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Composite Particles in a Separable Potential Model.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. SIMPLE SEPARABLE POTENTIAL MODEL</td>
<td>5</td>
</tr>
<tr>
<td>A. Model</td>
<td>5</td>
</tr>
<tr>
<td>B. Solution for the Light Boson</td>
<td>9</td>
</tr>
<tr>
<td>C. Transformation Formalism and the Heavy Boson</td>
<td>16</td>
</tr>
<tr>
<td>D. Discussion</td>
<td>19</td>
</tr>
<tr>
<td>III. PHYSICAL PROCESSES</td>
<td>23</td>
</tr>
<tr>
<td>A. Scattering of a Light Boson by n Heavy Bosons</td>
<td>23</td>
</tr>
<tr>
<td>B. Scattering of a Light Boson by a Composite Particle</td>
<td>28</td>
</tr>
<tr>
<td>C. Discussion</td>
<td>32</td>
</tr>
<tr>
<td>IV. SEPARABLE POTENTIAL MODEL WITH TWO KINDS OF LIGHT BosONS</td>
<td>35</td>
</tr>
<tr>
<td>A. Model and Solution</td>
<td>35</td>
</tr>
<tr>
<td>B. Physical States and Physical Processes</td>
<td>40</td>
</tr>
<tr>
<td>C. Discussion</td>
<td>44</td>
</tr>
<tr>
<td>V. SEPARABLE POTENTIAL MODEL WITH TWO KINDS OF STATIC HEAVY BosONS</td>
<td>45</td>
</tr>
<tr>
<td>A. Model and Special Solution</td>
<td>45</td>
</tr>
<tr>
<td>B. Physical States and Processes</td>
<td>54</td>
</tr>
<tr>
<td>C. Discussion</td>
<td>58</td>
</tr>
<tr>
<td>VI. CONCLUSION</td>
<td>60</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>Page</td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
</tr>
<tr>
<td>Appendix A. MISCELLANEOUS CALCULATIONS</td>
<td>65</td>
</tr>
<tr>
<td>Appendix B. TRANSFORMATION-THEORY FORMALISM FOR MODEL WITH TWO KINDS OF HEAVY BOSONS WITH EQUAL FORM FACTORS</td>
<td>73</td>
</tr>
<tr>
<td>VITA</td>
<td>76</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Fig. Page
1. Dispersion Graphs for $T_n(\omega(q))$ given in Eq. (111-4) 26
2. Dispersion Graphs for $T'_n(\omega(q))$ given in Eq. (111-21) 31
Composite particles (bound states) are investigated in three separable potential models--(1) the model with one kind of static heavy neutral scalar boson interacting with one kind of light neutral scalar boson, (2) the model with one kind of static heavy boson and two (or any number) kinds of light bosons, (3) the model with two (or any number) kinds of static heavy bosons and one kind of light boson. These models are solved exactly, the last one only in the case of equal interaction form factors. The S-matrix elements are calculated both directly and dispersion-theoretically. The allowed physical processes are found to be elastic scatterings only; the composite particles can neither be created, destroyed, nor altered in any way within the framework of the models studied. The S-matrix elements are shown to be independent of whether the scattering target is a number of heavy bosons or any composite particle which could be formed by the same number (and kinds) of heavy bosons. The composite particle states are identified as those eigenstates of the Hamiltonian for which the total number of heavy bosons is nonzero, the number of free heavy bosons is zero, and the total number of light bosons is greater than the number of free light bosons. The energy of such states is less than the rest energy of its components, i.e., the states are bound states.
Chapter I

INTRODUCTION

The concept of composite particle is less fundamental than that of elementary particle yet is as necessary in a comprehensive theory of matter. However, an understanding of the composite particle faces the same sort of difficulties as found in understanding elementary particles, the first difficulty being the lack of an acceptable definition of elementary particle. In one sense, though, we are in a better position regarding composite particles; at least, we know that certain objects — the deuteron, the hydrogen atom, etc. — are composite particles whereas we cannot be certain that other subjects — the pion, the lambda particle, etc. which are usually regarded as elementary particles — actually are elementary. This is a dilemma for which there are several different approaches to bypass; two of these are

1. the "democratic formalism" of Chew\(^1\) in which essentially no distinction is made between elementary and composite particles,

2. the approach, e.g. Vaughn, Aaron, and Amado\(^2\), in which the goal is the development of theoretical criteria for distinguishing elementary from composite particles (essentially without having to define either).

We object to the first of these, at least in the limit of the absurd, because the idea of taking, say, the electron and the iron-57 nucleus
on the same footing of elementarity is not particularly appealing. The second approach certainly has merit; the development of a theoretical understanding of the differences between elementary and composite particles would greatly aid in formulating a definition of elementarity of a particle.

In the following, we investigate the nature of composite particles which occur in a particular type of field-theoretic model, namely, the two-body separable potential model. We find out how the composite particle formalism develops in this model and how the composite particles are (or are not) manifested in the physical processes of this model. This type of model is chosen because it has bound states of several particles which satisfies the simplest, most intuitive idea of a composite particle.

The separable potential model is itself of interest because (1) it is one of the few exactly soluble field-theoretic models and thus (2) is useful for testing various formalisms and calculational methods. This model has been studied often, see Henley and Thirring\(^3\) and the early work cited therein. Of particular interest here is the recent work of Kazes\(^4\) in which the model is treated in the Lehmann-Symanzik-Zimmermann\(^5\) asymptotic condition formalism. Also of interest is the analysis of Ghirardi and Rimini\(^6\) of the properties of some generalized separable potentials. Somewhat surprisingly, since it is usually regarded as unphysical, the separable potential model is being applied to physical problems. Yamaguchi\(^7\) and Yamaguchi and Yamaguchi\(^8\) use this
model to describe nucleon-nucleon scattering at low energies. More recently, various authors\textsuperscript{9-13} have tackled the three-nucleon problem by means of two-body separable interactions. Perhaps auspicious for future applications to many-body theory is the fact that use of separable potentials "equivalent" to suitably chosen local potentials results in wavefunctions systematically smaller in the inner region of the potential than the wavefunctions of the local potentials.\textsuperscript{14,15} The foregoing is not intended as a complete survey of work in separable potential models but does show the measure of active interest in them.

The theory of the simple separable potential with one kind of light neutral scalar boson is given in Chapt.II. This discussion follows, more or less, that of Kazes\textsuperscript{4} but avoids some of Kazes' awkwardnesses, e.g. in the handling of the integrating factor $\epsilon$. The physical processes in this model are analyzed in Chapt.III; we make the pertinent calculations both straightforwardly by use of the commutation relations of Chapt.II and dispersion-theoretically thus showing the application of this calculation technique to the model. We study in Chapt.IV an extended model with two (and any number) different kinds of light bosons interacting with a single kind of static heavy boson and in Chapt.V a model with one kind of light boson interacting with two (and more) kinds of static heavy bosons, a complete solution being obtained for the special case of
identical form factors. Finally, the nature of the composite particle in the separable potential model is discussed in Chapt. VI.
Chapter 11

SIMPLE SEPARABLE POTENTIAL MODEL

A. Model

The simple separable potential model in which light neutral scalar bosons of mass $\mu$ interact with static heavy neutral scalar bosons of mass $M$ is described by the Hamiltonian (units $\hbar = 1 = c$)

$$H = M \phi^\dagger \phi + \int d\vec{k} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k}) + H_I$$

where the interaction part $H_I$ is

$$H_I = \lambda \phi^\dagger \phi F,$$

$F$ and $\omega(\vec{k})$ are defined as

$$F = \int d\vec{k} f(\vec{k}) a(\vec{k}),$$

$$\omega(\vec{k}) = (k^2 + \mu^2),$$

and $\lambda$ is the interaction coupling constant. The function $f(\vec{k})$ is the shape factor (in momentum space) of the potential and is necessarily real (time-reversal invariance); $f(\vec{k})$ is assumed to be square-integrable

$$\int d\vec{k} f^2(\vec{k}) < \infty$$

such that all integrals encountered in the following are well-defined.

The operators $a^\dagger(\vec{k})$ and $a(\vec{k})$ are creation and annihilation operators, respectively, for light bosons of momentum $\vec{k}$. Similarly, $\phi^\dagger$ and $\phi$ are creation and annihilation operators for heavy bosons. The commutation relations for these operators are

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}'), \quad (5a)$$

$$[a(\vec{k}), a(\vec{k}')] = 0, \quad (5b)$$
The Heisenberg operators \( a(\mathbf{k}, t) \) and \( \varphi(t) \) have equations of motion, obtained in the usual manner, as follows:

\[
\begin{align*}
-i \frac{d}{dt} a^+(\mathbf{k}, t) &= [H, a^+(\mathbf{k}, t)] = \omega(\mathbf{k}) a^+(\mathbf{k}, t) - j_a^+(\mathbf{k}, t), \\
-i \frac{d}{dt} \varphi^+(t) &= [H, \varphi^+(t)] = \mathcal{M} \varphi^+(t) - j_{\varphi}^+(t)
\end{align*}
\]

where

\[
\begin{align*}
j_a^+(\mathbf{k}) &= -\lambda \varphi^+ \varphi(\mathbf{k}) F^+ \\
j_{\varphi}^+ &= -\lambda \varphi^+ F^+ F
\end{align*}
\]

(adjoint operator expressions are given here for later convenience).

The equation for \( a^+(\mathbf{k}, t) \) is solved formally as follows: We define the retarded Green's function

\[
\Delta_R(t; \omega(\mathbf{k})) = i \Theta(t) e^{i \omega(\mathbf{k}) t}
\]

which is a solution of the equation

\[
[i \frac{d}{dt} + \omega(\mathbf{k})] \Delta_R(t; \omega(\mathbf{k})) = -\delta(t),
\]

then obtain in the usual way the Yang-Feldman equation for \( a^+(\mathbf{k}, t) \)

\[
a^+(\mathbf{k}, t) = a_{in}^+(\mathbf{k}, t) - \int dt' \Delta_R(t-t'; \omega(\mathbf{k})) j_a^+(\mathbf{k}, t')
\]

where the in-operator \( a_{in}^+(\mathbf{k}, t) \) is defined in the Lehmann-Symanzik-Zimmermann sense, i.e.
\[ \langle \psi | a_{\text{in}}^\dagger (\mathbf{k}, t) | \psi' \rangle = e^{i\omega(k)(t-\tau)} \langle \psi | a_{\text{in}}^\dagger (\mathbf{k}, \tau) | \psi' \rangle \]
\[ = \lim_{\tau \to -\infty} e^{i\omega(k)(t-\tau)} \langle \psi | a^\dagger (\mathbf{k}, \tau) | \psi' \rangle, \]  

(13)

\[ | \psi \rangle \text{ and } | \psi' \rangle \text{ being any two normalizable state vectors of the Hilbert space of physical states.} \]

The integral equation for \( a^\dagger (\mathbf{k}, t) \) in terms of the out-operator \( a_{\text{out}}^\dagger (\mathbf{k}, t) \) is

\[ a^\dagger (\mathbf{k}, t) = a_{\text{out}}^\dagger (\mathbf{k}, t) - \int dt' \Delta_A(t; t'; \omega(k)) j_a^\dagger(k, t') \]  

(14)

where the advanced Green's function is

\[ \Delta_A(t; \omega(k)) = -i \Theta(-t) e^{i\omega(k)t} \]  

(15)

and \( a_{\text{out}}^\dagger (\mathbf{k}, t) \) is defined by

\[ \langle \psi | a_{\text{out}}^\dagger (\mathbf{k}, t) | \psi' \rangle = \lim_{\tau \to -\infty} e^{i\omega(k)(t-\tau)} \langle \psi | a^\dagger (\mathbf{k}, t) | \psi' \rangle \]  

(16)

in the same manner as in the \( a_{\text{in}}^\dagger (\mathbf{k}, t) \) definition. Further, \( a_{\text{out}}^\dagger (\mathbf{k}, t) \) can be written as

\[ a_{\text{out}}^\dagger (\mathbf{k}, t) = a_{\text{in}}^\dagger (\mathbf{k}, t) + \int dt' \Delta(t; t'; \omega(k)) j_a^\dagger(k, t) \]  

(17)

where

\[ \Delta(t; \omega(k)) = \Delta_A(t; \omega(k)) - \Delta_R(t; \omega(k)) = -ie^{i\omega(k)t} \]  

(18)

Clearly, Eqs.(12), (14) and (17) must be understood in terms of matrix elements. We could now obtain formal solutions for \( a^\dagger (\mathbf{k}, t) \) by iteration of each of Eqs.(12) and (14); however, these solutions would not be especially useful, so are omitted. Instead, we take a somewhat different approach in Sec. II-B.
The Yang-Feldman equation for $\varphi^\dagger(t)$ is
\[ \varphi^\dagger(t) = \varphi_{\text{in}}^\dagger(t) - \int dt' \Delta_R(t-t';M)j^\dagger(t') \] (19)
where the retarded Green's function
\[ \Delta_R(t;M) = i\Theta(t)e^{iMt} \] (20)
is a solution of
\[ [i(d/dt) + M]\Delta_R(t;M) = -\delta(t) \]
Similarly, $\varphi^\dagger(t)$ is given in terms of the advanced Green's function
\[ \Delta_A(t;M) = -i\Theta(-t)e^{iMt} \] (21)
as
\[ \varphi^\dagger(t) = \varphi_{\text{out}}^\dagger(t) - \int dt' \Delta_A(t-t';M)j^\dagger(t') \] (22)
Again, the in- and out-operators $\varphi_{\text{in}}^\dagger(t)$ and $\varphi_{\text{out}}^\dagger(t)$ are defined in
the Lehmann-Symanzik-Zimmermann sense as
\[ \langle \Psi | \varphi_{\text{in}}^\dagger(t) | \Psi' \rangle = \lim_{T \to -\infty} e^{iM(t-T)} \langle \Psi | \varphi^\dagger(t) | \Psi' \rangle, \] (23)
\[ \langle \Psi | \varphi_{\text{out}}^\dagger(t) | \Psi' \rangle = \lim_{T \to +\infty} e^{iM(t-T)} \langle \Psi | \varphi^\dagger(t) | \Psi' \rangle, \] (24)
$| \Psi \rangle$ and $| \Psi' \rangle$ being any two vectors of the Hilbert space. These
operators are related as
\[ \varphi_{\text{out}}^\dagger(t) = \varphi_{\text{in}}^\dagger(t) + \int dt' \Delta(t-t';M)j^\dagger(t') \] (25)
where
\[ \Delta(t;M) = \Delta_A(t;M) - \Delta_R(t;M) = -ie^{iMt}. \] (26)
Here also we emphasize that Eqs. (19), (22) and (25) must be interpreted
in terms of matrix elements.
The formal solution of Eq. (19) as obtained by iteration is

\[ \phi^\dagger(t) = e^{i\mathbf{M}\cdot t} \phi^\dagger_{\text{in}} + (i\lambda) \int dt' \Theta(t-t') F^\dagger(t') F(t) e^{i\mathbf{M}\cdot t} \phi^\dagger_{\text{in}} \]

\[ + (i\lambda)^2 \int dt' dt'' \Theta(t-t') \Theta(t'-t'') \]

\[ \times F^\dagger(t') F(t') F(t'') F(t') e^{i\mathbf{M}\cdot V_{1n}} \]

\[ + \ldots \]

\[ = \{ \mathcal{P} \exp[i\lambda \int dt' \Theta(t-t') F^\dagger(t') F(t')] \} e^{i\mathbf{M}\cdot t} \phi^\dagger_{\text{in}} . \quad (27) \]

Here $\mathcal{P}$ is the Wick time-ordering operator. In terms of $\phi^\dagger_{\text{out}}$, $\phi^\dagger(t)$ is

\[ \phi^\dagger(t) = \{ \mathcal{P}' \exp[-i\lambda \int dt' \Theta(t'-t) F^\dagger(t') F(t')] \} e^{i\mathbf{M}\cdot t} \phi^\dagger_{\text{out}} \quad (28) \]

where $\mathcal{P}'$ is a Wick-type time-ordering operator which orders operators (in each term of the expansion of the exponential) with later times to the right (recall that $\mathcal{P}$ orders later times to the left). Although not in easily interpretable form, these formal solutions are adequate for present purposes. Finally, these expressions must also be interpreted in terms of matrix elements (this reminder is generally omitted hereafter).

