(\pi,\mathcal{H}) \equiv \left(\int_\Gamma^\oplus \pi_\lambda d\mu(\lambda), \int_\Gamma^\oplus \mathcal{H}_\lambda d\mu(\lambda)\right)$ be a conical representation and suppose that $L : \mathcal{H} \to \mathcal{H}$ is a U_∞-intertwining operator. Consider the conical vector $v = \int_\Gamma^\oplus v_\lambda d\mu(\lambda)$. Then

Lv is a conical vector for a subrepresentation of π and can thus be written $Lv = fv$ for some $f \in L^2(\Gamma, \mu)$. It follows that

$$L(\pi(g)v) = \pi(g)(fv) = \int_{\Gamma}^{\oplus} \pi(g) f(\lambda) v_\lambda d\mu(\lambda) = f\pi(g)v$$

for all $g \in U_\infty$ and hence $Ly = fy$ for all $y \in \mathcal{H}$. In other words, intertwining operators for π may be identified with multiplier operators, and thus the ring of intertwining operators for π is commutative. Hence π is multiplicity-free. □

As promised before, we now show that there are typically a very large number of inequivalent conical representations of U_∞ with a given highest-weight support Γ. By Theorem 6.1, this problem is equivalent to finding Borel measures with full support on Γ that are absolutely discontinuous with respect to each other.

One way to define a measure $\mu_{\rm rec}$ on Γ is as follows: we assign $\mu_{\rm rec}(\Gamma^\nu) = \frac{1}{\#\Gamma_1}$ for each ν in Γ_1. Then, for $\nu \in \Gamma_{n+1}$, recursively define $\mu_{\rm rec}(\Gamma^\nu) = \frac{1}{\#\Gamma^\lambda_{n+1}} \mu_{\rm rec}(\Gamma^\lambda)$ if $\lambda \in \Gamma_n$ and $\nu \in \Gamma^\lambda$. One can see quite easily that the atoms of $\mu_{\rm rec}$ are precisely the isolated points of the topological space Γ; all other singleton sets will have measure zero.

We now show that for any point x in Γ we can construct a Borel measure μ_x of full support on Γ whose atoms are precisely the isolated points of Γ and x. Thus, if $x \neq y$ are non-isolated points in Γ, then μ_x, μ_y, and $\mu_{\rm rec}$ lie in distinct measure classes since their null sets do not agree:

$$\begin{array}{ll} \mu_x(\{x\}) > 0, & \mu_x(\{y\}) = 0 \\ \mu_y(\{x\}) = 0, & \mu_y(\{y\}) > 0 \\ \mu_{\rm rec}(\{x\}) = 0, & \mu_{\rm rec}(\{y\}) = 0 \end{array}$$

There are many ways to construct μ_x given $x \in \Gamma$, but we shall use the following method, which involves a simple modification to the recursively uniform measure. For $\lambda \in \Gamma_1$, define $\mu_x(\Gamma^\lambda) = \frac{3}{4}$ if $x|_{\mathfrak{a}_n} = \lambda$ and $\mu_x(\Gamma^\lambda) = \left(\frac{1}{\#\Gamma_1 - 1}\right)\frac{1}{4}$ otherwise. Next suppose that $\mu_x(\Gamma^\nu)$ has been defined for all $\nu \in \Gamma_n$. For $\lambda \in \Gamma_{n+1}$, we define

$$\mu_x(\Gamma^\lambda) = \begin{cases} \frac{1}{2} + \frac{1}{2^{n+1}} & \text{if } x \in \Gamma^\lambda \\ \left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) \frac{1}{(\#\Gamma_n^{\lambda|_{\mathfrak{a}_n}}) - 1} & \text{if } x \notin \Gamma^\lambda \text{ and } x \in \Gamma^{\lambda|_{\mathfrak{a}_n}} \\ \frac{1}{\#\Gamma_n^{\lambda|_{\mathfrak{a}_n}}} \mu(\Gamma^{\lambda|_{\mathfrak{a}_n}}) & \text{otherwise,} \end{cases}$$

where, as before, $\Gamma^\nu_n = \{\gamma \in \Gamma_n | \ \gamma|_{\mathfrak{a}_n} = \nu\}$. We have thus recursively defined a countably additive Borel measure μ_x on Γ. Note that μ_x has full support on Γ because $\mu_x(\Gamma^\lambda) > 0$ for every open basis set $\Gamma^\lambda \subseteq \Gamma$. Furthermore, one can easily check that $\mu_x(\{x\}) = \frac{1}{2}$ and that $\mu_x(\{y\}) = 0$ if $y \neq x$ and y is not an isolated point of Γ.

7. Closing Remarks and Further Research

We have managed to prove several results for the unitary conical representations of U_∞, including the classification of unitary smooth conical representations, which generalize the finite-dimensional conical representations of finite-dimensional symmetric spaces. However, the question remains of whether it is possible to construct unitary conical representations of G_∞. The most likely approach would be to construct a sort of unitary spherical principal series representation, perhaps by a direct limit of unitary principal series representations. See also [48] for one approach to constructing an analogue of the principal series for direct-limit groups.