B. Solution for the Light Boson

We write Eq. (12) as

\[ a^\dagger(\mathbf{k},t) = a^\dagger_{\text{in}}(\mathbf{k},t) + i\lambda \phi^\dagger \varphi f(\mathbf{k}) \int dt' \Theta(t-t') e^{i\omega(\mathbf{k})(t-t')} e^{i\mathbf{t'}^\dagger F^\dagger(t')} \]

\[ = a^\dagger_{\text{in}}(\mathbf{k},t) + \lambda \phi^\dagger \varphi f(\mathbf{k}) \int \frac{d\omega}{\omega} \frac{e^{i(\omega-i\epsilon)t}}{\omega - \omega(\mathbf{k}) - i\epsilon} \quad (29) \]

where the damping factor with $\epsilon$ a real, positive infinitesimal is introduced to define the integral and
\[ F^+(\omega) = \int \, d\vec{k} \, f(k) a^+(\vec{k}, \omega) . \]

\[ a^+(\vec{k}, \omega) = (2\pi)^{-1} \int \, dt \, e^{-i\omega t} a^+(\vec{k}, t) . \]

Since \( a^+_{in}(\vec{k}, t) = e^{i\omega(k)t} a^+_{in}(\vec{k}) \), we obtain an expression for \( a^+(\vec{k}, \omega) \) from Eq. (29) as

\[ a^+(\vec{k}, \omega) = a^+_{in}(\vec{k}) \delta(\omega - \omega(k)) + \lambda \phi^+ \phi f(k) F^+(\omega)/(\omega - \omega(k) - i\epsilon) \]

(30)

We then multiply the above by \( f(k) \), integrate over all \( k \)-space, and simplify to get

\[ D_-(\omega) F^+(\omega) = \int \, d\vec{k} \, f(k) \delta(\omega - \omega(k)) a^+_{in}(\vec{k}) \]

(31)

where

\[ D(z) = 1 + \lambda \phi^+ \phi \int \, d\vec{k} \, f^2(k)/(\omega(k) - z) , \]

(32a)

\[ D_{\pm}(\omega) = D(z = \omega \pm i\epsilon) . \]

(32b)

We use the definitions

\[ C^+(z) = \int \, dw \, F^+(\omega)/(\omega - z) . \]

(33a)

\[ C^+_\pm(\omega) = C^+(\omega = \omega \pm i\epsilon) , \]

(33b)

and Eq. (29) with \( t = 0 \)

\[ a^+(\vec{k}) = a^+_{in}(\vec{k}) + \lambda \phi^+ \phi f(k) C^+_+(\omega(k)) \]

(34)

to put Eq. (31) into the form

\[ D_-(\omega) F^+(\omega) = \int \, d\vec{k} \, f(k) \delta(\omega - \omega(k)) a^+(\vec{k}) \]

\[ - \lambda \phi^+ \phi \int \, d\vec{k} \, f^2(k) \delta(\omega - \omega(k)) C^+_+(\omega) . \]

(35)
The quantities \( D(z) \) and \( C^+(z) \) are functions, not operators (recall that practically everything herein must be interpreted in terms of matrix elements) of the complex variable \( z \). We then use the identity

\[
\mathcal{P} (\omega - \omega')^{-1} = (\omega - \omega' \pm i\epsilon)^{-1} \pm \pi i \delta(\omega - \omega')
\]

where \( \mathcal{P} \) indicates principal part, to obtain the relations

\[
D_+ (\omega) - D_- (\omega) = 2\pi i \lambda \phi \phi \int \frac{d\mathbf{k}}{2\pi} \hat{f}(\mathbf{k}) \delta(\omega - \omega(\mathbf{k})) ,
\]

\[
C^+_+ (\omega) - C^+_-(\omega) = 2\pi i F^+_+ (\omega)
\]

and in turn obtain

\[
D_+ (\omega) C^+_+ (\omega) - D_- (\omega) C^+_-(\omega) = 2\pi i \int \frac{d\mathbf{k}}{2\pi} f(\mathbf{k}) \delta(\omega - \omega(\mathbf{k})) a^+ (\mathbf{k})
\]

from Eq.(35). This is the discontinuity across the cut from \( \mu \) to infinity along the real axis of the function \( D(z)C^+(z) \). Since \( D(z \to \infty) \to 1 \) and \( C^+(z \to \infty) \to F^+/z \), \( D(z)C^+(z) \) is

\[
D(z)C^+(z) = \int \frac{d\mathbf{k}}{2\pi} f(\mathbf{k}) a^+ (\mathbf{k})/(\omega(\mathbf{k}) - \omega) ,
\]

possible contributions continuous across the cut being neglected.

We now can express \( a^+ (\mathbf{k}) \) by means of Eq.(37) as

\[
a^+ (\mathbf{k}) = a^+_{\text{in}} (\mathbf{k}) + \lambda \phi \phi f(\mathbf{k}) D^+_+ (\omega(\mathbf{k})) \int \frac{d\mathbf{k}' f(\mathbf{k}') a^+ (\mathbf{k}')}{\omega(\mathbf{k}') - \omega(\mathbf{k}) - i\epsilon},
\]

a form considerably more tractable than Eq.(34). In a similar way, \( a^+ (\mathbf{k}) \) can be obtained in terms of \( a^+_{\text{out}} (\mathbf{k}) \) as

\[
a^+ (\mathbf{k}) = a^+_{\text{out}} (\mathbf{k}) + \lambda \phi \phi f(\mathbf{k}) D^-_+ (\omega(\mathbf{k})) \int \frac{d\mathbf{k}' f(\mathbf{k}') a^+ (\mathbf{k}')}{\omega(\mathbf{k}') - \omega(\mathbf{k}) + i\epsilon}
\]

(39)
To relate $a_{\text{in}}(\vec{k})$ and $a_{\text{out}}(\vec{k})$, we make use of the expressions for

\[ C_+^\dagger(\omega) - C_-^\dagger(\omega) = 2\pi i D_0^{-1}(\omega) \int d\vec{k} f(\vec{k}) \delta(\omega - \omega(\vec{k})) a_{\text{in}}(\vec{k}), \]

\[ = 2\pi i D_0^{-1}(\omega) \int d\vec{k} f(\vec{k}) \delta(\omega - \omega(\vec{k})) a_{\text{out}}^\dagger(\vec{k}) \]

(see Appendix A for the derivations) to get

\[ a_{\text{out}}(\vec{k}) = a_{\text{in}}(\vec{k}) + 2\pi i \lambda \phi \phi f(k) D_0^{-1}(\omega(\vec{k})) \int d\vec{k}' f(\vec{k}') \delta(\omega(\vec{k}) - \omega(\vec{k}')) a_{\text{in}}(\vec{k}'), \]

\[ a_{\text{in}}(\vec{k}) = a_{\text{out}}(\vec{k}) - 2\pi i \lambda \phi \phi f(k) D_0^{-1}(\omega(\vec{k})) \int d\vec{k}' f(\vec{k}') \delta(\omega(\vec{k}) - \omega(\vec{k}')) a_{\text{out}}^\dagger(\vec{k}'). \]

Straightforward but not necessarily simple calculations yield the following commutation relations:

\[ [a_{\text{in}}(\vec{k}), a_{\text{in}}^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') - [a_{\text{out}}(\vec{k}), a_{\text{out}}^\dagger(\vec{k}')] \]

\[ [a_{\text{in}}(\vec{k}), a_{\text{in}}(\vec{k}')] = 0 = [a_{\text{out}}(\vec{k}), a_{\text{out}}(\vec{k}')] \]

\[ [a_{\text{in}}(\vec{k}), a_{\text{in}}^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') + \lambda \phi \phi f(k) f(k') D_0^{-1}(\omega(\vec{k})) \frac{\omega(k) - \omega(k')}{i \epsilon} \]

\[ [a_{\text{out}}(\vec{k}), a_{\text{out}}^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') + \lambda \phi \phi f(k) f(k') D_0^{-1}(\omega(\vec{k})) \frac{\omega(k) - \omega(k')}{i \epsilon} \]

\[ [a_{\text{in}}(\vec{k}), a(\vec{k}')] = 0 = [a_{\text{out}}(\vec{k}), a(\vec{k}')] \]

\[ [a_{\text{out}}(\vec{k}), a_{\text{in}}^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') - 2\pi i \lambda \phi \phi f(k') D_0^{-1}(\omega(\vec{k})) \delta(\omega(\vec{k}) - \omega(\vec{k}')) \]

\[ [a_{\text{out}}(\vec{k}), a_{\text{in}}(\vec{k}')] = 0. \]
\begin{align}
[a_{\text{in}}(k), \varphi^+] &= -\lambda \varphi^+ f(k) D^{-1}_-(\omega(k)) \int \frac{dk' f(k') a(k')}{\omega(k') - \omega(k) - i\epsilon}, \\
[a_{\text{out}}(k), \varphi^+] &= -\lambda \varphi^+ f(k) D_-^{-1}(\omega(k)) \int \frac{dk' f(k') a(k')}{\omega(k') - \omega(k) - i\epsilon}, \\
[a_{\text{in}}^+(k), \varphi^+] &= -\lambda \varphi^+ f(k) D_+^{-1}(\omega(k)) \int \frac{dk' f(k') a^+(k')}{\omega(k') - \omega(k) - i\epsilon}, \\
[a_{\text{out}}^+(k), \varphi^+] &= -\lambda \varphi^+ f(k) D_+^{-1}(\omega(k)) \int \frac{dk' f(k') a^+(k')}{\omega(k') - \omega(k) + i\epsilon}.
\end{align}

As an illustration of these calculations, Eq.\,(42a) is derived in Appendix A.

In these commutation relations, $D^{-1}(z)$ is an operator. To define it, we first note that $\varphi^+ \varphi$ commutes with the Hamiltonian. Consequently, the physical states of the system can be taken as simultaneous eigen­ 
states of $H$ and $\varphi^+ \varphi$. We must choose these states such that the eigen­ 
values of $\varphi^+ \varphi$ are integers; not surprisingly, we will interpret $\varphi^+ \varphi$ as 
the total number operator for heavy bosons. We now define the operator 
$D(z)$ as 
\begin{equation}
D(z) = \sum D_n(z) P_n
\end{equation}
where $P_n$ is the $n$ heavy boson projection operator and the function 
$D_n(z)$ is 
\begin{equation}
D_n(z) = 1 + \lambda n \int dk f^2(k)/(\omega(k) - z).
\end{equation}

For $\lambda < 0$, $D_n(z)$ has one and only one zero on the real axis below $\mu$ 
provided $n > [\lambda \int dk f^2(k)/(\omega(k) - \mu)]^{-1}$. Denoting this zero by $\mu_n$, we see 
from a Taylor series expansion of $D_n(z)$ about $\mu_n$ that 
\begin{equation}
D^{-1}_n(z - \mu_n) \rightarrow r_n/(z - \mu_n).
\end{equation}
and also
\[ \int d\mathbf{k} r^2(k)/(\omega(k) - \mu_n) = -1/\lambda n, \quad (45) \]
\[ \int d\mathbf{k} r^2(k)/(\omega(k) - \mu_n)^2 = 1/\lambda n r_n. \quad (46) \]

Using the expression for \( D_+(\omega) - D_-(\omega) \) given by Eq.(36a), we write
\[ D_{n-1}^{+1}(\omega) - D_{n-1}^{-1}(\omega) = -2\pi i\lambda n \Theta(\omega - \mu) \int d\mathbf{k} \left| D_{n+1}^{-1}(\omega(k)) \right|^2 r^2(k) \delta(\omega - \omega(k)). \quad (47) \]

\( D_{n-1}^{+1}(z) \) is then
\[ D_{n-1}^{+1}(z) = 1 - \lambda n \int d\mathbf{k} \left| D_{n+1}^{-1}(\omega(k)) \right|^2 r^2(k)/(\omega(k) - z) - r_n/(\mu_n - z). \quad (48) \]

Finally, the operator \( D_{n-1}^{+1}(z) \) is
\[ D_{n-1}^{+1}(z) = \Sigma D_{n-1}^{+1}(z) P_n. \quad (49) \]

By virtue of \( D_n(z) \) being a Wigner R-function\(^1\), i.e., the sign of \( \text{Im} D_n(z) \) being the same as the sign of \( \text{Im} z \), the integral
\[ \int d\mathbf{k} \left| D_{n+1}^{-1}(\omega(k)) \right|^2 r^2(k)/(\omega(k) - z) \]

is convergent.

Although neither particularly useful nor necessary, the following relations are included for the sake of completeness:
\[ a_{\mathbf{k}}^\dagger(\mathbf{k},t) = a_{\mathbf{k}}^\dagger(\mathbf{k},t) - \lambda \varphi^\dagger \phi f(k) \sum P_n A_n^\dagger e^{i\mu n_t}/(\omega(k) - \mu_n) \]
\[ - \lambda \varphi^\dagger \phi f(k) \int d\mathbf{k}' D_{n+1}^{-1}(\omega(k')) f(k') a_{\mathbf{k}}^\dagger(\mathbf{k}) e^{i(\omega(k') - 1\epsilon)t}/\omega(k') - \omega(k) - 1\epsilon, \quad (50a) \]
\[
a^+(\vec{k}, t) = a^+_{\text{out}}(\vec{k}, t) - \lambda \varphi^+ \varphi f(k) \sum \int_{r^+} A^+_n e^{i\mu_n t/(\omega(k) - \mu_n)}
- \lambda \varphi^+ \varphi f(k) \int dR^+D^+\Sigma^+(\omega(k'))f(k')a^+_{\text{out}}(k')e^{i(\omega(k') + i\epsilon)t},
\]

where \(A^+_n\) is defined as
\[
A^+_n = \int d\vec{k} f(k) a^+(\vec{k})/(\omega(k) - \mu_n). \tag{51}
\]

These are obtained from Eq.(29) by means of the expression for \(F^+(\omega)\) in Eq.(36b) and the complete expressions for \(C^+_\omega - C^-\omega\) given in Eq.(A-2).

The operators \(A^+_n\) relate to the composite particle states and are discussed further in Sec.II-D. Their commutation relations are calculated readily; they are
\[
\begin{align*}
[A_n, A^+_n] &= 1/\lambda n \varphi, \tag{52a} \\
[A_n, A^+_n] &= 0, \tag{52b} \\
[A_n, \varphi^+] &= 0 = [A_n, \varphi], \tag{52c} \\
[a^+_{\text{in}}(\vec{k}), A^+_n] &= 0 = [a_{\text{out}}(\vec{k}), A^+_n], \tag{52d} \\
[a^+_{\text{in}}(\vec{k}), A_n] &= 0 = [a_{\text{out}}(\vec{k}), A_n], \tag{52e} \\
[a(\vec{k}), A^+_n] &= f(k)/(\omega(k) - \mu_n), \tag{52f} \\
[a(\vec{k}), A_n] &= 0. \tag{52g}
\end{align*}
\]
Equation (52d) is derived in Appendix A as an illustration of these calculations.
C. Transformation Formalism and the Heavy Boson

The relevant commutation relations of $\varphi_{\text{in}}$ and $\varphi_{\text{out}}$ are required for later use. To obtain these from the formal solutions Eqs.(27) and (28) would be difficult, if not impossible. Instead, we take an indirect approach; we first cast the light boson results in a transformation-theory framework, then use this formalism for treating the heavy boson.

The transformation-theory formalism given here closely follows the work of Kazes. We write by comparison with Eqs.(38) and (39)

\[ a_{\text{in}}^+(k) = a_{\text{in}}^+(k')(1 + \zeta_{\text{in}}^+_{k')k}, \quad (53a) \]
\[ a_{\text{out}}^+(k) = a_{\text{out}}^+(k')(1 + \zeta_{\text{out}}^+_{k')k}, \quad (53b) \]

where repeated labels on the right signify integration and

\[ (\zeta_{\text{in}}^+_{k')k} = -\lambda \varphi_{\text{in}}^+\varphi_{\text{in}}(k)f(k')D^+(\omega(k))/\omega(k') - \omega(k) - i\epsilon), \quad (54a) \]
\[ (\zeta_{\text{out}}^+_{k')k} = -\lambda \varphi_{\text{out}}^+\varphi_{\text{out}}(k)f(k')D^+(\omega(k))/\omega(k') - \omega(k) + i\epsilon) \quad (54b) \]

(our $\zeta_{\text{in}}^+$ is the same as Kazes' $\alpha$). For brevity, we continue only the in-operator case, noting that

\[ (\zeta_{\text{in}}^+_{k')k} = (\zeta_{\text{out}}^+_{k')k}). \quad (54c) \]

We use the commutation relations Eq.(42) and calculate directly

\[ [(1 + \zeta_{\text{in}}^+) (1 + \zeta_{\text{in}}^+)]_{k')k} = \delta(k' - k'), \quad (55a) \]
\[ [(1 + \zeta_{\text{in}}^+) (1 + \zeta_{\text{in}}^+)]_{k')k} = \delta(k' - k') - \lambda f(k)f(k')\Sigma n_{p}p_{n}/(\omega(k) - \omega_{n})(\omega(k') - \omega_{n}). \quad (55b) \]

The derivation of Eq.(55b) is given in Appendix A. Clearly $(1 + \zeta_{\text{in}}^+)$ is
an isometric matrix whenever the second term on the right-hand side of Eq. (55b) is nonzero.