Several questions about harmonic analysis on the symmetric space G_∞/K_∞ and $G_\infty/M_\infty N_\infty$ remain. While neither of these infinite-dimensional spaces possess G_∞-invariant measures, there is a possibility of constructing G_∞-invariant measures on larger spaces. We briefly overview this construction now.

Consider a direct system $\{G_n\}_{n\in\mathbb{N}}$ of Lie groups and suppose that there are measurable (not necessarily continuous) projections $p_n : G_{n+1} \to G_n$ such that p_n is G_n-equivariant and $p_n(g) = g$ for $g \in G_n$. In other words, one has a projective system of σ-algebras dual to the direct system of groups. The resulting projective-limit space $\overline{G_\infty} = \varprojlim G_n$ is acted on by the direct-limit group $G_\infty = \varinjlim G_n$. Each group G_n possesses a G_n-quasi-invariant probability measure μ_n.

It is then possible to define a projective-limit probability measure $\mu_\infty = \varprojlim \mu_n$ on $\overline{G_\infty}$ using Kolmogorov's theorem. If this measure is quasi-invariant under the action of G_∞ on $\overline{G_\infty}$ then it is possible to define a unitary "regular representation" of G_∞ on $L^2(\overline{G_\infty}, \mu_\infty)$. This "regular representation" can then be decomposed into irreducible representations.

In fact, precisely this scheme was used by Doug Pickrell in [38] to study analysis on an infinite-dimensional Grassmannian space and later by Olshanski and Borodin in [4] to develop a theory of harmonic analysis on the infinite-dimensional unitary group $U(\infty)$. The role played by probability theory in the latter context was crucial. In fact, the problem was shown to be related to the study of infinite point processes. Most intriguingly, probabilistic models from statistical mechanics appeared in the decomposition.

It would be interesting to consider a similar analysis on the infinite-dimensional symmetric space G_∞/K_∞ and the horocycle space $G_\infty/M_\infty N_\infty$. That is, one would construct projective-limit spaces $\overline{G_\infty/K_\infty}$ and $\overline{G_\infty/M_\infty N_\infty}$ which possess G_∞-quasi-invariant measures. The problem, then, would be to decompose the corresponding unitary representations of G_∞ on $L^2(\overline{G_\infty/K_\infty})$ and $L^2(\overline{G_\infty/M_\infty N_\infty})$ into irreducible subrepresentations. One interesting question is whether those representations decompose into direct integrals of unitary spherical and conical representations of G_∞, respectively.

Also of interest is whether a sort of Radon transform may be constructed between functions on G_∞/K_∞ and functions on $G_\infty/M_\infty N_\infty$. In fact, for spaces of regular functions this has been done in the recent paper [19]. However, it would be interesting if it were possible to develop a Hilbert space analogue of the Radon transform, perhaps mapping between functions in $L^2(\overline{G_\infty/K_\infty})$ and functions in $L^2(\overline{G_\infty/M_\infty N_\infty})$.

8. Appendix:Admissiblility of Classical Direct Limits

The aim of this appendix is to show that each classical example is admissible. For the explicit matrix realizations of the compact-type Riemannian symmetric spaces, see [16, p. 446, 451–455].

The classical propagated direct systems of Riemannian symmetric spaces are listed in Table 8.1, where each row gives a noncompact-type symmetric space G_n/K_n and its simply-connected compact dual space U_n/K_n, and where the restricted roots exhibit the Dynkin diagram Ψ_n. For each row, the limit $G_\infty/K_\infty = \varinjlim G_n/K_n$ is propagated and also that it is possible to choose Cartan subalgebras of U_n for each $n \in \mathbb{N}$ so that $U_\infty = \varinjlim U_n$ is a propagated direct-limit group (see, for instance, [34, Section 2] or [45, Section 3]).

Note that in each row of Table 8.1, the symmetric space U_n/K_n is simply-connected. However, in certain rows the group U_n is not simply-connected. We may remove this obstruction simply by passing to the universal cover $\widetilde{U_n}$ of U_n. In fact, that the involution θ_n on $\mathfrak{u}_n$ integrates to an involution $\widetilde{\theta_n}$ on $\widetilde{U_n}$. Denote the fixed-point subgroup for $\widetilde{\theta_n}$ in $\widetilde{U_n}$ by $\widetilde{K_n}$. By simply-connectedness all of the inclusions on the Lie algebra level integrate to inclusions on the group level, so that $\widetilde{U_n}/\widetilde{K_n}$ forms a propagated direct system of compact-type symmetric spaces. Furthermore, one sees that if $p : \widetilde{U_n} \to U_n$ is the covering map, then $p\left(\widetilde{K_n}\right) \subseteq K_n$. Hence p factors to a covering map from $\widetilde{U_n}/\widetilde{K_n}$ to U_n/K_n (see [16, p. 213]). Since U_n/K_n is already simply-connected, we see that $\widetilde{U_n}/\widetilde{K_n}$ is diffeomorphic to U_n/K_n.

(8.1)

Classical direct systems of irreducible Riemannian symmetric spaces				
	G_n	U_n	K_n	Ψ_n
1	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SU}(n)\times\mathrm{SU}(n)$	$\mathrm{diag}\,\mathrm{SU}(n)$	A_{n-1}
2	$\mathrm{Spin}(2n+1,\mathbb{C})$	$\mathrm{Spin}(2n+1)\times$ $\mathrm{Spin}(2n+1)$	$\mathrm{diag}\,\mathrm{Spin}(2n+1)$	B_n
3	$\mathrm{Spin}(2n,\mathbb{C})$	$\mathrm{Spin}(2n)\times$ $\mathrm{Spin}(2n)$	$\mathrm{diag}\,\mathrm{Spin}(2n)$	D_n
4	$\mathrm{Sp}(n,\mathbb{C})$	$\mathrm{Sp}(n)\times\mathrm{Sp}(n)$	$\mathrm{diag}\,\mathrm{Sp}(n)$	C_n
5_1	$\mathrm{SU}(p,n-p)$	$\mathrm{SU}(n)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(n-p))$	C_p
5_2	$\mathrm{SU}(n,n)$	$\mathrm{SU}(2n)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))$	C_n
6_1	$\mathrm{SO}_0(p,n-p)$	$\mathrm{SO}(n)$	$\mathrm{SO}(p)\times\mathrm{SO}(n-p)$	B_p
6_2	$\mathrm{SO}_0(n,n)$	$\mathrm{SO}(2n)$	$\mathrm{SO}(n)\times\mathrm{SO}(n)$	B_n
7_1	$\mathrm{Sp}(p,n-p)$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(n-p)$	C_p
7_2	$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(2n)$	$\mathrm{Sp}(n)\times\mathrm{Sp}(n)$	C_n
8	$\mathrm{SL}(n,\mathbb{R})$	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	A_{n-1}
9	$\mathrm{SL}(n,\mathbb{H})$	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	A_{n-1}
10_1	$\mathrm{SO}^*(4n)$	$\mathrm{SO}(4n)$	$\mathrm{U}(2n)$	C_n
10_2	$\mathrm{SO}^*(2(2n+1))$	$\mathrm{SO}(2(2n+1))$	$\mathrm{U}(2n+1)$	C_n
11	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	C_n

8.1. **A General Strategy for Proving Admissibility.** The embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$A \mapsto \begin{pmatrix} I & & \\ & A & \\ & & I \end{pmatrix} \tag{8.2}$$

for the systems in rows 5_2, 6_2, and 7_2. In all other cases in Table 8.1, the embedding $G_n \hookrightarrow G_{n+1}$ takes the form

$$A \mapsto \begin{pmatrix} A & \\ & I \end{pmatrix}, \tag{8.3}$$

where I is a 1×1, 2×2, or 4×4 identity matrix.

Suppose we can choose $\mathfrak{a}_n$ for each n in such a way that

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} * & 0 & * \\ 0 & \mathfrak{a}_n & 0 \\ * & 0 & * \end{pmatrix} \tag{8.4}$$

or

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} \mathfrak{a}_n & 0 \\ 0 & * \end{pmatrix} \tag{8.5}$$

(depending on the type of embedding $G_n \hookrightarrow G_{n+1}$). In this case, since $\mathfrak{a}_n$ commutes with $M_n = Z_{K_n}(\mathfrak{a}_n)$ by definition, it follows from (8.4) and (8.5) that $\mathfrak{a}_{n+1}$ commutes with

$$M_n \cong \begin{pmatrix} I & 0 & 0 \\ 0 & M_n & 0 \\ 0 & 0 & I \end{pmatrix}$$

or

$$M_n \cong \begin{pmatrix} M_n & 0 \\ 0 & I \end{pmatrix},$$

respectively, depending on the type of embedding $G_n \hookrightarrow G_{n+1}$. In other words, $M_n \leq Z_{K_{n+1}}(\mathfrak{a}_{n+1}) = M_{n+1}$

Hence, in order to prove that a propagated direct limit is admissible, it is sufficient to show that either (8.4) or (8.5) holds. In most cases, our proof of admissibility will take this form.