We next define an operator $V_{in}^\dagger$ (identical to Kazes' $U$) such that

$$V_{in}^\dagger a^\dagger(\mathbf{k}) = a^\dagger_{in}(\mathbf{k})V_{in}^\dagger$$

(56)

from which follows with the help of Eq. (53a)

$$[V_{in}^\dagger, a^\dagger(\mathbf{k})] = a^\dagger(\mathbf{k}')(\zeta_{in}^{\dagger})_{k'k}V_{in}^{\dagger}$$

(57)

The $V_{in}^\dagger$ operator is constructed to satisfy the above equation as

$$V_{in}^\dagger = 1 + a^\dagger(\mathbf{k})a(\mathbf{k}')(\zeta_{in}^{\dagger})_{kk'}^{\dagger}$$

(58a)

$$+ (2!)^{-1} a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}')(\mathbf{k}')a(\mathbf{k}'')(\zeta_{in}^{\dagger})_{kk'}^{\dagger}(\zeta_{in}^{\dagger})_{k'k}^{\dagger} + ...$$

The out-operator $V_{out}^\dagger$ is obtained in the same way and is given by Eq. (58a) altered such that "in" labels are replaced everywhere by "out". The $V_{in}$ operator

$$V_{in} = 1 + a^\dagger(\mathbf{k})a(\mathbf{k}')(\zeta_{in}^{\dagger})_{kk'}^{\dagger}$$

(58b)

$$+ (2!)^{-1} a^\dagger(\mathbf{k})a^\dagger(\mathbf{k}')(\mathbf{k}')a(\mathbf{k}'')(\zeta_{in}^{\dagger})_{kk'}^{\dagger}(\zeta_{in}^{\dagger})_{k'k}^{\dagger} + ...$$

satisfies

$$[V_{in}, a^\dagger(\mathbf{k})] = a^\dagger(\mathbf{k}')(\zeta_{in}^{\dagger})_{k'k}V_{in}^{\dagger}$$

(59)

so that

$$[V_{in}^\dagger V_{in}, a^\dagger(\mathbf{k})] = 0 ,$$

(60a)

$$[V_{in}^\dagger V_{in}, a^\dagger(\mathbf{k})] = - \lambda f(k) \sum \frac{nnr \rho_n A_n^\dagger V_{in} V_{in}}{\omega(k) - \mu_n}$$

(60b)
The derivation of Eqs. (60a,b) is given in Appendix A. We see from the definitions and Eq. (55a) that

\[ V_{in}\dagger V_{in} = 1. \quad (61a) \]

\[ V_{in}\dagger V_{in} = 1 - \lambda \sum \Sigma r_n r_n A_A^\dagger A_A + (2!)^{-1} \lambda^2 \sum \Sigma r_n r_n r_n A_A^\dagger A_A A_A^\dagger A_A + \ldots. \quad (61b) \]

Similar relations hold for "in" labels replaced by "out". We note that \( V_{in} \) and \( V_{out} \) are not unitary for \( \lambda < 0 \); however, for \( \lambda > 0 \), all the terms except the first on the right-hand side of Eq. (61b) vanish and thereby make \( V_{in} \) and \( V_{out} \) unitary.

We add two useful relations

\[ V_{in} a_{in}^\dagger (k) = a_{in}^\dagger (k) V_{in}. \quad (62) \]

\[ [p_{in} A_A^\dagger V_{in}] = -V_{in}^\dagger p_{in} A_A^\dagger \quad (63) \]

(Eq. (63) is derived in Appendix A).

We can now tackle the heavy boson problem. We write by analogy to Eq. (56)

\[ \phi_{in}^\dagger V_{in} = \phi_{in}^\dagger V_{in}. \quad (64) \]

Kazes\(^4\) obtains this relation in a direct but somewhat lengthy derivation which we need not reproduce here. The properties of the \( V \)-operators allow the above to be written as

\[ \phi_{in}^\dagger = V_{in}^\dagger \phi_{in}^\dagger V_{in}. \quad (65) \]

We then obtain by direct calculation

\[ [\phi_{in}, \phi_{in}^\dagger] = V_{in}^\dagger V_{in}. \quad (66a) \]
The out-operator expressions corresponding to Eqs. (65) and (66) are obtained by replacing "in" everywhere by "out". Finally, we find

\[
[\phi_{\text{out}}, \phi_{\text{in}}^\dagger] = V_{\text{out}} V_{\text{out}} P \left[ \exp[\lambda \int_0^t dt F^\dagger(t) F(t)] \right]
\]  

by use of the formal solution for \( \phi^\dagger(t) \) given by Eq. (27) in the out-state limit.

The foregoing gives sufficient information about the heavy boson for later calculations. We remark however, that for \( \lambda < 0 \) the commutator \([\phi_{\text{in}}, \phi_{\text{in}}^\dagger] \) is an operator. This is considered further in the next section.

D. Discussion

With all the necessary tools in hand, we now analyze the simple separable potential model. We first note that (derivation in Appendix A)

\[
\int \Delta k \omega(k) a_{\text{in}}^\dagger(k) a(k)
\]

\[
= \int \Delta k \omega(k) a_{\text{in}}^\dagger(k) a_{\text{in}}(k) - \lambda \phi^\dagger F^\dagger F + \lambda \sum n r_n \mu_n P_n A_n^\dagger A_n ;
\]

hence, the Hamiltonian can be written as

\[
H = M \phi^\dagger \phi + \int \Delta k \omega(k) a_{\text{in}}^\dagger(k) a_{\text{in}}(k) + \lambda \sum n r_n \mu_n P_n A_n^\dagger A_n .
\]
We recall that we must use only representations in which $\phi^+ \phi$ is the total number operator for heavy bosons. Inspection of both expressions for $H$ shows that the operators $a^{\dagger}(\vec{k}) a(\vec{k}) d_k a^{\dagger}_{in}(\vec{k}) a_{in}(\vec{k})$, $\phi^+_{in} \phi_{in}$, and $\Sigma P_n A_{xn}$ are all constants of the motion, i.e., all commute with $H$. These should have some interpretation as physical observables, most likely in terms of number operators of some kind. Also, since we know this model should have composite particle states, we want to find out how such states appear.

We first consider the state

$$|n_M; n_1(\vec{k}_1), n_2(\vec{k}_2), ..., n_\ell(\vec{k}_\ell); \text{in}\rangle = \frac{1}{(n_1! n_2! ... n_\ell !)^{1/2}} a^{\dagger}_{in}(\vec{k}_1) a_{in}(\vec{k}_1) n_1 a^{\dagger}_{in}(\vec{k}_2) a_{in}(\vec{k}_2) n_2 ... a^{\dagger}_{in}(\vec{k}_\ell) a_{in}(\vec{k}_\ell) n_\ell |\Omega\rangle$$

(70)

where $|\Omega\rangle$ is the vacuum state and the $n$'s have positive integer or zero values. This state is an eigenstate of $H$ with eigenvalue $n_M + n_1 \omega(\vec{k}_1) + n_2 \omega(\vec{k}_2) + ... + n_\ell \omega(\vec{k}_\ell)$; of $\int d\vec{k} a^{\dagger}(\vec{k}) a(\vec{k})$ with eigenvalue $n_1 + n_2 + ... + n_\ell$; of $\phi^+_{in}\phi_{in}$ with eigenvalue $n_1 + n_2 + ... + n_\ell$; of $\phi^+_{in}\phi_{in}$ with eigenvalue $n_1 + n_2 + ... + n_\ell$; of $\phi^+_{in}\phi_{in}$ with eigenvalue $n$. The operator $\Sigma P_n A_{xn}$ acting on this state gives zero identically. We next consider the state

$$|(n_M, m); n_1(\vec{k}_1), n_2(\vec{k}_2), ..., n_\ell(\vec{k}_\ell); \text{in}\rangle = \left(\lambda_n r_n\right)^{m/2} (n_1! n_2! ... n_\ell !)^{-1/2} P_n (A_{xn})^m a^{\dagger}_{in}(\vec{k}_1) a_{in}(\vec{k}_1) n_1 a^{\dagger}_{in}(\vec{k}_2) a_{in}(\vec{k}_2) n_2 ... a^{\dagger}_{in}(\vec{k}_\ell) a_{in}(\vec{k}_\ell) n_\ell |\Omega\rangle$$

(71)

where the $n$'s and the $m$ have positive integer or zero values. This state
is an eigenstate of $H$ with eigenvalue $nM + m\mu_n + \sum n_i \omega(k_i) + \sum n_k \omega(k_k)$; of $\int d\vec{k} a_\uparrow(\vec{k}) a(\vec{k})$ with eigenvalue $m + \sum n_i + \sum n_k$; of $\int d\vec{k} a_{\downarrow n}(\vec{k}) a_{\uparrow n}(\vec{k})$ with eigenvalue $\sum n_i + \sum n_k$; of $\phi^\dagger \phi$ with eigenvalue $n$; of $\lambda \sum_n P_n A_n^\dagger A_n$ with eigenvalue $m$. The operator $\phi^\dagger \phi_{\downarrow n}$ acting on this state gives zero identically by virtue of Eq.(66f), i.e., since

$$\phi_{\downarrow n} P_n A_n^\dagger = 0.$$ 

All the above eigenvalues are calculated easily by means of the various commutation relations given in the preceding sections.

The foregoing allows the following interpretation:

1. $|nM; n_1(k_1), n_2(k_2), \ldots, n_k(k_k); \downarrow n\rangle$ is a state with $n$ free heavy bosons, with $n_i$ free light bosons each of energy $\omega(k_i)$ and momentum $\vec{k}_i$, etc.;

2. $|(nM, m\mu); n_1(\vec{k}_1), n_2(\vec{k}_2), \ldots, n_k(\vec{k}_k); \downarrow n\rangle$ is a state with $m$ light bosons bound to $n$ heavy bosons having a bound-state energy $nM + m\mu_n$, with $n_i$ free light bosons each of energy $\omega(k_i)$ and momentum $\vec{k}_i$, etc.;

3. $\phi^\dagger \phi$ is the number operator which tells the total number of heavy bosons, either bound or free;

4. $\phi_{\downarrow n}^\dagger \phi_{\downarrow n}$ is the number operator which tells the number of free heavy bosons only;

5. $\int d\vec{k} a_\uparrow(\vec{k}) a(\vec{k})$ is the number operator which tells the total number of light bosons, both bound and free;

6. $\int d\vec{k} a_{\downarrow n}(\vec{k}) a_{\uparrow n}(\vec{k})$ is the number operator which tells the number of free light bosons only;
7. $\lambda \sum_n n^2 a_n^+ a_n$ is the number operator which tells the number of bound light bosons only;

8. $a_{in}^+(\vec{k})$ and $a_{in}^-(\vec{k})$ are the creation and annihilation operators, respectively, for a free light boson of energy $\omega(k)$ and momentum $\vec{k}$;

9. $\phi_{in}^+$ and $\phi_{in}^-$ are the creation and annihilation operators, respectively, for a static free heavy boson of mass $M$ but only when acting on the vacuum state or states with no composite particles;

10. $\phi_{in}^+$ and $\phi_{in}^-$ give zero identically when acting on states with a composite particle;

11. $\sum_n n^2 p_n a_n^+ a_n$ and $\sum_n n^2 p_n a_n^+ a_n$ are the creation and annihilation operators, respectively, for a bound light boson (except when acting on the vacuum state where there is nothing to bind the light boson).

Statement (10) is verified by inspection from Eqs. (66e) and (66f). This statement is particularly significant; it says that states with more than one composite particle or states with both free heavy bosons and a composite particle are not allowed in the simple separable potential model.

How composite particles arise in this model is now evident. Their presence is attributable to the poles of $D^{-1}(z)$ on the real axis $\mu$ for $\lambda < 0$. If $\lambda > 0$, $D^{-1}(z)$ has no such poles; hence no composite particles exist in this case. Further, the commutator $[\phi_{in}^+, \phi_{in}^-]$ being operator-valued (for $\lambda < 0$) is due to the presence of composite particles. This is reflected in the properties of $\phi_{in}^+$ and $\phi_{in}^-$ given in statement (10).

Finally, corresponding results for outgoing states (and out-operators) are obtained by replacing 'in' labels everywhere in this section by 'out' labels.
Chapter III

PHYSICAL PROCESSES

A. Scattering of a Light Boson by n Heavy Bosons

The S-matrix element for the scattering of a light boson of momentum \( \vec{q} \) and energy \( \omega(q) \) by \( n \) static heavy bosons is

\[
\langle nM, \vec{k}; \text{out} | nM, \vec{q}; \text{in} \rangle = \langle nM; \text{out} | [a_{\text{out}}(\vec{k}), a_{\text{in}}^\dagger(\vec{q})] | nM; \text{in} \rangle \\
= \delta(\vec{k} - \vec{q}) + 2 \pi i \lambda n \delta(\omega(\vec{k}) - \omega(\vec{q})) f^2(q) \\
\times \left[ -1 + \lambda n \int \frac{dp}{\omega(p) - \omega(q) - i\epsilon} \right] \\
\times \left[ \int [D^{-1}_n(\omega(p))] \frac{f^2(p)}{\omega(p)} \right] + \frac{r_n}{(\mu_n - \omega(q))} \\
(1)
\]

as follows immediately from the expressions for \([a_{\text{out}}(\vec{k}), a_{\text{in}}^\dagger(\vec{k}')]\), Eq.(11-42f), and \(D^{-1}_n(z)\), Eq.(11-48), and the identity

\[
\langle nM; \text{out} | n'M; \text{in} \rangle = \langle nM; \text{in} | n'M; \text{in} \rangle = \delta_{nn'} . \\
(2)
\]

This identity is proven in Appendix A.

The dispersion-theoretic approach to the calculation of this S-matrix element is formulated as follows: We first write

\[
\langle nM, \vec{k}; \text{out} | nM, \vec{q}; \text{in} \rangle = \langle nM; \text{out} | a_{\text{out}}(\vec{k}) | nM, \vec{q}; \text{in} \rangle \\
= \lim_{T \to \infty} \langle nM; \text{out} | a(\vec{k}, t)e^{i\omega(k)t} | nM, \vec{q}; \text{in} \rangle \\
= \lim_{T \to \infty} \langle nM; \text{out} | a(\vec{k}, t)e^{i\omega(k)t} | nM, \vec{q}; \text{in} \rangle \\
= \int dt e^{i\omega(k)t} [-i \frac{d}{dt} + \omega(k)] \langle nM; \text{out} | a(\vec{k}, t) | nM, \vec{q}; \text{in} \rangle
\]
\[
\langle nM, \vec{k}; \text{out}|nM, \vec{q}; \text{in}\rangle = \langle nM; \text{out}|a_{\text{in}}(\vec{k})a_{\text{in}}^+(\vec{q})|nM; \text{in}\rangle \\
+ i \int \! dt \, e^{i\omega(k)t} \langle nM; \text{out}|j_a(k,t)|nM, \vec{q}; \text{in}\rangle
\]
\[
= \delta(\vec{k}-\vec{q}) + 2\pi i \delta(\omega(k) - \omega(q)) T_n(\omega(q))
\]
where \(T_n(\omega(q))\) is defined as
\[
T_n(\omega(q)) = \langle nM|j_a(q)a_{\text{in}}^+(\vec{q})|nM\rangle.
\]

We omit "in" and "out" labels on the heavy boson states here and hereafter since the distinction between the two has no significance for the matrix elements concerned (see the derivation of Eq.(2) in Appendix A). In a similar manner, we calculate \(T_n(\omega(q))\) as
\[
T_n(\omega(q)) = \lim_{\tau \to -\infty} \langle nM|j_a(q)a_{\text{in}}^+(\vec{q},t) e^{-i\omega(q)t}|nM\rangle
\]
\[
= i \int \! dt \, e^{-i\omega(q)t} \left( i \frac{d}{dt} + \omega(q) \right) \Theta(-t) \langle nM|j_a(q)a_{\text{in}}^+(\vec{q},t)|nM\rangle
\]
\[
= -\lambda nf^2(q) + i \int \! dt \Theta(-t) e^{-i\omega(q)t} \langle nM|[j_a(q),j_a^+(q,t)]|nM\rangle.
\]
Equation (5) defines a function analytic in the complex \(\omega(q)\)-plane cut along the real axis from \(\mu\) to \(\infty\) and satisfying the dispersion relation
\[
T_n(\omega(q)) = \frac{-\lambda nf^2(q) + \pi i \int_{\mu}^{\infty} d\omega(q') \Im T_n(\omega(q'))}{2\mu \frac{\omega(q') - \omega(q) - i\epsilon}{\omega(q') - \omega(q) - i\epsilon}}
\]
where

\[ \text{Im} T_n(\omega(q)) = \pi \lambda_r \langle nM| j_a(q)|S \rangle \langle S| j_a^\dagger(q)|nM \rangle \delta(E_s - nM - \omega(q)) \]  

and \( |S\rangle \) is the (intermediate) state with energy \( E_s \). Note that the completeness of asymptotic states is assumed. Equation (7) is obtained by inserting a complete set of states in Eq. (5), doing the time integration, and taking the imaginary part of the resultant expression for \( T_n(\omega(q)) \).