8.2. $U_n = L_n \times L_n$ **and** $K_n = \text{diag } L_n$**.** This case corresponds to the first four rows in Table 8.1. In this case, one sees that

$$\begin{aligned} \mathfrak{u}_n &= \mathfrak{l}_n \times \mathfrak{l}_n \\ \mathfrak{k}_n &= \{(X, X) \in \mathfrak{u}_n | X \in \mathfrak{l}_n\} \\ i\mathfrak{p}_n &= \{(X, -X) \in \mathfrak{u}_n | X \in \mathfrak{l}_n\}. \end{aligned}$$

Furthermore, if we fix a Cartan subalgebra $\mathfrak{h}_n \subseteq \mathfrak{l}_n$ for each n, then we can choose

$$i\mathfrak{a}_n = \{(X, -X) \in \mathfrak{u}_n | X \in \mathfrak{h}_n\}.$$

Now suppose that $g \in L_n$ and that $(g, g) \in M_n = Z_{K_n}(\mathfrak{a}_n)$. Then $g \in Z_{L_n}(\mathfrak{h}_n)$; that is, g centralizes the Cartan subalgebra $\mathfrak{h}_n$ of $\mathfrak{l}_n$. Since K_n is connected, it follows that $g \in H_n \equiv \exp(\mathfrak{h}_n)$. Thus $M_n = \text{diag } H_n$ for each n. It follows that $M_k \leq M_n$ for $k \leq n$.

8.3. $\text{Rank}(G_\infty/K_\infty) \equiv \dim \mathfrak{a}_\infty < \infty$**.** This case was already discussed in [19] and corresponds to rows 5_1, 6_1, and 7_1 in Table 8.1. If $\dim \mathfrak{a}_\infty < \infty$, then for k large enough, one has $\mathfrak{a}_k = \mathfrak{a}_\infty$. Suppose $k \leq n$ and $g \in M_k$. That is, $g \in K_k$ and g centralizes $\mathfrak{a}_k$. But $\mathfrak{a}_k = \mathfrak{a}_n = \mathfrak{a}_\infty$ and $K_k \leq K_n$. Thus $g \in M_n$.

8.4. $\text{Rank}(G_n/K_n) = \text{Rank}(G_n)$ **for all** $n \in \mathbb{N}$**.** This case corresponds to rows 8 and 11 in Table 8.1. One has that $\mathfrak{a}_n$ is a Cartan subalgebra for $\mathfrak{g}_n$. In particular, $Z_{\mathfrak{g}_n}(\mathfrak{a}_n) = \mathfrak{a}_n$ and so $\mathfrak{m}_n = \{0\}$ for all $n \in \mathbb{N}$.

For example, if we let $G_n = \text{SL}(n, \mathbb{R})$ and $K_n = \text{SO}(n)$ and make the standard choice of $\mathfrak{a}_n = \{\text{diag}(a_1, \ldots, a_n) | a_i \in \mathbb{R}\}$, then one has $M_n = \{\text{diag}(\pm 1, \ldots, \pm 1)\}$. Thus $M_k \leq M_n$ for $k \leq n$.

8.5. $U_n/K_n = \text{SU}(2n)/S(\text{SU}(n) \times \text{SU}(n))$**.** This case corresponds to row 5_2 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{su}(n, n)$, $\mathfrak{u}_n = \mathfrak{su}(2n)$, and $\mathfrak{k}_n = \mathfrak{s}(\mathfrak{su}(n) \oplus \mathfrak{su}(n))$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\begin{aligned} \mathfrak{u}_n &= \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \text{M}(2n, \mathbb{C}) \middle| \begin{array}{l} A^* = -A,\, D^* = -D, \\ \text{and } \text{Tr}(A) + \text{Tr}(D) = 0 \end{array} \right\} \\ \mathfrak{k}_n &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \text{M}(2n, \mathbb{C}) \middle| \begin{array}{l} A^* = -A,\, D^* = -D, \\ \text{and } \text{Tr}(A) + \text{Tr}(D) = 0 \end{array} \right\} \\ i\mathfrak{p}_n &= \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in \text{M}(2n, \mathbb{C}) \right\}. \end{aligned}$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| a_i \in \mathbb{R} \right\}$$

Thus condition (8.4) is satisfied and so G_∞/K_∞ is admissible.

8.6. $U_n/K_n = \text{SO}(2n)/(\text{SO}(n) \times \text{SO}(n))$**.** This case corresponds to row 6_2 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{so}(n, n)$, $\mathfrak{u}_n = \mathfrak{so}(2n)$, and $\mathfrak{k}_n = \mathfrak{so}(n) \oplus \mathfrak{so}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(2n,\mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}(2n,\mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^{\mathrm{T}} & 0 \end{pmatrix} \in \mathrm{M}(2n,\mathbb{R}) \right\}.$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

Thus condition (8.5) is satisfied and so G_∞/K_∞ is admissible.

8.7. $U_n/K_n = \mathrm{Sp}(2n)/(\mathrm{Sp}(n) \times \mathrm{Sp}(n))$**.** This case corresponds to row 7_2 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{sp}(n,n)$, $\mathfrak{u}_n = \mathfrak{sp}(2n)$, and $\mathfrak{k}_n = \mathfrak{sp}(n) \oplus \mathfrak{sp}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where

$$J_n = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}.$$

More explicitly, one has

$$\mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \mathrm{M}(2n,\mathbb{H}) \middle| A^* = -A \text{ and } D^* = -D \right\}$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathrm{M}(2n,\mathbb{H}) \middle| A^* = -A \text{ and } D^* = -D \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} \in \mathrm{M}(2n,\mathbb{H}) \right\}.$$

We choose

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccc|ccc} & & & & & a_n \\ & & & & \iddots & \\ & & & a_1 & & \\ \hline & & -a_1 & & & \\ & \iddots & & & & \\ -a_n & & & & & \end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

Thus condition (8.4) is satisfied and so G_∞/K_∞ is admissible.

8.8. $U_n/K_n = \mathrm{SU}(2n)/\mathrm{Sp}(n)$**.** This case corresponds to row 9 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{u}_n = \mathfrak{su}(2n)$ and $\mathfrak{k}_n = \mathfrak{sp}(n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where J_n is given by

$$J_n = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{8.6}$$

One can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where

$$\widetilde{J}_n = \left(\begin{array}{ccc|ccc} & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right). \tag{8.7}$$

The calculations will be easier if we use $\widetilde{\theta}_n$ instead of θ_n. However, we must use θ_n in order for the inclusions $U_n \to U_{n+1}$ to take the form of (8.3). We can move freely between these pictures, however, because $J_n = E_{\sigma_n} J_n E_{\sigma_n}^{-1}$, where $E_{\sigma_n} \in M(2n, \mathbb{C})$ is the permutation matrix corresponding to the permutation

$$\sigma = (1\ n)(2\ (n+1)) \cdots ((n-1)\ 2n) \in S_{2n}.$$

In other words, the rows and columns are interwoven, so that the first n basis elements of $\mathbb{C}^{2n}$ are mapped to odd-numbered basis elements and the final n basis elements of $\mathbb{C}^{2n}$ are sent to even-numbered basis elements.

We proceed by using $\widetilde{\theta}_n$. We have

$$\begin{aligned} \mathfrak{su}(2n) = \mathfrak{u}_n &= \left\{ \begin{pmatrix} A & B \\ -B^* & D \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \,\middle|\, \begin{array}{l} A^* = -A,\ D^* = -D, \text{ and} \\ \mathrm{Tr}(A) + \mathrm{Tr}(D) = 0 \end{array} \right\} \\ \mathfrak{sp}(n) \cong \mathfrak{k}_n &= \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \,\middle|\, \begin{array}{l} A^* = -A \\ \text{and } B^{\mathrm{T}} = B \end{array} \right\} \\ i\mathfrak{p}_n &= \left\{ \begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \,\middle|\, \begin{array}{l} A^* = -A, B^{\mathrm{T}} = -B, \\ \text{and } \mathrm{Tr}(A) = 0 \end{array} \right\}. \end{aligned}$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{2n} \end{pmatrix} \,\middle|\, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{2n} a_i = 0 \right\}$$

for $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{ccc|ccc} ia_1 & & & & & \\ & \ddots & & & & \\ & & ia_n & & & \\ \hline & & & ia_1 & & \\ & & & & \ddots & \\ & & & & & ia_n \end{array}\right) \,\middle|\, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{n} a_i = 0 \right\}.$$

We now proceed to the θ_n picture. Conjugation of $\widetilde{\mathfrak{h}}_n$ by E_{σ_n} (followed by renumbering the indices) yields the θ_n-stable Cartan subalgebra

$$\mathfrak{h}_n = \widetilde{\mathfrak{h}}_n = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{2n} \end{pmatrix} \,\middle|\, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^{2n} a_i = 0 \right\}.$$

Finally, conjugation of $\widetilde{\mathfrak{a}}_n$ by E_{σ_n} yields

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{ccccccc} ia_1 & & & & & & \\ & ia_1 & & & & & \\ & & ia_2 & & & & \\ & & & ia_2 & & & \\ & & & & \ddots & & \\ & & & & & ia_n & \\ & & & & & & ia_n \end{array} \right) \middle| \, a_i \in \mathbb{R} \text{ and } \sum_{i=1}^n a_i = 0 \right\}.$$

While condition (8.5) is not quite satisfied, we do have that

$$\mathfrak{a}_{n+1} \subseteq \begin{pmatrix} \mathfrak{a}_n + \mathbb{C}\mathrm{Id} & 0 \\ 0 & * \end{pmatrix}. \tag{8.8}$$

Since $\mathfrak{m}_n$ centralizes $\mathfrak{a}_n$, it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_n + \mathbb{C}\mathrm{Id}$. Thus by (8.8), it follows that $\mathfrak{m}_n$ commutes with $\mathfrak{a}_{n+1}$. Thus $\mathfrak{m}_m \subseteq \mathfrak{m}_n$ for $m \leq n$, and it follows that G_∞/K_∞ is admissible.

8.9. $U_n/K_n = \mathrm{SO}(4n)/\mathrm{U}(2n)$. This case corresponds to row 10_1 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{so}^*(4n)$, $\mathfrak{u}_n = so(4n)$ and $\mathfrak{k}_n = \mathfrak{u}(2n)$. The involution is given by $\theta_n : A \mapsto J_n A J_n^{-1}$, where J_n is given by (8.6). As in the previous example, one can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where $\widetilde{J}_n$ is given by (8.7).