The only states for which \( \langle nM| j_a(q)|s \rangle \) does not vanish forthwith are

\[ \langle nM| j_a(q)|s \rangle = \langle nM| f_n^2(q) \delta(\omega(q) - \mu_n^a) \]  

\[ + 4\pi^2 \Theta(\omega(q) - \mu_n^a)(\omega^2(q) - \mu_n^a)^{1/2} \omega(q) |T_n(\omega(q))|^2 \]  

(8)

To solve the dispersion relation Eq. (6), we define the function

\[ H_n(\omega(q)) = \lambda_r f_n^2(q) [(\mu_n^a - \omega(q)) T_n(\omega(q))]^{-1} \]  

(9)

which is analytic in the whole complex \( \omega(q) \)-plane except for a cut along the real axis from \( \mu_n^a \) to \( \infty \). \( H_n(\omega(q)) \) has no poles in the finite portion of the complex \( \omega(q) \)-plane since

\[ H_n(\omega(q) = \mu_n^a) = 1 ; \]  

(10)

thus \( H_n(\omega(q)) \) satisfies the once-subtracted dispersion relation

\[ H_n(\omega(q)) = 1 + \pi^{-1}(\mu_n^a - \omega(q)) \int_{\omega(q')}^{\infty} d\omega(q') \frac{\text{Im} H_n(\omega(q'))}{\mu_n^a - \omega(q') - i\epsilon} \]  

\[ \times \mu_n^a - \omega(q') \]  

(11)
Fig. 1. Dispersion graphs for $T_n(\omega(q))$ given in Eq. (III-4).
From Eqs. (8) and (9), we get

\[
\text{Im} \, H_n(\omega(q)) = \frac{\lambda n r_n f^2(q) \text{Im} T_n(\omega(q))}{(\mu_n - \omega(q)) |T_n(\omega(q))|^2}
\]

\[
= -4\pi^2 \lambda n r_n f^2(q) \Theta(\omega(q) - \mu)(\omega^2(q) - \mu^2)^{1/2} \omega(q)/(\mu_n - \omega(q))
\]

(12)

and, in turn, find

\[
H_n(\omega(q)) = 1 - \lambda n r_n (\mu_n - \omega(q))^2 \int dq' f^2(q')/(\omega(q') - \omega(q) - i\epsilon)(\mu_n - \omega(q'))^2
\]

(13)

\[H_n(\omega(q))\] has the properties

\[
\lim_{\omega(q) \to \infty} H_n(\omega(q)) = 1 - \lambda n r_n \int dq' f^2(q')/(\omega(q') - \mu_n)^2 \to 0. \quad (14a)
\]

\[
\lim_{\omega(q) \to \infty} [r_n^{-1}(\omega(q) - \mu_n)H_n(\omega(q))] \to D_n^{+}(\omega(q)) \to 1. \quad (14b)
\]

The above expression for \(H_n(\omega(q))\) allows \(T_n(\omega(q))\) to be written as

\[
T_n(\omega(q)) = \frac{\lambda n r_n f^2(q)}{\mu_n - \omega(q)} \left[1 - \lambda n r_n (\mu_n - \omega(q))\right]
\]

\[
\times \int dq' f^2(q')/(\omega(q') - \omega(q) - i\epsilon)(\mu_n - \omega(q'))^2 \right]^{-1}.
\]

(15)

Also we see from the limit properties of \(H_n(\omega(q))\) that

\[
\lim_{\omega(q) \to \infty} T_n(\omega(q)) \to -\lambda nf^2(q).
\]

(16)
The result for the S-matrix element, Eq. (3), is

\[ \langle nM, \vec{k}; \text{out} | nM, \vec{q}; \text{in} \rangle = \delta(\vec{k} - \vec{q}) + 2\pi i \delta(\omega(k) - \omega(q)) \lambda n r n \frac{f^2(q)}{(\mu_n - \omega(q))} \times \left[ 1 - \lambda n r n (\mu_n - \omega(q)) \int \frac{dq' f^2(q')}{(\omega(q') - \omega(q) - i\epsilon)(\mu_n - \omega(q'))} \right]^{-1}. \]

(17)

The remaining problem is to show that the two forms, Eqs. (1) and (17), are identical. This can be done by means of Eqs. (11.44), (11.45), and (11.48) and the partial-fraction expansion

\[ \frac{(\omega(q) - \mu_n)^2}{(\omega(q') - \omega(q) - i\epsilon)(\omega(q') - \mu_n)^2} = \frac{1}{\omega(q') - \omega(q) - i\epsilon} - \frac{1}{\omega(q') - \mu_n} - \frac{1}{(\omega(q') - \omega(q) - i\epsilon)(\omega(q') - \mu_n)^2}. \]

This calculation is lengthy but not difficult, hence is omitted.

B. Scattering of a Light Boson by a Composite Particle.

We now consider the scattering of a light boson of momentum \( \vec{q} \) and energy \( \omega(q) \) by a composite particle consisting of \( m \) light bosons bound to \( n \) heavy bosons. The composite particle is static and has an energy \( nM + m\mu_n \). The S-matrix element is
\[ \langle (nM, ma), k; \text{out} | (nM, ma), q; \text{in} \rangle \]

\[ = (\lambda n r_n)^m (m!)^{-1} \langle nM; \text{out} | a_{\text{out}}(\vec{k}) a_{\text{in}}^\dagger(q) (A_n)^m (A_n^\dagger)^m | nM; \text{in} \rangle \quad (18) \]

since \( A_n^\dagger \) and \( A_n \) commute with \( a_{\text{out}}(\vec{k}) \) and \( a_{\text{in}}^\dagger(\vec{k}) \). We use the \([A_n, A_n^\dagger]\)

expression in Eq. (11.52a) and the fact that

\[ P_n A_n | nM; \text{in} \rangle = 0 \]

to show directly that

\[ \langle (nM, ma), \bar{k}; \text{out} | (nM, ma), \bar{q}; \text{in} \rangle = \langle nM, \bar{k}; \text{out} | nM, \bar{q}; \text{in} \rangle . \quad (19) \]

This matrix element can also be obtained in a dispersion-theoretic calculation. Proceeding in detailed analogy to the previous calculation, we get

\[ \langle (nM, ma), \bar{k}; \text{out} | (nM, ma), \bar{q}; \text{in} \rangle \]

\[ = \delta(\bar{k} - \bar{q}) + 2\pi i \delta(\omega(k) - \omega(q)) T_n^\prime(\omega(q)) \quad (20) \]

where \( T_n^\prime(\omega(q)) \) is defined as

\[ T_n^\prime(\omega(q)) = \langle (nM, ma) | j_a(q) a_{\text{in}}^\dagger(\bar{q}) | (nM, ma) \rangle . \quad (21) \]

(We have necessarily been somewhat cavalier in omitting the "in" and "out" labels here). The development for \( T_n^\prime(\omega(q)) \) is essentially identical to that for \( T_n(\omega(q)) \) except that
\[ \text{Im} T'_n(\omega(q)) \]
\[ = \sum_s \langle (nM, ma) | j^+_a(q) | S \rangle \langle S | j^+_a(q) | (nM, ma) \rangle \delta(E_s - nM - \mu_n + \omega(q)) \]
\[ - \sum_s \langle (nM, ma) | j^+_a(q) | S' \rangle \langle S' | j^+_a(q) | (nM, ma) \rangle \delta(E_s, - nM - \mu_n + \omega(q)) \]
\[ = \sum_{s'} \langle (nM, ma) | j^+_a(q) | S' \rangle \langle S' | j^+_a(q) | (nM, ma) \rangle \delta(E_s - nM - \mu_n + \omega(q)) \]
\[ \text{(22)} \]

looks considerably different. The only intermediate states which contribute in the first sum above are

\[ |(nM, (m+1)a)\rangle, \]
\[ a^+_i(n|\bar{a}) (nM, ma) \rangle ; \]

in the second sum, only the state

\[ |(nM, (m-1)a)\rangle \]

contributes, see Fig. 2. Direct calculation then gives

\[ \text{Im} T'_n(\omega(q)) = \pi \lambda r_n \delta(\omega(q) - \mu_n) \]
\[ + 4\pi^2 (\omega^2(q) - \mu^2)^{1 \over 2} \omega(q) \| T'_n(\omega(q)) \|^2 \Theta(\omega(q) - \mu) \]
\[ \text{(23)} \]

which is exactly the same relationship as for \( \text{Im} T_n(\omega(q)) \), Eq.(8).

Clearly, since \( T'_n(\omega(q)) \) satisfies the same relations as \( T_n(\omega(q)) \), we have

\[ T'_n(\omega(q)) = T_n(\omega(q)). \]
\[ \text{(24)} \]

Consequently, the dispersion-theoretic calculation for

\[ \langle (nM, ma), \bar{k}; \text{out} | (nM, ma), \bar{q}; \text{in} \rangle \]

agrees with the direct calculation, Eq.(19).
Fig. 2. Dispersion graphs for $T_n'(\omega(q))$ given in Eq. (III-21).
C. Discussion

Since the total number of light bosons, the number of free light bosons, the number of bound light bosons, etc. are all separately conserved, reactions such as

\[ a + (nM,ma) \rightarrow a + a + (nM,(m-1)a), \quad (25a) \]

\[ a + a + (nM,ma) \rightarrow a + (nM,(m+1)a) \quad (25b) \]

are not allowed; that is, bombardment of a given composite particle by light bosons cannot produce a final state with a composite particle different than the composite particle in the initial state. To emphasize this point, we show by direct calculation that the S-matrix element for process Eq.(25a) vanishes

\[ \langle (nM,(m-1)a);k,k';\text{out}|(nM,ma),q;\text{in} \rangle \]

\[ \propto \langle nM;\text{out}|p_n^{(A_n)^{m-1}}(A_n^\dagger)^m a_{\text{out}}^{(k)}a_{\text{out}}^{(k')})a_{\text{in}}^{(q)}|nM;\text{in} \rangle = 0. \]

Although composite particles can exist, they cannot be altered in any way, in particular, can neither be created or destroyed. Thus, the only physical processes allowed in the simple separable potential model are elastic scatterings.

The results in Sec.III-A and III-B show that the elastic scattering of a light boson by \( n \) heavy bosons is identical to the elastic scattering by a composite particle of any number of light bosons bound to \( n \) heavy bosons; the significance is: Scattering
experiments cannot give any information which would permit \( n \) free heavy bosons to be distinguished from a composite particle with \( n \) heavy bosons binding any number of light bosons.

As lagniappe, we furnish scattering phase shifts. We define the \((S\text{-wave})\) scattering phase shift \( \delta_n(\omega(q)) \) for a target of \( n \) free heavy bosons by the relation

\[
e^{2i\delta_n(\omega(q))} = \int d\vec{k} \langle n\Phi, k; \text{out} | n\Phi, \vec{q}; \text{in} \rangle.
\]

Clearly, this is also the scattering phase shift for a target of any composite particle with \( n \) heavy bosons. Equation (3) allows this to be expressed as

\[
e^{2i\delta_n(\omega(q))} = 1 + 2\pi i \int d\vec{k} \delta(\omega(k) - \omega(q)) T_n(\omega(q))
\]

from which follows after integration

\[
e^{i\delta_n(\omega(q))} \sin \delta_n(\omega(q)) = 4\pi^2 \Theta(\omega(q) - \mu)(\omega^2(q) - \mu^2)^{1/2} \omega(q) T_n(\omega(q)).
\]

Since

\[
\tan \delta_n(\omega(q)) = \frac{\text{Im} T_n(\omega(q))}{\text{Re} T_n(\omega(q))},
\]

we get immediately

\[
\delta_n(\omega(q)) = \tan^{-1} \left\{ \frac{\pi \lambda n f^2(q) |D_{n\pm}^{-1}(\omega(q))|^2}{\lambda n \Theta \int d\vec{k} \frac{D_{n\pm}^{-1}(\omega(k)) 2\tau^2(k)}{\omega(k) - \omega(q)} + \frac{r_n}{\mu_n - \omega(q)} - 1} \right\}
\]

\[
(27)
\]
where the expression already obtained for $T_n(\omega(q))$ is used. Further, since

$$\delta_n(\omega(q)) = (2i)^{-1}\ln[\frac{T_n(\omega(q))}{T_n^*(\omega(q))}] ,$$

the limit for $T_n(\omega(q))$ given in Eq.(16) shows that

$$\lim_{\omega(q) \to \infty} \delta_n(\omega(q)) \to 0 .$$

(28)
Chapter IV
SEPARABLE POTENTIAL MODEL WITH
TWO KINDS OF LIGHT BOSONS

A. Model and Solution

We extend the model studied in Chap. 11 to describe two
different kinds of light neutral scalar boson, the first kind with
mass $\mu_1$ and the second kind with mass $\mu_2$, which interact via a
separable potential with a single kind of static heavy neutral scalar
boson with mass $M$. The Hamiltonian is

$$H = M \phi^\dagger \phi + \int \mathrm{d}k [\omega_1(k)a_1^\dagger(k)a_1(k) + \omega_2(k)a_2^\dagger(k)a_2(k)] + H_1$$

$$H_1 = \lambda_1 \phi^\dagger \phi F_1^\dagger F_1 + \lambda_2 \phi^\dagger \phi F_2^\dagger F_2$$

where for $i = 1, 2$

$$F_i = \int \mathrm{d}k f_i(k) a_i(k)$$

$$\omega_i(k) = (\k^2 + \mu_i^2)^{\frac{1}{2}}$$

$\lambda_i$ is the interaction coupling constant and $f_i(k)$ is the momentum-
space shape factor of the potential for the first kind of light boson.
The operators $a_i^\dagger(k)$ and $a_i(k)$ are creation and annihilation operators,
respectively, for light bosons of the first kind with momentum $k$. The
quantities with subscript 2 have the corresponding significance for
light bosons of the second kind. Hereafter, light bosons of the first
and second kinds are referred to as $a_1$-bosons and $a_2$-bosons, respectively. The $\varphi$ and $\varphi^\dagger$ operators have the same meaning as in Sec.II-A. We note that $f_1(k)$ is real and has the property
\[
\int d\kappa f_i^2(k) < 0 ; \quad i = 1, 2.
\]

The above operators obey the commutation relations
\[
[a_i(\vec{k}), a_j^\dagger(\vec{k}')] = \delta_{ij} \delta(\vec{k}-\vec{k}'),
\]
\[
[a_i(\vec{k}), a_j(\vec{k}')] = 0,
\]
\[
[a_i(\vec{k}), \varphi] = 0 = [a_i(\vec{k}), \varphi^\dagger],
\]
the commutation relations of $\varphi$, $\varphi^\dagger$ being given by Eqs.(II-5c,d).

The equations of motion for the Heisenberg operators are
\[
[i(d/dt) + \omega_i(k)] a_i^\dagger(\vec{k},t) = j_{ai}^\dagger(k,t),
\]
\[
[i(d/dt) + M] \varphi^\dagger(t) = J_\varphi^\dagger(t)
\]
where
\[
j_{ai}^\dagger(k) = -\lambda_i \varphi^\dagger \varphi f_i(k) F_i^\dagger
\]
\[
J_\varphi^\dagger = -\varphi^\dagger (\lambda_1 F_1^\dagger F_1 + \lambda_2 F_2^\dagger F_2)
\]
(the $J$ symbol is used here to avoid confusion with the quite similar equation in Sec.II-A). We omit here and hereafter explicit notation that $i$ takes both values 1 and 2.

Comparing Eqs.(6) and (8) with Eqs.(II-7) and (II-9), we see the only difference is that $i$ subscripts appear in the former but not
the latter. Thus we get the solution for Eq.(6) by taking all the light boson development in Secs.(II-A,B) and adding subscripts appropriately. We caution that there is only one kind of heavy boson projection operator $P_n$ since there is only one kind of heavy boson in this extended model.