We work first on the $\widetilde{\theta}_n$-side. We have

$$\mathfrak{so}(4n) = \mathfrak{u}_n = \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(4n,\mathbb{R}) \middle| A^{\mathrm{T}} = -A \text{ and } D^{\mathrm{T}} = -D \right\}$$

$$\mathfrak{u}(2n) \cong \mathfrak{k}_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{M}(4n,\mathbb{R}) \middle| \begin{array}{l} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = B \end{array} \right\}$$

$$i\mathfrak{p}_n = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathrm{M}(4n,\mathbb{R}) \middle| \begin{array}{l} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = -B \end{array} \right\}.$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \left(\begin{array}{ccccccc} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & a_{2n} \\ & & & & & -a_{2n} & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}$$

and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{ccccc|ccccc} 0 & a_1 & & & & & & & & \\ -a_1 & 0 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 0 & a_n & & & & & \\ & & & -a_n & 0 & & & & & \\ \hline & & & & & 0 & -a_1 & & & \\ & & & & & a_1 & 0 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & 0 & -a_n \\ & & & & & & & & a_n & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}.$$

Moving to the θ_n-picture, we conjugate everything by E_{σ_n} and renumber the indices to arrive at the θ_n-stable Cartan algebra

$$\mathfrak{h}_n = \left\{ \left(\begin{matrix} \begin{matrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \\ -a_1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \\ -a_3 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 0 & a_{2n-1} & 0 \\ 0 & 0 & 0 & a_{2n} \\ -a_{2n-1} & 0 & 0 & 0 \\ 0 & -a_{2n} & 0 & 0 \end{matrix} \end{matrix} \right) \middle| \; a_i \in \mathbb{R} \right\},$$

and finally

$$i\mathfrak{a}_n = \left\{ \left(\begin{matrix} \begin{matrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & -a_1 \\ -a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & -a_2 \\ -a_2 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 0 & a_n & 0 \\ 0 & 0 & 0 & -a_n \\ -a_n & 0 & 0 & 0 \\ 0 & a_n & 0 & 0 \end{matrix} \end{matrix} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

Hence $\mathfrak{a}_n$ is block-diagonal, and moving from $\mathfrak{a}_n$ to $\mathfrak{a}_{n+1}$ is simply a matter of adding another 4×4 block. Thus we see that condition (8.5) is satisfied and hence G_∞/K_∞ is admissible.

8.10. $U_n/K_n = \mathrm{SO}(2(2n+1))/\mathrm{U}(2n+1)$**.** This case corresponds to row 10_2 in Table 8.1. One has $\mathfrak{g}_n = \mathfrak{so}^*(2(2n+1))$, $\mathfrak{u}_n = so(4n)$ and $\mathfrak{k}_n = \mathfrak{u}(2n)$. As in the previous example, one can also obtain the same symmetric space by using the involution $\widetilde{\theta}_n : A \mapsto \widetilde{J}_n A \widetilde{J}_n^{-1}$, where $\widetilde{J}_n$ is given by (8.7).

We first work on the $\widetilde{\theta}_n$ side. We then have

$$\begin{aligned} \mathfrak{so}(2(2n+1)) = \mathfrak{u}_n &= \left\{ \begin{pmatrix} A & B \\ -B^{\mathrm{T}} & D \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{matrix} A^{\mathrm{T}} = -A \\ \text{and } D^{\mathrm{T}} = -D \end{matrix} \right\} \\ \mathfrak{u}(2n+1) \cong \mathfrak{k}_n &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{matrix} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = B \end{matrix} \right\} \\ i\mathfrak{p}_n &= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathrm{M}(2(2n+1), \mathbb{R}) \middle| \begin{matrix} A^{\mathrm{T}} = -A \\ \text{and } B^{\mathrm{T}} = -B \end{matrix} \right\}. \end{aligned}$$

There is a $\widetilde{\theta}_n$-stable Cartan subalgebra

$$\widetilde{\mathfrak{h}}_n = \left\{ \left(\begin{array}{cccc|cccc} 0 & & & & a_1 & & & \\ & \begin{matrix} 0 & a_2 \\ -a_2 & 0 \end{matrix} & & & & & & \\ & & \ddots & & & & & \\ & & & \begin{matrix} 0 & a_{n+1} \\ -a_{n+1} & 0 \end{matrix} & & & & \\ \hline -a_1 & & & & 0 & & & \\ & & & & & \begin{matrix} 0 & a_{n+2} \\ -a_{n+2} & 0 \end{matrix} & & \\ & & & & & & \ddots & \\ & & & & & & & \begin{matrix} 0 & a_{2n+1} \\ -a_{2n+1} & 0 \end{matrix} \end{array} \right) \middle| \; a_i \in \mathbb{R} \right\}.$$