Since all the operators related to the $a_1$-boson commute with all those related to the $a_2$-boson, we write the commutation relations with two subscripts, $i$ and $j$, and insert a $\delta_{ij}$ as needed. For the sake of convenience, those commutation relations necessary for the calculation of matrix elements are listed below:

\begin{align*}
[a_i, \nu_{\bar{\kappa}}, a_j, \nu_{\bar{\kappa}'},] &= \delta_{ij} \delta(\bar{\kappa} - \bar{\kappa}') = [a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'}}], \\
[a_i, \nu_{\bar{\kappa}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= 0 = [a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'}}] \quad (10a) \\
[a_i, \nu_{\bar{\kappa}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= \delta_{ij} \{\delta(\bar{\kappa} - \bar{\kappa}') + \frac{\lambda_1 \phi f_i(k) f_j(k') D_{i-1}^{-1}(\omega_i(k))}{\omega_i(k) - \omega_i(k') - i\epsilon}\}, \\
[a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= \delta_{ij} \{\delta(\bar{\kappa} - \bar{\kappa}') + \frac{\lambda_1 \phi f_i(k) f_j(k') D_{i+1}^{-1}(\omega_i(k))}{\omega_i(k) - \omega_i(k') + i\epsilon}\}, \\
[a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= 0 = [a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'}}] \quad (10d) \\
[a_i, \nu_{\bar{\kappa}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= \delta_{ij} \{\delta(\bar{\kappa} - \bar{\kappa}') - 2\pi \lambda_1 \phi f_i(k) f_j(k') D_{i+1}^{-1}(\omega_i(k)) \delta(\omega_i(k) - \omega_i(k'))\}, \\
[a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= \delta_{ij} \{\delta(\bar{\kappa} - \bar{\kappa}') - 2\pi \lambda_1 \phi f_i(k) f_j(k') D_{i+1}^{-1}(\omega_i(k)) \delta(\omega_i(k) - \omega_i(k'))\}, \\
[a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'},}] &= 0 = [a_i, \nu_{\nu_{\bar{\kappa}}}, a_j, \nu_{\nu_{\bar{\kappa}'}}] \quad (10f)
\end{align*}
\[ [a_{i, \text{out}}(\vec{k}), a_{j, \text{in}}(\vec{k}')] = 0 , \quad (10g) \]
\[ [a_{i, \text{in}}(\vec{k}), \phi^+] = -\lambda_i \phi^+ f_i(k) D_{i-1}^{-1}(\omega_i(k)) \int \frac{d \vec{k}' f_i(k') a_i(\vec{k}')}{\omega_i(k') - \omega_i(k) + i\epsilon} , \quad (10h) \]
\[ [a_{i, \text{out}}(\vec{k}), \phi^+] = -\lambda_i \phi^+ f_i(k) D_{i+1}^{-1}(\omega_i(k)) \int \frac{d \vec{k}' f_i(k') a_i(\vec{k}')}{\omega_i(k') - \omega_i(k) - i\epsilon} , \quad (10i) \]
\[ [a_{i, \text{in}}^+(\vec{k}), \phi^+] = -\lambda_i \phi^+ f_i(k) D_{i+1}^{-1}(\omega_i(k)) \int \frac{d \vec{k}' f_i(k') a_i^+(\vec{k}')}{\omega_i(k') - \omega_i(k) + i\epsilon} , \quad (10j) \]
\[ [a_{i, \text{out}}^+(\vec{k}), \phi^+] = -\lambda_i \phi^+ f_i(k) D_{i-1}^{-1}(\omega_i(k)) \int \frac{d \vec{k}' f_i(k') a_i^+(\vec{k}')}{\omega_i(k') - \omega_i(k) - i\epsilon} , \quad (10k) \]
\[ [A_i, n^+ A_j^+, n] = \delta_{ij} / \lambda_i n r_{i, n} , \quad (10l) \]
\[ [A_i, n^+ A_j, n] = 0 , \quad (10m) \]
\[ [A_i, n^+ \phi^+] = 0 = [A_i, n^+ \phi] , \quad (10n) \]
\[ [a_{i, \text{in}}(\vec{k}), A_j^+, n] = 0 = [a_{i, \text{out}}(\vec{k}), A_j^+, n] , \quad (10o) \]
\[ [a_{i, \text{in}}(\vec{k}), A_j, n] = 0 = [a_{i, \text{out}}(\vec{k}), A_j, n] , \quad (10p) \]
\[ [a_i(\vec{k}), A_j^+] = \delta_{ij} f_i(k) / (\omega_i(k) - \mu_i, n) , \quad (10q) \]
\[ [a_i(\vec{k}), A_j, n] = 0 . \quad (10r) \]

The formal solution for the heavy boson presents no difficulty; we simply have to replace \( \lambda F^+ F \) in Eqs. (11-27) and (11-28) by
\[ \lambda_1 F_1^\dagger F_1 + \lambda_2 F_2^\dagger F_2. \] The result for \( \varphi^\dagger(t) \) in terms of \( \varphi_{in}^\dagger \) is

\[ \varphi^\dagger(t) = \{ \mathcal{P} \exp[i\lambda_1 \int dt' \Theta(t-t')F_1^\dagger(t')F_1(t')] \} \]

\[ \times \{ \mathcal{P} \exp[i\lambda_2 \int dt' \Theta(t-t')F_2^\dagger(t')F_2(t')] \} e^{iMt} \varphi_{in}^\dagger \quad (11) \]

written in a form useful in the following.

At first sight, the transformation-theory formalism would appear to give some difficulty; there are now two \( V \)'s, namely, \( V_{1in} \) for the \( a_1 \)-bosons and \( V_{2in} \) for the \( a_2 \)-bosons. However, study of Eq.(11) and Eq.(11.64) show that the proper transformation relation is

\[ (V_{1in}V_{2in})^\dagger \varphi^\dagger = \varphi_{in}^\dagger (V_{1in}V_{2in})^\dagger \quad (12) \]

We note that the order of writing the \( V_{1in}V_{2in} \) product has no significance since \( V_{1in} \) and \( V_{2in} \) commute. Now we see that the transformation-theory formalism for the heavy boson develops just as in Sec.II-C, the only difference being that \( V_{in} \) is replaced by the product \( (V_{1in}V_{2in}) \). The resulting useful commutation relations are

\[ [\varphi_{in}, \varphi_{in}^\dagger] = (V_{1in}V_{2in})^\dagger (V_{1in}V_{2in}) \quad (13a) \]

\[ [\varphi_{in}, \varphi_{in}^\dagger] = 0 \quad (13b) \]

\[ [\varphi^\dagger, \varphi_{in}] = \varphi_{in}^\dagger \quad (13c) \]

\[ [a_{1in}(k), \varphi_{in}^\dagger] = 0 = [a_{1in}(\bar{k}), \varphi_{in}] \quad (13d) \]

\[ [p_{n1in}, \varphi_{in}^\dagger] = p_{nA_{1in}}^\dagger \varphi_{in}^\dagger \quad (13e) \]
\[ [P_n A^\dagger_{i,n}, \phi_{in}] = P_n A^\dagger_{i,n} \phi_{in}, \quad (13f) \]
\[ [\phi_{out}, \phi_{in}^\dagger] = (V_{1out} V_{2out})^\dagger (V_{1out} V_{2out}) \]
\[ \times \mathcal{P} \{ \exp \lambda_1 \int dt F_1(t) \} \mathcal{P} \{ \exp \lambda_2 \int dt F_2(t) \} \mathcal{P} \{ \exp \lambda_3 \int dt F_3(t) \} \mathcal{P} \{ \exp \lambda_4 \int dt F_4(t) \}. \quad (13g) \]

Since \( V_{1in} \) and \( V_{2in} \) are essentially independent, we also have
\[ V_i, \ln V_i^\dagger, \ln = 1, \quad (14a) \]
\[ V_i^\dagger V_i, \ln = 1 - \lambda_i \Delta_n r_i, \ln P_n A^\dagger_{i,n} A_{i,n} \]
\[ + (2^i)^{-1} \lambda_i \Delta_n r_i, \ln P_n A^\dagger_{i,n} A_{i,n} A_i, \ln A_i, \ln + \ldots. \quad (14b) \]

For \( \lambda_i < 0 \), \( V_i, \ln \) is not unitary. If either (or both) \( \lambda_1 \) or \( \lambda_2 \) is negative, then the commutator \( [\phi_{out}, \phi_{in}^\dagger] \) is operator-valued. This indicates an additional richness of composite particles in this extended model.

B. Physical States and Physical Processes

Just as in the simple model, the Hamiltonian of this extended model can be written as
\[ H = M \phi^\dagger \phi + \int d\vec{k} [\omega_1(\vec{k}) a^\dagger_{1ln}(\vec{k}) a_{1in}(\vec{k}) + \omega_2(\vec{k}) a^\dagger_{2ln}(\vec{k}) a_{2in}(\vec{k})] \]
\[ + \sum_n \mu_n [\lambda_1 r_1 \mu_{1n} A^\dagger_n A_{1n} + \lambda_2 r_2 \mu_{2n} A^\dagger_n A_{2n}] \quad (15) \]

we then see that \( a_i^\dagger(\vec{k}) a_i(\vec{k}) \), \( a_i, \ln_a_{1i, \ln a_{1i, \ln} \), \( \phi_{in}^\dagger \phi_{in} \), and \( P_n A^\dagger_{i,n} A_{i,n} \) are all constants of the motion and therefore have physical interpretation in terms of number operators. The most general (in-) state
of this model is

$$\left| (nM, m_1a_1, m_2a_2), n_1(\vec{k}_1), n_2(\vec{k}_2), \ldots, n_\ell(\vec{k}_\ell) \right>$$

$$= (\lambda_1 n_{1r})^{m_1/2} (\lambda_2 n_{2r})^{m_2/2} (n_1m_1!m_2!n_1^!n_2^!\ldots n_\ell^!)$$

$$\times n_1!! n_2!! \ldots n_\ell!!$$

$$\times p_n(A_{1n}^+)^{m_1} (A_{2n}^+)^{m_2} [a_{11n}^+(\vec{k}_1)]^{n_1} [a_{12n}^+(\vec{k}_2)]^{n_2} \ldots [a_{1\ell n}^+(\vec{k}_\ell)]^{n_\ell}$$

$$\times [a_{21n}^+(\vec{k}_1)]^{n_1^!} [a_{22n}^+(\vec{k}_2)]^{n_2^!} \ldots [a_{2\ell n}^+(\vec{k}_\ell)]^{n_\ell^!} (\phi_{1n}^+)^n |\Omega> \quad (16)$$

This state is an eigenstate of $H$ with eigenvalue

$$nM + m_1M_{1n} + m_2M_{2n} + n_1\omega_1(k_1) + n_2\omega_1(k_2) + \ldots + n_\ell\omega_1(k_\ell)$$

$$+ n_1^!\omega_2(k_1^!) + n_2^!\omega_2(k_2^!) + \ldots + n_\ell^!\omega_2(k_\ell^!)$$

of $\int dk a_1^+(\vec{k}) a_1(\vec{k})$ with eigenvalue $m_1 + n_1 + n_2 + \ldots + n_\ell$,

of $\int dk a_2^+(\vec{k}) a_2(\vec{k})$ with eigenvalue $m_2 + n_1^! + n_2^! + \ldots + n_\ell^!$,

of $\int dk a_{11n}^+(\vec{k}) a_{11n}(\vec{k})$ with eigenvalue $n_1 + n_2 + \ldots + n_\ell$,

of $\int dk a_{21n}^+(\vec{k}) a_{21n}(\vec{k})$ with eigenvalue $n_1^! + n_2^! + \ldots + n_\ell^!$,

of $\sum \lambda_{1r} n_{1r} p a_{1n}^+ A_{1n}$ with eigenvalue $m_1$,

of $\sum \lambda_{2r} n_{2r} p a_{2n}^+ A_{2n}$ with eigenvalue $m_2$,

and finally, of $\phi_{1n}^+ \phi_{1n}$ with eigenvalue $\delta(m_1,0)\delta(m_2,0)$
(note that these are Kronecker deltas). Thus we see that this state contains a composite particle of \( m_1 a_1\)-bosons and \( m_2 a_2\)-bosons bound to \( n \) heavy bosons; it also contains \( n_1 + n_2 + \ldots + n_f \) free \( a_1\)-bosons and \( n_1' + n_2' + \ldots + n_r' \) free \( a_2\)-bosons.

We can now see that the various operators in this model have the following meaning:

1. \( \int d\vec{k} a_i^\dagger(\vec{k}) a_i(\vec{k}) \) is the number operator which tells the total number of light bosons of \( i \) kind, both bound and free;

2. \( \int d\vec{k} a_i,_{\text{in}}(\vec{k}) a_i,_{\text{in}}(\vec{k}) \) is the number operator which tells the number of free light bosons of \( i \) kind only;

3. \( \lambda_i \sum \Lambda_{n,\text{in}}^i,_{\text{in}} P A_i^n A_i^n \) is the number operator which tells the number of bound light bosons of \( i \) kind only;

4. \( a_i,_{\text{in}}(\vec{k}) \) and \( a_i,_{\text{in}}(\vec{k}) \) are the creation and annihilation operators, respectively, for a free light boson of \( i \) kind, energy \( \omega_i(\vec{k}) \) and momentum \( \vec{k} \);

5. \( \Sigma A_i^n A_i^n \) and \( \Sigma A_i^n A_i^n \) are the creation and annihilation operators, respectively, for a bound light boson of \( i \) kind (except when acting on the vacuum state).

\( \phi^\dagger \phi \), \( \phi_{\text{in}}^\dagger \phi_{\text{in}} \), \( \phi_{\text{in}}^\dagger \) and \( \phi_{\text{in}} \) have the same interpretation here as for the simple model in Sec.II-D. In this extended model as well, states with more than one composite particle or states with both free heavy bosons and a composite particle are not allowed.
The $S$-matrix element for the elastic scattering of a light boson of kind, momentum $\vec{q}$ and energy $\omega_i(q)$, by static heavy bosons is

$$
\langle nM,(\vec{q})_i; \text{out}\mid nM,(\vec{q})_i; \text{in} \rangle = \langle nM; \text{out}\mid [a_{i,\text{out}}(\vec{q}), a_{i,\text{in}}^\dagger(\vec{q})] \mid nM; \text{in} \rangle
$$

$$
= \delta(\vec{k}-\vec{q}) + 2\pi i \delta(\omega_i(k)-\omega_i(q)) T_{i,n}(\omega_i(q))
$$

(17)

where

$$
T_{i,n}(\omega_i(q)) = \lambda_i n f_i^2(q) \int \frac{dp}{D_{i,n}(\omega_i(p))} \frac{|f_i^2(p)|^2}{\omega_i(p)-\omega_i(q)-i\epsilon}
$$

$$
+ \frac{r_{i,n}}{\mu_{i,n}-\omega_i(q)} - 1
$$

(18)

This is obviously the same matrix element as would obtain were there one kind of light boson contained in the model. For elastic scattering by a composite particle, the $S$-matrix element is

$$
\langle (nM,m_1 a_1,m_2 a_2),(\vec{k})_i; \text{out}\mid (nM,m_1 a_1,m_2 a_2),(\vec{q})_i; \text{in} \rangle
$$

$$
= (\lambda_1 n r_{1n})^m_1 (\lambda_2 n r_{2n})^m_2 (m_1! m_2!)^{-1} \langle nM; \text{out}\mid a_{i,\text{out}}(\vec{k}) a_{i,\text{in}}^\dagger(\vec{q})
$$

$$
\times P_n(A_{1n})^m_1 (A_{2n})^m_2 (A_{1n}^\dagger)^m_1 (A_{2n}^\dagger)^m_2 \mid nM; \text{in} \rangle
$$

$$
= \langle nM,(\vec{k})_i; \text{out}\mid nM,(\vec{q})_i; \text{in} \rangle,
$$

(19)

just as before.
Processes such as
\[ a_1 + (n_H, m_1 a_1, m_2 a_2) \rightarrow a_1 + a_1 + (n_H, (m_1 - 1) a_1, m_2 a_2) \]  (20a)
and
\[ a_1 + (n_H, m_1 a_1, m_2 a_2) \rightarrow a_1 + a_2 + (n_H, m_1 a_1, (m_2 - 1) a_2) \]  (20b)
are not allowed, all particle number quantities being separately conserved. Thus, this model has only elastic scatterings as physical processes.

C. Discussion

Inspection of the foregoing shows that this model can be extended further to include any number of different kinds of light bosons. No mathematical or conceptual difficulties would arise in this generalization of the formalism of Secs.(IV-A,B). A rich variety of composite particles would exist in the generalized model: \( n \) heavy bosons could bind any number of all different kinds of light bosons provided only that the pertinent coupling constants be negative, etc.

Finally, we draw the following conclusions:
1. Scattering experiments cannot distinguish between \( n \) free heavy bosons and any composite particle formed by \( n \) heavy bosons;
2. Composite particles cannot be formed, destroyed, or altered in any way by the physical processes allowed in the (generalized) separable potential model.

Note that these are straight-forward generalizations of the conclusions drawn from the simple separable potential model.
Chapter V  
SEPARABLE POTENTIAL MODEL WITH TWO KINDS OF STATIC HEAVY BOSONS

A. Model and Special Solution

We now study a separable potential model with a single kind of light boson of mass $\mu$ interacting with two kinds of static heavy bosons, the first kind with mass $M_1$ and the second kind with mass $M_2$. The Hamiltonian of this system is

\[
H = M_1 \varphi_1^\dagger \varphi_1 + M_2 \varphi_2^\dagger \varphi_2 + \int dk \, \omega(k) a^\dagger (k) a(k) + H_1,
\]

where for $i = 1, 2$

\[
H_1 = \lambda_1 \varphi_1^\dagger \varphi_1 F_1 F_1 + \lambda_2 \varphi_2^\dagger \varphi_2 F_2 F_2
\]

and

\[
F_i = \int dk f_i(k) a(k),
\]

\[
\omega(k) = (\vec{k}^2 + \mu^2)^{\frac{1}{2}}
\]

$\lambda_1$ is the interaction coupling constant and $f_i(k)$ is the momentum-space shape factor of the potential for the first kind of heavy boson. The operators $\varphi_1^\dagger$ and $\varphi_1$ are creation and annihilation operators, respectively, for static heavy bosons of the first kind. The quantities with subscript 2 have the corresponding significance for static heavy bosons of the second kind. Hereafter, we refer to static heavy bosons of the first and those of the second kind as $\varphi_1$-bosons and $\varphi_2$-bosons, respectively. The $a^\dagger (k)$ and $a(k)$ operators are the light boson creation and annihilation...
operators, respectively. The functions \( f_i(k) \) are real and defined such that
\[
\int dk f_i^2(k) < \infty ; \quad i = 1, 2.
\]
We hereafter omit explicit notation that \( i \) takes both values 1 and 2.