and we can choose

$$i\widetilde{\mathfrak{a}}_n = \left\{ \left(\begin{array}{ccccccc|ccccccc} 0 & & & & & & & & & & & & & \\ & 0 & a_1 & & & & & & & & & & & \\ & -a_1 & 0 & & & & & & & & & & & \\ & & & \ddots & & & & & & & & & & \\ & & & & & 0 & a_n & & & & & & & \\ & & & & & -a_n & 0 & & & & & & & \\ \hline & & & & & & & 0 & & & & & & \\ & & & & & & & & 0 & -a_1 & & & & \\ & & & & & & & & a_1 & 0 & & & & \\ & & & & & & & & & & \ddots & & & \\ & & & & & & & & & & & & 0 & -a_n \\ & & & & & & & & & & & & a_n & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}.$$

Moving to the θ_n-picture, we conjugate everything by E_{σ_n} and renumber the indices to arrive at the θ_n-stable Cartan algebra

$$\mathfrak{h}_n = \left\{ \left(\begin{array}{cccccccccccc} 0 & a_1 & & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & & \\ & & 0 & 0 & a_2 & 0 & & & & & & \\ & & 0 & 0 & 0 & a_3 & & & & & & \\ & & -a_2 & 0 & 0 & 0 & & & & & & \\ & & 0 & -a_3 & 0 & 0 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & & 0 & 0 & a_{2n-1} & 0 \\ & & & & & & & & 0 & 0 & 0 & a_{2n} \\ & & & & & & & & -a_{2n-1} & 0 & 0 & 0 \\ & & & & & & & & 0 & -a_{2n} & 0 & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\},$$

and finally

$$i\mathfrak{a}_n = \left\{ \left(\begin{array}{cccccccccccc} 0 & 0 & & & & & & & & & & \\ 0 & 0 & & & & & & & & & & \\ & & 0 & 0 & a_1 & 0 & & & & & & \\ & & 0 & 0 & 0 & -a_1 & & & & & & \\ & & -a_1 & 0 & 0 & 0 & & & & & & \\ & & 0 & a_1 & 0 & 0 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & & 0 & 0 & a_n & 0 \\ & & & & & & & & 0 & 0 & 0 & -a_n \\ & & & & & & & & -a_n & 0 & 0 & 0 \\ & & & & & & & & 0 & a_n & 0 & 0 \end{array} \right) \middle| \, a_i \in \mathbb{R} \right\}.$$

Hence $\mathfrak{a}_n$ is block-diagonal, and moving from $\mathfrak{a}_n$ to $\mathfrak{a}_{n+1}$ is simply a matter of adding another 4×4 block. Thus we see that condition (8.5) is satisfied and hence G_∞/K_∞ is admissible.

References

[1] D. Arnal, *Symmetric nonself-adjoint operators in an enveloping algebra.* J. Functional Analysis **21**, 432–447 (1975).

[2] D. Beltiţă, *Functional analytic background for a theory of infinite–dimensional Lie groups*, in "Developments and Trends in Infinite Dimensional Lie Theory." Progress in Math. **288**, Birkhäuser 2011, 367–392.

[3] D. Beltiţă, K-H. Neeb. *A nonsmooth continuous unitary representation of a Banach-Lie group.* J. Lie Theory **18** (2008), No. 4, 933–936.

[4] A. Borodin and G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes.* Ann. of Math. **161** (2005), No. 3, 1319–1422.

[5] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer 1985.

[6] A. Danilenko. *Grding domains for unitary representations of countable inductive limits of locally compact groups*, Mat. Fiz. Anal. Geom. **3** (1996) 231–260.

[7] M. Dawson, G. Ólafsson, and J. A. Wolf, *Direct systems of spherical functions and representations.* Journal of Lie Theory **23** (2013), No. 3, 711–729.

[8] G. van Dijk, *Introduction to Harmonic Analysis and Generalized Gelfand Pairs.* Walter de Gruyter, 2009.

[9] J. Dixmier, *C^*-Algebras.* North-Holland, 1977.

[10] I. Dmitrov, I. Penkov, and J. A. Wolf, *A Bott-Borel-Weil theory for direct limits of algebraic groups*, Amer. J. Math. **124** (2002), No. 5, 955–998.

[11] J.Faraut, *Infinite dimensional spherical analysis.* COE Lecture Notes Vol. 10 (2008), Kyushu University.

[12] H. Glöckner, *Direct limit Lie groupos and manifolds.* J. math. Kyoto Univ. **3** (2003), 1–26.

[13] R. Goodman and N. Wallach, *Symmetry, Representations, and Invariants.* Springer 2009.

[14] F. Greenleaf and M. Moskowitz, *Cyclic vectors for representations of locally compact groups*, Mathematische Annalen **190** (1971), 265–268.

[15] S. Helgason, *A duality for symmetric spaces with applications to group representations.* Adv. Math. **5** (1970), 1–154.

[16] ______, *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, 1978.

[17] ______, *Groups and Geometric Analysis.* Academic Press, 1984.

[18] ______, *Geometric Analysis on Symmetric Spaces*, Second Edition. AMS 2008.

[19] J. Hilgert, G. Ólafsson, *The Radon transform and its dual for limits of symmetric spaces*, {arXiv:1310.3668}.

[20] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group.* Invent. Math. *158* (2004), No. 3, 551–642

[21] A.W. Knapp, *Lie Groups: Beyond an Introduction,* Second Edition. Birkhäuser 2002.

[22] V. I. Kolomytsev, Yu. S. Samoilenko, *Irreducible representations of inductive limits of groups.* Ukrainian Mathematical Journal **29** (1977), No. 4, 402–405.

[23] R. Lipsman, *The Plancherel formula for the horocycle space and generalizations,* J. Australian Math. Soc. Series A **61** (1996), No. 1, 42–56.

[24] G. Mackey, *Harmonic analysis as the exploitation of symmetry,* Bull. Amer. Math. Soc. (N.S.) **3** (1980), No. 1, 543–698.

[25] ______ *The Theory of Unitary Group Representations*, U. Chicago 1976.

[26] L. Natarajan, E. Rodríguez-Carrington, and J. A. Wolf, *Differentiable structure for direct limit groups*, Lett. Mat. Physics **23** (1991), 99–109.

[27] ______, *Locally convex Lie groups.* Nova J. Algebra and Geometry, **2** (1993), 59–87.

[28] ______, *The Bott-Borel-Weil theorem for direct limit groups.* Trans. Amer. Math. Soc. **353** (2001), 4583–4622.

[29] K-H. Neeb. *On differentiable vectors for representations of innite dimensional Lie groups.* J. Funct. Anal. **259** (2010), No. 11, 2814–2855.

[30] ______, personal correspondence.

[31] G. Ólafsson and H. Schlichtkrull, *Representation theory, Radon transform and the heat equation on a Riemannian symmetric space,* in "Group Representations, Ergodic Theory, and Mathematical Physics: A Tribute to George W. Mackey." Contemporary Mathematics **449**, AMS 2008, 315–344.

[32] G. Ólafsson and J. A. Wolf, *Extension of symmetric spaces and restriction of Weyl groups and invariant polynomials.* New Developments in Lie Theory and Its Applications, Contemporary Mathematics **544** (2011), pp. 85–100.

[33] ______, *Separating vector bundle sections by invariant means.* arXiv:1210.5494

[34] ______, *The Paley-Wiener theorem and limits of symmetric spaces.* J. Geom. Anal. 24 (2014), No. 1, 1–31.

[35] G. Olshanski, *Unitary representations of the infinite-dimensional classical groups $U(p,\infty)$, $SO_0(p,\infty)$, $Sp(p,\infty)$ and the corresponding motion groups.* Funct. Anal. Appl. **12** (1978), No. 3, 185–195.

[36] ______, *Infinite-dimensional classical groups of finite R-rank: description of representations and asymptotic theory.* Functional Anal. Appl. **18** (1984), No. 1, 22–34.

[37] ______, *Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe.* In Representations of Lie Groups and Related Topics, Adv. Stud. Contemp. Math. **7**, Gordon and Breach 1990.

[38] D. Pickrell, *Measures on infinite-dimensional Grassmann manifolds.* J. Funct. Anal. **70** (1987), No. 2, 323–356.

[39] ______, *Decomposition of regular representations for $U(H)_\infty$.* Pacific J. Math. **128** (1987), No. 2, 319–332.

[40] ______, *The separable representations of $U(H)$.* Proc. Amer. Math. Soc. **102** (1988), No. 2, 416–420.

[41] F. Riesz and B. Sz.-Nagy, *Functional Analysis.* Ferderick Ungar Publishing Co., 1955.

[42] W. Rudin, *Functional Analysis*, Second Edition. McGraw-Hill 1991.

[43] Y. S. Samoilenko, *Spectral Theory of Families of Self-Adjoint Operators.* Springer 1991.

[44] M. Sugiura, *Representations of compact groups realized by spherical functions on symmetric spaces.* Proc. Japan Acad. **38** (1962) 111–113.

[45] J. A. Wolf, *Infinite dimensional multiplicity free spaces I: Limits of compact commutative spaces.* In "Developments and Trends in Infinite Dimensional Lie Theory", ed. K.-H. Neeb and A. Pianzola, Progress in Math. **288**, Birkhäuser 2011, pp. 459–481.

[46] ______, *Infinite dimensional multiplicity free spaces II: Limits of commutative nilmanifolds*, Contemp. Math. **491** (2009), pp. 179–208.

[47] ______, *Infinite dimensional multiplicity free spaces III: Matrix coefficients and regular function*, Mathematische Anallen **349** (2011), pp. 263–299.

[48] ———, *Principal series representations of infinite dimensional Lie groups, II: Construction of induced representations.* In "Geometric Analysis and Integral Geometry," Contemp. Math. **598**, AMS 2013.

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