These particle operators obey the commutation relations
\[
\begin{align*}
[a(k), e^+(k')] &= \delta(k-k'), \quad (5a) \\
[a(k), a(k')] &= 0, \quad (5b) \\
[\varphi_i, \varphi_j^+] &= \delta_{ij}, \quad (5c) \\
[\varphi_i, \varphi_j] &= 0, \quad (5d) \\
[a(k), \varphi_i] &= 0 = [a(k), \varphi_i^+]. \quad (5e)
\end{align*}
\]

The equations of motion for the Heisenberg operators are then
\[
\begin{align*}
[i\frac{d}{dt} + \omega(k)] a^+(k, t) &= J_a^+(k, t) \quad (6) \\
[i\frac{d}{dt} + M\varphi_i(t)] \varphi_i^+(t) &= J_{\varphi_i}^+(t) \quad (7)
\end{align*}
\]
where
\[
\begin{align*}
J_a^+(k) &= -\lambda_1 \varphi_i^+ \varphi_1 f_1(k) F_1^+ - \lambda_2 \varphi_2^+ \varphi_2 f_2(k) F_2^+, \quad (8) \\
J_{\varphi_i}^+ &= -\lambda_1 \varphi_i^+ F_1 F_i. \quad (9)
\end{align*}
\]
(the J symbol is used here to avoid confusion with Eq. (11-7)).

Comparing Eqs. (6) and (8) with Eqs. (11-7) and (11-9), we see that the solution for the light boson in this case presents some difficulty.
We first obtain the expression

$$a^\dagger(k, t) = a^\dagger_{in}(k, t) + \int \frac{d\omega \left[ \lambda_1 \phi^\dagger \phi^\dagger f_1(k) f_1^\dagger(\omega) + \lambda_2 \phi^\dagger \phi^\dagger f_2(k) f_2^\dagger(\omega) \right] e^{i(\omega - \omega(k) - i\epsilon)}}{\omega - \omega(k) - i\epsilon}$$

(10)

In a derivation that follows step-by-step the one for Eq.(11-29). We then can write

$$a^\dagger(k, \omega) = a^\dagger_{in}(k) \delta(\omega - \omega(k)) + \frac{\lambda_1 \phi^\dagger \phi^\dagger f_1(k) f_1^\dagger(\omega) + \lambda_2 \phi^\dagger \phi^\dagger f_2(k) f_2^\dagger(\omega)}{\omega - \omega(k) - i\epsilon}.$$  

(11)

We do the same manipulations as before to get

$$D_{1-}(\omega) F_1^\dagger(\omega) + \lambda_2 \phi^\dagger \phi^\dagger G_{-}(\omega) F_2^\dagger(\omega) = \int d\tilde{k} f_1(k) a^\dagger_{in}(k) \delta(\omega - \omega(k)),$$  

(12a)

$$D_{2-}(\omega) F_2^\dagger(\omega) + \lambda_1 \phi^\dagger \phi^\dagger G_{-}(\omega) F_1^\dagger(\omega) = \int d\tilde{k} f_2(k) a^\dagger_{in}(k) \delta(\omega - \omega(k))$$  

(12b)

where

$$D_1(z) = 1 + \lambda_1 \phi^\dagger \phi^\dagger \int dk f_1^2(k)/(\omega(k) - z),$$  

(13a)

$$G(z) = \int dk f_1(k) f_2(k)/(\omega(k) - z).$$  

(13b)

Defining

$$C^\dagger_i(z) = \int d\omega F_i^\dagger(\omega)/(\omega - z)$$  

(14)

such that

$$C^\dagger_{i+}(\omega) - C^\dagger_{i-}(\omega) = 2\pi i F_i^\dagger(\omega).$$  

(15)
and noting that
\[ D_{1+}(\omega) - D_{1-}(\omega) = 2\pi i \lambda \phi_1^+ \phi_1 \int d\vec{k} f_1^2(k) \delta(\omega - \omega(k)) \]  
we substitute \( a_n^+(k) \) as obtained from Eq.(10) with \( t = 0 \) into Eqs.(12a,b) and use these definitions and relations to get
\[
[D_{1+}(\omega) \ C_{1+}^+(\omega) + \lambda_2 \phi_2^{+}\phi_2 G_+(\omega) \ C_{2+}^+(\omega)] \\
- [D_{1-}(\omega) \ C_{1-}^-(\omega) + \lambda_2 \phi_2^{+}\phi_2 G_-(\omega) \ C_{2-}^-(\omega)] \\
= 2\pi i \int d\vec{k} f_1(k) a_1^+(k) \delta(\omega - \omega(k)) ,
\]
(17a)
\[
[D_{2+}(\omega) \ C_{2+}^+(\omega) + \lambda_1 \phi_1^{+}\phi_1 G_+(\omega) \ C_{1+}^+(\omega)] \\
- [D_{2-}(\omega) \ C_{2-}^-(\omega) + \lambda_1 \phi_1^{+}\phi_1 G_-(\omega) \ C_{1-}^-(\omega)] \\
= 2\pi i \int d\vec{k} f_2(k) a_2^+(k) \delta(\omega - \omega(k)) .
\]
(17b)

We recall that all these equations must be interpreted in terms of matrix elements by virtue of the Lehmann-Symanzik-Zimmermann asymptotic conditions; hence, each of the above is an expression for the discontinuity across the cut along the real axis from \( \mu \) to infinity of a complex function. These functions are then
\[
D_1(z) C_1^+(z) + \lambda_2 \phi_2^{+}\phi_2 G(z) C_2^+(z) = \int d\vec{k} f_1(k) a_1^+(k)/(\omega(k) - z) ,
\]
(18a)
\[
D_2(z) C_2^+(z) + \lambda_1 \phi_1^{+}\phi_1 G(z) C_1^+(z) = \int d\vec{k} f_2(k) a_2^+(k)/(\omega(k) - z) \]
(18b)
since \( D_1(z - \infty) \rightarrow 1, D_2(z - \infty) \rightarrow 1, G(z - \infty) \rightarrow 0, C_1^+(z - \infty) \rightarrow - F_1^+ / z \), and \( C_2^+(z - \infty) \rightarrow - F_2^+ / z \).
Solving for $C^+_1(z)$ and $C^+_2(z)$

$$
C^+_1(z) = [D_1(z)D_2(z) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 G_2(z)]^{-1} \times \int \frac{d\vec{k}}{\omega(\vec{k}) - z} [D_2(z)f_1(k) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 G(z)f_2(k)] a^+(\vec{k}) \, , (19a)
$$

$$
C^+_2(z) = [D_1(z)D_2(z) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 G_2(z)]^{-1} \times \int \frac{d\vec{k}}{\omega(\vec{k}) - z} [D_1(z)f_2(k) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 G(z)f_1(k)] a^+(\vec{k}) \, , (19b)
$$

and putting these into Eq.(10) with $t = 0$, we get

$$
a^+_{\text{in}}(\vec{k}) = a^+_{\text{out}}(\vec{k}) + [D_{1+}(\omega)D_{2+}(\omega) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 G_+^\omega(\omega)]^{-1} \times \int d\vec{k}' [\lambda_1 \varphi_1 \varphi_1 f_1(k)f_1(k')D_{2+}(\omega(k)) + \lambda_2 \varphi_2 \varphi_2 f_2(k)f_2(k')D_{1+}(\omega(k)) - \lambda_1 \lambda_2 \varphi_1 \varphi_2 f_1(k)f_2(k')D_{1+}(\omega(k))]

\times \frac{a^+(\vec{k}')}{\omega(k') - \omega(k) - i\epsilon} \, . (20)
$$

The expression in terms of $a^+_{\text{out}}(\vec{k})$ is omitted for the sake of brevity.

Equation (20) is obviously a great deal more complicated than Eq.(11-38). Rather than treat this general case we consider the more tractable special case

$$
f_1(k) = f_2(k) = f(k) ;
$$
We get for this case results quite similar (except for important differences) to those in Sec. (11-8). These results are the following:

1. expressions for $a^\dagger(\vec{k})$

$$a^\dagger(\vec{k}) = a^\dagger_{\text{in}}(\vec{k}) + (\lambda_1 \phi_1^\dagger \phi_1 + \lambda_2 \phi_2^\dagger \phi_2) f(k) \mathcal{D}_-^{-1}(\omega(k)) \int \frac{d\vec{k}'}{2\pi} f(k') a^\dagger(\vec{k}')$$

$$= a^\dagger_{\text{out}}(\vec{k}) + (\lambda_1 \phi_1^\dagger \phi_1 + \lambda_2 \phi_2^\dagger \phi_2) f(k) \mathcal{D}_-^{-1}(\omega(k)) \int \frac{d\vec{k}'}{2\pi} f(k') a^\dagger(\vec{k}')$$

(22a)

(22b)

2. an expression for $a^\dagger_{\text{out}}(k)$ in terms of $a^\dagger_{\text{in}}(k)$

$$a^\dagger_{\text{out}}(\vec{k}) = a^\dagger_{\text{in}}(\vec{k}) + 2\pi i (\lambda_1 \phi_1^\dagger \phi_1 + \lambda_2 \phi_2^\dagger \phi_2) f(k) \mathcal{D}_-^{-1}(\omega(k))$$

$$\times \int d\vec{k}' f(k') \delta(\omega(k) - \omega(k')) a_{\text{in}}(\vec{k}')$$

(23)

3. commutation relations

$$[a_{\text{in}}(\vec{k}), a^\dagger_{\text{in}}(\vec{k}')] = \delta(\vec{k} - \vec{k}') = [a_{\text{out}}(\vec{k}), a^\dagger_{\text{out}}(\vec{k}')]$$

(24a)

$$[a_{\text{in}}(\vec{k}), a_{\text{in}}(\vec{k}')] = 0 = [a_{\text{out}}(\vec{k}), a_{\text{out}}(\vec{k}')]$$

(24b)

$$[a_{\text{in}}(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') + (\lambda_1 \phi_1^\dagger \phi_1 + \lambda_2 \phi_2^\dagger \phi_2) \frac{f(k) f(k') \mathcal{D}_-^{-1}(\omega(k))}{\omega(k) - \omega(k') + i\epsilon}$$

(24c)

$$[a_{\text{out}}(\vec{k}), a^\dagger(\vec{k}')] = \delta(\vec{k} - \vec{k}') + (\lambda_1 \phi_1^\dagger \phi_1 + \lambda_2 \phi_2^\dagger \phi_2) \frac{f(k) f(k') \mathcal{D}_-^{-1}(\omega(k))}{\omega(k) - \omega(k') - i\epsilon}$$

(24d)

$$[a_{\text{in}}(\vec{k}), a(\vec{k}')] = 0 = [a_{\text{out}}(\vec{k}), a(\vec{k}')]$$

(24e)
\[ [a_{\text{out}}(k), a_{\text{in}}(k')] = 0, \] (24h)

\[ [a_{\text{out}}(k), \varphi_i^+] = -\lambda_i \varphi_i^+ f(k) D_+^{-1}(\omega(k)) \int \frac{dk' f(k') a(k')}{\omega(k') - \omega(k) + i\epsilon}, \] (24j)

\[ [a_{\text{in}}(k), \varphi_i^+] = -\lambda_i \varphi_i^+ f(k) D_+^{-1}(\omega(k)) \int \frac{dk' f(k') a(k')}{\omega(k') - \omega(k) - i\epsilon}, \] (24k)

where

\[ D'(z) = 1 + (\lambda_1 \varphi_1^+ \varphi_1 + \lambda_2 \varphi_2^+ \varphi_2) \int dk f^2(k)/(\omega(k) - z). \] (25)

Continuing to follow the pattern of Sec.(11-B), we first note that we must choose states such that $\varphi_1^+ \varphi_1$ is the number operator for heavy bosons of kind; we then can write

\[ D'(z) = \sum D_{nm}'(z) P_{1n} P_{2m} \] (26a)

where $P_{1n}$ is the $n$ $\varphi_1$-boson projection operator, $P_{2m}$ is the $m$ $\varphi_2$-boson projection operator, and

\[ D_{nm}'(z) = 1 + (\lambda_1 n + \lambda_2 m) \int dk f^2(k)/(\omega(k) - z). \] (26b)
For $(\lambda_1 n + \lambda_2 m) < 0$, $D_{nm}^\prime$ can have a single zero on the real axis below $\mu$.

Denoting this zero by $\mu_{nm}$, we have

$$D_{nm}^{\prime-1}(z - \mu_{nm}) = r_{nm}/(z - \mu_{nm}) ,$$

$$\int d\vec{k} f^2(k)/(\omega(k) - \mu_{nm}) = -1/(\lambda_1 n + \lambda_2 m) ,$$

$$\int d\vec{k} f^2(k)/(\omega(k) - \mu_{nm})^2 = 1/(\lambda_1 n + \lambda_2 m) r_{nm} .$$

Following then by detailed analogy with the previous work is the expression for $D_{nm}^{\prime-1}(z)$, we get

$$D_{nm}^{\prime-1}(z) = \sum D_{nm}^{\prime-1}(z) \rho_{1n} \rho_{2m}$$

with

$$D_{nm}^{\prime-1}(z) = 1 - (\lambda_1 n + \lambda_2 m) \int \frac{d\vec{k} \left| D_{nm}^{\prime-1}(\omega(k)) \right|^2 f^2(k)}{\omega(k) - z} - \frac{r_{nm}}{\mu_{nm} - z}$$

we note that $\mu_{nm}$ (and $r_{nm}$) has no particular relationship to $\mu_{n0}$ and $\mu_{0m}$ (and $r_{n0}$ and $r_{0m}$), i.e., the solutions of the model with two kinds of heavy bosons can not be considered in any sense in terms of the solutions of models with only one kind of heavy boson.

We now define the operators

$$A_{nm}^\dagger = \int d\vec{k} f(k) a_{nm}^\dagger(\vec{k})/(\omega(k) - \mu_{nm})$$

but omit as unnecessary the explicit writing out of expressions analogous to Eqs.(11-50a,b). However, we give for convenience the commutation relations

$$[A_{nm}, A_{nm}^\dagger] = 1/(\lambda_1 n + \lambda_2 m) r_{nm} ,$$
corresponding to those of Eqs. (11-52a)-(11-52g).

To obtain results for the \( \phi_i \)-bosons, we compare Eqs. (7) and (9) with Eqs. (11-8) and (11-10) and note that the only difference is the appearance of \( i \) subscripts in the former but not the latter. Hence, the formal solution for \( \phi_i^\dagger(t) \) obtains from the appropriate introduction of \( i \) subscripts in Eqs. (11-27) and (11-28) as
\[
\phi_i^\dagger(t) = [\mathcal{P} \exp[\lambda_i \int_{t'}^t dt' \Theta(t-t')F_i^\dagger(t')F_i(t')]] e^{i M t_i^\dagger} \phi_i,_{\text{in}}, \tag{33a}
\]
\[
= [\mathcal{P}' \exp[-\lambda_i \int_{t'}^t dt' \Theta(t-t')F_i^\dagger(t')F_i(t')]] e^{i M t_i^\dagger} \phi_i,_{\text{out}}. \tag{33b}
\]

The transformation-theory development is essentially identical to that given in Sec. II-C; details including the definition of the operators \( V'_{\text{in}} \) and \( V'_{\text{out}} \) are given in Appendix B. The results of this analysis is the following set of commutation relations:
\[ [\phi_i, \text{in}, \phi_j, \text{in}] = \delta_{ij} V_{\text{in}}^+ V_{\text{in}}, \quad (34a) \]
\[ [\phi_i, \text{in}, \phi_j, \text{in}] = 0, \quad (34b) \]
\[ [\phi_i^+, \phi_i, \text{in}] = \delta_{ij} \phi_i^+, \text{in}, \quad (34c) \]
\[ [a_{\text{in}}(\vec{k}), \phi_i, \text{in}] = 0 = [a_{\text{in}}(\vec{k}), \phi_j, \text{in}], \quad (34d) \]
\[ [P_{\text{in}} P_{2m} A_{nm}^+, \phi_i, \text{in}] = P_{\text{in}} P_{2m} A_{nm}^+ \phi_i, \text{in}, \quad (34e) \]
\[ [P_{\text{in}} P_{2m} A_{nm}^+, \phi_i, \text{in}] = P_{\text{in}} P_{2m} A_{nm}^+ \phi_i, \text{in}, \quad (34f) \]
\[ [\phi_i, \text{out}, \phi_j, \text{in}] = \delta_{ij} V_{\text{out}}^+ V_{\text{out}} \{ \mathbb{P} \exp \left[ i \lambda_i \int \text{d}t F_i^+(t) F_i(t) \right] \}. \quad (34g) \]

The foregoing completes the solution of the model with two kinds of static heavy bosons in the special case of identical form factors.

**B. Physical States and Physical Processes**

In analogy to the work in the previous models, we can write the Hamiltonian for the (special case) separable potential model with two kinds of heavy bosons as

\[ H = M_1 \phi_1^+ \phi_1 + M_2 \phi_2^+ \phi_2 + \int \text{d}k \omega(k) a_{\text{in}}^+(\vec{k}) a_{\text{in}}(\vec{k}) \]
\[ + \sum (\lambda_1 n + \lambda_2 m) p_{nm} p_{nm} P_{\text{in}} P_{2m} A_{nm}^+ A_{nm}. \quad (35) \]

The quantities \( a_{\text{in}}^+(\vec{k}) a(\vec{k}), a_{\text{in}}^+(\vec{k}) a_{\text{in}}(\vec{k}), \phi_i^+, \text{in} \phi_i, \text{in} \) and \( P_{\text{in}} P_{2m} A_{nm}^+ A_{nm} \)
are all constants of the motion and therefore have physical interpretation in terms of number operators. The most general (in-)state of this model is

\[ |(nM_1, n'M_2, \alpha \rangle; n_1(\vec{k}_1), n_2(\vec{k}_2), \ldots, n_\lambda(\vec{k}_\lambda); \Omega \rangle \]

\[ = [((\lambda_1 n + \lambda_2 n') r_{nn'} - 1)^{m/2}(n!n'!m!n_1!n_2! \ldots n_\lambda)!]^{1/2} \]

\[ \times p_{nn'}^m (A_{nn'}^\dagger)^{m_1} [a_{in}(\vec{k}_1)]^{n_1} [a_{in}(\vec{k}_2)]^{n_2} \ldots [a_{in}(\vec{k}_\lambda)]^{n_\lambda} \]

\[ \times (\varphi_{1in}^\dagger)^{n}(\varphi_{2in}^\dagger)^{n'}|\Omega\rangle \] (36)

This state is an eigenstate of \( H \) with eigenvalue \( m + n_1 + n_2 + \ldots + n_\lambda + \sum_\lambda (\lambda_1 n + \lambda_2 n') r_{nn'}^m P_{nn'}^m A_{nn'}^\dagger A_{nn'} \), with eigenvalue \( m \), of \( \varphi_{1in}^\dagger \varphi_{1in} \) with eigenvalue \( n6(m, 0) \), and finally of \( \varphi_{2in}^\dagger \varphi_{2in} \) with eigenvalue \( n'6(m, 0) \). Evidently this general state contains a composite-particle consisting of \( m \) light bosons bound to \( n \varphi_1 \)-bosons and \( n' \varphi_2 \)-bosons; it also contains \( n_1 + n_2 + \ldots + n_\lambda \) free light bosons.

We now see that the various operators in this model have the following meaning:

1. \( \int d\vec{k} a^\dagger(\vec{k}) a(\vec{k}) \) is the number operator which tells the total number of light bosons, both bound and free;
2. $\int d\vec{k} a_{\vec{k}}^\dagger a_{\vec{k}}$ is the number operator which tells the number of free light bosons only;
3. $\Sigma (\lambda_1 n + \lambda_2 n') r_{nn'} P_{1n} P_{2n'} A_{nn'}^\dagger A_{nn'}$ is the number operator which tells the number of bound light bosons only;
4. $\varphi_i^\dagger \varphi_i$ is the number operator which tells the total number of heavy bosons of $i$ kind, either bound or free;
5. $\varphi_{i, in}^\dagger \varphi_{i, in}$ is the number operator which tells the number of free heavy bosons of $i$ kind only;
6. $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ are the creation and annihilation operators, respectively, for a free light boson of energy $\omega (\vec{k})$ and momentum $\vec{k}$;
7. $\Sigma P_{1n} P_{2n'} A_{nn'}^\dagger$ and $\Sigma P_{1n} P_{2n'} A_{nn'}$ are the creation and annihilation operators, respectively, for a bound light boson (except when acting on the vacuum state);
8. $\varphi_{i, in}^\dagger$ and $\varphi_{i, in}$ are the creation and annihilation operators, respectively, for a free heavy boson of $i$ kind but only when applied to the vacuum state or a state with no composite particle;
9. $\varphi_{i, in}^\dagger$ and $\varphi_{i, in}$ give zero identically when applied to states with a composite particle.

Here, as well as in the previous models, states with more than one composite particle or states with both free heavy bosons and a composite particle are not allowed.

The $S$-matrix element for the elastic scattering of a light boson of momentum $\vec{q}$ and energy $\omega (q)$, by $n \phi_1$-bosons and $n' \phi_2$-bosons is
\[ \langle nM_1, n'M_2, k; \text{out}\mid nM_1, n'M_2, q; \text{in} \rangle = \delta(k-q) + 2\pi i \delta(\omega(k) - \omega(q)) T_{nn'}(\omega(q)), \quad (37) \]

where

\[ T_{nn'}(\omega(q)) = (\lambda_1 n + \lambda_2 n') f^2(q) \int \frac{dp}{\mu_{nn'} - \omega(q) - i\epsilon} \left[ (\lambda_1 n + \lambda_2 n') \frac{\partial}{\partial \omega(\omega(p))} \right]^2 f^2(p) \]

\[ + \frac{r_{nn'}}{\mu_{nn'} - \omega(q) - 1}. \quad (38) \]

This agrees with Eq. (111-2) in the cases where either \( n \) or \( n' \) is zero. For elastic scattering by a composite particle, the S-matrix element is

\[ \langle (nM_1, n'M_2, m) \mid k; \text{out}\rangle \langle nM_1, n'M_2, m, q; \text{in} \rangle = \left[ (\lambda_1 n + \lambda_2 n') r_{nn'} \right]^{m(m')}^{-1} \]

\[ \times \langle nM_1, n'M_2; \text{out}\rangle a_{\text{out}}(k) a_{\text{in}}(q) \rho_{1n} \rho_{2n} \langle A_{nn'} \rangle^{m(A_{nn'})}^{m} |nM_1, n'M_2; \text{in}\rangle \]

\[ = \langle nM_1, n'M_2, k; \text{out}\rangle \langle nM_1, n'M_2, q; \text{in} \rangle, \quad (39) \]

the familiar and now expected result.

Again, since the various particle numbers are conserved, all processes in which a composite particle would be altered in any way are forbidden. Thus, the only physical process is elastic scattering.
C. Discussion

The separable potential model with two kinds of heavy bosons with the same form factors is similar to the simple model and the model with more than one kind of light boson in that

1. scattering experiments can not distinguish between n free heavy bosons and any composite particle which could be formed by the same n heavy bosons (any mixture of two kinds of heavy bosons allowed);

2. composite particles cannot be formed, destroyed, or altered in any way by the physical processes allowed in the model.

The present model differs, however, with respect to the conditions under which composite particles can exist, the condition being that $D'(z)$ have a zero, i.e.,

$$\lambda_1 n + \lambda_2 m < 0,$$

and

$$|\lambda_1 n + \lambda_2 m| \int d\vec{k} f^2(k)/(\omega(k) - \mu) > 1,$$

where $n$ is the number of $\varphi_1$-bosons and $m$ the number of $\varphi_2$-bosons.

For $\lambda_1 < 0$ and $\lambda_2 < 0$, composite particles can exist for any values of $n$ and $m$ sufficiently large that the above relation is satisfied.

The case with $\lambda_1 < 0$ but $\lambda_2 > 0$ can have composite particles for given $m$ only for $n$ such that

$$n > m|\lambda_2|/|\lambda_1| + [i|\lambda_1| \int d\vec{k} f^2(k)/(\omega(k) - \mu)]^{-1}.$$
contrariwise, for $\lambda_1 > 0$ and $\lambda_2 < 0$, $m$ for given $n$ must satisfy

$$m > n|\lambda_1|/|\lambda_2| + \left[|\lambda_2|\int d\mathbf{k}^2/(\omega(k)-\mu)\right]^{-1}$$

in order for composite particles to be possible.

Finally, we remark that this type of separable potential model can be generalized to the case of any number of different kinds of static heavy bosons. This generalized model can be solved in a manner essentially identical to the foregoing provided that all the form factors are equal.
Chapter VI

CONCLUSION

The discussions in Secs. II-D, IV-C, and V-C emphasize by repetition that the composite particles in the separable potential models studied herein cannot be created, destroyed, or altered by the allowed physical processes. Further, composite particles with a given number of static heavy bosons cannot be distinguished by scattering experiments from the same number of free heavy bosons or composite particles which differ only in the number of bound light bosons (this fact is noted without comment in a footnote on p. 107 of Henley and Thirring\(^3\)). We see, however, that the scattering of a light boson by a target (heavy bosons) does tell whether or not the target can form bound states, there being a contribution in the scattering amplitude due to the possibility of composite particles (bound states).

The composite-particle formalism as developed for each of the models is essentially the following: Certain states, for example \(|(nM, ma)\rangle\), are defined as in Eq. (II-71) by means of particular operator (or operator-function) products applied to the vacuum state. These states have such energies and particle numbers as to require identification as composite-particle states. We understand these states in terms of our simplest idea of what a composite particle should be, that is, two or more particles bound together such that the rest
energy of the total is less than the sum of the rest energies of the
individual (free) particles.

These composite particles could not be considered elementary.

In the Hamiltonian (or Lagrangian) formalism of field theory, the
usual practice is to associate an elementary particle with each type
of field operator which appears in the Hamiltonian (or Lagrangian),
all interaction coupling constants being set equal to zero. An exa-
mination of the Hamiltonian expressions in Eqs. (11-1), (IV-1), and
(V-1) shows that these composite particles cannot be regarded as
elementary with respect to this criterion. The same conclusion follows
from the alternate expressions Eqs. (11-69), (IV-15), and (V-35).

Although an operator can be defined (after Kazes)

\[ B^\dagger(nM,ma) = (\lambda n)_{\frac{m}{2}}(m!n!)^{-\frac{1}{2}}p_n(A_n^\dagger)^m \]

\[ \times \sum_{n'}^{}[(n-n')!]^{-\frac{1}{2}}(\phi_{in})^{n'} + n'[(n-n')!]^{-\frac{1}{2}}(\phi_{in})^{n'}] \]

with the properties

1. \( B^\dagger(nM,ma) \) applied to any in-state not containing a composite
   particle produces the composite particle \( (nM,ma) \) and
2. \( B^\dagger(nM,ma) \) applied to any in-state already containing a
   composite particle gives zero identically;

this certainly could not be mistaken for an operator associated with
an elementary particle. We note also that a state containing \( (nM,ma) \)
could also be produced from a state containing \( (nM,m'a) \), \( m' \neq m \), by the
suitable applications of either $A_n$ or $A_n^\dagger$ operator products, a possibility not incorporated in $B^{\dagger}(nM, ma)$ — this could be included but the resulting expression would be exceedingly awkward. Finally, $B^{\dagger}(nM, ma)$ vanishes when $\lambda$ is set equal to zero; it obviously cannot be regarded as the operator for an elementary particle.

The present study also shows that separable potential models with any number of different kinds of static heavy neutral scalar bosons and any number of different kinds of light neutral scalar bosons are exactly solvable provided that all the different kinds of heavy bosons have identical form factors with respect to the interaction with any single kind of light boson.

Although the $S$-matrix elements in the separable potential model are readily calculable once the commutation relations for the in- and out-operators are known, the calculation of the same results by dispersion-theoretic techniques shows the applicability of such calculational methods in this problem. However, the direct calculation provides information not obtainable by dispersion theory, namely, expressions for the pole $\mu_n$ and its residue $r_n$ as follows from Eqs. (11-45) and (11-46), respectively.
REFERENCES


Appendix A

MISCELLANEOUS CALCULATIONS

I. Derivation of Eqs.(11-40a,b)

From the definition of $C_{\pm}^\dagger(z)$ and $C_{\pm}^\dagger(\omega)$ given by Eqs.(11-37) and (11-33b), respectively, we write

$$C_{\pm}^\dagger(\omega) - C_{\mp}^\dagger(\omega) = D_{\pm}^{-1}(\omega) \int \frac{dkf(k)\phi^\dagger(k)}{\omega(k)-\omega-\frac{i\epsilon}{\Theta}} - D_{\mp}^{-1}(\omega) \int \frac{dkf(k)\phi^\dagger(k)}{\omega(k)-\omega+\frac{i\epsilon}{\Theta}}$$

$$= 2\pi i D_{\mp}^{-1}(\omega)\int dkf(k)\phi^\dagger(k)\delta(\omega-\omega(k))$$

$$+ [D_{\pm}^{-1}(\omega)- D_{\mp}^{-1}(\omega)] \int \frac{dkf(k)\phi^\dagger(k)}{\omega(k)-\omega\mp i\epsilon}$$

(A-1)

by use of the relation

$$(\omega(k)-\omega\pm i\epsilon)^{-1} = (\omega(k)-\omega\mp i\epsilon)^{-1} \mp 2\pi i \delta(\omega-\omega(k)).$$

With the help of Eqs.(11-48) and (11-49) we see that Eq.(A-1) can be developed for two cases, i.e.,

a. $\omega > \mu$

$$C_{\pm}^\dagger(\omega) - C_{\mp}^\dagger(\omega) = 2\pi i D_{\mp}^{-1}(\omega)\int dkf(k)$$

$$x \{ a^\dagger(k) - \lambda \phi^\dagger \phi f(k) D_{\pm}^{-1}(\omega(k)) \int \frac{dk^\prime f(k^\prime) a^\dagger(k^\prime)}{\omega(k^\prime)-\omega(k)\mp i\epsilon} \} \delta(\omega-\omega(k)),$$

$$= 2\pi i \times \{ a^\dagger(k) - \lambda \phi^\dagger \phi f(k) D_{\pm}^{-1}(\omega(k)) \int \frac{dk^\prime f(k^\prime) a^\dagger(k^\prime)}{\omega(k^\prime)-\omega(k)\mp i\epsilon} \} \delta(\omega-\omega(k)),$$

$$= 2\pi i D_{\pm}^{-1}(\omega)\int dkf(k)\phi^\dagger(k)\phi(\omega-\omega(k))$$

(A-2b)

where Eqs.(11-38) and (11-39) and the property $|D_{\pm}^{-1}(\omega)|^2 = D_{\pm}^{-1}(\omega)D_{\mp}^{-1}(\omega)$ are used;
b. $\omega < \mu$

\[
C_\uparrow(\omega) - C_\downarrow(\omega) = (D_+^{-1}(\omega) - D_-^{-1}(\omega)) \int \frac{d\k^2 f(k) a^\dagger(\k)}{\omega(k) - \omega}
\]

\[
= -2\pi i \sum_n P_n A_n^\dagger \delta(\omega - \mu_n)
\]

(A-2c)

where the definition of $A_n^\dagger$, Eq.(11-51), is used.

2. Derivation of Eq.(11-42a)

Using the definition of $a_{in}^\dagger(\k)$, Eq.(11-38), and the $a(\k)$ commutation relations, we write

\[
[a_{in}(\k), a_{in}^\dagger(\k')] = \delta(\k - \k')
\]

\[
\lambda \varphi^\dagger \varphi f(k) f(k') [D_+^{-1}(\omega(k')) - D_-^{-1}(\omega(k))]
\]

\[
- \frac{\omega(k) - \omega(k')}{\omega(k) - \omega(k') - i\epsilon}
\]

\[
+ (\lambda \varphi^\dagger \varphi)^2 f(k) f(k') D_-^{-1}(\omega(k)) D_+^{-1}(\omega(k'))
\]

\[
\times \int (\omega(k') - \omega(k) + i\epsilon)(\omega(k') - \omega(k') - i\epsilon)
\]

(A-3)

The property

\[
D_+^{-1}(\omega(k')) - D_-^{-1}(\omega(k)) = D_-^{-1}(\omega(k)) D_+^{-1}(\omega(k')) [D_-(\omega(k)) - D_+(\omega(k'))]
\]

and Eq.(11-36a) allow the second term in Eq.(A-3) to be written as

\[
- (\lambda \varphi^\dagger \varphi)^2 f(k) f(k') D_-^{-1}(\omega(k)) D_+^{-1}(\omega(k'))
\]

\[
\times \int \frac{d\k^2 f(k')}{(\omega(k') - \omega(k) + i\epsilon)(\omega(k') - \omega(k') - i\epsilon)}
\]

which cancels the third term in Eq.(A-3).
3. Derivation of Eq. (11-52d)

From the definition of \( a_{\text{in}}(\overrightarrow{k}) \) given by Eq. (11-38) and the commutation relation

\[
[a(\overrightarrow{k}), A_n] = f(k)/(\omega(k) - \mu_n)
\]

we write

\[
[a_{\text{in}}(\overrightarrow{k}), A_n^+] = \frac{f(k)}{\omega(k) - \mu_n} \left[ 1 - \lambda \eta D^{-1}_-(\omega(k)) (\omega(k) - \mu_n) \right]
\]

\[
\quad \times \int \frac{dk'}{\omega(k') - \omega(k) + i\epsilon}(\omega(k') - \mu_n)
\]

Equations (11-44) and (11-45) and the relation

\[
\frac{\omega(k) - \mu_n}{(\omega(k') - \omega(k) + i\epsilon)(\omega(k') - \mu_n)} = \frac{1}{\omega(k') - \omega(k) + i\epsilon} - \frac{1}{\omega(k) - \mu_n}
\]

are then used to write Eq. (A-4) as

\[
[a_{\text{in}}(\overrightarrow{k}), A_n^+] = \frac{f(k)}{\omega(k) - \mu_n} \left[ 1 - \lambda \eta D^{-1}_-(\omega(k)) \left( \frac{D_-(\omega(k)) - 1}{\lambda \eta} + (\lambda \eta)^{-1} \right) \right]
\]

\[= 0.\]

In the same way, we get \([a_{\text{out}}(\overrightarrow{k}), A_n^+] = 0\).

4. Derivation of Eq. (11-55b)

From Eqs. (11-54a, b, c) we write

\[
\left[ (1 + \xi_{\text{in}}^+) (1 + \xi_{\text{in}}) \right]_{\overrightarrow{k}, \overrightarrow{k}'} = \delta(\overrightarrow{k} - \overrightarrow{k}') + (\xi_{\text{in}}^+)_{\overrightarrow{k}, \overrightarrow{k}'} + (\xi_{\text{in}})_{\overrightarrow{k}', \overrightarrow{k}} + (\xi_{\text{in}}^+)_{\overrightarrow{k}', \overrightarrow{k}} + (\xi_{\text{in}})_{\overrightarrow{k}, \overrightarrow{k}'}
\]
By means of the definition of $D^{-1}(z)$, Eq. (11.48), this can be written as

$$
\left[ (1+\xi_{in}^{+})(1+\xi_{in}) \right]_{k'k}^{-1}
$$

$$
= \delta(k-k') - \frac{\lambda \phi^+ \phi f(k) f(k')}{\omega(k')-\omega(k) - i\epsilon} \sum_{n} r_{n} p_{n} \left[ \frac{1}{\omega(k) - \mu_n + i\epsilon} - \frac{1}{\omega(k') - \mu_n - i\epsilon} \right]
$$

$$
- \left( \frac{\lambda \phi^+ \phi}{\omega(k')-\omega(k) - i\epsilon} \right)^2 \int d\overrightarrow{k'} |D_{-1}^{-1}(\omega(k'))|^2 f^2(k')
$$

$$
\times \left[ \frac{1}{\omega(k) - \omega(k') + i\epsilon} - \frac{1}{\omega(k) - \omega(k') - i\epsilon} \right]
$$

$$
+ \left( \frac{\lambda \phi^+ \phi}{\omega(k')-\omega(k) - i\epsilon} \right)^2 \int d\overrightarrow{k'} |D_{-1}^{-1}(\omega(k'))|^2 f^2(k')
$$

$$
\frac{1}{(\omega(k')-\omega(k') - i\epsilon)(\omega(k) - \omega(k') + i\epsilon)}
$$

$$
= \delta(k-k') - \lambda f(k) f(k') \Sigma r_{n} p_{n} / (\omega(k) - \mu_n)(\omega(k') - \mu_n).
$$

(11-55b)

5. Derivation of Eqs. (11-60a,b)

With the help of Eqs. (11.57) and (11.59), we calculate as

$$
[V_{in}, V_{in}^{\dagger}, a^{\dagger}(\overrightarrow{k})] = [V_{in}, a^{\dagger}(\overrightarrow{k})] V_{in}^{\dagger} + V_{in} [V_{in}^{\dagger}, a^{\dagger}(\overrightarrow{k})]
$$

$$
= a^{\dagger}(\overrightarrow{k'}) (\xi_{in})^{+}_{k'k} V_{in}^{\dagger} + V_{in} a^{\dagger}(\overrightarrow{k'}) (\xi_{in})^{+}_{k'k} V_{in}
$$

$$
= a^{\dagger}(\overrightarrow{k'}) (\xi_{in})^{+}_{k'k} V_{in}^{\dagger} + a^{\dagger}(\overrightarrow{k'}) V_{in} (\xi_{in})^{+}_{k'k} V_{in}
$$

$$
+ a^{\dagger}(\overrightarrow{k'}) (\xi_{in})^{+}_{k'k} V_{in} (\xi_{in})^{+}_{k'k} V_{in}^{\dagger}
$$

(11-55b)
by virtue of Eq. (11-55a).

With the help of Eqs. (11-57) and (11-59), we write

\[ [V_{in}^\dagger, a^{\dagger}(\vec{k})] = [V_{in}^\dagger a^{\dagger}(\vec{k}') (\xi_{in})_{\vec{k}'\vec{k}}^{-1} + a^{\dagger}(\vec{k}') V_{in}^\dagger (\xi_{in})_{\vec{k}'\vec{k}}^{-1}] V_{in} \]

\[ = a^{\dagger}(\vec{k}') V_{in}^\dagger \left[ (\xi_{in})_{\vec{k}'\vec{k}}^{-1} + (\xi_{in})_{\vec{k}'\vec{k}}^{-1} + (\xi_{in})_{\vec{k}'\vec{k}}^{-1} - (\xi_{in})_{\vec{k}'\vec{k}}^{-1} \right] V_{in} \]

This can then be expressed by means of Eq. (11-55b) and (11-51) as

\[ [V_{in}^\dagger, a^{\dagger}(\vec{k})] = -\lambda f(k) \left[ \sum_{n} \int \frac{dk_{1} f(k') a^{\dagger}(\vec{k})}{(\omega(k') - \mu_{n})(\omega(k') - \mu_{n})} \right] V_{in}^\dagger V_{in} \]

\[ = -\lambda f(k) \left[ \sum_{n} \frac{P_{n} A_{n}^{\dagger}}{(\omega(k') - \mu_{n})} \right] V_{in}^\dagger V_{in} \quad (11-60b) \]

6. Derivation of Eq. (11-63)

From Eq. (11-51) and the Hermitian conjugate of Eq. (11-59) we write

\[ [P_{n} A_{n}, V_{in}^\dagger] = V_{in}^\dagger \int \frac{dk_{1} f(k) a^{\dagger}(\vec{k})}{\omega(k') - \mu_{n}} \]

\[ = V_{in}^\dagger \left\{ -\lambda n \int \frac{dk_{1} f^{2}(k)}{\omega(k') - \mu_{n}} \int \frac{dk_{1} f(k') a(k')}{\omega(k) - \omega(k') - i\epsilon} D_{+}^{-1}(\omega(k')) \right\} \]

where \((\xi_{in})_{\vec{k}'\vec{k}}\) is given by Eq. (11-54a) with \(k\) and \(k'\) interchanged.

Inserting the relation

\[ -\left[ (\omega(k') - \mu_{n})(\omega(k') - \omega(k') - i\epsilon) \right]^{-1} = (\omega(k') - \mu_{n})^{-1} \left[ \frac{1}{\omega(k') - \mu_{n}} - \frac{1}{\omega(k) - \omega(k') - i\epsilon} \right] \]

in the above we obtain
\[ [p_n A_n, v_{in}^+] = v_{in}^+ p_n \{ \lambda n \int \frac{dk f^2(k)}{\omega(k) - \mu_n} \int \frac{dk' f(k') a(k')}{\omega(k') - \mu_n} D_+^{-1}(\omega(k')) - \lambda n \int \frac{dk' f(k') a(k')}{\omega(k') - \mu_n} D_+^{-1}(\omega(k')) \int \frac{dk f^2(k)}{\omega(k) - \omega(k') - i\epsilon} \} \]

\[ = -v_{in}^+ p_n \int \frac{dk' f(k') a(k')}{\omega(k') - \mu_n} \]

\[ = -v_{in}^+ p_n A_n \quad (11-63) \]

After use of the definitions \(D(z)\), Eq.(11-44), and \(A_n\), Eq.(11-51).

Finally, from Eq.(11-63) we note

\[ p_n A_n v_{in} = 0 = v_{in}^+ p_n A_n, \]

an extremely significant relation with respect to possible states containing composite particles.

7. Derivation of Eq.(11-68)

From the definition of \(a_{in}^+(k)\), Eq.(11-38), we write

\[ \int d\tilde{k} \omega(k) a_{in}^+(k) a_{in}(\tilde{k}) \]

\[ = \int d\tilde{k} \omega(k) a^+(k) a(k) \]

\[ + \lambda \varphi^+ \varphi \int d\tilde{k}' d\tilde{k}'' f(k') f(k'') a^+(k') a(k'') [\omega(k'') D_+^{-1}(\omega(k'')) - \omega(k') D_+^{-1}(\omega(k'))] \]

\[ + (\lambda \varphi^+ \varphi)^2 \int d\tilde{k} \omega(k) |D_+^{-1}(\omega(k))|^2 f^2(k) \]

\[ \times \int \frac{dk' f(k') a(k')}{\omega(k') - \omega(k) - i\epsilon} \int \frac{dk'' f(k'') a(k'')}{\omega(k'') - \omega(k) + i\epsilon} \]

The definition of $D^{-1}(z)$, Eq. (11-48), allow this to be written as

$$
\int d\vec{k} \omega(k) a^\dagger(\vec{k}) a(\vec{k})
= \int d\vec{k} \omega(k) a^\dagger(\vec{k}) a(\vec{k})
+ \lambda \varphi^\dagger \varphi \sum \frac{d\vec{k}'}{d\vec{k}'} f(k') a^\dagger(k') a(k') [\omega(k') - \omega(k''') + i\epsilon]
+ \lambda \varphi^\dagger \varphi \sum \frac{d\vec{k}'}{d\vec{k}'} f(k') a^\dagger(k') a(k')
\times \Sigma \frac{1}{\omega(k') - \mu_n + i\epsilon} - \frac{1}{\omega(k') - \mu_n - i\epsilon}
- (\lambda \varphi^\dagger \varphi)^2 \int d\vec{k} |D^{-1}_+(\omega(k))|^2 f^2(k) \int d\vec{k} d\vec{k}' f(k') f(k') a^\dagger(k') a(k')
\times \frac{\omega(k')}{\omega(k) - \omega(k') + i\epsilon} - \frac{\omega(k')}{\omega(k) - \omega(k') - i\epsilon}
+ (\lambda \varphi^\dagger \varphi)^2 \int d\vec{k} \omega(k) |D^{-1}_-(\omega(k))|^2 f^2(k)
\times \int \frac{d\vec{k}' f(k') a^\dagger(k')}{\omega(k') - \omega(k) - i\epsilon} \int \frac{d\vec{k}' f(k') a(k')}{\omega(k') - \omega(k) + i\epsilon}
$$

The fourth and fifth terms cancel so that

$$
\int d\vec{k} \omega(k) a^\dagger(\vec{k}) a(\vec{k})
= \int d\vec{k} \omega(k) a^\dagger(\vec{k}) a(\vec{k}) + \lambda \varphi^\dagger \varphi F^\dagger F - \lambda \varphi^\dagger \varphi \Sigma \mu_n \Gamma_n a^\dagger(n) a(n).
$$

(11-68)
8. Derivation of Eq. (III-2)

From the relation

\[ V_{in}^\dagger V_{in} = \phi = V_{out}^\dagger V_{out} \]

we write

\[
\begin{align*}
\langle nM; out | n'M; in \rangle &= (n!)^{-\frac{1}{2}} \langle \Omega | (\phi_{out})^n | n'M; in \rangle \\
&= (n!)^{-\frac{1}{2}} \langle \Omega | V_{out} (\phi_{out})^n V_{out}^\dagger | n'M; in \rangle \\
&= (n!)^{-\frac{1}{2}} \langle \Omega | V_{in} (\phi_{in})^n V_{in}^\dagger | n'M; in \rangle \\
&= (n!)^{-\frac{1}{2}} \langle \Omega | (\phi_{in})^n | n'M; in \rangle \\
&= \langle nM; in | n'M; in \rangle \\
&= \delta_{n'n'}
\end{align*}
\]

where we have used the fact that

\[ V_{in} | nM \rangle = V_{in}^\dagger | nM \rangle = | nM \rangle = V_{out} | nM \rangle = V_{out}^\dagger | nM \rangle \]

evident from definitions. We note also that a similar procedure applies to composite-particle states as

\[ \langle (nM, m) ; out | (n'M, m') ; in \rangle \]

\[
\begin{align*}
&= (\lambda n_r)^{m/2} \langle nM; out | P_n (A_n)^m | n'M, m'; in \rangle \\
&= (\lambda n_r)^{m/2} \langle nM; in | P_n (A_n)^m | n'M, m'; in \rangle \\
&= \delta_{n'n'} \delta_{m'm'}
\end{align*}
\]
Appendix B

TRANSFORMATION-THEORY FORMALISM FOR MODEL
WITH TWO KINDS OF HEAVY BOSONS WITH EQUAL FORM FACTORS

The transformation-theory formalism given here closely follows
that of Sec. I I -C. We write by comparison with Eq. (V-22a)

\[ a_{in}^{\dagger}(k) = a_{in}(k') (1 + \zeta_{in}^{\dagger})_{k'k} \]  \hspace{1cm} (B-1)

where repeated labels on the right imply integration, and

\[ (\zeta_{in}^{\dagger})_{k'k} = \frac{(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) f(k) f(k') D_{-1}^{-1}(\omega(k))}{\omega(k') - \omega(k) - i\epsilon} . \]  \hspace{1cm} (B-2)

From the commutation relation for \( a_{in}(k) \) and \( a_{in}^{\dagger}(k) \), Eq. (V-24a), and
the above expressions we obtain

\[ [(1 + \zeta_{in}^{\dagger}) (1 + \zeta_{in}^{\dagger})]_{k'k} = \delta(k - k') , \]  \hspace{1cm} (B-3a)

\[ [(1 + \zeta_{in}^{\dagger}) (1 + \zeta_{in}^{\dagger})]_{k'k} = \delta(k - k') - f(k) f(k') \sum_{nm} \frac{(\lambda_1 + \lambda_2 \varphi_2) r_{nm} p_{1m} p_{2n}}{\omega(k) - \mu_{nm} (\omega(k') - \mu_{nm})} . \]  \hspace{1cm} (B-3b)

Next we define an operator \( V_{in}^{\dagger} \) such that

\[ V_{in}^{\dagger} a_{in}(k) = a_{in}(k) V_{in}^{\dagger} \]  \hspace{1cm} (B-4)

from which follows, with the help of Eq. (B-1), that

\[ [V_{in}^{\dagger}, a_{in}(k)] = a_{in}(k') \zeta_{in}^{\dagger}_{k'k} V_{in}^{\dagger} \]  \hspace{1cm} (B-5)

with solution
Taking the Hermitean conjugate of Eq. (B-6), we obtain

\[ V_{in}' = 1 + a^\dagger(k)a(k')\xi_{in}'_{kk'}^{\dagger}, \]

\[ + (2!)^{-1} a^\dagger(k) a^\dagger(k') a(k') a(k^\dagger) \xi_{in}'_{kk'}^{\dagger}(\xi_{in}'_{kk'}^{\dagger})^{(i5)}_{j5,j5} + ... \]

which satisfies the relation

\[ [V_{in}', a^\dagger(k)] a^\dagger(k')\xi_{in}'_{kk'}^{\dagger}V_{in}' \]

From Eq. (B-8) follows

\[ V_{in}' a^\dagger_{in}(k) = a^\dagger(k)V_{in}' \]

Finally, with the help of Eqs. (B-4), (B-5), (B-8), and (B-9), we get

\[ [V_{in}', V_{in}', a^\dagger(k)] = 0, \]

\[ [V_{in}', V_{in}', a^\dagger(k)] = -f(k) \Sigma \frac{(\lambda_1 n + \lambda_2 m)}{\omega(k) - \mu_{nm}} r_{nm} P_{1n} P_{2m} A^\dagger a^\dagger \cdot \]

As in the previous work, we have

\[ V_{in}' V_{in}' = 1, \]
\[ V_{in}^\dagger V_{in} = 1 - \sum (\lambda_1 + \lambda_2^m) r_{nm} p_{ln} p_{2m} A_{nm}^\dagger A_{nm} \]

\[ + (2!)^{-1} \sum (\lambda_1 + \lambda_2^m)^2 r_{nm} p_{ln} p_{2m} A_{nm}^\dagger A_{nm}^\dagger A_{nm} A_{nm} + \ldots \]  

(B-11b)

We note that the relation Eq.(11-65) also applies to this extended model, namely

\[ \phi_{i, in}^\dagger = V_{in}^\dagger \phi_{i, in}^\dagger V_{in} ; \quad i = 1, 2. \]  

(B-12)

Note also, that both \( \phi_1^\dagger \) and \( \phi_2^\dagger \) are transformed by the same operator \( V_{in}^\dagger \); however, the structure of the operator \( V_{in}^\dagger \) as evident from Eq.(8-7) shows that \( V_{in}^\dagger \) can act differently on \( \phi_1^\dagger \) than on \( \phi_2^\dagger \).
VITA

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Major Field: Physics

Title of Thesis: Composite Particles in a Separable Potential Model

Approved:

[Signatures of Major Professor and Chairman, Dean of the Graduate School]

EXAMINING COMMITTEE:

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